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A Lie Transformation Group Attached to a Second Order Elliptic Operator

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Abstract. Given a second order elliptic differential operator L on a compact C^∞ manifold, we prove that the group of transformations which preserve the sheaf of functions annuled by L is a Lie transformation group under the compact-open topology.

Contents

1. Introduction

It is well known that for a given manifold, the group of diffeomorophisms which preserve some geometric structure is often a Lie transformation group. For instance, in [3], the authors deduce from a famous theorem of Palais a large list of results of this kind. In this note, where all objects are assumed to be of class C^∞, we consider, on a compact manifold M, a partial differential operator which has in any local coordinates the form

$$L = a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + b^k \frac{\partial}{\partial x^k} + c$$

with the following conditions:

- the coefficients a^{ij}, b^k and c are C^∞ functions,
- the matrix (a^{ij}) is symmetric and positive definite,
- the function c has negative values.

Using the Bochner-Montgomery Theorem [2] on the group of differentiable transformations, we study the case where the geometric structure considered is the sheaf $KerL$. More precisely, for $i = 1, 2$, let M_i be an m_i-manifold provided with a partial differential operator L_i satisfying the three conditions above. We introduce the

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subset $E_L(M_1, M_2)$ of all maps $f \in C^0(M_1, M_2)$ such that for any open subset U_2 of M_2 and $\varphi \in C^\infty(U_2)$ satisfying $L_2\varphi = 0$ on U_2, the composite function $\varphi \circ f$ also satisfies $L_1(\varphi \circ f) = 0$ on $f^{-1}(U_2)$. In the case $M_1 = M_2 = M$, we shall consider the group $E_L(M)$ of all homeomorphisms of M such that for any pair (U_2, φ) as above, $L\varphi = 0$ on U_2 if and only if $L(\varphi \circ f) = 0$ on $f^{-1}(U_2)$. In the next section we shall prove the following results:

Proposition 1.1. *When M_1 is compact, $E_L(M_1, M_2)$ provided with the compact-open topology is locally compact and contained in $C^\infty(M_1, M_2)$.*

Proposition 1.2. *$E_L(M)$ is a Lie transformation group under the compact-open topology.*

2. Proofs

First, we give two technical lemmas.

Lemma 2.1. *For any point p there exists a coordinate system $\{x^i\}_{i=1}^m$ defined on a neighborhood U such that*

$$Lx^i \equiv 0 \quad \textit{for } i = 1, \dots, m.$$

Proof. By the imposition of a suitable coordinate system, we may assume that $p = 0$, M is a neighborhood of p in $\mathbf{R}^m$ and $L = a^{kl}\dfrac{\partial^2}{\partial x^k \partial x^l} + b^j \dfrac{\partial}{\partial x^j} + c$ with $a^{ll}(x) \neq 0$ for all x in M and $l = 1, \dots, m$. For $i = 1, \dots, m$, we prove the existence of a function u^i such that $Lu^i = 0$, $u^i(0) = 0$ and $d_0 u^i = dx^i$. To do this, we apply the theorem of Hörmander given in the appendix of [4] and since L is elliptic, the solutions u^i are C^∞ and give a coordinate system. □

The next lemma seems classical under other forms.

Lemma 2.2. *Let $L = a^{kl}\dfrac{\partial^2}{\partial x^k \partial x^l} + b^j \dfrac{\partial}{\partial x^j} + c$ be an elliptic differential operator on an open subset W of $\mathbf{R}^m$, with C^∞ coefficients, and suppose that the function c is negative. Let $\{f_n\}_{n \in \mathbf{N}}$ be a sequence of continuous functions on W satisfying*

(a) *$Lf_n = 0$ for all $n \in \mathbf{N}$.*
(b) *There exists $C > 0$ such that $|f_n(x)| \leq C$ for all $x \in W$ and $n \in \mathbf{N}$.*

Then

(i) *$f_n. \in C^\infty(W)$ for all $n \in \mathbf{N}$.*
(ii) *One can extract a subsequence $\{f_{n_k}\}_{k \geq 1}$ which converges in $C^\infty(W)$.*

Proof. Assertion (i) follows from the ellipticity of L. Let us prove assertion (ii). Let K be a compact subset of W and $\varepsilon > 0$ such that the closed ball $\overline{B(p, \varepsilon)} = \{|x - p| \leq \varepsilon\}$ is contained in W whenever $p \in K$. Given a point $p \in K$, we consider the linear map S from $C^0(\partial B(p, \varepsilon))$ into $C^\infty(B(p, \varepsilon))$ which sends $\lambda \in C^0(\partial B(p, \varepsilon))$ on the unique solution of the Dirichlet problem:

$$Lu = 0 \text{ on } B(p, \varepsilon); \quad u = \lambda \text{ on } \partial B(p, \varepsilon).$$

In fact S is continuous. Indeed, if $\{\lambda_n\}$ converges to λ in $C^0(\partial B(p, \varepsilon))$ then, by the maximum principle $\{S\lambda_n\}$ converges to $S\lambda$ uniformly on $B(p, \varepsilon)$; this makes sure that S is closed and one can apply the closed graph theorem. Now, by our hypothesis the sequence $\{\lambda_n = f_n |_{\partial B(p,\varepsilon)}\}$ is bounded and $f_n = S\lambda_n$; thus, the sequence $\{f_n\}$ is bounded in the Montel space $C^\infty(B(p,\varepsilon))$ and one can extract a converging subsequence. Since K is compact, one can find a neighborhood U of K and a subsequence which converges in $C^\infty(U)$. But W is a countable union of compact subsets, and so the classical diagonal method gives the conclusion. □

Proofs of Propositions 1.1 and 1.2. Since the two manifolds are metrizable, the compact-open topology is equivalent to the one of the uniform convergence on M_1. By Lemma 2.1, we have an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of M_2 where the U_λ are relatively compact, such that on any U_λ there is a coordinate system $y_\lambda = \left\{y_\lambda^\beta\right\}_{1\leq\beta\leq m_2}$ satisfying

$$L_2 y_\lambda^\beta = 0 \ \text{ on } \ U_\lambda \quad \text{and there exists } C_\lambda > 0 \ \text{ with } \ \left|y_\lambda^\beta\right| \leq C_\lambda \ \text{ on } \ U_\lambda.$$

By classical topology, there exists a second open covering $\{U'_\lambda\}_{\lambda\in\Lambda}$ of the paracompact manifold M_2 with the inclusions $\overline{U'_\lambda} \subseteq U_\lambda$. For $\overline{f} \in E(M_1, M_2)$ we put $V_\lambda = \overline{f}^{-1}(U'_\lambda)$; from the compactness of M_1 we deduce a finite part Λ' of Λ and an open covering $\{V'_\lambda\}_{\lambda\in\Lambda'}$ of M_1 such that $\overline{V'_\lambda} \subseteq V_\lambda$ for all $\lambda \in \Lambda'$. Clearly, the subset

$$\Omega = \left\{f \in E_L(M_1, M_2) \mid f(\overline{V'_\lambda}) \subseteq U'_\lambda \text{ for all } \lambda \in \Lambda'\right\}$$

is a neighborhood of $\overline{f}$ in $E_L(M_1, M_2)$ in the compact-open topology. Let $\{h_n\}_{n\geq 1}$ be a sequence in Ω. Let d be a metric on M_1 and $\varepsilon > 0$ such that for any $p \in M_1$ the ball $\overline{B(p,\varepsilon)} = \{d(x,p) \leq \varepsilon\}$ is contained in the intersection of some chart domain with some $V'_{\lambda(p)}$ ($\lambda(p) \in \Lambda'$). Given a point $p \in M_1$, we apply Lemma 2.2 to each of sequences $\left\{y^\beta_{\lambda(p)} \circ h_n\right\}_{n\geq 1}$ and we obtain a subsequence $\{y_{\lambda(p)} \circ h_{n_k}\}$ which converges uniformly on $B(p,\varepsilon)$. As constructed, the limit has its values in $y_{\lambda(p)}(U_{\lambda(p)})$ and by composition we get a subsequence of $\{h_n\}$ which converges uniformly on $B(p,\varepsilon)$. The first part of the proposition results from the compactness of M_1 and the obvious fact that $E_L(M_1, M_2)$ is closed in $C^0(M_1, M_2)$ provided with the compact-open topology. The ellipticity of L_1 gives the second part.

For Proposition 1.2, $E_L(M)$ is obviously a group. Furthermore, we know from a result of Arens [1] that it is a topological group under the compact-open topology. Moreover, we can deduce easily from Proposition 1.1 that it is locally compact and the Bochner-Montgomery Theorem [2] gives the conclusion.

Corollary 2.3. *Suppose that the manifold M is compact. Then the group $E'_L(M)$ of all homeomorphisms $f \in C^0(M,M)$ such that for any open subset U of M and $\varphi \in C^\infty(U)$ we have:*

$$(L\varphi) \circ f = L(\varphi \circ f) \ \textit{on} \ f^{-1}(U)$$

is a Lie transformations group under the compact-open topology.

Proof. Since the linear differential operators are continous on the distributions, one can verify that $E'_L(M)$ is a closed subgroup in $E_L(M)$ and use the Cartan Theorem. □

Remark 2.4. 1. In Proposition 1.2, the compactness of M is not necessary.
2. $E'_L(M)$ may be a proper subgroup of $E_L(M)$

Firstly, if L is the Laplace-Beltrami operator of a Riemannian manifold M, $E_L(M)$ is the group of conformal transformations which is a Lie transformation group when $m \geq 3$ (see [5, p. 310]). Secondly, when M is the euclidian space $\mathbf{R}^m$, $E'_L(M)$ is the group of isometries.

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