Lecture Notes: Elementary Point-Set Topology

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Chapter 1

Introduction

Welcome to topology! Topology is a *fundamental* branch of mathematics in that it depends on a relatively small amount of other mathematics, yet a relatively large amount of mathematics requires topology to some degree (at least implicitly). To be sure, this doesn't imply that the study of topology is easier (or harder) than other mathematical subjects! What this does imply is that a smaller variety of prerequisite coursework is required. For any mathematics course there are two types of prerequisite: coursework and "mathematical maturity". The first type is the requirement that the prospective student has successfully taken courses in subjects that the prospective course builds upon. The second type of prerequisite is harder to qualify and quantify. In brief, it is clear that as one studies mathematics, one develops a greater capacity and ability to deal with abstract ideas, and in a technical manner (e.g. understanding definitions and theorems, proving theorems, generalization, etc.). This capacity and ability is what is meant by mathematical maturity. Strictly speaking, no prerequisite coursework is required. Even the necessary set theory and logic will be covered within the course, albeit briefly. However, the minimum level of mathematical maturity required is closely approximated by that of an average American undergraduate mathematics major having successfully taken the standard calculus sequence, and 2 or more upper level courses requiring theorem proving (e.g. modern algebra, linear algebra, or especially advanced calculus).

Topology plays a role in a majority of mathematical subjects, but it has also played a major role in many of the most modern and innovative ideas in other fields, such as physics (e.g. superstring theory) and biochemistry (e.g. the structure of DNA). Topology is also a very interesting in its own right. For at least these reasons, it is becoming more and more important for undergraduate mathematics majors to be introduced to this subject, whether the student's ultimate goal is mathematics education (at any level), graduate school, or to provide support for a non-mathematical field.

The goal of this course is to provide a rigorous introduction to the basic ideas of *point-set* topology. As such, there will be no discussion of the fascinating topic of *algebraic topology*, as is often the case in more "conceptual" undergraduate courses in topology that discuss, e.g., "knots". A rigorous course in algebraic topology best fits in the current mathematical curriculum at the graduate level.

We will first cover the basic ideas in logic, set theory, and category theory that will be important for the subsequent course material. Since the "common denominator" of student mathematical background will include calculus, we next motivate the study of topology by reexamining some key results from calculus, such as the intermediate value theorem. At this stage we will also single out familiar ideas about the real number line (e.g. intervals) and functions (e.g. continuity), whose generalization leads directly to abstract topology.

Many of these ideas from calculus, as we learned them, depend heavily on the use of a *metric*, that is, a notion of "distance". For example, the open interval $(-1, 1)$ may be described as the set of points on the real line whose distance from 0 is less than 1. Recall that a function $f(x)$ is continuous at x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

and that this limit is defined in terms of the beloved ϵ's and δ's (cf. A.2) that appear in inequalities (e.g. $0 < |x - x_0| < \delta$) involving *absolute values of differences* (that is *distances*, or a *metric*!). Therefore, in order to provide a bridge from calculus to abstract topology, we next study *metric topology.* Topology does not require a metric, but if, like the real number line $\mathbb{R}$, there is a metric available, we may define our topology in terms of the metric. This is an advantage, since it allows us to take advantage of familiar calculus notions.

After metric topology, we are finally primed for the abstract study of topology. We first define the objects of the topological category: topological spaces, a fundamental notion being the *basic open set*, which is the generalization of the open interval in $\mathbb{R}$. In point-set topology, we wish to describe sets of points in detail, and the rest of the chapter introduces definitions to accomplish this.

With the basic idea of a topological space and various types of subsets of points, we finish off the basic description of the topological category by defining the morphisms of the category, and by learning how to create new topological spaces from given ones. These constructions are one of the most important and useful parts of a first course in point-set topology.

The remaining chapters study the various properties of topological spaces such as separation, compactness, and connectivity. Coverage of the remaining chapters will depend largely on the pace of the course, whose length will be assumed to be one semester. These topics are very important; they are not optional to the study of point-set topology. Nevertheless, due to time constraints, some portion will likely have to be left to a subsequent course.

Lastly, we note that there are appendices at the end that serve as a workbook for you fill in, allowing you to summarize, sort, and digest material as it is presented. Also, there is a list of symbols that will be used freely in the course, along with some examples of their usage.

Chapter 2

Foundations

The only background material we need is the standard foundational material, namely set theory and enough logic to communicate precisely. Additionally, we will take the opportunity to learn the simplest ideas of category theory, which, at the very least, will help us organize mathematical stuctures in general, and topological structures in particular.

Although much of the logic and set theory may be familiar from other coursework, it will be very important that this material is fresh in the mind. Furthermore, it will provide an opportunity to point out the results that play the most important role later on.

2.1 Logic

From logic, we will need only notation and implication.

2.1.1 Quantifiers

The symbol $\forall$ is a logical quantifier that is equivalent to any of the phrases "for every", "for each", and "for all". For example, we could write

"For every student, assign a grade of A"

as

"$\forall$ student, assign a grade of A".

The important point is that $\forall$ *strictly* means *for every,* not “all but one”, or “some”.

The existential quantifier $\exists$, on the other hand, is equivalent to either the phrase “there exists” or “there is”. So,

“there is a future topologist among us”

may be written as

“$\exists$ a future topologist among us”.

The important point here is that the statment says there is definitely one potential topologist among us, *maybe* more, but definitely more than zero potential topologists.

2.1.2 Implication

If P and Q are statements, then we write ”P implies Q” as

$$\mathsf{P} \Rightarrow \mathsf{Q}.$$

For example, “$x = |x|$ implies $x \geq 0$” could be written simply as

$$x = |x| \Rightarrow x \geq 0.$$

Similarly, “P does not imply Q” is written as

$$\mathsf{P} \nRightarrow \mathsf{Q}.$$

NOTE: If $\mathsf{P} \Rightarrow \mathsf{Q}$ is a true statement, then the *converse*, $\mathsf{Q} \Rightarrow \mathsf{P}$, may or may not be a true statement. In the absolute value example above, it is true that $\mathsf{Q} \Rightarrow \mathsf{P}$, (that is, $x \geq 0 \Rightarrow x = |x|$). In contrast, consider the statement

$$f(x) \text{ is a polynomial} \Rightarrow f(x) \text{ is continuous.}$$

However the converse is certainly not true:

$$f(x) \text{ is continuous } \nRightarrow f(x) \text{ is a polynomial.}$$

For example, $\sin x$ is not a (finite order) polynomial, but it is continuous.

In the case where $\mathsf{P} \Rightarrow \mathsf{Q}$ *and* $\mathsf{Q} \Rightarrow \mathsf{P}$, we write either

$$\mathsf{P} \Leftrightarrow \mathsf{Q} \text{ or } \mathsf{Q} \Leftrightarrow \mathsf{P}.$$

Another common notation for $\Leftrightarrow$ is "iff", both for which we say "if and only if". This is since for $\mathsf{P} \Rightarrow \mathsf{Q}$ we may say "P only if Q," and for $\mathsf{Q} \Rightarrow \mathsf{P}$ we may say "P if Q."

Exercise 2.1.1 *Come up with two implications* $\mathsf{P} \Rightarrow \mathsf{Q}$ *from any previous math course (e.g. calculus, algebra, etc.) such that the converses* $\mathsf{Q} \Rightarrow \mathsf{P}$ *are also true.*

Exercise 2.1.2 *Come up with two implications* $\mathsf{P} \Rightarrow \mathsf{Q}$ *from any previous math course such that the converses are false.*

2.1.3 Negation

The negation $\neg(\mathsf{P} \Rightarrow \mathsf{Q})$ of a statement $\mathsf{P} \Rightarrow \mathsf{Q}$ can be a subtle issue. Consider the statement:

$$\forall x \in [0,1], f(x) > 0.$$

This says that f is a positive function on the closed unit interval. One might think the negation of this statement may have something to do with having a strictly negative function on $[0,1]$ or something else. Logically, the negation is simply that "f is not a positive function on $[0,1]$". The symbolic phrase changes more substantially then the english:

$$\neg(\forall x \in [0,1], f(x) > 0.)$$

$$\Leftrightarrow$$

$$\exists x \in [0,1] \ni f(x) \leq 0.$$

NOTE: that $\forall$ was replaced by $\exists$, and $>$ was replaced by $\leq$. These "substitutions" happen quite often in these kinds of statements, but one must check that the substitutions result in the desired statement. Making substitutions blindly leads to logical errors.

Exercise 2.1.3 *Write out the* $\epsilon - \delta$ *definition of a limit symbolically. Negate it.*

2.1.4 Contrapositive

Logically, the statement

$$\mathsf{P} \Rightarrow \mathsf{Q}$$

is equivalent to its *contrapositive*:

$$\neg\mathsf{Q} \Rightarrow \neg\mathsf{P}.$$

The absolute value example above,

$$x = |x| \Rightarrow x \geq 0,$$

has as contrapositive

$$x < 0 \Rightarrow x \neq |x|,$$

which is reasonable!

Suppose we wish to prove (or disprove) a mathematical statement $\mathsf{P} \Rightarrow \mathsf{Q}$. It occasionally happens that it is easier to prove (disprove) the contrapositive. Since the contrapositive of a statement is the same as the statement, we are free to work with the contrapositive. This is a useful technique for proving (disproving) theorems, along with proof by contradiction and disproof by counterexample.

Exercise 2.1.4 *Write out symbolically the polynomial/continuity example above. Write out the contrapositive. Is it reasonable?*

2.2 Sets

The basic notions and notations of set theory are likely familiar, and as a result, our review will be rapid and nontrivial examples will be left to the text. One can go deeper into set theory and the foundations of mathematics in general. It is a very deep and interesting subject. However, we will not concern ourselves with questions such as "is the set of all sets a set?" [1]

[1] This is technically relevant, for example, in category theory, where we will consider the class of all topological spaces. In fact, the answer is negative(!), which is why "the *set* of all topological spaces" was avoided. We won't go very deeply into category theory, so we can safely use basic set theory. The interested reader is invited to read about the *Theory of Classes* or the *Gödel-Bernays axioms* for set theory.

2.2.1 Set

A set will be understood as a collection of objects (points, elements). In particular, there needs to be enough data about the collection to determine whether or not a given object is in the set or not. Thus,

"the collection of trees"

(written $\{trees\}$) is a set, while

"the collection of all trees of a certain type"

would not be a set. As stated, there is no way to tell if a particular tree is in the collection or not. Two sets A and B are called *equal,* and we write $A = B$, provided that they have the same elements:

$$s \in A \Leftrightarrow s \in B.$$

The standard set notation is of the form

$$\{x \in \mathbb{R} | 0 < x < 1\},$$

which, alternatively, can be written as $(0, 1)$ in interval notation. The generic format is

$$\{\text{element type} \mid \text{conditions on elements}\}.$$

In the $(0, 1)$ example, the elements of the set x were of the real number type, and the conditions on the real number elements were that they be greater than 0 and less than 1.

NOTE: A set is defined by membership of its elements. There is no order imposed on its elements. Thus, $\{1, 2, 3\} = \{2, 3, 1\} = \{3, 2, 1\}$. Imposing some sort of ordering amongst the elements of a set would be *additional* structure on the set. Furthermore, repetition of elements is not recognized, so that $\{1, 2, 3\} = \{2, 3, 1, 1\} = \{3, 3, 1, 2, 2, 1, 2\}$.

2.2.2 Subset

Definition 2.2.1 *Let A be a set. We say that B is a subset of A, $B \subset A$, provided that $b \in B \Rightarrow b \in A$.*

Example 2.2.1

$$(0,1) \subset [0,1],$$

$$\{f(x)|f \text{ is continuous}\} \subset \{f(x)|f \text{ is differentiable}\}.$$

The empty set, $\emptyset := \{\}$, is the set with no elements. It is a subset of every set.

NOTE: Two sets A and B have the same set of elements iff

$$A \subset B \text{ and } B \subset A.$$

This is provides a standard way to prove that two sets are equal.

2.2.3 Power Set

Definition 2.2.2 *Let A be a set. The power set of A, 2^A, is the set of all subsets:*

$$2^A := \{a \mid a \subset A\}.$$

Example 2.2.2

$$A = \{1,2,3\} \Rightarrow 2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

NOTE: In this example, A has 3 elements, and 2^A has $8 = 2^3$ elements. For sets having finitely many elements, this will always be true. This is the origin of the notation 2^A. Of course, in the case that A has infinitely many elements (e.g. $[0,1]$), the notation can no longer be connected to the "number" of elements. We will take up the issue of "number" of elements in a set properly in §2.4.

2.2.4 Operations on Sets

Complement

Definition 2.2.3 *Let A and B be two sets. The complement of A in B, written $B - A$, is obtained be removing those elements from B that are also elements of A :*

$$B - A := \{b \in B \mid b \notin A\}.$$

Example 2.2.3 *We illustrate the definition of the complement with a trivial example. Let*

$$A = \{1,2,3\},\ B = \{2,3,4,5\}.$$

Then,

$$B - A = \{4,5\}.$$

Union and Intersection

Definition 2.2.4 *The union $A\bigcup B$ of two sets is the set consisting of all elements from A and B:*

$$a \in C = A \bigcup B \Leftrightarrow a \in A \text{ or } a \in B.$$

Example 2.2.4 *Let A and B be as in example 2.2.3. Then,*

$$A \bigcup B = \{1,2,3\} \bigcup \{2,3,4,5\} = \{1,2,3,4,5\}.$$

Example 2.2.5 *(slightly less trivial)*

$$(0,1) \bigcup (\frac{1}{2}, 12) = (1, 12).$$

Definition 2.2.5 *The intersection $A\bigcap B$ of two sets is the set consisting of all elements common to A and B:*

$$a \in C = A \bigcap B \Leftrightarrow a \in A \text{ and } a \in B.$$

Example 2.2.6 *Let A and B be as in example 2.2.3. Then,*

$$A \bigcup B = \{1,2,3\} \bigcap \{2,3,4,5\} = \{2,3\}.$$

Example 2.2.7 *(slightly less trivial)*

$$(0,1) \bigcap (\frac{1}{2}, 12) = (\frac{1}{2}, 1),$$

$$[0,1] \bigcap [\frac{1}{2}, 2] = [\frac{1}{2}, 1].$$

Exercise 2.2.1 *Consider open, closed, half open/closed intervals on the real line. Determine when intersection of a pair of these intervals results in an open, closed, or half open/closed interval.*

It is straightforward to generalize the definition of union and intersection to an arbitrary number of sets as follows (see also the text). Let A_α be some collection of sets indexed by α (we will treat index sets more carefully later). For example, α may range over the integers $\mathbb{Z}$, or some finite set: $\alpha \in \{1, 2, 3, 4\}$. Then, the union $\bigcup_\alpha A_\alpha$ is the set of all elements from each of the sets A_α :

$$\bigcup_\alpha A_\alpha = \{a \mid \exists \alpha \ni \ a \in A_\alpha\}.$$

Similarly, the intersection $\bigcap_\alpha A_\alpha$ is defined to be the set of elements common to all of the A_α :

$$\bigcap_\alpha A_\alpha = \{a \mid \forall \alpha, \ a \in A_\alpha\}.$$

(looking ahead) We will see that a "topology" $\mathcal{T}(A)$ on a set A is, roughly, the specification of which subsets $\tau \in 2^A$ are to be considered the "open" subsets of A. Therefore, $\mathcal{T}(A) \subset 2^A$.

For the standard topology on the real line $\mathbb{R}$, a subset $\tau \in 2^{\mathbb{R}}$ is open iff it is the union of arbitrarily many open intervals. If this is the case for some subset τ, then $\tau \in \mathcal{T}(A)$.

A closed set $\beta \in 2^{\mathbb{R}}$ is, by definition, the complement of an open set:

$$\exists \ \tau \in \mathcal{T}(\mathbb{R}) \ni \ \beta = \mathbb{R} - \tau.$$

NOTE: It is important to recognize that in general, a subset $\mu \in 2^A$ may be neither open nor closed. In particular, just because μ isn't open (i.e. $\mu \in \mathcal{T}(A)$) $\mu \in 2^A - \mathcal{T}(A)$, we cannot conclude that μ is closed. For $\mathbb{R}$, the interval notation makes this relatively obvious. As we will see later, a subset can be open *and* closed!

Exercise 2.2.2 *Characterize, in terms of the union, intersection, and interval notation, those subsets $a \in 2^{\mathbb{R}}$ that are open, and those that are closed. Which subsets are neither open nor closed? Are there any subsets that are open and closed? (Word to the wise: Make strict use of the definition.)*

The following theorem describes how union, intersection, and complement behave with respect to one another. See the text for a partial, and slightly incorrect, proof (can you find the error?).

Theorem 2.2.1 *DeMorgan's Laws*
For each index α, *let* $S_\alpha \subset T$. *Then,*

$$\bigcup_\alpha (T - S_\alpha) = T - \bigcap_\alpha S_\alpha$$

and

$$\bigcap_\alpha (T - S_\alpha) = T - \bigcup_\alpha S_\alpha.$$

Cartesian Product

The Cartesian product of sets is a very fundamental and useful operation in mathematics, and is present every time we see, for example, the real plane $\mathbb{R}^2$. We will now view the basic definitions and examples, but further investigation will have to wait for our study of functions in §2.3.2.

Definition 2.2.6 *Let* S *be a set. The "Cartesian product"* $S \times T$ *of* S *with* T *is the set whose elements consist of all ordered pairs of elements from* S *and* T :

$$\{(a, b) \mid a \in S, b \in T\}.$$

NOTE: Unlike for sets, the ordering of the entries in the pairs is important, even in the case of $S \times S$.

Example 2.2.8 *Let* $(a, b) \in \ S \times S$. *Then,*

$$(a, b) \neq (b, a),$$

while

$$\{(a, b), (b, a)\} = \{(b, a), (a, b)\} \subset S \times T.$$

Since $S \times S$ is a set, we may consider the product $(S \times S) \times S$. A typical element of $(S \times S) \times S$ has the form

$$((a, b), c), a, b, c \in S.$$

Since all the information is in the *ordering* of the entries a, b, c, the inner pair of parenthesis are superfluous, and we write

$$((a, b), c) = (a, b, c).$$

Continuing this way, we may conclude

$$(S \times S) \times S = S \times S \times S \quad =: \quad S^3, \tag{2.1}$$
$$S \times S \times S = S \times (S \times S) \quad =: \quad S^3, \tag{2.2}$$
$$(S \times T) \times U = S \times (T \times U) \quad =: \quad S \times T \times U. \tag{2.3}$$

Similarly, we define Cartesian products with any number "factors" $S_1, S_2, S_3, \ldots$ as

$$S_1 \times S_2 \times S_3 \times \cdots := \{(a_1, a_2, a_3, \ldots) \mid a_i \in S_i \ \forall i = 1, 2, 3, \ldots\}.$$

NOTE: There are "projection" functions π_i naturally defined on any Cartesian product by

$$\pi_i \quad : \quad S_1 \times S_2 \times S_3 \times \cdots \to S_i \tag{2.4}$$
$$: \quad (a_1, a_2, a_3, \ldots) \mapsto a_i \tag{2.5}$$

Example 2.2.9 *Real 3-space and real n-space.*

$$\mathbb{R}^3 \quad := \quad \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\} \tag{2.6}$$
$$\mathbb{R}^n \quad := \quad \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots a_n \in \mathbb{R}\} \tag{2.7}$$

$$\pi_2 \ : \ \mathbb{R}^3 \to \mathbb{R} \quad : \quad (x, y, z) \mapsto y \tag{2.8}$$
$$\pi_i \ : \ \mathbb{R}^n \to \mathbb{R} \quad : \quad (a_1, a_2, a_3, \ldots, a_i, \ldots, a_n) \mapsto a_i. \tag{2.9}$$

Example 2.2.10 *Complex 2-space*[2] *and complex n-space.*

$$\mathbb{C}^2 \quad := \quad \mathbb{C} \times \mathbb{C} = \{(z, w) \mid z \in \mathbb{C}, w \in \mathbb{C}\} \tag{2.10}$$
$$\mathbb{C}^n \quad := \quad \mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C} = \{(z_1, z_2, \ldots, z_n) \mid z_1, z_2, \ldots z_n \in \mathbb{C}\} \tag{2.11}$$

$$\pi_2 \ : \ \mathbb{C}^2 \to \mathbb{R} \quad : \quad (z, w) \mapsto w \tag{2.12}$$
$$\pi_i \ : \ \mathbb{C}^n \to \mathbb{C} \quad : \quad (z_1, z_2, z_3, \ldots, z_i, \ldots, z_n) \mapsto z_i. \tag{2.13}$$

[2] We resist the terminology "complex plane" for $\mathbb{C}^2$ to avoid confusion with the "Argand plane," the latter being the representation of $\mathbb{C}$ as the real plane $\mathbb{R}^2$ via the identification $(x, y) \mapsto z = x + iy$.

2.3 Relations and Functions

The notion of a relation may be thought of as a generalization of a function $f : S \to T$, which assigns a *unique* element $f(s) = t \in T$ to each $s \in S$. The presence of "unique" in this definition is crucial, that is, if $t_1 = f(s)$ and $t_2 = f(s)$, then $t_1 = t_2$. Now, consider a typical calculus function such as

$$g \;:\; \mathbb{R} \to \mathbb{R} \;:\; x \mapsto g(x) = x^2.$$

An equivalent (and currently more useful) definition of a function is as a *subset* $F \subset S \times T$, where the connection between the assignment f and the subset F is given by:

$$F = \{(s, f(s)) \mid s \in S\} \subset S \times T.$$

We then see that for a subset F to contain the same information as the assignment f, the subset F must have the property that *every* $s \in S$ must appear as the first entry of a pair $(s,t) \in F$, and moreover, only in *one* such pair. For the quadratic function above, we have

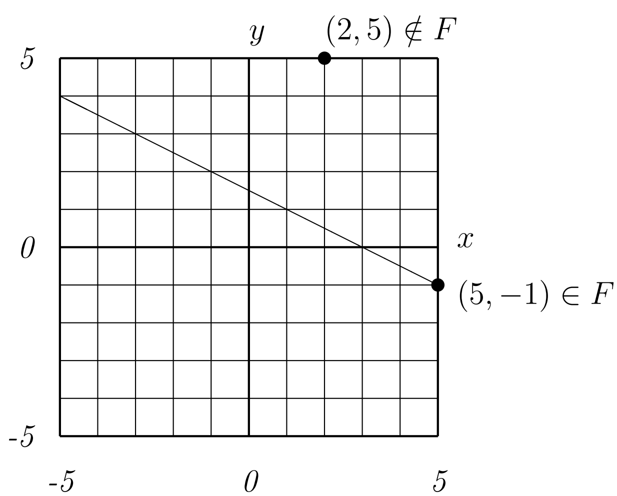
$$G = \{(x, x^2) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2.$$

NOTE: Recall from calculus that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is the one-dimensional subset of points in the real plane $\mathbb{R}^2$

$$graph(f) = \{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

Thus, the graph corresponds to the subset definition of a function and $F = graph(f)$.

Example 2.3.1 *Consider the function/assignment* $f(x) = -\frac{1}{2}x + \frac{3}{2}$. *The elements of the subset* $F \subset \mathbb{R}^2$ *may be visualized by the graph of* f :



2.3.1 Relations

A relation generalizes the subset definition of a function by removing all restrictions on the subset $\tilde{f}$.

Definition 2.3.1 *A relation between two sets S and T is simply a subset*

$$R \subset S \times T.$$

If $(s, t) \in R$, then we say that s and t are R-related.

NOTE: Since a relation is defined as a subset of a Cartesian product, the order of the entries is important, as we shall see, even in relations between S and S.

Example 2.3.2 *All functions $f : S \to T$ are relations, however, the converse is not true. The unit circle $\mathcal{S}^1 = \{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ defines a relation between $\mathbb{R}$ and $\mathbb{R}$, but not a function. Using the vertical line test, we note that the largest subsets of the unit circle that define functions are the upper and lower closed semi-circles:*

$$\mathcal{S}^1_+ := \mathcal{C} \bigcap \{(x, y) \mid y \geq 0\}$$

$$\mathcal{S}^1_- := \mathcal{C} \bigcap \{(x, y) \mid y \leq 0\}$$

NOTE: As in the above example, there are many relations that do not define functions, but that can be naturally broken up into further subsets, each of which define a function. This isn't always the case, since the whole of $\mathbb{R}^2$, being a subset of itself, defines a relation between $\mathbb{R}$ and $\mathbb{R}$, and it is completely unclear how to obtain a function from this relation.

Inverse relations

Associated with any relation R between S and T is a relation R^{-1} between T and S, called the *inverse* relation:

$$R^{-1} := \{(t, s) \in T \times S \mid (s, t) \in R\}.$$

Exercise 2.3.1 *What is the inverse relation R^{-1} associated to R, if R is the relation defining the unit circle (see example 2.3.2)? As always, justify your answer with discussion.*

For our purposes, and with such a simple definition, there is not much more to say about relations in general. However, as in the case of relations that are also functions, if we are willing to impose conditions on the relations, we obtain important kinds of relations.

Equivalence relations

By a relation on a set S, we mean a relation between S and S. An equivalence relation abstracts the notion of "=", as in equality numbers. In many contexts, including topology, it is useful to take a set and consider certain elements to be "equal", even if they are distinct initially. A ready example from algebra is integer modular arithmetic. Consider $\mathbb{Z}_2$, the integers modulo 2. In this case, all even integers are "equal", and all odd integers are "equal". Thus, in $\mathbb{Z}_2$, there are only two elements, the even element [0] and the odd element [1].

Definition 2.3.2 *An "equivalence relation" R on a set S is a relation on S satisfying the following conditions:* $\forall\, x, y, z \in S$,

1. xRx.
2. $xRy \Rightarrow yRx$.
3. $xRy,\ yRz \Rightarrow xRz$.

Example 2.3.3 *Consider*[3] *the relation = of equality in* $\mathbb{R}$

$$= \quad \text{``} = \text{''} \quad \{(x, x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

It is trivial to see that definition 2.3.2 is satisfied, since the 3 conditions amount to

$$x = x$$

$$x = x \Rightarrow x = x$$

$$x = x,\ x = x \Rightarrow x = x$$

This is certainly true for all real numbers x.

[3] Since we will be dealing with two different uses of the symbol =, we will add quotes to the equality of sets symbol, and leave the equality of real numbers sign as is.

The notion of a partition formalizes the idea of making elements of a set equal.

Definition 2.3.3 *A partition $\mathcal{P} \subset 2^S$ of a S is a choice of disjoint subsets of S (called "classes") whose union is S :*

$$\mathcal{P} = \{\mathcal{P}_\alpha\},$$

where

$$\mathcal{P}_\alpha \in 2^S,\ \mathcal{P}_\alpha \bigcap \mathcal{P}_\beta = \emptyset, \bigcup_\alpha \mathcal{P}_\alpha = S.$$

An equivalence relation os S defines a partition of S. Roughly, s and t belong to the same class $\mathcal{P}_\alpha$ iff $s \sim t$. In this case, the classes $\mathcal{P}$ are called "equivalence classes". It is also standard to denote the set of equivalence classes of a set S and an equivalence relation $\sim$ as

$$\mathcal{P} = S/\sim .$$

See the text for a partial proof.

Example 2.3.4 $\mathbb{Z}_3$
Consider the integers $\mathbb{Z}$. We may use the division algorithm to express any integer b as

$$b = 3n + a,\ a \in \{0, 1, 2\}.$$

That is, divide 3 into b, and denote the remainder by a. Since this remainder is unique, we may define a relation $\sim$ on $\mathbb{Z}$ by relating integers that have the same remainder upon division by 3.

In particular, $\forall\ b, \hat{b} \in \mathbb{Z}$, write $b = 3n + a, \hat{b} = 3\hat{n} + \hat{a}$. Then we define

$$b \sim \hat{b} \Leftrightarrow a = \hat{a}.$$

1. *$b \sim b$ since the remainder a is unique as we defined it.*

2. *Suppose $b \sim \hat{b}$. Then, $a = \hat{a}$ and since equality of integers defines an equivalence relation, $\hat{a} = a$, and it follows that $\hat{b} \sim b$. Thus, $b \sim \hat{b} \Rightarrow \hat{b} \sim b$.*

3. *Suppose $b \sim \hat{b}$ and $\hat{b} \sim \hat{\hat{b}}$. This implies $a = \hat{a}$ and $\hat{a} = \hat{\hat{a}}$. Again, since $=$ is an equivalence relation, we have $a = \hat{\hat{a}}$, and therefore, $b \sim \hat{\hat{b}}$.*

There are 3 equivalence classes determined by the three possible remainders 0,1,2. We denote these classes by $[0], [1], [2]$, *respectively. For example,* $6 \in [0], 43 \in [1], 335 \in [2]$. *One can go on to define addition and multiplication on the equivalence classes to make the ring* $\mathbb{Z}_3 = \{[0], [1], [2]\}$.

Example 2.3.5 $\mathbb{RP}^1$
Define a relation $\sim$ *on* $\mathbb{R}^2 - (0,0)$ *as follows.*

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow \exists \lambda \in \mathbb{R}, \lambda \neq 0 \ni \lambda(x_1, y_1) = (x_2, y_2).$$

Therefore, in terms of partitions, $[(x,y)] = \{(\hat{x}, \hat{y}) \in \mathbb{R}^2 - (0,0) \mid \lambda(\hat{x}, \hat{y}) = (x,y), \lambda \neq 0\}$. *We can get a geometric picture of the equivalence classes as follows. From elementary algebra, we know that a line* l *through the origin is completely determined by its slope* $m = \frac{\Delta y}{\Delta x}$, *and the special case of a vertical slope corresponds to* $m = 0$ *or* $m =$ "∞". *Given such a line* l, *its points* (x_1, y_1) *are characterized by the property that the ratio of its coordinates equals the slope of* l :

$$(x_1, y_1) \in l \ \Leftrightarrow \frac{y_1}{x_1} = m.$$

Now, suppose[4] $(x_1, y_1) \in l$, $m \neq 0$ *and* $m \neq \infty$, *so that* $\frac{y_1}{x_1} = m$. *Consider another point* $(x_2, y_2), x_2 \neq 0 \neq y_2$. *Set* $\lambda = \frac{y_2}{y_1} \neq 0, \infty$, *so that* $y_2 = \lambda y_1$. *Then,*

$$\begin{aligned}
(x_2, y_2) \in l \quad &\Leftrightarrow \quad \frac{y_2}{x_2} = m = \frac{y_1}{x_1} && (2.14)\\
&\Leftrightarrow \quad x_2 = \lambda x_1 && (2.15)\\
&\Leftrightarrow \quad \lambda(x_1, y_1) = (x_2, y_2) && (2.16)\\
&\Leftrightarrow \quad (x_1, y_1) \sim (x_2, y_2). && (2.17)
\end{aligned}$$

Therefore, an equivalence class $[(x_1, y_1)]$ *is a line through (but not including) the origin. Thus,*

$$\mathbb{RP}^1 := \mathbb{R}^2 - (0,0)/\sim \; = \{\text{lines through the origin }\}.$$

NOTE: A line through the origin in $\mathbb{R}^2$ corresponds to exactly a pair of points on the unit circle $\mathbb{S}^1$, namely the points on $\mathbb{S}^1$ that the line intersects. We will later see that $\mathbb{RP}^1$ is in fact *topologically indistinguishable* from $\mathbb{S}^1$.

[4] The special cases $m \neq 0$ and $m \neq \infty$ are handled similarly and more simply.

Example 2.3.6 *On* $\mathbb{R}$, *define the relation*

$$x \sim y \Leftrightarrow \exists\, z \in \mathbb{Z} \ni y = x + z.$$

This may be described as "integer translations". This is an equivalence relation. We may consider the interval of representatives $[0, 1]$, *noting that the only duplicate is* $0 \sim 1$. *Geometrically,* $\mathbb{R}/\mathbb{Z}$, *the space of equivalence classes, can be thought of as the unit interval with the endpoints 0 and 1 "glued" together. This is again the circle* $\mathbb{S}^1$.

2.3.2 Functions

Inverse functions

Set morphisms

Inverse relations

Index sets

2.4 Cardinality

2.4.1 Cardinality

finite

Infinite Sets

countably/uncountably infinite

2.5 Categories

- category (object,morphism)
- mono/epi/isomorphism
- examples
- metric and topological categories

Chapter 3

Motivation for Abstract Topology

3.1 Calculus

- Intermediate Value Theorem
- Maximum Value Theorem
- Uniform Continuity Theorem

3.2 The character of open and closed sets: intervals

- open and closed intervals in $\mathbb{R}$
- Euclidean metric
- unions and intersections
- continuity
- the strange from the familiar: the Cantor fractal set

Chapter 4

Metric Topology

- metric
- ϵ-ball ($D-p$ neighborhood)
- d-open set
- d-closed set
- convergent sequence
- continuous function

Definition 4.0.1 *Let $f\colon (X,d) \to (Y,\tilde{d})$ be a map between metric spaces. f is continuous if it satisfies one of the following equivalent conditions.*

Theorem 4.0.1 *Let $f\colon (X,d) \to (Y,\tilde{d})$ be a map between metric spaces. The following are equivalent:*

1. *Let $x_o \in X$ be fixed but arbitrary. Let $\epsilon > 0$ be given. There exists $\delta > 0\colon d(x,x_o) < \delta \Rightarrow \tilde{d}(f(x), f(x_o)) < \epsilon$*
2. *Let $x_o \in X$ be fixed but arbitrary. Let $\epsilon > 0$ be given. There exists $\delta > 0\colon x \in B^d_\delta(x_o) \Rightarrow f(x) \in B^{\tilde{d}}_\epsilon(f(x_o))$*
3. *Let $x_o \in X$ be fixed but arbitrary. Let $\epsilon > 0$ be given. There exists $\delta > 0$ such that $f(B^d_\delta(x_o)) \subset B^{\tilde{d}}_\epsilon(f(x_o))$, or equivalently, $B^d_\delta(x_o) \subset f^{-1}(B^{\tilde{d}}_\epsilon(f(x_o)))$.*

4. $U \overset{op}{\subset} Y \Rightarrow f^{-1}(U) \overset{op}{\subset} X$. *("Inverse images of open sets are open.")*
5. *For each sequence* $x_n \to x \in X$, $f(x_n) \to f(x) \in Y$.

Proof:

- $(1 \Leftrightarrow 2 \Leftrightarrow 3)$:
  Let $x_o \in X$ be fixed but arbitrary. Let $\epsilon > 0$ be given.
  Recall the definition of a δ-ball:

  $$B_\delta^d(x_o) = \{x \in X | d(x, x_o) < \delta\},$$

  and similarly for ϵ-ball.
  Then:
  $$\text{“}\exists\, \delta > 0 \colon d(x, x_o) < \delta \Rightarrow \tilde{d}(f(x), f(x_o)) < \epsilon\text{”}$$
  is equivalent to:

  $$\text{“}\exists\, \delta > 0 \colon x \in B_\delta^d(x_o) \Rightarrow f(x) \in B_\epsilon^{\tilde{d}}(f(x_o)),\text{”}$$

  which, since functions preserve inclusions, is equivalent to condition:

  $$\text{“}\exists \delta > 0 : f(B_\delta^d(x_o)) \subset B_\epsilon^{\tilde{d}}(f(x_o)),\text{”}$$

  which, since inverses preserve inclusions, is equivalent to:

  $$\text{“}\exists\, \delta > 0 \colon B_\delta^d(x_o) \subset f^{-1}(B_\epsilon^{\tilde{d}}(f(x_o))).\text{”}$$

- $(3 \Rightarrow 4)$
  Assume:
  For $x_o \in X$ fixed but arbitrary, for any $\epsilon > 0$, there exists $\delta > 0$ such that $B_\delta^d(x_o) \subset f^{-1}(B_\epsilon^{\tilde{d}}(f(x_o)))$.

  We need to show: $U \overset{op}{\subset} Y \Rightarrow f^{-1}(U) \overset{op}{\subset} X$.

To this end, let $U \overset{op}{\subset} Y$ $x \in f^{-1}(U)$. Since $U \overset{op}{\subset} Y$, U contains an ϵ-ball: $\exists \epsilon > 0$ such that $B_\epsilon^{\tilde{d}}(f(x)) \subset U$. By our assumption $\exists \delta > 0:$

$$B_\delta^d(x) \subset f^{-1}(B_\epsilon^{\tilde{d}}(f(x))). \tag{4.1}$$

But since f^{-1} preserves subsets, $B_\epsilon^{\tilde{d}}(f(x)) \subset U \Rightarrow f^{-1}(B_\epsilon^{\tilde{d}}(f(x)) \subset f^{-1}(U)$, and so togeher with (4.1) we have $B_\delta^d(x) \subset U$. In other terms, for an arbitrary point x in $f^{-1}(U)$, we can always find a δ-ball centered at x that is contained in $f^{-1}(U)$ — the very definition that $f^{-1}(U)$ be an open set.

– $(4 \Rightarrow 5)$

Assume:
$U \overset{op}{\subset} Y \Rightarrow f^{-1}(U) \overset{op}{\subset} X$.

We need to show:
For each sequence $x_n \to x \in X$, $f(x_n) \to f(x) \in Y$.
To this end, let $x_n \to x \in X$, so that for *any* δ-ball $B_\delta^d(x)$, a.b.f.m. $x_n \in B_\delta^d(x)$. We need to show that $f(x_n) \to f(x) \in Y$, i.e., that for any ϵ-ball, a.b.f.m. $f(x_n) \in B_\epsilon^{\tilde{d}}(f(x))$. But, any such $B_\epsilon^{\tilde{d}}(f(x))$ is an open set, and so by assumption, $f^{-1}(B_\epsilon^{\tilde{d}}(f(x)))$ is open (and therefore is a neighborhood of each of its points such as x). In particular, there is a δ-ball $B_\delta^d(x)$ which, by assumption, must contain a.b.f.m. x_n. In other terms, $S = \{x_n | x_n \in B_\delta^d(x)\} \subset B_\delta^d(x)$ contains a.b.f.m. x_n, and so $f(S) \subset f(B_\delta^d(x)) \subset f^{-1}(B_\epsilon^{\tilde{d}}(f(x)))$ contains a.b.f.m. $f(x_n)$. That is, $f(x_n) \to f(x) \in Y$.

– $(5 \Rightarrow 3)$

Assume:
For each sequence $x_n \to x \in X$, $f(x_n) \to f(x) \in Y$.

We need to show:
For $x_o \in X$ fixed but arbitrary, for $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta^d(x_o)) \subset B_\epsilon^{\tilde{d}}(f(x_o))$.
To this end, suppose by way of contradiction that $\exists f(x_o) \in Y$ and $\epsilon > 0$ such that $\forall \delta > 0$, $f(B_\delta^d(x_o)) \not\subseteq B_\epsilon^{\tilde{d}}(f(x_o))$. We now build a sequence $x_n \to x_o \in X$, such that $f(x_n) \nrightarrow f(x_o) \in Y$, that gives us the desired contradiction.

For each $n \in \mathbb{Z}^+$, choose an element $x_n \in B^d_{\frac{1}{n}}(x_o)$ such that $f(x_n) \notin B^{\tilde{d}}_{\epsilon}(f(x_o)) \neq \emptyset$. Then, $x_n \to x_o$, yet $f(x_n) \not\to f(x_o)$.
Q.E.D.

- set distance
- closure of a set

Chapter 5

The Category of Topological Spaces

Definition 5.0.2 *The topological category, $\mathcal{TOP}$, is the category whose objects are topological spaces and whose morphisms are continuous functions. The equivalences are homeomorphisms.*

5.1 Topologies and Topological Spaces

Definition 5.1.1 *Let X be a set, a collection of subsets $\tau \subset 2^X$ of X forms a topology on X iff it satisfies the following 3 conditions:*

1. $X \in \tau, \emptyset \in \tau$.
2. *Closure w.r.t. intersection:*
$$U \cap V \in \tau \text{ whenever } U, V \in \tau.$$
3. *Closure w.r.t. arbitrary unions:*
$$\bigcup_{\alpha \in \mathcal{I}} U_\alpha \in \tau \text{ whenever } \forall \alpha \in \mathcal{I}, U_\alpha \in \tau.$$

The pair (X, τ) is called a topological space.

5.1.1 Basic Examples

Lower limit topology, discrete, trivial topology, etc.

5.1.2 Bases

Ultimately, the key assumption used in proving the theorem that continuous functions preserve convergent sequences is the existence of the countable family

$$\{D\!-\!B_{\frac{1}{n}}(x)|n \in \mathbb{Z}^+\}$$

of open sets about any point x in the metric space (X, D). In fact, no other aspect of the metric D is required. This suggests that the assumption that X and Y be metric spaces is stronger than necessary for the aforementioned theorem. Thus, we abstract the salient property by defining the notion of a "neighborhood basis".

Definition 5.1.2 *Neighborhood Basis*
Let (X, τ) be a topological space. A neighborhood basis at $x \in X$" is a collection $\eta_x \subset \tau$ of open sets such that:

1. $x \in U \overset{op}{\subset} X,\ \forall\ U \in \eta_x$.
2. $\forall\ V \overset{op}{\subset} X,\ x \in V,$ *there is a member $U \in \eta_x$ such that $U \subset V$.*

Definition 5.1.3 *A subcollection $\mathcal{B} \subset \tau$ constitutes a basis for τ provided every open set $U \in \tau$ is a union of basis elements:*

$$\forall\ U \in \tau,\ U = \bigcup_{\alpha \in \mathcal{I}} B_\alpha \text{ for some } \{B_\alpha \,|\, \alpha \in \mathcal{I}\} \subset \mathcal{B}.$$

In other terms, given $U \in \tau$, we may associate a basis element $B_x \subset U$ with each $x \in U$, and then:

$$U = \bigcup_{x \in U} B_x \tag{5.1}$$

Example 5.1.1 *delta-balls form a basis.*

Definition 5.1.4 *The Countability Properties:*

1. *Let (X, τ) be a topological space. If each $x \in X$ has a countable neighborhood basis, then we say that X is first countable (written 1°).*
2. *Let (X, τ) be a topological space. If each $x \in X$ has a countable basis for τ, then we say that X is second countable (written 2°).*

Second countability is stronger than first countability. That is,

$$(X, \tau)\ \ 2^\circ \Rightarrow (X, \tau)\ \ 1^\circ,$$

but not conversely. For suppose (X, τ) is 2° with countable basis $\mathcal{B}$. Given any point $x \in X$,

$$\eta_x = \{U \in \mathcal{B} | x \in U\}$$

is a neighborhood basis for x (exercise). Thus, (X, τ) is 1°. However, the converse is false, as is proven by example 5.1.2.

Basis Recognition Theorem It is often the case that one starts not with a topology, but with a collection of subsets that one would like to have as a basis for a topology. For example, in the metric case, the δ-balls are a very natural collection of subsets to work with. However, not any collection of subsets may form a basis. For example, consider a basis $\mathcal{B}$ for a topology τ on a set X. Since basis elements $\{B_\alpha\}$ are themselves open sets, an intersection of basis elements must be open:

$$B, \tilde{B} \in \mathcal{B} \Rightarrow U := B \cap \tilde{B} \in \tau,$$

and an arbitrary union of basis elements must be open:

$$\{B_\alpha | \alpha \in \mathcal{I}\} \subset \mathcal{B} \Rightarrow V := \bigcup_{\alpha \in \mathcal{I}} B_\alpha \in \tau.$$

As with any open sets, U and V must, in turn, be expressible as a union of basis elements. V is already expressed as a union of basis elements, but for U, there must exist $\{B_\alpha | \alpha \in \mathcal{I}\} \subset \mathcal{B}$ such that:

$$U := B \cap \tilde{B} = \bigcup_{\alpha \in \mathcal{I}} B_\alpha.$$

It is this condition that is key to determining whether or not a collection $\mathcal{B}$ of subsets will or will not determine a topology for which it will be a basis. This is the content of the *basis recognition theorem* :

Theorem 5.1.1 *Let X be a set, $\mathcal{B}$ a collection of subsets of X satisfying*

1. *$\mathcal{B}$ covers X : $X = \bigcup_{B \in \mathcal{B}} B$;*

2. $\forall B, \tilde{B} \in \mathcal{B}, B \cap \tilde{B} = \bigcup_{\alpha \in \mathcal{I}} B_\alpha$, *for some* $\{B_\alpha \in \mathcal{B} | \alpha \in \mathcal{I}\}$.

Then $\mathcal{B}$ *is a basis for the topology defined by*

$$\tau = \{U \subset X | U = \bigcup_{\alpha \in \mathcal{I}} B_\alpha, \text{for some} \{B_\alpha \in \mathcal{B} | \alpha \in \mathcal{I}\}\}.$$

Proof: Fill in.

Example 5.1.2 *Consider the real line* $\mathbb{R}$ *together with the collection of "half-open" intervals*

$$\mathcal{B}_{ll} = \{[a, b) \subset \mathbb{R} \mid a < b\}.$$

1. (a) *Since*

$$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [-n, n) \subset \bigcup_{B \in \mathcal{B}_{ll}} B,$$

   $\mathcal{B}_{ll}$ *is a cover for* $\mathbb{R}$.

   (b) *There are 4 cases for the intersection* $U := [a, b) \cap [c, d)$:

      i. $[a, b) \subset [c, d) \Rightarrow U = [a, b)$.

      ii. $[c, d) \subset [a, b) \Rightarrow U = [c, d)$.

      iii. $a < c,\ b < d \Rightarrow U = [c, b)$.

      iv. $a > c,\ b > d \Rightarrow U = [a, d)$. *In each case* U *is again a half-open interval. Thus, by the basis recognition theorem,* $\mathcal{B}_{ll}$ *is the basis for a topology, the lower limit topology* τ_{ll}.

2. (X, τ_{ll}) *is* 1°, *since* $\forall x \in \mathbb{R}$,

$$\{[x, x + \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$$

   is a countable neighborhood basis for x.

3. *Let* $\tilde{\mathcal{B}}$ *be any basis for* τ_{ll}. *For each* $x \in \mathbb{R}$, *choose a basis element* B_x *satisfying* $x \in B_x \subset [x, x + 1)$. *If* $x \neq y$, *then* $B_x \neq B_y$. *Thus,* $\tilde{\mathcal{B}}$ *must be uncountable and* (X, τ) *is not* 2°.

Example 5.1.3 *Consider* $(\mathbb{R}^n, \tau)$, *where* τ *is the standard topology (e.g. defined by the pythagorian metric). This space is* 2° *(hence* 1°*), since* τ *has the countable basis*

$$\mathcal{B} = \{B_{\frac{1}{n}}(P) \mid n \in \mathbb{Z}^+, P \in \mathbb{Q}^n \subset \mathbb{R}^n\}.$$

5.1.3 Comparing Topologies

Definition 5.1.5 *Let X be a set. Let τ and $\tilde{\tau}$ be two (possibly different) topologies on X. We call τ coarser than $\tilde{\tau}$ iff $\tau \subset \tilde{\tau}$. In the case that $\tau \subsetneq \tilde{\tau}$, we call τ strictly coarser than $\tilde{\tau}$. Alternatively, we call $\tilde{\tau}$ finer, respectively strictly finer, than τ.*

Example 5.1.4 $\mathbb{R}^n$ *with trivial vs. standard vs. discrete topologies. Increasingly finer.*

The next theorem provides a way to compare two topologies by comparing their bases.

Theorem 5.1.2 *Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be bases for topologies τ and $\tilde{\tau}$ (respectively) on a set X. Then, τ is coarser than $\tilde{\tau}$ iff*

$$\forall B \in \mathcal{B}, \forall x \in B, \exists \tilde{B} \in \tilde{\mathcal{B}} \ni x \in \tilde{B} \subset B.$$

Proof:
($\Rightarrow$) Consider an arbitrary point x in an arbitrary basic open set $B \in \mathcal{B}$. If we suppose τ is coarser than $\tilde{\tau}$ so that $\tau \subset \tilde{\tau}$, we have that $B \in \tilde{\tau}$. It follows that we may write (cf. equation 5.1)

$$B = \bigcup_{x \in U} \tilde{B}_x,$$

Therefore, for each $x \in B$ there exists $\tilde{B}_x \in \tilde{\mathcal{B}}$ so that $x \in \tilde{B} \subset B$.
($\Leftarrow$) Suppose that $\forall x \in B \in \mathcal{B}, \exists \tilde{B} \in \tilde{\mathcal{B}} \ni x \in \tilde{B} \subset B$, and suppose that $U \in \tau$. We need to show $U \in \tilde{\tau}$. Since $\mathcal{B}$ generates τ, we may write

$$U = \bigcup_{x \in U} B_x$$

for some basis elements $B_x \in \mathcal{B}$ By assumption we may find $\tilde{B}_x \in \tilde{\mathcal{B}}$ for each $x \in U$ such that $x \in \tilde{B}_x \subset B_x$. Then,

$$U = \bigcup_{x \in U} \tilde{B}_x.$$

It follows that $U \in \tilde{\tau}$.
Q.E.D.

Example 5.1.5 *The lower limit topology τ_{ll} on $\mathbb{R}$ is strictly finer than the standard topology τ.*

Consider the standard bases $\mathcal{B}_{ll}$ (cf. example 5.1.2) and $\mathcal{B}$ for τ_{ll} and τ. $\mathcal{B}$ consists of all open intervals. Given such an open interval (a, b) and $x \in (a, b)$, we have $[x, b) \subset (a, b)$ and $[x, b) \in \mathcal{B}_{ll}$. Therefore τ_{ll} is finer than τ. In contrast, given $[x, y) \in \mathcal{B}_{ll}$, any $(a, b) \in \mathcal{B}$ containing x must contain some $c < x$. Therefore it must be that $(a, b) \not\subseteq [x, y)$. Thus, τ_{ll} is strictly finer than τ.

Consequences for continuity- fine domains and coarse codomains improve "chances for continuity" and conversely: identity map on X - theorem. Pictures of 2^X, etc.

5.2 Derived sets

We now make some definitions in order to analyze subsets of points topologically derived from a given subset of a topological space. For the following definitions, A is a subset of a topological space (X, τ). Each definition will be followed by simple, illustrative examples involving the real line with the standard open interval topology $(\mathbb{R}, \tau)$.

Definition 5.2.1 *The interior of A, denoted $\mathring{A}$, is defined to be the union of all open sets contained in A :*

$$\mathring{A} = \bigcup U, \text{where } U \overset{op}{\subset} X, U \subset A.$$

Example 5.2.1 *Let $A = (2, 3]$. Then $\mathring{A} = (2, 3)$. Note that every open set contained in A is contained in a basis element of the form $(2 + \frac{1}{n}, 3 - \frac{1}{n})$, $n \in \mathbb{Z}^+, n > 2$ and*

$$(2, 3) = \bigcup_{n \in \mathbb{Z}^+, n>2} (2 + \frac{1}{n}, 3 - \frac{1}{n}).$$

Example 5.2.2 *Let $A = (2, 3] \cup [4, 5] \cup \{8\}$. Then $\mathring{A} = (2, 3) \cup (4, 5)$.*

Definition 5.2.2 *The closure of A, denoted $\bar{A}$ or $\text{Cl}A$, is defined to be the intersection of all closed sets containing A :*

$$\bar{A} = \bigcap F, \text{where } F \overset{cl}{\subset} X, A \subset F.$$

Example 5.2.3 *Let* $A = (2, 3]$. *Then* $\bar{A} = [2, 3]$. *Note that every closed set* $F \neq [2, b], [a, 3]$ *containing* A *contains a basic closed set of the form* $[2 - \frac{1}{n}, 3 + \frac{1}{n}]$, $n \in \mathbb{Z}^+$, *and*

$$[2, 3] = \bigcap_{n \in \mathbb{Z}^+} (2 - \frac{1}{n}, 3 + \frac{1}{n}).$$

Example 5.2.4 *Let* $A = (2, 3] \cup [4, 5] \cup \{8\}$. *Then* $\bar{A} = [2, 3] \cup [4, 5] \cup \{8\}$.

Definition 5.2.3 *A point* $x \in X$ *is a limit point of* A *if and only if every neighborhood* U *of* x *intersects* A *in a point other than* x *itself:*

$$\forall\, U \overset{op}{\subset} X,\ x \in U,\ \ A \cap (U - \{x\}) \neq \emptyset$$

The set of all limit points of A *is denoted* A'.

NOTE: A limit point of A may or may not be a point of A.

Example 5.2.5 *Both* 2 *and* 3 *are limit poins of* $(1, 3)$. $A' = [1, 3] = \bar{A}$.

Example 5.2.6 *Let* $A = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. *Then* $A' = \{0\}$, *since every open set containing 0 contains* $\frac{1}{n}$ *for large enough* n. *More surprisingly,* $A' \cap A = \emptyset$. *This is because for every* $\frac{1}{n}$, *there is an open set containing* $\frac{1}{n}$ *but containing no other point of* A, *for example*

$$(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon),$$

where $\epsilon = \frac{1}{n} - \frac{1}{n+1}$.

Definition 5.2.4 A *is dense is* X *if and only if* $\bar{A} = X$.

Example 5.2.7 *Let* $A = \mathbb{Q}$. *Then* $\bar{A} = \mathbb{R}$. *That is,* $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

Chapter 6

Derived Spaces and Morphisms

6.1 Subspaces

- subspace topology
- subspaces

6.2 Morphisms

- morphism/continuous map
- equivalence/homeomorphism
- open/closed map

6.3 Quotient spaces

- quotient map
- quotient topology
- quotient space
- projective spaces

6.4 Product spaces

- projection maps
- product topology
- box topology
- product space

Chapter 7

Separation Axioms

7.1 Hausdorff spaces

- T_0 space
- T_1 space
- T_2 (Hausdorff) space
- pseudometric space vs. metric space
- inheritance

7.2 Regular spaces

- T_3 space
- regular space
- inheritance

7.3 Normal spaces

- T_4 space
- normal space
- inheritance

7.4 Function extensions

- Urysohn's Lemma
- Tietze's Extension Theorem

Chapter 8

Covering Properties and Metrization

8.1 Countability and Metrization

- open cover
- subcover
- refinement (of an open cover)
- Lindelöf spaces
- 1st countability
- 2nd countability
- separability
- Urysohn Metrization Theorem

8.2 Compactness

- compact spaces
- compactness and the separation axioms
- compactness and derived spaces

- compactness and morphisms
- local compactness
- compactness in Euclidean spaces

8.3 1-pt. Compactification

Chapter 9

Topological Connectivity

9.1 Connectedness

- connected sets
- path connected sets
- connectedness and derived spaces

9.2 Local connectedness

- locally connected sets
- connected components

Appendix A

Notation

A.1 Glossary

SYMBOL	MEANING
$\{\}$	"the set of" (set braces)
$\emptyset$	"the empty set"
$\forall$	"for every"
$\exists$	"there exists"
$\exists!$	"there exists a unique"
$\ni$	"such that"
$\Rightarrow$	"implies"
$\Longleftrightarrow$	"if and only if" or "is equivalent to"
$\in$	"is a member of"
$\bigcup$	"intersection"
$\bigcap$	"union"
$\subset$	"is a subset of"
$\propto$	"is proportional to"
$\longmapsto$	"maps to (the element...)"
$\longrightarrow$	"maps into (the set...)"
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^2$	The set of ordered pairs of real numbers (i.e. the Cartesia
$\mathbb{R}^n$	The set of ordered n-tuples of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{M}^{m \times n}$	The set of $m \times n$ matrices.
Q.E.D.	End of proof.
(Latin: Quod Erat Demonstrandum)	(lit. That which was to be shown)

A.2 Examples

•$\mathbb{R}^2$ is the set of ordered pairs of real numbers:

$$\mathbb{R}^2 = \{(x, y) \,|\, x \,\epsilon\, \mathbb{R}, y \,\epsilon\, \mathbb{R}\,\}.$$

• A parametric equation for the unit circle:

$$\alpha \,:\, \mathbb{R} \longrightarrow \mathbb{R}^2 \,:\, t \longmapsto (\cos t,\, \sin t).$$

• The definition of a limit of a function (real valued, one real variable): The limit of $f(x)$ as x approaches $x_)$ exists and equals $L \iff$

$$\forall \epsilon \,>0\, \exists \delta \,>\, 0 \,\ni\, |f(x) - L| \,<\, \epsilon$$

whenever

$$0 < |x - x_0| < \delta.$$

• Interval notation:

$$(0, 1) = (-\infty, 1) \bigcap (0, \infty) = (0, \frac{1}{2}) \bigcup [\frac{1}{2}, 1).$$

• Existence and uniqueness of the standard matrix of a linear transformation:
$\forall$ linear transformation $T : \mathbb{R}^p \longrightarrow \mathbb{R}^n,\ \exists!\, A \in \mathbb{M}^{n\times p} \,\ni\, \forall\, \vec{x} \,\in\, \mathbb{R}^p$,
$T(\vec{x}) = A\vec{x}$.

Appendix B

Exercises

Here are some hyperlinks to the exercises:

2.1.1	2.1.2	2.1.3	2.1.4
2.2.1	2.2.2		

Appendix C

Definitions

The remaining appendices form a workbook, wherein the student can assemble and organize the many definitions, important examples, theorems, interrelatioinships, that arise in point-set topology. It is basically a skeleton outline of the (more than) the course, and could therefore help organize any other items the student finds noteworthy.

C.1 Foundations

etc.

C.1.1 Logic

- quantifiers ($\forall, \exists$)
- implication ($\Rightarrow, \Leftarrow, \Leftrightarrow$)

- negation
- contrapositive

C.1.2 Sets

- set
- subset
- power set
- complement
- union
- intersection
- Cartesian product of sets

C.1.3 Relations

- relation
- inverse relation
- equivalence relation
- partition
- function
- restriction of a function

- inverse function
- composition of functions
- injective/sujective/bijective
- indexing function

C.1.4 Cardinality

- cardinality
- finite
- countably infinite
- uncountably infinite

C.1.5 Categories

- category (object,morphism)
- mono/epi/isomorphism
- the category $\mathcal{MET}$
- the category $\mathcal{TOP}$

C.2 Motivation for Abstract Topology

C.2.1 Calculus

C.2.2 The character of open and closed sets: intervals

- open/closed intervals in $\mathbb{R}$
- Euclidean metric
- union
- intersection
- continuity

C.3 Metric Topology

- metric
- ϵ-ball ($D - p$ neighborhood)
- d-open set
- d-closed set
- convergent sequence
- continuous function
- set distance
- closure of a set

C.4 The Category of Topological Spaces

C.4.1 Topologies

- topological space
- basis
- fineness/coarseness of a topology

C.4.2 Derived sets

- closure
- interior
- frontier
- exterior
- derived set
- limit point
- dense set

C.4.3 Derived Spaces and Morphisms

C.4.4 Subspaces

- subspace topology
- subspace

- open subspace
- closed subspace
- hereditary property

Definition C.4.1 *Hereditary*

Let $\mathcal{P}$ be a property of a topological space, such as being a "Hausdorff" space. $\mathcal{P}$ is called a *hereditary* property provided every subspace $Y \subset^{sspc} X$ has the property $\mathcal{P}$ whenever the "parent space" X does.

C.4.5 Morphisms

- morphism/continuous map
- equivalence/homeomorphism
- open/closed map

C.4.6 Quotient spaces

- quotient map
- quotient topology
- quotient space

C.4.7 Product spaces

- projection map
- product topology
- box topology
- product space

C.5 Separation Axioms

C.5.1 Hausdorff spaces

- T_0 space
- T_1 space
- T_2 (Hausdorff) space

C.5.2 Regular spaces

- T_3 space
- regular space

C.5.3 Normal spaces

- T_4 space
- normal space

C.5.4 Function extensions

C.6 Covering Properties and Metrization

C.6.1 Countability and Metrization

- open cover
- subcover
- refinement (of an open cover)
- Lindelöf space
- 1st countability
- 2nd countability
- separability

C.6.2 Compactness

- compact spaces
- locally compact

C.6.3 1-pt. Compactification

- 1-pt. compactification

C.7 Topological Connectivity

C.7.1 Connectedness

- disconnected set
- connected set
- path connected set

C.7.2 Local connectedness

- locally connected set
- connected component

Appendix D

Important Results

D.1 Foundations

etc.

D.1.1 Logic

- De Morgan's Laws

D.1.2 Sets

D.1.3 Relations

D.1.4 Cardinality

- a countable union of sets is countable
- a finite product of countable sets is countable
- a countable product of countable sets is *not* countable

- the power set of a countably infinite set is uncountable

D.1.5 Categories

D.2 Motivation for Abstract Topology

D.2.1 Calculus

- Intermediate Value Theorem
- Maximum Value Theorem
- Uniform Continuity Theorem

D.2.2 The character of open and closed sets: intervals in the real line

- arbitrary unions and finite intersections of open sets are open
- arbitrary intersections and finite unions of closed sets are closed
- $\mathbb{R}$ and $\emptyset$ are both open and closed sets
- one-point sets are closed
- $\mathbb{Q}$ is neither open nor closed

- the Cantor fractal set is uncountable yet has zero length, has no isolated points yet contains no interval, and is closed

D.3 Metric Topology

- ϵ-balls are open sets
- every open set is a union of ϵ-balls
- arbitrary unions and finite intersections of open sets are open
- arbitrary intersections and finite unions of closed sets are closed
- if $x_n \to x$, then every open set containing x contains all but finitely many x_n.
- sequence characterization of continuity: $f : X \to Y$ is continuous iff $f(x_n) \to f(x)$ is a convergent sequence in Y whenever $x_n \to x$ is a convergent sequence in X.
- metric independent characterization of continuity: inverse images of open sets are open

D.4 The Category of Topological Spaces

D.4.1 Topologies

-

D.4.2 Derived sets

-

D.4.3 Derived Spaces and Morphisms

D.4.4 Subspaces

-

D.4.5 Morphisms

-

D.4.6 Quotient spaces

-

D.4.7 Product spaces

-

D.5 Separation Axioms

D.5.1 Hausdorff spaces

-

D.5.2 Regular spaces

-

D.5.3 Normal spaces

-

D.5.4 Function extensions

- Urysohn's Lemma
- Tietze's Extension Theorem

D.6 Covering Properties and Metrization

D.6.1 Countability and Metrization

- Urysohn Metrization Theorem

D.6.2 Compactness

-

D.6.3 1-pt. Compactification

D.7 Topological Connectivity

D.7.1 Connectedness

-

D.7.2 Local connectedness

-

Appendix E

Subspace Inheritance

E.1 Metric Topology

- The restriction of a metric to a subset of a metric space is a metric. The restricted metric endows the subset with the structure of a metric space.
- A metric-open set restricted to a metric subspace is open.
- A metric-closed set restricted to a metric subspace is closed.
- Consider a metric subspace $Y \subset^{sspc} X$, and let Z be any metric space. The restiction $f|_Y : Y \to Z$ of a continuous function $f : X \to Z$ is continuous.

E.2 The Category of Topological Spaces

E.2.1 Topologies

- metric topology is inherited
- the restriction of basis elements to a subspace are basis elements for the subspace topology
- the restriction of an open set to a subsapce is open
- the restriction of a closed set to a subspace is closed

E.2.2 Derived sets

E.2.3 Morphisms

- Consider a topological subspace $Y \subset^{sspc} X$, and let Z be any topological space. The restiction $f|_Y : Y \to Z$ of a continuous function $f : X \to Z$ is continuous.

E.2.4 Quotient spaces

E.2.5 Product spaces

E.3 Separation Axioms

E.3.1 Hausdorff spaces

- a subspace of a T_0 space is T_0

- a subspace of a T_1 space is T_1
- a subspace of a T_2 (Hausdorff) space is T_2 (Hausdorff)

E.3.2 Regular spaces

- a subspace of a regular space is regular

E.3.3 Normal spaces

- only closed subspaces of normal spaces are normal

E.4 Covering Properties and Metrization

E.4.1 Countability and Metrization

- closed subspaces of Lindelöf spaces are Lindelöf
- subspaces of 1st countable spaces are 1st countable
- subspaces of 2nd countable spaces are 2nd countable
- open subspaces of separable spaces are separable

E.4.2 Compactness

- a closed subset of a compact space is compact
- only closed and bounded subsets of $\mathbb{R}^n$ are compact

- every open or closed subspace of a locally compact Hausdorff space is locally compact Hausdorff

E.4.3 1-pt. Compactification

- 1-pt. compactification

E.5 Topological Connectivity

E.5.1 Connectedness

- disconnected set
- connected set
- path connected set

E.5.2 Local connectedness

Appendix F

Special examples/counterexamples

F.1 Foundations

etc.

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- negation
- contrapositive

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- power set
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- path connected set

F.7.2 Local connectedness

- locally connected set
- connected component

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