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A THEOREM ON GLOBALLY SYMMETRIC FINSLER SPACES

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Abstract. In this paper, we study the Symmetric of this paper, we study the Symmetric of Finsler spaces. We talk about globally Symmetric Finsler spaces. Then we prove that this space can be written as a coset space of Lie group with an invariant Finsler metric. Finally, we prove that such a space must be Berwaldian.

1. Introduction

The study of Finsler spaces has important in physics and Biology ([cf, [2]]), In particular there are several important books about such spaces (see, [4], [7], ate). For example recently D. Bao, C. Robels, Z. Shen used the Randers metric in Finsler on Riemannian manifolds ([5],[7]page 20). We must also point out there was only little study about symmetry of such spaces ([9],[12]). For example E. Cartan has been showed symmetry plays very important role in Riemannian geometry ([7],page 9).

Now here, we first explain that general properties of a globally Symmetric Finsler space and finally, we prove that each such space is Berwaldian.

Definition 1.1. *Let V be a n-dimensional real vector space. A Minkowski norm on V is a functional F on V which is smooth on $V-\{0\}$ and satisfies the following conditions.*

(1) $F(u) \geq 0, \forall u \in V$;
(2) $F(\lambda u) = \lambda F(u), \forall \lambda > 0$;

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(3) *For any basis* $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ *of* V, *write* $F(y) = F(y^1, y^2, ..., y^n)$ *for* $y = y^j \in \varepsilon_j$. *Then the Hessian matrix*

$$(g_{ij}) := \left([\frac{1}{2}F^2]y^i y^j\right)$$

is positive-definite at any point of $V - \{0\}$.

Definition 1.2. *A smooth reversible Finsler metric is said to be Parallel if and only if* $D \times R = 0$.

Definition 1.3. *A Finsler space is locally symmetric if, for any* $p \in M$, *the geodesic reflection* s_p *is a local isometry of the Finsler metric.*

Definition 1.4. *A reversible Finsler space* (M, F) *is called globally Symmetric if for any* $p \in M$ *the exists an involutive isometry* σ_x (*that is,* $\sigma_x^2 = I$ *but* $\sigma_x \neq I$) *of such that* x *is an isolated fixed point of* σ_x.

Definition 1.5. *Let* G *be a Lie group and* K *is a closed subgroup of* G. *Then the coset space* G/K *is called symmetric if there exists an involutive automorphism* σ *of* G *such that*

$$G_\sigma^0 \subset K \subset G_\sigma,$$

where G_σ *is the subgroup consisting of the fixed points of* σ *in* G *and* G_σ^0 *denotes the identity component of* G_σ.

Theorem 1.6. *Let* G/K *be a symmetric coset space. Then any* G*-invariant reversible Finsler metric* (*if exists*) F *on* G/K *makes* $(G/K, F)$ *a globally symmetric Finsler space* ([8] *, page 8*).

Theorem 1.7. *Let* (M, F) *be a globally Symmetric Finsler space. For* $p \in M$, *denote the involutive isometry of* (M, F) *at* p *by* S_x. *Then we have*

(*a*) *For any* $p \in M, (dS_x)_x = -I$. *In particular,* F *must be reversible.*

(*b*) (M, F) *is forward and backward complete;*

(*c*) (M, F) *is homogenous. This is, the group of isometries of* $(M, F), I(M, F)$, *acts transitively on* M.

(*d*) *Let* $\widetilde{M}$ *be the universal covering space of* M *and* π *be the projection mapping. Then* $(\widetilde{M}, \pi^*(F))$ *is a globally Symmetric Finsler space, where* $\pi^*(F)$ *is define by*

$$\pi^*(F)(q) = F((d\pi)_{\widetilde{p}}(q)), \; q \in T_{\widetilde{p}}(\widetilde{M}),$$

(See [8] *to prove).*

Corollary 1.8. *Let* (M, F) *be a globally Symmetric Finsler space. Then for any* $p \in M, s_p$ *is a local geodesic Symmetry at* p. *The Symmetry* s_p, *is unique.* (*See prove of Theorem 1.2 and Cf.* [4])

2. A theorem on globally Symmetric Finsler spaces

Theorem 2.1. *Let (M, F) be a globally Symmetric Finsler space. Then exits a Riemannian Symmetric pair (G, K) such that M is diffeomorphic to G/K and F is invariant under G.*

Proof. The group $I(M, F)$ of isometries of (M, F) acts transitively on M ((C) of theorem 1.2). We proved that $I(M, F)$ is a Lie transformation group of M and for any $p \in M$ ([7], page 78), the isotropic subgroup $I_p(M, F)$ is a compact subgroup of $I(M, F)$ ([4]). Since M is connected ([10]) and the subgroup K of G which p fixed is a compact subgroup of G. Furthermore, M is diffeomorphic to G/K under the mapping $gH \to g.p$, $g \in G$ ([10], Theorem 2.5).
As in the Riemannian case in page 209 of [10], we define a mapping s of G into G by $s(g) = s_p g s_p$, where s_p donote the (unique) involutive isometry of (M, f) with p as an isolated fixed point. Then it is easily seen that s is an involutive automorphism of G and the group K lies between the closed subgroup K_s of fixed points of s and the identity component of K_s. Furthermore, the group K contains no normal subgroup of G other than $\{e\}$. That is, (G, K) is symmetric pair. (G, K) is a Riemannian symmetric pair, because K is compact. □

The following results will be useful in the proof of our aim of this paper.

Proposition 2.2. *Let $(M, \bar{F})$ be a Finsler space, $p \in M$ and H_p be the holonomy group of $\bar{F}$ at p. If F_p is a H_p invariant MinKowski norm on $T_p(M)$, then F_p can be extended to a Finsler metric F on M by parallel translations of $\bar{F}$ such that F is affinely equivalent to $\bar{F}$* ([7]*, proposition 4.2.2*)

Proposition 2.3. *A Finsler metric F on a manifold M is a Berwald metric if and only if it is affinely equivalent to a Riemannian metric g. In this case, F and g have the same holonomy group at any point $p \in M$ (see proposition 4.3.3 of* [7]*).*

Now the main aim

Theorem 2.4. *Let (M, F) be a globally symmetric Finsler space. Then (M, F) is a Berwald space. Furthermore, the connection of F coincides with the Levi-civita connection of a Riemannian metric g such that (M, g) is a Riemannian globally symmetric space.*

Proof. We first prove F is Beraldian. By Theorem 2.1, there exsits a Riemannian symmetric pair (G, K) such that M is diffeomorphic to G/K and F is invariant under G. Fix a G- invariant Riemannian metric g on G/K. Without losing generality, we can assume that (G, K) is effective. Since being a Berwald space is a local property, we can assume further that G/K is simple connected. Then we have a decomposition (page 244 of [10]):

$$G/K = E \times G_1/K_1 \times G_2/K_2 \times ... \times G_n/K_n,$$

where E is a Euclidean space, G_i/K_i are simply connected irreducible Riemannian globally symmetric spaces, $i = 1, 2, ..., n$. Now we determine the holonomy groups of g at the origin of G/K. According to the de Rham decomposition theorem (Cf. [4]), it is equal to the product of the holonomy groups of E and G_i/K_i at the origin. Now E has trivial holonomy group. For G_i/K_i, by the holonomy theorem of Ambrose and Singer ([1], page 231, it shows, for any connection, how the curvature from generats the holonomy group), we know that the lie algebra η_i of the holonomy group H_i is spanned by the linear mappings of the form $\{\widetilde{\tau}^{-1}R_0(X,Y)\widetilde{\tau}\}$, where τ denotes any piecewise smooth curve starting from o, $\widetilde{\tau}$ denotes parallel displacements (with respect to the restricted Riemannian metric) a long $\widetilde{\tau}$, $\widetilde{\tau}^{-1}$ is the inverse of $\widetilde{\tau}$, R_0 is the curvature tensor of G_i/K_i of the restricted Riemannian metric and $X, Y \in T_0(G_i/K_i)$. Since G_i/K_i is a globally symmetric space, the curvature tensor is invariant under parallel displacements (page 201 of [10]). So

$$\eta_i = span\{R_0(X,Y)|X,Y \in T_0(G_i/K_i)\},$$

(see page 243 of [10]).
On the other hand, Since G_i is a semisimple group. We know that the Lie algebra of $K_i^* = Ad(K_i) \simeq K$ is also equal to the span of $R_0(X,Y)$ ([10]).
The groups H_i, K_i^* are connected (because G_i/K_i) is simply connected) ([10] and [11]). Hence we have $H_i = K_i^*$. Consequently the holonomy group H_0 of G/K at the origin is

$$K_1^* \times K_2^* \times ... \times K_n^*$$

Now F defines a Minkwoski norm F_0 on $T_0(G/K)$ which is invariant by H_0 ([4]). By proposition 2.2, we can construct a Finsler metric $\bar{F}$ on G/K by parallel translations of g. By proposition 2.3, $\bar{F}$ is Berwaldian. Now for any point $p_0 = aK \in G/K$, there exists a geodesic of the Riemannian manifold $(G/K, g)$, say $\gamma(t)$ such that $\gamma(0) = 0, \gamma(1) = p_0$. Suppose the initial vector of γ is X_0 and take $X \in p$ such that $d\pi(X) = X_0$. Then it is known that $\gamma(t) = \exp tX.p_0$ and $d\tau(\exp tX)$ is the parallel translate of $(G/K, g)$ along γ ([10], page 208). Since F is G- invariant, it is invariant under this parallel translate. This means that F and $\bar{F}$ concide at $T_{p_0}(G/K)$. Consequently they concide everywhre. Thus F is a Berwald metric.
For the next assertion, we use a result of Szabo' ([4],page 278) which asserts that for any Berwald metric on M there exists a Riemannian metric with the same connection. We have proved that (M, F) is a Berwald space. Therefore there exists a Riemannian metric g_1 on M with the same connection as F. In Cf.[10], we showed

that the connection of a globally symmetric Berwald space is affine symmetric. So (M, F) is a Riemannian globally symmetric space ([10]). □

From the proof of theorem 2.4, we have the following corollary.

Corollary 2.5. *Let $(G/K, F)$ be a globally symmetric Finsler space and $g = \ell + p$ be the corresponding decomposition of the Lie algebras. Let π be the natural mapping of G onto G/K. Then $(d\pi)_e$ maps p isomorphically onto the tangent space of G/K at $p_0 = eK$. If $X \in p$, then the geodesic emanating from p_0 with initial tagent vector $(d\pi)_e X$ is given by*

$$\gamma_{d\pi.X}(t) = \exp tX.p_0.$$

Furthermore, if $y \in T_{p_0}(G/K)$, then $(d \exp tX)_{p_0}(Y)$ is the parallel of Y along the geodesic (see [10], proof of theorem 3.3).

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