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Matrix Realizations of Four-Dimensional Lie Algebras and Corresponding Invariant Spaces

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We have performed classification of nonequivalent realizations of solvable four-dimensional Lie algebras. Furthermore, the finite-dimensional invariant spaces are obtained which can be utilized for construction of exactly solvable matrix models of one-dimensional Schrödinger equation.

This paper is devoted to the application of realizations of four-dimensional Lie algebras for the construction of exactly solvable matrix models of one-dimensional Schrödinger equation.

The paper is organized as follows. In first section we perform the construction of realizations of solvable four-dimensional Lie algebras. Then from the realizations we pick out those for which we can construct a model. In second section we describe the procedure of obtaining the invariant spaces admitted by realizations of Lie algebras, which was found in first part of this paper, and present the invariant spaces.

1 Realizations of four-dimensional Lie algebras

We will construct nonequivalent realizations of four-dimensional real Lie algebras in class of matrix differential operators

$$Q = \xi(x)\partial_x + \eta(x), \tag{1}$$

where $\xi(x)$ is smooth real function, $\eta(x)$ is a complex matrix. Here and below $\partial_x = \frac{d}{dx}$.

Note that the classification of realizations of three-dimensional Lie algebras was done by R. Zhdanov in [1].

Abstract Lie algebras of dimension $n \leq 5$ have been classified by G.M. Mubarakzyanov in [2]. There are twelve algebras $L_{4,j}$ which are not direct sums of algebras of lower dimensions. Let us consider the algebra $L_{4,6}$ with non-zero commutation relations

$$L_{4,6}: \quad [Q_1, Q_4] = aQ_1, \quad [Q_2, Q_4] = bQ_2 - Q_3, \quad [Q_3, Q_4] = Q_2 + bQ_3, \quad (a \neq 0,\ b \geq 0).$$

From [3] we know that any one of the operators Q_i $(i = 1, \dots 4)$ may be equal to ∂_x or $\eta(x)$. Let $Q_1 = \partial_x$ and other of operators have the form (1):

$$Q_i = \xi_i(x)\partial_x + \eta_i(x), \qquad i = 2, 3, 4.$$

As $[Q_1, Q_2] = [Q_1, Q_3] = 0$, $[Q_1, Q_4] = aQ_1$ then $\xi_i = \alpha_i$, $\eta_i = A_i$, $i = 2, 3$, $\alpha_4 = ax$, $\eta_4 = A_4$, where $\alpha_i \in R, A_i$ are arbitrary constant matrices $r \times r$. Substituting $Q_1 = \partial_x$, $Q_2 = \alpha_2\partial_x + A_2$, $Q_3 = \alpha_3\partial_x + A_3, Q_4 = ax\partial_x + A_4$ into the commutation relations we obtain $(a - b)^2 = -1$. That is why if $Q_1 = \partial_x$ then there exist no realizations of algebra $L_{4,6}$ in class of operators (1).

Let $Q_1 = \eta(x)$ and other operators have the form (1). Then all $\xi_i(x)$ can not be equal to zero simultaneously.

If $\xi_2(x) \neq 0$, then the operator Q_2 may be reduced to the operator $Q_2 = \partial_x$. In this case from commutation relations it follows that $Q_3 = \alpha_3\partial_x + A_3$, $Q_4 = (b-\alpha_3)x\partial_x - A_3x + A_4$, where $\alpha_3 \in R$, A_i are arbitrary constant matrices. The check of the relation $[Q_3, Q_4] = Q_2 + bQ_3$ gives $\alpha_3^2 = -1$. Hence, in this case algebra $L_{4,6}$ has no realizations in the class of operators (1) too.

If $\xi_2(x) = 0$, then or $\xi_3(x) \neq 0$ or $\xi_3(x) = 0$ and $\xi_4(x) \neq 0$. The checking of commutation relations shows that in this case algebra the $L_{4,6}$ has no realizations in the class of operators (1).

If $\xi_2(x) = \xi_3(x) = 0$ and $\xi_4(x) \neq 0$, then operator Q_4 may be reduced to the operator $Q_4 = \partial_x$ and the checking of commutation relations for the algebra $L_{4,6}$ shows that $Q_1 = A\exp(-ax)$, $Q_2 = \exp(-bx)(B\cos x + C\sin x)$, $Q_3 = \exp(-bx)(C\cos x - B\sin x)$, where A, B, C are arbitrary non-zero $r \times r$ matrices which satisfy the commutation relations

$$[A, B] = [A, C] = [B, C] = 0.$$

Below we give the list of nonequivalent realizations of the four-dimensional Lie algebras $L_{4,j}$.

$L^1_{4,1}:$ $Q_1 = A, \quad Q_2 = B, \quad Q_3 = \partial_x, \quad Q_4 = Bx + C,$
$[A, B] = [A, C] = 0, \quad [B, C] = A.$

$L^2_{4,1}:$ $Q_1 = A, \quad Q_2 = -Ax + B, \quad Q_3 = \frac{1}{2}Ax^2 - Bx + C, \quad Q_4 = \partial_x,$
$[A, B] = [B, C] = [A, C] = 0.$

$L^1_{4,2}:$ $Q_1 = \partial_x, \quad Q_2 = \partial_x + A, \quad Q_3 = \beta\partial_x + B, \quad Q_4 = x\partial_x + C,$
$[A, B] = 0, \quad [A, C] = A, \quad [B, C] = A + B.$

$L^2_{4,2}:$ $Q_1 = A, \quad Q_2 = B, \quad Q_3 = \partial_x, \quad Q_4 = x\partial_x + Bx + C,$
$[A, B] = 0, \quad [B, C] = B, \quad [A, C] = aA.$

$L^1_{4,3}:$ $Q_1 = \partial_x, \quad Q_2 = A, \quad Q_3 = B, \quad Q_4 = x\partial_x + C,$
$[A, B] = [A, C] = 0, \quad [B, C] = A.$

$L^2_{4,3}:$ $Q_1 = A, \quad Q_2 = B, \quad Q_3 = \partial_x, \quad Q_4 = Bx + C,$
$[A, B] = [B, C] = 0, \quad [A, C] = A.$

$L^3_{4,3}:$ $Q_1 = Ae^{-x}, \quad Q_2 = B, \quad Q_3 = -Bx + C, \quad Q_4 = \partial_x,$
$[A, B] = [B, C] = [A, C] = 0.$

$L^1_{4,4}:$ $Q_1 = A, \quad Q_2 = B, \quad Q_3 = \partial_x, \quad Q_4 = x\partial_x + Bx + C,$
$[A, B] = 0, \quad [A, C] = A, \quad [B, C] = A + B.$

$L^2_{4,4}:$ $Q_1 = Ae^{-x}, \quad Q_2 = e^{-x}(Ax + B), \quad Q_3 = e^{-x}\left(\frac{1}{2}Ax^2 - Bx + C\right), \quad Q_4 = \partial_x,$
$[A, B] = [A, C] = [B, C] = 0.$

$L^1_{4,5}:$ $Q_1 = \partial_x, \quad Q_2 = \alpha\partial_x + A, \quad Q_3 = \beta\partial_x + B, \quad Q_4 = x\partial_x + C,$
$[A, B] = 0, \quad [A, C] = A, \quad [B, C] = B.$

$L^2_{4,5}:$ $Q_1 = A, \quad Q_2 = \partial_x + \epsilon B, \quad Q_3 = \partial_x + (1-\epsilon)B, \quad Q_4 = (\epsilon b + (1-\epsilon)a)x\partial_x + C,$
$[A, B] = 0, \quad [A, C] = A, \quad [B, C] = (\epsilon a + (1-\epsilon)b)B.$

$L^3_{4,5}:$ $Q_1 = Ae^{-x}, \quad Q_2 = e^{-ax}(\alpha\partial_x + B), \quad Q_3 = e^{-bx}(\beta\partial_x + C), \quad Q_4 = \partial_x,$
$[A, B] = -\alpha A, \quad [A, C] = -\beta B, \quad [B, C] = \alpha bC - \beta aB, \quad a = b.$

$L_{4,6}:$ $Q_1 = Ae^{-ax}, \quad Q_2 = e^{-bx}(B\cos x + C\sin x), \quad Q_3 = e^{-bx}(C\cos x - B\sin x),$
$Q_4 = \partial_x, \quad [A, B] = [A, C] = [B, C] = 0.$

$L^1_{4,7}:$ $Q_1 = A, \quad Q_2 = -Ax + B, \quad Q_3 = \partial_x, \quad Q_4 = x\partial_x - \frac{1}{2}Ax^2 + Bx + C,$
$[A, B] = 0, \quad [A, C] = 2A, \quad [B, C] = B.$

$L^2_{4,7}:$ $Q_1 = Ae^{-2x}, \quad Q_2 = Be^{-x}, \quad Q_3 = e^{-x}(Bx - C), \quad Q_4 = \partial_x,$
$[A, B] = [A, C] = 0, \quad [B, C] = -A.$

$L^1_{4,8}:$ $Q_1 = A, \quad Q_2 = \epsilon\partial_x + (\epsilon - 1)(Ax - B), \quad Q_3 = (1 - \epsilon)\partial_x + \epsilon(Ax + B),$
$Q_4 = (2\epsilon - 1)x\partial_x + C, \quad [A, B] = [A, C] = 0, \quad [B, C] = (1 - 2\epsilon)B.$

$L^2_{4,8}:$ $Q_1 = A, \quad Q_2 = e^{-x}(\alpha\partial_x + B), \quad Q_3 = e^{x}(\beta\partial_x + C), \quad Q_4 = \partial_x,$
$[A, B] = [A, C] = 0, \quad [B, C] = -\beta B - \alpha C + A, \quad \alpha\beta = 0.$

$L^1_{4,9}:$ $Q_1 = A, \quad Q_2 = \partial_x - \epsilon(Ax + B), \quad Q_3 = \partial_x + (1 - \epsilon)(Ax + B), \quad Q_4 = x\partial_x + C,$
$[A, B] = 0, \quad [A, C] = 2A, \quad [B, C] = (1 - 2\epsilon)B.$

$L^2_{4,9}:$ $Q_1 = Ae^{-(1+b)x}, \quad Q_2 = e^{-x}(\epsilon\alpha\partial_x + B), \quad Q_3 = e^{-bx}((1 - \epsilon)\beta\partial_x + C),$
$Q_4 = \partial_x, \quad [A, B] = -\epsilon(1 + b)A\alpha, \quad [A, C] = (\epsilon - 1)(1 + b)\beta A,$
$[B, C] = A + \epsilon\alpha bC + (\epsilon - 1)\beta B.$

$L_{4,10}:$ $Q_1 = A, \quad Q_2 = B\cos x + C\sin x, \quad Q_3 = C\cos x - B\sin x, \quad Q_4 = \partial_x,$
$[A, B] = [A, C] = 0, \quad [B, C] = A.$

$L_{4,11}:$ $Q_1 = Ae^{-2ax}, \quad Q_2 = e^{-ax}(B\cos x + C\sin x), \quad Q_3 = e^{-ax}(C\cos x - B\sin x),$
$Q_4 = \partial_x, \quad [A, B] = [A, C] = 0, \quad [B, C] = A.$

$L^1_{4,12}:$ $Q_1 = Ae^{-x}, \quad Q_2 = Be^{-x}, \quad Q_3 = \partial_x, \quad Q_4 = C,$
$[A, B] = 0, \quad [A, C] = -B, \quad [B, C] = A.$

$L^2_{4,12}:$ $Q_1 = A\cos x + B\sin x, \quad Q_2 = B\cos x - A\sin x, \quad Q_3 = \alpha\partial_x + C, \quad Q_4 = \partial_x,$
$[A, B] = 0, \quad [A, C] = A + \alpha B, \quad [B, C] = B - \alpha A.$

Here A, B, C are arbitrary constant $r \times r$ matrices, α, β are arbitrary constants, $\epsilon = 0, 1$.

In what follows we shall consider only 2×2 matrices. It is known [4], that any matrix may be reduced to one of the forms $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. After corresponding procedure we conclude that realizations of algebras $L^1_{4,1}$, $L^1_{4,2}$, $L^1_{4,3}$, $L^1_{4,4}$, $L^1_{4,7}$, $L^2_{4,7}$, $L^1_{4,9}$, $L_{4,10}$, $L_{4,11}$, $L^1_{4,12}$ has no models. Thus we will seek models for realizations of Lie algebras $L^2_{4,1}$, $L^2_{4,2}$, $L^2_{4,3}$, $L^3_{4,3}$, $L^2_{4.4}$, $L^1_{4,5}$, $L^2_{4,5}$, $L^3_{4,5}$, $L_{4,6}$, $L^1_{4,8}$, $L^2_{4,8}$, $L^2_{4,9}$, $L^2_{4,12}$.

2 Invariant spaces admitted by four-dimensional Lie algebras

The second step in construction of matrix models is description of invariant spaces for each of obtained realizations of four-dimensional Lie algebras. This step we will show for an example of realization of the Lie algebra $L_{4,6}$.

It is known [3], that invariant space corresponding to the operator $Q_4 = \partial_x$ have such form:

$$\Pi = \Pi^1 \oplus \Pi^2 = \sum_j \exp(\lambda_j x) P^{[m_j]} \vec{e}_1 + \sum_j \exp(\lambda_j x) R^{[n_j]} \vec{e}_2,$$

where $P^{[m_j]}$, $R^{[n_j]}$ are m_j, n_j-th degree polynomials in x.

Acting on Π by the operator $Q_1 = A\exp(-ax)$ we get

$$Q_1\Pi = A\exp(-ax)\sum_j \exp(\lambda_j x)P^{[m_j]}\vec{e}_1 + A\exp(-ax)\sum_j \exp(\lambda_j x)R^{[n_j]}\vec{e}_2$$
$$= \sum_j \exp((\lambda_j - a)x)P^{[m_j]}\lambda\vec{e}_1 + \sum_j \exp((\lambda_j - a)x)R^{[n_j]}(\lambda\vec{e}_2 + \vec{e}_1)$$
$$= \sum_j \exp((\lambda_j - a)x)\left(\lambda P^{[m_j]} + R^{[n_j]}\right)\vec{e}_1 + \sum_j \exp((\lambda_j - a)x)R^{[n_j]}\lambda\vec{e}_2.$$

Let $\lambda \neq 0$. Fix the minimum λ_1. Then $\lambda_1 - a < \lambda_1$. But this inequality is impossible. Hence, a polynomial near $\exp(\lambda_1 - a)$ must be $R^1 = 0$, and respectively $P^1 = 0$. Thus all polynomials are zero, and invariant space is empty. This case is not interesting for us, that is why we do not consider the case $\lambda = 0$. So, the result of the action Q_1 on Π has such form:

$$Q_1\Pi = \sum_j \exp((\lambda_j - a)x)R^{[n_j]}\vec{e}_1.$$

The invariant space will have the form:

$$\Pi_1 = \sum_k \left(\exp((\lambda_k - a)x)P^{[m_k]}\vec{e}_1 + \exp(\lambda_k x)R^{[n_k]}\vec{e}_2\right), \qquad n_k \leq m_k.$$

We act on Π_1 by the operator $Q_2 = \exp(-bx)(B\cos x + C\sin x)$:

$$Q_2\Pi_1 = \sum_k \exp((\lambda_k - b - i)x)R^{[n_k]}((b_2 + ic_2)\vec{e}_1 + (b_1 + ic_1)\vec{e}_2)$$
$$+ \sum_k \exp((\lambda_k - (a+b) - i)x)P^{[m_k]}(b_1 + ic_1)\vec{e}_1.$$

Again we fix minimum λ_1. Then degrees $\lambda_k - b - i < \lambda_1$, $\lambda_k - (a+b) - i < \lambda_k$. That is why $R^1 = P^1 = 0$ or $b_1 + ic_1 = 0$. In the first case the invariant space is empty. Thus we take the case for which $b_1 + ic_1 = 0$. Hence, the invariant space admitted by the operators Q_1, Q_2, Q_4 should have such form:

$$\Pi_2 = \sum_k \left(\exp((\lambda_k - a)x)P^{[m_k]}\vec{e}_1 + \exp((\lambda_k - b - i)x)S^{[r_k]}\vec{e}_1 + \exp(\lambda_k x)R^{[n_k]}\vec{e}_2\right),$$
$$m_k, r_k \geq n_k.$$

Finally, we act on the space Π_2 by operator $Q_3 = \exp(-bx)(C\cos x - B\sin x)$:

$$Q_3\Pi_2 = \sum_k \Big(\exp((\lambda_k - b + i)x)R^{[n_k]}((b_2 - ic_2)\vec{e}_1 + (b_1 - ic_1)\vec{e}_2)$$
$$+ \exp((\lambda_k - a - b + i)x)P^{[m_k]}(b_1 - ic_1)\vec{e}_1 + \exp((\lambda_k - 2b)x)S^{[r_k]}(b_1 - ic_1)\vec{e}_1\Big).$$

After similar actions we obtain that $b_1 - ic_1 = 0$. This equality is possible when $b_1 = c_1 = 0$. Hence invariant space admitted by the Lie algebra $L_{4,6}$ has the following form:

$$\Pi = \sum_k \Big(\exp((\lambda_k - a)x)P^{[m_k]}\vec{e}_1 + \exp((\lambda_k - b - i)x)S^{[r_k]}\vec{e}_1$$
$$+ \exp((\lambda_k - b + i)x)T^{[q_k]}\vec{e}_1 + \exp(\lambda_k x)R^{[n_k]}\vec{e}_2\Big), \qquad q_k, m_k, r_k \geq n_k.$$

Moreover, matrices A, B, C are of the following form:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}.$$

Below we adduce invariant spaces for rest of four-dimensional Lie algebras.

$$L^2_{4,1}: \quad \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2, \quad m_k \geq n_k + 2,$$
$$A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_1 E + c_2 \sigma_0.$$

$$L^3_{4,3}: \quad \Pi = \sum_k \exp((\lambda_k - 1)x) S^{[r_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2,$$
$$r_k \geq n_k + 1, \quad A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_1 E + c_2 \sigma_0.$$

$$L^2_{4,4}: \quad \Pi = \sum_k \exp((\lambda_k - 1)x) S^{[r_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2,$$
$$r_k \geq n_k + 2, \quad A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_2 \sigma_0.$$

$$L^3_{4,5}: \quad \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{c_1}{\beta} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{c_1}{\beta} + 1 + k\right) x\right) d_k^* \vec{e}_2,$$
$$K_1 \geq K_2 + 2, \quad d_k, d_k^* = \text{const}, \quad A = \lambda E + \sigma_0, \quad B = a\sigma_0,$$
$$C = \frac{c_1}{\beta} E + \frac{c_2}{\beta} \sigma_0 - a\sigma_+, \quad a = \pm 1.$$

$$L_{4,6}: \quad \Pi = \sum_k \exp((\lambda_k - a)x) P^{[m_k]} \vec{e}_1 + \sum_k \exp((\lambda_k - b - i)x) S^{[r_k]} \vec{e}_1$$
$$+ \sum_k \exp((\lambda_k - b + i)x) T^{[q_k]} \vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]} \vec{e}_2,$$
$$q_k, m_k, r_k \geq n_k, \quad A = \sigma_0, \quad B = b_2 \sigma_0, \quad C = c_2 \sigma_0.$$

$$L^2_{4,8}: \quad 1.\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{c_1}{\beta} - k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{c_1}{\beta} - k\right) x\right) d_k^* \vec{e}_2,$$
$$K_1 > K_2, \quad d_k, d_k^* = \text{const}, \quad A = \sigma_0, \quad B = \frac{1}{\beta} \sigma_0, \quad C = \frac{c_1}{\beta} E.$$
$$2.\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{b_1}{\alpha} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{b_1}{\alpha} + k\right) x\right) d_k^* \vec{e}_2,$$
$$K_1 > K_2, \quad d_k, d_k^* = \text{const}, \quad A = \sigma_0, \quad B = \frac{b_1}{\alpha} E, \quad C = \frac{1}{\alpha} \sigma_0.$$

$$L^2_{4,9}: \quad 1.\ \epsilon = 0,\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{c_1}{\beta} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{c_1}{\beta} + 2 + k\right) x\right) d_k^* \vec{e}_2,$$
$$K_1 \geq K_2, \quad d_k, d_k^* = \text{const}, \quad A = \sigma_0, \quad B = \frac{1}{b} \sigma_0, \quad C = c_1 E + c_2 \sigma_0 - 2\sigma_+.$$
$$2.\ \epsilon = 1,\ \Pi = \sum_{k=0}^{K_1} \exp\left(\left(-\frac{b_1}{\alpha} + k\right) x\right) d_k \vec{e}_1 + \sum_{k=0}^{K_2} \exp\left(\left(-\frac{b_1}{\alpha} + 1 + b + k\right) x\right) d_k^* \vec{e}_2,$$
$$K_1 \geq K_2, \quad d_k, d_k^* = \text{const}, \quad A = \sigma_0, \quad B = \frac{b_1}{\alpha} E + \frac{b_2}{\alpha} \sigma_0 - (1 + b)\sigma_+, \quad C = \frac{1}{\alpha} \sigma_0.$$

$$L_{4,12}^2: \quad 1.\ \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]}\vec{e}_1 + \sum_k \exp((\lambda_k - i)x) S^{[r_k]}\vec{e}_1 + \sum_k \exp((\lambda_k + i)x) W^{[s_k]}\vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]}\vec{e}_2 + \sum_k \exp((\lambda_k - i)x) T^{[q_k]}\vec{e}_2 + \sum_k \exp((\lambda_k + i)x) V^{[t_k]}\vec{e}_2,$$

$$n_k, t_k, q_k \le m_k, r_k, s_k, \quad A = \lambda E, \quad B = \alpha\lambda E, \quad C = c_1 E + (c_2 - c_1)\sigma_+.$$

$$2.\ \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]}\vec{e}_1 + \sum_k \exp((\lambda_k - i)x) S^{[r_k]}\vec{e}_1 + \sum_k \exp((\lambda_k + i)x) W^{[s_k]}\vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]}\vec{e}_2 + \sum_k \exp((\lambda_k - i)x) T^{[q_k]}\vec{e}_2 + \sum_k \exp((\lambda_k + i)x) V^{[t_k]}\vec{e}_2,$$

$$n_k, t_k, q_k \le m_k, r_k, s_k, \quad A = \lambda E, \quad B = \alpha\lambda E, \quad C = \nu E + \sigma_0.$$

$$3.\ \Pi = \sum_k \exp(\lambda_k x) P^{[m_k]}\vec{e}_1 + \sum_k \exp((\lambda_k - i)x) S^{[r_k]}\vec{e}_1 + \sum_k \exp((\lambda_k + i)x) W^{[s_k]}\vec{e}_1 + \sum_k \exp(\lambda_k x) R^{[n_k]}\vec{e}_2,$$

$$q_k, m_k, r_k \ge n_k, \quad A = \sigma_0, \quad B = b_2\sigma_0, \quad C = c_1 E + c_2\sigma_0 + \sigma_+.$$

Here $P^{[m_k]}$, $R^{[n_k]}$, $S^{[r_k]}$, $W^{[s_k]}$, $T^{[q_k]}$, $V^{[t_k]}$ are m_k, n_k, r_k, s_k, q_k, t_k-th degree polynomials in x correspondingly, $\sigma_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, λ, α, β, ν, b_1, b_2, c_1, c_2 are arbitrary constants, $\alpha, \beta \neq 0$.

3 Conclusions

The above realizations of Lie algebras and the corresponding invariant spaces will be used for construction of exactly solvable 2×2 matrix Schrödinger models in future works. What is more, Hermitian models present special interest since they describe physical models with real eigenvalues of Hamiltonians.

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