

# Contact and equivalence of submanifolds of homogeneous spaces

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## 1 Introduction

The problem of equivalence of submanifolds of homogeneous spaces of Lie groups was extensively treated by F. Cartan by his method of moving frames [2]. A basic idea of Cartan's method is that for sufficiently high  $k$ ,  $G$ -contact of order  $k$  (see §4) implies  $G$ -equivalence. In other words, for each homogeneous space  $M$  there exists an integer  $k$ , depending on the dimension  $p$ , such that if two submanifolds  $S$  and  $\bar{S}$  of same dimension  $p$  have  $G$ -contact of order  $k$  then, there exists  $g \in G$  such that  $gS = \bar{S}$ . Cartan treated several important geometrical examples and proved in each case the existence of  $k$ .

Essentially, Cartan's method of proving the existence of the element  $g \in G$  consists in using the uniqueness of solution of a system of first order differential equations as in Frobenius theorem. Cartan's theory has been the subject of attention of a great number of authors (see for example [4], [5]). However, they all reduce the proof of the existence of the element  $g \in G$  to the uniqueness of solution of a first order differential system whereas it seems more natural and geometrical to deal directly with a higher order differential system.

The notion of contact element as defined by Ehresmann [3] allows a geometrical formulation of the theorem of existence and uniqueness of solution of higher order

completely integrable differential systems which is a straight forward generalization of Frobenius theorem (theorem 1). It is the uniqueness of this theorem that we use to solve the problem of  $G$ -equivalence. As a result, the regularity conditions on the submanifolds  $S$  and  $\bar{S}$ , which are necessary for the theorem of equivalence to hold (theorem 3), can be given a simple and geometrical definition, valid in any homogeneous space  $M$ . Also, in the method of moving frames, the invariants of a submanifold  $S$  of  $M$  are defined attaching special higher order frames to the points of  $S$ , [2], [5]. These frames are constructed by subtle geometrical arguments valid for a fixed homogeneous space whereas we construct the invariants of  $S$  as the elements of a complete set of invariants of the orbits of  $G$  acting on a manifold of higher order contact elements.

The equivalence problem may be posed for two immersions  $f, h : S \rightarrow M$  of a differentiable manifold  $S$ .  $f$  and  $g$  are equivalent if there exists  $g \in G$  such that  $h = L_g \circ f$  where  $L_g(x) = gx, x \in S$ . This fixed parametrization theorem has been treated by J.A.Vederesi [7] by means of a higher order differential system defined in a manifold of jets.

The paper ends with a necessary and sufficient condition for a submanifold  $S$  of  $M$  to be an open set of an orbit of a Lie subgroup  $K$  of  $G$ .

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## 2 Contact elements

All manifolds and maps considered in this paper are assumed to be differentiable of class  $C^\infty$ . If  $M$  and  $N$  are manifolds and  $f : M \rightarrow N$  is a map, the induced map on tangent spaces at points  $a \in M$  and  $b = f(a) \in N$  will be denoted by

$f_a : T_a M \rightarrow T_b N$ . Given integers  $p, k \geq 0$ ,  $p \leq \dim M$ ,  $J^{k,p} M$  denotes the manifold of all  $k$ -jets of rank  $p$  whose source is the origin of  $\mathbb{R}^p$  and whose target is any point of  $M$ . Let  $GL^k \mathbb{R}^p$  be the Lie group of invertible  $k$ -jets whose source and target are at the origin of  $\mathbb{R}^p$ . By definition, a contact element of order  $k$  and dimension  $p$  of  $M$  is an equivalence class of  $J^{k,p} M$  under the equivalence relation: for  $X, Y \in J^{k,p} M$ ,  $X \sim Y$  if there exists  $Z \in GL^k \mathbb{R}^p$  such that  $Y = X \circ Z$ . The set of contact elements of order  $k$  and dimension  $p$  of  $M$  is a differentiable manifold denoted by  $C^{k,p}(M)$ .  $C^{0,p}(M)$  identifies naturally with  $M$  [ 3 ].

For  $0 \leq k' \leq k$  there is a natural projection  $\pi_{k'}^k : C^{k,p}(M) \rightarrow C^{k',p}(M)$ . If  $k' = 0$ , we write  $\pi^k : C^{k,p} M \rightarrow M$  instead of  $\pi_0^k$ . The fiber of  $C^{k,p} M$  over  $a \in M$  is denoted by  $C_a^{k,p} M$ . If  $p$  is the dimension of  $M$ ,  $C_a^{k,p} M$  has only one element which is denoted by  $C_a^k M$  and is called the contact element of order  $k$  of  $M$  at the point  $a \in M$ .

Given a submanifold  $S$  of  $M$ ,  $S \subset M$ , and an integer  $p$ ,  $0 \leq p \leq \dim S$ , there is a natural injection of  $C^{k,p} S$  into  $C^{k,p} M$ . If  $p$  is the dimension of  $S$ , composing the map  $a \in S \rightarrow C_a^{k,p} S \in C^{k,p} M$  with the injection  $C^{k,p} S \rightarrow C^{k,p} M$ , we define an injection  $C^k : a \in S \rightarrow C_a^k S \in C^{k,p} M$ . The image of this injection is denoted by  $C^k S \subset C^{k,p} M$ . Two submanifolds  $S$  and  $\bar{S}$  of  $M$  of same dimension  $p$  have contact of order  $k$  at a common point  $a$  if  $C_a^k S = C_a^k \bar{S}$ .

### 3 Completely integrable differential systems of higher order

A differential system of order  $k \geq 1$  and dimension  $p$  defined over a manifold  $M$  is a submanifold  $\Omega^k$  of  $C^{k,p} M$  such that the projection  $\pi^k : \Omega^k \rightarrow M$  is of rank equal to the dimension of  $M$ . An integral manifold of  $\Omega^k$  is a submanifold  $S$  of  $M$  of dimension  $p$  such that  $C_x^k S \in \Omega^k$  for all  $x \in S$ . For  $X \in C^{k,p} M$ , let  $F_X$  be the fiber

of  $X$  by the projection  $\pi_{k-1}^k : C^{k,p}M \rightarrow C^{k-1,p}M$ . The symbol  $\sigma(X)$  of  $\Omega^k$  at the point  $X \in \Omega^k$  is by, definition, the vector space

$$\sigma(X) = T_X\Omega^k \cap T_XF_X.$$

Let  $X^{k+1} \in C^{k+1,p}M$ ,  $X^k = \pi_k^{k+1}(X)$ , and let  $S$  be a submanifold of  $M$  such that  $X^{k+1} \in C_a^{k+1}S$ ,  $a \in S$ . Then,  $C_{X^k}^1(C^kS)$  depends only on  $X^{k+1}$  and not on the choice of  $S$ . Hence, there is a natural imbedding

$$\bar{\Lambda}^{k,1} : C^{k+1,p}M \rightarrow C^{1,p}(C^{k,p}M)$$

which maps  $X^{k+1}$  into  $C_{X^k}^1(C^kS)$ . By definition, the first prolongation of the differential system  $\Omega^k$  is the subset  $\Omega^{k,1}$  of  $C^{k+1,p}M$  defined by

$$\Omega^{k,1} = (\bar{\Lambda}^{k,1})^{-1}[C^{1,p}(\Omega^k) \cap \bar{\Lambda}^{k,1}(C^{k+1,p}M)].$$

Since  $\pi_k^{k+1} = \pi_0^1 \circ \Lambda^{k,1}$ , it follows that  $\pi_k^{k+1}$  maps  $\Omega^{k,1}$  into  $\Omega^k$ . If  $S$  is an integral manifold of  $\Omega^k$  then,  $C_x^{k+1}S \in \Omega^{k,1}$  for every  $x \in S$ . Hence, a necessary condition for the existence of an integral manifold of  $\Omega^k$  going through every point of  $\Omega^k$  is that the projection  $\pi_k^{k+1} : \Omega^{k,1} \rightarrow \Omega^k$  be surjective.

**Theorem 1:** *Let  $\Omega^k \subset C^{k,p}M$  be a differential system of order  $k \geq 1$  and let  $X \in \Omega^k$  be a contact element such that*

- 1)  $\sigma(X) = \{0\}$ ;
- 2) *The image of  $\Omega^{k,1}$  by the projection  $\pi_k^{k+1} : \Omega^{k,1} \rightarrow \Omega^k$  is a neighborhood of  $X$  in  $\Omega^k$ .*

*Then, there exists an integral manifold  $S$  of  $\Omega^k$  such that  $X \in C^kS$ . Moreover, if  $S$  and  $S'$  are integral manifolds of  $\Omega^k$  such that  $X \in C^kS \cap C^kS'$ , there exists a set  $W$  which is an open neighborhood of  $X$  in  $C^kS$  and  $C^kS'$ .*

Theorem 1 is a geometrical version of the theorem of existence and uniqueness of solutions of completely integrable systems of partial differential equations of order  $k \geq 1$ . Taking suitable coordinates in  $C^{k+1,p}M$  and  $C^{k,p}M$ , the existence of integral manifolds of  $\Omega^k$  reduces to the existence of solutions of a completely integrable system of partial differential equations [6].

## 4 Contact of submanifolds

Let  $G$  be a Lie group acting transitively on the manifold  $M$ . Two submanifolds  $S$  and  $\bar{S}$  of  $M$  of same dimension  $p$ , have  $G$ -contact of order  $p$  at points  $a \in S$  and  $\bar{a} \in \bar{S}$  if there exists  $g \in G$  such that  $ga = \bar{a}$  and  $gS$  and  $\bar{S}$  have contact of order  $k$  at the point  $\bar{a}$ .  $S$  and  $\bar{S}$  have  $G$ -contact of order  $k \geq 0$  if there exists a diffeomorphism  $\phi : S \rightarrow \bar{S}$  such that for all  $x \in S$ ,  $S$  and  $\bar{S}$  have contact of order  $k$  at points  $x$  and  $\phi(x) = g(x)x$ . We say in this case that  $\phi$  makes contact of order  $k$  of  $S$  onto  $\bar{S}$ .  $S$  and  $\bar{S}$  are  $G$ -equivalent if there exists  $g \in G$  such that  $gS = \bar{S}$ .  $S$  and  $\bar{S}$  are locally  $G$ -equivalent at points  $a \in S$  and  $\bar{a} \in \bar{S}$  if there are open neighborhoods of  $a$  and  $\bar{a}$  in  $S$  and  $\bar{S}$  which are  $G$ -equivalent.

The action of  $G$  on  $M$  extends to an equivariant action on the manifold  $C^{k,p}M$  of contact elements of order  $k$  and dimension  $p$  of  $M$ . For a point  $x \in M$ , let  $C_x^k S$ ,  $G_x^k$  and  $d^k(x)$  denote respectively the contact element of order  $k$  of  $S$  at the point  $x$ , the isotropy subgroup of  $G$  at the point  $C_x^k S$  and the dimension of  $G_x^k$ . We call  $G_x^k$  the isotropy subgroup of order  $k$  of the point  $x$  of  $S$ . Put  $X = C_x^k S$  and let  $h^k(x)$  be the dimension of the vector space  $T_X(GX) \cap T_X C^k S$  where  $C^k S$  is the submanifold of  $C^{k,p}M$  of all contact elements of order  $k$  of  $S$  and  $T_X(GX)$  and  $T_X C^k S$  are the tangent spaces of the orbit  $GX$  and of  $C^k S$  at the point  $X$ .

For  $k' \leq k$ ,  $d^k(x) \leq d^{k'}(x)$  and  $h^k(x) \leq h^{k'}(x)$ . Hence, there exists an integer

$k \geq 1$  such that  $d^k(x) = d^{k-1}(x)$  and  $h^k(x) = h^{k-1}(x)$ . We say that  $a \in S$  is a  $k$ -regular point of  $S$  under the action of  $G$  if there exists  $k \geq 1$  such that

- 1)  $d^k(a) = d^{k-1}(a)$  and  $h^k(a) = h^{k-1}(a)$ ;
- 2)  $d^k(x)$  and  $h^k(x)$  are constant for  $x$  varying in a neighborhood of  $a$  in  $S$ .

The order of  $a$  is the least integer satisfying conditions above . If  $a$  is a  $k$ -regular point of  $S$  then  $ga$  is a  $k$ -regular point of  $gS$ .

**Theorem 2:** *Let  $S, \bar{S}$  be two submanifolds of  $M$  of same dimension  $p$ . Let  $a \in S$  and  $\bar{a} \in \bar{S}$  be two points. Assume that  $\bar{a}$  is a  $k$ -regular point of  $\bar{S}$  and that there exists a continuous map  $\varphi : V \rightarrow G$ , defined in a neighborhood  $V$  of  $a$  in  $S$ , such that  $\varphi(a).a = \bar{a}$ ,  $\varphi(x).x \in \bar{S}$  and  $\varphi(x).C_x^k S = C_{\varphi(x)}^k \bar{S}$  for all  $x \in V$ . Then, there exist open neighborhoods  $W$  and  $\bar{W}$  of  $a$  and  $\bar{a}$  in  $S$  and  $\bar{S}$  which are  $G$ -equivalent.*

The proof of theorem 2 is based on the uniqueness statement of theorem 1.

In theorems 3, 4, 5, 6, 8 below we assume that  $G$  is a compact Lie group and  $H$  is a closed subgroup of  $G$ . Let  $L$  be the union of all  $G$ -orbits of  $C^{k,p}M$  of type  $H$  that is, orbits whose isotropy subgroups are conjugate to  $H$ . Denote by  $L/G$  the quotient space of  $L$  by the orbits and by  $\pi : L \rightarrow L/G$  the natural projection. It is known [1] that  $(L, L/G, \pi, G/H, G)$  is a differentiable fiber bundle with structural group  $G$  and standard fiber  $G/H$ .

Let  $f : S \rightarrow \bar{S}$  be a diffeomorphism such that  $S$  and  $\bar{S}$  have  $G$ -contact of order  $k \geq 1$  at corresponding points  $x \in S$  and  $\bar{x} = f(x) \in \bar{S}$  and let  $a \in S$  and  $\bar{a} = f(a) \in \bar{S}$  be two points. Considering suitable cross sections of the fiber bundle  $(L, L/G, \pi, G/H, G)$  one can prove the existence of a neighborhood  $V$  of  $a$  in  $S$  and of a differentiable map  $\varphi : V \rightarrow G$  such that  $\varphi(x).x = f(x)$  and  $\varphi(x).C_x^k S = C_{\bar{x}}^k \bar{S}$ . Hence, theorem 2 can be restated as follows:

**Theorem 3** *Assume that  $G$  is compact and that there exists  $k \geq 1$  such that*

1.  $\bar{a} \in \bar{S}$  is a  $k$ -regular point.
2. The isotropy subgroups of  $C_x^k \bar{S}$  are conjugate in  $G$  for all  $\bar{x} \in \bar{S}$ .
3. There exists a diffeomorphism  $f : S \rightarrow \bar{S}$  such that  $S$  and  $\bar{S}$  have  $G$ -contact of order  $k$  at corresponding points.

*Let  $a \in S$  be such that  $f(a) = \bar{a}$ . Then  $S$  and  $\bar{S}$  are locally  $G$ -equivalent at points  $a$  and  $\bar{a}$ .*

**Theorem 4** *Assume that  $S$  and  $\bar{S}$  are connected and that there exists an integer  $k \geq 1$  such that:*

1.  $\bar{x} \in \bar{S}$  is a  $k$ -regular point of  $\bar{S}$  and  $h^k(\bar{x}) = 0$  for all  $\bar{x} \in \bar{S}$ .
2. The isotropy subgroups of  $C_x^k \bar{S}$  are conjugate in  $G$  for all  $\bar{x} \in \bar{S}$ .
3. There exists a diffeomorphism  $f : S \rightarrow \bar{S}$  such that  $S$  and  $\bar{S}$  have  $G$ -contact of order  $k$  at corresponding points.

*Then,  $f$  is the restriction to  $S$  of the translation by an element  $g$  of  $G$  :  $f = L_g|_S$ .*

Consider again the fiber bundle  $(L, L/G, \pi, G/H, G)$ . There exists a finite number of real valued differentiable functions  $\tilde{\rho}_i$ ,  $1 \leq i \leq r$ , defined in  $L$ , such that two contact elements  $X, \bar{X} \in L$  are in the same fiber of  $L$  if and only if  $\tilde{\rho}_i(X) = \tilde{\rho}_i(\bar{X})$ ,  $1 \leq i \leq r$ . Given a submanifold  $S$  of  $M$  of dimension  $p$ , and assuming that the orbits of  $C_x^k S$  are of type  $H$  for all  $x \in S$ , one can pull back the functions  $\tilde{\rho}_i$  by the map  $\sigma^k : x \in S \rightarrow C_x^k S \in L$ . The set of functions  $\rho_i = \tilde{\rho}_i \circ \sigma^k$ ,  $1 \leq i \leq r$ , is a complete set of  $G$ -invariants of order  $k$  of the submanifold  $S$  of  $M$ . Often the invariants can

be defined in a natural way and have deep geometrical meaning as for instance, the curvature and torsion of curves and the principal curvatures of surfaces in  $\mathbb{R}^3$ .

Assuming that the isotropy subgroups of  $C_x^k S$  and  $C_{\bar{x}}^k \bar{S}$  are of type  $H$  for all  $x \in S$  and  $\bar{x} \in \bar{S}$ , complete sets of invariants of order  $k$ ,  $\rho_i$  and  $\bar{\rho}_i$  can be defined in  $S$  and  $\bar{S}$ . Condition  $h^k(\bar{x}) = 0$  in theorem 4 is then clearly equivalent to stating that the rank of differentials  $d\bar{\rho}_i$ ,  $1 \leq i \leq r$ , is  $p$  at every point  $\bar{x} \in \bar{S}$ . One can then restate theorems 3 and 4 in the following way.

**Theorem 5 :** *Let  $\bar{a} \in \bar{S}$  be a  $k$ -regular point of  $\bar{S}$ ,  $k \geq 1$ . Assume following conditions are satisfied:*

1. *The isotropy subgroups of  $C_x^k S$  and  $C_{\bar{x}}^k \bar{S}$  are conjugate for all  $x \in S$  and  $\bar{x} \in \bar{S}$ .*
2. *There exists a diffeomorphism  $f : S \rightarrow \bar{S}$  such that*

$$\bar{\rho}_i = \rho_i \circ f, \quad 1 \leq i \leq r.$$

*Then,  $S$  and  $\bar{S}$  are locally  $G$ -equivalents at points  $a = f^{-1}(\bar{a})$  and  $\bar{a}$ .*

**Theorem 6 :** *Let  $S, \bar{S}$  be two submanifolds of  $M$  and let  $k \geq 1$  be such that*

1. *Every point  $\bar{x} \in \bar{S}$  is  $k$ -regular.*
2. *The isotropy subgroups of  $C_x^k S$  and  $C_{\bar{x}}^k \bar{S}$  are conjugate for all  $x \in S$  and  $\bar{x} \in \bar{S}$ .*
3. *There exists a diffeomorphism  $f : S \rightarrow \bar{S}$  such that*

$$\rho_i = \bar{\rho}_i \circ f, \quad 1 \leq i \leq r.$$

4. *The rank of differentials  $d\bar{\rho}_i$ ,  $1 \leq i \leq r$ , is  $p$  at every point  $\bar{x} \in \bar{S}$ .*



Then,  $f$  is the restriction to  $S$  of the left translation by an element of  $G$ :  $f = L_g|_S$ .

Let us assume that  $S$  is an open set of an orbit of a Lie subgroup  $K$  of  $G$ . Then,  $h^k(x) = p$  and the isotropy subgroups of  $C_x^k$  are conjugate for all  $x \in S$  and  $k \geq 0$ . Hence there exists  $k \geq 1$  such that every  $x \in S$  is a  $k$ -regular point of  $S$ . Conversely,

**Theorem 7 :** *A necessary and sufficient condition for a connected submanifold  $S$  of  $M$  to be an open set of an orbit of a Lie subgroup  $K$  of  $G$  is the existence of  $k \geq 1$  such that for all  $x \in S$ ,  $x$  is a  $k$ -regular point of  $S$  and  $h^k(x) = p$ .*

Assuming  $G$  compact and the isotropy subgroups of order  $k$  of points of  $S$  conjugate, a complete set of invariants of order  $k$  can be defined on  $S$ . Clearly,  $h^k(x) = p$  for every  $x \in S$  if and only if the invariants are constant on  $S$ . Therefore, the following corollary to theorem 7 holds.

**Theorem 8 :** *Assume  $G$  compact and  $S$  connected. Assume also that for some integer  $k \geq 1$ , every point of  $S$  is  $k$ -regular and all isotropy subgroups of order  $k$  of points of  $S$  are conjugate. Then, a necessary and sufficient condition for  $S$  to be an open set of an orbit of a Lie subgroup of  $G$ , is that the invariants of order  $k$  of  $S$  be constant.*

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