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A study on the PDEs with power-law nonlinearity

A. Molabahrami [a,b,*], A. Shidfar [a]

[a] Department of Mathematics, Iran University of Science and Technology, Tehran, Iran

[b] Department of Mathematics, Ilam University, Ilam, Iran

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ABSTRACT

In this paper, we introduce two types of PDEs with power-law nonlinearity which contain many types of the linear and nonlinear PDEs. For solving the Cauchy problems of the introduced PDEs, we apply an analytic technique, namely the homotopy analysis method (HAM).

1. Introduction

There are many types of PDEs, that they have power-law nonlinearity, such as Burgers, Fisher, Burgers–Fisher, Burgers–Huxley, Schrödinger, Fitzhugh–Nagumo, Klein–Gordon equations and so on. On the other hand, there are some PDEs which are reduced to the PDEs with power-law nonlinearity by using a suitable transformation, such as sin-Gordon and sinh-Gordon equations and so on. Thus, it is very useful to introduce the PDEs with power-law nonlinearity in general, and solve them by using an effective, convenient and promising method. In this paper, we introduce two types of PDEs with power-law nonlinearity in general. We prefer to use the HAM to solve the problems with power-law nonlinearity, because the HAM contains the convergence-parameter, which provides us with a simple way to adjust and control the convergence region of solution series. First Liao in 1992 employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) [1–6]. After this, many types of nonlinear problems were solved with HAM by others [7–11].

In this paper, the basic idea of the HAM is introduced and then its application for some problems with power law nonlinearity is studied. In this paper, we shall apply HAM to find the approximate analytical solution of some problems with power law nonlinearity. Comparisons with the exact solution shall be performed.

Now, we introduce the equation

$$u_t \prod_{i=1}^{M} \left(\eta_i + \lambda_i u^i\right) + \sum_{n=0}^{j} \left(u^{p_n} \, (\bar{u})^{l_n} \, \gamma_n \prod_{k=1}^{q_n} \left(u_x^{(k)}\right)^{p_k} \right) = \prod_{i=1}^{l} \left(\alpha_i + \beta_i u^i\right), \tag{1}$$

where $u_x^{(k)} = \frac{\partial^k u}{\partial x^k}$ and $\bar{u}$ is conjugate of u, with initial condition

$$u(x, 0) = f(x), \tag{2}$$

and the equation

$$u_{tt} \prod_{i=1}^{M} \left(\eta_i + \lambda_i u^i\right) + \sum_{n=0}^{j} \left(u^{p_n} \, (\bar{u})^{l_n} \, (u_t)^{m_n} \, \gamma_n \prod_{k=1}^{q_n} \left(u_x^{(k)}\right)^{p_k} \right) = \prod_{i=1}^{l} \left(\alpha_i + \beta_i u^i\right), \tag{3}$$

* Corresponding author. Tel.: +98 914 143 6187; fax: +98 841 222 7022.
E-mail addresses: a_m_bahrami@yahoo.com (A. Molabahrami), shidfar@iust.ac.ir (A. Shidfar).

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with initial conditions

$$\begin{cases} u(x,0)=f(x), \\ u_t(x,0)=g(x). \end{cases} \tag{4}$$

The Eq. (1) contains the many types of linear and nonlinear PDEs, such as Burgers, Fisher, Burgers–Fisher, Burgers–Huxley, Schrödinger with a power-law nonlinearity, Fitzhugh–Nagumo and so on. Also, the Eq. (3) contains the Klein–Gordon equation with a power-law nonlinearity and so on.

In this paper, the HAM is applied for solving the Cauchy problems of the Eqs. (1) and (3) on the infinite line. We prove that, if the solution series given by HAM is convergent, it must be an exact solution of the considered nonlinear problem. Furthermore, the so-called homotopy-Pade technique (HPT) is applied to accelerate the convergence of solution series.

2. A review of the HAM

In this section the basic idea of the homotopy analysis method [1] is introduced. Here, we first give the following definition.

Definition. Let ϕ be a function of the homotopy-parameter q, then

$$D_m(\phi) = \frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}\bigg|_{q=0}, \tag{5}$$

is called the mth-order homotopy-derivative of ϕ, where $m \geq 0$ is an integer [12].

To show the basic idea of the HAM, let us consider the following nonlinear equation in a general form

$$N[u(r,t)] = 0,$$

where N is a nonlinear operator, $u(r,t)$ is an unknown function, and r and t denote spatial and temporal independent variables, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, more details about homotopy technique and its applications are found in [13–17], Liao [1] constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] = q\hbar H(r,t)N[\phi(r,t;q)], \tag{6}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar$ is non-zero and called convergence-parameter, $H(r,t)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(r,t)$ is an initial guess of $u(r,t)$, and $\phi(r,t;q)$ is an unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(r,t;0) = u_0(r,t) \quad \text{and} \quad \phi(r,t,1) = u(r,t),$$

respectively. Thus as q increases from 0 to 1, the solution $\phi(r,t;q)$ varies from the initial guess $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(r,t;q)$ in Taylor series with respect to q, one has

$$\phi(r,t;q) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t)q^m, \tag{7}$$

where

$$\psi_m(r,t) = D_m[\phi(r,t;q)]. \tag{8}$$

If the auxiliary linear operator, the initial guess, the convergence-parameter, and the auxiliary function are so properly chosen, the series (7) converges at $q = 1$, one has

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t),$$

which must be one of solutions of original nonlinear equation, as proved by Liao [1]. As $\hbar = -1$ and $H(r,t) = 1$, Eq. (6) becomes

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] + qN[\phi(r,t;q)] = 0,$$

which is used mostly in the homotopy perturbation method (HPM), whereas the solution is obtained directly, without using Taylor series. The comparison between HAM and HPM can be found in [18,19]. As $H(r,t) = 1$, Eq. (6) becomes

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] = q\hbar N[\phi(r,t;q)], \tag{9}$$

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which is used in the HAM when it is not introduced in the set of base functions. According to definition (8), the governing equation can be deduced from the zero-order deformation equation (6). Define the vector

$$\vec{u}_n = \{u_0(r,t), u_1(r,t), \ldots, u_n(r,t)\}.$$

Operating on both sides of Eq. (6) with D_m, we have the so called mth-order deformation equation

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t) R_m(\vec{u}_{m-1}, r, t), \tag{10}$$

where

$$R_m(\vec{\psi}_{m-1}, r, t) = D_{m-1}\left(N[\phi(r,t;q)]\right), \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 2. \end{cases}$$

Substituting (7) into the (11), we have

$$R_m(\vec{u}_{m-1}, r, t) = D_{m-1}\left(N\left[\sum_{n=0}^{+\infty} u_n(r,t) q^n\right]\right). \tag{12}$$

It should be emphasized that $u_m(r,t)$ for $m \ge 1$ is governed by the linear equation (10) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as *Mathematica* and *Maple*.

3. The HAM solution

In this section, we obtain the series pattern solutions of the Eqs. (1) and (3)with initial conditions (2) and (4) respectively. To apply the HAM to the equations with power-law nonlinearity, it is necessary to introduce the *Molabahrami and Khani's Theorems*. Molabahrami and Khani proved the following theorems.

Theorem 1. *For homotopy-series $\phi = \sum_{n=0}^{+\infty} u_n q^n$, it holds*

$$D_m(\phi^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} u_{r_{k-1}},$$

where $m \ge 0$ and $k \ge 0$ are positive integers.

Proof. Please refer to [8]. □

Corollary 1. *Let $\psi = (\bar{\phi})^{p_{n+1}} \phi^{p_0} \prod_{k=1}^{n} \left(\phi^{(k)}\right)^{p_k}$ where $\bar{\phi}$ is conjugate of ϕ, from Theorem 1, we have*

$$D_m(\psi) = \sum_{r_1=0}^{m} w_{m-r_1} \sum_{r_2=0}^{r_1} w_{r_1-r_2} \sum_{r_3=0}^{r_2} w_{r_2-r_3} \cdots \sum_{r_{A-2}=0}^{r_{A-3}} w_{r_{A-3}-r_{A-2}} \sum_{r_{A-1}=0}^{r_{A-2}} w_{r_{A-2}-r_{A-1}} w_{r_{A-1}},$$

where $A = \sum_{k=0}^{n+1} p_k$ and $w_{p_{k_r}}$ is substituted by $u_{p_{k_r}}^{(k)}$ for $r = 1, 2, \ldots, p_k$ and by $\bar{u}_{p_{k_r}}$ for $r = 1, 2, \ldots, p_{n+1}$, where $k = 0, 1, 2, \ldots, n$ and $u_{p_{k_r}}^{(0)} = u_{p_{k_r}}$.

Theorem 2. *Let $s = \sum_{n=0}^{+\infty} u_n$. Then, recall the assumptions in Theorem 1, we have*

$$\sum_{m=0}^{+\infty} D_m(\phi^k) = s^k.$$

Proof. Please refer to [8]. □

Corollary 2. *According to the Theorem 2, recall the assumptions in Corollary 1, we have*

$$\sum_{m=0}^{+\infty} D_m(\psi) = (\bar{s})^{p_{n+1}} s^{p_0} \prod_{k=1}^{n} \left(s^{(k)}\right)^{p_k}.$$

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To apply the Theorems 1 and 2 to the $\prod_{i=1}^{n}\left(\alpha_i+\beta_i\phi^i\right)$, we set

$$\prod_{i=1}^{n}\left(\alpha_i+\beta_i\phi^i\right)=\sum_{i=0}^{\frac{n(n+1)}{2}} a_i\phi^i,$$

where $a_0=\prod_{i=1}^{n}\alpha_i,\ldots,a_{\frac{n(n+1)}{2}}=\prod_{i=1}^{n}\beta_i$.

From initial conditions (2) and (4), it is reasonable to express the solution by a set of base functions

$$\{g_n(x)e_n(t)|n\geq 0\},\tag{13}$$

in the form

$$u(x,t)=\sum_{n=0}^{+\infty}a_n g_n(x)e_n(t),\tag{14}$$

where a_n is a coefficient, $g_n(x)$ as a coefficient is a function with respect to x and $e_n(t)$ is determined to provide the so-called *rule of solution expression* $u(x,t)$. Under the *rule of solution expression* denoted by (14) and from Eqs. (1) and (3), it is straightforward to choose

$$L[\phi(x,t,q)]=\gamma_1(x,t)\frac{\partial^2\phi(x,t,q)}{\partial t^2}+\gamma_2(x,t)\frac{\partial\phi(x,t,q)}{\partial t}+\gamma_3(x,t)\phi(x,t,q)\tag{15}$$

as an auxiliary linear operator, where $\gamma_i(x,t)$ is a function. For the Eq. (1), we assume $\gamma_1(x,t)\equiv 0$ and $\gamma_2(x,t)\neq 0$. For the Eq. (3), we assume $\gamma_1(x,t)\neq 0$. The solution given by HAM denoted by (14) can be represented by many different base functions. In this work we set $e_n(t)=t^n$. Thus, we choose the auxiliary linear operators for the Eqs. (1) and (3) as follows

$$L[\phi(x,t,q)]=\frac{\partial\phi(x,t,q)}{\partial t},\tag{16}$$

and

$$L[\phi(x,t,q)]=\frac{\partial^2\phi(x,t,q)}{\partial t^2},\tag{17}$$

respectively. The auxiliary linear operator (16) has the property $L[C]=0$, where C as a coefficient is a function with respect to x, and the auxiliary linear operator (17) has the property $L[C_1t+C_2]=0$, where C_i as a coefficient is a function with respect to x.

According to Eqs. (1) and (3), we define the nonlinear operators

$$\begin{aligned}N[\phi(x,t,q)]&=\frac{\partial\phi(x,t,q)}{\partial t}\prod_{i=1}^{M}\left(\eta_i+\lambda_i\left(\phi(x,t,q)\right)^i\right)\\&\quad+\sum_{n=0}^{j}\left(\left(\phi(x,t,q)\right)^{p_n}\left(\bar{\phi}(x,t,q)\right)^{l_n}\gamma_n\prod_{k=1}^{q_n}\left(\frac{\partial^k\phi(x,t,q)}{\partial x^k}\right)^{p_k}\right)-\prod_{i=1}^{l}\left(\alpha_i+\beta_i\left(\phi(x,t,q)\right)^i\right),\end{aligned}\tag{18}$$

and

$$\begin{aligned}N[\phi(x,t,q)]&=\frac{\partial^2\phi(x,t,q)}{\partial t^2}\prod_{i=1}^{M}\left(\eta_i+\lambda_i\left(\phi(x,t,q)\right)^i\right)\\&\quad+\sum_{n=0}^{j}\left(\phi(x,t,q)\left(\frac{\partial\phi(x,t,q)}{\partial t}\right)^{m_n}\left(\bar{\phi}(x,t,q)\right)^{l_n}\gamma_n\prod_{k=1}^{q_n}\left(\frac{\partial^k\phi(x,t,q)}{\partial x^k}\right)^{p_k}\right)^{p_n}\\&\quad-\prod_{i=1}^{l}\left(\alpha_i+\beta_i\left(\phi(x,t,q)\right)^i\right),\end{aligned}\tag{19}$$

respectively. From Eqs. (11) and (18), and Theorem 1 and Corollary 1, we have

$$R_m[\vec{u}_{m-1}(x,t)]=\sum_{n=0}^{j}D_{m-1}\left(\frac{1}{j+1}u_t\prod_{i=1}^{M}\left(\eta_i+\lambda_iu^i\right)+u^{p_n}\left(\bar{u}\right)^{l_n}\gamma_n\prod_{k=1}^{q_n}\left(u_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_iu^i\right)\right),\tag{20}$$

and from Eqs. (11) and (19), and Theorem 1 and Corollary 1, we have

$$R_m[\vec{u}_{m-1}(x,t)]=\sum_{n=0}^{j}D_{m-1}\left(\frac{1}{j+1}u_{tt}\prod_{i=1}^{M}\left(\eta_i+\lambda_iu^i\right)+u^{p_n}\left(\bar{u}\right)^{l_n}\left(u_t\right)^{m_n}\gamma_n\prod_{k=1}^{q_n}\left(u_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_iu^i\right)\right).\tag{21}$$

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Let $u_m^*(x,t)$ denote a special solution of the equation

$$L[u_m^*(x,t)] = \hbar H(x,t) R_m[\vec{u}_{m-1}(x,t)]. \tag{22}$$

Now, the solution of the mth-order deformation equation (10) under the Eq. (18) with initial condition $u_m(x,0) = 0$, for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + u_m^*(x,t) - u_m^*(x,0), \tag{23}$$

and, the solution of the mth-order deformation equation (10) under the Eq. (19) with initial conditions $u_m(x,0) = 0$ and $\frac{\partial u_m}{\partial t}(x,0) = 0$, for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + u_m^*(x,t) - t u_m^{*\prime}(x,0) - u_m^*(x,0), \tag{24}$$

where the prime denotes differentiation with respect to the similarity variable t. According to (1) and (2) and the *rule of solution expression* (14), it is straightforward that the initial approximation can be in the form

$$u_0(x,t) = (1 + h_1(t)) f(x) + h_2(t), \tag{25}$$

where $h_1(0) = h_2(0) = 0$, the functions $h_1(t)$ and $h_2(t)$ are chosen subject to obey the *rule of solution expression* (14). According to (3) and (4) and the *rule of solution expression* (14), it is straightforward that the initial approximation can be in the form

$$u_0(x,t) = (1 + g_1(t)) f(x) + g_2(t) g(x) + g_3(t), \tag{26}$$

where $g_1(0) = g_2(0) = g_3(0) = 0$ and $g_1'(0) = g_3'(0) = 0$ and $g_2'(0) = 1$, the functions $g_1(t)$ and $g_2(t)$ and $g_3(t)$ are chosen subject to obey the *rule of solution expression* (14). According to the *rule of solution expression* denoted by (14) and from Eqs. (23) and (24) to obey the *rule of coefficient ergodicity* the auxiliary function $H(x,t)$ is chosen.

3.1. Convergence theorem

In this subsection, we prove that, if the solution series given by HAM is convergent, it must be an exact solution of the considered nonlinear problem.

Theorem 3. *If the series*

$$u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t),$$

it converges, where $u_m(x,t)$ is governed by

1. *The Eq.* (23) *under the definition* (18)*, it must be an exact solution of Eq.* (1) *with initial condition* (2)*.*
2. *The Eq.* (24) *under the definition* (19)*, it must be an exact solution of Eq.* (3) *with initial condition* (4)*.*

Proof. If the series is convergent, we can write

$$s(x,t) = \sum_{m=0}^{+\infty} u_m(x,t),$$

and it holds

$$\lim_{n\to+\infty} u_n(x,t) = 0.$$

Then, using (10) and (15), we have

$$\begin{aligned}
\hbar H(x,t) \sum_{m=1}^{+\infty} R_m[\vec{u}_{m-1}(x,t)] &= \lim_{n\to+\infty} \sum_{m=1}^{n} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] \\
&= L\left\{ \lim_{n\to+\infty} \sum_{m=1}^{n} [u_m(x,t) - \chi_m u_{m-1}(x,t)] \right\} \\
&= L[\lim_{n\to+\infty} u_n(x,t)] \\
&= 0,
\end{aligned}$$

which gives, since $\hbar \neq 0$ and $H(x,t) \neq 0$,

$$\sum_{m=1}^{+\infty} R_m[\vec{u}_{m-1}(x,t)] = 0. \tag{27}$$

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1. Substituting (18) into the (27), recall the Theorem 2 and Corollary 2, and simplifying it, we obtain

$$\begin{aligned}\sum_{m=1}^{+\infty} & R_m[\vec{u}_{m-1}(x,t)] \\ &= \sum_{m=1}^{+\infty}\left\{\sum_{n=0}^{j} D_{m-1}\left(\frac{1}{j+1}u_t\prod_{i=1}^{M}\left(\eta_i+\lambda_i u^i\right)+u^{p_n}\,(\bar{u})^{l_n}\,\gamma_n\prod_{k=1}^{q_n}\left(u_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_i u^i\right)\right)\right\} \\ &= \sum_{m=0}^{+\infty}\left\{\sum_{n=0}^{j} D_{m}\left(\frac{1}{j+1}u_t\prod_{i=1}^{M}\left(\eta_i+\lambda_i u^i\right)+u^{p_n}\,(\bar{u})^{l_n}\,\gamma_n\prod_{k=1}^{q_n}\left(u_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_i u^i\right)\right)\right\} \\ &= \sum_{n=0}^{j}\left(\frac{1}{j+1}s_t\prod_{i=1}^{M}\left(\eta_i+\lambda_i s^i\right)+s^{p_n}\,(\bar{s})^{l_n}\,\gamma_n\prod_{k=1}^{q_n}\left(s_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_i s^i\right)\right) \\ &= 0.\end{aligned}$$

From initial condition $u_m(x,0)=0$ and (25), it holds

$$s(x,0)=f(x).$$

So, $s(x,t)$ satisfies Eqs. (1) and (2), and therefore is an exact solution of the Eq. (1) with initial condition (2).

2. Substituting (19) into the (27), recall the Theorem 2 and Corollary 2, and simplifying it, we obtain

$$\begin{aligned}\sum_{m=1}^{+\infty} & R_m[\vec{u}_{m-1}(x,t)] \\ &= \sum_{m=1}^{+\infty}\left\{\sum_{n=0}^{j} D_{m-1}\left(\frac{1}{j+1}u_{tt}\prod_{i=1}^{M}\left(\eta_i+\lambda_i u^i\right)+u^{p_n}\,(\bar{u})^{l_n}\,\gamma_n\prod_{k=1}^{q_n}\left(u_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_i u^i\right)\right)\right\} \\ &= \sum_{m=0}^{+\infty}\left\{\sum_{n=0}^{j} D_{m}\left(\frac{1}{j+1}u_{tt}\prod_{i=1}^{M}\left(\eta_i+\lambda_i u^i\right)+u^{p_n}\,(\bar{u})^{l_n}\,\gamma_n\prod_{k=1}^{q_n}\left(u_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_i u^i\right)\right)\right\} \\ &= \sum_{n=0}^{j}\left(\frac{1}{j+1}s_{tt}\prod_{i=1}^{M}\left(\eta_i+\lambda_i s^i\right)+s^{p_n}\,(\bar{s})^{l_n}\,\gamma_n\prod_{k=1}^{q_n}\left(s_x^{(k)}\right)^{p_k}-\frac{1}{j+1}\prod_{i=1}^{l}\left(\alpha_i+\beta_i s^i\right)\right) \\ &= 0.\end{aligned}$$

From initial conditions $u_m(x,0)=\frac{\partial u_m}{\partial t}(x,0)=0$ and (26), it holds

$$s(x,0)=f(x) \quad \text{and} \quad s_t(x,0)=g(x).$$

So, $s(x,t)$ satisfies Eqs. (3) and (4), and therefore is an exact solution of the Eq. (3) with initial conditions (4). This ends the proof. □

To investigate the influence of $\hbar$ on the convergence of the solution series given by the HAM, we first plot the so-called $\hbar$-curves of $u_{tt}(0,0)$ and $u_{ttt}(0,0)$. According to the $\hbar$-curves, it is easy to discover the valid region of $\hbar$, which corresponds to the line segment nearly parallel to the horizontal axis.

4. Result analysis

In this section, four examples are presented. We use $(n+1)$ terms in evaluating approximate solution $u_{\text{approx}[n]}=\sum_{m=0}^{n}u_m$. We first plot the so-called $\hbar$-curves of $u''_{\text{approx}[n]}(0,0)$ and $u'''_{\text{approx}[n]}(0,0)$ to discover the valid region of $\hbar$, which corresponds to the line segment nearly parallel to the horizontal axis. Finally, the so-called homotopy-Pade technique is employed for Examples 1 and 4 to accelerate the convergence of solution series.

Example 1. Consider the one-dimensional Burgers problem

$$\begin{cases} u_t+uu_x=u_{xx}, & t>0,\ -\infty<x<+\infty, \\ u(x,0)=2x, & -\infty<x<+\infty, \end{cases} \tag{28}$$

with the exact solution

$$u(x,t)=\frac{2x}{1+2t}.$$

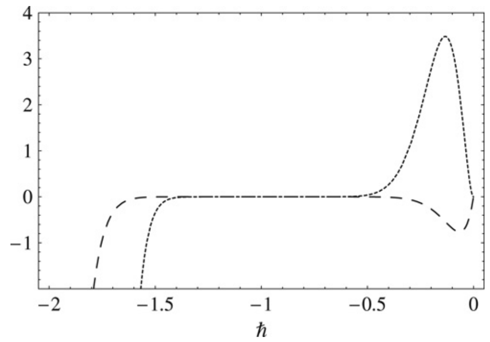

Fig. 1. The $\hbar$-curve of $u''_{\text{approx}[15]}(0,0)$ and $u'''_{\text{approx}[15]}(0,0)$.

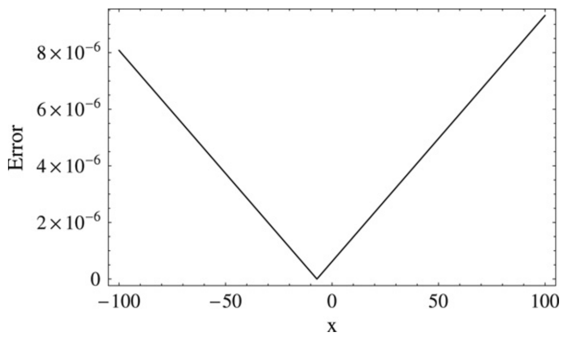

Fig. 2. The absolute error of $u_{\text{approx}[15]}$ for $t = 0.1$, at $\hbar = -0.53$.

with $u_0(x,t) = t + 2x$ and $H(x,t) = 1$, from (23), we have

$$u_1(x,t) = \hbar\,(1+4x)\,t + \hbar t^2,$$
$$u_2(x,t) = \hbar(1+\hbar)\,(1+4x)\,t + \hbar\,(1+2\hbar(1+4x))\,t^2 + 2\,\hbar^2\,t^3,$$
$$u_3(x,t) = \hbar(1+\hbar)^2\,(1+4x)\,t + \hbar(1+\hbar)\,(1+\hbar(3+16x))\,t^2 + 2\,\hbar^2\,(2+\hbar(3+8x))\,t^3 + 4\,\hbar^3\,t^4,$$
$$u_4(x,t) = \hbar(1+\hbar)^3\,(1+4x)\,t + \hbar(1+\hbar)^2\,(1+4\hbar(1+6x))\,t^2 + 6\,\hbar^2(1+\hbar)\,(1+2\hbar(1+4x))\,t^3 + 4\,\hbar^3\,(3+4\hbar(1+2x))\,t^4 + 8\,\hbar^4\,t^5,$$
$$\vdots$$

For $u_{\text{approx}[15]}$, from the $\hbar$-curves (Fig. 1), it is found that, when $-1.40 \le \hbar \le -0.50$, the solution series (14) converges to the exact solution (28). Fig. 2 shows the comparison between the exact solution of (28) and $u_{\text{approx}[15]}$.

Example 2. Consider the Burgers–Huxley problem [8]

$$\begin{cases} u_t + u^2 u_x - u_{xx} = \dfrac{2}{3}u(1-u^2)(u^2-1), & t > 0,\ 0 \le x \le 1, \\ u(x,0) = \left(\dfrac{1}{2} + \dfrac{1}{2}\tanh\left(\dfrac{1}{3}x\right)\right)^{\frac{1}{2}}, & 0 \le x \le 1, \end{cases} \tag{29}$$

with the exact solution

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{9}(3x+t)\right)\right)^{\frac{1}{2}}.$$

With $u_0(x,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{3}x\right)\right)^{\frac{1}{2}}$ and $H(x,t) = 1$, from (23), we have

$$u_1(x,t) = \frac{5}{18\sqrt{2}}\hbar\frac{\sec h^2\left(\frac{x}{3}\right)}{\sqrt{1+\tanh\left(\frac{x}{3}\right)}}t,$$

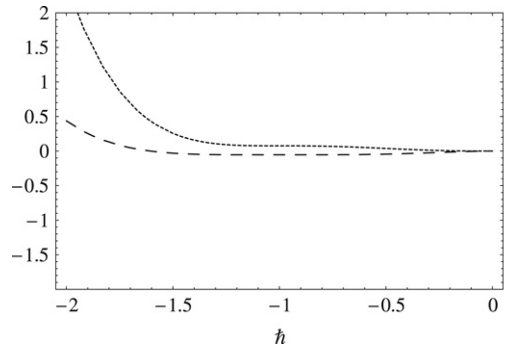

Fig. 3. The $\hbar$-curve of $u''_{\text{approx}[5]}(0, 0)$ and $u'''_{\text{approx}[5]}(0, 0)$.

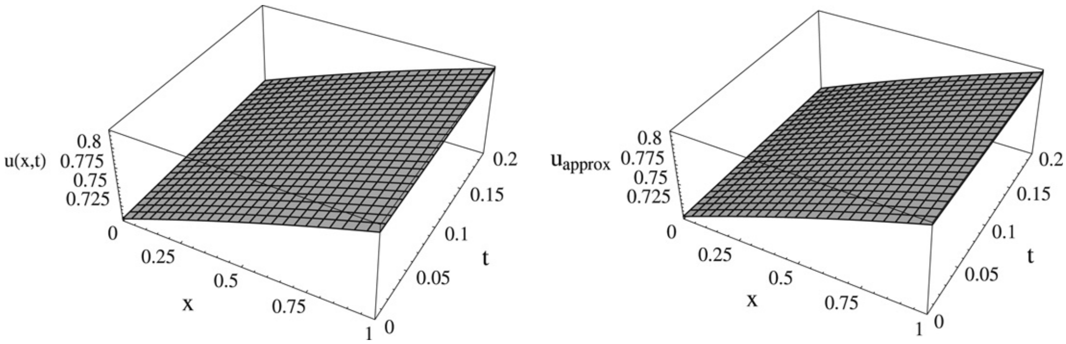

Fig. 4. The exact solution and the $u_{\text{approx}[5]}$ at $\hbar = -0.003$ respectively.

$$u_2(x,t) = (1+\hbar)u_1(x,t) - \frac{25}{162}\hbar^2 \frac{-1+2e^{\frac{2x}{3}}}{\left(1+e^{\frac{2x}{3}}\right)^2\sqrt{1+e^{-\frac{2x}{3}}}}t^2,$$

$$\begin{aligned} u_3(x,t) &= (1+\hbar)^2 u_1(x,t) - \frac{25}{81}\hbar^2(1+\hbar)\frac{-1+2e^{\frac{2x}{3}}}{\left(1+e^{\frac{2x}{3}}\right)^2\sqrt{1+e^{-\frac{2x}{3}}}}t^2 \\ &\quad + \frac{125}{34992\sqrt{2}}\hbar \frac{\sec h^4\left(\frac{x}{3}\right)\left(-10+5\cos h\left(\frac{2x}{3}\right)+3\sin h\left(\frac{2x}{3}\right)\right)}{\sqrt{1+\tanh\left(\frac{x}{3}\right)}}t^3, \end{aligned}$$

$$\vdots$$

For $u_{\text{approx}[5]}$, from the $\hbar$-curves (Fig. 3), it is found that, when $-1.3 \le \hbar < 0$, the solution series (14) converges to the exact solution (29). Fig. 4 shows the comparison between the exact solution of (29) and $u_{\text{approx}[5]}$.

Example 3. Consider the Cauchy problem for a Schrödinger equation with a power-law nonlinearity

$$\begin{cases} iu_t + \lambda u_{xx} + \gamma\,|u|^{10}\,u = 0, \quad i^2 = -1, \; -\infty < x < +\infty, \; t > 0, \\ u(x,0) = (\alpha + i\beta)a^{i(bx+c)+d}, \quad -\infty < x < +\infty. \end{cases} \tag{30}$$

With $u_0(x,t) = (\alpha + i\beta)a^{i(bx+c)+d}$ and $H(x,t) = 1$, from (23), at $\hbar = -1$, we have

$$u_1(x,t) = i(\alpha + i\beta)a^{i(bx+c)+d}\left(-\lambda\,(b\text{Ln}a)^2 + (\alpha^2+\beta^2)^3 a^{10d}\gamma\right)t,$$

$$u_2(x,t) = -\frac{1}{2}(\alpha + i\beta)a^{i(bx+c)+d}\left(-\lambda\,(b\text{Ln}a)^2 + (\alpha^2+\beta^2)^3 a^{10d}\gamma\right)^2 t^2,$$

$$u_3(x,t) = \frac{1}{6}(-i\alpha + \beta)a^{i(bx+c)+d}\left(-\lambda\,(b\text{Ln}a)^2 + (\alpha^2+\beta^2)^3 a^{10d}\gamma\right)^3 t^3,$$

$$\vdots$$

Thus

$$u(x,t) = (\alpha + \mathrm{i}\beta)a^{\mathrm{i}(\alpha x+c)+d}\Big(1 + \left(-\lambda\,(b\mathrm{Ln}a)^2 + (\alpha^2+\beta^2)^3 a^{10d}\gamma\right)\mathrm{i}t + \frac{1}{2!}\left(-\lambda\,(b\mathrm{Ln}a)^2 + (\alpha^2+\beta^2)^3 a^{10d}\gamma\right)^2(\mathrm{i}t)^2 + \frac{1}{3!}\left(-\lambda\,(b\mathrm{Ln}a)^2 + (\alpha^2+\beta^2)^3 a^{10d}\gamma\right)^3(\mathrm{i}t)^3 + \cdots\Big).$$

So, the exact solution of the problem (30) is

$$u(x,t) = (\alpha + \mathrm{i}\beta)a^{\mathrm{i}(bx+c)+d}\mathrm{e}^{\mathrm{i}\left(-\lambda(b\mathrm{Ln}a)^2+(\alpha^2+\beta^2)^3a^{10d}\gamma\right)t}.$$

Example 4. Consider the Cauchy problem for a sin-Gordon equation [20]

$$\begin{cases} u_{tt} - u_{xx} + \sin u = 0, \\ u(x,0) = 0, \qquad u_t(x,0) = 4\sec \mathrm{h}x, \end{cases} \tag{31}$$

with the exact solution $u(x,t) = 4\arctan(\sec \mathrm{h}(x)t)$. By using the transformation $w = \tan\frac{u}{2}$, the problem (31) becomes

$$\begin{cases} w_{tt}(1+w^2) - 2ww_t^2 - w_{xx}(1+w^2) + 2ww_x^2 + (1+w^2)w = 0, \\ w(x,0) = 0, \qquad w_t(x,0) = 2\sec \mathrm{h}x. \end{cases} \tag{32}$$

With $w_0(x,t) = 2t\sec \mathrm{h}x$ and $H(x,t) = 1$, from (24), we have

$$w_1 = -2\hbar\left(\sec \mathrm{h}^3(x)\right)t^3 + \frac{4}{5}\hbar\left(\sec \mathrm{h}^3(x)\right)t^5,$$

$$\begin{aligned} w_2 = &-\frac{1}{4}\hbar\sec \mathrm{h}^3(x)\left(8 + 3\hbar\sec \mathrm{h}^4(x) + 4\hbar\cos \mathrm{h}(2x)\sec \mathrm{h}^4(x) + \hbar\cos \mathrm{h}(4x)\sec \mathrm{h}^4(x)\right)t^3 \\ &+ \frac{1}{5}\hbar\sec \mathrm{h}^3(x)\left(4 + 8\hbar\sec \mathrm{h}^4(x) + 9\hbar\cos \mathrm{h}(2x)\sec \mathrm{h}^4(x) + \hbar\cos \mathrm{h}(4x)\sec \mathrm{h}^4(x)\right)t^5 \\ &- \frac{2}{105}\hbar^2\sec \mathrm{h}^7(x)\,(51 + 32\cos \mathrm{h}(2x) + \cos \mathrm{h}(4x))\,t^7 + \frac{2}{45}\hbar^2\sec \mathrm{h}^7(x)\,(5 + 3\cos \mathrm{h}(2x))\,t^9, \end{aligned}$$

$$\vdots$$

For $w_{\mathrm{approx}[4]}$, from the $\hbar$-curves (Fig. 5), it is found that, when $-1.50 \le \hbar \le -0.50$ the solution series (14) converges to the exact solution (32). Fig. 6 shows the comparison between the exact solution of (31) and $u_{\mathrm{approx}[4]}$, which $u_{\mathrm{approx}[4]} = 2\arctan\left(w_{\mathrm{approx}[4]}\right)$.

The so-called homotopy-Pade technique (HPT) can greatly enlarge the convergence region and rate of solution series given by the HAM. For details about the HPT, please refer to Liao ([12], Section 2.3.7). Employing the HPT to the solution series of Example 1, we find the [2, 2] HPT as follows

$$\mathrm{HPT}[2,2] = \frac{2\,\hbar^2\,x}{\hbar^2(1+2t)}.$$

It is found that the HPT[2,2] does not depend upon the convergence-parameter $\hbar$, since $\hbar \neq 0$, the HPT[2,2] is the exact solution of (28). So, the HPT can enlarge the convergence region of the solution series for larger values of t. Finally, Fig. 7 shows the comparison between the exact solution of (31) and $u_{\mathrm{HPT}[2,2]}$, which $u_{\mathrm{HPT}[2,2]} = 2\arctan(\mathrm{HPT}[2,2])$.

5. Conclusion

In this paper, two PDEs with power-law nonlinearity were introduced which contain many types of the linear and nonlinear PDEs. We obtained the series pattern solutions of the Cauchy problems for the introduced PDEs by using the homotopy analysis method (HAM). The results of the examples show that the HAM is a very effective and convenient tool to solve the Cauchy problems for the PDEs with power-law nonlinearity. On the other hand, example 4 shows that the HAM is a promising tool to the nonlinear Cauchy problems which have other than power-law nonlinearity. Also, it is found that the HPT can greatly enlarge the convergence region and rate of solution series given by the HAM.

This paper provides us with a new, suitable and simple way to solve the Cauchy problems for the nonlinear PDEs by using the HAM. Thus, such an approach can be used for the other nonlinear problems. In addition, this paper provides us with a convenient way to choose the auxiliary linear operator, auxiliary function, initial guess and the set of base functions suitable to represent the solution of a given nonlinear problem. The HAM is more suitable than other methods to solve the nonlinear

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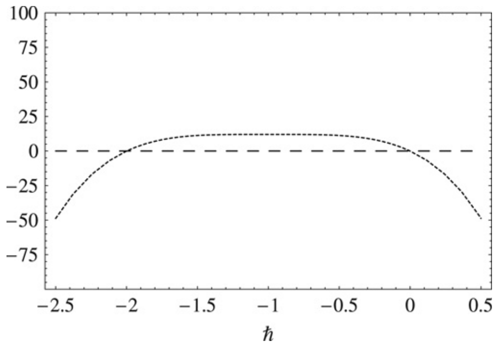

Fig. 5. The $\hbar$-curve of $w''_{\text{approx}[4]}(0,0)$ and $w'''_{\text{approx}[4]}(0,0)$.

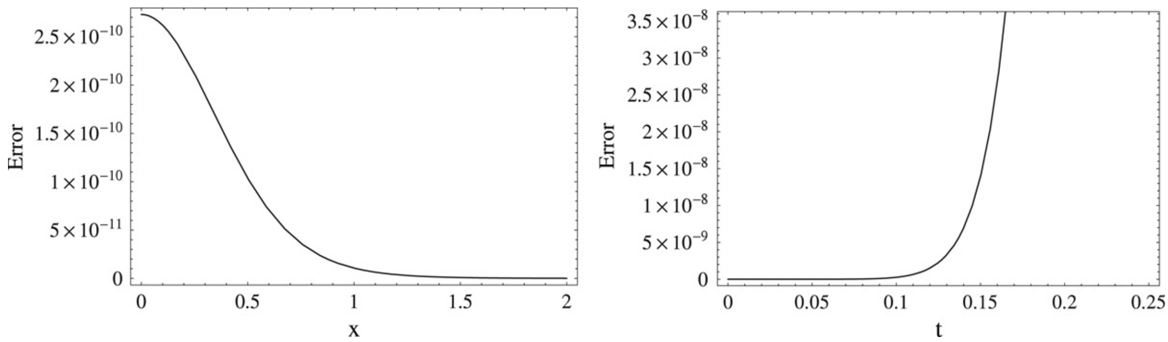

Fig. 6. The absolute error for $u_{\text{approx}[4]}$ at $\hbar = -0.997$, $t = 0.1$ and $x = 0.1$ respectively.

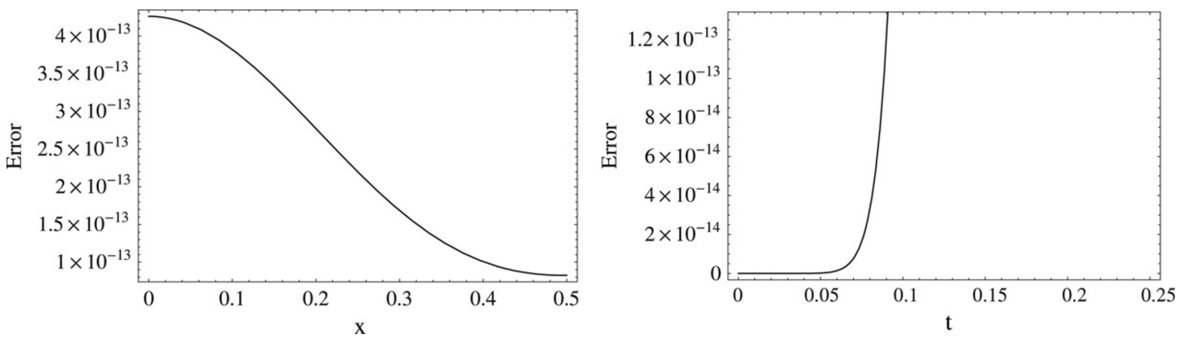

Fig. 7. The absolute error for $u_{\text{HPT}[2,2]}$ at $\hbar = -0.997$, $t = 0.1$ and $x = 0.1$ respectively.

problems, because the HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. Thus the convergence-parameter plays an important role within the frame of the HAM which can be determined by the so-called $\hbar$-curves.

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