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Application of the Group-Theoretical Method to Physical Problems

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Abstract

The concept of the theory of continuous groups of transformations has attracted the attention of applied mathematicians and engineers to solve many physical problems in the engineering sciences. Three applications are presented in this paper. The first one is the problem of time-dependent vertical temperature distribution in a stagnant lake. Two cases have been considered for the forms of the water parameters, namely water density and thermal conductivity. The second application is the unsteady free-convective boundary-layer flow on a non-isothermal vertical flat plate. The third application is the study of the dispersion of gaseous pollutants in the presence of a temperature inversion. The results are found in closed form and the effect of parameters are discussed.

1 Introduction

According to Seshadri and Na [1], different methods for carrying out similarity analysis of partial differential equations are classified into direct methods, where the concept of group invariance is not explicitly invoked, and group-theoretical methods that are based upon the invocation of invariance under groups of transformations of the partial differential equation and the auxiliary conditions.

The foundation of the group-theoretical method is contained in the general theory of continuous transformation groups that were introduced and treated extensively by Lie [2] in 1875. Group-theoretical methods provide a powerful tool because they are not based on linear operators, superposition, or any other aspects of linear solution techniques. Therefore, these methods are applicable to nonlinear differential models.

Throughout the history of similarity analysis, a variety of problems in science and engineering have been solved. Among these we find a general treatment of steady two-dimensional incompressible laminar flow of fluid into an infinite region of the same fluid by Abbott and Kline [3] in 1960, impact of thin long rods by Taulbee *et al* [4] in 1971, wave propagation in viscoelastic, viscoplastic and electrical transmission lines by Ames and Suliciu [5] in 1982, time-dependent free surface flows under gravity by Sachdev and Philip [6] in 1986, problem of ocean acoustics by Richards [7] in 1987, unsteady free-convective boundary-layer flow on a non-isothermal vertical flat plate by Abd-el-Malek *et al* [8] in 1990, steady free-convective boundary-layer flow on a non-isothermal vertical circular cylinder by Abd-el-Malek and Badran [9] in 1991, dispersion of gaseous pollutants in the presence of a temperature inversion by Badran and Abd-el-Malek [10] in 1993, and nonlinear temperature variation across the lake depth neglecting the effect of external heat sources by Abd-el-Malek [11] in 1997. Recently Boutros *et al* [12, 13] considered two problems namely, potential equation in triangular regions with temperature distribution along the boundaries in the form of polynomials, and the second is the time-dependent vertical temperature distribution in stagnant lake taking into consideration an external heat source. Many physical application are illustrated in Sedov [14], and Rogers and Ames [15].

In the present paper we will present three physical applications of the group-theoretical method namely, time-dependent vertical temperature distribution in a stagnant lake taking into consideration the effect of external heat source. The problem has been solved for two possible forms of the water parameters (water density and thermal conductivity). Second application is the unsteady free-convective boundary-layer flow on a non-isothermal vertical flat plate. As a third application we study the dispersion of gaseous pollutants in the presence of a temperature inversion. The obtained results are found in closed form and the effect of different parameters are discussed.

2 Application (I): Time-dependent vertical temperature distribution in a stagnant lake

2.1 Mathematical formulation

Consider the one dimensional heat transfer equation for heat flux in the vertical direction, neglecting the convective motion of the fluid and assuming the absolute value of the specific heat of the water as sensibly constant within the range of temperatures considered; take it to be unity. The vertical transport of heat in a deep lake is modelled by

$$\rho(T)T_t = [\kappa(T)T_z]_z + r(z,t), \qquad z > 0, \quad t > 0, \tag{2.1}$$

where "T" is the temperature; $r(z,t)$ is rate at which solar radiation is absorbed by the water; "t" is the time; "z" is the distance measured downward from the lake surface; "ρ" is the density; and "κ" is the thermal conductivity.

Boundary and initial conditions:
During early Spring, most of the lakes exhibit a nearly homothermal temperature distribution with a low degree of temperature extending all the way to the bottom (see Sundaram and Rehm [16]). In all of the calculations presented in this paper, the initial condition will be taken as that corresponding to the end of spring homothermy, i.e.,

$$T(z,0) = T_0, \tag{2.2.1}$$

where T_0 is the temperature of the lake at maximum Spring homothermy.

The boundary conditions are considered as follows:

$$\text{(i)} \quad T_z(0,t) = 0, \quad t > 0, \tag{2.2.2}$$

$$\text{(ii)} \quad T(h,t) = T_0 + \gamma t^{1/m}, \quad t > 0,\ 0 \le \gamma \ll 1,\ m > 0. \tag{2.2.3}$$

In our analysis we will consider two cases for the form of the density $\rho(T)$ and the thermal conductivity $\kappa(T)$, namely:

Case (1): $\rho = \alpha q(z)(T - T_0)^m, \quad \kappa = \beta g(z),$

Case (2): $\rho = \alpha q(z)(T - T_0)^s, \quad \kappa = \beta (T - T_0)^n,$

where α and β are constants, m, s and n are positive constants and $q(z)$ and $g(z)$ are arbitrary functions to be determined later on.

Write

$$T(z,t) = w(z,t) + T_0, \tag{2.3}$$

by which differential equation (2.1) takes the form:

$$\rho(w) w_t = [k(w) w_z]_z + r(z,t), \quad z > 0,\ t > 0, \tag{2.4}$$

and the initial and boundary conditions take the form:

(1) *initial condition*:

$$w(z,0) = 0, \tag{2.5.1}$$

(2) *boundary conditions*:

$$\text{(i)} \quad w_z(0,t) = 0, \quad t > 0, \tag{2.5.2}$$

$$\text{(ii)} \quad w(h,t) = \gamma t^{1/m}, \quad t > 0,\ 0 \le \gamma \ll 1,\ m > 0. \tag{2.5.3}$$

2.2 Case (1): $\rho = \alpha q(z) w^m$, $\kappa = \beta g(z)$

In this case differential equation (2.4) takes the form:

$$\beta g(z) w_{zz} + \beta g_z w_z - \alpha q(z) w^m w_t = -r(z,t). \tag{2.6}$$

2.2.1 Solution of the problem

Our method of solution depends on the application of a one-parameter group transformation to the partial differential equation (2.6).

2.2.1 (a) The group systematic formulation. The procedure is initiated with the group G; a class of transformation of one-parameter "a" of the form

$$G: \quad \overline{Q} = C^Q(a) Q + P^Q(a), \tag{2.7}$$

where Q stands for t, z, r, w, κ, ρ and the C's and P's are real-valued and at least differentiable in "a".

2.2.1 (b) The invariance analysis. To transform the differential equation, transformations of the derivatives of w, κ and ρ are obtained from G via chain-rule operations:

$$\overline{Q}_{\bar{i}} = \left(\frac{C^Q}{C^i}\right) Q_i, \qquad \overline{Q}_{\bar{i}\bar{j}} = \left(\frac{C^Q}{C^i C^j}\right) Q_{ij}; \qquad i = z, t;\ j = z, t. \tag{2.8}$$

Equation (2.6) is said to be invariantly transformed, for some function $H(a)$, whenever

$$\beta\overline{g}\,\overline{w}_{\bar{z}\bar{z}} + \beta\overline{g}_{\bar{z}}\,\overline{w}_{\bar{z}} - \alpha\overline{q}(\overline{w})^m\,\overline{w}_{\bar{t}} + \overline{r} = H(a)\left[\beta g w_{zz} + \beta g_z w_z - \alpha q w^m w_t + r\right]. \tag{2.9}$$

Substitution from (2.7) into (2.9) yields

$$\begin{aligned} &\beta\left[\frac{C^g C^w}{(C^z)^2}\right] g w_{zz} + \beta\left[\frac{C^g C^w}{(C^z)^2}\right] g_z w_z - \alpha w^m\left[\frac{C^q (C^w)^{m+1}}{C^t}\right] q w_t + [C^r] r + R(a) \\ &\quad = H(a)\left[\beta g w_{zz} + \beta g_z w_z - \alpha q w^m w_t + r\right], \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} R(a) = &\left[\beta\frac{P^g C^w}{(C^z)^2}\right] w_{zz} - \left[\alpha P^q (C^w w + P^w)^m \frac{C^w}{C^t}\right] w_t \\ &-\alpha (C^q q)\frac{C^w}{C^t} w_t \sum_{k=1}^{m}\binom{m}{k}(C^w w)^{m-k}(P^w)^k + P^r. \end{aligned}$$

The invariance of (2.9) implies $R(a) \equiv 0$. This is satisfied by putting

$$P^r = P^q = P^w = P^g = 0, \tag{2.11}$$

and

$$\left[\frac{C^q (C^w)^{m+1}}{C^t}\right] = \left[\frac{C^g C^w}{(C^z)^2}\right] = C^r = H(a). \tag{2.12}$$

Moreover, the boundary and initial conditions (2.5.2), (2.5.3) and (2.5.1) are also invariant in form, implying that

$$P^z = 0, \quad P^T = 0, \quad C^z = 1 \quad \text{and} \quad (C^w)^m = C^t. \tag{2.13}$$

Combining equations (2.12) and invoking the result (2.13), we get

$$C^r = C^q C^w = C^g C^w \qquad \text{which yields} \quad C^q = C^g. \tag{2.14}$$

Finally, we get the one-parameter group G which transforms invariantly the differential equation (2.1), as well as the boundary and initial conditions (2.2). The group G is of the form

$$G: \quad \overline{z} = z, \ \ \overline{t} = (C^w)^m t, \ \ \overline{q} = C^q q, \ \ \overline{r} = C^q C^w r, \ \ \overline{w} = C^w w, \ \ \overline{g} = C^q g. \tag{2.15}$$

2.2.1 (c) The complete set of absolute invariants. Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation. Then we have to proceed in our analysis to obtain a complete set of absolute invariants.

If $\eta \equiv \eta(z,t)$ is the absolute invariant of the independent variables, then

$$g_j(z, t; w, r, \kappa, \rho) = F_j[\eta(z,t)]; \qquad j = 1, 2, 3, 4 \tag{2.16}$$

are the four absolute invariants corresponding to w, r, κ and ρ. The application of a basic theorem in group theory, see [17], states that: *a function $g(z,t;w,r,\kappa,\rho)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation*

$$\sum_{i=1}^{6}(\alpha_i Q_i + \beta_i)\frac{\partial g}{\partial Q_i} = 0; \qquad Q_i = z, t, w, r, \kappa, \rho, \tag{2.17}$$

where

$$\alpha_i = \frac{\partial C^{Q_i}}{\partial a}(a^0), \qquad \beta_i = \frac{\partial P^{Q_i}}{\partial a}(a^0); \qquad i = 1, 2, \ldots, 6, \tag{2.18}$$

and a^0 denotes the value of "a" which yields the identity element of the group.

Owing to equation (2.17), $\eta(z,t)$ is an absolute invariant if it satisfies

$$(\alpha_1 z + \beta_1)\eta_z + (\alpha_2 t + \beta_2)\eta_t = 0, \tag{2.19}$$

Group (2.15) gives:

$$\alpha_1 = \beta_1 = \beta_2 = 0, \tag{2.20}$$

and hence from (2.17) and (2.20) we get

$$\eta_t = 0, \tag{2.21}$$

which gives

$$\eta(z,t) = F(z). \tag{2.22}$$

Without loss of generality we can use the identity function:

$$\eta(z,t) = z. \tag{2.23}$$

By a similar analysis the absolute invariants of the dependent variables w, r, κ and ρ are

$$\begin{aligned} &w(z,t) = \Gamma(t)F(\eta), \qquad r(z,t) = A(t)\Theta(\eta), \\ &q(z) = B(t)\Phi(\eta), \qquad g(z) = Y(t)\Psi(\eta). \end{aligned} \tag{2.24}$$

From which we conclude that

$$q(z) = \Phi(\eta), \tag{2.25}$$

$$g(z) = \Psi(\eta). \tag{2.26}$$

$$\text{At } t = 0: \quad \Gamma(0) = 0. \tag{2.27}$$

2.2.2 The reduction to ordinary differential equation

Substituting from (2.24) into (2.6) and dividing by Γ, we get

$$\beta\Psi F_{\eta\eta} + \beta\Psi_\eta F_\eta - \alpha\Phi\Gamma^{m-1}F^{m+1}\Gamma_t = -\frac{A(t)\Theta(\eta)}{\Gamma}. \tag{2.28}$$

For (2.28) to be reduced to an expression in the single independent invariant η, it is necessary that the coefficients should be constants or functions of η alone. Thus

$$\Gamma^{m-1}\Gamma_t = C_1, \qquad \frac{A(t)}{\Gamma} = C_2. \tag{2.29}$$

Take $C_1 = 1$:

$$\Gamma(t) = (mt)^{1/m}, \quad m \neq 0, \tag{2.30}$$

(2.29) and (2.30) yield:

$$A(t) = C_2(mt)^{1/m}, \quad m \neq 0. \tag{2.31}$$

Hence (2.28) may be rewritten as

$$\beta\Psi F_{\eta\eta} + \beta\Psi_\eta F_\eta - \alpha\Phi F^{m+1} = -C_2\Theta(\eta). \tag{2.32}$$

Following Girgis and Smith [18], we assume the heat source distribution in the form:

$$\Theta(\eta) = \mathrm{e}^{-\xi\eta}, \tag{2.33}$$

where ξ is the absorption coefficient for water, which has the value 0.048.

Take:

$$\Phi(\eta) = \frac{\Psi(\eta)}{F(\eta)}, \qquad F(\eta) \neq 0, \quad 0 \leq \eta \leq h. \tag{2.34}$$

According to (2.33) and (2.34), equation (2.32) takes the form

$$\beta\Psi F_{\eta\eta} + \beta\Psi_\eta F_\eta - \alpha\Psi F^m = -C_2\mathrm{e}^{-\xi\eta}. \tag{2.35}$$

Write:

$$\Psi = \mathrm{e}^{-\mu\eta}, \tag{2.36}$$

where μ is a constant, then (2.35) becomes

$$F_{\eta\eta} - \mu F_\eta - \frac{\alpha}{\beta}F^m = -\frac{C_2}{\beta}\mathrm{e}^{-(\xi-\mu)\eta}, \tag{2.37}$$

with the boundary conditions:

$$\text{(i)} \quad F(0) = 0, \tag{2.38}$$

$$\text{(ii)} \quad F(h) = \gamma/m^{(1/m)}. \tag{2.39}$$

2.2.3 Analytical solution for different forms of the parameters

For $m = 1$, differential equation (2.37) becomes

$$F_{\eta\eta} - \mu F_\eta - \sigma^2 F = -\frac{C_2}{\beta}\mathrm{e}^{-(\xi-\mu)\eta}; \qquad \sigma^2 = \frac{\alpha}{\beta} \tag{2.40}$$

and the boundary conditions become

$$\text{(i)} \quad F(0) = 0, \tag{2.41}$$

$$\text{(ii)} \quad F(h) = \gamma, \tag{2.42}$$

which has the exact solution

$$F(\eta) = a_1\mathrm{e}^{r_1\eta} + a_2\mathrm{e}^{r_2\eta} + a_3\mathrm{e}^{-(\xi-\mu)\eta}, \tag{2.43}$$

where

$$r_{1,2} = \frac{\mu \pm \sqrt{\mu^2 + 4\sigma^2}}{2},$$

r_1 for (+) sign and r_2 for (−) sign, and

$$a_3 = -\frac{C_2}{\beta(\mu-\xi)^2 - \mu\beta(\mu-\xi) - \alpha}.$$

For finite temperature, $a_1 = 0$, and applying condition (2.41), we get the solution in the form:

$$F(\eta) = a_3\left[\mathrm{e}^{-(\xi-\mu)\eta} + \frac{\xi-\mu}{r_2}\mathrm{e}^{r_2\eta}\right]. \tag{2.44}$$

Hence the temperature distribution across the lake, corresponding to case (1) is:

$$T(z,t) = T_0 - \frac{C_2 t}{\beta(\mu-\xi)^2 - \beta\mu(\mu-\xi) - \alpha}\left[\mathrm{e}^{-(\xi-\mu)z} + \frac{\xi-\mu}{r_2}\mathrm{e}^{r_2 z}\right]. \tag{2.45}$$

For $0 < t < 150$ (in days), following Girgis and Smith [18], we use the following values of the parameters: $C_2 = 2496$, $\beta = 12355$, $\mu = 1.439239\times10^{-4}$, $\xi = 0.048$, $h = 400$ meter, $T_0 = 4\,^\circ\mathrm{C}$. The obtained results are plotted in Fig.1 and Fig.2.

2.3 Case (2): $\rho = \alpha q(z)w^s$, $\kappa = \beta w^n$

Differential equation (2.4) takes the form

$$w^n w_{zz} + nw^{n-1}(w_z)^2 - \sigma^2 q(z)w^s w_t = -r(z,t). \tag{2.46}$$

Following the same analysis as in case (1), we get the following group G:

$$G: \quad \overline{z} = z, \quad \overline{t} = (C^w)^m t, \quad \overline{q} = (C^w)^{n+m-s}q, \quad \overline{r} = (C^w)^{n+1}r, \quad \overline{w} = C^w w. \tag{2.47}$$

Figure 1. Distribution of temperature T (time = 40 days) againts the lake depth "z" in meters, corresponding to case: $\rho = \alpha q(z)w$, $\kappa = \beta g(z)$ for different values of parameter "α".

Figure 2. Distribution of temperature for different times to constant "α" ($\alpha = 14095$), for $\rho = \alpha q(z)w$, $\kappa = \beta g(z)$.

The absolute invariants are:

$$\eta = z, \quad w(z,t) = \Gamma(t)F(\eta), \quad r(z,t) = A(t)\Theta(\eta), \quad q(z) = B(t)\Phi(\eta). \tag{2.48}$$

Again, it is clear that $B(t) = 1$; from which we get

$$q(z) = \Phi(\eta), \tag{2.49}$$

leading to

$$\overline{q} = q, \tag{2.50}$$

which is satisfied if and only if $(C^w)^{n+m-s} = 1$. That is $C^w = 1$ or $n = s - m$. C^w can not be unity. Hence we obtain the only possible case:

$$n = s - m. \tag{2.51}$$

Take

$$\Phi(\eta) = \frac{1}{F(\eta)}, \qquad F(\eta) \neq 0, \quad 0 \leq \eta \leq h, \tag{2.52}$$

$$\text{At} \quad t = 0: \quad \Gamma(0) = 0. \tag{2.53}$$

Figure 3. Distribution of temperature T (time = 40 days) againts the lake depth "z" in meters, corresponding to case: $\rho = \alpha q(z)w$, $\kappa = \beta$ for different values of parameter "α".

Figure 4. Distribution of temperature for different times to constant "α" ($\alpha = 13306$), for $\rho = \alpha q(z)w$, $\kappa = \beta g(z)$.

Following the same analysis as in section (2.2.2), we reach to the following ordinary differential equation:

$$F^n F_{\eta\eta} + nF^{n-1}(F_\eta)^2 - \sigma^2 F^s = -C_2 \mathrm{e}^{-\xi\eta}. \tag{2.54}$$

For the case when $m = s = 1$, and from (2.51), we find $n = 0$. Hence (2.54) becomes

$$F_{\eta\eta} - \sigma^2 F = -C_2 \mathrm{e}^{-\xi\eta}, \tag{2.55}$$

and the corresponding boundary conditions are:

$$\text{(i)} \quad F(0) = 0, \tag{2.56}$$

$$\text{(ii)} \quad F(h) = \gamma. \tag{2.57}$$

Applying boundary conditions (2.56) and (2.57), we get the solution

$$F(\eta) = a_1 \mathrm{e}^{\sigma\eta} + a_2 \mathrm{e}^{-\sigma\eta} + \frac{C_2}{\sigma^2 - \xi^2} \mathrm{e}^{-\xi\eta}. \tag{2.58}$$

For finite temperature, $a_1 = 0$, and applying boundary condition (2.56), we get the solution in the form:

$$F(\eta) = \frac{C_2}{\sigma^2 - \xi^2} \left(\mathrm{e}^{-\xi\eta} - \frac{\xi}{\sigma} \mathrm{e}^{-\sigma\eta} \right). \tag{2.59}$$

Hence the temperature distribution across the lake, corresponding to case (2) is:

$$T(z,t) = T_0 + \frac{C_2 t}{\sigma^2 - \xi^2} \left(\mathrm{e}^{-\xi z} - \frac{\xi}{\sigma} \mathrm{e}^{-\sigma z} \right). \tag{2.60}$$

For $0 < t < 150$ (in days), following Girgis and Smith [18], we use the following values of the parameters: $C_2 = 0.2014$, $\beta = 12391$, $\xi = 0.048$, $h = 400$ meter, $T_0 = 4\,^\circ\mathrm{C}$, the obtained results are plotted in Fig.3 and Fig.4.

3 Application (II): Unsteady free-convective boundary-layer flow on a non-isothermal vertical flat plate

3.1 Mathematical formulation

Consider a natural convective, laminar, boundary layer adjacent to a semi-infinite, vertical flat plate. The fluid is isothermal and of constant temperature $\overline{T}_\infty$, far from the plate. The plate has nonuniform surface temperature $\overline{T}_w > \overline{T}_\infty$ (heated plate case). The fluid has the following constant properties: "β" is the volumetric coefficient of thermal expansion, "ν" the kinematic viscosity, and "α"is the thermal diffusivity.

Along with the application of the Boussinesq and boundary-layer approximation, the equations of motion may be written as:

(1) Conservation of mass:

$$\overline{u}_{\overline{x}} + \overline{v}_{\overline{y}} = 0. \tag{3.1}$$

(2) Momentum equation:

$$\overline{u}_{\overline{t}} + \overline{u}\,\overline{u}_{\overline{x}} + \overline{v}\,\overline{u}_{\overline{y}} = g\beta(\overline{T} - \overline{T}_\infty) + \nu\overline{u}_{\overline{y}\,\overline{y}}. \tag{3.2}$$

(3) Energy equation:

$$\overline{T}_{\overline{t}} + \overline{u}\,\overline{T}_{\overline{x}} + \overline{v}\,\overline{T}_{\overline{y}} = \alpha\overline{T}_{\overline{y}\,\overline{y}}. \tag{3.3}$$

Boundary conditions:

$$\text{(i)} \quad \overline{v} = 0,\ \overline{u} = 0,\ \overline{T} = \overline{T}_w(\overline{x}, \overline{t}\,) \quad \text{at} \ \ \overline{y} = 0,\ \overline{t} > 0, \tag{3.4.1}$$

$$\text{(ii)} \quad \overline{u} = 0,\ \overline{T} = \overline{T}_\infty \quad \text{as} \ \ \overline{y} \to \infty. \tag{3.4.2}$$

Dimensionalize the variables according to:

$$\begin{aligned} &x = \frac{\overline{x}}{L}, \qquad y = (\mathrm{Gr})^{1/4}\frac{\overline{y}}{L}, \qquad T = \frac{\overline{T} - \overline{T}_\infty}{\Delta T}, \qquad \Theta = \frac{T}{T_w}, \\ &u = \frac{\overline{u}}{U}, \qquad v = (\mathrm{Gr})^{1/4}\frac{\overline{v}}{U}, \qquad t = U\frac{\overline{t}}{L}, \end{aligned} \tag{3.5}$$

where L is some arbitrary reference length, $\Delta T = \overline{T}_{\mathrm{ref}} - \overline{T}_\infty$, $\overline{T}_{\mathrm{ref}}$ is some arbitrary reference temperature, $U = (g\beta L\Delta T)^{1/2}$ is the Characteristic velocity, and $\mathrm{Gr} = \dfrac{g\beta L^3\Delta T}{\nu^2}$ is the Grashof number.

In dimensionalized form

The basic equations are:

$$u_x + v_y = 0, \tag{3.6}$$

$$u_t + uu_x + vu_y = T + u_{yy}, \tag{3.7}$$

$$T_t + uT_x + vT_y = \frac{1}{\mathrm{Pr}} T_{yy}; \qquad Pr = \frac{\nu}{\alpha} \text{ is the Prandtl number.} \tag{3.8}$$

The boundary conditions are:

$$\text{(i)} \quad v = 0, \quad u = 0, \quad T = T_w(x,t) \ \text{ at } \ y = 0, \ t > 0, \tag{3.9.1}$$

$$\text{(ii)} \quad u = 0, \quad T = 0, \quad \text{as} \quad y \to \infty. \tag{3.9.2}$$

From the continuity equation there exists a stream function $\Psi(x,y)$ such that

$$u = \frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\partial \Psi}{\partial x}. \tag{3.10}$$

Momentum and energy equations take the form:

$$\Psi_{yt} + \Psi_y \Psi_{yx} - \Psi_x \Psi_{yy} = \Theta T_w + \Psi_{yyy}, \tag{3.11}$$

$$T_w \Theta_t + \Theta (T_w)_t + T_w \Psi_y \Theta_x + \Theta \Psi_y (T_w)_x - T_w \Psi_x \Theta_y = \frac{1}{\mathrm{Pr}} T_w \Theta_{yy}. \tag{3.12}$$

Boundary conditions are:

$$\begin{aligned} &\Psi_x(x,0,t) = \Psi_y(x,0,t) = 0, \qquad \theta(x,0,t) = 1, \\ &\lim_{y\to\infty} \Psi_y(x,y,t) = 0, \qquad \lim_{y\to\infty} \Theta(x,y,t) = 0. \end{aligned} \tag{3.13}$$

3.2 Solution of the problem

A class of two-parameter group (a_1, a_2) has the form

$$\overline{S} = C^S(a_1,a_2) S + K^s(a_1,a_2), \tag{3.14}$$

where "S" stands for x, y, t; ψ, Θ, T_w, and C's and K's are real valued and differentiable functions with respect to a_1 and a_2.

The invariance analysis:

Invariance of the transformed momentum equation leads to the following group G:

$$G: \begin{cases} \overline{x} = (C^y C^\Psi) x + K^x, \quad \overline{y} = C^y y, \quad \overline{t} = (C^y)^2 t + K^t, \\ \overline{\Psi} = C^\Psi \Psi + K^\Psi, \quad \overline{T}_w = \left(\dfrac{C^\psi}{(C^y)^3} \right) T_w, \quad \overline{\Theta} = \Theta. \end{cases} \tag{3.15}$$

The absolute invariants:

A function $\eta(x,y,t)$ is an absolute invariant of a two-parameter group if η satisfies:

$$\begin{aligned} &(\alpha_1 x + \alpha_2)\eta_x + (\alpha_3 y + \alpha_4)\eta_y + (\alpha_5 t + \alpha_6)\eta_t = 0, \qquad \text{and} \\ &(\beta_1 x + \beta_2)\eta_x + (\beta_3 y + \beta_4)\eta_y + (\beta_5 t + \beta_6)\eta_t = 0, \end{aligned} \tag{3.16}$$

where

$$\alpha_1 = \frac{\partial C^x}{\partial a_1}(a_1^0, a_2^0), \quad \alpha_2 = \frac{\partial K^x}{\partial a_1}(a_1^0, a_2^0), \ \ldots,$$

$$\beta_1 = \frac{\partial C^x}{\partial a_2}(a_1^0, a_2^0), \quad \beta_2 = \frac{\partial K^x}{\partial a_2}(a_1^0, a_2^0), \ \ldots.$$

The only possible form for the absolute invariant η is:

$$\eta = \frac{y}{\sqrt{a_1 t + b_1}}, \tag{3.17}$$

where $a_1 = \alpha_5 = \beta_5$, $b_1 = \alpha_6 = \beta_6$ are constants.

Abd-el-Malek, *et al* [8] have shown that the case of $\eta = y/\sqrt{a_1 x + b_1 t + c}$ does not lead to any solution.

The complete set of absolute invariants:

The only possible forms for the absolute invariant Ψ and T_w are:

$$\Psi(x, y, t) = \Gamma(x, t) F(\eta), \tag{3.18}$$

$$T_w = T_0\,\omega(x, t). \tag{3.19}$$

3.2.1 Solution corresponding to the form of η in (3.17):

The corresponding differential equations are:

$$\begin{aligned} &F_{\eta\eta\eta} + \left(\frac{a_1}{2}\eta + F\right) F_{\eta\eta} - (F_\eta)^2 + a_1 F_\eta + \Theta = 0, \\ &\frac{1}{\mathrm{Pr}}\Theta_{\eta\eta} + \left(\frac{a_1}{2}\eta + F\right) \Theta_\eta + (2a_1 - F_\eta)\Theta = 0 \end{aligned} \tag{3.20}$$

with the boundary conditions:

$$F(0) = F_\eta(0) = 0, \quad \Theta(0) = 1, \quad F_\eta(\infty) = 0, \quad \Theta(\infty) = 0. \tag{3.21}$$

We get the following solution:

$$\begin{aligned} &T_w = \frac{x + b_2}{(0.4472 a_1 t + b_1)^2}, \qquad u = \frac{x + b_2}{a_1 t + b_1} F_\eta, \\ &v = -\frac{F}{\sqrt{a_1 t + b_1}}, \qquad q = (x + b_2)\frac{[-\Theta_\eta(0)]}{(a_1 t + b_2)^{5/2}}. \end{aligned} \tag{3.22}$$

3.3 Conclusion

It is clear from the obtained results (3.22) that the temperature profile overshoots in the region of the boundary layer near the plate. This phenomena occurs for values of $a_1 > 1$, and becomes stronger as a_1 increases. This means that this phenomenon is accomplished by those cases for which T_w decreases rapidly with time.

If we study the effect of Pr on the temperature profile we find that there is a rapid increase in θ near the plate. This becomes more evident for larger values of Pr. Also the thermal boundary-layer thickness decreases for increasing values of Pr.

4 Application (III): Dispersion of gaseous pollutants in the presence of a temperature inversion

4.1 Mathematical formulation

The gaseous pollutant is bounded from above by the ground surface and from below by the inversion layer, which is at height "h" from the ground surface. Assuming that the pollution, with concentration $C(x,y)$, is evenly distributed throughout the layer, and the mean concentration of the pollutant at $x = 0$ averaged over $0 \le y \le h$ is constant and equal to C_0. The diffusion of the pollutants takes place due to the wind that has a constant mean velocity $u = u(x)$ in the x-direction, and the eddy diffusivities κ_1 and κ_2 in the x and y-directions, respectively, are also independent of y.

The normalized steady state diffusion equation, that governs the dispersion of the gaseous pollutants is

$$u(x)C_x = \kappa_1(x)C_{xx} + \kappa_2(x)C_{yy}, \tag{4.1}$$

with the boundary conditions

$$\left.\begin{array}{ll} \kappa_1 C_y = \lambda\gamma C & \text{at } y = 0 \\ C_y = 0 & \text{at } y = 1 \\ C = 1 & \text{at } x = 0,\ 0 < y < 1 \\ C_x = 0 & \text{at } x \to \infty \end{array}\right\}, \qquad \gamma^2 = \frac{h}{u_0}, \tag{4.2}$$

where all x and y are scaled with respect to h, C with respect to C_0, u with respect to u_0, κ_1 and κ_2 with respect to $u_0 h$, and u_0 is a reference velocity. Values of λ classify two cases, case (1): $\lambda \ll 1$ corresponds to the case where no pollutant is absorbed by the ground, case (2): $\lambda \gg 1$ corresponds to the case where all pollutant is absorbed by the ground.

Introduce the non-dimensional function $\theta(x,y)$ and $C^*(x)$ such that

$$C(x,y) = \theta(x,y)C^*(x), \tag{4.3}$$

equation (4.1) becomes

$$u(C^*\theta_x + \theta C^*_x) = \kappa_1(C^*\theta_{xx} + 2\theta_x C^*_x + \theta C^*_{xx}) + \kappa_2 C^*\theta_{yy}. \tag{4.4}$$

4.2 Solution of the problem

Following the same analysis as we did in application (I), we find that the group G_1 which transforms invariantly the differential equation (4.4) and the boundary conditions (4.2), is in the form:

$$G_1 : \quad \begin{cases} \overline{x} = E^x(a)x, \quad \overline{y} = y, \quad \overline{u} = E^x(a)u, \\ \overline{\kappa}_1 = (E^x(a))^2\kappa_1, \quad \overline{\kappa}_2 = \kappa_2, \quad \overline{C}^* = E^{C^*}(a)C^*, \quad \overline{\theta} = \theta. \end{cases} \tag{4.5}$$

The absolute invariants of the independent and dependent variables are:

$$\begin{array}{lll} \eta = y, & F(x,u) = \dfrac{u}{x}, & G(x,\kappa_1) = \dfrac{\kappa_1}{x^2}, \\ \theta = \theta(y), & \kappa_2 = \kappa_2(x), & C^* = C^*(x). \end{array} \tag{4.6}$$

4.3 The reduction to ordinary differential equation

Substituting from (4.6) into (4.4) gives:

$$[\kappa_2 C^*]\theta_{yy} + [x^2 G C^*_{xx} - xFC^*_x]\theta = 0. \tag{4.7}$$

The requirement of reducing (4.7) to two ordinary differential equations, for some constant p^2, implies that:

$$[x^2 G C^*_{xx} - xFC^*_x] = p^2[\kappa_2 C^*]. \tag{4.8}$$

Under this assumption, (4.4) gives the ordinary differential equation of $\theta(y)$, namely $\theta_{yy} + p^2\theta = 0$, which has the solution

$$\theta(y) = A\cos p(y + \varepsilon). \tag{4.9}$$

Rearranging (4.8), we get

$$\alpha C^*_{xx} - 2\beta C^*_x - p^2 C^* = 0, \tag{4.10}$$

where, using (4.6), the constants α and β are

$$\alpha = \frac{x^2 G}{\kappa_2} = \frac{\kappa_1}{\kappa_2} \quad \text{and} \quad \beta = \frac{xF}{2\kappa_2} = \frac{u}{2\kappa_2}. \tag{4.11}$$

The ordinary differential equation (4.10) has the general solution

$$C^*(x) = A_1 \mathrm{e}^{m_1 x} + A_2 \mathrm{e}^{m_2 x}. \tag{4.12}$$

Applying the boundary conditions (4.2) as $x \to \infty$, for $\alpha > 0$, we get the solution

$$C^*(x) = A_1 \mathrm{e}^{mx}, \tag{4.13}$$

where

$$m = \beta - \sqrt{\beta^2 + \alpha p^2}. \tag{4.14}$$

Substituting (4.9) and (4.13) in (4.3), we get

$$C(x, y) = B\mathrm{e}^{mx}\cos p(y + \varepsilon). \tag{4.15}$$

Application of the boundary condition (4.2) at the inversion level determines $\varepsilon = -1$. Hence

$$C(x, y) = B\mathrm{e}^{mx}\cos p(y - 1). \tag{4.16}$$

Application of the boundary condition (4.2) at the ground surface yields

$$\tan p = \frac{\lambda\gamma}{\kappa_2 p}. \tag{4.17}$$

The constants "B" and "p" will be determined corresponding to the two limiting cases of λ.

4.4 Analytical solution corresponding to the two limiting cases of λ

Case (1): $\lambda \ll 1$, which corresponds to no pollutant absorbed by the ground.

Assuming $\dfrac{\lambda\gamma}{\kappa_2 p}$ to be a very small quantity leads to

$$p^2 = \frac{\lambda\gamma}{\kappa_2} \tag{4.18}$$

and hence, by applying the boundary condition (4.2), we find $B = 1$.

The concentration distribution for the case (1) is

$$C = \mathrm{e}^{mx}. \tag{4.19}$$

where "m" is given by (4.14).

Case (2): $\lambda \gg 1$, which corresponds to all pollutant absorbed by the ground.

In this case, from equation (4.17), we get

$$p = N\pi.$$

The concentration distribution model will be

$$C(x, y) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{mx} \cos[N\pi(y-1)], \tag{4.20}$$

where

$$m = \beta - \sqrt{\beta^2 + N^2\pi^2\alpha}, \qquad N = \frac{2n-1}{2}; \quad n = 1, 2, 3, \ldots\ . \tag{4.21}$$

The constants B_n will be determined as the cosine's Fourier coefficients of the expansion of the function: $C = 1$ in $0 < y < 1$ and $x = 0$, we get

$$B_n = 2\left(\frac{\sin N\pi}{N\pi}\right).$$

Hence the concentration distribution for the case (2) is

$$C(x, y) = 2\sum_{n=1}^{\infty} \left(\frac{\sin N\pi}{N\pi}\right) \mathrm{e}^{mx} \cos[N\pi(y-1)], \tag{4.22}$$

where "m" and "N" are given by (4.21).

4.5 Results and discussion

For the case where no pollutant is absorbed by the ground surface, it is found that the concentration distribution has the form

$$C = \exp\left\{\frac{1}{2\kappa_2}\left[u - \sqrt{u^2 + 4\lambda\kappa_1\gamma}\right] x\right\},$$

which is independent of "y" and mainly depends on u, κ_1, κ_2, h and λ. For very small λ or very large u, it is clear that no pollutant will be absorbed by the ground surface. Also we concluded that:

(1) As "u" increases, the absorbed pollutant by the ground surface will be less.

(2) As the eddy diffusivity ratio, α, increases the absorbed pollutant by the ground surface will be more.

For the case where all pollutant is absorbed by the ground surface, it is found that the concentration distribution has the form:

$$C = 2\sum_{n=1}^{\infty}\left(\frac{\sin N\pi}{N\pi}\right)\exp\left[\frac{1}{2\kappa_2}\left(u-\sqrt{u^2+4N^2\pi^2\kappa_1}\right)x\right]\cos[N\pi(y-1)].$$

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