Topics in Classical Algebraic Geometry. Part I

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Chapter 1

Polarity

1.1 Polar hypersurfaces

1.1.1 The polar pairing

Let E be a finite-dimensional vector space over a field K of characteristic 0 and E^* be its dual space of linear functions. We have a canonical bilinear pairing

$$\langle , \rangle : E \otimes E^* \to K. \tag{1.1}$$

which can be extended to a bilinear pairing

$$S^k E \otimes S^d E^* \to S^{d-k} E^*, \quad d \geq k, \tag{1.2}$$

where S^k denotes the symmetric power of a vector space. The proof follows easily by first extending (1.1) to tensor products and then using the universal property of symmetric product. Explicitly, it can be described as follows. Pick a basis $(\xi_0, \ldots, \xi_n)$ of E and let $(T_0, \ldots, T_n)$ be the dual basis in E^*. We can identify an elements $S^d E^*$ with a homogeneous polynomial F of degree d in the variables $T_0, \ldots, T_n$ and an element of $S^k E$ with a homogeneous polynomial Φ of degree k in variables ξ_i. Since

$$\langle \xi_i, T_j \rangle = \delta_{ij},$$

we can identify ξ_i with the partial derivative operator $\partial_i = \frac{\partial}{\partial T_i}$ and hence view any element $\Phi \in S^k E$ as a differential operator

$$P_\Phi = \Phi(\partial_0, \ldots, \partial_n).$$

, The pairing (1.2) becomes

$$\langle \Phi(\xi_0, \ldots, \xi_n), F(T_0, \ldots, T_n) \rangle = P_\Phi(F). \tag{1.3}$$

For any monomial $\boldsymbol{\partial}^{\mathbf{i}} = \partial_0^{i_0}\cdots\partial_n^{i_n}$ of degree k and any monomial $\mathbf{T}^{\mathbf{j}} = T_0^{j_0}\cdots T_n^{j_n}$ of degree m, we have

$$\boldsymbol{\partial}^{\mathbf{i}}(\mathbf{T}^{\mathbf{j}}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-)\mathbf{i}!}\mathbf{T}^{\mathbf{j}-\mathbf{i}} & \text{if } \mathbf{j}-\mathbf{i}\geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

Here and later we use the vector notation:

$$\mathbf{i}! = i_0!\cdots i_n!, \quad \mathbf{i} = (i_0,\ldots,i_n)\geq 0 \Leftrightarrow i_0,\ldots,i_n\geq 0, \quad ,|\mathbf{i}| = i_0+\ldots+i_n.$$

This gives an explicit expression for the pairing (1.2). Consider a special case when

$$\Phi = (a_0\partial_0+\ldots+a_n\partial_n)^k = k!\sum_{|\mathbf{i}|=k}(\mathbf{i}!)^{-1}\mathbf{a}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}.$$

Then

$$P_\Phi(F) = k!\sum_{|\mathbf{i}|=k}(\mathbf{i}!)^{-1}\mathbf{a}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}(F). \tag{1.5}$$

Let $\mathbb{P}(E) = \mathbb{P}^n$. We view any $F\in S^dE^*$ as a section of $\mathcal{O}_{\mathbb{P}^n}(d)$, and denote by $V(F)$ the hypersurface of degree d in $\mathbb{P}^n$ defined by F. It is considered as an effective divisor in $\mathbb{P}^n$, not necessary reduced. We can also view $\Phi\in S^kE$ as a polynomial defining a hypersurface of degree k in the dual projective space $\check{\mathbb{P}}^n = \mathbb{P}(E^*)$ (an *envelope* of *class* k).

We view $a_0\partial_0+\ldots+a_n\partial_n\neq 0$ as a point $a\in\mathbb{P}(E)$ with projective coordinates $(a_0,\ldots,a_n)$.

Definition 1.1. *The hypersurface $V(P_{a^k}(F))$ of degree $d-k$ is called k*th polar hypersurface *of the hypersurface $V(F)$ with respect to the point a. Abusing the notation we set*

$$P_a(V(F)) = V(P_{a^k}(F)).$$

Example 1.1.1. Let $d=2$, i.e.

$$F(T_0,\ldots,T_n) = \sum_{i=0}^n \alpha_{ii}T_i^2 + \sum_{0\leq i<j\leq n}\alpha_{ij}T_iT_j$$

is a quadratic form. Then

$$P_a(F) = \sum_{i=0}^n a_i\frac{\partial F_2}{\partial T_i} = 2\sum_{0\leq i,j\leq n}a_i\alpha_{ij}T_j, \quad a_{ji} = a_{ij}.$$

The linear map $a\mapsto\frac{1}{2}P_a(F)$ is a map from E to E^* which can be identified with an element of $E^*\otimes E^* = (E\otimes E)^*$ which is the polar bilinear form associated to F with matrix (a_{ij}).

Let us give another definition of the polar hypersurfaces. Choose two different points $a = (a_0, \ldots, a_n)$ and $b = (b_0, \ldots, b_n)$ in $\mathbb{P}^n$ and consider the line $l = \langle a, b\rangle$ spanned by the two points as the image of the map

$$\phi : \mathbb{P}^1 \to \mathbb{P}^n, \quad (t_0, t_1) \mapsto t_0 a + t_1 b$$

(a parametric equation of l). Then the intersection $l \cap V(F)$ is isomorphic to the positive divisor on $\mathbb{P}^1$ defined by the degree d homogeneous form

$$\phi^*(F) = F(t_0 a + t_1 b) = F(t_0 a_0 + t_1 b_0, \ldots, t_0 a_n + t_1 b_n).$$

Using the Taylor formula at $(0, 0)$, we can write

$$\phi^*(F) = \sum_{k+m=d} \frac{d!}{m!k!} t_0^m t_1^k A_{km}(a, b), \tag{1.6}$$

where

$$A_{km}(a, b) = \frac{\partial^d \phi^*(F)}{\partial t_0^k \partial t_1^m}(0, 0).$$

Using the Chain Rule, we get

$$\begin{aligned} A_{km}(a, b) &= k! \sum_{|\mathbf{i}|=k} (\mathbf{i}!)^{-1} \mathbf{a}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}}(F)(b) = P_{a^k}(F)(b) = P_{b^m a^k}(F) \qquad (1.7)\\ &= m! \sum_{|\mathbf{j}|=m} (\mathbf{j})^{-1} \mathbf{b}^{\mathbf{j}} \boldsymbol{\partial}^{\mathbf{j}}(F)(a) = P_{b^m}(F)(a) = P_{a^k b^m}(F). \end{aligned}$$

Note the symmetry

$$A_{km}(a, b) = A_{mk}(b, a). \tag{1.8}$$

When we fix a and let b vary in $\mathbb{P}^n$ we obtain a hypersurface of degree $d-k$ which is the kth polar hypersurface of $V(F)$ with respect to the point a. When we fix b and vary a, we obtain the mth polar hypersurface of $V(F)$ with respect to the point b. When we vary both a and b, we get a hypersurface

$$P_{a^k b^m}(V(F)) \subset \mathbb{P}^n \times \mathbb{P}^n \tag{1.9}$$

of bi-degree $(d-k, d-m)$.

Since we are in characteristic 0, $P_{a^m}(F) \neq 0$ for $m \leq d$. To see this we use the *Euler formula*:

$$dF = \sum_{i=0}^n T_i \frac{\partial F}{\partial T_i}.$$

Applying this formula to the partial derivatives we obtain

$$d(d-1)\dots(d-k+1)F = k!\sum_{|\mathbf{i}|=k}(\mathbf{i}!)^{-1}\mathbf{T}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}(F) \tag{1.10}$$

(also called the Euler formula). It follows from this formula that

$$a \in V(P_{a^k}(F)) \Leftrightarrow a \in V(F). \tag{1.11}$$

In view of (1.9), we have

$$b \in V(P_{a^k}(F)) \Leftrightarrow a \in V(P_{b^{d-k}}(F)). \tag{1.12}$$

1.1.2 First polars

Let us consider some special cases. Let $F \in S^dE^*$ be a homogeneous form of degree d. Obviously, any 0-th polar of $V(F)$ is equal to $V(F)$, and, by (1.12), the d-th polar $V(P_{a^d}(F))$ is empty if $a \notin V(F)$ and $\mathbb{P}^n$ if $a \in V(F)$. Now take $k = 1, d-1$. We have, using (1.7)

$$P_a(F) = \sum_{i=0}^{n} a_i\frac{\partial F}{\partial T_i},$$

$$P_{a^{d-1}}(F) = \sum_{i=0}^{n}\frac{\partial F}{\partial T_i}(a)T_i.$$

Together with (1.12) this implies the following.

Theorem 1.1.1. *For any $b \in V(F)$ let $PT(V(F))_b$ denote the (embedded) tangent hyperplane of $V(F)$ at b if b is a nonsingular point or $\mathbb{P}^n$ otherwise. Then*

$$PT(V(F))_b = V(P_{b^{d-1}}(F)).$$

For any $b \in F$,

$$b \in V(P_a(F)) \Leftrightarrow a \in PT(V(F))_b.$$

The set of first polars $V(P_a(F))$ defines a linear system in the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d-1)|$. The dimension of this system is $\leq n$.

Proposition 1.1.2. *The dimension of the linear system of first polars $\leq r$ if and only if, after a linear change of variables, the polynomial F becomes a polynomial in $r+1$ variables.*

Proof. Induction on n and $n-r$. The assertion is obvious if $r=n$. Assume $r=n-1$. Let $\sum c_i\partial_i(F)=0$ be a nontrivial linear relation between the first partial derivatives. Consider an invertible linear change of variables

$$T_i=\sum_{j=0}^{n} a_{ij}U_j, \quad i=0,\ldots,n,$$

where $a_{i0}=c_i, i=0,\ldots,n$. By the Chain Rule,

$$\frac{\partial F}{\partial U_0}=\sum_{i=0}^{n} c_i\frac{\partial F}{\partial T_i}=0.$$

This proves the assertion in this case. Assume $r<n-1$. By induction on $n-r$, we may assume that, after a linear change of variables, F depends only on the variables $U_0,\ldots,U_{r+2}$. By induction on n, after a further change of variables, we may assume that F depends only on variables $V_0,\ldots,V_{r+1}$. □

1.1.3 Second polars

For any $F\in S^dE^*$, a $(d-2)$- polar of $V(F)$ is a quadric, called the *polar quadric* of $V(F)$ with respect to a. It is defined by the quadratic form

$$Q=P_{a^{d-2}}(F)=(d-2)!\sum_{|\mathbf{i}|=d-2}(\mathbf{i}!)^{-1}a^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}(F).$$

By symmetry, $b\in V(P_{a^{d-2}}(F))\Leftrightarrow a\in V(P_{b^2}(F))$. Thus

$$Q=2\sum_{|\mathbf{i}|=2}(\mathbf{i}!)^{-1}\mathbf{T}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}(F)(a).$$

By (1.11), any $a\in F$ belongs to the polar quadric $V(P_{a^{d-2}}(F)$. Also, by Theorem 1.1.1,

$$PT(V(P_{a^{d-2}}(F))_a=V(P_a(P_{a^{d-2}}(F)))=V(P_{a^{d-1}}(F))=PT(V(F))_a. \tag{1.13}$$

This shows that the polar quadric is tangent to the hypersurface at the point a.

Let us see where $V(P_{a^2}(F))$ intersects $V(F)$. Consider the line ℓ through two points a,b. Let $\phi:\mathbb{P}^1\to\mathbb{P}^n$ be its parametric equation. The index of intersection $i(V(F),\ell)_a$ of ℓ with $V(F)$ at the point a is equal to the multiplicity μ of the polynomial $\phi^*(F)$ at the point $(1,0)$. It follows from (1.6) that

$$\mu\geq s+1\Leftrightarrow b\in V(P_{a^{d-i}}(F)), \quad i=0,\ldots,s. \tag{1.14}$$

For $s = 0$, by (1.11), this condition says that $a \in V(F)$, i.e. $i(V(F),\ell)_a \geq 1$. For $s = 1$, by Theorem 1.1.1, this condition says that b, and hence ℓ, belongs to the tangent plane $PT(V(F))_a$. For $s = 2$, this condition says that ℓ belongs to the second polar $V(P_{a^{d-2}}(F))$.

Definition 1.2. *A line is called a* flex tangent *to $V(F)$ at a point a if*

$$i(V(F)),\ell)_a > 2.$$

Proposition 1.1.3. *Let ℓ be a line through a point a. Then ℓ is a flex tangent to $V(F)$ at a if and only if it is contained in the intersection of $PT(V(F))_a$ with the polar quadric $V(P_{a^{d-2}}(F))$.*

Note that the intersection of a quadric hypersurface Q with its tangent hyperplane H at a point $a \in Q$ is a cone in H over the quadric $\bar{Q}$ in the image $\bar{H}$ of H in $\mathbb{P}(E/Ka)$. Let us explain what it means. By theorem 1.1.1,

$$PT(Q)_a = P_a(Q) := \{x \in \mathbb{P}(E) : B(a,x) = 0\},$$

where B is the polar symmetric bilinear form of Q. Since

$$B(a,b) = \tfrac{1}{2}\big(Q(a+b) - Q(a) - Q(b)\big),$$

we see that with each $b \in Q \cap PT(Q)_a$, the line $\langle a,b\rangle$ is contained in $Q \cap PT(Q)_a$. So, the intersection is a cone with vertex a. The linear projection $\mathbb{P}(E) \to \mathbb{P}(E/Ka)$ with center at a maps $H = PT(Q)_a$ to a subspace $\bar{H}$ of $\mathbb{P}(E/Ka)$ and maps $Q \cap H$ to a quadric $\bar{Q}$ in $\bar{H}$. The cone $H \cap Q$ is equal to the closure of the pre-image of $\bar{Q}$ in $H \setminus \{a\}$. This is what it means to be a cone over a quadric in $\bar{H} \cong \mathbb{P}^{n-2}$.

If $n = 2$, a cone over quadric in $\mathbb{P}^0$ is one point. If $n > 2$, then the dimension of the cone is positive.

Corollary 1.1.4. *Assume $n \geq 3$. For each $a \in F$ there exists a flex tangent line. The union of the flex tangent lines containing the point a is the cone $PT(V(F))_a \cap V(P_{a^{d-2}}(F))$ in $PT(V(F))_a$.*

Example 1.1.2. Assume a is a singular point of $V(F)$. By Theorem 1.1.1, this is equivalent to $V(P_{a^{d-1}}(F)) = \mathbb{P}^n$. By (1.13), the polar quadric Q is also singular at a and thus it is a cone over its image under the projection from a. The union of flex tangents is equal to Q.

Example 1.1.3. Assume a is a nonsingular point of a surface $X = F \subset \mathbb{P}^3$. A hyperplane which is tangent to X at a cuts out in X a curve C with a singular point

a. If a is an ordinary double point of C, there are two flex tangents corresponding to the two branches of C at a. The polar quadric Q is nonsingular at a. It is a cone over a quadric $\bar{Q}$ in $\mathbb{P}^1$. If $\bar{Q}$ consists of 2 points we have two flex tangents corresponding to two branches of C at a. If $\bar{Q}$ consists of one point (corresponding to non-reduced hypersurface in $\mathbb{P}^1$), then we have one branch. The latter case happens only if Q is singular at some point $b \neq a$.

1.1.4 The Hessian

Let $Q(a)$ be a polar quadric of F at some point $a \in \mathbb{P}^n$. The symmetric matrix defining the corresponding quadratic form is the *Hessian matrix* of second partial derivatives of F. It is equal to the matrix

$$\mathrm{He}(F) = \begin{pmatrix} F_{00}(a) & F_{01}(a) & \dots & F_{0n}(a) \\ F_{10}(a) & F_{11}(a) & \dots & F_{1n}(a) \\ \vdots & \vdots & \vdots & \vdots \\ F_{n0}(a) & F_{n1}(a) & \dots & F_{nn}(a) \end{pmatrix}, \quad F_{ij} = \frac{\partial^2 F}{\partial T_i \partial T_j}. \tag{1.15}$$

evaluated at the point a. The quadric $Q(a)$ is singular if and only if the determinant of the matrix is equal to zero (the singular points correspond to the null-space of the matrix). The hypersurface

$$\mathrm{He}(V(F)) = V(\det \mathrm{He}(F)) \tag{1.16}$$

describes the set of points $a \in \mathbb{P}^n$ such that the polar quadric $V(P_{a^{d-2}}(F))$ is singular. It is called the *Hessian hypersurface* of F. Its degree is equal to $(d-2)(n+1)$ unless it coincides with $\mathbb{P}^n$.

Proposition 1.1.5. *The following is equivalent:*

(i) $\mathrm{He}(F) = \mathbb{P}^n$*;*

(ii) there exists a nonzero polynomial $G(Z_0, \dots, Z_n)$ *such that*

$$G(\partial_0(F), \dots, \partial_n(F)) \equiv 0.$$

Proof. This is a special case of a more general result about the *jacobian* of $n+1$ polynomial functions $f_0, \dots, f_n$ defined by

$$J(f_0, \dots, f_n) = \det(\frac{\partial f_i}{\partial T_j}).$$

Suppose $J(f_0, \dots, f_n) \equiv 0$. Then the map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by the functions $f_0, \dots, f_n$ is degenerate at each point (i.e. df_x is of rank $< n+1$ at each point x). Thus the closure of the image is a proper closed subset of $\mathbb{C}^{n+1}$. Hence there is an irreducible polynomial which vanishes identically on the image.

Conversely, assume that $G(f_0, \ldots, f_n) \equiv 0$ for some polynomial G which we may assume to be irreducible. Then

$$\frac{\partial G}{\partial T_i} = \sum_{i=0}^{n} \frac{\partial G}{\partial Z_j}(f_0, \ldots, f_n)\frac{\partial f_j}{\partial T_i} = 0,\ i = 0, \ldots, n.$$

Since G is irreducible its set of zeroes is nonsingular on a Zariski open set U. Thus the vector

$$\big(\frac{\partial G}{\partial Z_0}(f_0(x), \ldots, f_n(x)), \ldots, \frac{\partial G}{\partial Z_n}(f_0(x), \ldots, f_n(x)\big)$$

is a nontrivial solution of the system of linear equations with matrix $(\frac{\partial f_i}{\partial T_j}(x))$, where $x \in U$. Thus the determinant of this matrix must be equal to zero. This implies that $J(f_0, \ldots, f_n) = 0$ on U hence is identically zero. □

Remark 1.1.1. It was claimed by O. Hesse that the vanishing of the Hessian implies that the partial derivatives are linearly dependent. Unfortunately, his attempted proof is completely wrong. The first counterexample was given by P. Gordan and E. Noether in [Gordan-Noether]. Consider the polynomial

$$F = T_2T_0^2 + T_3T_1^2 + T_4T_0T_1 = 0.$$

Note that the partial derivatives

$$\frac{\partial F}{\partial T_2} = T_0^2, \quad \frac{\partial F}{\partial T_3} = T_1^2, \quad \frac{\partial F}{\partial T_4} = T_0T_1$$

are algebraically dependent. This implies that the Hessian is identically equal to zero. We have

$$\frac{\partial F}{\partial T_0} = 2T_0T_2 + T_4T_1, \quad \frac{\partial F}{\partial T_1} = 2T_1T_3 + T_4T_0.$$

Suppose that a linear combination of the partials is equal to zero. Then

$$c_0T_0^2 + c_1T_1^2 + c_2T_0T_1 + c_3(2T_0T_2 + T_4T_1) + c_4(2T_1T_3 + T_4T_0) = 0.$$

Collecting the terms in which T_2, T_3, T_4 enters we get

$$2c_3T_0 + c_4T_4 = 0, \quad 2c_3T_1 = 0, \quad c_3T_1 + c_4T_0.$$

This gives $c_3 = c_4 = 0$. Since the polynomials T_0^2, T_1^2, T_0T_1 are linear independent we also get $c_0 = c_1 = c_2 = 0$.

The known cases when the assertion of Hesse is true are $d = 2$ (any n) and $n \leq 3$ (any d) (see [Gordan-Noether], [Lossen]).

Recall that the set of singular quadrics in $\mathbb{P}^n$ is the *discriminant hypersurface* $\mathcal{D}_2(n)$ in $\mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$ defined by the equation

$$\det\begin{pmatrix} T_{00} & T_{01} & \dots & T_{0n} \\ T_{01} & T_{11} & \dots & T_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ T_{0n} & T_{1n} & \dots & T_{nn} \end{pmatrix} = 0. \tag{1.17}$$

By differentiating, we easily find that its singular points are defined by the determinants of $n \times n$ minors of the matrix. This shows that the singular locus of $\mathcal{D}_2(n)$ parametrizes quadrics defined by quadratic forms of rank $\leq n-1$ (or corank ≥ 2). Abusing the terminology we say that a quadric is of rank k if the corresponding quadratic form is of this rank. Note that

$$\dim \text{Sing}(Q) = \text{corank}(Q) - 1.$$

Assume that $\text{He}(F) \neq 0$. Consider the rational map $p : \mathbb{P}^n = \mathbb{P}(E) \to \mathbb{P}(S^2(E^*)) = \mathbb{P}^{\binom{n+2}{2}-1}$ defined by $a \mapsto V(P_{a^{d-2}}(F))$. Note that $P_{a^{d-2}})(F) = 0$ implies $P_{a^{d-1}}(F) = 0$ and hence $\sum_{i=0}^n b_i\partial_i(F)(a) = 0$ for all b. This shows that a is a singular point of F. Thus p is defined everywhere except maybe at singular points of F. Thus, if F is nonsingular, the map p is regular, and the pre-image of the discriminant hypersurface is equal to the Hessian of F. The pre-image of the singular locus $\text{Sing}(\mathcal{D}_2(n))$ consists of the subset of points $a \in \text{He}(F)$ such that $\dim \text{Sing}(V(P_{a^{d-2}}(F))) > 0$. One expects that, in general case, this will be equal to the set of singular points of the Hessian hypersurface.

Here is another description of the Hessian hypersurface.

Theorem 1.1.6. *The Hessian hypersurface* $\text{He}(V(F))$ *is the locus of singular points of first polars of* $V(F)$.

Proof. Let $a \in \text{He}(V(F))$ and let $b \in \text{Sing}(V(P_{a^{d-2}}(F)))$. Then

$$P_b(P_{a^{d-2}}(F)) = P_{a^{d-2}}(P_b(F)) = 0.$$

Since $P_b(F)$ is of degree $d-1$, this means that the tangent hyperplane of $V(P_b(F))$ at a is equal to $\mathbb{P}^n$, i.e., $a \in \text{Sing}(V(P_b(F)))) = 0$.

Conversely, if $a \in \text{Sing}(V(P_b(F)))$ for $b \in \mathbb{P}^n$, then $P_{a^{d-2}}(P_b(F)) = 0$, hence $P_b(P_{a^{d-2}}(F)) = 0$. This means that b is a singular point of the polar quadric with respect to a. Hence $a \in \text{He}(V(F))$. $\square$

Let us find the affine equation of the Hessian hypersurface. Applying the Euler formula (1.10), we can write

$$T_0F_{0i} = (d-1)\partial_i(F) - T_1F_{1i} - \ldots - T_nF_{ni},$$

$$T_0\partial_0(F) = dF - T_1\partial_1(F) - \ldots - T_n\partial_n(F).$$

Plugging in the equation of the Hessian, we get

$$\det(\mathrm{He}(F)) = \frac{(d-1)^2}{T_0^2}\det\begin{pmatrix} \frac{d}{d-1}F & \partial_1(F_d) & \ldots & \partial_n(F_d) \\ \partial_1(F_d) & F_{11} & \ldots & F_{1n} \\ \vdots & \vdots & \vdots & \\ \partial_n(F_d) & F_{n1} & \ldots & F_{nn} \end{pmatrix}. \tag{1.18}$$

Let $f(Z_1,\ldots,Z_n)$ be the dehomogenization of F with respect to T_0, i.e.,

$$F(T_0,\ldots,T_d) = T_0^d f(\frac{T_1}{T_0},\ldots,\frac{T_n}{T_0}).$$

We have

$$\frac{\partial F}{\partial T_i} = T_0^{d-1}f_i(Z_1,\ldots,Z_n), \quad i = 1,\ldots,n,$$

$$\frac{\partial^2 F}{\partial T_i \partial T_j} = T_0^{d-2}f_{ij}(Z_1,\ldots,Z_n), \quad i,j = 1,\ldots,n,$$

where

$$f_i = \frac{\partial f}{\partial Z_i}, \quad f_{ij} = \frac{\partial^2 f}{\partial Z_i \partial Z_j}.$$

Plugging in these expressions in (1.18), we obtain, that up to a nonzero constant factor,

$$T_0^{-(n+1)(d-2)}\det(\mathrm{He}(F)) = \det\begin{pmatrix} \frac{d}{d-1}f(Z) & f_1(Z) & \ldots & f_n(Z) \\ f_1(Z) & f_{11}(Z) & \ldots & f_{1n}(Z) \\ \vdots & \vdots & \vdots & \\ f_n(Z) & f_{n1}(Z) & \ldots & f_{nn}(Z) \end{pmatrix}, \tag{1.19}$$

where $Z_i = T_i/T_0 . i = 1,\ldots,n$.

1.1.5 Parabolic points

Let us see where $\mathrm{He}(V(F))$ intersects $V(F)$. If $a \in V(F) \cap \mathrm{He}(V(F))$, then the polar quadric $V(P_{a^{d-2}}(F)$ is singular at some point b. If $n > 2$ this means that the cone of flex tangents is a cone over a singular cone in $\mathbb{P}^{n-2}$. Such a point called a *parabolic point* of $V(F)$. The set of parabolic points is the *parabolic hypersurface* in $V(F)$. In the case $n = 2$, a singular quadric is reducible. So, $V(P_{a^{d-2}}(F))$ is singular if and only if a lies on its line component. By Proposition 1.1.3, this line component must be a flex tangent.

A point a of a plane curve C such that there exists a flext tangent at a is called an *inflection point* or a *flex* of C. So we have proved

Theorem 1.1.7. *Let $C = V(F) \subset \mathbb{P}^2$. Then $\mathrm{He}(C) \cap C$ consists of inflection points of C. In particular, each curve of degree ≥ 3 has an inflection point, and the number of inflection points is either infinite or $\leq 3d(d-2)$.*

Let us see when C has infinitely many inflection points. Certainly, this happens when C contains a line component; each its point is an inflection point. It must be also an irreducible component of $\mathrm{He}(C)$. The set of inflection points is a closed subset of C. So, if C has infinitely many inflection points, it must have an irreducible component consisting of inflection points. Each such component is contained in $\mathrm{He}(C)$. Conversely, each common irreducible component of C and $\mathrm{He}(C)$ consists of inflection points.

Proposition 1.1.8. *Assume $n = 2$. Each common irreducible component of $V(F)$ and $P_a(V(F))$ is a line.*

Proof. Let R be an irreducible component of the curve $V(F)$ which is contained in the Hessian. Take a nonsingular point of R and consider an affine equation of R with coordinates (x, y). We may assume that $\mathcal{O}_{R,x}$ is included in $\hat{\mathcal{O}}_{R,x} \cong K[[T]]$ such that $x = T, y = T^r\epsilon$, where $\epsilon(0) = 1$. Thus the equation of R looks like

$$f(x, y) = y - x^r + g(x, y), \tag{1.20}$$

where $g(x, y)$ does not contain terms $ay, a \in K$. It is easy to see that $(0, 0)$ is an inflection point if and only if $r > 2$ with the flex tangent $y = 0$.

We use the affine equation of the Hessian (1.19), and obtain the image of

$$h(x, y) = \det \begin{pmatrix} \frac{d}{d-1}f & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}$$

in $K[[T]]$ is equal to

$$\det\begin{pmatrix} 0 & -rT^{r-1}+g_1 & 1+g_2 \\ -rT^{r-1}+g_1 & -r(r-1)T^{r-2}+g_{11} & g_{12} \\ 1+g_2 & g_{12} & g_{22}\end{pmatrix}.$$

Since every monomial entering in g is divisible by y^2, xy or $x^i, i > r$, we see that g_y is divisible by T and g_x is divisible by T^r. Also g_{11} is divisible by t^{r-1}. This shows that

$$h(x,y) = \det\begin{pmatrix} 0 & aT^{r-1}+\ldots & 1+\ldots \\ aT^{r-1}+\ldots & -r(r-1)T^{r-2}+\ldots & g_{12} \\ 1+\ldots & g_{12} & g_{22}\end{pmatrix},$$

where $\ldots$ denotes terms of higher degree in T. Computing the determinant, we get that it is equal to $r(r-1)T^{r-2}+\ldots$. This means that its image in $K[[T]]$ is not equal to zero, unless the equation of the curve is equal to $y = 0$, i.e. the curve is a line. □

In fact, we have proved more. We say that a nonsingular point of C is an inflection point of order $r-2$ and write it as $\mathrm{ordfl}(C)_x$if one can choose an equation of the curve as in (1.20) with $r \geq 3$. It follows from the previous proof that $r-2$ is equal to the multiplicity $i(C, \mathrm{He})_a$ of the intersection of the curve and its Hessian at the point a. Thus, if we denote the order of we see that for any nonsingular curve which is not a line

$$\sum_{x\in C}\mathrm{ordfl}(C)_x = 3d(d-2). \tag{1.21}$$

1.1.6 The Steinerian

The *Steinerian hypersurface* of $V(F)$ is defined by

$$\mathrm{St}(V(F)) = \{a \in \mathbb{P}^n : V(P_a(F)) \text{ is singular}\} \tag{1.22}$$

By Theorem 1.22, any singular point b of $P_a(F), a \in \mathrm{St}(F)$, lies on $\mathrm{He}(F)$, and hence the polar quadric $P_{b^{d-2}}(F)$ is singular. The point a must be its singular point. Thus we obtain an equivalent definition of the Steinerian hypersurface:

$$\mathrm{St}(V(F)) = \bigcup_{a\in \mathrm{He}(V(F))} \mathrm{Sing}(V(P_{a^{d-2}}(F))). \tag{1.23}$$

We also have

$$\mathrm{He}(V(F)) = \bigcup_{a\in \mathrm{St}(V(F))} \mathrm{Sing}(V(P_a(F))). \tag{1.24}$$

Assume that the matrix $\mathrm{He}(F)(a)$ is of coank 1. Then the quadric $V(P_{a^{d-2}}(F))$ has a unique singular point $b = (b_0, \ldots, b_n)$, whose coordinates can be chosen to be any column or a row of the adjugate matrix $\mathrm{adj}(\mathrm{He}(F))$ evaluated at the point a. Thus $\mathrm{St}(V(F))$ is the image of the Hessian hypersurface under the rational map

$$\mathrm{He}(V(F)) - \to \mathrm{St}(V(F)), \quad a \mapsto \mathrm{Sing}(P_{a^{d-2}}(F),$$

given by polynomials of degree $2(d-2)$. Also, if first polar $P_a(F)$ has an isolated singular point for a general point a, we get a rational map

$$\mathrm{St}(V(F)) - \to \mathrm{He}(V(F)), \quad a \mapsto \mathrm{Sing}(P_a(F)).$$

These maps are obvioulsy inverse to each other. It is a difficult question to determine the sets of indeterminacy points for both maps.

It is known that the locus of singular hypersurfaces of degree d in $\mathbb{P}(V)$ is a hypersurface

$$\mathcal{D}_n(d) \subset \mathbb{P}(S^d E^*)$$

of degree $(n+1)(d-1)^n$ defined by the *discriminant* of a general degree d homogeneous polynomial in $n+1$ variables (the *discriminant hypersurface*). Let L be the projective subspace of $\mathbb{P}(S^d E^*))$ which consists of first polars of $V(F)$. Assume that no polar $P_a(F)$ is equal to zero. Then

$$\mathrm{St}(V(F)) \cong L \cap \mathcal{D}_n(d).$$

Thus we obtain

$$\deg(\mathrm{St}(V(F))) = (n+1)(d-2)^n \tag{1.25}$$

Example 1.1.4. Let $d = 3$. Then the $\mathrm{He}(V(F)) = \mathrm{St}(V(F))$ is a hypersurface of degree $n+1$. We have a rational self-map

$$st : \mathrm{He}(V(F)) - \to \mathrm{He}(V(F)).$$

It assigns to a singular polar quadric its singular point.

1.2 The dual hypersurface

1.2.1 The polar map

The linear space of first polars $P_a(F)$ defines a linear subsystem of the complete linear system $\left|\mathcal{O}_{\mathbb{P}^n}(d-1)\right|$ of hypersurfaces of degree $d-1$ in $\mathbb{P}^n$. Its dimension is equal to n if the first partial derivatives of F are linearly independent. By Proposition 1.1.2 this happens if and only if F is not a cone. We assume that this is the case. Let us identify the linear system of first polars with $\mathbb{P}(E) = \mathbb{P}^n$ by assigning to each $a \in \mathbb{P}^n$ the polar hypersurface $V(P_a(F))$. Let $\mathcal{P} : \mathbb{P}^n - \to \check{\mathbb{P}}^n$ be the rational map defined by the linear system of polars. It is called the *polar map*). In coordinates, the polar map is given by

$$\mathcal{P} : (x_0, \dots, x_n) \mapsto (\frac{\partial F}{\partial T_0}(x), \dots, \frac{\partial F}{\partial T_n}(x)).$$

Recall that a hyperplane $H = V(\sum a_i \xi_i)$ in the dual projective space $\check{\mathbb{P}}^n$ is the point $a = (a_0, \dots, a_n) \in \mathbb{P}^n$. The pre-image of the hyperplane H under $\mathcal{P}$ is the polar $P_a(F) = V(\sum a_i \frac{\partial F}{\partial T_i}(x))$.

If F is nonsingular, the polar map is a regular map given by polynomials of degree $d-1$. Its degree is equal to $(d-1)^n$.

Proposition 1.2.1. *Assume F is nonsingular. The ramification divisor R of the polar map is equal to the Hessian hypersurface of F.*

Proof. Note for any finite map $f : X \to Y$ of nonsingular varieties, the ramification divisor R_f is defined locally by the determinant of the linear map of locally free sheaves $f^*(\Omega^1_Y) \to \Omega^1_X$. The image of R_f in Y is called the *branch divisor*. Both of the divisors may be nonreduced. We have the *Hurwitz formula*

$$K_X = f^*(K_Y) + R_f. \tag{1.26}$$

The map f is étale outside R_f, i.e., for any point $x \in X$ the homomorphism of local ring $\mathcal{O}_{Y,f(y)} \to \mathcal{O}_{X,x}$ defines an isomorphism of their formal completions. In particular, pre-image $f^{-1}(Z)$ of a nonsingular subvariety $Z \subset Y$ is nonsingular outside the support of R_f. Applying this to the polar map we see that the singular points of $V(P_a(F)) = \mathcal{P}^{-1}(H_a)$ are contained in the ramification locus R of the polar map. On the hand, we know that the set of singular points of first polars is the Hessian He$(V(F))$. This shows He$(V(F)) \subset R$. Applying the Hurwitz formula , we have $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, $K_{\check{\mathbb{P}}^n} = \mathcal{O}_{\check{\mathbb{P}}^n}(-n-1)$, $\mathcal{P}^{-1}(K_{\check{\mathbb{P}}^n}) = \mathcal{O}_{\mathbb{P}^n}((-n-1)(d-1))$. This gives $\deg(R) = (n+1)(d-2) = \deg(He(V(F))$. This shows that He$(V(F)) = R$. □

What is the branch divisor? One can show that the pre-image of a hyperplane H_a is singular if and only if it is tangent to the branch locus of the map. The pre-image of H_a is the polar hypersurface $V(P_a(F))$. Thus the set of hyperplanes tangent to the branch divisor is equal to the Steinerian St$(V(F))$. This shows that the branch locus contains the dual variety of St$(V(F))$.

1.2.2 Dual varieties

Recall that the *dual variety* X^* of a subvariety X in $\mathbb{P}^n = \mathbb{P}(E)$ is defined as the closure in the dual projective space $\check{\mathbb{P}}^n = \mathbb{P}(E^*)$ of the locus of hyperplanes in $\mathbb{P}^n$ which are tangent to X at some nonsingular point.

When X is a hypersurface F, we see that the dual variety is the image of F under the rational map given by the first polars. In fact, the point $(\partial_0(F)(x), \ldots, \partial_n(F)(x))$ in $\check{\mathbb{P}}^n$ is the hyperplane $V(\sum_{i=0}^n \partial_i(F)(x)T_i)$ in $\mathbb{P}^n$ which is tangent to F at the point x.

The following result is called the *projective duality*:

Theorem 1.2.2. *Assume that K is of characteristic 0 (as we always do). Then $(X^*)^* = X$.*

Proof. See [Gelfand-Kapranov-Zelevinsky] or [Harris]. □

It follows from the proof that for any nonsingular point $y \in X^*$, the dual of the embedded tangent space $PT(X^*)_y \subset \check{\mathbb{P}}^n$ is equal to the closure of the set of nonsingular points $x \in X$ such that the hyperplane y in $\mathbb{P}^n$ is tangent to X at x. In particular, the fibres of the *duality map*

$$d : X^{\text{ns}} \to X^*, \quad x \mapsto PT(X)_x \tag{1.27}$$

are open subsets of a projective subspace in $\mathbb{P}^n$. Here and later X^{ns} denotes the set of nonsingular points of a variety X. In particular, if X^* is a hypersurface, the dual space of $PT(X^*)_y$ must be a point, and hence the map d is birational.

Let us apply it to our case when $X = F$ is a nonsingular hypersurface. Then the map given by first polars is a regular map $\mathbb{P}^n \to \check{\mathbb{P}}^n$ given by homogeneous polynomials of degree $d-1$. It is a finite map (after applying the Veronese map it becomes a linear projection map). Therefore its fibres are finite sets. This shows that the dual of a nonsingular hypersurface is a hypersurface. Thus, the duality map, equal to the restriction of the polar map, is a birational isomorphism

$$d : F \cong_{\text{bir}} F^*.$$

Example 1.2.1. Let $\mathcal{D}_d(n)$ be the discriminant hypersurface in $\mathbb{P}(S^dE^*)$. We would like to describe explicitly the tangent hyperplane of $\mathcal{D}_d(n)$ at its nonsingular point. Let

$$\tilde{\mathcal{D}}_d(n) = \{(V, x) \in |\mathcal{O}_{\mathbb{P}^n}(d) \times \mathbb{P}^n : x \in \text{Sing}(V)\}.$$

Let us see that $\tilde{\mathcal{D}}_d(n)$ is nonsingular and the projection to the first factor

$$\pi : \tilde{\mathcal{D}}_d(n) \to \mathcal{D}_d(n)$$

is a resolution of singularities. In particular, π is an isomorphism over the open set $\mathcal{D}_d(n)^{\rm ns}$ of nonsingular points of $\mathcal{D}_d(n)$.

The fact that $\tilde{\mathcal{D}}_d(n)$ is nonsingular follows easily from considering the projection to $\mathbb{P}^n$. For any point $b \in \mathbb{P}^n$ the fibre of the projection is the projective space of hypersurfaces which have a singular point at b (this amounts to $n+1$ linear conditions on the coefficients). Thus $\tilde{\mathcal{D}}_d(n)$ is a projective bundle over $\mathbb{P}^n$ and hence is nonsingular.

Let us see where π is an isomorphism. Let us choose projective coordinates $A_{\mathbf{i}}, |\mathbf{i}| = d$, in $|\mathcal{O}_{\mathbb{P}^n}(d)| = \mathbb{P}(S^dE^*)$ corresponding to the coefficients of a hypersurface of degree d and coordinates $x_0, \ldots, x_n$ in $\mathbb{P}^n$. Then $\tilde{\mathcal{D}}_d(n)$ is given by $n+1$ bihomogeneous equations of bidegree $(1, d-1)$:

$$\frac{\partial G_d}{\partial T_s}(x) = \sum_{|\mathbf{i}|=d} i_s A_{\mathbf{i}} x^{\mathbf{i}-e_s} = 0, \quad s = 0, \ldots, n, \tag{1.28}$$

where $G_d = \sum_{|\mathbf{i}|=d} A_{\mathbf{i}} T^{\mathbf{i}}$ and e_s is the sth unit vector in $\mathbb{Z}^{n+1}$. We identify the tangent space of $\mathbb{P}(S^dE^*) \times \mathbb{P}(E)$ with the space $S^dE^*/KF \oplus E/Kb$. Then $T(\tilde{\mathcal{D}}_d(n))_{(V(F),b)}$ is defined by the $(n+1) \times \binom{n+d}{d}$ matrix M of partial derivatives of equations (1.28)

$$\begin{pmatrix} \ldots & i_0 b^{\mathbf{i}-e_0} & \ldots & \sum_{|\mathbf{i}=d|} i_0 i_0 a_{\mathbf{i}} b^{\mathbf{i}-e_0-e_0} & \ldots & \sum_{|\mathbf{i}=d|} i_0 i_n a_{\mathbf{i}} b^{\mathbf{i}-e_0-e_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \ldots & i_n b^{\mathbf{i}-e_n} & \ldots & \sum_{|\mathbf{i}=d|} i_n i_0 a_{\mathbf{i}} b^{\mathbf{i}-e_n-e_0} & \ldots & \sum_{|\mathbf{i}=d|} i_n i_n a_{\mathbf{i}} b^{\mathbf{i}-e_n-e_n} \end{pmatrix},$$

where $F = \sum_{|\mathbf{i}=d|} a_{\mathbf{i}} T^{\mathbf{i}}$ and it is underestood that $b^{\mathbf{i}-e_s} = 0$ if $\mathbf{i} - e_s$ is not a non-negative vector. The projection π is an isomorphism if its differential is injective and the map is one-to one. To make it one-to-one we must have $\dim \mathrm{Sing}(V) = 0$. Now look at the map π restricted to the set of pairs (V, x) with the property that x is a single singular point of V. Suppose $d\pi_{(V,b)}$ has nontrivial kernel. A vector in the kernel is a vector $C = (\ldots, c_{\mathbf{i}}, \ldots, c_0, \ldots, c_n)$ satisfying $M \cdot c = 0$ and $(\ldots, c_{\mathbf{i}}, \ldots) = \lambda(\ldots, a_{\mathbf{i}}, \ldots)$. The last $n+1$ columns of the matrix M form the Hessian matrix of F at b. Multiplying the matrices, we obtain

$$M \cdot C = \lambda \nabla(F)(b) + \mathrm{He}(F) \cdot c = 0,$$

where $\nabla(F)(b)$ is the vector of partial derivatives of F computed at the point b and $c = (c_0, \ldots, c_n)$. Since $\mathrm{Sing}(V(F)) = \{b\}$, we have $\nabla(F)(b) = 0$, Thus the differential has kernel if and only if the equation $\mathrm{He}(F)(b)X = 0$ has a solution not proportional to $b = (b_0, \ldots, b_n)$. We may assume that $b = (1, 0, \ldots, 0)$. Write

$F = T_0^{d-2}G(T_1, \ldots, T_n) + \ldots$ as in Exercise 1.1. Computing the Hessian matrix at the point b we see that it is equal to

$$\begin{pmatrix} 0 & \ldots & \ldots & 0 \\ 0 & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

where $G = \sum_{0 \le i,j \le n} a_{ij} T_i T_j$ (see Chapter 1, (1.19)). A solution not proportional to $(1, 0, \ldots, 0)$ exists if and only if $\det(a_{ij}) = 0$. By definition, this means that the singular point of F at b is not an ordinary double point.

The image of the tangent space under the differential $d\pi$ is equal to the set of $G_d \in T\mathbb{P}(S^d E^*)_F = S^d E^*/KF$ such that the linear system

$$\text{He}(F) \cdot X = -\nabla(G_d)(b)$$

has a solution. Since the corank of the Hessian matrix is equal to 1, any solution must be orthogonal to the kernel of the matrix, i.e. $b \cdot \nabla(G_d)(b) = 0$. By Euler's formula implies that $G_b(b) = 0$.

Now we are ready to compute the dual variety of $\mathcal{D}_d(n)$. The condition $G_d(b) = 0$, where $\text{Sing}(V(F)) = \{b\}$ is equivalent that $D(b^d)(F) = 0$. Thus the tangent hyperplane, considered, as a point in the dual space $\mathbb{P}(S^d E)$ corresponds to the envelope $b^d = (\sum_{s=0}^n b_s \partial_i)^d$. The set of such envelopes is the Veronese variety $\nu_d(\mathbb{P}^n(E^*))$, the image of $\check{\mathbb{P}}^n = \mathbb{P}^n(E^*) \to \mathbb{P}(S^d E) = \mathbb{P}(S^d E^*)$ under the map given by the complete linear system $|\mathcal{O}_{\check{\mathbb{P}}^n}(d)|$ of hypersurfaces of degree d in the dual projective space.

Thus

$$\mathcal{D}_d(n)^* \cong \nu_d(\mathbb{P}^n). \tag{1.29}$$

Example 1.2.2. Let Q be a nonsingular quadric in $\mathbb{P}^n$. Let $A = (a_{ij})$ be a symmetric matrix defining Q, i.e. $Q = \{x \in \mathbb{P}^n : x \cdot A \cdot x = 0\}$. A tangent hyperplane of Q at a point $x \in \mathbb{P}^n$ is the hyperplane

$$T_0 \sum_{j=0}^n a_{0j} x_j + \ldots + T_n \sum_{j=0}^n a_{nj} x_j = 0.$$

Thus the vector of coordinates $y = (y_0, \ldots, y_n)$ of the tangent hyperplane is equal to the vector $A \cdot x$. Since A is invertible, we can write $x = A^{-1} \cdot y$. We have

$$0 = x \cdot A \cdot x = (y \cdot^t A^{-1})A(A^{-1} \cdot y) = y \cdot^t A^{-1} \cdot y = 0.$$

Here we treat x or y as a row-matrix or as a column-matrix in order the matrix multiplication makes sense. Since $A^{-1} = \det(A)^{-1}\text{adj}(A)$, where adj$(A)$ is the *adjugate matrix* (no, it is not a mistake; the ancients used this word when they talked about the matrix which nowadays is called the adjoint matrix), we obtain that the dual variety of Q is also a quadric given by the adjugate matrix of the matrix defining Q.

1.2.3 The Plücker equations

Let C be a plane curve of degree d. If C is nonsingular, it first polar $P_a(C)$ with respect to a general point in $\mathbb{P}^2$ intersects C at $d(d-1)$ points b such that $a \in PT(C)_b$. This shows that the pencil of lines through a contains $d(d-1)$ tangent lines to C. A pencil of lines in $\mathbb{P}^2$ is the same as a line in the dual plane. Thus we see that the dual curve C^* has $d(d-1)$ intersection points with a general line. In other words

$$\deg(C^*) = d(d-1). \tag{1.30}$$

If C is singular, the degree of C^* must be smaller. In fact, all polars $P_a(C)$ pass through singular points of C and hence the number of nonsingular points b such that $a \in PT(C)_b$ is smaller than $d(d-1)$. The difference is equal to the sum of intersection numbers of a general polar and the curve at singular points

$$d(d-1) - \deg(C^*) = \sum_{x \in \text{Sing}(C)} i(C, P_a(C))_x. \tag{1.31}$$

Let us compute the intersection numbers assuming that C has only ordinary nodes and cusps. Assume x is an ordinary node. Choose a coordinate system such that $x = (1,0,0)$ and write the equation as in Exercise 1.1, i.e., $F = T_0^{d-2}F_2(T_1,T_2)+\ldots$. We may assume that $F_2(T_1,T_2) = T_1T_2$. Computing the partials and dehomogenizing the equations, we find that $P_a(F) = a_1f_x + a_2f_y$, where $f = xy + \ldots$ is the affine equation of the curve, and f_x, f_y its partials in x and y. Thus, we need to compute the dimension of the vector space

$$K[x,y]/(f, a_1f_x + a_2f_y) = K[x,y]/(xy + \ldots, a_1x + a_2y + \ldots),$$

where $\ldots$ denotes the terms of higher degree. It is easy to see that this number is equal to the intersection number at a node with a general line through the node. The number is equal to 2.

If x is an ordinary cusp, the affine equation of C is $y^2 + x^3 + \ldots$ and we have to compute the dimension of the vector space

$$K[x,y]/(f, a_1f_x + a_2f_y) = K[x,y]/(y^2 + x^3 + \ldots, a_1x^2 + a_2y + \ldots).$$

It is easy to see that this number is equal to the intersection number at a cusp with a parabola whose tangent is equal to the line $y = 0$. The number is equal to 3.

Thus we obtain

Theorem 1.2.3. *Let C be an plane curve of degree d with no line components. Assume that C has only ordinary double points and ordinary cusps as singularities. Then*

$$deg(C^*) = d(d-1) - 2\delta - 3\kappa,$$

where δ is the number of nodes and κ is the number of cusps.

Note that that the dual curve C^* of a nonsingular curve of degree $d > 2$ is always singular. This follows from the formula for the genus of a nonsingular plane curve and the fact that C and C^* are birational. The polar map $C \to C^*$ is equal to the normalization map. A singular point of C^* corresponds to a line which is either tangent to C at several points, or is a flex tangent. We skip a local computation which shows that a line which is a flex tangent at one point with ordfl $= 1$ (an *honest flext tangent*) gives an ordinary cusp of C^* and a line which is tangent at two points which are not inflection points (*honest bitangent*) gives a node. Thus we obtain that the number $\check{\delta}$ of nodes of C^* is equal to the number of honest bitangents of C and the number $\check{\kappa}$ of ordinary cusps of C^* is equal to the number of honest flex tangents to C^*.

Assume that C is nonsingular and C^* has no other singular points except ordinary nodes and cusps. We know that the number of inflection points is equal to $3d(d-2)$. Applying Theorem 1.2.3 to C^*, we get that

$$\check{\delta} = \frac{1}{2}\big(d(d-1)(d(d-1)-1) - d - 9d(d-2)\big) = \frac{d(d-2)(d^2-9)}{2}. \quad (1.32)$$

This is the (expected) number of bitangents of a nonsingular plane curve. For example, we expect that a nonsingular plane quartic has 28 bitangents.

We refer for discussions of Plücker formulas to many modern text-books (e.g. [Fischer], [Fulton], [Griffiths-Harris], [Gelfand-Kapranov-Zelevinsky]).

Exercises

1.1 Let

$$F(T) = T_0^{n-1}F_1(T_1,\ldots,T_n) + T_0^{d-2}F_2(T_1,\ldots,T_n) + \ldots + F(T_1,\ldots,T_n).$$

Show that the cone of flex tangents to F at the point $a = (1,0,\ldots,0)$ is isomorphic to the cone $V(F_1,F_2) \subset \mathbb{P}^n$. Assume that $(1,0,\ldots,0)$ does not belong to

He$(V(F))$. Show that, up to a linear change of variables, the affine equation of F is equal to $f = Z_1 + Z_2^2 + \ldots + Z_n^2 +$ terms of higher order.

1.2 Show that a line contained in a hypersurface F belongs to all polars of F with respect to any point on this line.

1.3 Let $X = F$ be a nonsingular surface in $\mathbb{P}^3$ of degree > 1. Show that the set of flex tangents to X is a surface in the Grassmannian $G_1(3)$ of bidegree $(3(d-2), d(d-1)(d-2))$ (Recall that $A^2(G_1(3)) \cong H^4(G_1(3), \mathbb{Z})$ is freely generated by the Schubert varieties σ_1 and σ_2, where σ_1 can be represented by lines through a general point, and σ_2 can be represented by lines in a general plane. So, the problem asks to show that the cohomology class of the locus of flex tangents is cohomologous to $3(d-2)\sigma_1 + d(d-1)(d-2))\sigma_2$).

1.4 Find the multiplicity of the intersection of a curve C with its Hessian at an ordinary double point and at an ordinary cusp of C.

1.5 Let X be a hypersurface which contains a plane through any point. Show that the Hessian of X contains X. Is the convese true?

1.6 Let $X = V(F_3)$ be a nonsingular cubic hypersurface. Show that any nonsingular point of He(X) defines a quadric of corank 1. Show that this assertion does not extend to hypersurfaces of degree > 3.

1.7 Let $n = 2$ and $d \leq 4$. Assume He$(V(F)) = \mathbb{P}^2$. Show that F is the union of concurrent lines.

1.8 Find the dual variety of a quadric of corank r in $\mathbb{P}^n$.

1.9 Let $\mathcal{D}_{m,n} \subset \mathbb{P}^{mn-1}$ be the image in the projective space of the variety of $m \times n$ matrices of rank $\leq \min\{m, n\} - 1$. Show that the variety

$$\tilde{\mathcal{D}}_{m,n} = \{(A, x) \in \mathbb{P}^{mn-1} \times \mathbb{P}^n : A \cdot x = 0\}$$

is a resolution of singularities of $\mathcal{D}_{m,n}$. Find the dual variety of $\mathcal{D}_{m,n}$.

1.10 Find the dual variety of the Segre variety $s_n(\mathbb{P}^n) \subset \mathbb{P}^{n^2+2n}$.

1.11 Prove that the degree of the dual variety of a nonsingular hypersurface of degree d in $\mathbb{P}^n$ is equal to $d(d-1)^{n-1}$.

1.12 Let C be the union of k nonsingular conics in general position. Show that C^* is also the union of k nonsingular conics in general position. Check the Plücker formulas in this case.

1.13 Let C has only δ ordinary nodes and κ ordinary cusps as singularities. Assume that the dual curve C^* has also only $\check{\delta}$ ordinary nodes and $\check{\kappa}$ ordinary cusps as singularities. Find $\check{\delta}$ and $\check{\kappa}$ in terms of d, δ, κ.

1.14 Give an example of a self-dual (i.e. $C^* \cong C$) plane curve of degree > 2.

Chapter 2

Conics

2.1 Self-polar triangles

2.1.1 The Veronese quartic surface

Recall that the *Veronese variety* is defined to be the image of the map

$$\mathbb{P}(E^*) \to \mathbb{P}(S^d E^*), \quad V(L) \mapsto V(L^d),$$

where L is a nonzero linear form on E. Replacing E with the dual space E^*, we obtain a map

$$\nu_d : \mathbb{P}(E) \to \mathbb{P}(S^d E), \quad Kv \to Kv^d, \tag{2.1}$$

where v is a nonzero vector in E. This map is called the *Veronese map*. Using the polarity pairing (1.2), one can identify the space $S^d E^*$ with the dual space $(S^d E)^*$. This allows one to identify (2.1) with the map given by the complete linear system $|\mathcal{O}_{\mathbb{P}(E)}(d)|$ of hypersurfaces of degree d. The Veronese variety is of dimension n and degree d^n. More generally, one defines a Veronese variety as the image of $\mathbb{P}^n$ in $\mathbb{P}^{N(d,n)} - 1, N(d,n) = \binom{n+d}{d}$ under the map given by the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$ and a choice of a basis in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$.

Let $d = n = 2$, this is the case of the *Veronese quartic surface* Ver_2^2. The pre-image of a hyperplane H in $\mathbb{P}(S^2 E) \cong \mathbb{P}^5$ is a conic C. There are three sorts of hyperplane corresponding to the cases C is nonsingular, C is a line-pair, C is a double line. In the first case H intersects the Veronese surface $\mathrm{Ver}_2^2 = v_2(\mathbb{P}^2)$ transversally, in the second case H is tangent to Ver_2^2 at a single point, and in the third case H is tangent to Ver_2^2 along a conic.

Choosing a basis in E we can identify the space $S^2 E$ with the space of symmetric 3×3 matrices. The Veronese surface is identified with matrices of rank 1. Its equations are given by 2×2 minors. The variety of matrices of rank ≤ 2 is the cubic hypersurface $\mathcal{D}_2(2)$ given by the determinant. It singular along the Veronese surface.

Since any nonzero matrix of rank ≤ 2 can be written as a sum of matrices of rank 1, we see that the discriminant cubic hypersurface $\mathcal{D}_2(2)$ is equal to the first secant variety of Ver_2^2.

A linear projection of Ver_2^2 from a point not lying in $\mathcal{D}_2(2)$ is an isomorphism onto a quartic surface V_4 in $\mathbb{P}^4$, called the *projected Veronese surface*.

The image of Ver_2^2 under a linear projection from a point Q lying in $\mathcal{D}_2(2)$ but not lying on the surface is a non-normal quartic surface V_4' in $\mathbb{P}^4$. To see this we may assume that $Q = T_0^2 + T_1^2$. The plane of conics $aT_0^2 + bT_0T_1 + cT_1^2 = 0$ contains Q and intersects Ver_2^2 along the conic of double lines $(\alpha T_0 + \beta T_1^2)^2$. The projection maps this conic two-to-one to a double line of the image of Ver_2^2.

The image of Ver_2^2 under a linear projection from its point is a cubic scroll, the image of $\mathbb{P}^2$ under a map given by the linear system of conics with one base point.

2.1.2 Polar lines

Let C be a nonsingular conic. For any point $a \in \mathbb{P}^2$ the first polar $P_a(C)$ is a line, called the *polar line* of a. For any line l there exists a unique point a such that $P_a(C) = l$. The point a is called the *pole* of l. The point a considered as a line in the dual plane is the polar line of the point l with respect to the dual conic $\check{C}$.

A set of three non-colinear lines ℓ_1, ℓ_2, ℓ_3 is called a *self-polar triangle* with respect to C if ℓ_i is the polar line of C with respect to the point of intersection of the other two lines.

Recall that two unordered pairs $\{a,b\},\{c,d\}$ of points in $\mathbb{P}^1$ are called *harmonic conjugate* if

$$-2\beta\beta' + \alpha\beta' + \alpha'\beta = 0, \tag{2.2}$$

$V(\alpha t_0^2 + 2\beta t_0 t_1 + \gamma t_1^2) = \{a,b\}$ and $V(\alpha' t_0^2 + 2\beta' t_0 t_1 + \gamma' t_1^2) = \{c,d\}$. It follows that this definition is independent or order in the pairs.

It is easy to check that (2.2) is equivalent to the polarity condition

$$P_{cd}(q) = P_{ab}(q') = 0, \tag{2.3}$$

where $V(q) = \{a,b\}, V(q') = \{c,d\}$.

Proposition 2.1.1. *Let l_1, l_2, l_3 be a self-polar triangle of C and $a = l_1 \cap l_2$. Assume $a \notin C$. Then the pairs $l_3 \cap C$ and $\{l_1 \cap l_2, l_2 \cap l_3\}$ are harmonic conjugate. Conversely, if $\{c,d\}$ is a pair of points on l_3 which is harmonic conjugate to the pair $C \cap l_3$, then the lines $\langle a,b\rangle, \langle a,c\rangle, l_3$ form a self-polar triangle of C.*

Proof. Consider the pair $C \cap l_3$ as a quadric q in l. We have $c \in P_b(C)$, thus $P_{ab}(C) = 0$. Restricting to l and using (2.3), we see that b, c form a harmonic pair with respect to q. Conversely, if $P_{bc}(q) = 0$, the line polar line $P_b(C)$ contains a and intersects l at c, hence coincides with $\langle a, c\}$. Similarly, $P_c(C) = \langle a, b\rangle$. □

The polar line $l = P_a(C)$ intersects the conic C at two points x, y such that $a \in \mathrm{PT}(C)_x \cap \mathrm{PT}(C)_y$.

Borrowing terminology from the Euclidean geometry, we call three non-collinear lines in $\mathbb{P}^2$ a *triangle*. The lines themselves will be called the *sides* of the triangle. The three intersection points of pairs of sides are called the *vertices* of the triangle.

Choose projective coordinates in $\mathbb{P}^2$ such that $\ell_i = \{T_i = 0\}$. Then

$$F_{(1,0,0)} = T_0 = \frac{\partial F}{\partial T_0} = a_{00}T_0 + a_{01}T_1 + a_{02}T_2, \tag{2.4}$$

$$F_{(0,1,0)} = T_1 = \frac{\partial F}{\partial T_1} = a_{11}T_1 + a_{01}T_0 + a_{12}T_2, \tag{2.5}$$

$$F_{0,0,1} = T_2 = \frac{\partial F}{\partial T_2} = a_{22}T_2 + a_{02}T_0 + a_{12}T_1$$

implies that

$$F = \frac{1}{2}(T_0^2 + T_1^2 + T_2^2). \tag{2.6}$$

Conversely, any conic with equation

$$F_2 = l_1^2 + l_2^2 + l_3^2 = 0,$$

where l_i are three linear independent linear forms, defines a self-polar triangle with sides $V(l_i)$.

Any triangle in $\mathbb{P}^2$ defines the dual triangle in the dual plane $\check{\mathbb{P}}^2$. Its sides are the pencils of lines with the base point of one of the vertices.

Proposition 2.1.2. *The dual of a self-polar triangle of a conic C is a self-polar triangle of the dual conic $\check{C}$*

Proof. Choose the coordinate system such that the self-polar triangle is the coordinate triangle. Then $C = V(T_0^2 + T_1^2 + T_2^2)$ and the assertion is easily verified. □

2.1.3 The variety of self-polar triangles

Let C be a nonsingular conic. The group of projective transformations of $\mathbb{P}^2$ leaving C invariant is isomorphic to the projective orthogonal group

$$\mathrm{PO}(3) = \mathrm{O}(3)/(\pm I_3) \cong \mathrm{SO}(3).$$

It is also isomorphic to the group PSL_2 via the Veronese map

$$\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2, \quad (t_0, t_1) \mapsto (t_0^2, t_0t_1, t_1^2).$$

Obviously $\mathrm{PO}(3)$ acts transitively on the set of self-polar triangles of C. We may assume that C is given by (2.6). The stabilizer subgroup of the self-polar triangle defined by the coordinate lines is equal to the subgroup generated by permutation matrices and orthogonal diagonal matrices. It is easy to see that it is isomorphic to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_3$. An easy exercise in group theory gives that this group is isomorphic to the permutation group S_4. Thus we obtain the following.

Theorem 2.1.3. *The set of self-polar triangles of a nonsingular conic has a structure of a homogeneous space SO$(3)/\Gamma$, where Γ is a finite subgroup isomorphic to S_4.*

Let us describe a natural compactification of the homogeneous space $\mathrm{SO}(3)/\Gamma$. Let V be a Veronese surface in $\mathbb{P}^5$. We view $\mathbb{P}^5$ as the projective space of conics in $\mathbb{P}^2$ and V as its subvariety of double lines. A trisecant plane of V is spanned by three linear independent double lines. A conic $C \in \mathbb{P}^5$ belongs to this trisecant if and only if the corresponding three lines form a self-polar triangle of C. Thus the set of self-polar triangles of C can be identified with the set of trisecant planes of the Veronese surface which contain C. The latter will also include *degenerate self-polar triangles* corresponding to the case when the trisecant plane is tangent to the Verones surface at some of its points of intersections. Projecting from C to $\mathbb{P}^4$ we will identify the set of self-polar triangles (maybe degenerate) with the set of trisecant lines of the projected Veronese surface $\bar{X}$. This is a closed subvariety of the Grassmann variety $G(2,5)$ of lines in $\mathbb{P}^4$.

Let E be a linear space of odd dimension $2k+1$ and let $G(2,E)$ be the Grassmannian of lines in $\mathbb{P}(E)$. Consider its Plücker embedding $G(2,E) \hookrightarrow \mathbb{P}(\Lambda^2 E)$. Any nonzero $\omega \in \Lambda^2 E^* = \Lambda^2 E^*$ defines a hyperplane H_ω in $\mathbb{P}(\Lambda^2 E)$. Consider ω as a linear map $\alpha_\omega : E \to E^*$ defined by $\alpha_\omega(v)(w) = \omega(v,w)$. The map α_ω is skew-symmetric in the sense that its transpose map coincides with $-\alpha_\omega$. Thus its determinant is equal to zero, and $\mathrm{Ker}(\alpha_\omega) \neq \{0\}$. Let v_0 be a nonzero element of the kernel. Then for any $v \in E$ we have $\omega(v_0, v) = \alpha_\omega(v)(v_0) = 0$. This shows that ω vanishes on all decomposable 2-vectors $v_0 \wedge v$. This implies that the intersection of the hyperplane H_ω with $G(2,E)$ contains all lines which intersect the linear subspace $C_\omega = \mathbb{P}(\mathrm{Ker}(\alpha_\omega)) \subset \mathbb{P}(E)$ which we call the *pole* of the hyperplane H_ω.

Now recall the following result from linear algebra (see Exercise 2.1). Let A be a skew-symmetric matrix of odd size $2k+1$. Its principal submatrices A_i of size $2k$ (obtained by deleting the i-th row and the i-th column) are skew-symmetric matrices of even size. Let Pf_i be the Pfaffians of A_i (i.e. $\det(A_i) = \mathrm{Pf}_i^2$). Assume that $\mathrm{rank}(A) = 2k$, or, equivalently, not all Pf_i vanish. Then the system of linear equations $A \cdot x = 0$ has one-dimensional null-space generated by the vector $(a_1, \ldots, a_{2k+1})$, where $a_i = (-1)^{i+1}\mathrm{Pf}_i$.

Let us go back to Grassmannians. Suppose we have a $s+1$-dimensional subspace W in $\Lambda^2 E^*$ spanned by $\omega_0, \ldots, \omega_s$. Suppose for any $\omega \in W$ we have $\mathrm{rank}(\alpha_\omega) = 2k$, or equivalently, the pole C_ω of H_ω is a point. It follows from the theory of determinant varieties that the subvariety

$$\{K\omega \in \mathbb{P}(\Lambda^2 E^*) : \mathrm{corank}(\alpha_\omega) \geq i\}$$

is of codimension $\binom{i}{2}$ in $\mathbb{P}(\Lambda^2 E^*)$ [Kleppe-Laksov, Harris-Tu]. Thus, if $s < 4$, a general W will satisfy the assumption. Consider a regular map $\Phi : \mathbb{P}(W) \to \mathbb{P}(E)$ defined by $\omega \mapsto C_\omega$. If we take $\omega = t_0\omega_0 + \ldots + t_s\omega_s$ so that $t = (t_0, \ldots, t_s)$ are projective coordinates in $\mathbb{P}(W)$, we obtain that Φ is given by $2k+1$ principal Pfaffians of the matrix A_t defining ω.

We shall apply the preceeding to the case $\dim E = 5$. Take a general 3-dimensional subspace W of $\Lambda^2 E^*$. The map $\Phi : \mathbb{P}(W) \to \mathbb{P}(E) \cong \mathbb{P}^4$ is defined by homogeneous polynomials of degree 2. Its image is a projected Veronese surface S. Any trisecant line of S passes through 3 points on S which are the poles of elements w_1, w_2, w_3 from W. These elements are linearly independent since otherwise their poles lie on the conic image of a line under Φ. But no trisecant line can be contained in a conic plane section of S. We consider $\omega \in W$ as a hyperplane in the Plücker space $\mathbb{P}(\Lambda^2 E)$. Thus any trisecant line is contained in all hyperplanes defined by W. Now we are ready to prove the following.

Theorem 2.1.4. *Let X be the closure in $G_1(4)$ of the locus of tri-secant lines of a projected Veronese surface. Then X is equal to the intersection of $G(2,5)$ with 3 linear independent hyperplanes. In particular, X is a Fano 3-fold of degree 5 with canonical sheaf $\omega_X \cong \mathcal{O}_X(-2)$.*

Proof. We have already shown that the locus of poles of a general 3-dimensional linear W space of hyperplanes in the Plücker space is a projected Veronese surface S and its tri-secant variety is contained in $Y = \cap_{w\in W} H_w \cap G(2,5)$. So, its closure X is also contained in Y. On the other hand, we know that X is irreducible and 3-dimensional (it contains an open subset isomorphic to a homogeneous space $\mathrm{SO}(3)/S_4$). By Bertini's Theorem the intersection of $G(2,5)$ with a general linear space of codimension 3 is an irreducible 3-dimensional variety. This proves that $Y = X$. By another Bertini's theorem, Y is smooth. The rest is the standard computation of the canonical class of the Grassmann variety and the adjunction formula. It is known that the canonical class of the Grassmannian $G = G(m+1, n+1)$ of m-dimensional subspaces of $\mathbb{P}^n$ is equal to

$$K_G = \mathcal{O}_G(-n-1) \tag{2.7}$$

(see Exercise 3.2). By the adjunction formula, the canonical class of $X = G(2,5) \cap H_1 \cap H_2 \cap H_3$ is equal to $\mathcal{O}_X(-2)$. □

Corollary 2.1.5. *The homogeneous space $X = \mathrm{SO}(3)/S_4$ admits a smooth compactification $\bar{X}$ isomorphic to the intersection of $G(2,5)$, embedded via Plücker in $\mathbb{P}^9$, with a linear subspace of codimension 3. The boundary $\bar{X} \setminus X$ is an anti-canonical divisor cut out by a hypersurface of degree 2.*

Proof. The only unproven assertion is about the boundary. We use that the 3-dimensional group $G = \mathrm{SL}(2)$ acts transitively on a 3-dimensional variety X minus the boundary. For any point $x \in X$ consider the map $\mu_x : G \to X, g \mapsto g \cdot x$. Its fibre over the point x is the isotropy subgroup G_x of x. The differential of this map defines a linear map $\mathfrak{g} = T(G)_e \to T(X)_x$. When we let x vary in X, we get a map of vector bundles

$$\phi : \mathfrak{g}_X = \mathfrak{g} \times X \to T(X).$$

Now take the determinant of this map

$$\Lambda^3\phi = \Lambda^3\mathfrak{g} \times X \to \Lambda^3 T(X) = K_X^*,$$

where K_X is the canonical line bundle of X. The left-hand side is the trivial line bundle over X. The map $\Lambda^3(\phi)$ defines a section of the anticanonical line bundle. The zeroes of this section is the set where the differential of the map μ_x is not injective, i.e., where $\dim G_x > 0$. But this is exactly the boundary of X. In fact, the boundary consists of orbits of smaller dimension than 3, hence the isotropy of each such orbit is of positive dimension. This shows that the boundary is contained in our anti-canonical divisor. Obviously, the latter is contained in the boundary. Thus we see that the boundary is equal to the intersection of $G(2,5)$ with a quadric hypersurface.

□

Remark 2.1.1. There is another construction of the variety $\bar{X}$ of self-polar triangles due to S. Mukai and H. Umemura [Lect. Notes in Math., vol. 1016] [**?**]. Let V_6 be the space of homogeneous binary forms $f(t_0, t_1)$ of degree 6. The group $\mathrm{SL}(2)$ has a natural linear representation in V_6 via linear change of variables. Let

$f = t_0t_1(t_0^4 - t_1^4)$. The zeroes of this polynomials can be taken as the vertices of a regular octahedron inscribed in $S^2 = \mathbb{P}^1$. The stabilizer subgroup of f in $\mathrm{SL}(2)$ is isomorphic to the binary octahedron group $\Gamma \cong S_4$. Consider the projective linear representation of $\mathrm{SL}(2)$ in $\mathbb{P}(V_6) \cong \mathbb{P}^5$. In the loc. cit. paper of Mukai-Umemura it is proven that the closure $\bar{X}$ of this orbit in $\mathbb{P}(V_6)$ is smooth and $B = \bar{X} \setminus X$ is the union of two orbits $Kt_0^5t_1$ and Kt_0^6. The first orbit is of dimension 2. The isotropy subgroup of the first orbit is isomorphic to the multiplicative group K^*. The second orbit is one-dimensional and is contained in the closure of the first one. The isotropy subgroup is isomorphic to the subgroup of upper triangular matrices. They also show that B is equal to the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under a $\mathrm{SL}(2)$-equivariant map given by a linear system of curves of bi-degree $(5, 1)$. Thus B is of degree 10, hence is cut out by a quadric. The image of the second orbit is a smooth rational rational curve in B and is equal to the singular locus of B. The fact that the two varieties are isomorphic follows from the the theory of Fano 3-folds. It can be showsn that there is a unique Fano threefold V with $\mathrm{Pic}(V) = \mathbb{Z}\frac{1}{2}K_V$ and $K_V^3 = 40$. We will discuss this variety in a later chapter.

2.1.4 Conjugate triangles

Let $C = V(F)$ be a nonsingular conic, and ℓ be a line in $\mathbb{P}^2$. Take any two points $a, b \in \ell$ and let $P_\ell(C)$ be the intersection point of the polar lines $P_a(C)$ and $P_b(C)$. It is called the *pole* of ℓ with respect to C. It is the unique point p such that the polar line of C with respect to p is equal to ℓ. From the point view of linear algebra, the one-dimensional subspace defining p is orthogonal to the two-dimensional subspace defining ℓ with respect to the symmetric bilinear form defined by F.

Given a triangle with sides ℓ_1, ℓ_2, ℓ_3, the poles of the sides are the vertices of the triangle which is called the *conjugate triangle*. Its sides are the polar lines of the vertices of the original triangle. It is clear that this defines a duality on the set of triangles. Clearly, a triangle is *self-conjugate* if and only if it is a self-polar triangle.

Let ℓ_1, ℓ_2, ℓ_3 be three tangents to C at the points p_1, p_2, p_3, respectively. They form a triangle which can be viewed as a *circumscribed triangle*. It follows from Theorem 1.1.1 that the conjugate triangle has vertices p_1, p_2, p_3. It can be viewed as a *inscribed triangle*. The lines $\ell_1' = \langle p_2, p_3\rangle, \ell_1' = \langle p_2, p_3\rangle, \ell_1' = \langle p_2, p_3\rangle$ are polar lines with respect to the points q_1, q_2, q_3, respectively.

Two lines in $\mathbb{P}^2$ are called *conjugate* with respect to C is the pole of one line belong to the second one. It is a reflexive relation on the set of lines. Obvioulsy, two triangles are conjugate if and only if each of the sides of the first triangle is conjugate to a side of the second triangle.

Now let us consider the following problem. Given two triangles without common sides, find a conic C such that the triangles are conjugate to each other with respect to the conic C. Assume that the first triangle is formed by the coordinate lines $T_i = 0$. Using equations (2.4) it is easy to get a necessary and sufficient condition for this to be true. Let A be the 3×3 matrix whose rows are the coefficients of the linear equations defining the sides of the second triangle. Then the two triangles are conjugate to each other if and only if there exists an invertible diagonal matrix D such that the matrix $D \cdot A$ is symmetric. Or, equivalently,

$$D \cdot A \cdot D^{-1} = A^t. \tag{2.8}$$

It is easy to see that this gives one condition on the set of triangles.

Remark 2.1.2. Consider a triangle (with an order on the set of sides) as a point in $(\mathbb{P}^2)^3$. Then the ordered pairs of conjugate triangles with respect to some conic is a hypersurface in an open subset of $(\mathbb{P}^2)^6$. Is it possible to find the equation of the closure X of this set? By symmetry, X is cut out by a hypersurface in the Segre embedding of the product.

Remark 2.1.3. Let $X = (\mathbb{P}^2)^{[3]}$ be the Hilbert scheme of $\mathbb{P}^2$ of 0-cycles of degree 3. It is a minimal resolution of singularities of the 3d symmetric product of $\mathbb{P}^2$. Consider the open subset of X formed by unordered sets of 3 non-collinear points. We may view a point of U as a triangle. Thus any nonsingular conic C defines an automorphism g_C of U of order 2. Its set of fixed points is equal to the variety of self-polar triangles of C. The automorphism of U can be viewed as a birational automorphism of X. The complement of U is a divisor. It is given by the determinant of the matrix formed by the coordinates of three general points. Since X is nonsingular, one can extend g_C to an open subset of codimension ≥ 2. What is this set and how g_C acts on it?

The variety X is rational, i.e. birationally isomorphic to $\mathbb{P}^6$ (see Exercise 3.1). One can also ask what is group of birational automorphisms of $\mathbb{P}^6$ generated by the involutions g_C.

2.2 Poncelet relation

2.2.1 Darboux's theorem

Let C be a conic, and let $T = \{\ell_1, \ell_2, \ell_3\}$ be a circumscribed triangle. A conic C' which has T as an inscribed triangle is called the *Poncelet related conic*. Since passing through a point impose one condition, we have ∞^2 Poncelet related conics corresponding to a fixed triangle T. Varying T we expect to get ∞^5 conics, so that any conic is Poncelet related to C with respect to some triangle. But surprisingly this is wrong! A theorem of Darboux asserts that there is a pencil of divisors $p_1+p_2+p_3$ such that the triangles T with sides tangent to C at the points p_1, p_2, p_3 define the same Poncelet related conic.

We shall prove it here. In fact, for the future use we shall prove a more general result.

Instead of circumscribed triangles we shall consider circumscribed n-polygons. An *n-polygon* P in $\mathbb{P}^2$ is an ordered set of $n \geq 3$ points $(p_1, \ldots, p_n)$ in $\mathbb{P}^2$ such that no three points p_i, p_{i+1}, p_{i+2} are colinear. The points p_i are the *vertices* of P, the lines $\langle p_i, p_{i+1}\rangle$ are called the *sides* of P (here $p_{n+1} = p_1$). We say that two polygons are equal if the sets of their sides are equal. The number of n-polygons with the same set of vertices is equal to $n!/2n = (n-1)!/2$.

We say that P is circumscribed around a nonsingular C if each side is tangent to C. Given any ordered set $(q_1, \ldots, q_n)$ of n points on C, let ℓ_i be the tangent lines to C at the points q_i. Then they are the sides of the n-polygon P with vertices $p_i = \ell_i \cap \ell_{i+1}, i = 1, \ldots, n$ ($\ell_{n+1} = \ell_1$). This polygon is circumscribed around C. This gives a one-to-one correspondence between n-polygons circumscribed around C and ordered sets of n points on C.

Let $P = (p_1, \ldots, p_n)$ be an n-polygon circumscribed around a nonsingular conic C. A conic S is called *Poncelet n-related* to C with respect to P if all points p_i lie on C.

Let us start with any two conics C and S. We choose a point p_1 on S and a tangent ℓ_1 to C passing through p_1. It intersects S at another point p_2. We repeat this construction. If the process stops after n steps (i.e. we are not getting new points p_i), we get an inscribed n-polygon in S which is circumscribed around C. In this case S is Poncelet related to C. The *Darboux Theorem* which will prove later says that if the process stops, then we can construct infinitely many n-polygons with this property starting from an arbitrary point on S.

Consider the following correspondence on $C \times S$:

$$R = \{(x, y) \in C \times S : \langle x, y\rangle \text{ is tangent to } C \text{ at } x\}.$$

Since, for any $x \in C$ the tangent to C at x intersects S at two points, and, for any $y \in S$ there are two tangents to C passing through y, we get that E is of bidegree $(2, 2)$. This means if we identify C, S with $\mathbb{P}^1$, then R is a curve of bidegree $(2, 2)$. As is well-known R is a curve of arithmetic genus 1.

Lemma 2.2.1. *The curve R is nonsingular if and only if the conics C and S intersect at four distinct points. In this case, R is isomorphic to the double cover of C (or S) ramified over the four intersection points.*

Proof. Consider the projection map $\pi_S : R \to S$. This is a map of degree 2. A branch point $y \in S$ is a point such that there only one tangent to C passing through

y. Obvioulsy, this is possible only if $y \in C$. It is easy to see that R is nonsingular if and only if the double cover $\pi_S : R \to S \cong \mathbb{P}^1$ has four branch points. This proves the assertion. □

Note that the second projection map $\pi_C : R \to C$ must also have 4 branch points, if R is nonsingular. A point $x \in C$ is a branch point if and only if the tangent of C at x is tangent to S. So we obtain that two conics intersect transversally if and only if there are four different common tangents.

Take a point $(x[0], y[0]) \in R$ and let $(x[1], y[1]) \in R$ be defined as follows: $y[1]$ is the second point on S on the tangent to $x[0]$, $x[1] \neq x[0]$ is the point where the tangent of C at $[x[1]$ contains $y[1]$. This defines a self-map $\tau_{C,S} : R \to R$. This map has no fixed points on R and hence, if we fix a group law on R, is a translation map t_a with respect to a point a. Obviously, we get an n-polygon if and only if t_a is of order n, i.e. the order of a in the group law is n. As soon as this happens we can use the automorphism for constructing n-polygons starting from an arbitrary point $(x[0], y[0])$. This is the Darboux Theorem which we have mentioned in above.

Theorem 2.2.2. *(G. Darboux) Let C and S be two nondegenerate conics intersecting transversally. Then C and S are Poncelet n-related if and only if the automorphism $\tau_{C,S}$ of the associated elliptic curve R is of order n. If C and S are Poncelet n related, then starting from any point $x \in C$ and any point $y \in S$ there exists an n-polygon with a vertex at y and one side tangent to C at y which is circumscribed about C and inscribed in S.*

In order to give a more explicit answer when two conics are Poncelte related one needs to recognize when the automorphism $\tau_{C,S}$ is of finite order. Let us choose projective coordinates such that C is the Veronese conic $T_0T_2 - T_1^2 = 0$, the image of $\mathbb{P}^1$ under the map $(t_0, t_1) \mapsto (t_0^2, t_0t_1, t_2^2)$. By using a projective transformation leaving C invariant we may assume that the four intersection points $p_1, \ldots, p_4$ of C and S correspond to the points $(1,0), (1,1), (0,1), (1,a) \in \mathbb{P}^1$, where $a \neq 0, 1$. Then R is isomorphic the elliptic curve given by the affine equation

$$y^2 = x(x-1)(x-a).$$

The conic S belongs to the pencil of conics with base points $p_1, \ldots, p_4$:

$$(T_0T_2 - T_1^2) + \lambda T_1(aT_0 - (1+a)T_1 + T_2) = 0.$$

We choose the zero point in the group law on R to be the point $(x[0], y[0]) = (p_4, p_4) \in C \times S$. Then the automorphism $\tau_{C,S}$ sends this point to $(x[1], y[1])$, where

$$y[1] = (\lambda a, \lambda(1+a) + 1, 0), \quad x[1] = ((a+1)^2\lambda^2, 2a(1+a)\lambda, 4a^2).$$

Thus $x[1]$ is the image of the point $(1, \frac{2a}{(a+1)\lambda}) \in \mathbb{P}^1$ under the Veronese map. The point $y[1]$ corresponds to one of the two roots of the equation

$$y^2 = \frac{2a}{(a+1)\lambda}(\frac{2a}{(a+1)\lambda} - 1)(\frac{2a}{(a+1)\lambda} - a).$$

So we need a criterion characterizing points $(x, \pm\sqrt{x(x-1)(x-a)}$ of finite order. Note that different choice of the sign corresponds to the inversion involution on the elliptic curve. So, the order of the points corresponding to two different choices of the sign are the same. We have the following:

Theorem 2.2.3. *(A. Cayley). Let R be an elliptic curve with affine equation*

$$y^2 = g(x),$$

where $g(x)$ is a cubic polynomial with three distinct nonzero roots. Write

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Then a point $(0, \sqrt{g(0)})$ is of order n if and only if

$$\begin{vmatrix} a_2 & a_3 & \dots & a_{m+1} \\ a_3 & a_4 & \dots & a_{m+2} \\ \vdots & \vdots & \vdots & \\ a_{m+1} & a_{m+2} & \dots & a_{2m} \end{vmatrix} = 0, \quad n = 2m+1,$$

$$\begin{vmatrix} a_3 & a_4 & \dots & a_{m+1} \\ a_4 & a_5 & \dots & a_{m+2} \\ \vdots & \vdots & \vdots & \\ a_{m+1} & a_{m+2} & \dots & a_{2m} \end{vmatrix} = 0, \quad n = 2m.$$

Proof. We fix a square root c_0 of $g(0)$ and consider the point $p = (0, c_0)$. A necessary and sufficient condition for p to be a n-torsion point is that there exists a rational function f on R with a zero of order n at p and a pole of order n at the infinity point $(\infty, 0)$. We shall assume that $n = 2k-1$ is odd. The other case is considered similarly. Since f is regular on the affine part, it must be a restriction of a polynomial $F(x, y)$ of some degree d. Since the infinity is an inflection point, the degree of F must be equal to $k-1$ and $F(x, y)$ must have a zero of order $2k-1$ at $(0, c_0)$ and a pole of order $k-2$ at infinity. Now we expand $y = \sum_{k=0}^{\infty} a_k x^k$ and put

$$y_m = a_0 + a_1 x + \ldots + a_{k-1} x^{k-1}.$$

We have

$$\begin{aligned} y - y_k &= a_k x^k + \ldots + a_{2k-2}x^{2k-2} + \ldots \\ x(y - y_{k-1}) &= a_{k-1}x^k + \ldots + a_{2k-3}x^{2k-2} + \ldots \\ \ldots &= \ldots \\ x^{k-2}(y - y_2) &= a_2 x^k + \ldots + a_k x^{2k-2} + \ldots . \end{aligned}$$

We can find $n-1$ coefficients $c_0, c_1, \ldots, c_{k-2}$ such that the polynomial

$$F(x,y) = c_0(y - y_k) + c_1 x(y - y_{k-1}) + \ldots + c_{k-2}x^{k-2}(y - y_2)$$

vanishes at $x = 0$ of order $2k-1$ if and only if

$$\begin{vmatrix} a_k & a_{k-1} & \ldots & a_2 \\ a_{k+1} & a_k & \ldots & a_3 \\ \ldots & \ldots & \ldots & \ldots \\ a_{2k-2} & a_{2k-3} & \ldots & a_k \end{vmatrix} = 0.$$

It is easy to see that this determinant is equal to one of the determinants from the assertion of the theorem. □

To apply the proposition we have to take

$$\alpha = \frac{2a}{(a+1)\lambda}, \quad \beta = 1 + \frac{2a}{(a+1)\lambda}, \quad \gamma = a + \frac{2a}{(a+1)\lambda}.$$

Let us consider the variety $\mathcal{P}_n$ of pairs of conics (C, S) such that S is Poncelet n-related to C. We assume that C and S intersect each other transversally. We already know that $\mathcal{P}_n$ is a hypersurface in $\mathbb{P}^5 \times \mathbb{P}^5$. Obviously $\mathcal{P}_n$ is invariant with respect to the diagonal action of the group $\mathrm{SL}(3)$ (acting on the space of conics). Thus the equation of $\mathcal{P}_n$ is an invariant of a pair of conics. This invariant was computed by F. Gerbradi [**?**]. It is of bidegree $(\frac{1}{4}T(n), \frac{1}{2}T(n))$, where $T(n)$ is equal to the number of elements of order n in the abelian group $(\mathbb{Z}/n\mathbb{Z})^2$.

Let us look at the quotient of $\mathcal{P}_n$ by $\mathrm{PSL}(3)$. Consider the rational map β : $\mathbb{P}^5 \times \mathbb{P}^5 \to (\mathbb{P}^2)^{(4)}$ which assigns to (C, S) the point set $C \cap S$. The fibre of β over a subset B of 4 points in general linear position is isomorphic to an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the pencil of conics with base point B. Since we can always transform such B to the set of points $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1),$ the group $\mathrm{PSL}(3)$ acts transitively on the open subset of such 4-point sets. Its stabilizer is isomorphic to the permutation group S_4 generated by the following matrices:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The orbit space $\mathcal{P}_n/\mathrm{PSL}(3)$ is isomorphic to a curve in an open subset of $\mathbb{P}^1 \times \mathbb{P}^1/S_4$, where S_4 acts diagonally. By considering one of the projection maps, we obtain that $\mathcal{P}_n/\mathrm{PSL}(3)$ is an open subset of a cover of $\mathbb{P}^1$ of degree N equal to the number of Poncelet n-related conics in a given pencil of conics with 4 distinct base points with respect to a fixed conic from the pencil. This number was computed by F. Gerbardi [**?**] and is equal to $\frac{1}{2}T(n)$. A modern account of Gerbardi's result is given in [**?**]. A smooth compactification of $\mathcal{P}_n/\mathrm{PSL}(3)$ is the modular curve $X^0(n)$ which parametrizes the isomorphism classes of the pairs (R, e), where R is an elliptic curve and e is a point of order n in R.

Proposition 2.2.4. *Let C and S be two nonsingular conics. Consider each n-polygon inscribed in C as a subset of its vertices, and also as positive divisor of degree n on C. The closure of the set of n-polygons inscribed in C and circumscribed about S is either empty, or a g_n^1, i.e. a linear system of divisors of degree n.*

Proof. First observe that two polygons inscribed in C and circumscribed around S which share a common vertex must coincide. In fact, the two sides passing through the vertex in each polygon must be the two tangents of S passing through the vertex. They intersect C at another two common vertices. Continuing in this way we see that the two polygons have the same set of vertices. Now consider the Veronese embedding v_n of $C \cong \mathbb{P}^1$ in $\mathbb{P}^n$. An effective divisor of degree n is a plane section of the Veronese curve $\mathrm{Ver}_n^1 = v_n(\mathbb{P}^1)$. Thus the set of effective divisors of degree n on C can be identified with the dual projective space $\check{\mathbb{P}}^n$. A hyperplane in $\check{\mathbb{P}}^n$ is the set of hyperplanes in $\mathbb{P}^n$ which pass through a fixed point in $\mathbb{P}^n$. The degree of an irreducible curve $X \subset \check{\mathbb{P}}^n$ of divisors is equal to the cardinality of the set of divisors containing a fixed general point of Ver_n^1. In our case it is equal to 1. □

2.2.2 Invariants of pairs of conics

The Poncelet theorem is an example of *porism* which can be loosely stated as follows. If one can find one object satisfying a certain special property then there are infinitely many such objects. In case of Darboux's theorem this is the property of the existence of a polygon inscribed in one conic and circumscribed about the other conic. Here we consider another example of a porism between two conics. This time the relation is the following.

Given two nonsingular conics C and S there exists a self-conjugate triangle with respect to C which is inscribed in S.

Proposition 2.2.5. *Let S and C be two nonsingular conics defined by symmetric matrices A and B respectively. Then C admits a self-conjugate triangle which is inscribed in S if and only if*

$$\mathrm{Tr}(AB^{-1}) = 0.$$

Moreover, if this condition is satisfied, for any point $x \in S \setminus (S \cap C)$ there exists a self-conjugate triangle inscribed in S with vertex at x.

Proof. Let Q be an invertible 3×3 matrix. Replacing A with $A' = Q^T AQ$ and B with $B' = Q^T BQ$ we check that

$$\mathrm{Tr}(A'B'^{-1}) = \mathrm{Tr}(Q^T AB^{-1}(Q^T)^{-1}) = \mathrm{Tr}(AB^{-1}).$$

This shows that trace condition is invariant with respect to a linear change of variables. Thus we may assume that $C = V(T_0^2 + T_1^2 + T_2^2)$. Suppose there is a self-conjugate triangle with respect to C which is inscribed in S. Since the orthogonal group of C acts transitively on the set of self-conjugate triangles, we may assume that the tringle is the coordinate triangle. Then the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ must be on S. Hence

$$S = V(aT_0T_1 + bT_0T_2 + cT_1T_2),$$

and the condition $\mathrm{Tr}(AB^{-1})$ is verified.

Let us show the sufficiency of the trace condition. Choose coordinates as above. Let

$$S = V(aT_0^2 + bT_1^2 + cT_2^2 + dT_0T_1 + eT_0T_2 + fT_1T_2).$$

The trace condition is

$$a + b + c = 0.$$

Let $x = (x_0, x_1, x_2)$ be any point on S and $l = V(x_0T_0 + x_1T_1 + x_2T_2)$ be the polar line $P_x(C)$. Without loss of generality, we may assume that $x_2 = -1$ so that we can write $T_2 = x_0T_0 + x_1T_1$ and take T_0, T_1 as coordinates on l. The line l intersects S at two points (c_0, c_1) and (d_0, d_1) which are the zeroes of the binary form

$$q = aT_0^2 + bT_1^2 + c(x_0T_0 + x_1T_1)^2 + dT_0T_1 + (eT_0 + fT_1)(x_0T_0 + x_1T_1)$$

$$= (a + ex_0 + cx_0^2)T_0^2 + (b + ex_1 + cx_1^2)T_1^2 + 2(d + ex_1 + fx_0 + cx_0x_1)T_0T_1.$$

The line l intersects C at the points $y = (a_0, a_1, x_0a_0+x_1a_1)$ and $z = (b_0, b_1, b_0x_0+b_1x_1)$ such that its coordinates on l are the zeroes of the binary form

$$q' = T_0^2 + T_1^2 + (x_0T_0 + x_1T_1)^2 = (1 + x_0^2)T_)^2 + 2x_0x_1T_0T_1 + (1 + x_1^2)T_1^2.$$

It follows from Proposition 2.2.5 that the points x, y, z are the vertices of a self-polar triangle if and only if (2.3) hold. To check this condition we will use that $a+b+c=0$ and $ax_0^2+bx_1^2+c+dx_0x_1-ex_0-fx_1=0$. We have

$$(a+ex_0+cx_0^2)(1+x_1^2)+(b+ex_1+cx_1^2)(1+x_0^2)-2(d+ex_1+fx_0+cx_0x_1)x_0x_1$$

$$= a+b+ex_0+cx_0^2+ex_1+cx_1^2+(a+ex_0+cx_0^2)x_1^2+(b+ex_1+cx_1^2)x_0^2$$
$$-2(d+ex_1+fx_0+cx_0x_1)x_0x_1$$

Replacing $a+b$ with $-c=-ax_0^2-bx_1^2-dx_0x_1+ex_0+fx_1$ we check that the sum is equal to zero. Thus starting from any point x on S we find that the triangle with vertices x, y, z is self-conjugate with respect to C. □

Remark 2.2.1. Let $\mathbb{P}^2 = \mathbb{P}(E)$ and $C = V(Q), S = V(F)$, where $Q, F \in S^2E^*$. Let $\check{C} = V(\Phi)$, where $\Phi \in S^2(E)$. Then the trace condition from the previous theorem is

$$\langle \Phi, F \rangle = 0,$$

where the pairing is the polarity pairing (1.2).

Consider the set of self-polar triangles with respect to C inscribed in S. We know that this set is either empty or of dimension ≥ 1. We consider each triangle as a set of its 3 vertices, i.e. as an effective divisor of degree 3 on S.

Proposition 2.2.6. *The closure X of the set of self-polar triangles with respect to C which are inscribed in S, if not empty, is a g_3^1, i.e. a linear system of divisors of degree 3.*

Proof. First we use that two self-polar triangles with respect to C and inscribed in S which share a common vertex must coincide. In fact, the polar line of the vertex must intersect S at the vertices of the triangle. Then the assertion is proved using the argument from the proof of Proposition 2.2.4. □

Note that a general g_3^1 contains 4 singular divisors corresponding to ramification points of the corresponding map $\mathbb{P}^1 \to \mathbb{P}^1$. In our case these divisors correspond to 4 intersection points of C and S.

Another example of a poristic statement is the following.

Theorem 2.2.7. *Let T and T' be two different triangles. The following assertions are equivalent:*

(i) the exists a conic S containing the vertices of the two triangles;

(ii) there exists a conic Σ touching the sides of the two triangles;

(iii) there exists a conic C with respect to which each of the triangles is self-polar.

Moreover, when one of the conditions is satisfied, there is an infinite number of triangles inscribed in S, circumscribed around Σ, and all of these triangles are self-polar with respect to C.

Proof. (iii)$\Leftrightarrow$ (ii) Let l_1, l_2, l_3 and m_1, m_2, m_3 be the sides of the two triangles considered as points in the dual plane $\check{\mathbb{P}}^2$. Consider the linear systems $V = |\mathcal{O}_{\check{\mathbb{P}}^2}(2) - l_1 - l_2 - l_3|$ and $W = |\mathcal{O}_{\check{\mathbb{P}}^2}(2) - m_1 - m_2 - m_3|$ of conics passing through the corresponding points. Let $l_i = V(L_i), m_i = V(M_i)$ for some linear forms L_i, M_i. Let $C = V(F)$. We can write

$$F = a_1 L_1^2 + a_1 L_2^2 + a_3 L_3^2 = b_1 M_1^2 + b_2 M_2^2 + b_3 M_3^2$$

for some scalars a_i, b_i. For any $V(\Phi) \in V \cup W$ we have $\langle \Phi, F \rangle = 0$. This shows that the span of V and W in $|\mathcal{O}_{\check{\mathbb{P}}^2}(2)|$ is contained in a hyperplane orthogonal to F. Thus $V \cap W \neq \emptyset$ and a common conic vanishes at all l_i's and m_i's. Hence the dual conic Σ is touching the sides of the two triangles. Reversing the arguments, we find that condition (ii) implies that there exists a conic $V(F)$ such that $\langle \Phi, F \rangle = 0$ for any $V(\Phi) \in V \cup W$. Since, for any $V(\Phi) \in V \cup W$, $\langle \Phi, L_i^2 \rangle = \langle \Phi, M_i^2 \rangle = 0$, we obtain that F belongs to the linear span of L_1^2, L_2^2, L_3^2, and also to the linear span of M_1^2, M_2^2, M_3^2. This proves the equivalence of (ii) and (iii). More details for this argument can be seen in the later chapter about the apolarity theory.

(iii)$\Leftrightarrow$ (i) This follows from Proposition 2.1.2.

Let us prove the last assertion. Suppose one of the conditions of the theorem is satisfied. Then we have the conics C, S, Σ with the asserted properties with respect to the two triangles T, T'. By Proposition 2.2.6, the set of self-polar triangles with respect to C inscribed in S is a g_3^1. By Proposition 2.2.4, the set of triangles inscribed in S and circumscribed around Σ is also a g_3^1. Two g_3^1's with 2 common divisors coincide. □

Let $C = V(F)$ and $S = V(G)$ be two conics (not necessary nonsingular). Consider the pencil $V(t_0 F + t_1 G)$ of conics spanned by C and S. The zeroes of the discriminant equation $D = \text{discr}(t_0 F + t_1 G) = 0$ correspond to singular conics in the pencil. In coordinates, if F, G are defined by symmetric matrices $A = (a_{ij}), B = (b_{ij})$, respectively, then $D = \det(t_0 A + t_1 B)$ is a homogeneous polynomial of degree ≤ 3. Choosing different system of coordinates replaces A, B by $Q^T A Q, Q^T B Q$, where Q is an invertible matrix. This replaces D with $\det(Q)^2 D$. Thus the coefficients of D are invariants on the space of pairs of quadratic forms on $\mathbb{C}^3$ with respect to the action of the group $\text{SL}(3)$. To compute

D explicitly, we use the following formula for the determinant of the sum of two $n \times n$ matrices $X + Y$:

$$\det(X+Y) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \ldots < i_k \le n} \Delta_{i_1,\ldots,i_k}, \tag{2.9}$$

where $\Delta_{i_1,\ldots,i_k}$ is the determinant of the matrix X in which the columns $X_{i_1}, \ldots, X_{i_k}$ are replaced with the columns $Y_{i_1}, \ldots, Y_{i_k}$. Applying this formula to our case, we get

$$D = \Delta t_0^3 + \Theta t_0^2 t_1 + \Theta' t_0 t_1^2 + \Delta' t_1^3, \tag{2.10}$$

where

$$\begin{aligned} \Delta &= \det A \qquad (2.11)\\ \Theta &= \det(A_1A_2B_1) + \det(A_1B_2A_3) + \det(B_1A_2A_3) = \text{Tr}(B \cdot \text{adj}(A))\\ \Theta' &= \det(B_1B_2A_1) + \det(B_1A_2B_3) + \det(A_1B_2B_3) = \text{Tr}(A \cdot \text{adj}(B))\\ \Delta' &= \det(B) \end{aligned}$$

where adj means the *adjugate matrix* of complementary minors.

We immediately recognize the geometric meanings of vanishing of the coefficients of D.

The coefficient Δ (resp. Δ') vanishes if and only if C (resp. S) is a singular conic.

If Δ, Δ' are nonzero, then the coefficient Θ (resp. Θ') vanishes if and only if there exists a self-polar triangle of C inscribed in S (resp. a self-polar triangle of S inscribed in C). This follows from Proposition 2.2.5.

We can also express the condition that the two conics are Poncelet related.

Proposition 2.2.8. *Let C and S be two nonsingular conics. A triangle inscribed in C and circumscribed around S exists if and only if*

$$\Theta'^2 - 4\Theta\Delta' = 0.$$

Proof. Choose a coordinate system such that $C = V(2T_0T_1 + 2T_1T_2 + 2T_0T_2)$. Suppose there is a triangle inscribed in C and circumscribed around S. Applying an orthogonal transformation, we may assume that the vertices of the triangle are the references points $(1,0,0), (0,1,0)$ and $(0,0,1)$. Let $S = V(G)$, where

$$G = aT_0^2 + bT_1^2 + cT_2^2 + 2dT_0T_1 + 2eT_0T_2 + 2fT_1T_2. \tag{2.12}$$

The condition that the triangle is circumscribed in S is that the points $(1,0,0), (0,1,0)$, and $(0,0,1)$ lie on the dual conic $\check{S}$. This implies that the diagonal entries $bc -$

$f^2, ac - e^2, ab - d^2$ of the matrix $\mathrm{adj}(B)$ are equal to zero. Therefore we may assume that

$$G = \alpha^2T_0^2 + \beta^2T_1^2 + \gamma^2T_2^2 - 2\alpha\beta T_0T_1 - 2\alpha\gamma T_0T_2 - 2\beta\gamma T_1T_2. \tag{2.13}$$

We get

$$\Theta' = \mathrm{Tr}\Big(\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \cdot \begin{pmatrix}0 & 2\alpha\beta\gamma^2 & 2\alpha\gamma\beta^2\\ 2\alpha\beta\gamma^2 & 0 & 2\beta\gamma\alpha^2\\ 2\alpha\gamma\beta^2 & 2\beta\gamma\alpha^2 & 0\end{pmatrix}\Big) = 4\alpha\beta\gamma(\alpha+\beta+\gamma),$$

$$\Theta = \mathrm{Tr}\Big(\begin{pmatrix}\alpha^2 & -\alpha\beta & -\alpha\gamma\\ -\alpha\beta & \beta^2 & -\beta\gamma\\ -\alpha\gamma & -\beta\gamma & \gamma^2\end{pmatrix} \begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}\Big) = -(\alpha+\beta+\gamma)^2,$$

$$\Delta' = -4(\alpha\beta\gamma)^2.$$

This checks that $\Theta'^2 - 4\Theta\Delta' = 0$.

Let is prove the sufficiency of the condition. Take a tangent l_1 to S intersecting C at two points x, y and consider a tangent l_2 to S passing through x and a tangent l_3 to S passing through y. The triangle with sides l_1, l_2, l_3 is circumscribed around S and has 2 vertices on C. Choose coordinates such that this triangle is the coordinate triangle. Then, we may assume that $C = V(aT_0^2 + 2T_0T_1 + 2T_1T_2 + 2T_0T_2)$ and $S = V(G)$, where G is as in (2.13). Computing $\Theta'^2 - 4\Theta\Delta'$ we find that it is equal to zero if and only if $a = 0$. Thus the coordinate triangle is inscribed in C. □

Remark 2.2.2. Choose coordinate system such that $C = V(T_0^2 + T_1^2 + T_2^2)$. Then the condition that S is Poncelet related to C with respect to triangles is easily seen to be equal to

$$c_2^2 - c_1c_3 = 0,$$

where

$$\det(A - tI_3) = (-t)^3 + c_1(-t)^2 + c_2(-t) + c_3$$

is the characteristic polynomial of the symmetric matrix defining S. This is a quartic hypersurface in the space of conics. The polynomials c_1, c_2, c_3 generate the algebra of invariants of $S^2(\mathbb{C}^3)^*$ with respect to the group $\mathrm{SL}(3)$.

Exercises

2.1 Let E be a vector space of even dimension $n = 2k$ over a field F of characteristic 0 and $(e_1, \dots, e_n)$ be a basis in E. Let $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j \in \Lambda^2 E^*$ and $A = (a_{ij})_{1 \le i \le j \le n}$ be the skew-symmetric matrix defined by the coefficients a_{ij}. Let $\Lambda^k(\omega) = \omega \wedge \cdots \wedge \omega = ak! e_1 \wedge \cdots \wedge e_n$ for some $a \in F$. The element a is called the *Pfaffian* of A and is denoted by $\text{Pf}(A)$.

(i) Show that
$$\text{Pf}(A) = \sum_{S \in \mathcal{S}} \epsilon(S) \prod_{(i,j) \in S} a_{ij},$$
where S is a set of pairs $(i_1, j_1), \dots, (i_k, j_k)$ such that $1 \le i_s < j_s \le 2k, s = 1, \dots, k$, $\{i_1, \dots, i_k, j_1, \dots, j_k\} = \{1, \dots, n\}$, $\mathcal{S}$ is the set of such sets S, $\epsilon(S) = 1$ if the permutation $(i_1, j_1, \dots, i_k, j_k)$ is even and -1 otherwise.

(ii) Compute $\text{Pf}(A)$ when $n = 2, 4, 6$.

(iii) Show that, for any invertible matrix C,
$$\text{Pf}({}^t C \cdot A \cdot C) = \det(C)\text{Pf}(A).$$

(iv) Using (iii) prove that
$$\det(A) = \text{Pf}(A)^2.$$

(iv) Show that
$$\text{Pf}(A) = \sum_{i=1}^{n} (-1)^{i+j-1} \text{Pf}(A_{ij}) a_{ij},$$
where A_{ij} is the matrix of order $n - 2$ obtained by deleting the ith and jth rows and columns of A.

(v) Let B be a skew-symmetric matrix of odd order $2k - 1$ and B_i be the matrix of order $2k - 2$ obtained from B by deleting the ith row and ith column. Show that the vector $(\text{Pf}(B_1), \dots, (-1)^{i+1}\text{Pf}(B_i), \dots, \text{Pf}(B_{2k-1}))$ is a solution of the equation $B \cdot x = 0$.

(vi) Show that the rank of a skew-symmetric matrix A of any order n is equal to the largest m such that there exists $i_1 \dots < i_m$ such that the matrix $A_{i_1 \dots i_m}$ obtained from A by deleting i_jth rows and columns, $j = 1, \dots, m$, has nonzero Pfaffian.

2.2 Let $G_m(n) = G(m+1, n+1)$ be the Grassmannian of m-dimensional subspaces in $\mathbb{P}^n = \mathbb{P}(E)$.

(i) Show that, for any subspace $\pi = \mathbb{P}(L) \in G_m(n)$, the tangent space $TG_m(n)_\pi$ is canonically isomorphic to Hom$(L, E/L)$.

(ii) Using (i) show that the canonical class of $G_m(n)$ is isomorphic to the line bundle $\Lambda^{(m+1)(n-m)}(\mathcal{E})$, where $\mathcal{E} = \mathcal{S} \otimes \mathcal{Q}^*$, $\mathcal{S}$ is the tautological rank $m+1$-subbundle, and $\mathcal{Q}$ is the tautological quotient bundle of rank $n-m$.

(iii) Use the fact that $\Lambda^{n-m}(\mathcal{Q}) \cong \mathcal{O}_{G_m(n)}(1)$ to prove that the canonical class of $G_m(n)$ is isomorphic to $\mathcal{O}_{G_m(n)}(-n-1)$.

2.3 Let P be a trisecant plane in the space of conics to the Veronese variety of double lines. Consider it as a point in the Grassmannian $G_2(\mathbb{P}(S^2(E^*))) \cong G_2(5)$. Show that the plane of hyperplanes through P, considered as a point in the dual Grassmannian $G_2(\mathbb{P}(S^2(E)))$, is a 2-dimensional linear system of conics in the dual plane $\mathbb{P}(E^*)$ with 3 base points corresponding to the double lines in P.

2.4 Let $V = \nu_2(\mathbb{P}^2)$ be a Veronese surface in $\mathbb{P}^5$.

(i) Show that a general 3-dimensional subspace L intersects V at 4 points.

(ii) Let π be a plane in $\mathbb{P}^5$ and L_π be the 2-dimensional linear system (a *net*) of conics $\nu_2^*(H)$ in $\mathbb{P}^2$, where H is a hyperplane in $\mathbb{P}^5$ containing π. Show that π is a trisecant plane if and only the set of base points of L_π consists of 3 points (counting with multiplicities). Conversely, the linear system of conics through 3 points defines a unique trisecant plane.

(iii) Show that the set of nets of conics with three base points (a subvariety of the Grassmannian of 2-planes in the space of conics) contains an irreducible divisor parametrizing nets with 3 distinct collinear points and an irreducible divisor parametrizing nets with 2 base points, one of them is infinitely near.

(iv) Using (iii) show that the anticanonical divisor of degenerate triangles is irreducible.

(v) Show that the trisecant planes intersecting the Veronese plane at one point (corresponding to net of conics with one base point of multiplicity 3) define a smooth rational curve in the boundary of the variety of self-polar triangles. Show that this curve is equal to the set of singular points of the boundary.

2.5 Let $U \subset (\mathbb{P}^2)^{(3)}$ be the subset of the symmetric product of $\mathbb{P}^2$ parametrizing the sets of three distinct points. For each set $Z \in U$ let L_Z be the linear system of conics containing Z. Consider the map $f : U \to G_2(5), Z \mapsto L_Z \in |\mathcal{O}_{\mathbb{P}^2}(2)|$.

(i) Consider the divisor D in U parametrizing sets of 3 distinct collinear points. Show that $f(D)$ is a closed subvariety of $G_2(5)$ isomorphic to $\mathbb{P}^2$.

(ii) Show that the map f extends to the Hilbert scheme $(\mathbb{P}^2)^{[3]}$ of 0-cycles Z with $h^0(\mathcal{O}_Z) = 3$ (which admits a natural map $\pi : (\mathbb{P}^2)^{[3]} \to (\mathbb{P}^2)^{(3)}$ which is a resolution of singularities).

(iii) Show that the closure $\bar{D}$ of $\pi^{-1}(D)$ in the Hilbert scheme is isomorphic to a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$ and the restriction of f to $\bar{D}$ is the projection map to its base.

(iv) Let $\mathbb{P}(\mathcal{S})$ be the projectivization of the tautological rank 3 vector bundle over the Grassmannian $G_2(5)$ and $p : \mathcal{P} \to (\mathbb{P}^2)^{[3]}$ be its pull-back under the map f. Show that its fibre over a point Z is the linear system of conics with base scheme equal to Z.

(v) Define the map $\tilde{f} : \mathcal{P} \to |\mathcal{O}_{\mathbb{P}^2}(2)|$ which assigns to a point in the fibre $p^{-1}(Z)$ the corresponding conic in the net of conics though Z. Show that the fibre of $\tilde{f}$ over a nonsingular conic C is isomorphic to the Fano variety of self-polar triangles of the dual conic C^*.

(vi) Let $\mathcal{P}^s = \tilde{f}^{-1}(\mathcal{D}_2(2))$ be the pre-image of the hypersurface of singular conics. Describe the fibres of the projections $p : \mathcal{P}^s \to (\mathbb{P}^2)^{[3]}$ and $\tilde{f} : \mathcal{P}^s \to \mathcal{D}_2(2)$.

2.6 Prove that the nth symmetric product of $\mathbb{P}^n$ is a rational variety.

2.7 Prove directly that there are 4 common tangents to two conics which intersect at 4 points.

2.8 Extend Darboux's Theorem to the case of two conics which do not intersect transversally.

2.9 Given two nonsingular conics C and S, find a necessary and sufficient condition in order that C admits a self-polar triangle circumscribed around S.

2.10 Find the geometric interpretation of vanishing of the invariants Θ, Θ' from (2.10) in the case when C or S is a singular conic.

2.11 Express the condition that two conics are tangent in terms of the invariants $\Delta, \Delta', \Theta, \Theta'$.

2.12 Let p_1, p_2, p_3, p_4 be four distinct points on a nonsingular conic C. Let $\langle p_i, p_j \rangle$ denote the line through the points p_i, p_j. Show that the triangle with the vertices $A = \langle p_1, p_3 \rangle \cap \langle p_2, p_4 \rangle$, $B = \langle p_1, p_2 \rangle \cap \langle p_3, p_4 \rangle$ and $C = lap_1, p_4 \rangle \cap \langle p_2, p_3 \rangle$ is a self-conjugate triangle with respect to C.

2.13 show that two pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ are harmonic conjugate if and only the cross-ratio $(a, b, c, d) = \frac{(d-b)(c-a)}{(d-c)(b-a)}$ is equal to -1.

Chapter 3

Plane cubics

3.1 Equations

3.1.1 Weierstrass equation

Let X be a nonsingular projective curve of genus 1. By Riemann-Roch, for any divisor D of degree $d > 0$, we have $\dim H^0(X, \mathcal{O}_X(D)) = d$. The complete linear system $|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$ defines an isomorphism $X \cong C$, where C is a curve of degree d in $\mathbb{P}^{d-1}$ (see [Hartshorne], Chapter IV, Corollary 3.2). We consider here the case $d = 3$, i.e., a plane cubic model C of X. Let $C = V(F(T_0, T_1, T_2)$. By Theorem 1.22, C has an inflection point p_0. Without loss of generality we may assume that $p_0 = (0, 0, 1)$ and the tangent line at this point has the equation $T_0 = 0$. This implies that $F = T_0 G(T_0, T_1, T_2) + aT_1^3$, where G is a quadratic polynomial. We may assume that $G = bT_2^2 + T_2 L(T_0, T_1) + Q(T_0, T_1)$ for some quadratic polynomial Q and a linear polynomial L. Notice that $b \neq 0$, otherwise, we can express T_2 as a rational function in T_0, T_1 and obtain that E is a rational curve. So, we may assume that $b = 1$. Replacing T_2 with $T_2 + \frac{1}{2}L(T_0, T_1)$ we may assume that $L = 0$. Now the equation looks as

$$F = T_0 T_2^2 + aT_1^3 + bT_1^2 T_0 + cT_1 T_0^2 + dT_0^3 = 0.$$

By scaling, we may assume that $a = 1$. Replacing T_1 with $T_1 + \frac{b}{3}T_0 \neq 0$, we may assume that $b = 0$. This gives us the *Weierstrass equation* of a nonsingular cubic:

$$T_0 T_2^2 + T_1^3 + \alpha T_1 T_0^2 + \beta T_0^3 = 0 \tag{3.1}$$

It is easy to see that C is nonsingular if and only if the polynomial $x^3 + \alpha x + \beta$ has no multiple roots, or, equivalently, its discriminant $\Delta = 4\alpha^3 + 27\beta^2$ is not equal to zero.

The projection $(t_0, t_1, t_2) \mapsto (t_0, t_1)$ exhibits C as the double cover of $\mathbb{P}^1$ with branch points $(1, x), (0, 1)$, where $x^3 + \alpha x + \beta = 0$. The corresponding points $(1, x, 0), (0, 1, 0)$ on C are the ramification points. If we choose $p_0 = (0, 1, 0)$ to be the zero point in the group law on C, then $2p \sim 2p_0$ for any ramification point p implies that p is a 2-torsion point. Any 2-torsion point is obtained in this way. Here we use that the group law on a cubic curve with the distinguished point O chosen as the zero point is given by the formula

$$P \oplus Q \in \big|P + Q - O\big|. \tag{3.2}$$

Note that, by Riemann-Roch, the complete linear system $\big|P + Q - O\big|$ consists of one point.

3.1.2 Hesse equation

Since any flex tangent line intersects the curve with multiplicity 3 (not more!), applying formula(1.21), we obtain that the curve has exactly 9 inflection points. Using the group law on an elliptic curve with an inflection point as the zero, we can interpret any inflection point p as a 3-torsion point. This follows from (3.2) since the divisor of the rational function $L/L_0 \mod (F(T_0, T_1, T_2)$, where $L = 0$ is the equation of the inflection tangent at p and $L_0 = 0$ is the equation of the inflection tangent at p_0, is equal to $3p - 3p_0$. This of course agrees with the fact the group $X[3]$ of 3-torsion points on an elliptic curve X is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$.

Let H be a subgroup of order 3 of X. Since the sum of elements of this group add up to 0, we see that the corresponding 3 inflection points p, q, r satisfy $p + q + r \sim 3p_0$. It is easy to see the rational function on C with the divisor $p + q + r - 3p_0$ can be obtain as the restriction of the rational function $M(T_0, T_1, T_2)/L_0(T_0, T_1, T_2)$, where $M = 0$ defines the line containing the points p, q, r. There are 3 cosets with respect to each subgroup H. Since the sum of elements in each conjugacy class is again equal to zero, we get 12 lines, each containing three inflection points. Conversely, if a line contains three inflection points, the sum of these points is zero, and it is easy to see that the three points is a conjugacy class with respect to some subgroup H. Each element of $(\mathbb{Z}/3\mathbb{Z})^3$ is contained in 4 cosets (it is enough to check this for the zero element). Thus we obtain a configuration of 12 lines and 9 points, each line contains 3 points, and each point is contained in 4 lines. This is called a *Hesse configuration*.

Let p_0 be an inflection point taken as the zero element in the group law. For any $p \in X[3]$, any line section $D = q_1 + q_2 + q_3$ of C_3, we have

$$D \oplus p = (q_1 \oplus p) + (q_2 \oplus p) + (q_3 \oplus p) \sim D + 3p - 3p_0 \sim D.$$

This shows that the translation automorphism $t_p : x \mapsto x \oplus p$ is induced by a projective transformation g_p (because our curve is embedded in $\mathbb{P}^2$ by the linear system $|D|$). . Let l_1, l_2, l_3 be three lines corresponding to a subgroup H and its two cosets. For any $p \in H$, the projective automorphism g_p leaves each line l_i invariant. If the lines are collinear, the group H acts on the pencil of lines through the point $q = l_1 \cap l_2 \cap l_3$ leaving the three lines fixed. However, it is known (and easy) that any cyclic group acting nontrivially on $\mathbb{P}^1$ has two fixed points. Thus, H acts identically on the pencil. Consider the set of fixed points of g_p. It is a closed subset of $\mathbb{P}^2$ and intersects with each line of the pencil at 2 points. One of them is the base point of the pencil. This immediately implies that the fixed set consists of the base point and a line not containing this point. This line intersects the cubic. But the cubic does not contain fixed points because the translation automorphism does not have fixed points.

This contradiction shows that l_1, l_2, l_3 are non-collinear lines. Thus we may assume that the equation of l_i is $T_i = 0, i = 0, 1, 2$. Now a generator τ of H acts by the formula $\tau : (t_0, t_1, t_2) \mapsto (at_0, bt_1, ct_2)$, where $a^3 = b^3 = c^3 = \lambda$ for some $\lambda \neq 0$. Multiplying the coordinates by $\lambda^{-1/3}$ (which is the identity transformation on $\mathbb{P}^2$) we may assume that $\lambda = 1$ and $a = 1$. Note that no two of a, b, c are equal. If, say $a = b$, then τ acts identically on the pencil of lines through the point $(0, 0, 1)$, and we saw already that this is impossible. Thus the action of τ is given by

$$\tau : (t_0, t_1, t_2) \mapsto (t_0, \epsilon t_1, \epsilon^2 t_2), \quad \epsilon = e^{2\pi i/3}. \tag{3.3}$$

Consider another subgroup H' of order 3 and let σ be its generator. Since σ permutes the cosets of H we see that σ permutes cyclically the coordinate lines. Thus we may assume that

$$\sigma : (t_0, t_1, t_2) \mapsto (at_1, bt_2, ct_0).$$

After linear change of variables defined by a diagonal matrix, we may assume that $a = b = c = 1$. Thus σ acts by the formula

$$\sigma : (t_0, t_1, t_2) \mapsto (t_1, t_2, t_0). \tag{3.4}$$

The equations (3.3) and (3.4) define a projective linear representation of the group $(\mathbb{Z}/3\mathbb{Z})^3$ in $\mathbb{P}^2$. It is called the *Schrödinger representation*. It is obtained from a linear irreducible representation of a non-commutative group $\mathcal{H}_3$ whose center is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and the quotient by the center is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$.

We know that the curve $C = V(F)$ is invariant with respect to the action of $(\mathbb{Z}/3\mathbb{Z})^3$ in $\mathbb{P}^2$. Thus τ transforms each monomial $T^{\mathbf{i}}$ entering in F into a monomial $\epsilon^k T^{\mathbf{i}}$ for some $k = 0, 1, 2$. If $k \neq 0$, no monomial $T_i^3, T_0T_1T_2$ enters in F_3. Assume $k = 1$. Then no monomial $T_0^2T_2, T_0T_1^2, T_1T_2^2$ enters either. Thus the equation of C_3 is given by $AT_0^2T_1 + BT_1^2T_2 + CT_2^2T_0 = 0$. Intersecting with the line $V(T_0)$, we see that we have only two fllex points on it. But the original curve has three distinct flex points. Similarly, we consider the case $k = 2$. Thus $k = 0$ and the equation becomes $AT_0^3 + BT_1^3 + CT_2^3 + DT_0T_1T_2 = 0$. To make it σ-invariant, we must have $A = B = C$. Obviously $A \neq 0$. After multiplying the equation by a nonzero scalar, we arrive at the *Hesse equation* of a nonsingular plane cubic:

$$F = V(T_0^3 + T_1^3 + T_2^3 + 6mT_0T_1T_2) \tag{3.5}$$

Here the expression for the last coefficient is given to simplify future computations. The condition that the curve is nonsingular is

$$1 + 8m^3 \neq 0. \tag{3.6}$$

3.1.3 The Hesse pencil

Consider a pencil of plane cubics defined by the equation

$$\lambda(T_0^3 + T_1^3 + T_2^3) + \mu T_0T_1T_2 = 0. \tag{3.7}$$

It is called the *Hesse pencil*. Its base points are

$$\begin{aligned} &(0,1,-1), \quad (0,1,-\epsilon), \quad (0,1,-\epsilon^2), \\ &(1,0,-1), \quad (1,0,-\epsilon), \quad (1,0,-\epsilon^2), \\ &(1,-1,0), \quad (1,-\epsilon,0), \quad (1,-\epsilon^2,0), \end{aligned} \tag{3.8}$$

where

$$\epsilon = e^{2\pi i/3}.$$

As is easy to see they are the nine inflection points of any nonsingular member of the pencil. The singular members of the pencil correspond to the values of the parameters

$$(\lambda, \mu) = (0, 1),\ (1, -3),\ (1, -3\epsilon),\ (1, -3\epsilon^2).$$

The last three values correspond to the three values of m for which the Hesse equation defines a singular curve.

Any triple of lines containing the nine base points belong to the pencil and define a singular member. Here they are:

$$\begin{gathered} V(T_0), \quad V(T_1), \quad V(T_2), \\ V(T_0+T_1+T_2),\ V(T_0+\epsilon T_1+\epsilon^2 T_2),\ V(T_0+\epsilon^2 T_1+\epsilon T_2) \\ V(T_0+\epsilon T_1+T_2),\ V(T_0+\epsilon^2 T_1+\epsilon^2 T_2),\ V(T_0+T_1+\epsilon T_2) \\ V(T_0+\epsilon^2 T_1+T_2),\ V(T_0+\epsilon T_1+\epsilon T_2),\ V(T_0+T_1+\epsilon^2 T_2) \end{gathered} \tag{3.9}$$

We leave to a suspicious reader to check that

$$(T_0+T_1+T_2)(T_0+\epsilon T_1+\epsilon^2 T_2)(T_0+\epsilon^2 T_1+\epsilon T_2) = T_0^3+T_1^3+T_2^3-3T_0T_1T_2,$$

$$(T_0+\epsilon T_1+T_2)(T_0+\epsilon^2 T_1+\epsilon^2 T_2)(T_0+T_1+\epsilon T_2) = T_0^3+T_1^3+T_2^3-3\epsilon T_0T_1T_2,$$

$$(T_0+\epsilon^2 T_1+T_2)(T_0+\epsilon T_1+\epsilon T_2)(T_0+T_1+\epsilon^2 T_2) = T_0^3+T_1^3+T_2^3-3\epsilon^2 T_0T_1T_2.$$

The 12 lines (3.9) and 9 inflection points (3.8) form the Hesse configuration corresponding to any nonsingular member of the pencil.

Choose $(0,1,-1)$ to be the zero point in the group law on C_3. Then we can define an isomorphism of groups $\phi : (\mathbb{Z}/3\mathbb{Z})^2 \to X[3]$ by sending $(1,0)$ to $(0,1,-\epsilon)$, $(0,1)$ to $(1,0,-1)$. The points of the first row is the subgroup H generated by $\phi((1,0))$. The points of the second row is the coset of H containing $\phi((0,1))$.

Remark 3.1.1. Note that varying m in $\mathbb{P}^1 \setminus \{-\frac{1}{2}, -\frac{\epsilon}{2}, -\frac{\epsilon^2}{2}, \infty\}$ we obtain a family of elliptic curves X_m with a fixed isomorphism $\phi_m : (\mathbb{Z}/3\mathbb{Z})^2 \to X_m[3]$. By blowing up the 9 base points we obtain a rational surface $S(3)$ together with a morphism

$$f : S(3) \to \mathbb{P}^1$$

obtained from the rational map $\mathbb{P}^2 - \to \mathbb{P}^1, (t_0,t_1,t_2) \mapsto (t_0t_1t_2, t_0^3+t_1^3+t_2^3)$ by resolving (minimally) the indeterminacy points. The fibre of f over a point $(a,b) \in \mathbb{P}^2$ is isomorphic to the member of the Hesse pencil corresponding to $(\lambda,\mu) = (-b,a)$. One can show that this is a *modular family* of elliptic curves with 3-level, i.e. the universal object for the fine moduli space of pairs (X,ϕ), where X is an elliptic curve and $\phi : (\mathbb{Z}/3\mathbb{Z})^2 \to X[3]$ is an isomorphism of groups. If $K = \mathbb{C}$, there is a canonical isomoprhism $\mathbb{P}^1 \cong Y$, where Y is the modular curve of level 3, i.e. the nonsingular compactification of the quotient of the upper half-plane $\mathcal{H} = \{a+bi \in \mathbb{C} : b > 0\}$ by the group

$$\Gamma(3) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : A \equiv I_3 \mod 3\}$$

which acts on $\mathcal{H}$ by Moebius transformations $z \mapsto \frac{az+b}{cz+d}$. The boundary of $H/\Gamma(3)$ in Y consists of 4 points (the cusps). They correspond to the singular members of the Hesse pencil.

3.2 Polars of a plane cubic

3.2.1 The Hessian of a cubic hypersurface

Let $V = V(F)$ be a cubic hypersurface in $\mathbb{P}^n$. We know that the Hessian $\mathrm{He}(V)$ is the locus of points $a \in \mathbb{P}^n$ such that the polar quadric $P_a(V))$ is singular. Also we know that, for any $a \in \mathrm{He}(V)$,

$$\mathrm{Sing}(P_a(V)) = \{b \in \mathbb{P}^2 : P_b(P_a(V)) = 0\}.$$

Since $P_b(P_a(V)) = P_a(P_b(V))$ we obtain that $b \in \mathrm{He}(V)$.

Theorem 3.2.1. *The Hessian* $\mathrm{He}(V)$ *of a cubic hypersurface* V *contains the Steinerian* $\mathrm{St}(V)$. *If* $\mathrm{He}(V) \neq \mathbb{P}^n$, *then*

$$\mathrm{He}(V) = \mathrm{St}(V).$$

For the last assertion one only needs to compare the degrees of the hypersurfaces. They are equal to $n+3$ (see Chapter 1).

In particular, the rational map, if defined,

$$\mathrm{He}(V) \to \mathrm{St}(V), a \mapsto \mathrm{Sing}(P_a(V))$$

is a birational automorphism of the Hessian hypersurface. We have noticed this already in Chapter 1.

Proposition 3.2.2. *Assume* V *has only isolated singularities. Then* $\mathrm{He}(V) = \mathbb{P}^n$ *if and only if* V *is a cone over a cubic hypersurface in* $\mathbb{P}^{n-1}$.

Proof. Consider the subvariety $P_{a,b^2}(V) \subset \mathbb{P}^n \times \mathbb{P}^n$ defined in (1.9). For each $a \in \mathbb{P}^n$, the fibre of the first projection over the point a is equal to the polar $P_a(V)$. For any $b \in \mathbb{P}^n$, the fibre of the second projection over the point b is equal to the second polar $P_{b^2}(V) = V(\sum \partial_i(F_3)(b)T_i)$. Let $U = \mathbb{P}^n \setminus \mathrm{Sing}(V)$. For any $b \in U$, the fibre of the second projection is a hyperplane in $\mathbb{P}^n$. This shows that $p_2^{-1}(U)$ is nonsingular. The restriction of the first projection to U is a morphism of nonsingular varieties. The general fibre of this morphism is a regular scheme over the general point of $\mathbb{P}^n$. Since we are in characteristic 0, it is a smooth scheme. Thus there exists an open subset $W \subset \mathbb{P}^n$ such that $p_1^{-1}(W) \cap U$ is nonsingular.

If $\mathrm{He}(V) = 0$ all polar quadrics $P_a(V)$ are singular, and a general polar must have singularities inside of $p_2^{-1}(\mathrm{Sing}(V)$. This means that $p_1(p_2^{-1}(\mathrm{Sing}(V))) = \mathbb{P}^n$. For any $x \in \mathrm{Sing}(V)$, all polar quadrics contain x and either all of them are singular at x or there exists an open subset $U_x \subset \mathbb{P}^n$ such all quadrics $P_a(V)$ are nonsingular at x for $a \in U_x$. Suppose that for any $x \in \mathrm{Sing}(V)$ there exists a polar quadric which is nonsingular at x. Since the number of isolated singular points is finite, there will be an open set of points $a \in \mathbb{P}^n$ such that the fibre $p_1^{-1}(a)$ is nonsingular in $p_2^{-1}(\mathrm{Sing}(V)$. This is a contradiction. Thus, there exists a point $c \in \mathrm{Sing}(V)$ such that all polar quadrics are singular at x. This implies that c is a common solution of the systems of linear equations $\mathrm{He}(F_3)(a) \cdot X = 0, a \in \mathbb{P}^n$. Thus the first partials of F_3 are linearly dependent. Now we apply Proposition 1.1.2 from Chapter 1 to obtain that V is a cone. □

Remark 3.2.1. The example of a cubic hypersurface in $\mathbb{P}^4$ which we considered in Remark 1.1.1 shows that the assumption of the theorem cannot be weakened. Its singular locus is a plane $T_0 = T_1 = 0$.

3.2.2 The Hessian of a plane cubic

The partials of F_3 are

$$T_0^2 + 3T_1T_2, \quad T_1^2 + 3T_0T_2, \quad T_2^2 + 3T_0T_1 \tag{3.10}$$

Thus the Hessian of C_3 has the following equation:

$$\mathrm{He}(C_3) = \begin{vmatrix} T_0 & mT_2 & mT_1 \\ mT_2 & T_1 & mT_0 \\ mT_1 & mT_0 & T_2 \end{vmatrix} = (1+2m^3)T_0T_1T_2 - m^2(T_0^3+T_1^3+T_2^3). \tag{3.11}$$

In particular, the Hessian of the member of the Hesse pencil corresponding to the parameter $(\lambda, \mu) = (1, 6m)$ is

$$H(m) = T_0^3 + T_1^3 + T_2^3 - \frac{1+2m^3}{m^2} T_0T_1T_2 = 0, \quad m \neq 0, \tag{3.12}$$

or, if $m = 0$

$$T_0T_1T_2 = 0.$$

If $(\lambda, \mu) = (0, 1)$, the Hessian is the whole $\mathbb{P}^2$.

Lemma 3.2.3. *Let C be a nonsingular cubic. The following assertions are equivalent:*

(i) $\dim \mathrm{Sing}(P_a(C)) > 0$*;*

(ii) $a \in \mathrm{Sing}(\mathrm{He}(C))$*;*

(iii) $\mathrm{He}(C)$ *is the union of three nonconcurrent lines;*

(iv) C *is isomorphic to a Fermat cubic* $T_0^3 + T_1^3 + T_1^3 = 0$*;*

(v) $\mathrm{He}(C)$ *is a singular cubic;*

Proof. Use the Hesse equation for a cubic and for its Hessian. We see that $\mathrm{He}(C)$ is singular if and only if either $m = 0$ or $1 + 8(-\frac{1+2m^3}{6m^2})^3 = 0$. Obviously, $m = 1$ is a solution of the second equation. Other solutions are ϵ, ϵ^2. This corresponds to $\mathrm{He}(C_3)$, where C_3 os of the form $T_0^3 + T_1^3 + T_1^3 = 0$, or

$$T_0^3 + T_1^3 + T_1^3 + 6\epsilon^i T_0T_1T_2 = (\epsilon^i T_0 + \epsilon T_1 + T_2)^3 + (T_0 + \epsilon^i T_1 + T_2)^3$$
$$+ (T_0 + T_1 + \epsilon^i T_2)^3 = 0,$$

where $i = 1, 2$, or

$$T_0^3 + T_1^3 + T_1^3 + 6T_0T_1T_2 = (T_0 + T_1 + T_2)^3 + (T_0 + \epsilon T_1 + \epsilon^2 T_2)^3$$
$$+ (T_0 + \epsilon^2 T_1 + \epsilon T_2)^3 = 0.$$

This computation proves the equivalence of (iii), (iv),(v).

Assume (i) holds. Then the rank of the Hessian matrix He is equal to 1. It is easy to see that the first two rows are proportional if and only if $m(m^3 - 1) = 0$. It follows from the previous computation that this implies (iv). The corresponding point a is one of the three intersection points of the lines such that the cubic is equal to the sum of the cubes of linear forms defining these lines. Direct computation shows that (ii) holds. This shows the implication (i) ⇒ (ii).

Assume (ii) holds. Again the previous computations show that $m(m^3 - 1) = 0$ and the Hessian curve is the union of three lines. Again (i) is directly verified. □

Corollary 3.2.4. *Assume that* C *is not isomorphic to a Fermat cubic. Then the Hessian cubic is not singular, and the map* $a \mapsto \mathrm{Sing}(P_a(C))$ *is an involution on* C *without fixed points.*

Proof. The only unproved assertion is that the involution does not have fixed points. A fixed point a has the property that $P_a(P_a(C)) = P_{a^2}(C) = 0$. It follows from Theorem 1.1.1 that this implies that $a \in \mathrm{Sing}(C)$. □

Remark 3.2.2. Consider the Hesse pencil of cubics with parameters $(\lambda, \mu) = (m_0, 6m)$

$$C(m_0, m) = V(m_0(T_0^3 + T_1^3 + T_2^3) + 6m_1T_0T_1T_2 = 0).$$

Taking the Hessian of each curve from the pencil we get the pencil

$$H(\lambda) = V(\lambda_0 T_0^3 + T_1^3 + T_2^3 + 6\lambda_1 T_0 T_1 T_2 = 0).$$

The map $C(m_0, m) \to \mathrm{He}(C(m_0, m))$ defines a regular map

$$H : \mathbb{P}^1 \to \mathbb{P}^1, \quad (m_0, m_1) \mapsto (t_0, t_1) = (-m_0 m_1^2, m_0^3 + 2m_1^3) \tag{3.13}$$

This map is of degree 3. For a general value of the inhomogeneous parameter $\lambda = t_1/t_0$, the pre-image consists of three points with inhomogeneous coordinate $m = m_1/m_0$ satisyfing the cubic equation

$$6\lambda m^3 - 2m^2 + 1 = 0. \tag{3.14}$$

We know that the points

$$(\lambda_0, \lambda_1) = (0, 1), (1, -\tfrac{1}{2}), (1, -\tfrac{\epsilon}{2}), (1, -\tfrac{\epsilon^2}{2})$$

correspond to singular members of the λ-pencil. These are the branch points of the map H. Over each branch point we have two points in the pre-image. The points

$$(m_0, m_1) = (1, 0), (1, 1), (1, \epsilon), (1, \epsilon^2).$$

are the ramification points corresponding to cubiucs isomorphic to the Ferma cubic. A non-ramication point in the pre-image corresponds to a singular member.

Let $C(m) = C(1, m)$. If we fix a group law on a $H(m) = \mathrm{He}(C(m))$, we can identify the involution described in Corollary 3.2.4 with the translation with respect to a non-trivial 2-torsion point η. Given a nonsingular cubic curve $H(m)$ together with a fixed point free involution τ there exists a unique nonsingular cubic $C(m)$ such that $H(m) = \mathrm{He}(C(m))$ and the involution τ is the involution described in the corollary. Thus the 3 roots of the equation (3.14) can be identified with 3 non-trivial torsion points on $H(m)$. We refer to Exercises for a reconstruction of $C(m)$ from the the pair $(H(m), \eta)$.

3.2.3 The dual cubic

Write the equation of a general line in the form $T_2 = t_0 T_0 + t_1 T_1$ and plug in the Hesse equation (3.12). The corresponding cubic equation has a multiple root if and only if the line is a tangent. We have

$$(t_0 T_0 + t_1 T_1)^3 + T_0^3 + T_1^3 + 6m T_0 T_1 (t_0 T_0 + t_1 T_1)$$

$$= (t_0^3 + 1) T_0^3 + (t_1^3 + 1) T_1^3 + (3t_0^2 t_1 + 6m t_0) T_0^2 T_1 + (3t_0 t_1^2 + 6m t_1) T_0 T_1^2 = 0.$$

The condition that there is a multiple root is that the discriminant of the homogeneous cubic form in T_0, T_1 is zero. The discriminant of the cubic form $aT_0^3 + bT_0^2T_1 + cT_0T_1^2 + dT_1^3$ is equal to

$$D = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2.$$

After plugging in, we obtain

$$(3t_0^2t_1+6mt_0)^2(3t_0t_1^2+6mt_1)^2+18(3t_0^2t_1+6mt_0)(3t_0t_1^2+6mt_1)(t_0^3+1)(t_1^3+1)$$
$$-4(t_0^3+1)(3t_0t_1^2+6mt_1) - 4(t_1^3+1)(3t_1t_0^2+6mt_0) - [27(t_0^3+1)^2(t_1^3+1)^2$$
$$= -27 + 864t_0^3t_1^3m^3 + 648t_0^2t_1^2m - 648m^2t_0t_1^4 - 648m^2t_0^4t_1 + 648m^2t_0t_1$$
$$+1296m^4t_0^2t_1^2 - 27t_1^6 - 27t_0^6 + 54t_0^3t_1^3 - 864t_1^3m^3 - 864t_0^3m^3 - 54t_1^3 - 54t_0^3.$$

It remains to homogenize the equation and divide by (-27) to obtain the equation of the dual curve

$$X_0^6 + X_1^6 + X_2^6 - (2+32m^3)(X_0^3X_1^3 + X_0^3X_2^3 + X_2^3X_1^3)$$
$$-24m^2X_0X_1X_2(X_0^3+X_1^3+X_2^3) - (24m+48m^4)X_0^2X_1^2X_2^2 = 0. \quad (3.15)$$

One checks that the points (3.8) are ordinary cusps of the dual cubic. This agrees with the Plücker formulas.

Exercises

3.1 Let $L \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ be a linear system of hypersurfaces of degree d. Prove that a general member of L is nonsinguar outside the set of base points of L (*The Second Bertini Theorem*).

3.2 Find the Hessian cubic of a nonsingular cubic given by the Weierstrass equation.

3.3 Find the Hessian cubic of a singular irreducible plane cubic curve.

3.4 Let $H = \mathrm{He}(C)$ be the Hessian cubic of a nonsingular plane cubic curve C. not isomorphic to a Fermat cubic. Let $\tau : H \to H$ be the Steinerian automorphism of H which assigns to $a \in H$ the unique singular point of $P_a(C)$. Fix a group law on H defined by the zero point α_0 and let $\tau : x \mapsto x + \alpha$, where α is a 2-torsion point on H.

(i) Show that for any $a \in H$ the line $\langle a, \tau(a)\rangle$ is a component of a unique polar conic $P_z(C)$.

(ii) Show that z is the intersection point of the tangent lines of C at a and $\tau(a)$ and the singular point of $P_z(C)$ is the third intersection point of C and the line $\langle a, \tau(a)\rangle$.

(iii) Show that the map $a \mapsto z$ defines an unramified double cover $H \to C$ whose fibres are the orbits of τ.

(iv) Let $\tilde{H} = \{(a, \ell) \in H \times \check{\mathbb{P}}^2 : \ell \subset P_a(C)\}$. Show that the projection $p_1 : \tilde{H} \to H$ is an unramified double cover.

(v) Show that $\tilde{H} \cong C$ [Hint: for any $(x, \ell) \in \tilde{H}$, $\ell = \langle a, \tau(a)\rangle$ for a unique point $a \in C$].

(vi) Show that the composition map $C \to \tilde{H} \to H \to C = H/(\tau)$ has fibres equal to the orbits of the group $C[2]$ of 2-torsion points on C.

(vii) Show that the irreducible components of polar conics $P_a(C), a \in H$, is a cubic curve in the dual plane (the *Caylerian curve* of C).

3.5 Let $C = V(F) \subset \mathbb{P}^2$ be a nonsingular cubic.

(i) Show that the set of second polars of C with respect to points on a fixed line ℓ is a conic in the dual plane. Its dual conic $C(\ell)$ in $\mathbb{P}^2$ is called the *polar conic* of the line.

(ii) Show that $C(\ell)$ is equal to the set of poles of ℓ with respect to polar conics $P_x(C)$, where $x \in \ell$.

(iii) Show that $C(\ell)$ is circumscribed in the triangle defined by intersection points of C and ℓ.

(iv) What happens to the conic $C(\ell)$ when the line ℓ is tangent to C?

(v) Show that the polar conic $C(\ell)$ of a nonsingular cubic C coincides with the locus of points x such that $P_x(C)$ is tangent to ℓ.

(vi) Show that the set of lines ℓ such that $C(\ell)$ is tangent to ℓ is the dual curve of C.

(vii) Let $\ell = V(a_0T_0 + a_1T_1 + a_2T_2)$. Show that $C(\ell)$ can be given by the equation

$$G(a, T) = \det \begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ a_0 & \frac{\partial^2 F}{\partial T_0^2} & \frac{\partial^2 F}{\partial T_0 \partial T_1} & \frac{\partial^2 F}{\partial T_0 \partial T_2} \\ a_1 & \frac{\partial^2 F}{\partial T_1 \partial T_0} & \frac{\partial^2 F}{\partial T_1^2} & \frac{\partial^2 F}{\partial T_1 \partial T_2} \\ a_2 & \frac{\partial^2 F}{\partial T_2 \partial T_0} & \frac{\partial^2 F}{\partial T_2 \partial T_1} & \frac{\partial^2 F}{\partial T_2^2} \end{pmatrix} = 0$$

(viii) Show that the dual curve C^* of C can be given by the equation (the *Schläfli equation*)

$$\det\begin{pmatrix} 0 & \xi_0 & \xi_1 & \xi_2 \\ \xi_0 & \frac{\partial^2 G(a,T)}{\partial T_0^2}(\xi) & \frac{\partial^2 G(a,T)}{\partial T_0 \partial T_1}(\xi) & \frac{\partial^2 G(a,T)}{\partial T_0 \partial T_2}(\xi) \\ \xi_1 & \frac{\partial^2 G(a,T)}{\partial T_1 \partial T_0}(\xi) & \frac{\partial^2 G(a,T)}{\partial T_1^2}(\xi) & \frac{\partial^2 G(a,T)}{\partial T_1 \partial T_2}(\xi) \\ \xi_2 & \frac{\partial^2 G(a,T)}{\partial T_2 \partial T_0}(\xi) & \frac{\partial^2 G(a,T)}{\partial T_2 \partial T_1}(\xi) & \frac{\partial^2 G(a,T)}{\partial T_2^2}(\xi) \end{pmatrix}$$

3.6 Let $C \subset \mathbb{P}^{d-1}$ be an elliptic curve embedded by the linear system $\left|\mathcal{O}_C(dp_0)\right|$, where p_0 is a point in C. Assume $d = p$ is prime.

(i) Show that the image of any p-torsion point is an osculating point of C, i.e., a point such that there exists a hyperplane (an *osculating hyperplane*) which intersects the curve only at this point.

(ii) Show that there is a bijective correspondence between the sets of cosets of $(\mathbb{Z}/p\mathbb{Z})^2$ with respect to subgroups of order p and hyperplanes in $\mathbb{P}^{p-1}$ which cut out in C the set of p osculating points.

(iii) Show that the set of p-torsion points and the set of osculating hyperplanes define a $(p^2_{p+1}, p(p+1)_p)$-configuration of p^2 points and $p(p+1)$ hyperplanes (i.e. each point is contained in $p+1$ hyperplanes and each hyperplane contains p points).

(iv) Find a projective representaion of the group $(\mathbb{Z}/p\mathbb{Z})^2$ in $\mathbb{P}^{p-1}$ such that each osculating hyperplane is invariant with respect to some cyclic subgroup of order p of $(\mathbb{Z}/p\mathbb{Z})^2$.

Chapter 4

Determinantal equations

4.1 Plane curves

4.1.1 The problem

Here we will try to solve the following problem. Given a homogeneous polynomial $F(T_0, \ldots, T_n)$ find a $d \times d$ matrix $A = (L_{ij}(T))$ with linear forms as its entries such that

$$F(T_0, \ldots, T_n) = \det(L_{ij}(T)). \tag{4.1}$$

We will also try to find in how many different ways one can do it.

First let us reinterpret this problem geometrically and coordinate free. Let E be a vector space of dimension $n+1$ and let W, V be vector spaces of dimension d. A square matrix corresponds to a linear map $W \to V$, or an element of $W^* \otimes V$. A matrix with linear forms corresponds to an element of $E^* \otimes W^* \otimes V$, or a linear map $\phi : E \to W^* \otimes V$.

We shall assume that the map ϕ is injective (otherwise the hypersurface $V(F)$ is a cone). Let

$$\phi : \mathbb{P}(E) \to \mathbb{P}(W^* \otimes V) \tag{4.2}$$

be the regular map of the associated projective spaces Let $\mathcal{D}_d \subset \mathbb{P}(W^* \otimes V)$ be the hypersurface parametrizing non-invertible linear maps. If we choose bases in W, V, then $\mathcal{D}_d$ is given by the determinant of a square matrix (whose entries will be coordinates in $W^* \otimes V$). The pre-image of $\mathcal{D}_d$ in $\mathbb{P}(E)$ is a hypersurface $V(F)$ of degree d. Our problem is to construct such a map ϕ in order that a given hypersurface is obtained in this way.

Note that the singular locus $\mathcal{D}_d^s$ of the determinantal variety $\mathcal{D}_d$ corresponds to matrices of corank ≥ 2. It is easy to see that its codimension in $\mathbb{P}(W^* \otimes V)$ is equal to 4. If the image of $\mathbb{P}(E)$ intersects $\mathcal{D}_d^s$, then $\phi^{-1}(\mathcal{D}_d)$ will be a singular hypersurface. So, a nonsingular hypersurface of dimension ≥ 3 cannot be given by a determinantal equation.

4.1.2 Plane curves

Let us first consider the case of nonsingular plane curves $C = V(F) \subset \mathbb{P}^2$. Assume that $F(T_0, T_1, T_2)$ admits a determinantal form. As we have explained, the image of the map ϕ does not intersect $\mathcal{D}_d^s$. Thus, for any $x \in C$, the corank of the matrix $\phi(x)$ is equal to 1 (here we consider a matrix up to proportionality since we are in the projective space). The kernel of this matrix is a one-dimensioanl subspace of W, i.e., a point in $\mathbb{P}(W)$. This defines a regular map

$$r : C \to \mathbb{P}(W), \quad x \mapsto \text{Ker}(\phi(x)).$$

Now let ${}^t\phi(x) : V^* \to W$ be the transpose map. In coordinates, it corresponds to the transpose matrix. Its kernel is isomorphic to $\text{Im}(\phi(x))^\perp$ and is also one-dimensional. So we have another regular map

$$l : C \to \mathbb{P}(V^*), \quad x \mapsto \text{Ker}({}^t\phi(x)).$$

Let

$$\mathcal{L} = r^*(\mathcal{O}_{\mathbb{P}(W)}(1)), \quad \mathcal{M} = l^*(\mathcal{O}_{\mathbb{P}(V)}(1)).$$

These are invertible sheaves on the curve C. We can identify W with a subspace of $H^0(C, \mathcal{L})^*$ and V with a subspace of $H^0(C, \mathcal{M})$. Consider the composition of regular maps

$$\psi : C \xrightarrow{r \times l} \mathbb{P}(W) \times \mathbb{P}(V^*) \xrightarrow{s_2} \mathbb{P}(W \otimes V^*), \tag{4.3}$$

where s_2 is the Segre map. It follows from the definition of the Segre map, that the tensor $\psi(x)$ is equal to $r(x) \otimes l(x)$. It can be viewed as a linear map $W^* \to V^*$. In coordinates, the matrix of this map is the product of the column vector defined by $\text{Ker}(\phi(x))$ and the row vector defined by $\text{Ker}({}^t\phi(x))$. It is a rank 1 matrix equal to the adjugate matrix of the matrix $A = \phi(x)$. Recall that a square matrix of rank 1 has a solution defined by any column of the adjugate matrix (since we have $A \cdot \text{adj}(A) = 0$). Similarly, the kernel of the transpose of A is given by any row of the adjugate matrtix. Thus the entries of the matrix $\psi(x)$ are $(d-1) \times (d-1)$ minors of the matrix $\phi(x)$. This implies that the map ψ is the restriction of the map

$$\Psi : \mathbb{P}(E) \to \mathbb{P}(W \otimes V^*), \quad x \to \text{adj}(\phi(x)).$$

Note that coordinate-free way to think about the adjugate matrix is to consider it as the matrix of the map $\Lambda^{d-1}(\phi(x))$ with respect to the bases of $\Lambda^{d-1}W, \Lambda^{d-1}(V)$

corresponding to the bases of W, V in which the matrix of $\phi(x)$ was computed. Now observe that

$$\Lambda^{d-1}(W) \cong W^* \otimes \Lambda^d(W), \quad \Lambda^{d-1}(V) \cong V^* \otimes \Lambda^d(V),$$

and we have a canonical isomorphism

$$\mathbb{P}(\text{Hom}(\Lambda^{d-1}(W), \Lambda^{d-1}(V))) \cong \mathbb{P}(\text{Hom}(W^* \otimes \Lambda^d(W), V^* \otimes \Lambda^d(V))$$

$$\cong \mathbb{P}(\text{Hom}(W^*, V^*)).$$

Here we use that the spaces $\Lambda^d(W), \Lambda^d(V)$ are one-dimensional. Let

$$\text{Adj} : \mathbb{P}(W^* \otimes V) - \to \mathbb{P}(W \otimes V^*) \tag{4.4}$$

be the rational map defined by assigning to a linear map $f : W \to V$ the linear map $\Lambda^{d-1}(f)$. It is not defined on the variety $\mathcal{D}_d^s$. It follows from above that our map (4.3) is the restriction of Adj to the curve C. Since Adj is defined by polynomials of degree $d-1$ (after we choose bases in W, V), we have

$$(\text{Adj} \circ \phi)^*\big(\mathcal{O}_{\mathbb{P}(W \otimes V^*)}(1)\big) = \mathcal{O}_{\mathbb{P}(E)}(d-1).$$

This gives

$$\psi^*\big(\mathcal{O}_{\mathbb{P}(W \otimes V^*)}(1)\big) = \mathcal{O}_{\mathbb{P}(E)}(d-1) \otimes \mathcal{O}_C = \mathcal{O}_C(d-1).$$

On the other hand, we get

$$\psi^*\big(\mathcal{O}_{\mathbb{P}(W \otimes V^*)}(1)\big) = (s_2 \circ (r \times l))^*\big(\mathcal{O}_{\mathbb{P}(W \otimes V^*)}(1)\big)$$

$$= (r \times l)^*\Big(s_2^*\big(\mathcal{O}_{\mathbb{P}(W \otimes V^*)}(1)\big)\Big) = (r \times l)^*\big(p_1^*(\mathcal{O}_{\mathbb{P}(W)}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}(V^*)}(1))\big)$$

$$= r^*\big(\mathcal{O}_{\mathbb{P}(W)}(1)\big) \otimes l^*\big(\mathcal{O}_{\mathbb{P}(V)}(1)\big) = \mathcal{L} \otimes \mathcal{M}.$$

Here $p_1 : P(W) \times \mathbb{P}(V^*) \to \mathbb{P}(W)$, $p_2 : P(W) \times \mathbb{P}(V^*) \to \mathbb{P}(V^*)$ are the projection maps. Comparing the two isomorphisms, we obtain

Lemma 4.1.1.

$$\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_C(d-1)$$

Remark 4.1.1. In coordinates, the rational map (4.4) is given by the polars of the determinantal hypersurface. In fact, if $A = (T_{ij})$ is a matrix with independent variables as entries, then

$$\frac{\partial \det(A)}{\partial T_{ij}} = M_{ij},$$

where M_{ij} is the ijth cofactor of the matrix A. The map Adj is a birational map since Adj$(A) = A^{-1}\det(A)$ and the map $A \to A^{-1}$ is obviously invertible. So, the determinantal equation is an example of a homogeneous polynomial such that the corresponding polar map is a birational map. Such a polynomial is called a *homaloidal polynomial* (see [Dolgachev], the Fulton's Volume.

Observe also that the map $\Psi : \mathbb{P}(E) \to \mathbb{P}(W \otimes V^*)$ is given by the linear systems of polars of the curve C.

Lemma 4.1.2. *Let $g = \frac{1}{2}(d-1)(d-2)$ be the genus of the curve C. Then*

(i) $\deg(\mathcal{L}) = \deg(\mathcal{M}) = \frac{1}{2}d(d-1) = g - 1 + d$*;*

(ii) $H^0(C, \mathcal{L}) = W^*,\ H^0(C, \mathcal{M}) = V$*;*

(iii) $H^i(\mathcal{L}(-1)) = H^i(\mathcal{M}(-1)) = \{0\}, i = 0, 1.$

Proof. Let us first first prove (iii). A nonzero section of $H^0(C, \mathcal{L}(-1))$ is a section of $\mathcal{L}$ which defines a hyperplane in $\mathbb{P}(W)$ which intersects the image $r(C)$ of the curve C along a divisor $r(D)$, where D is cut out in C by a line. Since all such divisors D are linear equivalent, we see that for any line ℓ the divisor $r(\ell \cap C)$ is cut out by a hyperplane in $\mathbb{P}(W)$. Choose ℓ such that it intersects C at d distinct points $x_1, \ldots, x_d$. Choose bases in W and V. The image of $\phi(\ell)$ in $\mathbb{P}(W^* \otimes V) = \mathbb{P}(\mathrm{Mat}_d)$ is a pencil of matrices $\lambda A + \mu B$. We know that there are d distinct values of (λ, μ) such that the corresponding matrix is of corank 1. Without loss of generality we may assume that A and B are nonsingular matrices. So we have d distinct λ_i such that the matrix $A + \lambda_i B$ is singular. Let $v_1, \ldots, v_d$ be the generators of Ker$(A + \lambda_i B)$. The corresponding points in $\mathbb{P}(W)$ are of course equal to the points $r(x_i)$. We claim that the vectors $v_1, \ldots, v_d$ are linearly independent vectors in W. The proof is by induction on d. Assume $a_1v_1 + \cdots + a_dv_d = 0$. Then $Av_i + \lambda_i Bv_i = 0$ for each $i = 1, \ldots, d$ gives

$$0 = A\Big(\sum_{i=1}^{d} a_iv_i\Big) = \sum_{i=1}^{d} a_iAv_i = -\sum_{i=1}^{d} a_i\lambda_iBv_i.$$

We also have

$$0 = B\Big(\sum_{i=1}^{d} a_iv_i\Big) = \sum_{i=1}^{d} a_iBv_i.$$

Multiplying the second equality by λ_d and adding to the first one, we obtain

$$\sum_{i=1}^{d-1} a_i(\lambda_d - \lambda_i)Bv_i = B\big(\sum_{i=1}^{d-1} a_i(\lambda_d - \lambda_i)v_i\big) = 0.$$

Since B is invertible this gives

$$\sum_{i=1}^{d-1} a_i(\lambda_i - \lambda_d)v_i = 0.$$

By induction, the vectors $v_1, \ldots, v_{d-1}$ are linearly independent. Since $\lambda_i \neq \lambda_d$, we obtain $a_1 = \ldots = a_{d-1} = 0$. Since $v_d \neq 0$, we also get $a_d = 0$.

Since $v_1, \ldots, v_d$ are linearly independent, the points $r(x_i)$ span $\mathbb{P}(W)$. Hence no hyperplane contains these points. This proves $H^0(C, \mathcal{L}(-1)) = 0$. Similarly, we prove that $H^0(C, \mathcal{M}(-1)) = 0$. Applying Lemma 4.1.1 we get

$$\mathcal{L}(-1) \otimes \mathcal{M}(-1) \cong \mathcal{O}_C(d-3) = \omega_C, \tag{4.5}$$

where ω_C is the canonical sheaf on C. By duality,

$$H^i(C, \mathcal{M}(-1)) \cong H^{1-i}(C, \mathcal{L}(-1)), i = 0, 1.$$

This proves (iii). Let us prove (i) and (ii). Let H be a section of $\mathcal{O}_C(1)$. The exact sequence

$$0 \to \mathcal{L}(-1) \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_H \to 0$$

gives, by passing to cohomology and applying (iii),

$$H^1(C, \mathcal{L}) = 0.$$

Replacing $\mathcal{L}$ with $\mathcal{M}$ and repeating the argument, we obtain that $H^1(C, \mathcal{L}) = 0$. We know that $\dim H^0(C, \mathcal{L}) \geq \dim W^* = d$. Applying Riemann-Roch, we obtain

$$\deg(\mathcal{L}) = \dim H^0(C, \mathcal{L}) + g - 1 \geq d + g - 1,$$

Similarly, we get

$$\deg(\mathcal{M}) \geq d + g - 1.$$

Adding up, and applying Lemma 4.1.1, we obtain

$$d(d-1) = \deg \mathcal{O}_C(d-1) = \deg(\mathcal{L}) + \deg(\mathcal{M}) \geq 2d + 2g - 2 = d(d-1).$$

Thus all the inequalities in above are the equalities, and we get assertions (i) and (ii).

□

Now we would like to prove the converse. Given a line bundle $\mathcal{L}$ on C satisfying the properties of the lemma, there exists a map $\phi : C \to \mathbb{P}(W^* \otimes V)$ such that $C = \phi^{-1}(\mathcal{D}_d)$. Here $W = H^0(C, \mathcal{L})^*, V = H^0(C, \mathcal{M})$, where $\mathcal{M} = \mathcal{O}_C(d-1) \otimes \mathcal{L}^{-1}$. It follows from the proof of Lemma 4.1.2 that $\mathcal{M}$ also satisfies the assertions of the Lemma.

Let $r : C \to \mathbb{P}(W), l : C \to \mathbb{P}(V)$ be the maps given by the complete linear systems $\mid \mathcal{L} \mid$ and $\mid \mathcal{M} \mid$. We define $\psi : C \to \mathbb{P}(W \otimes V^*)$ to be the composition of $r \times l$ and the Segre map s_2. By property (i) of Lemma 4.1.2, the map ψ is the restriction of the map $\Psi : \mathbb{P}(E) \to \mathbb{P}(W \otimes V^*)$ given by a linear system of plane curves of degree $d-1$. We can view this map as a tensor in $S^{d-1}(E^*) \otimes W \otimes V^*$. In coordinates it is a $d \times d$ matrix $A(T)$ with entries from the space of homogeneous polynomials of degree $d-1$. Since $\Psi|_C = \phi$, for any point $x \in C$, we have rank$A(x) = 1$. Let M be a 2×2 submatrix of $A(T)$. Since $\det M(x) = 0$ for $x \in C$, we have $F \mid \det M$. Consider a 3×3 submatrix N of $A(T)$. We have $\det$ adj$(N) = \det(N)^2$. Since the entries of adj(N) are determinants of 2×2 submatrices, we see that $F^3 \mid \det(N)^2$. Since F is irreducible, this immediately implies that $F^2 \mid \det(N)$. Continuing in this way we obtain that F^{d-2} divides all cofactors of the matrix A. Thus $B = F^{1-d}$adj(A) is a matrix with entries in E^*. Since rank$(B) =$ rank(adj(A)), and rank$(A(x)) = 1$, we get that rank$B(x) = d-1$ for any $x \in C$. So, if $\det B$ is not identically zero, we obtain that $\det(B)$ is a hypersurface of degree d vanishing on C, hence must coincide with cF for some $c \in K^*$. This shows that $C = V(\det(B))$. To see that $\det(B) \neq 0$, we have to use property (iii) of Lemma 4.1.2. Reversing the proof of this property, we see that for a general line ℓ in $\mathbb{P}(E)$ the images of the points $x_i \in \ell \cap C$ in $\mathbb{P}(W) \times \mathbb{P}(V^*)$ are the points (a_i, b_i) such that the a_i's span $\mathbb{P}(W)$ and the b_i's span $\mathbb{P}(V^*)$. The images of the x_i's in $\mathbb{P}(W \otimes V^*)$ under the map Ψ span a subspace L of dimension $d-1$. If we choose coordinates so that the points a_i and b_i are defined by the unit vectors $(0, \ldots, 1, \ldots, 0)$, then L corresponds to the space of diagonal matrices. The image of the line ℓ under Ψ is a Veronese curve of degree $d-1$ in L. A general point $\Psi(x), x \in \ell$ on this curve does not belong to any hyperplane in L spanned by $d-1$ points x_i's, thus it can be written as a linear combination of the points $\Psi(x_i)$ with nonzero coefficients. This represents a matrix of rank d. This shows that $\det A(x) \neq 0$ and hence $\det(B(x)) \neq 0$.

To sum up, we have proved the following theorem.

Theorem 4.1.3. *(A. Dixon) Let $C \subset \mathbb{P}^2$ be a nonsingular plane curve of degree d. Let* Pic$(C)^{g-1}$ *be the Picard variety of isomorphism classes of invertible sheaves on C of degree $g-1$ (or divisor classes of degree $g-1$). Let $\Theta \subset$* Pic^{g-1} *be the subset parametrizing sheaves $\mathcal{F}$ with $H^0(C, \mathcal{F}) \neq \{0\}$ (or effective divisors* *of degree $g-1$). Let $\mathcal{L}_0 \in \mathrm{Pic}^{g-1} \setminus \Theta$, and $\mathcal{M}_0 = \omega_C \otimes \mathcal{L}_0^{-1}$. Then $W = H^0(C, \mathcal{L}_0(1))^*$ and $V = H^0(C, \mathcal{M}_0(1))$ have dimension d and there is a unique regular map $\phi : \mathbb{P}^2 \to \mathbb{P}(W^* \otimes V)$ such that C is equal to the pre-image of the determinantal hypersurface $\mathcal{D}_d$ and the maps $r : C \to \mathbb{P}(W)$ and $l : C \to \mathbb{P}(V^*)$ given by the complete linear systems $\mid \mathcal{L}_0(1) \mid$ and $\mid \mathcal{M}_0(1) \mid$ coincide with the maps $x \mapsto \mathrm{Ker}(\phi(x))$ and $x \mapsto \mathrm{Ker}({}^t\phi(x))$, respectively. Conversely, given a map $\phi : \mathbb{P}^2 \to \mathbb{P}(W^* \otimes V)$ such that $C = \phi^{-1}(\mathcal{D}_d)$ there exists a unique $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$ such that $W \cong H^0(C, \mathcal{L}_0(1))^*$, $V \cong H^0(C, \omega_C(1) \otimes \mathcal{L}_0)^{-1}$ and the map ϕ is defined by $\mathcal{L}$ as above.*

Remark 4.1.2. Let X be the set of $d \times d$ matrices $A(T)$ with entries in E^* such that $F = \det A(T)$. The group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ acts on the set by

$$(g_1, g_2) \cdot A = g_1 \cdot A \cdot g_2^{-1}.$$

It follows from the Theorem that the orbit space X/G is equal to $\mathrm{Pic}^{g-1}(C) \setminus \Theta$.

We map $\mathcal{L}_0 \mapsto \mathcal{M}_0 = \omega_C \otimes \mathcal{L}_0^{-1}$ is an involution on $\mathrm{Pic}^{g-1} \setminus \Theta$. It corresponds to the involution on X defined by taking the transpose of the matrix.

4.1.3 Examples

Take $d = 2$. Then $\mathrm{Pic}^{g-1}(C_2)$ is one point represented by the divisor class of degree -1. It is obviously non-effective. Thus there is unique (up to the equivalence relation defined in Remark 4.1.2) representation of a conic as a determinant. For example,

$$T_0T_1 - T_2^2 = \det \begin{pmatrix} T_0 & T_2 \\ T_2 & T_1 \end{pmatrix}.$$

Take $d = 3$. Then $\mathrm{Pic}^{g-1}(C_3) = \mathrm{Pic}^0(C_3)$. If we fix a point $x_0 \in C_3$, then $x \mapsto [x - x_0]$ defines an isomorphism from $\mathrm{Pic}^0(C_3)$ and the curve C_3. The divisor $x - x_0$ is effective if and only if $x = x_0$. Thus we obtain that

$$\mathrm{Pic}^0(C_3) \setminus \Theta = C_3 \setminus \{x_0\}.$$

Let $\mathcal{L}_0 = \mathcal{O}_{C_3}(D)$, where D is a divisor of degree 0. Then $\mathcal{L} = \mathcal{L}_0(1) = \mathcal{O}_{C_3}(H + D)$, where H is a divisor of 3 collinear points. Similarly, $\mathcal{M}_0 = \mathcal{O}_{C_3}(-D)$ and $\mathcal{M} = \mathcal{M}_0(1) = \mathcal{O}_{C_3}(H - D)$. Note that any positive divisor of degree 3 is linearly equivalent to $H + D$ for some degree 0 divisor D. Thus any line bundle $\mathcal{L} = \mathcal{L}_0(1)$, where $\mathcal{L}_0 \in \mathrm{Pic}^0(C_3) \setminus \Theta$ corresponds to a positive divisor of degree 3 not cut out by a line. The linear system $|\mathcal{L}|$ gives a reembedding $C_3 \to C_3' \subset \mathbb{P}^2$ which is not projectively equivalent to the original embedding.

The map $r \times l$ maps C_3 to a curve $C \subset \mathbb{P}^2 \times \mathbb{P}^2$. Consider the restriction homomorphism

$$\alpha : W^* \otimes V \cong H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)) \otimes H^0(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(1)) \cong$$
$$H^0(\mathbb{P}(W) \times \mathbb{P}(V^*), p_1^*(\mathcal{O}_{\mathbb{P}(W)}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}(V^*)}(1))) \to$$
$$\to H^0(C_3, p_1^*(\mathcal{O}_{P(W)}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}(V^*)}(1)) \otimes \mathcal{O}_{C_3}) \cong$$
$$\cong H^0(C_3, \mathcal{L} \otimes \mathcal{M}) \cong H^0(C_3, \mathcal{O}_{C_3}(2)).$$

Lemma 4.1.4. *The kernel of the restriction map α is of dimension 3. Let*

$$\sum_{i,j=0}^{2} a_{ij}^{(k)} X_i Y_j = 0, \quad k = 1, 2, 3, \tag{4.6}$$

be the sections of bidegree $(1, 1)$ which span the kernel. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the variety defined by these equations. Then

$$X = C = (r \times l)(C_3).$$

Proof. The target space of α is of dimension $6 = \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. The domain of α is of dimension 9. In coordinates, an element of the kernel is a matrix A such that $xAy = 0$ for any $(x, y) \in C$. Since the image of C under the Segre map is equal to the image of an elliptic curve under a map defined by the complete linear system of degree 6, it must span $\mathbb{P}^5$. Thus we have 6 linearly independent conditions on A. This shows that the kernel is of dimension 3. The projection of X to the first factor (with coordinates X_0, X_1, X_2) is equal to the locus of points (t_0, t_1, t_2) such that the system

$$\sum_{i,j=0}^{2} a_{ij}^{(k)} t_i Y_j = \sum_{j=0}^{2} (\sum_{i=0}^{2} a_{ij}^{(k)} t_i) Y_j = 0, \quad k = 1, 2, 3$$

has a nontrivial solution. The condition for this is

$$\det \begin{pmatrix} \sum_{i=0}^{2} a_{i0}^{(1)} t_i & \sum_{i=0}^{2} a_{i1}^{(1)} t_i & \sum_{i=0}^{2} a_{i2}^{(1)} t_i \\ \sum_{i=0}^{2} a_{i0}^{(2)} t_i & \sum_{i=0}^{2} a_{i1}^{(2)} t_i & \sum_{i=0}^{2} a_{i2}^{(2)} t_i \\ \sum_{i=0}^{2} a_{i0}^{(3)} t_i & \sum_{i=0}^{2} a_{i1}^{(3)} t_i & \sum_{i=0}^{2} a_{i2}^{(3)} t_i \end{pmatrix} = 0 \tag{4.7}$$

Thus, replacing (t_0, t_1, t_2) with unknowns T_0, T_1, T_2, we obtain that the projection is either a cubic curve C' or the whole plane. Assume that the second case occurs. Since the determinant of a matrix does not change after taking the transpose of the matrix, we see that the projection of X to the second factor is also the whole plane. This easily implies that X is a graph of a projective automorphism $\mathbb{P}^2 \to \mathbb{P}^2$. In appropriate coordinates X becomes the diagonal, and hence C_3 embeds in $\mathbb{P}^2 \times \mathbb{P}^2$ by means of the diagonal map $\mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$. But this means that $\mathcal{L} \cong \mathcal{M} \cong \mathcal{O}_{C_3}(1)$. This contradicts our choice of $\mathcal{L}$. Thus the projection of X and of C to the first factor is the cubic curve C' which is C_3 reembedded by $|\mathcal{L}|$. Similarly, the projection of X and of C to the second factor is the cubic curve C'' which is C_3 reembedded by $|\mathcal{M}|$. This implies that $X = (r \times l)(C_3) = C$. □

Since any matrix $A(T)$ can be written in the form (4.7), we see that a determinantal equation of a plane cubic defines a model of the cubic as a complete intersection of three bilinear hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$.

4.1.4 Quadratic Cremona transformations

Note that (4.7) gives a determinantal equation for the reembedded curve $C' = r(C_3)$. Let us see that different plane models of the same elliptic curve differ by a birational transformation of the plane.

Let $f : X \to \mathbb{P}^n$ be a regular map from a variety X to a projective space. Recall that it is defined by an invertible sheaf $\mathcal{F} = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and a set of $n+1$ sections $(s_0, \ldots, s_n)$. Two different maps differ by a projective automorphism of $\mathbb{P}^n$ if and only if they are defined by isomorphic sheaves and isomorphic sets of sections. Suppose we have an automorphism $g : X \to X$. Then the composition $f \circ g : X \hookrightarrow \mathbb{P}^n$ is defined by the invertible sheaf $g^*(\mathcal{L})$ and sections $g^*(s_0), \ldots, g^*(s_n))$. Of course, the images of both maps f and $f \circ g$ are the same, but there is no projective automorphism of $\mathbb{P}^n$ which induces the automorphism g. However, in some cases one can find a birational automorphism T of $\mathbb{P}^n$ which does this job. Recall that, although T may be not defined on a closed subset $Z \subset \mathbb{P}^n$, it could be defined on the whole X. This happens, for example, when X is a nonsingular curve and $Z \cap X$ is a set of points. In fact, we know that any rational map of nonsingular projective curve to a projective variety extends to a regular map. Assume T is given by a linear system $|V|$ of hypersurfaces of degree m such that none of them vanish identically on the curve X. Let $x_1, \ldots, x_k$ be the points on $X \cap Z$. All polynomials $P \in V$ intersect X with some multiplicity. Let m_i be the minimal multiplicity (it is enough to compute it for a basis of V). Then it is easy to see that the restriction of T to X is given by a linear system defined by the line bundle

$\mathcal{F} = \mathcal{O}_X(m) \otimes \mathcal{O}_X(-m_1x_1 - \ldots - m_kx_k)$. This is the invertible sheaf which defines the regular map $T \circ f : X \to \mathbb{P}^n$. Sometimes this map is a new embedding of X.

Let us apply this to our situation. Fix a group law on an elliptic curve X with the zero point x_0. Let $g = t_x$ be the translation automorphism defined by a point x. Recall that

$$t_x(y) = x \oplus y \sim x + y - x_0.$$

For any divisor $D = \sum n_i x_i$, we have

$$t_x^*(D) = \sum n_i t_x^{-1}(x_i) = \sum n_i(x_i \ominus x) \sim \sum n_i(x_i + x_0 - x)$$

$$= \sum n_i x_i + \deg(D)(x_0 - x).$$

In particular, we see that t_x acts identically on divisors of degree 0 and in particular on divisors of functions. This allows one to define the action of t_x on the divisor classes.

Suppose we have two divisors D_1, D_2 of the same degree $m \neq 0$. Then $D_1 - D_2$ is of degree 0. Thus we can find a degree 0 divisor G such that $mG \sim D_1 - D_2$ (we use that the endomorphism of algebraic groups $[m] : X \to X, x \mapsto m \cdot x$ is surjective). Let $G \sim x_G - x_0$ for a unique point x_G. Then

$$t_{x_G}^*(D_1) = D_1 + m(x_0 - x_G) = D_1 - mG \sim D_2. \tag{4.8}$$

This shows that translations act transitively on divisor classes of the same positive degree.

Now suppose we have two embeddings of an elliptic curve $\phi_i : X \to \mathbb{P}^n, i = 1, 2$, which are given by a complete linear systems defined by the corresponding invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$. By the above we can find a point $x \in X$ such that $t_x^*(\mathcal{L}_1) = \mathcal{L}_2$ (recall that for any divisor D and any regular map $f : X \to Y$ we have $f^*(\mathcal{O}_Y(D)) = \mathcal{O}_X(f^*(D))$). This shows that the embeddings $\phi_2 : X \to \mathbb{P}^n$ and $\phi_1 \circ t_x : X \to \mathbb{P}^n$ are defined by the same invertible sheaf, and hence their images are projectively equivalent. But the image of $\phi_1 \circ t_x$ is obviously equal to the image of ϕ_1. Thus there exists a projective transformation g which sends $\phi_1(X)$ to $\phi_2(X)$ such that, for any $y \in X$,

$$g(\phi_1(t_x(y)) = \phi_2(y).$$

Thus if we change ϕ_1 by $g \circ \phi_1$ (by choosing different basis of the linear system defining ϕ_1), we find that one can always choose bases in linear systems $|\mathcal{L}_1|$ and $|\mathcal{L}_2|$ such that the corresponding maps have the same image. In particular, any plane nonsingular cubic can be obtained as the image of an elliptic curve under a map defined by any complete linear system of degree 3.

Now let us see how this implies that a translation automorphism of a nonsingular plane cubic can be realized by a certain Cremona transformation of the plane.

Let

$$T : \mathbb{P}^2 - \to \mathbb{P}^2, \quad (t_0, t_1, t_2) \mapsto (P_0(t_0, t_1, t_2), (P_0(t_0, t_1, t_2), (P_0(t_0, t_1, t_2))$$

be a rational map of $\mathbb{P}^2$ to itself given by polynomials of degree 2. The pre-image of a line $V(a_0T_0 + a_1T_1 + a_2T_2)$ is the conic $V(a_0P_0 + a_1P_1 + a_2P_2)$. The pre-image of a general point is equal to the intersection of the pre-images of two general lines, thus the intersection of two conics from the net L of conics spanned by P_0, P_1, P_2. If we want T to define a birational map we need the intersection of two general conics to be equal to 1. This can be achieved if all conics pass through the same set of three points p_1, p_2, p_3 (*base points*). These points must be non-collinear, otherwise all polynomials have a common factor, after dividing, we get a projective transformation. Birational automorphisms of $\mathbb{P}^2$ (*Cremona transformations*) which are obtained by nets of conics through three non-collinear points are called *quadratic transformations*. If we choose a basis in $\mathbb{P}^2$ such that $p_1 = (1, 0, 0), p_2 = (0, 1, 0), p_3 = (0, 0, 1)$ and a basis in L given by the conics $V(T_1T_2), V(T_0T_2), V(T_0T_1)$, then the transformation is given by the formula

$$T : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1). \tag{4.9}$$

This is called the *standard Cremona transformation*. In affine coordinates, it is given by

$$T : (x, y) \mapsto (x^{-1}, y^{-1}).$$

Let C_3 be a nonsingular cubic curve containing the base points p_1, p_2, p_3 of a quadratic transformation T. Then the restriction of T to C_3 is given by the complete linear system $|2H - p_1 - p_2 - p_3|$, where H is a line section of C. It is of degree 3, and hence defines an embedding of $i : C_3 \hookrightarrow \mathbb{P}^2$ such that $i^*(\mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathcal{O}_{C_3}(2H - p_1 - p_2 - p_3)$. Since $H = (2H - p_1 - p_2 - p_3) - (H - p_1 - p_2 - p_3)$, it follows from (4.8) that

$$i_x^*(\mathcal{O}_{C_3}(1)) \cong \mathcal{O}_{C_3}(2H - p_1 - p_2 - p_3),$$

where $3(x - x_0) \sim p_1 + p_2 + p_3 - H$. As we have explained earlier, this implies that there exists a projective automorphism g such that $T' = g \cdot T$ induces the translation automorphism t_x on C_3.

It follows from this that the group of translations acts transitively on the set of determinantal equations of C_3. One can change one discriminant equation to any other one by applying a quadratic transformation of $\mathbb{P}^2$ which leaves the curve invariant and induces a translation automorphism of the curve.

Example 4.1.1. Let

$$F_3 = T_0^2T_1 + T_1^2T_2 + T_2^2T_0 = \det\begin{pmatrix} T_0 & T_2 & T_2 \\ -T_1 & T_0 & 0 \\ -T_2 & 0 & T_1 \end{pmatrix} \tag{4.10}$$

Apply the Cremona transformation

$$T : (T_0, T_1, T_2) \mapsto (T_0T_1, T_0T_2, T_1T_2). \tag{4.11}$$

We have

$$(T_0T_1)^2T_0T_2 + (T_0T_2)^2T_1T_2 + (T_1T_2)^2T_0T_1 = T_0T_1T_2(T_0^2T_1 + T_1^2T_2 + T_2^2T_0).$$

Thus T transforms the curve to itself. Substituting (4.11) in the entries of the matrix $A(T)$ from (4.10), we get

$$\det\begin{pmatrix} T_0T_1 & T_1T_2 & T_1T_2 \\ -T_0T_2 & T_0T_1 & 0 \\ -T_1T_2 & 0 & T_0T_2 \end{pmatrix} = T_0T_1T_2 \det\begin{pmatrix} T_0 & T_2 & T_2 \\ -T_2 & T_1 & 0 \\ -T_1 & 0 & T_0 \end{pmatrix}.$$

Thus the new determinantal equation is

$$F_3 = \det\begin{pmatrix} T_0 & T_2 & T_2 \\ -T_2 & T_1 & 0 \\ -T_1 & 0 & T_0 \end{pmatrix}. \tag{4.12}$$

However, it is projectively equivalent to the old one.

$$\begin{pmatrix} T_0 & T_2 & T_2 \\ -T_2 & T_1 & 0 \\ -T_1 & 0 & T_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} T_0 & T_2 & T_2 \\ -T_1 & T_0 & 0 \\ -T_2 & 0 & T_1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Let r be the right kernel map defined by the second matrix. We have

$$r((0,1,0)) = (0,0,1),\ r((0,0,1)) = (0,1,-1),\ r((1,0,0)) = (0,1,0).$$

Since the points $(0,1,-1), (0,1,0), (0,0,1)$ are on a line, we get

$$\mathcal{L} = r^*(\mathcal{O}_{C'}(1)) = \mathcal{O}_{C_3}(x_1 + x_2 + x_3),$$

where $x_1 = (0,1,0), x_2 = (0,0,1), x_3 = (1,0,0)$. Thus the second determinantal equation corresponds to $\mathcal{L}_0 = \mathcal{L}(-1) = \mathcal{O}_{C_3}(x_1 + x_2 + x_3 - H)$, where H is a line section. Doing the same for the first matrix we find the same invertible sheaf $\mathcal{L}_0$. Note that $3H \sim 3(x_1 + x_2 + x_3)$ since $V(T_0T_1T_2)$ cuts out the divisor $3x_1 + 3x_2 + 3x_3$. This shows that the Cremona transformation induces an automorphism of the curve C equal to translation t_x, where x is a 3-torsion point. But we know from Lectrure 4 that such automorphism is induced by a projective transformation. This explains why we are not getting an essentially new determinantal equation.

4.1.5 Moduli space

Let us consider the moduli space of pairs $(C, A(T))$, where C is a nonsingular plane curve of degree d, $A(T)$ is a matrix of linear forms such that $C = V(\det(A(T)))$. We say that two pairs $(C, A(T))$ and $(C, B(T))$ are isomorphic if there exists invertible matrices C and D such that $B(T) = CA(T)D$. Equivalently we consider the space $\mathbb{P}(E^* \otimes W^* \otimes V)$ modulo the natural action of the group $G = GL(W) \times \mathrm{GL}(V)$ $((g_1, g_2)(x \otimes y \otimes z) = x \otimes g_1^*(y) \otimes g_2(z))$. The determinant map $A(T) \to \det(A(T))$ is obvioulsy invariant and defines a map

$$\det : \mathbb{P}(E^* \otimes W^* \otimes V)/G \to |\mathcal{O}_{\mathbb{P}^2}(d)|.$$

We consider this map as a map of sets since there is an issue here whether the orbit space exists as an algebraic variety. But let us restrict this map over the subset $|\mathcal{O}_{\mathbb{P}^2}(d)|^{ns}$ of nonsingular plane curves of degree d. When we know that the fibre of the map $\det$ over the curve C is bijective to $\mathrm{Pic}^{g-1}(C) \setminus \Theta_C$. There is an algebraic variety $\mathcal{P}ic_d^{g-1}$ (the *relative Picard scheme*) and a divisor $\Theta_d \subset \mathcal{P}ic_d$ which admits a morphism p

$$p : \mathcal{P}ic_d^{g-1} \setminus \Theta_d \to |\mathcal{O}_{\mathbb{P}^2}(d)|^{ns}$$

with fibres isomorphic to $\mathrm{Pic}^{g-1}(C) \setminus \Theta_C$. One can show that there exists a Zariski open subset $\mathbb{P}(E^* \otimes W^* \otimes V)^{ns}$ of $\mathbb{P}(E^* \otimes W^* \otimes V)$ such that its quotient by G is isomorphic to $\mathcal{P}ic_d^{g-1}$ and the determinant map agrees with the projection p.

Since $\mathcal{P}ic_d^{g-1}$ contains an open subset which is covered by an open subset of a projective space, the variety $\mathcal{P}ic_d^{g-1}$ is unirational. It is a very difficult question to decide whether the variety $\mathcal{P}ic_d^{g-1}$ is rational. It is known only for $d = 3$ and $d = 4$ (Formanek, 1979). Let us give a beautiful proof of the rationality in the case $d = 3$ due to Michelle Van den Bergh.

Theorem 4.1.5. *Assume $d = 3$. Then $\mathcal{P}ic_3^0$ is a rational variety.*

Proof. We give only a sketch of the proof. A point of $\mathcal{P}ic_3^0$ is a pair $(C, \mathcal{L})$, where C is a nonsingular plane cubic and $\mathcal{L}$ is the isomorphism class of an invertible sheaf of degree 0. Let D be a divisor of degree 0 such that $\mathcal{O}_C(D) \cong \mathcal{L}$. Choose a line

ℓ and let $H = \ell \cap C = p_1 + p_2 + p_3$. Let $p_i + D \sim q_i, i = 1, 2, 3$, where q_i is a point. Since $p_i - q_i \sim p_j - q_j$, we have $p_i + q_j \sim p_j + q_i$. This shows that the lines $\langle p_i, q_j\rangle$ and $\langle p_j, q_i\rangle$ intersect at the same point r_{ij} on C. Thus we have 9 points: $p_1, p_2, p_3, q_1, q_2, q_3, r_{12}, r_{23}, r_{13}$. We have

$$p_1 + p_2 + p_3 + q_1 + q_2 + q_3 + r_{12} + r_{23} + r_{13} \sim$$

$$\sim (p_1+p_2+p_3)+(q_1+q_2+q_3)+(H-p_1-q_2)+(H-p_1-q_3)+(H-p_2-q_3) \sim 3H$$

This easily implies that there is a cubic curve which intersects C at the nine points. Together with C we get a pencil of cubics with the nine points as the set of its base points. Let $U = \ell^3 \times (\mathbb{P}^2)^3/S_3$, where S_3 acts by

$$\sigma : \big((p_1, p_2, p_3), (q_1, q_2, q_3)\big) = \big((p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}), (q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)})\big).$$

The variety U is easily seen to be rational. The projection to $\ell^3/S_3 \cong \mathbb{P}^3$ defines an birational isomorphism between the product of $\mathbb{P}^3$ and $(\mathbb{P}^2)^3$. For each $u = (\mathcal{P}, \mathcal{Q}) \in U$, let $c(u)$ be the pencil of cubics through the points $p_1, p_2, p_3, q_1, q_2, q_3$ and the points $r_{ij} = \langle p_i, q\rangle$, where $(ij) = (12, (23), (13)$. Consider the set U' of pairs $(u, C), C \in c(u)$. The projection $(u, C) \mapsto u$ has fibres isomorphic to $\mathbb{P}^1$. Thus the field of rational functions on U' is isomorphic to the field of rational functions on a conic over the field $K(U)$. But this conic has a rational point. It is defined by fixing a point in $\mathbb{P}^2$ and choosing a member of the pencil passing though this point. Thus the conic is isomorphic to $\mathbb{P}^1$ and $K(U')$ is a purely transendental extension of $K(U)$. Now we define a birational map from $\mathcal{P}ic_3^0$ to U'. Each $(C, \mathcal{L})$ defines a point of U' by ordering the set $\ell \cap C$, then defining q_1, q_2, q_3 as above. The member of the corresponding pencil through p_i's, q_i's and r_{ij}'s is the curve C. Conversely, a point $(u, C) \in U'$ defines a point $(C, \mathcal{L})$ in $\mathcal{P}ic_3^0$. We define $\mathcal{L}$ to be the invertible sheaf corresponding to the divisor $q_1 + q_2 + q_3$. it is easy that these map are inverse to each other. $\square$

Remark 4.1.3. If we choose a basis in each space E, V, W, then a map $\phi : E \to \mathrm{Hom}(W, V)$ is determined by the matrices $A_i = \phi(e_i)$, where $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. Our moduli space is the space of triples (A_1, A_2, A_3) of $d \times d$ matrices up to the action of the group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ simultaneously by left and right multiplication

$$(g_1, g_2) \cdot (A_1, A_2, A_3) = (g_1^{-1} A_1 g_2, g_1^{-1} A_2 g_2, g_1^{-1} A_3 g_2).$$

Consider an open subset of maps ϕ such that A_1 is an invertible matrix. Taking $(g_1, g_2) = (1, A_1^{-1})$, we may assume that $A_1 = I_d$ is the identity matrix. The stabilizer subgroup of (I_d, A_2, A_3) is the subgroup of (g_1, g_2) such that $g_1 g_2 = 1$.

Thus our orbit space is equal to the orbit space of pairs of matrices (A, B) up to simultaneous conjugation. The determinantal curve has the affine equation

$$\det(I_d + XA + YB) = 0.$$

Compare this space with the space of matrices up to conjugation. As above this is reduced to the problem of description of the maps $E \to \mathrm{Hom}(V, W)$, where $\dim E = 2$ instead of 3. The determinantal curve is replaced with a determinantal hypersurface in $\mathbb{P}^1$ given by the equation

$$\det(I_d + XA) = 0.$$

Its roots are $(-\lambda^{-1})$, where λ are eigenvalues of the matrix A. If all roots are distinct (this corresponds to the case of a nonsingular curve!), a matrix is determined uniquely up to conjugacy by its eigenvalues, or equivalently by its characteristic polynomial. In the case of pairs of matrices, we need additional information expressed in terms of a point in $\mathrm{Pic}^{g-1} \setminus \Theta$.

4.2 Determinantal equations for hypersurfaces

4.2.1 Cohen-Macauley sheaves

Recall that a finitely generated module M over a local Noetherian commutative ring A is called *Cohen-Macaulay module* if there exists a sequence $a_1, \ldots, a_n$of elements in the maximal ideal of A such that n is equal to the dimension of the ring $A/\mathrm{Ann}(M)$ and $a_i \not\in \mathrm{Ann}(M/(a_1, \ldots, a_{i-1})M), i = 1, \ldots, n$.

If A is a Noetherian commutative ring, not necessary local, a finitely generated A-module is called Cohen-Macaulay if for any prime ideal $\mathfrak{p}$ the localization $M\mathfrak{p}$ is a Cohen-Macaulay module over $A_{\mathfrak{p}}$. A Noetherian commutative ring is called a *Cohen-Macaulay ring* if, considered as a module over itself, it is a Cohen-Macaulay module.

These definitions are globilized and give the notions of a Cohen-Macaulay scheme and a Cohen-Macaulay coherent sheaf.

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called an *arithmetically Cohen-Macaulay sheaf* (ACM-sheaf) if the corresponding module $\Gamma_*(\mathcal{F}) = \sum_{i\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j))$ is a graded Cohen-Macaulay module over the ring of polynomials $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$. Using a local cohomology characterization of Cohen-Macaulay modules one shows that $\mathcal{F}$ is a ACM-sheaf if and only if the following conditions are satisfied

(i) $\mathcal{F}_x$ is a Cohen-Macaulay module over $\mathcal{O}_{\mathbb{P}^n,x}$ for each $x \in \mathbb{P}^n$;

(ii) $H^k(\mathbb{P}^n, \mathcal{F}(j)) = 0$, for $j \in \mathbb{Z}, 1 \leq k \leq n - \dim \mathrm{Supp}(\mathcal{F})$, where $\mathrm{Supp}(\mathcal{F})$ denotes the support of $\mathcal{F}$.

It is known that for any Cohen-Macaulay module over a regular ring A

$$\mathrm{depth}(M) + \mathrm{proj}(M) = \dim A,$$

where proj denotes the projective dimension, the minimal length of a free resolution of M. A global analog of this equality for ACM-sheaves is

$$\dim \mathrm{Supp}(\mathcal{F}) + \mathrm{proj}(\mathcal{F}) = n,$$

where $\mathrm{proj}(\mathcal{F})$ denotes the projective dimension of $\mathcal{F}$, the minimal length of a projective graded resolution for the module $\Gamma_*(\mathcal{F})$.

Theorem 4.2.1. *Let $\mathcal{F}$ be an ACM-sheaf over $\mathbb{P}^n$ such that $\dim \mathrm{Supp}(\mathcal{F}) = 1$. Then there exists an exact sequence*

$$0 \to \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^n}(f_i) \xrightarrow{A} \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^n}(e_i) \to \mathcal{F} \to 0. \tag{4.13}$$

Proof. Since $\mathrm{proj}(\mathcal{F}) = 1$, and hence we get a resolution of graded S-modules

$$0 \to \bigoplus_{i=0}^{r} S(f_i) \to \bigoplus_{i=0}^{r} S(e_i) \to \Gamma_*(\mathcal{F}) \to 0.$$

Passing to the corresponding sheaves in $\mathbb{P}^n$ we obtain the exact sequence from the assertion. □

The map A is given by a $r \times r$ matrix whose ij entry is a homogeneous polynomial of degree $e_i - f_j$. We may assume that the resolution is minimal. To achieve this we must have $a_{ij} = 0$ whenever $e_i = f_j$.

Clearly, the support $\mathcal{F}$ is given by the determininant of the matrix A. It is a hypersurface of some degree d. We must have

$$d = (e_1 + \cdots + e_r) - (f_1 + \cdots + f_r). \tag{4.14}$$

Conversely if $V = V(F)$ is given as a determinant of a matrix A whose entries a_{ij} are homogeneous polynomials of degree $e_i - f_j$ such that the equality (4.14) hold, then we get a resolution (4.13) defined by the matrix. The cokernel $\mathcal{F}$ will be an ACM-sheaf.

Example 4.2.1. Take $\mathcal{F} = i_*\mathcal{O}_V(k)$. Then, the minimal resolution is of course

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d+k) \to \mathcal{O}_{\mathbb{P}^n}(k) \to \mathcal{F} \to 0. \tag{4.15}$$

Here $r = 1, f_1 = -d+k, e_1 = k$. The equation is the tautological one $F = \det((F))$, where (F) is the 1×1 matrix with entry F. Note that according to the Lefschetz Theorem on Hyperplane Sections, $\mathrm{Pic}(V) = \mathbb{Z}\mathcal{O}_V(1)$ if $n > 3$. Thus (4.13) reduces to (4.15) and we cannot get any nontrivial determinantal equations for nonsingular hypersurfaces of dimension ≥ 3.

4.2.2 Determinants with linear entries

Let V be a hypersurface of degree d in $\mathbb{P}^n$. Let $\mathcal{M}$ be an invertible sheaf on V. We will take $\mathcal{F} = i_*(\mathcal{M})$, where $i : V \hookrightarrow \mathbb{P}^n$ denotes the natural closed embedding. Then the condition (i) for a ACM-sheaf will be always satisfied (since $\mathcal{F}_x$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n,x}/(t_x)$, where $t_x = 0$ is a local equation of V). Condition (ii) reads as

$$H^i(V, \mathcal{M}(j)) = 0, \quad 1 \leq i \leq n-1,\ j \in \mathbb{Z}. \tag{4.16}$$

Assume the following additional conditions are satisfied.

$$H^0(V, \mathcal{M}(-1)) = H^{n-1}(V, \mathcal{M}(1-n)) = 0. \tag{4.17}$$

Consider the resolution (4.13), twist it by -1 and apply the exact sequence of cohomology. We must get

$$0 \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-f_i-1)) \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-e_i-1)) \to H^0(\mathbb{P}^n, \mathcal{F}(-1)) \to 0,$$

$$0 \to H^{n-1}(\mathbb{P}^n, \mathcal{F}(1-n)) \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-f_i-2)) \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-e_i-2)) \to 0.$$

Here we used the standard facts (see [Hartshorne]) that

$$H^k(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) = 0,\ k \neq 0, n,\ j \in \mathbb{Z},$$

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-j)).$$

Since $f_i < e_i$, (4.17) gives $e_i - 1 < 0$ and $-f_i - 2 < 0$, hence $e_i \leq 0, f_i \geq -1$. This implies $e_i = 0, f_i = -1$ for all $i = 1, \ldots, r$. Applying (4.14), we get $r = d$. So, we obtain a resolution

$$0 \to \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{A} \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F} \to 0. \tag{4.18}$$

This gives a determinantal expression of F as a $d \times d$ determinant with entries linear forms.

It is convenient to rewrite the exact sequence in the form

$$0 \to W_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{T} W_2 \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F} \to 0, \tag{4.19}$$

where W_1, W_2 are some linear spaces of dimension d, and T is a linear map

$$T : V \to \mathrm{Hom}(W_1, W_2),$$

where $\mathbb{P}^n = \mathbb{P}(V)$. The determinantal hypersurface $V(F)$ is the pre-image in $\mathbb{P}(V)$ of the variety of linear operators $W_1 \to W_2$ of rank less than d.

Applying the cohomology, we obtain a natural isomorphism

$$H^0(\mathbb{P}^n, \mathcal{F}) \cong W_2. \tag{4.20}$$

Twisting (4.18) by $\mathcal{O}_{\mathbb{P}^n}(-n)$ and applying the cohomology, we find a a natural isomorphism

$$H^{n-1}(\mathbb{P}^n, \mathcal{F}(-n)) \cong W_1 \otimes H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong W_1. \tag{4.21}$$

It follows from (4.19) and (4.20) that the invertibel sheaf $\mathcal{M}$ is generated by global sections and defines a morphism

$$l_T : V \to \mathbb{P}(W_2^*) = |\mathcal{M}|^*.$$

For any $x \in V$ the point $l_T(x)$ is the projectivization of the dual space of the cokernel of the matrix $T(x^*)$, where $x = Kx^*$ for some $x^* \in V$.

Twisting exact sequence (4.19) by $\mathcal{O}_{\mathbb{P}^n}(1)$ and applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(-, \mathcal{O}_{\mathbb{P}^n}))$ to the exact sequence (4.18) we obtain an exact sequence

$$0 \to W_2^* \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{T^*} W_1^* \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}(1), \mathcal{O}_{\mathbb{P}^n})) \to 0, \tag{4.22}$$

where $T^* : V \to \mathrm{Hom}(W_2^*, W_1^*)$ is defined by $T^*(v) = {}^tT(v)$. Now we apply Grothendieck's duality theorem (see [Conrad]) to obtain a natural isomorphism of sheaves

$$\mathcal{F}' = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}(1), \mathcal{O}_{\mathbb{P}^n})) \cong i_*(\mathcal{H}om_{\mathcal{O}_V}(\mathcal{F}(1), \mathcal{O}_{\mathbb{P}^n}(d))) \cong i_*(\mathcal{M}^*(d-1)). \tag{4.23}$$

Let

$$\mathcal{L} = \mathcal{M}^*(d-1). \tag{4.24}$$

We can rewrite (4.22) in the form

$$0 \to W_2^* \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{T^*} W_1^* \otimes \mathcal{O}_{\mathbb{P}^n} \to i_*(\mathcal{L}) \to 0, \tag{4.25}$$

Applying cohomology we see that the sheaf $\mathcal{L}$ satisfies the same condition (4.17) as $\mathcal{M}$.

It follows from (4.19) and (4.20) that the invertible sheaf $\mathcal{L}$ is generated by global sections and defines a morphism

$$r_T : VV \to \mathbb{P}(W_1) = |\mathcal{L}|^*.$$

For any $x \in V$ the point $r_T(x)$ is the projectivization of the kernel of the matrix $T(x)$, where $x = Kx^*$ for some $x^* \in V$.

4.2.3 The case of curves

Assume $n = 2$, i.e. V is a plane curve C of degree d. Then the condition (ii) for a ACM-sheaf is vacuous. The condition (4.17) becomes

$$H^0(V, \mathcal{M}(-1)) = H^1(V, \mathcal{M}(-1)) = 0.$$

We will assume that C is irreducible and a reduced curve and $\mathcal{M}$ is an invertible sheaf on C satisfying the condition

$$H^0(C, \mathcal{M}(-1)) = H^1(C, \mathcal{M}(-1)) = 0.$$

Let ω_C be the canonical sheaf of C and

$$p_a(C) = \dim H^1(C, \mathcal{O}_C) = \dim H^0(C, \omega_C)$$

be the *arithmetic genus* of C. By Riemann-Roch (the reader unfamiliar with the Riemann-Roch on a singular curve may assume that C is nonsingular),

$$\deg \mathcal{M}(-1)) = \dim H^0(C, \mathcal{M}(-1)) - \dim H^1(C, \mathcal{M}(-1)) + p_a(C) - 1 = p_a(C) - 1.$$

We also get that

$$\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_{\mathcal{O}_{\mathbb{P}^2}}(d-1).$$

If C is a nonsingular curve, everything agrees with the theory from the previous section.

Example 4.2.2. Let C be a plane irreducible cubic curve. Then $\mathcal{M}(-1)$ must be an invertible sheaf of degree 0 with no nonzero sections. It is known that $\text{Pic}^0(C) =\cong C \setminus \text{Sing}(C)$ and has a structure of an algebraic group isomorphic to $\mathbb{G}_m$ if C is a nodal cubic and isomorphic to $\mathbb{G}_a$ if C is a cuspidal cubic. Any nonzero element of this group defines a determinantal representation of C. For any nonzero $a \in K$, we have

$$T_0T_2^2 + 2T_1^3 = \det \begin{pmatrix} \frac{1}{a^2}T_0 & T_1 & T_1 \\ T_1 & -\frac{1}{a^2}T_0 & T_1 - aT_2 \\ T_1 & T_1 + aT_2 & 0 \end{pmatrix} \tag{4.26}$$

Note that, for any $t = (t_0, t_1, t_2) \in C$ the rank of the matrix is equal to 2, as it should be because the sheaf $\mathcal{L}$ is invertible. We cannot get a symmetric determinannt representation in this way because $\mathcal{L} \cong \mathcal{M}$ would imply that $\mathcal{L}$ is a non-trivial 2-torsion point of $\text{Pic}(C)$. However, the additive group does not have non-trivial torsion elements. On the other hand, we have

$$T_0T_2^2 + T_1^3 = \det \begin{pmatrix} -T_1 & 0 & -T_2 \\ 0 & -T_0 & -T_1 \\ -T_2 & -T_1 & 0 \end{pmatrix} \tag{4.27}$$

The matrices have rank 1 at the singular point $(1, 0, 0)$ of the curve. This shows that C admits symmetric determinantal representations not defined by an invertible sheaf on C. We refer to [Beauville] for the theory of symmetric determinantal representaions of singular plane curves with certain type of singularities. One can show that any determinantal representation of a cuspidal cubic is equivalent either to one given in (4.26) or to one given in (4.27). The latter one corresponds to a non-invertible ACM sheaf $\mathcal{E}$ on C satisfying

$$\mathcal{E} \cong \mathcal{H}om_{\mathcal{O}_C}(\mathcal{E}, \omega_C).$$

This sheaf "compactifies" the Picard scheme of C.

4.2.4 The case of surfaces

Let V be a normal surface of degree d in $\mathbb{P}^3$. We are looking for an invertible sheaf $\mathcal{M}$ on V such that $\mathcal{F} = i_*(\mathcal{M})$ is a ACM-sheaf satisfying an additional assumption (4.17). It will give us a resolution (4.18). It follows from this resolution that $\mathcal{M}$ is generated by global sections. By Bertini's theorem (in characteristic 0), a general section of $\mathcal{M}$ is a nonsingular curve C. Thus we can write $\mathcal{M} \cong \mathcal{O}_V(C)$ for some nonsingular curve C.

Since V is a hypersurface in $\mathbb{P}^3$, its local ring is a Cohen-Macaulay module over the corresponding local ring of $\mathbb{P}^3$. Thus the first condition for an ACM sheaf is satisfied. Let us interpret the second ACM condition $H^1(V, \mathcal{M}(j)) = 0, j \in \mathbb{Z}$. Recall that a subvariety $X \subset \mathbb{P}^n$ is called *projectively normal* if the restriction map $r : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \to H^0(X, \mathcal{O}_X(j))$ is surjective for all j. If X is nonsingular in codimension 1, one can show that it is the same as requiring that the projective coordinate ring of X is normal. Suppose $X \subset V$ for some hypersurface V. Then the restriction homomorphism r is the composition of the homomorphisms $r_1 : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \to H^0(V, \mathcal{O}_V(j))$ and $r_2 : H^0(V, \mathcal{O}_V(j)) \to H^0(X, \mathcal{O}_X(j))$. It is easy to see that V is projectively normal (the cokernel of r_1 is $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d)) = 0$). Thus X is projectively normal if r_2 is surjective. The exact sequence

$$0 \to \mathcal{J}_X(j) \to \mathcal{O}_V(j) \to \mathcal{O}_X(j) \to 0,$$

where $\mathcal{J}_X$ is the sheaf of ideals of X in V, shows that r_2 is surjective if and only if $H^1(X, \mathcal{J}_X(j)) = 0$ for all j. Applying this to our case, where $X = C \subset \mathbb{P}^3$, we get that C is projectively normal if and only if

$$H^1(V, \mathcal{O}_V(-C)(j)) = H^1(V, \omega_V(-j) \otimes \mathcal{O}_V(C))$$

$$= H^1(V, \mathcal{O}_V(C)(d-4-j)) = 0, \quad j \in \mathbb{Z}.$$

Here we used the adjunction formula for the canonical sheaf and the Serre Duality theorem. Thus we see that the ACM-condition is the condition for the projective normality of C.

To get a resolution (4.18) we need the additional conditions

$$H^0(V, \mathcal{O}_V(C)(-1)) = H^2(V, \mathcal{O}_V(C)(-2)) = 0. \tag{4.28}$$

Together with ACM condition this is equivalent to

$$\chi(\mathcal{O}_V(C)(-1)) = \chi(\mathcal{O}_V(C)(-2)) = 0. \tag{4.29}$$

Let $\mathcal{O}_V(1) = \mathcal{O}_V(H)$. Consider the exact sequence

$$0 \to \mathcal{O}_V(C)(-2) \to \mathcal{O}_V(C)(-1) \to \mathcal{O}_H(C-H) \to 0.$$

It gives

$$\chi(\mathcal{O}_H(C-H)) = \chi(\mathcal{O}_V(C)(-1)) - \chi(\mathcal{O}_V(C)(-2)).$$

By Bertini's Theorem we may assume that H is a nonsingular plane curve of degree d. By Riemann-Roch on H, we get

$$\deg(C) - d = \deg(\mathcal{O}_H(C-H)) = d(d-3)/2.$$

This gives

$$\chi(\mathcal{O}_V(C)(-1)) - \chi(\mathcal{O}_V(C)(-2)) \Longleftrightarrow \deg(C) = \tfrac{1}{2}d(d-1). \tag{4.30}$$

The exact sequence

$$0 \to \mathcal{O}_V(-1) \to \mathcal{O}_V(C)(-1) \to \mathcal{O}_C(C-H) \to 0$$

gives

$$\begin{aligned} \chi(\mathcal{O}_C(C-H)) &= \chi(\mathcal{O}_V(C)(-1)) - \chi(\mathcal{O}_V(-1)) \\ &= \chi(\mathcal{O}_V(C)(-1)) - \chi(\mathcal{O}_{\mathbb{P}^3})(-1) + \chi(\mathcal{O}_{\mathbb{P}^3})(-d-1) \\ &= \chi(\mathcal{O}_V(C)(-1)) - \binom{d}{3}. \end{aligned}$$

Applying Riemann-Roch, on C, we get

$$\chi(\mathcal{O}_C(C-H)) = \deg \mathcal{O}_C(C-H) + \chi(\mathcal{O}_C) = \deg \mathcal{O}_C(C+K_V-(d-3)H) + \chi(\mathcal{O}_C)$$
$$= \deg K_C - (d-3)\deg(C) + \chi(\mathcal{O}_C) = -\tfrac{1}{2}(d-3)d(d-1) + g(C) - 1.$$

Thus we see that

$$\chi(\mathcal{O}_V(C)(-1)) = 0 \Longleftrightarrow g(C) = \frac{1}{6}(d-2)(d-3)(2d+1).$$

Together with (4.30) we see that condition (4.28) is equivalent to the conditions

(i) C is a projectively normal curve;

(ii) $\deg(C) = \frac{1}{2}d(d-1)$,

(iii) $g(C) = \frac{1}{6}(d-2)(d-3)(2d+1)$.

Example 4.2.3. Take $d = 3$. We get $\deg(C) = 3, g(C) = 0$. We also have $h^0(C) = \dim H^0(V_2, \mathcal{O}_{V_2}(C)) = 3$. The linear system $|C|$ maps S to $\mathbb{P}^2$. This is a birational morphism which is inverse to the blow-up of 6 points in $\mathbb{P}^2$. We will see later when we will be discussing cubic surfaces, that there are 72 such linear systems. Thus a cubic surface can be written in 72 essentially different ways as a 3×3 determinant.

Example 4.2.4. Take $d = 4$. We get $\deg(C) = 6, g(C) = 3$. The projective normality is equivalent to the condition that C is not hyperelliptic (Exercise 5.11). We also have $h^0(C) = \dim H^0(V_2, \mathcal{O}_{V_2}(C)) = 4$. According to Noether's Theorem, the Picard group of a general surface of degree ≥ 4 is generated by a plane section. Since a plane section of a quartic surface is of degree 4, we see that a general quartic surface does not admit a determinantal equation. The condition that V_4 contains a curve C as above is one algebraic condition on the coefficients of a quartic surface.

Remark 4.2.1. Let $V = V(\det(A(T))$ be a determinantal equation of a nonsingular surface of degree d in $\mathbb{P}^3$. Let $C \subset H$ be a nonsingular plane section of V. Then we obtain a determinantal equation of C. The left kernel sheaf for C is the restriction of the sheaf $\mathcal{M}$ to C, where $\mathcal{M} = \mathcal{O}_V(C))$ is defined by the resolution (4.18). Since we know that $\deg(\mathcal{M} \otimes \mathcal{O}_C) = d(d-1)/2$, we obtain another proof that $\deg(C) = d(d-1)/2$.

Exercises

4.1 Show that any irreducible cubic curve admits a determinantal equation.

4.2 Let $(T_0(T_0-T_1), (T_0-T_2)(T_0-T_1), T_0(T_0-T_2))$ define a rational map from $\mathbb{P}^2$ to $\mathbb{P}^2$. Show that it is a birational map and find its inverse.

4.3 Let $C_3 = V(F_3)$ be a nonsingular plane cubic, p_1, p_2, p_3 be three non-collinear points. Let (A_0, A_1, A_2) define a quadratic Cremona transformation with fundamental points p_1, p_2, p_3. Let q_1, q_2, q_3 be another set of three points such that the six points $p_1, p_2, p_3, q_1, q_2, q_3$ are cut out by a conic. Let (B_0, B_1, B_2) define a quadratic Cremona transformation with fundamental points q_1, q_2, q_3. Show that

$$F_3^{-3} \det \mathrm{adj} \begin{pmatrix} A_0B_0 & A_0B_1 & A_0B_2 \\ A_1B_0 & A_1B_1 & A_1B_2 \\ A_2B_2 & A_2B_1 & A_2B_2 \end{pmatrix}$$

is a determinantal equation of C_3.

4.4 Find a determinantal equation of the cubic curve from Example 4.1.1 which is not equivalent to the equation from the example.

4.5 Find a determinantal equation for the *Klein quartic* $V(T_0^3T_1 + T_1^3T_2 + T_2^3T_0)$.

4.6 Find determinantal equations for a nonsingular quadric surface in $\mathbb{P}^3$.

4.7 Let $C_4 = V(F)$ be a nonsingular quartic curve. Show that F can be written as a 2×2 determinant of a matrix whose entries are quadratic polynomials. Find in how many essentially different ways it can be done.

4.8 Let $V \subset \mathrm{Mat}_d$ be a linear subspace of dimension 3 of the space of $d \times d$ matrices. Show that the locus of points $x \in \mathbb{P}^{d-1}$ such that there exists $A \in V$ for which $x \in \mathrm{Ker}(A)$ is defined by $\binom{d}{3}$ equations of degree d. In particular, for any determinantal equation of a curve C, the images of C under the maps $r : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ and $l : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ are defined by such a system of equations.

4.9 Let $V_4 = V(\det(A(T)))$ be a 4×4- determinantal equation of a nonsingular quartic surace and $\mathcal{O}_{V_4}(C)$ be the corresponding invertible sheaf represented by a non-hyperelliptic curve C of genus 3 and degree 6. Show that $\mathcal{L} = \mathcal{O}_{V_4}(-C)(3)$

is isomorphic to $\mathcal{O}_{V_4}(C')$ for some other curve of genus 3 and degree 6. Find the interpretation of the sheaf $\mathcal{L}$ in terms of the determinantal equation.

4.10 Let C be a non-hyperelliptic curve of genus 3 and degree 6 in $\mathbb{P}^3$.

(i) Show that the homogeneous ideal of C in $\mathbb{P}^3$ is generated by four cubic polynomials F_0, F_1, F_2, F_3.

(ii) Show that the equation of any quartic surface containing C can be written in the form $\sum L_i F_i = 0$, where L_i are linear forms.

(iii) Show that (F_0, F_1, F_2, F_3) define a birational map f from $\mathbb{P}^3$ to $\mathbb{P}^3$. The image of any quartic containing C is another quartic surface.

(iv) Show that the map f is the right kernel map for the determinantal representation of the quartic defined by the curve C.

4.11 Show that a curve of degre 6 and genus 3 in $\mathbb{P}^3$ is projectively normal if and only if it is not hyperelliptic.

4.12 Let C be a nonsingular plane curve of degree d and $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$. Assume that $h^0(\mathcal{L}_0) \neq 0$. Show that the image of C under the map given by the linear system $\mathcal{L}_0(1)$ is a singular curve.

Chapter 5

Theta characteristics

5.1 Odd and even theta characteristics

5.1.1 Symmetric determinants

We know from the previous chapter that a determinantal equation of a plane nonsingular curve C of degree d is defined by a pair of invertible sheaves $\mathcal{L}_0$ and $\mathcal{M}_0$ in $\text{Pic}^{g-1}(C) \setminus \Theta(C)$ such that $\mathcal{L}_0 \otimes \mathcal{M}_0 = \omega_C$. The sheaf $\mathcal{L} = \mathcal{L}_0(1)$ (resp. $\mathcal{M} = \mathcal{M}_0(1)$ defines the embedding $C \hookrightarrow \mathbb{P}^{d-1}$ corresponding to the right (resp. left) kernel of the matrix. Of course, if we are interested in determinantal equations defined by a symmetric matrix, we must have $\mathcal{L}_0 \cong \mathcal{M}_0$. Thus

$$\mathcal{L}_0^{\otimes 2} \cong \omega_C. \tag{5.1}$$

The isomorphism class of an invertible sheaf on a nonsingular curve C with this property is called a *theta characteristic*. In the language of divisors, a theta characteristic is a divisor class ϑ such that

$$2\vartheta = K_C.$$

The set of theta characteristics is an affine space over the subgroup $\text{Pic}(C)[2]$ of the Picard group of C whose elements are isomorphism classes of invertible sheaves $\mathcal{N}$ such $\mathcal{N}^{\otimes 2} \cong \mathcal{O}_C$. The order of this group is known to be equal to 2^{2g}.

In this chapter we will show that there are exactly $2^{g-1}(2^g+1)$ theta characteristics ϑ on a nonsingular curve C of genus g which have the property $h^0(\vartheta) = \dim H^0(C, \mathcal{O}(\vartheta))$ is even. Since 0 is even, the number of non-isomorphic symmetric determinantal equations for a plane curve of degree d is at most $2^{g-1}(2^g+1)$, where $g = (d-1)(d-2)/2$. It is strictly less for any d such that $d \equiv 3 \mod 4$. Indeed, the theta characteristic $\mathcal{F} = \mathcal{O}_C(\frac{1}{2}(d-3))$ satisfies $\dim H^0(C, \mathcal{F}) = \frac{1}{2}(d^2-1)$ is even but not zero.

Remark 5.1.1. For nonsingular hypersurfaces V in $\mathbb{P}^3$ the condition on $\mathcal{M}$ defining a symmetric determinant is that $\mathcal{M}$ is isomorphic to the sheaf $\mathcal{L}$ defined as the cokernel of the transpose of the matrix twisted by -1. Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^3}}(-, \mathcal{O}_V)$ to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^d \to \mathcal{O}_{\mathbb{P}^3}^d \to i_*\mathcal{M} \to 0,$$

we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}^d \to \mathcal{O}_{\mathbb{P}^3}(1)^d \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(i_*\mathcal{M}, \mathcal{O}_{\mathbb{P}^3}) \to 0.$$

Twisting by -1, we get

$$i_*\mathcal{L} = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(i_*\mathcal{M}, \mathcal{O}_{\mathbb{P}^3})(-1).$$

By the duality, we have

$$\omega_V \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(i_*\mathcal{O}_V, \omega_{\mathbb{P}^3}) \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(i_*\mathcal{O}_V, \mathcal{O}_{\mathbb{P}^3}(-4)).$$

By standard properties of the sheaves $\mathcal{E}xt^i$ we have

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(i_*\mathcal{M}, \mathcal{O}_{\mathbb{P}^3})(-1) \cong i_*(\mathcal{M}^* \otimes \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(i_*\mathcal{O}_V, \mathcal{O}_{\mathbb{P}^3}(-4))) \otimes \mathcal{O}_{\mathbb{P}^3}(3).$$

This gives

$$\mathcal{L} = \mathcal{M}^* \otimes \omega_V(3).$$

Thus, if $\mathcal{M} \cong \mathcal{L}$, we must have

$$\mathcal{M}^{\otimes 2} \cong \omega_V(3) = \mathcal{O}_V(d-1).$$

We also must have $h^0(\mathcal{M}(-1)) = 0$. Note that

$$\text{Pic}(V)[2] = \{0\}$$

for a nonsingular surface in $\mathbb{P}^3$. Thus there is at most one square root of $\mathcal{O}_V(d-1)$. When $d = 2k+1$ is odd, the square root is isomorphic to $\mathcal{M} = \mathcal{O}_V(k)$ but it does not satisfy the condition $h^0(\mathcal{M}(-1)) = 0$. So, there are no symmetric determinantal equations. When $d = 2k$ we have no contradiction. However, in both cases the nonexistence of symmetric determinantal equations follows from the general fact that the determinantal variety of symmetric $d \times d$ matrices is singular in codimension 2. Thus any linear projective space of dimension 3 intersects it the singular locus and cuts out a singular surface. So, only singular surfaces admit a symmetric determinantal equation. We will return to this later.

5.1.2 Quadratic forms over a field of characteristic 2

Recall that a quadratic form on a vector space V over a field F is a map $q : V \to F$ such that $q(av) = a^2 q(v)$ for any $a \in F$ and any $v \in V$ and the map

$$b_q : V \times V \to F, \quad (v, w) \mapsto q(v + w) - q(v) - q(w)$$

is bilinear (it is called the *polar bilinear form*). We have $b_q(v, v) = 2q(v)$ for any $v \in V$. In particular, q can be reconstructed from b_q if char$(F) \neq 2$. In the case when char$(F) = 2$, we get $b_q(v, v) \equiv 0$, hence b_q is a symplectic bilinear form. Two quadratic forms q, q' have the same polar bilinear form if and only if $q - q' = l$, where $l(v + w) = l(v) + l(w), l(av) = a^2 l(v)$ for any $v, w \in V, a \in F$. If F is a finite field of characteristic 2, $\sqrt{l}$ is a linear form on V, and we obtain

$$b_q = b_{q'} \Longleftrightarrow q = q' + \ell^2, \tag{5.2}$$

for a unique linear form $\ell : V \to F$.

Let $e_1, \ldots, e_n$ be a basis in V and $A = (a_{ij}) = (b_q(e_i, e_j))$ be the matrix of the bilinear form b_q. It is a symmetric matrix with zeros on the diagonal if char$(F) = 2$. It follows from the definition that

$$q(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i^2 q(e_i) + \sum_{1 \le i < j \le n} x_i x_j a_{ij}.$$

The *rank* of a quadratic form is the rank of the matrix A of the polar bilinear form. A quadratic form is called *nondegenerate* if the rank is equal to $\dim V$. In coordinate-free way this is the rank of the linear map $V \to V^*$ defined by b_q. The kernel of this map is called the *radical* of b_q. The restriction of q to the radical is identically zero. The quadratic form q arises from a nondegenerate quadratic form on the quotient space. In the following we assume that q is nondegenerate.

A subspace E of V is called *singular* if $q|E \equiv 0$. Each singular subspace is an *isotropic subspace* with respect to b_q, i.e., $b_q(v, w) = 0$ for any $v, w \in E$. The converse is true only if char$(F) \neq 2$.

Assume char$(F) = 2$. Since b_q is a nondegenerate symplectic form, $n = 2k$, and there exists a basis $e_1, \cdots, e_n$ such that the matrix of b_q is equal to

$$J_k = \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix}. \tag{5.3}$$

Thus

$$q(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i^2 q(e_i) + \sum_{i=1}^k x_i x_{i+k}.$$

Assume additionally that $F^* = F^{*2}$, i.e., each element in F is a square (i.e. F is a finite or algebraically closed field). Then, we can further reduce q to the form

$$q(\sum_{i=1}^{2k} x_i e_i) = (\sum_{i=1}^{n} \alpha_i x_i)^2 + \sum_{i=1}^{k} x_i x_{i+k}, \tag{5.4}$$

where $q(e_i) = \alpha_i^2,\ i = 1, \ldots, n$. This makes (5.2) more explicit. Fix a nondegenerate symplectic form $\langle , \rangle : V \times V \to F$. Each linear function on V is given by $\ell(v) = \langle v, \eta \rangle$ for a unique $\eta \in V$. By (5.2), two quadratic forms q, q' with polar bilinear form equal to $\langle , \rangle$ satisfy

$$q(v) = q'(v) + \langle v, \eta \rangle^2$$

for a unique $\eta \in V$. Choose a standard symplectic basis (i.e. the matrix of the bilinear form with respect to this basis is equal to (5.3)). The quadratic form defined by

$$q_0(\sum_{I=1}^{2k} x_i e_i) = \sum_{i=1}^{k} x_i x_{i+k}$$

has the polar bilinear form equal to the standard symplectic form. Any other form with the same polar bilinear form is defined by

$$q(v) = q_0(v) + \langle v, \eta_q \rangle^2,$$

where

$$\eta_q = \sum_{i=1}^{2k} \sqrt{q(e_i)} e_i.$$

From now on we assume that $F = \mathbb{F}_2$. In this case $a^2 = a$ for any $a \in \mathbb{F}_2$. The formula (5.2) shows that the set $Q(V)$ of quadratic forms associated to the standard symplectic form is an affine space over V with addition $q + \eta, q \in Q(V), \eta \in V$, defined by

$$(q + \eta)(v) = q(v) + \langle v, \eta \rangle = q(v + \eta) + q(\eta). \tag{5.5}$$

The number

$$\text{Arf}(q) = \sum_{i=1}^{k} q(e_i) q(e_{k+i}) \tag{5.6}$$

is called the *Arf invariant* of q. One can show that it is independent of the choice of a standard symplectic basis. A quadratic form $q \in Q(V)$ is called *even* (resp. *odd*) if $\text{Arf}(q) = 0$ (resp. $\text{Arf}(q) = 1$).

If we choose a standard symplectic basis for b_q and write q in the form $q_0+\eta_q$, then we obtain

$$\mathrm{Arf}(q)=\sum_{i=1}^{k}\alpha_i\alpha_{i+k}=q_0(\eta_q)=q(\eta_q). \tag{5.7}$$

In particular, if $q'=q+v=q_0+\eta_q+v$,

$$\mathrm{Arf}(q+v)+\mathrm{Arf}(q)=q_0(\eta_q+v)+q_0(\eta_q)=q_0(v)+\langle v,\eta_q\rangle=q(v) \tag{5.8}$$

It follows from (5.7) that the number of even (resp. even) quadratic forms is equal to the cardinality of the set $q_0^{-1}(0)$ (resp. $q_0^{-1}(1)$). We have

$$|q_0^{-1}(0)|=2^{k-1}(2^k+1),\quad |q_0^{-1}(1)|=2^{k-1}(2^k-1). \tag{5.9}$$

This is easy to prove by using induction on k.

Let $\mathrm{Sp}(V)$ be the group of linear automorphisms of the symplectic space V. If we choose a standard symplectic basis then

$$\mathrm{Sp}(V)\cong\mathrm{Sp}(2k,\mathbb{F}_2)=\{X\in\mathrm{GL}_{2k}(\mathbb{F}_2):{}^tX\cdot J_k\cdot X=J_k.\}$$

It is easy to see by induction on k that

$$|\mathrm{Sp}(2k,\mathbb{F}_2)|=2^{k^2}(2^{2k}-1)(2^{2k-2}-1)\cdots(2^2-1). \tag{5.10}$$

The group $\mathrm{Sp}(V)$ has 2 orbits in $Q(V)$, the set of even and the set of odd quadratic forms. An even quadratic form is equivalent to the form q_0 and an odd quadratic form is equivalent to the form

$$q_1=q_0+e_k+e_{2k},$$

where $(e_1,\ldots,e_{2k})$ is the standard symplectic basis. Explicitly,

$$q_1(\sum_{i=1}^{2k}x_ie_i)=\sum_{i=1}^{k}x_ix_{i+k}+x_k^2+x_{2k}^2.$$

The stabilizer subgroup $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) of an even quadratic form (resp. an odd quadratic form) is a subgroup of $\mathrm{Sp}(V)$ of index $2^{k-1}(2^k+1)$ (resp. $2^{k-1}(2^k-1)$). If $V=\mathbb{F}_2^{2k}$ with the symplectic form defined by the matrix J_k, then $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) is denoted by $\mathrm{O}(2k,\mathbb{F}_2)^+$ (resp. $\mathrm{O}(2k,\mathbb{F}_2)^-$).

5.2 Hyperelliptic curves

5.2.1 Equations of hyperelliptic curves

Let us first explicitly describe theta characteristics on hyperelliptic curves. Recall that a hyperelliptic curve of genus g is a nonsingular projective curve X admitting a degree 2 map $f : C \to \mathbb{P}^1$. By Hurwitz formula, there are $2g+2$ branch points $p_1, \ldots, p_{2g+2}$ in $\mathbb{P}^1$. Let $F_{2g+2}(T_0, T_1)$ be a binary form of degree $2g+2$ whose zeroes are the branch points. The equation of C in $\mathbb{P}(1, 1, g+1)$ is

$$T_2^2 + F_{2g+2}(T_0, T_1) = 0. \tag{5.11}$$

Recall that a weighted projective space $\mathbb{P}(\mathbf{q}) = \mathbb{P}(q_0, \ldots, q_n)$ is defined as the quotient of $K^{n+1} \setminus \{0\}/K^*$, where K^* acts by

$$t : (z_0, \ldots, z_n) \mapsto (t^{q_0} z_0, \ldots, t^{q_n} z_n).$$

Here $\mathbf{q} = (q_0, \ldots, q_n)$ are integers ≥ 1. In more scientific way, $\mathbb{P}(q_0, \ldots, q_n) =$ Proj$(K[T_0, \ldots, T_n])$, where $K[T_0, \ldots, T_n]$ is the polynomial algebra graded by the condition $\deg(T_i) = q_i$. We refer to [Dolgachev, Lecture Notes in Math., vol. 956] for the theory of weighted projective spaces and their subvarieties. Note that a hypersurface in $\mathbb{P}(\mathbf{q})$ is defined by a homogeneous polynomial where the unknowns are homogeneous of degree q_i. Thus the equation (5.11) defines a hypersurface of degree $2g+2$. Although $\mathbb{P}(\mathbf{q})$ is a singular variety in general, it has a canonical bundle

$$\omega_{\mathbb{P}(\mathbf{q})} = \mathcal{O}_{\mathbb{P}(\mathbf{q})}(-|\mathbf{q}|),$$

where $|\mathbf{q}| = q_0 + \ldots + q_n$. Here the Serre sheaves are understood in the sense of theory of projective spectrums of graded algebras. There is also the adjunction formula for a hypersurface of degree d

$$\omega_V = \mathcal{O}_V(d - |\mathbf{q}|). \tag{5.12}$$

In the case of a hyperelliptic curve, we have

$$\omega_C = \mathcal{O}_C(2g + 2 - g - 3) = \mathcal{O}_C(g-1).$$

The morphism $f : C \to \mathbb{P}^1$ corresponds to the projection $(t_0, t_1, t_2) \to (t_0, t_1)$ and we obtain that

$$\omega_C = f^*(\mathcal{O}_{\mathbb{P}^1}(g-1)).$$

The weighted projective space $\mathbb{P}(1, 1, g+1)$ is isomorphic to the projective cone in $\mathbb{P}^{g+2}$ over the Veronese curve $v_{g+1}(\mathbb{P}^1) \subset \mathbb{P}^{g+1}$. The hyperelliptic curve is isomorphic to the intersection of this cone and a quadric hypersurface in $\mathbb{P}^{g+1}$ not passing through the vertex of the cone. The projection from the vertex to the Veronese curve is the double cover $f : C \to \mathbb{P}^1$. The canonical linear system $|K_C|$ maps C to $\mathbb{P}^g$ with the image equal to the Veronese curve $v_{g-1}(\mathbb{P}^1)$.

5.2.2 2-torsion points on a hyperelliptic curve

Let $c_1, \ldots, c_{2g+2}$ be the ramification points of the map f. We assume that $f(c_i) = p_i$. Obviously, $2c_i - 2c_j \sim 0$, hence the divisor class of $c_i - c_j$ is of order 2 in Pic(C). Also, for any subset S of the set $B_g = \{1, \ldots, 2g+2\}$, we have

$$\alpha_S = \sum_{i \in S} c_i - |S| c_{2g+2} = \sum_{i \in S} (c_i - c_{2g+2}) \in \text{Pic}(C)[2].$$

Now observe that

$$\alpha_{B_g} = \sum_{i \in B_g} c_i - (2g+2) c_{2g+2} = \text{div}(\phi) \sim 0, \tag{5.13}$$

where $\phi = T_2/(bT_0 - aT_1)^{g+1}$, where $p_{2g+2} = (a, b)$ (we consider the fraction modulo the equation (5.11) defining C). Thus

$$c_i - c_j \sim 2c_i + \sum_{k \in B_g \setminus \{j\}} c_k - (2g+2) c_{2g+2} \sim \alpha_{B_g \setminus \{i,j\}}.$$

Adding to α_S the zero divisor $c_{2g+2} - c_{2g+2}$ we can always assume that $|S|$ is even. Also adding the principal divisor α_{B_g}, we obtain that $\alpha_S = \alpha_{\bar{S}}$, where $\bar{S}$ denotes $B_g \setminus S$.

Let $\mathbb{F}_2^{B_g} \cong \mathbb{F}_2^{2g+2}$ be the $\mathbb{F}_2$-vector space of functions $B_g \to \mathbb{F}_2$, or, equivalently, subsets of B_g. The sum is defined by the symmetric sum of subsets

$$S + S' = S \cup S' \setminus (S \cap S').$$

The subsets of even cardinality form a hyperplane. It contains the subsets $\emptyset$ and B_g as a subspace of dimension 1. Let E_g denote the factor space. The elements of E_g are represented by subsets of even cardinality up to the complementary set. We have

$$E_g \cong \mathbb{F}_2^{2g},$$

hence the correspondence $S \mapsto \alpha_S$ defines an isomorphism

$$I : E_g \cong \text{Pic}(C)[2]. \tag{5.14}$$

Note that E_g carries a natural symmetric bilinear form

$$e : E_g \times E_g \to \mathbb{F}_2, \quad e(S, S') = |S \cap S'| \mod 2. \tag{5.15}$$

This form is symplectic (i.e. $e(S, S) = 0$ for any S) and nondegenerate. If we choose a basis represented by the subsets

$$S_i = \{2i-1, 2i\}, \quad S_{g+i} = \{2i, 2i+1\}, \quad i = 1, \ldots, g, \tag{5.16}$$

then the matrix of the bilinear form e will be equal to J_g from (5.3)

Remark 5.2.1. Under the isomorphism (5.14), this bilinear form corresponds to the Weyl pairing on 2-torsion points of the Jacobian variety of C. It is defined as follows. Let $\alpha, \beta \in \text{Pic}(C)[2]$. Write $2\alpha = \text{div}(f), 2\beta = \text{div}(g)$ for some rational functions f, g on C. Then

$$e_w(\alpha, \beta) = \log(f(\beta)/g(\alpha)),$$

where $\log$ is an isomorphism of groups $\{-1, 1\} \to \mathbb{F}_2$ defined by sending -1 to 1. This definition does not depend on the choice of representatives of divisor classes and for any divisor $D = \sum x_i$ and a rational function ϕ we set $\phi(D) = \prod_i \phi(x_i)$.

Remark 5.2.2. The symmetric group S_{2g+2} acts on E_g via its action on B_g and preserves the symplectic form e. This defines a homomorphism

$$s_g : S_{2g+2} \to \text{Sp}(2g, \mathbb{F}_2)$$

If $g = 1$, $\text{Sp}(2, \mathbb{F}_2) \cong S_3$, and the homomorphism s_1 has the kernel isomorphic to the Klein group $(\mathbb{Z}/2\mathbb{Z})^2$. If $g = 2$, the homomorphism s_2 is an isomorphism. If $g > 2$, the homomorphism s_g is injective but not surjective.

5.2.3 Theta characteristics on a hyperelliptic curve

For any subset T of B_g set

$$\vartheta_T = \sum_{i \in T} c_i + (g - 1 - |T|)c_{2g+2} = \alpha_T + (g-1)c_{2g+2}.$$

We have

$$2\vartheta_T \sim 2\alpha_T + (2g-2)c_{2g+2} \sim (2g-2)c_{2g+2}$$

It follows from the proof of the Hurwitz formula that

$$K_C = f^*(K_{\mathbb{P}^1}) + \sum_{i \in B_g} c_i.$$

Choose a representative of $K_{\mathbb{P}^1}$ equal to $-2p_{2g+2}$ and use (5.13) to obtain

$$K_C \sim (2g-2)c_{2g+2}.$$

Thus we obtain that ϑ_T is a theta characteristic. Again adding and subtracting c_{2g+2} we may assume that $|T| \equiv g + 1 \mod 2$. Since T and $\bar{T}$ define the same theta characteristic, we will consider the subsets up to taking the complementary set. We obtain a set Q_g which has a natural structure of an affine space over E_g, the addition is defined by

$$\vartheta_T + \alpha_S = \vartheta_{T+S}.$$

Thus all theta characteristics are uniquely represented by the divisor classes ϑ_T, where $T \in Q_g$.

An example of an affine space over $V = \mathbb{F}_2^{2g}$ is the space of quadratic forms $q : \mathbb{F}_2^{2g} \to \mathbb{F}_2$ whose associated symmetric bilinear form b_q coincides with the standard symplectic form defined by the matrix (5.3). We identify V with its dual V^* by means of b_0 and set $q + l = q + l^2$ for any $l \in V^*$.

For any $T \in Q_g$ we define the quadratic form q_T on E_g by

$$q_T(S) = \tfrac{1}{2}(|T+S| - |T|) = |T \cap S| + \tfrac{1}{2}|S| = \tfrac{1}{2}|S| + e(S,T) \mod 2.$$

We have (all equalities are modulo 2)

$$q_T(S+S')+q_T(S)+q_T(S') = \tfrac{1}{2}(|S+S'|+|S|+|S'|)+e(S+S',T)+e(S,T)+e(S',T)$$
$$= \tfrac{1}{2}(2|S| + 2|S'| - 2|S \cap S'|) = |S \cap S'|.$$

Thus each theta characteristic can be identified with an element of the space $Q_g = Q(E_g)$ of quadratic forms on E_g with polar form e.

Also notice that

$$(q_T + \alpha_S)(A) = q_T(A) + e(S,A) = \tfrac{1}{2}|A| + e(T,A) + e(S,A)$$
$$= \tfrac{1}{2}|A| + e(T+S,A) = q_{T+S}(A).$$

Lemma 5.2.1. *Let ϑ_T be a theta characteristic on a hyperelliptic curve C of genus g identified with a quadratic form on E_g. Then the following properties are equivalent*

(i) $|T| = g+1 \mod 4$*;*

(ii) $h^0(\vartheta_T) = 0 \mod 2$*;*

(iii) q_T *is even.*

Proof. Without loss of generality we may assume that p_{2g+2} is the infinity point $(0,1)$ in $\mathbb{P}^1$. Then the field of rational functions on C is generated by the function $y = T_2/T_0$ and $x = T_1/T_0$. We have

$$\vartheta_T = \sum_{i \in T} c_i + (g-1-|T|)c_{2g+2} \sim (g-1+|T|)c_{2g+2} - \sum_{i \in T} c_i.$$

Any function ϕ from the space $L(\vartheta_T) = \{\phi : \text{div}(\phi) + \vartheta_T \geq 0\}$ has a unique pole at c_{2g+2} of order $< 2g+1$. Since the function y has a pole of order $2g+1$ at c_{2g+2}, we see that $\phi = f^*(p(x))$, where $p(x)$ is a polynomial of degree $\leq \frac{1}{2}(g-1+|T|)$

in x. Thus $L(\vartheta_T)$ is isomorphic to the linear space of polynomials $p(x)$ of degree $\leq \frac{1}{2}(g-1+|T|)$ with zeroes at $p_i, i \in T$. The dimension of this space is equal to $\frac{1}{2}(g+1-|T|)$. This proves the equivalence of (i) and (ii).

Let

$$U = \{1, 3, \ldots, 2g+1\} \subset B_g \tag{5.17}$$

be the subset of odd numbers in B_g. If we take the standard symplectic basis in E_g defined in (5.16), then we obtain that $q_U = q_0$ is the standard quadratic form associated to the standard symplectic basis. It follows from (5.7) that q_T is an even quadratic form if and only if $T = U + S$, where $q_U(S) = 0$. Let S consists of k even numbers and s odd numbers. Then $q_U(S) = |U \cap S| + \frac{1}{2}|S| = m + \frac{1}{2}(k+m) = 0 \mod 2$. Thus $|T| = |U+S| = |U| + |S| - 2|U \cap S| = (g+1) + (k+m) - 2m = g+1+k-m$. Then $m + \frac{1}{2}(k+m)$ is even, hence $3m + k \equiv 0 \mod 4$. This implies that $k - m \equiv 0 \mod 4$ and $|T| \equiv g+1 \mod 4$. Conversely, if $|T| \equiv g+1 \mod 4$, then $k - m \equiv 0 \mod 4$ and $q_U(S) = 0$. This proves the lemma. $\square$

5.3 Theta functions

5.3.1 Jacobian variety

Recall the definition of the Jacobian variety of a nonsingular projective curve C of genus g over $\mathbb{C}$. We consider C as a compact oriented 2-dimensional manifold of genus g. The homology group $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ and the cap product $\cap : H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \cong \mathbb{Z}$ equips it with a nondegenerate symplectic form. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be a standard symplectic basis, i.e.,

$$(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0, \quad (\alpha_i, \beta_j) = \delta_{ij}.$$

We choose a basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-differentials on C such that

$$\int_{\alpha_i} \omega_j = \delta_{ij}. \tag{5.18}$$

Let

$$\tau_{ij} = \int_{\beta_i} \omega_i.$$

The complex matrix $\Omega = (\tau_{ij})$ is called the *period matrix*. It satisfies

$${}^t\Omega = \Omega, \quad \Im(\Omega) > 0.$$

Let

$$\Lambda_\Omega = \begin{bmatrix} \Omega & I_g \end{bmatrix} \mathbb{Z}^{2g} \subset \mathbb{C}^g.$$

This is a free abelian subgroup of rank $2g$ of the additive group of $\mathbb{C}^g$. The quotient

$$\mathrm{Jac}(C) = \mathbb{C}^g/\Lambda_\Omega$$

is a complex g-dimensional torus. It is called the *Jacobian variety* of C.

We consider any divisor $D = \sum n_x x$ on C as a 0-cycle on C. The divisors of degree 0 are boundaries, i.e. $D = \partial\gamma$ for some 1-chain γ. Consider the vector

$$p(D;\gamma) = \Big(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g\Big) \in \mathbb{C}^g.$$

If $D = \partial\gamma'$, then $\gamma - \gamma'$ is a 1-cycle and hence homologous to an integral linear combination of the cycles α_i, β_i. This gives

$$p(D;\gamma) - p(D;\gamma') = \Big(\int_{\gamma-\gamma'} \omega_1, \ldots, \int_{\gamma-\gamma'} \omega_g\Big) \in \Lambda_\Omega.$$

This shows that $p(D;\gamma)$ modulo Λ_Ω depends only on D. This defines a homomorphism of groups $p : \mathrm{Div}(X)^0(C) \to \mathrm{Jac}(C)$. The *Abel-Jacobi Theorem* asserts that p is zero on principal divisors, and surjective. This defines an isomorphism

$$\mathrm{aj} : \mathrm{Pic}^0(C) \to \mathrm{Jac}(C) \tag{5.19}$$

which is called the *Abel-Jacobi map*. Fix a point $c_0 \in C$ and let

$$\mathrm{aj}_{c_0}^n : \mathrm{Pic}^n(C) \to \mathrm{Jac}(C) \tag{5.20}$$

be the composition of the homomorphism $\mathrm{Pic}^n(C) \to \mathrm{Pic}^0(C)$ defined by $[D] \mapsto [D - nc_0]$ and the Abel-jacobi map aj.

Let

$$W_{g-1}^r = \{[D] \in \mathrm{Pic}^{g-1}(C) : h^0(D) \geq r+1\}.$$

In particular, W_{g-1}^0 was denoted by Θ in Chapter 5, where we showed that the invertible sheaves $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$ defining a determinantal equation of a plane curve of genus g belong to the set $\mathrm{Pic}^{g-1}(C) \setminus W_{g-1}^0$. We will explicitly describe the image of W_g^r under the Abel-Jacobi map $\mathrm{aj}_{c_0}^{g-1}$.

5.3.2 Riemann's theta function

Let

$$\Theta(\mathbf{z};\Omega) = \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i(\mathbf{r}\Omega\mathbf{r} + 2\mathbf{z}\cdot\mathbf{r})}.$$

One can show that this function represents a holomorphic function on $\mathbb{C}^g$ in the variables $\mathbf{z} = (z_1, \ldots, z_g)$. It is called *Riemann's theta function*. Write any vector from Λ_Ω in the form $\Omega \cdot \mathbf{m} + \mathbf{n}$, where $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^g$. We have

$$\Theta(\mathbf{z} + \Omega\mathbf{m} + \mathbf{n}; \Omega) = \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i(\mathbf{r}\Omega\mathbf{r}+2(\mathbf{z}+\Omega\mathbf{m}+\mathbf{n})\cdot\mathbf{r})} =$$

$$e^{-\pi i(\mathbf{m}\Omega\mathbf{m}+2\mathbf{z}\cdot\mathbf{m})} \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i((\mathbf{r}+\mathbf{m})\Omega(\mathbf{r}+\mathbf{m})+2\mathbf{z}\cdot(\mathbf{r}+\mathbf{m}))} = e^{-\pi i(\mathbf{m}\Omega\mathbf{m}+2\mathbf{z}\cdot\mathbf{m})}\Theta(\mathbf{z}; \Omega).$$

This shows that although $\Theta(\mathbf{z}; \Omega)$ is not invariant function under translations by the lattice vectors, but it defines an invariant section of the trivial vector bundle $\mathbb{C}^g \times \mathbb{C}$ on which Λ_Ω acts by the formula

$$\omega \cdot (\mathbf{z}, t) = (\mathbf{z} + \omega, e_\omega(\mathbf{z})t),$$

where $e_\omega(\mathbf{z}) = e^{-\pi i(\mathbf{m}\Omega\mathbf{m}+2\mathbf{z}\cdot\mathbf{m})}$. The fact that this formula defines an action follows from the "1-cocycle" property of $e_\omega(\mathbf{z})$:

$$e_{\omega+\omega'}(\mathbf{z}) = e_\omega(\mathbf{z} + \omega')e_{\omega'}(\mathbf{z}).$$

Since the group Λ_Ω acts freely on $\mathbb{C}^g \times \mathbb{C}$, the orbit space $\mathbb{C}^g \times \mathbb{C}/\Lambda_\Omega$ is a complex manifold equipped with a structure of a line bundle $L \to \mathbb{C}^g/\Lambda_\Omega = \text{Jac}(C)$. The theta function $\Theta(\mathbf{z}, \Omega)$ descends to a holomorphic section of L which we continue to denote in the same way. Let Ω be the hypersurface of zeroes of this section. It is known that L is an ample line bundle and the linear system $|3\Theta|$ gives an embedding of $\text{Jac}(C)$ in $\mathbb{P}^N$, where $N = 3^g - 1$. The divisor Θ is the unique element of the linear system $|\Theta|$. A complex torus which admits an ample line bundle with one-dimensional space of holomorphic sections is called a *principally polarized abelian variety*.

The relation between the varieties W^r_{g-1} and the Riemann theta function is given by the following.

Theorem 5.3.1. *(Riemann's Singularities Theorem) There exists a unique theta characteritic κ_{c_0} in $\text{Pic}^{g-1}(C)$, the Riemann constant, such that*

$$\text{aj}^{g-1}_{c_0}(W^r_{g-1}) = \{x \in \text{Jac}(C) : \text{mult}_x\Theta \geq r + 1\} + \text{aj}^{g-1}_{c_0}(\kappa_{c_0}).$$

Corollary 5.3.2.

$$\text{aj}^{g-1}_{c_0}(W^0_{g-1}) = \Theta + \text{aj}^{g-1}_{c_0}(\kappa_{c_0}),$$

$$\text{aj}^{g-1}_{c_0}(W^1_{g-1}) = \text{Sing}(\Theta) + \text{aj}^{g-1}_{c_0}(\kappa_{c_0}).$$

5.3.3 Theta functions with characteristics

Let $\vartheta \in \mathrm{Pic}^{g-1}(C)$ be a theta characteristic. Since $\alpha = \vartheta - \kappa_{c_0} \in \mathrm{Pic}(C)[2]$ we see that

$$\vartheta \in W^r_{g-1} \Longleftrightarrow \mathrm{mult}_{\mathrm{aj}(\alpha)}\Theta \geq r+1. \tag{5.21}$$

The choice of Ω determines explicitly the group of 2-torsion points in the Jacobian:

$$\mathrm{Jac}(C)[2] = \tfrac{1}{2}\Lambda_\Omega/\Lambda_\Omega.$$

We can choose to represent elements of this group by vectors $v_{\epsilon,\eta} = \frac{1}{2}\epsilon + \frac{1}{2}\Omega\eta$, where ϵ, η are binary vectors. It is clear that $\Theta(\mathbf{z};\Omega)$ vanishes at a 2-torsion point (ϵ,η) if only if the function $\Theta(\mathbf{z}+v_{\epsilon,\eta};\Omega)$ vanishes at zero. For any vector $\Omega\mathbf{x}+\mathbf{y}$, we have

$$\Theta(\mathbf{z}+\Omega\mathbf{x}+\mathbf{y};\Omega) = \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i(\mathbf{r}\Omega\mathbf{r}+2(z+\Omega\mathbf{x}+\mathbf{y})\cdot\mathbf{r}} =$$

$$= e^{-\pi i(\mathbf{x}\cdot\Omega\cdot\mathbf{x}+2\mathbf{x}\cdot(\mathbf{z}+\mathbf{y})}\sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i\big((\mathbf{r}+\mathbf{x})\cdot\Omega\cdot(\mathbf{r}+\mathbf{x})+2(\mathbf{z}+\mathbf{y})\cdot(\mathbf{r}+\mathbf{x})\big)}.$$

Thus the translate of $\Theta(\mathbf{z};\Omega)$ differs from the function

$$\sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i\big((\mathbf{r}+\mathbf{x})\cdot\Omega\cdot(\mathbf{r}+\mathbf{x})+2(\mathbf{z}+\mathbf{y})\cdot(\mathbf{r}+\mathbf{x})\big)} \tag{5.22}$$

only by a nowhere vanishing factor. We take $x = \frac{1}{2}\epsilon, y = \frac{1}{2}\eta$, where ϵ,η are binary vectors. The function

$$\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega) = \sum_{r\in\mathbb{Z}^g} e^{\pi i\big((\mathbf{r}+\frac{1}{2}\epsilon)\cdot\Omega\cdot(\mathbf{r}+\frac{1}{2}\epsilon)+2(\mathbf{z}+\frac{1}{2}\eta)\cdot(\mathbf{r}+\frac{1}{2}\epsilon)\big)} \tag{5.23}$$

is called the *theta function with characteristic*. We denote by $\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$ the corresponding hypersurface in $\mathrm{Jac}(C)$. So, we can restate (5.21) in the form

$$\vartheta \in W^r_{g-1} \Longleftrightarrow \mathrm{mult}_0\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right] \geq r+1, \tag{5.24}$$

where $\mathrm{aj}(\vartheta-\kappa_{c_0}) = v_{\epsilon,\eta}$. Let us see now how the function $\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega)$ changes under the transformation $\mathbf{z}\mapsto -\mathbf{z}$. We have

$$\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](-\mathbf{z};\Omega) = \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i\big((\mathbf{r}+\frac{1}{2}\epsilon)\cdot\Omega\cdot(\mathbf{r}+\frac{1}{2}\epsilon)+2(-\mathbf{z}+\frac{1}{2}\eta)\cdot(\mathbf{r}+\frac{1}{2}\epsilon)\big)}$$

$$= \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i\big((-\mathbf{r}-\frac{1}{2}\epsilon)\cdot\Omega\cdot(-\mathbf{r}-\frac{1}{2}\epsilon)+2(\mathbf{z}-\frac{1}{2}\eta)\cdot(-\mathbf{r}-\frac{1}{2}\epsilon)\big)} =$$

$$e^{\pi i \epsilon\cdot\eta} \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i\left((\mathbf{r}-\frac{1}{2}\epsilon)\cdot\Omega\cdot(\mathbf{r}-\frac{1}{2}\epsilon)+2(\mathbf{z}+\frac{1}{2}\eta)\cdot(\mathbf{r}-\frac{1}{2}\epsilon)\right)}.$$

Now write $-\frac{1}{2}\epsilon = \frac{1}{2}\epsilon + \mathbf{m}$, where $\mathbf{m} \in \mathbb{Z}^g$. Then we can rewrite the previous expression in the form

$$e^{\pi i \epsilon\cdot\eta} \sum_{\mathbf{r}\in\mathbb{Z}^g} e^{\pi i\left((\mathbf{r}+\mathbf{m}+\frac{1}{2}\epsilon)\cdot\Omega\cdot(\mathbf{r}+\mathbf{m}+\frac{1}{2}\epsilon)+2(\mathbf{z}+\frac{1}{2}\eta)\cdot(\mathbf{r}+\mathbf{m}+\frac{1}{2}\epsilon)\right)} =$$

$$= e^{\pi i \epsilon\cdot\eta}\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega).$$

This gives

$$\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](-\mathbf{z};\Omega) = e^{\pi i \epsilon\cdot\eta}\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega). \tag{5.25}$$

Thus we see that

$$\mathrm{mult}_0(\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]) \text{ is } \begin{cases}\text{even} & \text{if } \epsilon\cdot\eta = 0 \mod 2\\ \text{odd} & \text{if } \epsilon\cdot\eta = 1 \mod 2.\end{cases}$$

Recall that $\epsilon\cdot\eta$ is equal to the value of the standard quadratic form q_0 on $\mathbb{F}_2^{2g}$ defined by $q_0(\epsilon,\eta) = \epsilon\cdot\eta$ associated to the standard symplectic form on $\mathbb{F}_2^{2g}$. This quadratic form is even, and hence we obtain the following.

Theorem 5.3.3. *Let* $\mathrm{TChar}(C)$ *be the set of theta characteristics on* C. *Set*

$$\mathrm{TChar}(C)^{\mathrm{ev}} = \{\vartheta \in \mathrm{TChar}(C) : h^0(\vartheta) \textit{ is even}\},$$

$$\mathrm{TChar}(C)^{\mathrm{odd}} = \{\vartheta \in \mathrm{TChar}(C) : h^0(\vartheta) \textit{ is odd}\}.$$

Then

$$\left|\mathrm{TChar}(C)^{\mathrm{ev}}\right| = 2^{g-1}(2^g+1),$$

$$\left|\mathrm{TChar}(C)^{\mathrm{odd}}\right| = 2^{g-1}(2^g-1).$$

An even theta characteristic ϑ with $h^0(\vartheta) > 0$ is called a *vanishing theta characteristic*. The reason for this name is clear. The corresponding theta function with characteristics vanishes at zero.

We also can see how a theta characteristic defines a quadratic form on $\mathrm{Pic}(C)[2]$. For any $\vartheta = [D] \in \mathrm{TChar}(C)$ set

$$q_\vartheta(\alpha) = h^0(\vartheta+\alpha) + h^0(\vartheta) \mod 2.$$

Let ϑ corresponds to the theta function $\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega)$ and $\alpha = \frac{1}{2}\Omega a + \frac{1}{2}b$, where a, b are binary vectors. By the Riemann-Kempf theorem, we see that

$$q_\theta(\alpha) = \text{mult}_0\Theta\left[\begin{smallmatrix}\epsilon+a\\ \eta+b\end{smallmatrix}\right](\mathbf{z},\Omega) + \text{mult}_0\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z},\Omega).$$

This number is even if and only if

$$(\epsilon + a)\cdot(\eta + b) + \epsilon\cdot\eta = \epsilon\cdot b + \eta\cdot a + a\cdot b = 0 \mod 2$$

This shows that the values of q_θ coincide with the values of the quadratic form on $\mathbb{F}_2^{2g}$ defined by

$$q(a,b) = \epsilon\cdot b + \eta\cdot a + a\cdot b. \tag{5.26}$$

This quadratic form is equal to $q_0 + (\epsilon,\eta)$, where (ϵ,η) is the vector in $\mathbb{F}_2^{2g}$ defined by the binary vectors ϵ and η. The Arf invariant of this quadratic form is equal to

$$\text{Arf}(q_\theta) = \epsilon\cdot\eta \mod 2 = q_0(\epsilon,\eta).$$

5.3.4 Hyperelliptic curves again

In this case we can compute the Riemann constant explicitly. Recall that we identify 2-torsion points with subsets of even cardinality of the set $B_g = \{1,\ldots,2g+2\}$ which we can identify with the set of ramification or branch points. Let us choose c_{2g+2} to be the point c_0 which is used to define the Abel-Jacobi map. Let us define a standard symplectic basis in C by choosing the 1-cycle α_i to be the path which goes from c_{2i-1} to c_{2i} along one sheet of the Riemann surface C and returns to c_{2i-1} along the other sheet. Similarly we define the 1-cycle β_i by choosing the points c_{2i} and c_{2i+1}. Choose g holomorphic forms ω_j normalized by the condition (5.18). Let Ω be the corresponding period matrix. Notice that each holomorphic 1-form changes sign when we switch the sheets. This gives

$$\begin{aligned}\tfrac{1}{2}\delta_{ij} = \tfrac{1}{2}\int_{\alpha_i}\omega_j &= \int_{c_{2i}}^{c_{2i-1}}\omega_j = \int_{c_{2i-1}}^{c_{2g+2}}\omega_j - \int_{c_{2i}}^{c_{2g+2}}\omega_j \\ &= \int_{c_{2i-1}}^{c_{2g+2}}\omega_j + \int_{c_{2i}}^{c_{2g+2}}\omega_j - 2\int_{c_{2i}}^{c_{2g+2}}\omega_j.\end{aligned}$$

Since

$$2\Big(\int_{c_{2i}}^{c_{2g+2}}\omega_1,\ldots,\int_{c_{2i}}^{c_{2g+2}}\omega_g\Big) = \text{aj}(2c_i - 2c_{2g+2}) = 0,$$

we obtain

$$\text{aj}(c_{2i-1} + c_{2i} - 2c_{2g+2}) = \tfrac{1}{2}e_i \mod \Lambda_\Omega,$$

where, as usual, e_i denotes the ith unit vector. Let S_i, S_{g+i} be defined as in (5.16). We obtain that

$$\mathsf{aj}(\alpha_{S_i}) = \tfrac{1}{2}e_i \mod \Lambda_\Omega.$$

Similarly, we find that

$$\mathsf{aj}_{c_{2g+2}}(\alpha_{S_{g+i}}) = \tfrac{1}{2}\Omega \cdot e_i \mod \Lambda_\Omega.$$

Now we can match the set Q_g with the set of theta functions with characteristics. Recall that the set $U = \{1, 3, \ldots, 2g+1\}$ plays the role of the standard quadratic form. We have

$$q_U(S_i) = q_U(S_{g+i}) = 0, \quad i = 1, \ldots, g.$$

Comparing it with (5.26), we see that the theta function $\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega)$ corresponding to $\mathsf{aj}_{c_{2g+2}}(\vartheta_U)$ must be the function $\Theta(\mathbf{z};\Omega)$. This shows that

$$\mathsf{aj}^{g-1}_{c_{2g+2}}(\vartheta_U) = \mathsf{aj}_{c_{2g+2}}(\vartheta_U - \kappa_{c_{2g+2}}) = 0.$$

Thus the Riemann constant $\kappa_{c_{2g+2}}$ corresponds to the theta characteristic ϑ_U. This allows one to match theta characteristics with theta functions with theta characteristics.

Write any subset S of E_g in the form

$$S = \sum_{i=1}^{g} \epsilon_i S_i + \sum_{i=1}^{g} \eta_i S_{g+i},$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_g),\ \eta = (\eta_1, \ldots, \eta_g)$ are binary vectors. Then

$$\vartheta_{U+S} \longleftrightarrow \Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega).$$

In particular,

$$\vartheta_{U+S} \in \mathrm{TChar}(C)^{\mathrm{ev}} \Longleftrightarrow \epsilon \cdot \eta = 0 \mod 2.$$

One employs the notation $\vartheta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right]$ for theta characteristics whose associated theta function is $\Theta\left[\begin{smallmatrix}\epsilon\\ \eta\end{smallmatrix}\right](\mathbf{z};\Omega)$. Of course it depends on the choice of the point c_0. If C is a hyperlliptic curve with an ordered set of ramification points and $c_0 = c_{2g+2}$ this agrees with the above correspondence.

Example 5.3.1. We give the list of theta characteristics for small genus. We also list 2-torsion points at which the corresponding theta function vanishes.

$g = 1$

3 even "thetas":

$$\vartheta_{12} = \vartheta\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right] \quad (\alpha_{12}),$$

$$\vartheta_{13} = \vartheta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] \quad (\alpha_{13}),$$
$$\vartheta_{14} = \vartheta \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \quad (\alpha_{14}).$$

1 odd theta

$$\vartheta_{\emptyset} = \vartheta \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}).$$

$g = 2$
10 even thetas:

$$\begin{aligned}
\vartheta_{123} &= \vartheta \left[\begin{smallmatrix} 01 \\ 10 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{23}, \alpha_{13}, \alpha_{45}, \alpha_{46}, \alpha_{56}), \\
\vartheta_{124} &= \vartheta \left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{24}, \alpha_{14}, \alpha_{35}, \alpha_{36}, \alpha_{56}), \\
\vartheta_{125} &= \vartheta \left[\begin{smallmatrix} 00 \\ 11 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{25}, \alpha_{15}, \alpha_{34}, \alpha_{36}, \alpha_{46}), \\
\vartheta_{126} &= \vartheta \left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix} \right] \quad (\alpha_{12}, \alpha_{16}, \alpha_{26}, \alpha_{34}, \alpha_{35}, \alpha_{45}), \\
\vartheta_{234} &= \vartheta \left[\begin{smallmatrix} 10 \\ 01 \end{smallmatrix} \right] \quad (\alpha_{23}, \alpha_{34}, \alpha_{24}, \alpha_{15}, \alpha_{56}, \alpha_{16}), \\
\vartheta_{235} &= \vartheta \left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix} \right] \quad (\alpha_{23}, \alpha_{25}, \alpha_{35}, \alpha_{14}, \alpha_{16}, \alpha_{46}), \\
\vartheta_{236} &= \vartheta \left[\begin{smallmatrix} 01 \\ 00 \end{smallmatrix} \right] \quad (\alpha_{23}, \alpha_{26}, \alpha_{36}, \alpha_{14}, \alpha_{45}, \alpha_{15}), \\
\vartheta_{245} &= \vartheta \left[\begin{smallmatrix} 11 \\ 00 \end{smallmatrix} \right] \quad (\alpha_{24}, \alpha_{25}, \alpha_{13}, \alpha_{45}, \alpha_{16}, \alpha_{36}), \\
\vartheta_{246} &= \vartheta \left[\begin{smallmatrix} 00 \\ 00 \end{smallmatrix} \right] \quad (\alpha_{26}, \alpha_{24}, \alpha_{13}, \alpha_{35}, \alpha_{46}, \alpha_{15}), \\
\vartheta_{256} &= \vartheta \left[\begin{smallmatrix} 00 \\ 01 \end{smallmatrix} \right] \quad (\alpha_{26}, \alpha_{25}, \alpha_{13}, \alpha_{14}, \alpha_{34}, \alpha_{56}).
\end{aligned}$$

6 odd thetas

$$\begin{aligned}
\vartheta_{1} &= \vartheta \left[\begin{smallmatrix} 01 \\ 01 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}), \\
\vartheta_{2} &= \vartheta \left[\begin{smallmatrix} 11 \\ 01 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}, \alpha_{12}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}), \\
\vartheta_{3} &= \vartheta \left[\begin{smallmatrix} 11 \\ 01 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}, \alpha_{13}, \alpha_{23}, \alpha_{34}, \alpha_{35}, \alpha_{36}), \\
\vartheta_{4} &= \vartheta \left[\begin{smallmatrix} 10 \\ 10 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}, \alpha_{14}, \alpha_{24}, \alpha_{34}, \alpha_{45}, \alpha_{46}), \\
\vartheta_{5} &= \vartheta \left[\begin{smallmatrix} 10 \\ 11 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}, \alpha_{15}, \alpha_{35}, \alpha_{45}, \alpha_{25}, \alpha_{56}), \\
\vartheta_{6} &= \vartheta \left[\begin{smallmatrix} 01 \\ 11 \end{smallmatrix} \right] \quad (\alpha_{\emptyset}, \alpha_{16}, \alpha_{26}, \alpha_{36}, \alpha_{46}, \alpha_{56}).
\end{aligned}$$

$g = 3$
36 even thetas $\vartheta_{\emptyset}, \vartheta_{ijkl}$,
28 odd thetas ϑ_{ij}.

$g = 4$
136 even thetas $\vartheta_{i}, \vartheta_{ijklm}$
120 odd thetas ϑ_{ijk}.

Exercises

5.1 Find 3 non-equivalent symmetric determinant expressions for the cubic curve given by a Weierstrass equation $T_0T_2^2 + T_1^3 + aT_1T_0^2 + bT_0^3 = 0$.

5.2 Find a symmetric determinant expression for the Fermat quartic $V(T_0^4 + T_1^4 + T_2^4)$.

5.3 Let C be an irreducible plane curve of degree d with a $(d-2)$-multiple point. Show that its normalization is a hyperelliptic curve of genus $g = d-2$. Conversely, show that any hyperelliptic curve of genus g admits such a plane model.

5.4 Show that a nonsingular curve of genus 2 has a vanishing theta characteristic but a nonsingular curve of genus 3 has a vanishing theta characteristic if and only if it is a hyperelliptic curve.

5.5 Show that a nonsingular plane curve of degree 5 does not have a vanishing theta characteristic.

5.6 Find the number of vanishing theta characteristics on a hyperelliptic curve of genus g.

5.7 Let $C^{(n)}$ be the symmetric product of a nonsingular projective curve C, the orbit space of C^n with respect to the natural action of the symmetric group S_n.

(i) Show that $C^{(n)}$ is a nonsingular variety.

(ii) Compute the numerical invariants of the surface $C^{(2)}$.

(iii) Show that $C^{(g-1)}$ is isomorphic to a resolution of singularities of W_{g-1}^0.

(iv) Show that the surface $C^{(2)}$, where C is a hyperelliptic curve of genus 3, contains a nonsingular rational curve with self-intersection -2.

Chapter 6

Plane Quartics

6.1 Odd theta characteristics

6.1.1 28 bitangents

A nonsingular plane quartic C is a non-hyperelliptic genus 3 curve embedded by its canonical linear system $|K_C|$. It contains $28 = 2^2(2^3 - 1)$ odd theta characteristics ϑ of degree 2. We have $\vartheta \sim p + q$ for a unique pair of points p, q. The line $l_\vartheta = \langle p, q\rangle$ joining these points is a *bitangent*. It cuts out in C the divisor $2p + 2q$. Thus we obtain

Theorem 6.1.1. *A nonsingular plane quartic has exactly 28 bitangents.*

Note that it could happen that $p = q$, then the bitangent cuts out the divisor $4p$. Such a bitangent is called a *flex bitangent*. The fact that we expect 28 honest bitangents for a plane quartics also follows from the Plücker formulas (Theorem 2.1.3 from Chapter 2).

Definition 6.1. *Three (resp. four) odd theta characteristics on a nonsingular curve C of genus g are called* syzygetic *or* asyzygetic *depending upon whether they add up to $2K_X$.*

When C is a plane quartic and we identify odd characteristics with bitrangents, the condition is that the six (resp. eight) points of contact do or do not lie on a conic. Here, if two contact points coincide we require that the conic be tangent to the bitangent at the contact point.

Note that if 3 odd theta characteristics $\ell_{\vartheta_1}, \ell_{\vartheta_2}, \ell_{\vartheta_3}$ are syzygetic then $2K_C - \vartheta_1 - \vartheta_2 - \vartheta_3$ is an odd theta characteristic. Thus a set of 3 syzygetic odd theta characteristics is a subset of four syzygetic bitangents.

In the following we will often identify an odd theta characteristic with the corresponding bitangent.

Let $\ell_i = V(L_i),\ i = 1, \dots, 4$, be four syzygetic bitangents and $K = V(Q)$ be the corresponding conic. Since $V(L_1L_2L_3L_4)$ and Q^2 cut out the same divisor on C we obtain that C can be given by an equation

$$F = L_1L_2L_3L_4 + Q^2 = 0. \tag{6.1}$$

Conversely, if F can be written in the form (6.1), the linear forms L_i define four syzygetic bitangents. So we see that F can be written as in (6.1) in only finitely many ways. This is confirmed by "counting constants". We have 12 constants for the linear forms and 6 constants for quadratic forms, they are defined up to scaling by $\lambda_1, \dots, \lambda_5$ subject to the condition $\lambda_1 \cdots \lambda_4 = \lambda_5^2$. Thus we have 14 parameters for quartic curves represented in the form (6.1). This is the same as the number of parameters for plane quartics.

Let us now compute the number of syzygetic tetrads of bitangents. Recall that a theta characteristic ϑ defines a quadratic form on 2-torsion points

$$q_\vartheta(\alpha) = h^0(\vartheta) + h^0(\vartheta + \alpha) \mod 2.$$

Lemma 6.1.2. *Let $\vartheta_1, \vartheta_2, \vartheta_3$ be odd theta characteristics on a nonsingular curve C of genus g. The following properties are equivalent:*

(i) $\vartheta_1, \vartheta_2, \vartheta_3$ *is a syzygetic triad;*

(ii) $\vartheta_1 + \vartheta_2 - \vartheta_3$ *is an odd theta characteristic;*

(iii) $q_{\vartheta_1}(\vartheta_2 - \vartheta_3) = 0$.

Proof. $(i) \Leftrightarrow (ii)$. Use that

$$2K_C - \vartheta_1 - \vartheta_2 - \vartheta_3 = \vartheta_1 + \vartheta_2 - \vartheta_3.$$

$(ii) \Leftrightarrow (iii)$ We have

$$q_{\vartheta_1}(\vartheta_2 - \vartheta_3) = h^0(\vartheta_1 + \vartheta_2 - \vartheta_3) + h^0(\vartheta_1)$$
$$= h^0(\vartheta_1 + \vartheta_2 - \vartheta_3) + 1 = 0 \Leftrightarrow h^0(\vartheta_1 + \vartheta_2 - \vartheta_3) = 1.$$

□

Theorem 6.1.3. *Let $(\vartheta_1, \vartheta_2)$ be a pair of two odd theta characteristics on a nonsingular curve C of genus g. The number of ways in which the pair can be extended to a syzygetic triad of odd theta characteristics is equal to $2^{2g-2} - 2^{g-1} - 2$.*

Proof. Let ϑ_3 be an odd theta characteristic and let $\epsilon = \vartheta_3 - \vartheta_2$. Assume that $\vartheta_1, \vartheta_2, \vartheta_3$ is a syzygetic triad. By the previous lemma,

$$q_{\vartheta_1}(\epsilon) = 0, \quad q_{\vartheta_2}(\epsilon) = h^0(\vartheta_3) + h^0(\vartheta_2) = 1 + 1 = 0.$$

Conversely, assume a 2-torsion divisor class ϵ satisfies $q_{\vartheta_1}(\epsilon) = q_{\vartheta_2}(\epsilon) = 0$. Set $\vartheta_3 = \vartheta_2 + \epsilon$. We have

$$h^0(\vartheta_3) = h^0(\vartheta_2) + q_{\vartheta_2}(\epsilon) = 1,$$

hence ϑ_3 is an odd theta characteristic. Also

$$q_{\vartheta_1}(\vartheta_2 - \vartheta_3) = q_{\vartheta_1}(\epsilon) = 0,$$

hence $\vartheta_1, \vartheta_2, \vartheta_3$ form a syzygetic triad.

Let $\eta = \vartheta_1 - \vartheta_2$. By (5.8), for any 2-torsion divisor class ϵ,

$$q_{\vartheta_2}(\epsilon) = q_{\vartheta_1}(\epsilon) + \langle \epsilon, \eta \rangle. \tag{6.2}$$

Thus the number of the ways we can extend ϑ_1, ϑ_2 to a syzygetic triad $\vartheta_1, \vartheta_2, \vartheta_3$ is equal to the cardinality of the set

$$Z = q_{\vartheta_1}^{-1}(0) \cap q_{\vartheta_2}^{-1}(0) \setminus \{0, \eta\}.$$

It follows from (6.2) that $\epsilon \in Z$ satisfies $\langle \epsilon, \eta \rangle = 0$. By definition, $q_{\vartheta_2}(\eta) = 0$, and hence, by (6.2), $q_{\vartheta_1}(\eta) = 0$. Thus any $\epsilon \in Z$ is a nonzero element in $V = \mathbb{F}_2\eta^\perp/\mathbb{F}_2\eta \cong \mathbb{F}_2^{2g-2}$. It is clear that q_{ϑ_1} and q_{ϑ_2} induce the same quadratic form q on V. It is an odd quadratic form. Indeed, we can choose a symplectic basis in $\mathrm{Pic}(C)[2]$ by taking as a first vector the vector η. Then computing the Arf invariant of q_{ϑ_1} we see that it is equal to the Arf invariant of the quadratic form q. Thus we get

$$\#Z = 2(2^{g-2}(2^{g-1} - 1) - 1) = 2^{2g-2} - 2^{g-1} - 2.$$

A half of elements of Z are of the form $\eta + \alpha$ and and a half are of the form α, where α is a lift of a nonzero vector from V for which $q(\alpha) = 0$. Now we are done. For each $\epsilon \in Z$ we define ϑ_3 such that $\vartheta_1, \vartheta_2, \vartheta_3$ are syzygetic. Thus there exists a conic through the corresponding 6 points. As we remarked before, the residual pair of points is the fourth odd theta characteristic. Conversely, if the residual divisor of a conic through p_1, q_1, p_2, q_2 contains an odd theta characteristic ϑ_3, the difference $\vartheta_3 - \vartheta_2$ belongs to Z. □

Corollary 6.1.4. *Let t be the the number of syzygetic tetrads of odd theta characteristics on a nonsingular curve of genus g. Then*

$$t = (2^{2g-3} - 2^{g-2} - 1)(2^{2g-1} - 2^{g-1} - 1)(2^{2g-1} - 2^{g-1})/12.$$

In particular, $t = 315$ if $g = 3$.

Proof. Let I be the set of triples $(\vartheta_1, \vartheta_2, T)$, where ϑ_1, ϑ_2 is a pair of odd theta characteristics, and T be a syzygetic tetrad containing ϑ_1, ϑ_2. We count $|I|$ in two ways by projecting I to the set of pairs of odd theta characteristics and to the set of syzygetic tetrads. Since each tetrad contains 6 pairs of odd theta characteristics, and each pair can be extened in $2^{2g-3} - 2^{g-2} - 1$ ways to a syzygetic tetrad, we get

$$|I| = (2^{2g-3} - 2^{g-2} - 1)\binom{2^{g-1}(2^g-1)}{2} = 6t.$$

This gives

$$t = (2^{2g-3} - 2^{g-2} - 1)(2^{2g-1} - 2^{g-1} - 1)(2^{2g-1} - 2^{g-1})/12.$$

□

Corollary 6.1.5. *Let $\vartheta_1 \sim p_1 + q_1, \vartheta_2 \sim p_2 + q_2$ be two bitangents on a plane quartic C. The pencil of conics through the points p_1, q_1, p_2, q_2 contains exactly five conics cutting out in C a divisor of degree 8 equal to the sum of 4 even theta characteristics.*

Let V be a vector space with a symplectic or symmetric bilinear form. Recall that a linear subspace L is called isotropic if the restriction of the bilinear form to L is identically zero.

Corollary 6.1.6. *Let $\vartheta_i, i = 1, 2, 3, 4,$ be a syzygetic tetrad of odd theta characteristics. Then $L = \{\vartheta_1 - \vartheta_i, \ldots, \vartheta_4 - \vartheta_i$ is an isotropic 2-dimensional subspace in* Pic$(C)[2]$ *which does not depend on the choice of ϑ_i.*

Proof. Let $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$ be an syzygetic tetrad. It follows from the proof of Theorem 6.1.3 that for any two of the theta characteristics ϑ_1, ϑ_2 we have

$$\langle \vartheta_3 - \vartheta_1, \vartheta_2 - \vartheta_1 \rangle = q_{\vartheta_1}(\vartheta_3 - \vartheta_2) + q_{\vartheta_1}(\vartheta_2 - \vartheta_1) + q_{\vartheta_1}(\vartheta_3 - \vartheta_1) = 0.$$

This shows that the 2-torsion points $\vartheta_i - \vartheta_1$ form an isotropic plane.

Subtracting $2\vartheta_i + 2\vartheta_j, i \neq j$, from both sides of the equality

$$\vartheta_1 + \ldots + \vartheta_4 = 2K_C$$

we get

$$\vartheta_k - \vartheta_l\ \vartheta_i - \vartheta_j, \tag{6.3}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. This shows that the subspace L of Pic$(C)[2]$ formed by the differences $\vartheta_j - \vartheta_i, j = 1, \ldots, 4$ is independent on the choice of i. □

Remark 6.1.1. It is known that the group $\mathrm{Sp}(2g, \mathbb{F}_2)$ acts transitively on the set of of isotropic planes in $\mathbb{F}_2^6$. The stabilizer group is a certain maximal parabolic subgroup whose order can be computed. Its index is equal to t. Thuse we see that any isotropic plane arises from a syzygetic tetrad of odd theta characteristics.

6.1.2 Steiner complexes

Let $\mathcal{P}$ be the set of unordered pairs of distinct odd theta characteritiscs on a nonsingular curve C of genus g. Consider the map

$$s : \mathcal{P} \to \mathrm{Pic}(C)[2] \setminus \{0\}, \quad \{\vartheta, \vartheta'\} \mapsto \vartheta - \vartheta'.$$

Definition 6.2. *The union of pairs from the same fibre of the map s is called a* Steiner compex of odd theta characteristics. .

It follows from (6.3) that any two pairs from a syzygetic tetrad are mapped to the same 2-torsion point and thus the corresponding four theta characteristics belong to the same Steiner complex. Conversely, let $\{\vartheta_1, \vartheta_1'\}, \{\vartheta_2, \vartheta_2'\}$ be two pairs from $s^{-1}(\epsilon)$.

We have

$$(\vartheta_1 + \vartheta_1') + (\vartheta_2 + \vartheta_2') = 2(2K_C - \vartheta_1 - \vartheta_1') = 4K_C - 2K_C = 2K_C. \quad (6.4)$$

This shows that the tetrad $(\vartheta_1, \vartheta_1', \vartheta_2, \vartheta_2')$ is syzygetic.

Theorem 6.1.7. *There are $\#\mathrm{Pic}(C)[2] = 2^{2g} - 1$ Steiner complexes of odd theta characteristics. Each Steiner complex consists of $2^{g-2}(2^{2g-1} - 2^{g-1} - 1)/(2^g + 1)$ pairs of odd theta characteristics.*

Proof. It suffices to show that the map $s : \mathcal{P} \to \mathrm{Pic}(C)[2] \setminus \{0\}$ is surjective. Choose a symplectic basis in $\mathrm{Pic}(C)[2]$ and consider the natural action of $G = \mathrm{Sp}(2g, \mathbb{F}_2)$ on the set $\mathrm{Pic}(C)[2] \setminus \{0\}$. This action is obviously transitive. On the other hand G also acts on the set $\mathcal{P}$ via its natural action on the set of theta characteritics. It is easy to see that the map s is G-invariant. Thus its image is a non-empty G-invariant subset of $\mathrm{Pic}(C)[2] \setminus \{0\}$. It must coincide with the whole set. □

Corollary 6.1.8. *There are 63 Steiner complexes of bitangents of a plane quartic curve. Each Steiner complex consists of 6 pairs of bitangents.*

In the rest of this section C is a plane quartic.

Let $l = 0, m = 0, p = 0, q = 0, r = 0, s = 0$ be the equations of 6 bitangents such that (l, m, p, q) and (l, m, r, s) form an syzygetic tetrad. By (6.1), we can write

$$F = lmpq - Q^2 = lmrs - R^2$$

for some quadratic forms Q, R. Subtracting we have

$$lm(pq - rs) = (Q + R)(Q - R).$$

If l divides $Q+R$ and m divides $Q-R$, then the quadric $Q = \frac{1}{2}[(Q+R)+(Q-R)]$ passes through the point $l \cap m$. But this is impossible since no two bitangents intersect at a point on the quartic. Thus we obtain that lm divides either $Q + R$ or $Q - R$. Without loss of generality, we get $lm = Q + R, \quad pq - rs = Q - R$, and hence $Q = \frac{1}{2}(lm + pq - rs)$. Thus we define the quartic by the equation

$$-4F = -4lmpq + (lm + pq - rs)^2$$

$$= (lm)^2 - 2lmpq - 2lmrs - 2pqrs + (pq)^2 + (rs)^2 = 0.$$

It is easy to see that this is equivalent to the equation

$$\sqrt{lm} + \sqrt{pq} + \sqrt{rs} = 0. \tag{6.5}$$

Thus we see that a nonsingular quartic can be written in 315 ways in the form of (6.1) and in $1260 = \binom{6}{3} \cdot 63$ ways in the form of (6.5).

Remark 6.1.2. Consider the orbit space $X = (\mathbb{C}^3)^6/T$, where

$$T = \{(z_1, z_2, z_3, z_4, z_5, z_6) \in (\mathbb{C}^*)^6 : z_1z_2 = z_3z_4 = z_5z_6\}.$$

Its dimension is equal to 14. Any orbit $T(l, m, p, q, r, s) \in X$ defines the quartic curve $V(\sqrt{lm} + \sqrt{pq} + \sqrt{rs})$. We have shown that the map $X \to |\mathcal{O}_{\mathbb{P}^2}(4)| \cong \mathbb{P}^{14}$ is of degree 1260. The group $\mathrm{PGL}(3)$ acts naturally on both spaces. One can show that $X/PGL(3)$ is a rational variety and we get a map $X/\mathrm{PGL}(3) \to |\mathcal{O}_{\mathbb{P}^2}(4)|/\mathrm{PGL}(3) \cong \mathcal{M}_3$ of degree 1260.

Now let us see how two Steiner complexes of bitangents intersect.

Lemma 6.1.9. *Let S, S' be two Steiner complexes of 6 pairs of bitangents, and ϵ, ϵ' be the corresponding elements of* $\mathrm{Jac}(C)[2]$. *Then $\#S \cap S' = 4$ (resp. 6) if and only if $\langle \epsilon, \epsilon' \rangle = 0$ (resp. 1).*

Proof. Let $\vartheta \in S \cap S'$. Then we have $\vartheta - \vartheta' = \epsilon, \vartheta - \vartheta'' = \eta$ for some odd theta characteristics $\vartheta' \in S, \vartheta'' \in S'$. This implies that

$$q_\vartheta(\epsilon) = q_\vartheta(\eta) = 0. \tag{6.6}$$

Conversely, if these equalities hold, then $\vartheta + \epsilon, \vartheta + \eta$ are odd theta characteristics and $\vartheta, \vartheta' \in S, \vartheta, \vartheta'' \in S'$. Thus we have reduced our problem to linear algebra. We want to show that the number of odd quadrics (associated to $\langle , \rangle$) which vanish at 2 nonzero vectors $\epsilon, \eta \in \mathbb{F}_2^6$ is equal to 4 or 6 depending on whether $\langle \epsilon, \eta \rangle = 0$ or 1. Let q be one such quadric. Then $q(\epsilon) = q(\eta) = 0$. Suppose we have an odd quadric q' with this property. Write $q' = q + \alpha$ for some α. We have $q(\alpha) = 0$ since q' is odd and

$$\langle \alpha, \epsilon \rangle = \langle \alpha, \eta \rangle = 0.$$

Let V be the plane spanned by ϵ, η. Assume $\langle \eta, \epsilon \rangle = 1$, then we can include ϵ, η in a standard symplectic basis. Computing the Arf invariant, we find that the restriction of q to $V^\perp$ is an odd quadratic form. Thus it has 6 zeroes. Each zero gives us a solution for α. Assume $\langle \eta, \epsilon \rangle = 0$. Then V is a singular plane for q since $q(\epsilon) = q(\eta) = q(\epsilon + \eta) = 0$. Consider $W = V^\perp / V \cong \mathbb{F}_2^2$. The quadraic form q induces an odd quadratic form on W. It does not vanish on nonzero vectors. Thus the only solutions for α are $0, \eta, \epsilon, \eta + \epsilon$. □

Definition 6.3. *Two Steiner complexes are called* syzygetic *(resp.* asyzygetic *if they have 4 (resp. 6) common bitangents.*

Theorem 6.1.10. *For any two syzygetic Steiner complexes S_1, S_2 there exists a unique Steiner complex S_3 which is syzygetic to S_1 and S_2 such that any bitangent is contained in one of the three complexes. The 2-torsion point corresponding to S_3 is equal to the sum of the 2-torsion points corresponding to the complexes S_1, S_2.*

Proof. Let S_1 be associated to ϵ and S_2 be associated to η. Then $\langle \eta, \epsilon \rangle = 0$. Since

$$\langle \epsilon + \eta, \eta \rangle = \langle \epsilon + \eta, \epsilon \rangle = 0,$$

we obtain that the Steiner complex S_3 associated to $\epsilon + \eta$ is syzygetic to S_1 and S_2. Suppose an odd theta characteristic ϑ belongs to $S_1 \cap S_2$. Then

$$q_\vartheta(\epsilon + \eta) = q_\vartheta(\epsilon) + q_\vartheta(\eta) + \langle \epsilon, \eta \rangle = 0 + 0 + 0 = 0.$$

This implies that $S_1 \cap S_2 \subset S_3$ and hence S_1, S_2, S_3 share the same set of 4 bitangents. This gives $|S_1 \cup S_2 \cup S_3| = 4 + 8 + 8 + 8 = 28$. Conversely, if S_3 has 4 common bitangents with S_1 and 4 common bitangents with S_2, then $|S_1 \cap S_2 \subset S_3| = 28$ implies $S_1 \cap S_2 \cap S_3 \neq \emptyset$. Then any $\vartheta \in S_1 \cap S_2 \cap S_3$ satisfies $q_\vartheta(\epsilon + \eta) = 0$, and we obtain that S_3 is associated to $\epsilon + \eta$. □

Definition 6.4. *Three syzygetic Steiner complexes such that any bitangent is contained in one of them is called a* syzygetic triad of Steiner complexes.

It follows from the proof that the set of syzygetic triads of Steiner complexes is bijective to the set of isotropic planes in Jac$(C)[2]$. Their number is 315.

Let S_1, S_2 be a pair of asyzygetic Steiner complexes corresponding to two non-orthogoanl 2-torsion points ϵ_1, ϵ_2. They define 12 bitangents which are not common to both S_1 and S_2. These 12 bitangents form a third Steiner complex S_3 which is asyzygetic to S_1 and S_2. The 2-torsion point corresponding to S_3 is the sum $\epsilon_1 + \epsilon_2$. We leave it to the reader to check this to the reader. A triple of Steiner complexes formed in this way is called *asyzygetic triad of Steiner complexes*. We have $63 \cdot 32/6 = 336$ asyzygetic triads of Steiner complexes.

6.1.3 Aronhold sets

Definition 6.5. *An ordered (resp. unordered) set of 7 bitangents $(\ell_1, \ldots, \ell_7)$ is called an* ordered Aronhold set *(resp.* unordered Aronhold set*) if any subset of three is asyzygetic.*

Theorem 6.1.11. *There is a natural bijection between the set of ordered Aronhold sets and symplectic isomorphisms $\mathbb{F}_2^6 \to$ Jac$(C)[2]$.*

Proof. Let $(\vartheta_1, \ldots, \vartheta_7)$ be the set of odd theta characteristics corresponding to an ordered Aronhold set of bitangents. Let

$$\eta_i = \vartheta_i - \vartheta_1, \quad i = 2, \ldots, 7.$$

Since $\vartheta_i, \vartheta_j, \vartheta_1$ is a asyzygetic triad, the plane spanned by η_i, η_j is not isotropic. Thus, for any $i \neq j$,

$$\langle \eta_i, \eta_j \rangle = 1 \tag{6.7}$$

Now let us define a standard symplectic basis in Jac$(C)[2])$ by setting

$$\epsilon_1 = \eta_1,\ \epsilon_2 = \eta_2 + \eta_3,\ \epsilon_3 = \eta_4 + \eta_5,$$

$$\epsilon_4 = \eta_1 + \ldots + \eta_6,\ \epsilon_5 = \eta_3 + \eta_4 + \eta_5 + \eta_6,\ \epsilon_6 = \eta_5 + \eta_6. \tag{6.8}$$

One checks immediately that, for any $1 \leq i < j \leq 6$,

$$\langle \epsilon_i, \epsilon_j \rangle = \delta_{3,j-i}. \tag{6.9}$$

Conversely, given a standard symplectic basis $(\epsilon_1, \ldots, \epsilon_6)$ satisfying (6.9) we solve equations (6.8) for η_i's. Then we define a quadratic form Q on Jac$(C)[2]$ by

$$q(a_1\eta_1 + \ldots + a_6\eta_6) = \sum_{1 \leq i < j \leq 6} a_i a_j.$$

Since, for any $i \neq j$, $q(\eta_i + \eta_j) + q(\eta_i) + q(\eta_j) = 1$, we see that the polar symmetric bilinear form b_q is equal to the one defined by (6.7) which is the Weil pairing. Thus q corresponds to some theta characteristic ϑ_1. It is easy to see that q vanishes at 28 points in Jac$(C)[2]$ corresponding to the coordinates $(a_1, \ldots, a_6)$ with 1,2,5 or 6 zero coefficients. Thus ϑ_1 is an odd theta characteristic. Now we set $\vartheta_{i+1} = \vartheta_1 + \eta_i, i = 1, \ldots, 6$ and check that $(\vartheta_1, \ldots, \vartheta_7)$ defines an Aronhold set of bitangents. We leave to the reader to see that this correspondence is the asserted bijection between the two sets. $\square$

Corollary 6.1.12. *There are 288 unordered Aronhold sets of bitangents.*

Proof. The symplectic group Sp$(6, \mathbb{F}_2)$ acts simply transitively on the set of symplectic bases. So their number is #Sp$(6, \mathbb{F}_2) = 2^9 \cdot 3^4 \cdot 5 \cdot 7$. The number of unordered bases is this number divided by 7! which us 288. $\square$

Remark 6.1.3. One can generalize the definition of an Aronhold set to any nonsingular curve C of genus g. Let $V =$ Pic$(C)[2]$ and $Q(V)$ be the space of quadratic forms on V. We consider the disjoint sum $W = V \coprod Q(V)$ as a vector space of dimension $2g+1$ for which V is a subspace and $Q(V)$ is the nontrivial coset. Let B be a basis of W formed by $2g+1$ elements of $Q(V)$. For any vector $v \in W$ we consider the coordinates of v in the basis B as a binary vector. We denote by $|v|$ the sum of the coordinates. An Aronhold set is a basis B as above such that any odd theta characteristic can be written as a vector v with $|v| \equiv 1 \mod 4$.

Remark 6.1.4. A natural question is whether the set of bitangents determines the quartic, i.e. whether two quartics with the same set of bitangents coincide. Surprizingly it has not been answered by the ancients. Only recently it was proven that the answer is yes (Caporaso-Sernesi, D. Lehavi).

6.2 Quadratic determinant equations

6.2.1 Coble's construction

Consider the following construction due to A. Coble.

Let C be a nonsingular plane quartic. Let $a \in$ Pic$^0(C) \setminus \{0\}$. Consider the natural bilinear map

$$m : H^0(C, \mathcal{O}_C(K_C + a)) \times H^0(C, \mathcal{O}_C(K_C - a)) \to H^0(C, \mathcal{O}_C(2K_C))$$

defined by the tensor multiplication of the sections. The associated map of complete linear systems

$$f : |K_C + a| \times |K_C - a| \to |2K_C| = |\mathcal{O}_{\mathbb{P}^2}(2)| \cong \mathbb{P}^5. \tag{6.10}$$

assigns to a pair of divisors $D \in |K_C + a|$ and $D' \in |K_C - a|$ the divisor $D + D' \in |2K_C|$. If we choose a basis (s_1, s_2) of $H^0(C, \mathcal{O}_C(K_C + a))$ and a basis (s_1', s_2') of $H^0(C, \mathcal{O}_C(K_C - a))$, then the map m is given by

$$(\lambda s_1 + \mu s_2, \lambda' s_1' + \mu' s_2') \mapsto \lambda\lambda' A_{11} + \lambda\mu' A_{12} + \lambda'\mu A_{21} + \mu\mu' A_{22}, \tag{6.11}$$

where $A_{11}, A_{12}, A_{21}, A_{22} \in H^0(C, \mathcal{O}_C(2K_C)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ are identified with homogeneous polynomials of degree 2 in variables T_0, T_1, T_2. Consider the variety

$$W = \{(D_1, D_2, x) \in |K_C + a| \times |K_C - a| \times \mathbb{P}^2 : x \in f(D_1, D_2)\}. \tag{6.12}$$

If we identify $|K_C + a|$ and $|K_C - a|$ with $\mathbb{P}^1$, we see that

$$W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$$

is a hypersurface defined by the multi-homogeneous equation

$$\lambda\lambda' A_{11} + \lambda\mu' A_{12} + \lambda'\mu A_{21} + \mu\mu' A_{22} = 0 \tag{6.13}$$

of multi-degree $(1, 1, 2)$.

Consider the projections

$$p_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1, \quad p_2 : W \to \mathbb{P}^2. \tag{6.14}$$

The fibre of p_1 over a point $((\lambda, \mu), (\lambda', \mu'))$ is isomorphic (under p_2) to a conic. It is singular if and only if the discriminant of the conic (6.13) is equal to zero. It is easy to see that this is a bihomogeneous polynomial in the variables $(\lambda, \mu), (\lambda', \mu')$ of bidegree $(3, 3)$. Thus the locus Δ_1 of points $(\lambda, \mu), (\lambda', \mu')$ such that p_1^{-1} is a reducible conic is a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(3, 3)$.

The fibre of p_2 over a point $x \in \mathbb{P}^2$ is isomorphic (under the first projection) to a curve of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Under the Segre isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and a quadric in $\mathbb{P}^3$, such a curve is isomorphic to a conic. This conic is reducible if and only if the equation

$$\lambda\lambda' A_{11}(x) + \lambda\mu' A_{12}(x) + \lambda'\mu A_{21}(x) + \mu\mu' A_{22}(x) = 0$$

is a "cross" on $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e. the union of two lines belonging to different rulings). We can rewrite the equation in the form

$$(\lambda, \mu) \cdot \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{pmatrix} \cdot \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix}.$$

It defines a reducible curve if and only if there exists (a, b) such that, after plugging in $\lambda = \lambda_0 \mu = \mu_0$, any (λ', μ') will satisfy the equation. The condition for this is of course

$$\det \begin{pmatrix} A_{00}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{pmatrix} = 0. \tag{6.15}$$

This defines a homogeneous equation of degree 4 in variables T_0, T_1, T_2. It is not identically equal to zero otherwise the entries A_{11}, A_{21} must have a common linear factor. The corresponding conics cut out the divisors $D_1 + D_1', D_2 + D_1'$, where $D_i = \mathrm{div}(s_i), D_i' = \mathrm{div}(s_i')$. Their common points form a divisor $D_1' \in |K_C + a|$. Since $a \neq 0$, D_1' cannot be cut out by a line. Thus (6.15) defines a quartic curve. It must coincide with our curve C. To see this it is enough to show that each point of C satisfies the equation. Let $x \in C$, then we choose a unique $D \in |K_C + a|$ containing x and take any $D' \in |K_C - a|$. We obtain a subset of the conic $p_2^{-1}(x)$ isomorphic to $\mathbb{P}^1$. This shows that $p_2^{-1}(x)$ is a reducible conic.

Conversely, suppose C is given by a determinantal equation as above. For every $x \in C$ we have the left and the right kernel of the corresponding matrix. These are one-dimensional vector spaces. The corresponding maps $\phi_i : C \to \mathbb{P}^1, i = 1, 2$, are defined by quadratic polynomials $(-A_{21}, A_{11})$ and $(-A_{12}, A_{11})$, respectively. Note that the common zeroes of both coordinates belong to the curve C . Thus the linear system defined by the two conics has 4 base points on C and hence ϕ_i is given by a linear system V_i of degree 4. We may assume that V_i is contained in $|K_C + a_i|$ for some divisor classes a_i of degree 0. It is easy to see that the base loci of the two linear systems add up to the zeroes of A_{11}. This immediately implies that $a_1 + a_2 = 0$.

Thus we have proved:

Theorem 6.2.1. *An equation of a nonsingular plane quartic C can be written in the form*

$$\begin{vmatrix} A_1 & A_2 \\ A_3 & A_4 \end{vmatrix} = 0,$$

where A_i's are homogeneous forms of degree 2. Let X be the set of matrices A with quadratic forms as its entries such that $C = V(\det A)$ modulo the equivalence relation defined by $A \sim B$ if $A = CBC'$ for some constant nonsingular matrices. The set X is bijective to the set $\mathrm{Pic}^0(C) \setminus \{0\}$.

Remark 6.2.1. The previous theorem agrees with the general theory developed in Chapter 5. To define a quadratic determinant one considers the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^2 \to \mathcal{O}_{\mathbb{P}^2}^2 \to i_*(\mathcal{M}) \to 0.$$

We have

$$h^0(\mathcal{M}) = 2, \quad h^0(\mathcal{M}(-1)) = h^1(\mathcal{M}) = 0, \quad h^1(\mathcal{M}(-1)) = 1.$$

By Riemann-Roch, $\deg(\mathcal{M}) = 4$, hence $\mathcal{M} = \mathcal{O}_C(K_C + a)$, for some $a \in$ Pic$^0(C)$. Since $h^0(\mathcal{M}(-1)) = 0$, we obtain $a \neq 0$.

Remark 6.2.2. (for a reader familiar with the theory of 3-folds) The variety W which we associated to $a \in$ Pic$^0(C) \setminus \{0\}$ is a Fano 3-fold. Its canonical sheaf is equal to

$$\omega_W \cong \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1).$$

In the Segre embedding it is equal to $\mathcal{O}_W(-1)$. It has two structures of a conic bundle, induced by the projections $p_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1$ and $p_2 : W \to \mathbb{P}^2$. The degeneration locus of the first map is a curve Δ_1 of arithmetic genus 4, and the degeneration locus Δ_2 of the second map is the curve C of genus 3. Note that each curve has a double cover defined by considering the irreducible components of the fibres. The double cover over Δ_2 splits since each component corresponds to one of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. The double cover $\tilde{\Delta}_1 \to \Delta_1$ does not split. For a general C, the curve Δ_1 is nonsingular and the double cover over it is unramified. One shows that the intermediate Jacobian variety of W is isomorphic to the Prym variety of the cover $\tilde{\Delta}_1 \to \Delta_1$. It is aslo isomorphic to the Prym variety of the trivial cover over Δ_2 which is the Jacobian of C. Thus we obtain that the intermediate Jacobian of W is isomorphic to Jac(C).

Remark 6.2.3. Let

$$V = H^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2)).$$

It is a vector space of dimension 24. Let U be an open subset in $\mathbb{P}(V)$ which consists of sections (6.13) such that the corresponding determinant (6.15) defines a nonsingular quartic curve. The group $G =$ SL$(2) \times$ SL(2) acts naturally on U and the orbit space is isomorphic to the space $\mathcal{P}ic_4^0 \setminus \{$zero section$\}$. Let W be the 3-fold (6.14) defined by a section from U. The projection $W \to \mathbb{P}^1 \times \mathbb{P}^1$ defines a curve Δ_1 of bidegree $(3, 3)$ parametrizing singular fibres. It comes with a double cover defined by choosing a component of a reducible fibre. In this way we see that U/G is birationally isomorphic to the space of nonsingular curves of bidegree $(3, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ together with an unramified double cover. If we further act by $G' =$ SL(3) the orbit space is birationally isomorphic to the universal Jacobian space over $\mathcal{M}_3$ and, on the other hand, to the moduli space $\mathcal{R}_4$ of curves of genus 4 together with a nontrivial 2-torsion divisor class. It was proven by F. Catanese that the latter space is a rational variety.

6.2.2 Symmetric quadratic determinants

Assume now that $a = \epsilon$ is a 2-torsion divisor class. Then $H^0(C, \mathcal{O}_C(K_C + \epsilon)) = H^0(C, \mathcal{O}_C(K_C - \epsilon))$ and the bilinear map m is symmetric.

The determinantal equation of C corresponding to ϵ must be given by a symmetric quadratic determinant

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}^2. \tag{6.16}$$

Thus we obtain the following.

Theorem 6.2.2. *An equation of a nonsingular plane quartic can be written in the form*

$$\begin{vmatrix} A_1 & A_2 \\ A_2 & A_3 \end{vmatrix} = 0,$$

where A_1, A_2, A_3 are homogeneous forms of degree 2. Let X be the set of symmetric 2×2 matrices A with quadratic forms as its entries such that $C = V(\det A)$ modulo the equivalence relation defined by $A \sim B$ if $A = CBC'$ for some constant nonsingular matrices. The set X is bijective to the set $\mathrm{Pic}^0(C)_2 \setminus \{0\}$.

Since $f(D_1, D_2) = f(D_2, D_1)$, the map f factors through a linear map

$$\bar{f} : \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^2}(2)|.$$

Here we identify $|K_C + \epsilon|$ with $\mathbb{P}^1$ and the symmetric square $\mathbb{P}^1 \times \mathbb{P}^1/S_2$ with $\mathbb{P}^2$ (a set of k unordered points in $\mathbb{P}^1$ is a positive divisor of degree k, i.e. an element of $|\mathcal{O}_{\mathbb{P}^1}(k)| \cong \mathbb{P}^k$). Explicitly, we view $\mathbb{P}^2$ as $\mathbb{P}(S^2(H^0(C, \mathcal{O}_C(K_C + \epsilon)))$. The corresponding linear map $S^2H^0(C, \mathcal{O}_C(K_C + \epsilon)) \to S^2H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ defines a regular map

$$\phi : \mathbb{P}^1 = |K_C + \epsilon| \to |\mathcal{O}_{\mathbb{P}^2}(2)|$$

which is quadratic. Explicitly,

$$\phi((t_0, t_1)) = V(t_0^2 A_{11}(T) + 2t_0t_1A_{12}(T) + t_1^2 A_{22}(T)).$$

Let $L(\epsilon)$ be a net of conics equal to the image of the map $\bar{f}$. By choosing a basis (s_0, s_1) of $H^0(C, K_C + \epsilon)$ we may assume that $L(\epsilon)$ is spanned by the conics

$$V(A_{11}) = f(D_1, D_1), \quad V(A_{12}) = f(D_1, D_2), \quad V(A_{22}) = f(D_2, D_2),$$

where $D_1 = \mathrm{div}(s_0), D_2 = \mathrm{div}(s_1)$. In particular, we see that $L(\epsilon)$ has no base points (since C has no pencils of divisors of degree 3).

The set $B(\epsilon)$ of singular conics in $L(\epsilon)$ is a plane cubic isomorphic to a plane section of the discriminant hypersurface $\mathcal{D}_2(2)$. Its pre-image under the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is the degeneration curve Δ_1 of the conic bundle $W \to \mathbb{P}^1 \times \mathbb{P}^1$ from (6.14).

Remark 6.2.4. We can view ϕ as the composition of the Veronese map $v_2 : \mathbb{P}^1 \to \mathbb{P}^2$ and the map $\bar{f}$. Let $v_2(\mathbb{P}^1)$ be the Veronese curve. The pre-image of $\mathcal{D}_2(2)$ under the map ϕ is the locus of zeroes of a binary sextic. It corresponds to the intersection scheme of the cubic $B(\epsilon)$ and the conic $v_2(\mathbb{P}^1)$. The curve C can be defined as the locus of point $x \in \mathbb{P}^2$ such that, viewed as hyperplanes in $|\mathcal{O}_{\mathbb{P}^2}(2)|$, the pre-image $\phi^{-1}(x)$ is a degenerate binary quadratic form.

Lemma 6.2.3. *The cubic curve $B(\epsilon)$ is nonsingular if and only if the linear system $|K_C + \epsilon|$ does not contain a divisor of the form $2a + 2b$.*

Proof. The plane section $L(\epsilon) \cap \mathcal{D}_2(2)$ is singular if and only if $L(\epsilon)$ contains a singular point of $\mathcal{D}_2(2)$ represented by a double line, or if it is tangent to $\mathcal{D}_2(2)$ at a nonsingular point. We proved in Chapter 2, section 2.1.2 that the tangent hypersurface of $\mathcal{D}_2(2)$ at a nonsingular point represented by a reducible conic Q is equal to the space of conics passing through the singular point q of Q. If L is contained in the tangent hyperplane, then all conics from $L(\epsilon)$ pass through q. But we have seen already that L is base-point free. This shows that $L(\epsilon)$ intersects transversally the nonsingular locus of $\mathcal{D}_2(2)$.

In particular, $B(\epsilon)$ is singular if and only if $L(\epsilon)$ contains a double line. Assume that this happens. Then we get two divisors $D_1, D_2 \in |K_C + \epsilon|$ such that $D_1 + D_2 = 2A$, where $A = a_1 + a_2 + a_3 + a_4$ is cut out by a line ℓ. Let $D_1 = p_1 + p_2 + p_3 + p_4, D_2 = q_1 + q_2 + q_3 + q_4$. Then the equality of divisors (not the divisor classes)

$$p_1 + p_2 + p_3 + p_4 + q_1 + q_2 + q_3 + q_4 = 2(a_1 + a_2 + a_3 + a_4)$$

implies that either D_1 and D_2 share a point x, or $D_1 = 2p_1 + 2p_2$, $D_2 = 2q_1 + 2q_2$. The first case is impossible, since $|K_C + \epsilon - x|$ is of dimension 0. The second case happens if and only if $|K_C + \epsilon|$ contains a divisors $D_1 = 2a + 2b$. The converse is also true. For each such divisor the line $\langle a, b\rangle$ defines a residual pair of points c, d such that $D_2 = 2c + 2d \in |K_C + \epsilon|$ and $f(D_1, D_2)$ is a double line. □

Let

$$I = \{(x, \ell) \in B(\epsilon) \times \check{\mathbb{P}}^2 : \ell \subset f(x)\}. \tag{6.17}$$

The first projection $p_1 : I \to B(\epsilon)$ is a double cover ramified at singular points of $B(\epsilon)$. The image $\tilde{B}(\epsilon)$ of the second projection is locus of lines in $\mathbb{P}^2$ which are irreducible components of reducible conics from $L(\epsilon)$. It is a plane curve of some degree d in the dual plane or the whole plane.

Lemma 6.2.4. *The curve $\tilde{B}(\epsilon) \subset \check{\mathbb{P}}^2$ parametrizing irreducible components of reducible conics from the linear system $L(\epsilon)$ is a plane cubic. If $B(\epsilon)$ is nonsingular, then $\tilde{B}(\epsilon)$ is also nonsingular and is isomorphic to an unramified double cover of $B(\epsilon)$.*

Proof. Let us see that $d = \deg(\tilde{B}(\epsilon)) = 3$. A line in the dual plane is the pencil of lines in the original plane. Thus d is equal to the number of line components of reducible conics in L which pass through a general point q in $\mathbb{P}^2$. Since q is a general point, we may assume that q is not a singular point of any reducible conic from $L(\epsilon)$. Then there are d different reducible conics passing through q.

We know that $L(\epsilon)$ has no base points. Then q must be a base point of a pencil of conics in $L(\epsilon)$. Note that a general pencil of conics in $L(\epsilon)$ has 4 distinct base points. To see this we consider the regular map $\mathbb{P}^2 \to |L(\epsilon)|^*$ defined by the linear system $|L(\epsilon)|$. Its degree is equal to 4, hence it general fibre consists of 4 distinct points. It is easy to check that a pencil of conics with 4 distinct base points contains 3 reducible conics. This shows that $d = 3$. If $B(\epsilon)$ is nonsingular, its double cover $p_1 : I \to B(\epsilon)$ is unramified, hence I is an elliptic curve. Its image $\tilde{B}(\epsilon) = p_2(I)$ in $\check{\mathbb{P}}^2$ is a plane cubic.

Note that two reducible conics $f(D_1, D_2)$ and $f(D_3, D_4)$ in $B(\epsilon)$ share a common irreducible component if and only if $D_1 + D_2$ is cut out by two lines ℓ and ℓ' and $D_3 + D_4$ is cut out by two lines ℓ and ℓ''. Let A be the divisor on C cut out by ℓ. We know that no two divisors from $|K_C + \epsilon|$ share a common point. Also no divisor is cut out by a line. This easily implies that $D_i \cap \ell$ consists of one point for each $i = 1, \ldots, 4$. Since $D_1 + D_2 \geq A$, $D_3 + D_4 \geq A$, we see that ℓ contains at least 2 ramification points of the map $C \to \mathbb{P}^1$ defined by the linear system $|K_C + \epsilon|$. Since we have only finitely many such points, we see that there are only finitely many such lines ℓ. In particular, the second projection $p_2 : I \to \tilde{B}(\epsilon)$ is an isomorphism over a dense Zariski subset of $\tilde{B}(\epsilon)$.

If $B(\epsilon)$ is nonsingular, then $p_2 : I \to \tilde{B}(\epsilon)$ is a birational map of an elliptic curve to a cubic. Obvioulsy, this cubic must be nonsingular. □

Theorem 6.2.5. *Let $S = \{(\ell_1, \ell_1), \ldots, (\ell_6, \ell_6')\}$ be a Steiner complex of 12 bitangents associated to a 2-torsion divisor class ϵ. Then the 12 bitangents, considered as points in the dual plane, lie on the cubic curve $\tilde{B}(\epsilon)$. If we assume that $|K_C + \epsilon|$ does not contain a divisor of the form $2p + 2q$, then the cubic curve is nonsingular.*

Proof. Let $(\vartheta_i, \vartheta_i')$ be a pair of odd theta characteristics corresponding to a pair (ℓ_i, ℓ_i') of bitangents from S. They define a divisor $D = \vartheta_i + \vartheta_i' \in |K_C + \epsilon|$ which is cut out by two lines. Thus $f(D, D) \in B(\epsilon)$ and the bitangents ℓ_i, ℓ_i' belong to $\tilde{B}(\epsilon)$. The rest of the assertions follow from the previous lemmas. □

Remark 6.2.5. Let S_1, S_2, S_3 be a syzygetic triad of Steiner complexes. They define three cubic curves $\tilde{B}(\epsilon), \tilde{B}(\eta), \tilde{B}(\eta+\epsilon)$ which have 4 points in common.

Remark 6.2.6. The cubic $\tilde{B}(\epsilon)$ has at most ordinary nodes as singularities. We know that the projection $p_1 : I \to B(\epsilon)$ is a double cover unramified outside singular points of $B(\epsilon)$ corresponding to double lines. If $B(\epsilon)$ is an irreducible cuspidal cubic, the complement of the cusp is isomorphic to $\mathbb{C}$ and hence does not admit nontrivial unramified covers. If $B(\epsilon)$ is the union of a conic and a line touching it at some point, then, again the complement of the singular point is the disjoint union of two copies of $\mathbb{C}$ and hence does not admit an unramified cover. Finally, if $B(\epsilon)$ is the union of 3 concurrent lines, then the complement to the singular point is the disjoint union of three copies of $\mathbb{C}$, no unramified covers again. Thus $B(\epsilon)$ is nonsingular or a nodal cubic. It is easy to see that its cover I is again nonsingular or a nodal curve of arithmetic genus 1. The second projection $p_2 : I \to \tilde{B}(\epsilon)$ is an isomorphism over the complement of finitely many points. It is easy to see that the image of a nodal curve is a nodal curve.

Remark 6.2.7. Let Δ_1 be a curve of bidegree $(3,3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ parametrizing singular fibres of the projection $p_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 / S_2 = \mathbb{P}^2$ be the quotient map. The curve Δ_1 is equal to $\pi^{-1}(B(\epsilon))$. The cover $\pi|_{\Delta_1} : \Delta_1 \to B(\epsilon)$ is a double cover ramified along the points where Δ_1 intersects the diagonal. It consists of pairs (D, D), where $D \in |K_C + \epsilon|$ such that $2D$ is cut out by a reducible conic. It is easy to see that this conic must be the union of two bitangents which form one of 6 pairs of bitangents from the Steiner complex associated to ϵ. The branch locus of $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is a conic C. It can be identified with the Veronese curve $v_2(\mathbb{P}^1)$ which we discussed in Remark 6.2.4. The cubic $B(\epsilon)$ intersects it transversally at 6 points. If it is nonsingular, Δ_1 is nonsingulat curve of genus 4. If $B(\epsilon)$ is an irreducible cubic with a node (by the previous remark it cannot have a cusp), Δ_1 is an irreducible curve of arithmetic genus 2 with two nodes. If $B(\epsilon)$ is the union of a conic and a line intersecting each other transversally, then Δ_1 is the union of a nonsingular elliptic curve of bidegree $(2,2)$ and a nonsingular rational curve of bidgegree $(1,1)$ which intersect each other transversally. If $B(\epsilon)$ is the union of three lines, then Δ_1 is the union of three nonsingular rational curves of bidegree $(1,1)$, each pair intersect transversally.

Remark 6.2.8. Let

$$V = H^0(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2)).$$

It is a vector space of dimension 18. Let U be an open subset in $\mathbb{P}(V)$ which consists of sections $t_0^2 A_{11}(T) + 2t_0t_1A_{12}(T) + t_1^2 A_{22}(T)$ such that the corresponding determinant (6.16) defines a nonsingular quartic curve. The group $G = \mathrm{SL}(2)$ acts naturally on U via its action on $\mathbb{P}^1$ and the orbit space X is a cover of degree 63 of the space $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{ns}}$ of nonsingular plane quartics. The fibre over C_4 is naturally identified with the set of nonzero 2-torsion divisor classes on C_4. Since X is obviously irreducible and of dimension 14, we obtain that X is an irreducible unramified finite cover of degree 63 of $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{ns}}$. Let Z be the closed subset of U of sections such that the linear system of quadrics spanned by $A_{11}(T), A_{12}(T), A_{22}(T)$ contains a double line. Its image in $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{ns}}$ is a closed set. Thus a general quartic satisfies the assumption of Lemma 6.2.3 for any ϵ. I do not know whether there exists a nonsingular quartic which does not satisfy these assumptions for any ϵ.

If we further act on X by $G' = \mathrm{SL}(3)$ via its natural action on $\mathbb{P}^2$ we obtain the orbit space birationally isomorphic to the space $\mathcal{R}_3$ of isomorphism classes of genus 3 curves together with a nontrivial divisor class of order 2. This space is known to be rational. This was proven by me (unpublished, see the paper on my web page) and independently by P. Katsylo. This space is also birationally isomorphic to the space of bielliptic curves of genus 4 (see Exercise 7.10).

Let $F = A_1A_3 - A_2^2$ be an expression of F as a symmetric determinant. Consider a quadratic pencil of conics

$$Q(\lambda,\mu) := \lambda^2 A_1 + 2\lambda\mu A_2 + \mu^2 A_3 = 0. \tag{6.18}$$

Then the condition that $Q(\lambda,\mu)$ is tangent to C is the vanishing of the discriminant $D = -A_1A_3 + A_2^2$ on C. Since it is identically vanishes on C, we see that every conic from the pencil is tangent to our quartic C. Thus we obtain

Corollary 6.2.6. *A nonsingular plane quartic can be in 63 different ways represented as an evolute of a quadratic pencil of conics.*

Remark 6.2.9. A quadratic pencil of conics (6.18) can be thought as a subvariety X of $\mathbb{P}^1 \times \mathbb{P}^2$ given by a bi-homogeneous equation of bi-degree $(2,2)$. The projection to $\mathbb{P}^1$ is a conic bundle with 6 degenerate fibres corresponding to six pairs of bitangents in the Steiner complex corresponding to the pencil. The projection to $\mathbb{P}^2$ is a double cover branched along the quartic C. Later on we will identify X with the Del Pezzo surface of degree 2 associated to a nonsingulart plane quartic.

6.3 Even theta characteristics

6.3.1 Contact cubics

A nonsingular plane quartic has $36 = 2^2(2^3+1)$ even theta characteristic. None of them vanishes since C is not hyperelliptic. For any even theta characteristic ϑ the linear system $|K_C+\vartheta|$ defines a symmetric determinant expression for C. Let $\mathbb{P}^2 = \mathbb{P}(E)$ and $V^* = H^0(C, \mathcal{O}_C(K_C+\vartheta)$. Recall that the symmetric determinantal expression for C corresponding to ϑ defines a linear map

$$E \to S^2V^* \subset \mathrm{Hom}(V, V^*)$$

which, after projectivization, defines a linear map of projective spaces

$$s : \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^3}(2)|,$$

where $\mathbb{P}^3 = \mathbb{P}(V) = |K_C+\theta|^*$. The image of s is a net of quadrics $\mathcal{N}$ in $\mathbb{P}^3$ whose locus of singular quadrics is equal to C. The set of singular points of quadrics from $\mathcal{N}$ is a sextic model S of C, the image of C under a map given by the linear system $|K_C+\vartheta|$.

The pre-image under s of a hyperplane cuts out a divisor from $D \in |K_C+\theta|$. The divisor $2D \in |3K_C|$ is cut out by a unique cubic. This cubic is called a *contact cubic*. When we vary D in $|K_C+\theta|$ we get a 3-dimensional variety of contact cubics isomorphic to $\mathbb{P}^3$. Thus we obtain 36 irreducible families of contact cubics.

Remark 6.3.1. Consider the set of nets of quadrics in $\mathbb{P}^3$ as the Grassmannian $G(3,10)$ of 3-dimensional subspaces in $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. Let U be an open subset defining nets $\mathcal{N}$ of conics such that the locus of singular conics defines a nonsingular plane quartic curve $C \subset \mathcal{N}$ together with an even theta characteristic ϑ. The group $\mathrm{SL}(4)$ acts $G(3,10)$ via its natural action in $\mathbb{P}^3$. The orbit space is birationally isomorphic to the unramified cover of degree 36 of $\mathcal{M}_3$ parametrizing isomorphism classes of pairs (C,ϑ), where C is a nonsingular non-hyperelliptic curve of genus 3 and ϑ is an even theta characteristic. We will show later that this space is birationally isomorphic to $\mathcal{M}_3$.

6.3.2 Cayley octads

The image of s is a net $\mathcal{N}$ (i.e. two-dimensional linear system) of quadrics. Take a basis Q_1, Q_2, Q_3 of $\mathcal{N}$. The base locus of $\mathcal{N}$ is the complete intersection of these quadrics. One expects to get 8 distinct points. Let us see that this is indeed true.

Theorem 6.3.1. *The set of base points of the net of quadrics $\mathcal{N}$ consists of 8 distinct points, no three of which are collinear, no four are coplanar.*

Proof. Suppose three points are on a line ℓ. This includes the case when two points coincide. This implies that ℓ is contained in all quadrics from $\mathcal{N}$. Take a point $x \in \ell$. For any quadric $Q \in \mathcal{N}$, the tangent plane of Q at x contains the line ℓ.

Thus the tangent planes form a pencil of planes through ℓ. Since $\mathcal{N}$ is a net, there must be a quadric which is singular at x. Thus each point of ℓ is a singular point of some quadric from $\mathcal{N}$. However, the set of singular points of quadrics from $\mathcal{N}$ is equal to the nonsingular sextic S. This shows that no three points are collinear.

Suppose that 4 points lie in a plane π. Restricting quadrics from $\mathcal{N}$ to π defines a linear system of conics through 4 points no three of which are collinear. It is of dimension 1. Thus, there exists a quadric in $\mathcal{N}$ which contains π. However, since C is nonsingular all quadrics in $\mathcal{N}$ are of corank ≤ 1. $\square$

Definition 6.6. *A set of 8 distinct points in $\mathbb{P}^3$ which is a complete intersection of 3 quadrics is called a Cayley octad.*

Let $f : C \to S \subset \mathbb{P}^3$ be the map defined by the linear system $|K_C + \theta|$. Its image is a sextic model of C given by the right kernel of the matrix defining the determinantal equation.

Theorem 6.3.2. *Let $q_1, \ldots, q_8$ be the Cayley octad defined by the net of quadrics $\mathcal{N}$. Each line $\langle q_i, q_j\rangle$ intersects S at two point $f(p_i), f(p_j)$. The line $\langle p_i, p_j\rangle$ is a bitangent of C.*

Proof. Fix a point q on $\ell = \langle q_i, q_j\rangle$ different from q_i, q_j. Each quadric from $\mathcal{N}$ vanishing at q has 3 common points with ℓ. Hence it contains ℓ. Since vanishing at a point is one linear condition on the coefficients of a quadric, we obtain a pencil of quadrics in $\mathcal{N}$ such that ℓ is contained in its set of base points. Two quadrics intersect along a curve of degree 4. Thus the base locus of the pencil is a reducible curve of degree 4 which contains a line component. The residual curve is a twisted cubic. Take a nonsingular quadric in the pencil. Then the cubic is a curve of bidegree $(2, 1)$ and a line is a curve of bidegree $(0, 1)$. Thus they intersect at 2 points x, y (not necessary distinct). Any two nonsingular quadrics from the pencil do not intersect at these points transversally. Hence they have a common tangent plane. For each point, an appropriate linear combination of these quadrics will be singular at this point. The pencil does not have any other singular quadrics. Indded, a singular point of such a quadric must lie on ℓ and hence define a singular point of the base locus. So it must be one of the two singular points x, y. No two quadrics from the same pencil share a singular point since the set C of singular quadrics does not contain a line. This shows that $x, y \in S$ and the pencil of quadrics is equal to the image of a line l in $\mathbb{P}^2$ under the map s which intersects C at 2 points p_i, p_j such that $f(p_i) = x, f(p_j) = y$. Thus l is a bitangent. $\square$

Note that since $28 = \binom{8}{2}$ all bitangents are accounted for in the sextic model of C.

We can also see all even theta characteristics.

Theorem 6.3.3. *Let $q_1, \ldots, q_8$ be the Cayley octad associated to an even theta characteristic ϑ. Let ϑ_{ij} be the odd theta characteristics corresponding to the lines $\langle q_i, q_j \rangle$. Then any even theta characteristics different from ϑ can be represented by the divisor class*

$$\vartheta_{i,jkl} = \vartheta_{ij} + \vartheta_{ik} + \vartheta_{il} - K_C.$$

for some distinct i, j, k, l.

Proof. Suppose that $\vartheta_{i,jkl}$ is an odd theta characteristic ϑ_{mn}. Consider the plane π which contains the points q_i, q_j, q_k. It intersects S at six points corresponding to the theta characteristics $\vartheta_{ij}, \vartheta_{ik}, \vartheta_{jk}$. Since the planes cuts out divisors from $|K_C + \theta|$, we obtain

$$\vartheta_{ij} + \vartheta_{ik} + \vartheta_{jk} \sim K_C + \vartheta$$

This implies that

$$\vartheta_{jk} + \vartheta_{il} + \vartheta_{mn} \sim K_C + \vartheta.$$

Hence $\langle q_j, q_k \rangle$ and $\langle q_i, q_l \rangle$ lie in a plane π'. The intersection point of the lines $\langle q_j, q_k \rangle$ and $\langle q_i, q_l \rangle$ is a base point of two pencils in $\mathcal{N}$ and hence is a base point of $\mathcal{N}$. However, it does not belong to the Cayley octad. This contradiction proves the assertion. □

Remark 6.3.2. Note that

$$\vartheta_{i,jkl} = \vartheta_{j,ikl} = \vartheta_{k,ijl} = \vartheta_{l,ijk}.$$

Thus $\vartheta_{i,jkl}$ depends only on the choice of a subset of four in $\{1, \ldots, 8\}$. Also it is easy to check that the complementary sets define the same theta characteristic. This shows that we get $35 = \binom{8}{4}/2$ different even theta characteristics. Together with $\vartheta = \vartheta_\emptyset$ we obtain 36 even theta characteristics. Observe now that the notations ϑ_{ij} for odd thetas and $\vartheta_{ijkl}, \vartheta_\emptyset$ agrees with the notation we used for theta characteristics on curves of genus 3 in Chapter 6.

6.4 Automorphisms of plane quartic curves

6.4.1 Automorphisms of finite order

Since an automorphism of a nonsingular plane quartic curve C leaves K_C invariant, it is defined by a projective transformation.

Lemma 6.4.1. *Let g be an automorphism of order $n > 1$ of a nonsingular plane quartic $C = V(F)$. Then one can choose coordinates in such a way that a generator of the cyclic group (g) is represented by a diagonal matrix* $\mathrm{diag}[1, \zeta_n^a, \zeta_n^b]$, *where ζ_n is a primitive nth root of unity, and F is given in the following list.*

(i) $(n = 2), (a, b) = (0, 1)$,

$$T_2^4 + T_2^2 L_2(T_0, T_1) + L_4(T_0, T_1).$$

(ii) $(n = 3), (a, b) = (0, 1)$,

$$T_2^3 L_1(T_0, T_1) + L_4(T_0, T_1).$$

(iii) $(n = 3), (a, b) = (1, 2)$,

$$F = T_0^4 + \alpha T_0^2 T_1 T_2 + T_0 T_1^3 + T_0 T_2^3 + \beta T_1^2 T_2^2.$$

(iv) $(n = 4), (a, b) = (0, 1)$,

$$T_2^4 + L_4(T_0, T_1).$$

(v) $(n = 4), (a, b) = (1, 2)$,

$$T_0^4 + T_1^4 + T_2^4 + \alpha T_0^2 T_2^2 + \beta T_0 T_1^2 T_2.$$

(vi) $(n = 6), (a, b) = (3, 2)$,

$$T_0^3 + T_1^3 + \alpha T_0^2 T_1^2 + T_0 T_2^3.$$

(vii) $(n = 7), (a, b) = (3, 1)$,

$$T_0^3 T_1 + T_1^3 T_2 + T_2^3 T_0 + \alpha T_0 T_1^2 T_2.$$

(viii) $(n = 8), (a, b) = (3, 7)$,

$$T_0^4 + T_1^3 T_2 + T_1 T_2^3.$$

(ix) $(n = 9), (a, b) = (3, 2)$,

$$T_0^4 + T_0 T_1^3 + T_2^3 T_1.$$

(x) $(n = 12), (a, b) = (3, 4)$,

$$F = T_0^4 + T_1^4 + T_0 T_2^3.$$

Here the subscripts in polynomials L_i indicates the degree.

Proof. Let us first choose coordinates such that g acts by the formula $g:(x_0,x_1,x_2)\mapsto (x_0,\zeta_n^a x_1,\zeta_n^b x_2)$. We will frequently use that F is of degree ≥ 3 in each variable. This follows from the assumption that F is nonsingular.

Case 1: One of a,b is equal to zero. We may assume that $a=0$.
Write F as a polynomial in T_2.

$$F=\alpha T_2^4+T_2^3L_1(T_0,T_1)+T_2^2L_2(T_0,T_1)+T_2L_3(T_0,T_1)+L_4(T_0,T_1). \quad (6.19)$$

If $\alpha\neq 0$, we must have $4b=0 \mod n$. This implies that $n=2$ or 4. In the first case $L_1=L_3=0$, and we get case (i). If $n=4$, we must have $L_1=L_2=L_3=0$, and we get case (iv).

If $\alpha=0$, then $3b=0 \mod n$. This implies that $n=3$ and $L_2=L_3=0$. This gives case (ii).

Case 2: $0,a,b$ are all distinct. In particular, $n>2$. Note the case when $a=b\neq 0$ is reduced to Case 1 by scaling the matrix of the transformation and permuting the variables. Let $P_1=(1,0,0),P_2=(0,1,0),P_3=(0,0,1)$ be the reference points.

Case 2a: All reference points lie on the surface.

This implies that the degree of F in each variable is equal to 3. We can write F in the form

$$F=T_0^3A_1(T_1,T_2)+T_1^3B_1(T_0,T_2)+T_2^3C_1(T_0,T_1)+$$
$$T_0^2A_2(T_1,T_2)+T_1^2B_2(T_0,T_2)+T_2^2C_2(T_0,T_1).$$

Since F is invariant, it is clear that any T_i cannot enter in two different coefficients A_1,B_1,C_1. Without loss of generality, we may assume that

$$F=T_0^3T_1+T_1^3T_2+T_2^3T_0+T_0^2A_2(T_2,T_3)+T_1^2B_2(T_0,T_2)+T_2^2C_2(T_0,T_1).$$

Now we have $a=3a+b=3b \mod n$. This easily implies $n=7$ and we can take a generator of (g) such that $(a,b)=(3,1)$. By checking the eigenvalues of other monomials, we verify that only the monomial $T_0T_1^2T_2$ can enter in F. This is case (vii). If $A_2=B_2=C_2=0$, we get case (vii).

Case 2b: Two reference points lie on the surface.

By normalizing the matrix and permuting the coordinates we may assume that $P_1=(1,0,0)$ does not lie on C. Then we can write

$$F=T_0^4+T_0^2L_2(T_1,T_2)+T_0L_3(T_1,T_2)+L_4(T_1,T_2),$$

where T_1^4,T_2^4 do not enter in L_4.

Without loss of generality we may assume that $T_1^3T_2$ enters in L_4. This gives $3a+b=0 \mod n$. Suppose $T_1T_2^3$ enters in L_4. Then $a+3b=0 \mod n$. This gives $n=8$, and we may take a generator of (g) corresponding to $(a,b)=(3,7)$. This is case (viii). If $T_1T_2^3$ does not enter in L_4, then T_2^3 enters in L_3. This gives $3b=0 \mod n$. Together with $3a+b=0 \mod n$ this gives $n=3$ and we take g with $(a,b)=(1,2)$ or $n=9$ and $(a,b)=(3,2)$. These are cases (iii) and (ix).

Case 2c: One reference point lies on the surface.

By normalizing the matrix and permuting the coordinates we may assume that $P_1=(1,0,0), P_2=(0,1,0)$ do not lie on C. Then we can write

$$F = T_0^4 + T_1^4 + T_0^2L_2(T_1,T_2) + T_0L_3(T_1,T_2) + L_4(T_1,T_2),$$

where T_1^4, T_2^4 do not enter in L_4. This immediately gives $4a=0 \mod n$. Suppose T_2^3 enters in L_3. Then $3b=0 \mod n$, hence $n=6$ or $n=12$. It is easy to see that

$$F = T_0^3 + T_1^3 + \alpha T_0^2T_1^2 + T_0T_2^3.$$

If $n=6$, then $(a,b)=(3,2)$ and α may be different from 0. This is case (vi). If $n=12$, then $(a,b)=(3,4)$ and $\alpha=0$. This is case (x).

Case 2d: None of the reference point lies on the surface.

In this case

$$F = T_0^4 + T_1^4 + T_2^4 + T_0^2L_2(T_1,T_2) + T_0L_3(T_1,T_2) + \alpha T_1^3T_2 + \beta T_1T_2^3.$$

Obviously, $4a=4b=0 \mod n$. This gives $n=4$ and $(a,b)=(1,2),(1,3)$. These cases are isomorphic and give (v). □

6.4.2 Automorphism groups

Recall some standard terminology from the theory of linear groups. Let G be a subgroup of the general linear group GL(V) of a complex vector space of dimension n. The group G is called *intransitive* if the representation of G in GL(V) is reducible. Otherwise it is called *transitive*. The group G is called *imprimitive* if G contains a intransitive normal subgroup G'. In this case V decomposes into a direct sum of G'-invariant proper subspaces, and elements from G permutes them.

We employ the notation from [ATLAS]:a cyclic group of order n is denoted by n, the semi-direct product $A \rtimes B$ is denoted by $A:B$, a central extension of a group A with kernel B is denoted by $B.A$.

Theorem 6.4.2. *The following is the list of all possible groups of automorphisms of a nonsingular plane quartic.*

Type	Order	Structure	Equation	Parameters
I	168	$L_2(7)$	$T_0^3T_1+T_1^3T_2+T_2^3T_0$	
II	96	$4^2:S_3$	$T_0^4+T_1^4+T_2^4$	
III	48	$4.A_4$	$T_0^4+T_1T_2^3+T_1T_2^3$	
IV	24	S_4	$T_0^4+T_1^4+T_2^4+a(T_0^2T_1^2+T_0^2T_2^2+T_1^2T_2^2)$	$a\neq\frac{-1\pm\sqrt{-7}}{2}$
V	16	4×4	$T_0^4+\alpha(T_1^4+T_2^4)+\beta T_0^2T_2^2$	$\alpha,\beta\neq 0,$
VI	9	9	$T_0^4+T_0T_1^3+T_1T_2^3$	
VII	8	$2.D_2$	$T_0^4+\alpha T_0^2(T_1^2+T_2^2)+T_1^4+T_2^4+\beta T_1^2T_2^2$	$\alpha\neq\beta$
VIII	7	7	$T_0^3T_1+T_1^3T_2+T_2^3T_0+aT_0T_1^2T_2$	$a\neq 0$
IX	6	6	$T_0^4+aT_0^2T_1^2+T_1^4+T_1T_2^3$	$a\neq 0$
X	6	S_3	$T_0^4+\alpha T_0^2T_1T_2+T_0(T_1^3+T_2^3)+\beta T_1^2T_2^2$	$a\neq 0$
XI	4	2×2	$T_0^4+T_0^2(\alpha T_1^2+\beta T_2^2)+T_1^4+T_2^4+\gamma T_1^2T_2^2$	$\alpha\neq\beta$
XII	3	3	$T_0^4+\alpha T_0^2T_1T_2+T_0T_1^3+T_0T_2^3+\beta T_1^2T_2^2$	$\alpha,\beta\neq 0$
XIII	2	2	$T_0^4+T_0^2L_2(T_1,T_2)+L_4(T_1,T_2)$	$\not\cong$ VII,X,XI

Proof. Case 1. Let G be an intransitive group realized as a group of automorphisms of a nonsingular plane quartic. Since in our case $n=3$, V must be the direct sum of one-dimensional subspaces V_i, or a one-dimensional subspace V_1 and a 2-dimensional subspace V_2.

Case 1a. $V=V_1\oplus V_2\oplus V_3$.

Choose coordinates (T_0,T_1,T_2) such that V_1 is spanned by $(1,0,0)$ and so on. Let $g\in G$ be an element of order n. After scaling the action, we may assume that g acts with eigenvalue 1 in V_1 and F as in Lemma 6.4.1.

Assume $n=12$. Suppose $G\neq(g)$ and let g' be an element of order m not in (g). It acts via a diagonal matrix diag$[1,\zeta_m^p,\zeta_m^q]$. We have $4p=3q=0\mod m$. It is easy to see that this implies that m divides 12 and g' belongs to (g). We will see later than the full automorphism group in this case is bigger and leads to case III.

Similar argument shows in all cases (v)-(x), any automorphism leaving each V_i invariant belongs to (g).

Assume now that g is as in case (i). If $G=(g)$, we get Type XII. Suppose there exists $g'\notin(g)$. We choose represent it by a diagonal matrix diag$[\zeta_m^p,\zeta_m^q,1]$. Since L_2 is invariant with respect to g', we must have $m=2$. Thus G is generated by g and g' which acts by diag$[1,-1,1]$. The group G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The binary quartic L_4 does not contain monomials $T_0^3T_1,T_0T_1^3$ and L_2 does not contain T_0T_1. We get Type XI.

Assume g is as in case (ii). In this case $G=(g)$ as in Type XII since any

$g' \in G$ must leave L_1.

Assume g is as in case (iv). If $G = (g)$, we get Type XIII. Suppose there exists $g' \notin (g)$. Let

$$L_4 = aT_0^4 + bT_1^4 + cT_0^3T_1 + dT_0T_1^3 + eT_0^2T_1^2.$$

Assume $a \neq 0, b \neq 0$. Then $m = 4$ and we may assume that $(p,q) = (1,3), (1,2), (2,1)$. (otherwise $g' \in (g)$). If $(p,q) = (1,3)$, then g'^3 acts via $(p,q) = (3,1)$ and hence the monomials $T_0^3T_1$ and $T_0T_1^3$ are not invariant. This gives $c = d = 0$ and we obtain Type V. If $(p,q) = (1,2)$ or $(2,1)$, the monomials $T_0^3T_1, T_0T_1^3, T_0^2T_1^2$ are not invariant and we get $c = d = e = 0$. This is a special case of Type V.

Now assume that one of $a \neq 0, b = 0$. Then $d \neq 0$ and we get $4p = p+3q = 0 \mod m$. In this case we easily see that $c = e = 0$. This gives that $G = \mathbb{Z}/12\mathbb{Z}$ and we get case (x) considered before. Similar result is obtained if $a = 0, b \neq 0$.

Finally we assume that $a = b = 0$. Then $c, d \neq 0$ and we get $3p + q = p + 3q = 0 \mod m$. This gives $m = 8$, and we are reduced to the case (viii) considered before.

Case 1b. $V = V_1 \oplus V_2, \dim V_2 = 2$, where V_2 is an irreducible representation of G. In particular, G is not abelian.

Choose coordinates such that $(1,0,0) \in V_1$ and V_2 is spanned by $(0,1,0)$ and $(0,0,1)$. We choose to represent the restriction $\bar{g}$ of any $g \in G$ to $W = V(T_0) = \mathbb{P}(V_2)$ by an element from $\mathrm{SL}(2)$. We will freely use the well-known classification of finite subgroups of $\mathrm{SL}(2)$.

Write

$$F = \alpha T_0^4 + T_0^3L_1(T_1,T_2) + T_0^2L_2(T_1,T_2) + T_0L_3(T_1,T_2) + L_4(T_1,T_2).$$

If $L_1 \neq 0$, then V_2 is reducible contradicting the assumption. Thus $L_1 = 0$. Assume $L_2 \neq 0$, then G leaves invariant $V(L_2)$. Let $\bar{G} \subset \mathrm{Aut}(W)$ be the restriction of G to W. It follows from the classification of finite subgroups of $\mathrm{Aut}(\mathbb{P}^1)$ that this is possible only if $\bar{G}$ is a subgroup of the dihedral group $D_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$. We can change a basis to assume that $\bar{G}$ is generated by transformations $g_1 : (t_1,t_2) \mapsto (-t_2,t_1)$ and $g_2 : (t_1,t_2) \mapsto (it_1,-it_2)$. Then $L_2 = a(T_0^2 + T_1^2)$ and $L_4 = b(T_0^4 + T_1^4) + cT_0^2T_1^2$. The group G is generated by transformations $(t_0,t_1,t_2) \mapsto (-t_0,-t_2,t_1)$, $(t_0,t_1,t_2) \mapsto (it_0,it_1,-it_2)$ and $(t_0,t_1,t_2) \mapsto (t_0,-t_1,-t_2)$. It is isomorphic to the binary dihedral group $2.D_2$ (also isomorphic to the quaternion group of order 8). This is Type VII.

Assume $L_1 = L_2 = 0$ but $L_3 \neq 0$. The only non-abelian subgroup of $\mathrm{SL}(2)$ which leaves $V(L_3)$ invariant is the dihedral group D_3 isomorphic to the symmetric group S_3. The locus of zeros of L_3 must consist of three distinct points in $\mathbb{P}^1$

forming an orbit. Choose coordinates such that $L_3 = T_1T_2(-T_1 + T_2)$, where the image of G in Aut($\mathbb{P}^1$) is generated by the transformation $g_1 : (t_1, t_2) \mapsto (t_2, -t_1 + t_2)$ of order 3, and the transformation $g_2 : (t_1, t_2) \mapsto (it_2, it_1)$ of order 2. It is directly verified that L_4 must be the square of a quadratic polynomial with zeros equal to the fixed points of g_1. The corresponding curve C is singular, so this case does not occur.

Finally we can consider the case $L_1 = L_2 = L_3 = 0$. The group $\bar{G}$ must leave $V(L_4)$ invariant. This easily implies that $\bar{G}$ is the tetraedral group T of order 12 (isomorphic to the alternating group A_4). We choose coordinates in W such that $\bar{G}$ is generated by an element of g_1 of order 2 and an element g_2 of order 3 represented by the unimodular matrices

$$S = \frac{1}{\sqrt{-3}}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}, \quad T = \begin{pmatrix}\rho & 0\\ 0 & \rho^2\end{pmatrix}, \ \rho = e^{2\pi i/3}.$$

The only invariant set of 4 distinct points in $\mathbb{P}^1$ is $V(L_4)$, where $L_4 = T_1(T_1^3+T_2^3)$ (an *anharmonic binary quartic*). The action of the subgroup $\bar{G}'$ of $\mathrm{SL}(2)$ generated by S and T on L_4 is given by the character $\chi : \bar{G}' \to \mu_3$ defined by $\chi(S) = 1, \chi(T) = \rho$. Thus G is equal to the group

$$G = \{(\alpha, g) \in \mu_{12} \times \bar{G}' : \alpha^4 = \chi(g)\}/(\pm 1).$$

Projecting to the second factor we see that G is isomorphic to $4.A_4$. This is Type III. Observe that, after switching the coordinates T_0 and T_1, the curve becomes isomorphic to the curve from case (x) of Lemma 6.4.1. This isomorphism is equivariant with respect to a cyclic subgroup of G of order 12. There are 4 such subgroups in G.

Case 3. The group G contains a normal transitive imprimitive subgroup H. The group H contains a subgroup from case 1 and the quotient by this subgroup permutes cyclically the coordinates. It follows from the list in Lemma 6.4.1 that it can happen only if

$$F = F = T_0^4 + \alpha T_0^2T_1T_2 + T_0(T_1^3 + T_2^3) + \beta T_1^2T_2^2 \qquad (6.20)$$
$$F = T_0^3T_1 + T_1^3T_2 + T_2^3T_0 \qquad (6.21)$$
$$F = T_0^4 + T_1^4 + T_2^4 \qquad (6.22)$$
$$F = T_0^4 + T_1^4 + T_2^4 + a(T_0^2T_1^2 + T_0^2T_2^2 + T_1^2T_2^2). \qquad (6.23)$$

In the first curve we have the additional automorphism of order 2 interchanging T_1 and T_2. This gives Type X.

The second curve is the *Klein quartic* which will be discussed in the next section.

The thired curve is the *Fermat quartic*. Its Hessian is equal to $V(T_0^2T_1^2T_2^2)$. Obviously it is invariant with respect to Aut(C). It is easy to see that the group of projective transformation leaving the the coordinate triangle $V(T_0T_1T_2)$ invariant is generated by permutations of coordinates and scaling of coordinate. This easily implies that Aut(C) is a group of order 96 isomorphic to the semi-direct product $(\mathbb{Z}/4\mathbb{Z})^4 \ltimes S_3$. This is Type II.

In the forth case Aut(C) permutations and sign changes of coordinates. it is easy to see that this defines a subgroup G of Aut(C) of order 24. It acts by permutations on the set of 4 bitangents $V(T_0 \pm T_1 \pm T_2)$ of C. This easily shows that G is isomorphic to the permutation group S_4 (or the octahedron group). One can show that the full automorphism group of the curve coincides with S_4 unless $a = \frac{1}{2}(-1 \pm \sqrt{7}$. This is Type IV. In the latter case the curve is isomorphic to the Klein curve (see [Fricke, B.2].

Case 4. G is a simple group.

Here we use the classification of simple nonabelian finite subgroups of PGL(3) (see [Blichfeldt]). There are only two transitive simple groups. One is the group $G$of order 168 isomorphic to the group of automorphisms of the Klein quartic. It contains an element g of order 7 and element of order 3 from the normalizer of the group (g). Thus G contains a imprimitive subgroup of order divisible by 7. It follows from the previous classification that C must be as in case (x) with $\alpha = 0$, so it is the Klein quartic. This is Type I.

The other group is the *Valentiner group* of order 360 isomorphic to the alternating group A_6. It is known that latter group does not admit a 3-dimensional linear representation (a certain central extension of degree 3 does). Since any automorphism group of a plane quartic acts on the 3-dimensional linear space $H^0(C, \omega_C)$ the Valentiner group cannot be realized as an automorphism group of a plane quartic.

□

6.4.3 The Klein quartic

Recall the following well-known result of A. Hurwitz.

Theorem 6.4.3. *Let X be a nonsingular connected projective curve of genus $g > 1$. Then*

$$\#\text{Aut}(X) \leq 84(g-1).$$

Proof. See [Hartshorne], Chapter IV. ex. 2.5. □

For $g = 3$, the bound gives $\#\text{Aut}(X) \leq 168$ and it is achieved for the Klein quartic

$$C = V(T_0^3 T_1 + T_1^3 T_2 + T_2^3 T_0).$$

Recall that we know that its group of automorphisms contains an element S of order 7 acting by the formula

$$S : (t_0, t_1, t_2) \mapsto (t_0 \epsilon, \epsilon^2 t_1, \epsilon^4 t_2), \quad \epsilon = e^{2\pi i/7},$$

where we scaled the action to represent the transformation by a matrix from SL(3). Another obvious symmetry is an automorphism G_2 of order 3 given by cyclic permutation U of the coordinates. It is easy to check that

$$U^{-1} S U = S^2, \tag{6.24}$$

so that the subgroup generated by S, U is a group of order 21 isomorphic to the semi-direct product $7 : 3$.

By a direct computation one checks that the unimodular matrix defines an automorphism T of C of order 2:

$$\frac{i}{\sqrt{7}} \begin{pmatrix} \epsilon - \epsilon^6 & \epsilon^2 - \epsilon^5 & \epsilon^4 - \epsilon^3 \\ \epsilon^2 - \epsilon^5 & \epsilon^4 - \epsilon^3 & \epsilon - \epsilon^6 \\ \epsilon^4 - \epsilon^3 & \epsilon - \epsilon^6 & \epsilon^2 - \epsilon^5 \end{pmatrix} \tag{6.25}$$

We have

$$TUT = U^2, \tag{6.26}$$

so that the subgroup generated by U, T is the dihedral group of order 6. One checks that the 49 products $S^a T S^b$ are all distinct. In particularly the cyclic subgroup (S) is not normal in the group G generated by S, T, U. Since the order of G is divisible by $2 \cdot \cdot 3 \cdot 7 = 42$, we see that $\#G = 42, 84, 126, or 168$. It follows from the Sylow theorem that the subgroup group (S) must be normal in the first three cases, so $\#G = 168$, and by Hurwitz's theorem

$$\text{Aut}(C) = G = \langle S, U, T \rangle.$$

Lemma 6.4.4. *The group $G = \text{Aut}(C)$ is a simple group of order 168.*

Proof. Suppose H is a nontrivial normal subgroup of G. Assume that its order is divisible by 7. Since its Sylow 7-subgroup cannot be normal in H, we see that H contains all Sylow 7-subgroups of G. By Sylow's Theorem, their number is equal to 8. This shows that $\#H = 56$ or 84. In the first case H contains a Sylow 2-subgroup of order 8. Since H is normal, all its conjugates are in H, and, in particular, $T \in H$. The quotient group G/H is of order 3. It follows from (6.26) that the coset of U must be trivial. Since 3 does not divide 56, we get a contradiction. In the second case, H contains S, T, U and hence coincide with G. So, we have shown that H cannot contain an element of order 7. Suppose it contains an element of order 3. Since all such elements are conjugate, H contains U. It follows from (6.24), that the coset of S in G/H is trivial, hence $S \in H$ contradicting the assumption. It remains to consider the case when H is a 2-group. Then $\#G/H = 2^a \cdot 3 \cdot 7$, with $a \leq 2$. It follows from Sylow's Theorem that the image of the Sylow 7-subgroup in G/H is normal. Thus its pre-image in G is normal. This contradiction finishes the proof that G is simple.

□

Remark 6.4.1. One can show that

$$G \cong \mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{PSL}_3(\mathbb{F}_2).$$

The first isomorphism has a natural construction via the theory of automorphic functions. The Klein curve is isomorphic to a compactification of the modular curve $X(7)$ corresponding to the principal congruence subgroup of of level 7 (see [Modular forms]). The second isomorphism has a natural construction via considering a model of the Klein curve over a finite field of 2 elements (see [Eightfold]).

Exercises

6.1 Show that two syzygetic tetrads of bitangents cannot have two common bitangents.

6.2 Let $C_t = V(tF + Q^2)$ be a family of plane quartics over $\mathbb{C}$ depending on a parameter t. Assume that $V(F)$ is nonsingular and $V(F)$ and $V(Q)$ intersect transversally at 8 points $p_1, \ldots, p_8$. Show that C_t is nonsingular for all t in some open neighborhood of 0 in usual topology and the limit of 28 bitangents when $t \to 0$ is equal the set of 28 lines which joins pairs of intersection points p_i, p_j.

6.3 Show that the locus of nonsingular quartics which admit a flex bitangent is a hypersurface in the space of all nonsingular quartics.

6.4 Consider the Fermat quartic $V(T_0^4 + T_1^4 + T_2^4)$. Find all bitangents and all Steiner complexes. Show that it admits 12 flex bitangents.

6.4 An open problem: what is the maximal possible number of flex bitangents on a nonsingular quartic?

6.5 Show that the threefold $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ defined in (6.14) is nonsingular.

6.6 Show that a nonsingular plane quartic C admits 63 conics which are tangent to C at 4 points.

6.7 Let $S = \{(\ell_1, \ell_1), \ldots, \ell_6, \ell_6')\}$ be a Steiner complex of 12 bitangents. Prove that the six intersection points $\ell_i \cap \ell_i'$ lie on a conic. [Hint: the conic is the Veronese curve from Remark 6.2.4].

6.8 Show that the linear system $L(\epsilon)$ of conics associated to a nonzero 2-torsion divisor class is equal to the linear system of first polars of the cubic $B(\epsilon)$.

6.9 Show that a choice of $\epsilon \in \text{Jac}(C)[2]$ defines a conic Q and a cubic B such that C is equal to the locus of points x such that the polar $P_x(B)$ is touching Q.

6.10 A nonsingular curve is called *bielliptic* if it admits a double cover to an elliptic curve. Show that the moduli space of bielliptic curves of genus 4 is birationally isomorphic to the moduli space of isomorphism classes of genus 3 curves together with a nonzero 2-torsion divisor class.

6.11 Let C be the Klein quartic. For any subgroup H of its automorphism group of C determine the genus of H and the ramification scheme of the cover $C \to C/H$.

6.12 Analyze the action of the automorphism group of the Klein quartic C on the set of even theta characteristics. Show that there is only one which is invariant with respect to the whole group. Find the corresponding determinantal representation of C.

Chapter 7

Planar Cremona transformations

7.1 Homaloidal linear systems

7.1.1 Cremona transformations

A birational map $T : \mathbb{P}^n - \to \mathbb{P}^n$ is called a *Cremona transformation*. As any rational map, it is given by a n-dimensional linear system $L \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ for some $d \geq 1$. A choice of a basis gives an explicit formula:

$$T : (x_0, \ldots, x_n) \mapsto (P_0(x_0, \ldots, x_n), \ldots, P_n(x_0, \ldots, x_n)),$$

where $P_i(T_0, \ldots, T_n)$ are homogeneous polynomials of degree d. A linear system defining a Cremona transformation is called a *homaloidal linear system*. Let

$$B_L = \bigcap_{D \in L} D \tag{7.1}$$

be the *base locus* of L or T, considered as a closed subscheme of $\mathbb{P}^n$. By canceling the polynomials P_i's by the common divisor, we assume that B_L has no divisorial components. Let $\pi : X \to \mathbb{P}^n$ be the blow-up of the subscheme B_T. There is a morphism $f : X \to \mathbb{P}^n$ such that $f = T \circ \pi$ as rational maps, i.e. the following diagram is commutative

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow f \\ \mathbb{P}^n & \overset{T}{- \to} & \mathbb{P}^n \end{array} \tag{7.2}$$

In fact, let $L = \mathbb{P}(L')$, where $L' \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)$, and let $\mathcal{I}$ be the ideal sheaf defining B_L. Then there is a canonical surjection of the trivial bundle

$$\mathcal{O}_{\mathbb{P}^n} \otimes L' \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^n}(d).$$

This defines a surjection of the symmetric Algebras $\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^n}(-d) \otimes L') \to \mathrm{Sym}(\mathcal{I})$ and hence a closed embedding

$$X \hookrightarrow \mathrm{Proj}(\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^n}(-d) \otimes L') \cong \mathbb{P}^n \times \mathbb{P}(L')^*.$$

The composition with the second projection is our map f. It follows from this that

$$f^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \pi^*(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}_X(-E), \tag{7.3}$$

where the image of $\pi^*(\mathcal{I})$ in $\mathcal{O}_X$ is the ideal sheaf $\mathcal{O}_X(-E)$ of some effective divisor E on X (the *exceptional divisor* of π). Since f is birational, $f_*(\mathcal{O}_X) = \mathcal{O}_{\mathbb{P}^n}$, and hence by the projection formula

$$f_*(f^*(\mathcal{O}_{\mathbb{P}^n}(1))) \cong \mathcal{O}_{\mathbb{P}^n}(1) \otimes f_*(\mathcal{O}_X) \cong \mathcal{O}_{\mathbb{P}^n}(1).$$

This implies that

$$H^0(X, f^*(\mathcal{O}_{\mathbb{P}^n}(1))) \cong H^0(X, f_*(f^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)).$$

This shows that the morphism f is given by the complete linear system

$$|f^*(\mathcal{O}_{\mathbb{P}^n}(1))| = |\pi^*(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_X(-E)|.$$

The map T is birational, if and only if f is a birational morphism. Assume X is nonsingular. This can be achieved by composing π with resolution of singularities. Let H be a hyperplane in $\mathbb{P}^n$ so that $\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(H)$. Then T is birational if and only if

$$f^*(H)^n = (\pi^*(dH) - E)^n = 1. \tag{7.4}$$

7.1.2 Exceptional configurations

From now on we assume that $n = 2$. In this case a homaloidal linear system is called a homaloidal net.

We know (see [Hartshorne]) that any birational morphism of nonsingular projective surfaces is a composition of blow-ups with centers at a closed point. Let

$$\pi : X = X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2 \tag{7.5}$$

be such a composition. Here $\pi_i : X_i \to X_{i-1}$ is the blow-up of a point $x_i \in X_{i-1}$. Let

$$E_i = \pi_i^{-1}(x_i), \quad \mathcal{E}_i = (\pi_{i+1} \circ \dots \pi_N)^{-1}(E_i). \tag{7.6}$$

Define inductively the numbers m_i and the linear systems L_i on X_i as follows. First we set

$$m_1 = \min_{D \in L} \text{mult}_{x_1} D$$

The linear system $\pi_1^*(L)$ on X_1 has the divisor $m_1 E_1$ as a fixed component. Let

$$L_1 = \pi_1^*(L) - m_1 E_1.$$

This is a linear system on X_1 without fixed components. Suppose $m_1, \ldots, m_i$ and $L_1, \ldots, L_i$ have been defined. Then we set

$$m_{i+1} = \min_{D \in L_i} \text{mult}_{x_{i+1}} D,$$

$$L_{i+1} = \pi_{i+1}^*(L_i) - m_i E_{i+1}.$$

It follows from the definition that

$$L_N = \pi^*(L) - \sum_{i=1}^N m_i \mathcal{E}_i \subset |\pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_X(-\sum_{i=1}^N m_i \mathcal{E}_i)|$$

has no fixed components and defines a rational map $T_N : X - \to \mathbb{P}^2$ which coincides with the regular map f on the complement of the divisor $\mathcal{E}_1 + \ldots + \mathcal{E}_N$. By commutativity of the diagram (7.2), the divisors of the linear system $|L_N|$ coincide with divisors of the complete linear system $|f^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ on an Zariski open set. Since L_N has no fixed divisorial components, a general divisor from L_N must coincide with a divisor from $|f^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Thus $L_N \subset |f^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ and since they have the same dimension, we obtain that

$$L_N = |f^*(\mathcal{O}_{\mathbb{P}^n}(1))|.$$

In particular, we have

$$f^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_X(-\sum_{i=1}^N m_i \mathcal{E}_i)$$

and hence

$$E = \sum_{i=1}^N m_i \mathcal{E}_i.$$

Lemma 7.1.1.

$$\mathcal{E}_i \cdot \mathcal{E}_j = -\delta_{ij},$$

$$\mathcal{E}_i \cdot K_X = -1.$$

Proof. This follows from the standard properties of the intersection theory on surfaces. For any morphism of nonsingular projective surfaces $\phi : S' \to S$ and two divisors D, D' on S, we have

$$\phi^*(D) \cdot \phi^*(D') = \deg(\phi) D \cdot D'. \tag{7.7}$$

Also, if C is a curve such that $\phi(C)$ is a point, we have

$$C \cdot \phi^*(D) = 0. \tag{7.8}$$

Applying (7.7), we have

$$-1 = E_i^2 = (\pi_{i+1} \circ \ldots \circ \pi_N)^*(E_i)^2 = \mathcal{E}_i^2.$$

Assume $i < j$. Applying (7.8) by taking $C = E_j$ and $D = (\pi_{i+1} \circ \ldots \circ \pi_{j-1})^*(E_i)$, we obtain

$$0 = E_j \cdot \pi_j^*(D) = (\pi_{j+1} \circ \ldots \circ \pi_N)^*(E_j) \cdot (\pi_{j+1} \circ \ldots \circ \pi_N)^*(D) = \mathcal{E}_j \cdot \mathcal{E}_i.$$

This proves the first assertion.

To prove the second assertion, we use that

$$K_{X_{i+1}} = \pi_i^*(K_{X_i}) + E_i.$$

By induction, this implies that

$$K_X = \pi^*(K_{X_0}) + \sum_{i=1}^{N} \mathcal{E}_i. \tag{7.9}$$

Intersecting with both sides and using (7.8), we get

$$K_X \cdot \mathcal{E}_j = \big(\sum_{i=1}^{N} \mathcal{E}_i\big) \cdot E_j = \mathcal{E}_j^2 = -1.$$

□

Definition 7.1. *The divisors $\mathcal{E}_i$ are called the* exceptional configurations *of the resolution $\pi : X \to \mathbb{P}^2$ of the birational map T.*

Corollary 7.1.2. *We have the following equalities.*

(i) $d^2 - E^2 = d^2 - \sum_{i=1}^{N} m_i^2 = 1$.

(ii) $-3d + \sum_{i=1}^{N} m_i^2 = -3$.

Proof. Let $\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{\mathbb{P}^2}(l)$, where ℓ is a line. We know that the linear system $|\pi^*(d\ell) - \sum_{i=1}^N m_i\mathcal{E}_i|$ defines a birational morphism $f : X \to \mathbb{P}^2$. It follows from (8.3.1) that

$$1 = (\pi^*(d\ell) - \sum_{i=1}^N m_i\mathcal{E}_i)^2 = d^2 + (\sum_{i=1}^N m_i\mathcal{E}_i)^2$$

$$= d^2 - E^2 = d^2 - \sum_{i=1}^N m_i^2.$$

Since the linear system $|\pi^*(d\ell) - E|$ defines a regular map, by Bertini's Theorem, its general member D is nonsingular. Since D is equal to the pre-image of a line under f it must be a smooth rational curve. We have

$$K_X = \pi^*(K_{\mathbb{P}^2} + \sum_{i=1} \mathcal{E}_i = \pi^*(-3\ell) + \sum_{i=1} \mathcal{E}_i.$$

By adjunction,

$$K_X \cdot (\pi^*(d\ell) - E) = -3d + \sum_{i=1}^N m_i = -3.$$

□

7.1.3 The bubble space

Let S be a nonsingular projective surface and $B(S)$ be the category of birational morphisms $\pi : S' \to S$ of nonsingular projective surfaces. Recall that a morphism from $(S' \stackrel{\pi'}{\to} S)$ to $(S'' \stackrel{\pi''}{\to} S)$ in this category is a regular map $\phi : S' \to S''$ such that $\pi'' \circ \phi = \pi'$.

Definition 7.2. *A* bubble space S^{bb} *of a nonsingular surface* S *is the factor set*

$$S^{\text{bb}} = \Big(\bigcup_{(S' \stackrel{\pi'}{\to} S \in B(S)} S' \Big)/R,$$

where R *is the following equivalence relation:* $x' \in S'$ *is equivalent to* $x'' \in S''$ *if the rational map* $\pi''^{-1} \circ \pi' : S' - \to S''$ *maps isomorphically an open neighborhood of* x' *to an open neighborhood of* x''.

It is clear that for any $\pi : S' \to S$ from $B(S)$ we have an injective map $i_{S'} : S' \to S^{\text{bb}}$. We will identify points of S' with their images. If $\phi : S'' \to S'$

is a morphism in $B(S)$ which is isomorphic in $B(S')$ to the blow-up of a point $x' \in S'$, any point $x'' \in \phi^{-1}(x')$ is called *infinitely near point* to x' of the first order. This is denoted by $x'' \succ x'$. By induction, one defines an infinitely near point of order k, denoted by $x'' \succ_k x'$. This defines a partial order on S^{bb}.

We say that a point $x \in S^{\mathrm{bb}}$ is of height k, if $x \succ_k x_0$ for some $x_0 \in S$. This defines the *height function* on the bubble space

$$\mathrm{ht}_S : S^{\mathrm{bb}} \to \mathbb{N}.$$

Clearly, $S = \mathrm{ht}^{-1}(0)$.

Let $\mathbb{Z}^{S^{\mathrm{bb}}}$ be the free abelian group generated by the set S^{bb}. Its elements are integer valued functions on S^{bb} with finite support. They added up as functions with values in $\mathbb{Z}$. We write elements of $\mathbb{Z}^{S^{\mathrm{bb}}}$ as finite linear combinations $\sum m(x)x$, where $x \in S^{\mathrm{bb}}$ and $m(x) \in \mathbb{Z}$ (similar to divisors on curves). Here $m(x)$ is the value of the corresponding function at x.

Definition 7.3. *A* bubble cycle *is an element $\eta = \sum m(x)x$ of $\mathbb{Z}^{S^{\mathrm{bb}}}$ satisfying the following additional properties*

(i) $m(x) \geq 0$ *for any* $x \in S^{\mathrm{bb}}$*;*

(i) *if* $m(x) \neq 0$ *and* $x \succ x'$ *then* $m(x') \neq 0$*;*

(ii) $x \succ x' \Rightarrow m(x) \leq m(x')$.

We denote the subgroup of bubble cycles by $\mathcal{Z}(S^{\mathrm{bb}})$.

Clearly, any bubble cycle η can be written in a unique way as a sum of bubble cycles Z_k such that the support of η_k is contained in $\mathrm{ht}^{-1}(k)$.

We can describe a bubble cycle by a weighted graph, called the *Enriques diagram*, by assigning to each point from its support a vertex, and joining two vertices by an ordered edge if one of the points is infinitely near to another point of the first order. the edge points to the point of lower height. We weight each vertex by the corresponding multiplicity. It is clear that the Enriques diagram is a tree.

Let $\xi = \sum m_x x$ be a bubble cycle. We order the points from the support of η such that $x_i \succ x_j$ implies $j < i$. We write $\xi = \sum_{i=1}^N m_i x_i$. Then we represent x_1 by a point on S and define $\pi_1 : S_1 \to S$ to be the blow-up of S with center at x_1. Then x_2 can be represented by a point on S_1 as either infinitely near of order 1 to x_1 or as a point equivalent to a point on S. We blow up x_2. Continuing in this way, we get a sequence of birational morphisms:

$$\pi : S_\xi = S_N \xrightarrow{\pi_N} S_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S, \tag{7.10}$$

where $\pi_{i+1} : S_{i+1} \to S_i$ is the blow-up of a point $y_i \in S_i$. Clearly, the bubble cycle η is equal to the bubble cycle $\sum_{i=1}^N m_i y_i$.

Let $\mathcal{L} \cong \mathcal{O}_S(D)$ be a line bundle on S and η be a bubble cycle. Order it as above and consider the sequence of birational morphisms (7.10). Let E_i and $\mathcal{E}_i$ be defined as in section 8.3.2. Let

$$|D - \eta| = \{D' \in |D| : \pi^*(D) - \sum_{i=1}^N m_i \mathcal{E}_i \geq 0\}.$$

This is a linear subsystem of $|D|$. Its elements satisfy the following linear conditions on divisors from $|D|$. For any $x \in \eta$ with $\mathrm{ht}(x) = 0$ we must have $\mathrm{mult}_x(D) \geq m(x)$. This condition depends only on the equivalence class of x. Let $y \in \eta$ with $\mathrm{ht}(y) = 1$ and $y \succ x$ for some $x \in \eta$. Then we must have $\mathrm{mult}_y(\phi^*(D) - m(x)E) \geq m_y$, where y is represented by a point on the exceptional curve E of the blow-up $\phi : S' \to S$ with center at x. Then we go to level 2 and so on.

Definition 7.4. *A bubble cycle $\eta \in \mathcal{Z}(\mathbb{P}^{2\mathrm{bb}})$ is called homaloidal of degree d if the linear system $|d\ell - \eta|$ is homaloidal.*

Theorem 7.1.3. *A bubble cycle $\eta = \sum_{i=1}^N m_i x_i$ is homaloidal of degree d if and only if $|d\ell - \eta|$ contains an irreducible divisor and the following numerical conditions are satisfied:*

$$d^2 - \sum_{i=1}^N m_i^2 = 1 \tag{7.11}$$

$$3d - \sum_{i=1}^N m_i = 3 \tag{7.12}$$

Proof. We have already proved the necessity of the conditions. Let $\pi : X = (\mathbb{P}^2)_\eta \to \mathbb{P}^2$ be the birational morphism which we constructed for any bubble cycle. By assumtion, the linear system $|\pi^*(d\ell) - \sum m_i \mathcal{E}_i|$ contains an irreducible divisor D. The arithmetical conditions give $D^2 = 1$ and $D \cdot K_X = -3$. By Riemann-Roch and Serre's duality,

$$h^0(D) \geq \tfrac{1}{2}(D^2 - D \cdot K_X) + 1 - h^0(K_X - D) = 3 - h^0(K_X - D).$$

Since X is a rational surface, $|K_X| = \emptyset$, and hence $h^0(K_X - D) = 0$. Thus $n = \dim |D| \geq 2$. Consider a rational map $\phi : X - \to \mathbb{P}^n$ defined by the linear system $|D|$. Since D is irreducible and $D^2 = 1$, we have at most one base point and any divisor from D has an ordinary point at the base point. Suppose, there is a base point. Let $p : X' \to X$ be the blow-up with center at the base point and E be the exceptional curve. The linear system $|D'| = |p^*(D) - E|$ has no base points and defines a regular map $f : X' \to \mathbb{P}^n$ which resolves ϕ. Since $D'^2 = 0$, the image of f is a curve C, and the pre-image of a general hyperplane H under f is equal to the pre-image of $H \cap C$. Since a general member of $|D'|$ is irreducible, $\deg(C) = \#(C \cap H) = 1$. Then C is a $\mathbb{P}^1$ embedded linearly, and hence $n = 1$. This contradiction proves the assertion. $\square$

The vector $(d : m_1, \ldots, m_N)$ is called the *characteristic* of the homaloidal net. It is defined up to an order of the numbers m_i. One usually orders points of height 0 with non-increasing coefficients $m_i's$, then order the points of height 1 in the same way, and so on.

Remark 7.1.1. There are further necessary conditions for the charcateristic $(d; m_1, \ldots, m_N)$ for a homaloidal net. For example, if m_1, m_2 correspond to points of height 0 of largest multiplicity, a line through the points should intersect a general member of the net non-negatively. This gives the inequality

$$d \geq m_1 + m_2.$$

Next we take a conic through 5 points with maximal multiplicities. We get

$$2d \geq m_1 + \ldots + m_5.$$

Then we take cubics through 9 points, quartics through 14 points and so on. The first case which can be ruled out in this way is $(5; 3, 3, 1, 1, 1, 1)$. It satisfies the equalities from the theorem but does not satisfy the condition $m \geq m_1 + m_2$.

As we will prove in the next chapter, the number of solutions of the equations (**??**) for fixed N is finite for $N \leq 8$ and is infinite for $N \geq 9$.

Example 7.1.1. Take $d = 2$. We find $\sum m_i^2 = 1, \sum_i m_i = 3$. This easily implies $m_1 = m_2 = m_3 = 1, N = 3$. The birational transformation of this type is called a *quadratic transformation*. The homaloidal linear system consists of conics passing through a bubble cycle $x_1 + x_2 + x_3$.

Assume $\eta = \eta_0$. The three points are not collinear, since otherwise all conics have a common line component. Let g be a projective transformation which sends the points x_1, x_2, x_3 to the points $p_1 = (0, 0, 1), p_2 = (0, 1, 0), p_3 = (1, 0, 0)$. Then $T \circ g^{-1}$ is given by the linear system of conics through the points p_1, p_2, p_3. We can choose a basis formed by the conics $V(T_1T_2), V(T_0T_2), V(T_0T_1)$. The corresponding Cremona transformation is given by the formula

$$T_1 : (t_0, t_1, t_2) \mapsto (t_1t_2, t_0t_2, t_0t_1). \tag{7.13}$$

In affine coordinates $Z_1 = T_1/T_0, Z_2 = T_2/T_0$, the transformation is given by

$$(x, y) \mapsto (1/x, 1/y).$$

Thus any quadratic transformation with no infinitely near base points is equal to $g \circ T_1 \circ g'$ for some projective transformations g, g'. Note that

$$T_1 \circ T_1 : t_0, t_1, t_2) \mapsto (t_0t_2t_0t_1, t_1t_2t_0t_1, t_1t_2t_0t_2) = t_0t_1t_2(t_0, t_1, t_2) = (t_0, t_1, t_2).$$

Thus T_1 is an involutive transformation. However, in general, $g \circ T_1 \circ g'$ is not an involution.

Assume now that $x_3 \succ_1 x_1$. Again, by a linear change of variables we may assume that $x_1 = (0, 0, 1), x_2 = (1, 0, 0)$ and x_2 corresponds to the tangent direction $T_0 = 0$. The homaloidal linear system consists of conics which pass through x_1, x_3 and have a common tangent $T_0 = 0$ at x_1. We can take a basis formed by the conics $V(T_0T_2), V(T_0T_1), V(T_1^2)$. The corresponding Cremona transformation is given by the formula

$$T_2 : (t_0, t_1, t_2) \mapsto (t_1^2, t_0t_1, t_0t_2). \tag{7.14}$$

Any quadratic transformation with one infinitely near base point is equal to $g \circ T_2 \circ g'$ for some projective transformations g, g'.

In affine coordinates $Z_1 = T_1/T_0, Z_2 = T_2/T_0$, the transformation is given by

$$(x, y) \mapsto (1/x, y/x^2).$$

Assume now that $x_3 \succ x_2 \succ x_1$. By a linear change of variables we may assume that $x_1 = (0, 0, 1)$, x_2 corresponds to the tangent direction $T_0 = 0$, and x_3 lies on the proper transform of the line $T_2 = 0$. The homaloidal linear system consists of conics which pass through x_1 and have a common tangent $T_0 = 0$, and after the blow-up x_1 still intersect at one point. We can take a basis formed by the conics $V(T_0T_2 - T_1^2), V(T_0^2), V(T_0T_1)$. The corresponding Cremona transformation is given by the formula

$$T_3 : (t_0, t_1, t_2) \mapsto (t_0^2, t_0t_1, t_1^2 - t_0t_2). \tag{7.15}$$

Any quadratic transformation with one infinitely near base point is equal to $g \circ T_2 \circ g'$ for some projective transformations g, g'.

In affine coordinates $Z_1 = T_1/T_0, Z_2 = T_2/T_0$, the transformation is given by

$$(x, y) \mapsto (x, x^2 - y).$$

Note that the case $x_2 \succ_1 x_1, x_3 \succ_1 x_1$ is not realized since a general member of the linear system is nonsingular at x_1.

It is easy to see each standard Cremona transformation T_i is involutive, i.e., satisfies $T_i^2 = \text{id}$.

Definition 7.5. *A Cremona transformation T_i is called the standard quadratic transformation of type i.*

Example 7.1.2. Assume that η consists of point taken with equal multiplicity m. Then the necessary conditions are

$$d^2 - Nm^2 = 1, \quad 3d - Nm = 3.$$

Multiplying the second equality by m and subtracting from the first one, we obtain $d^2 - 3dm = 1 - 3m$. This gives $(d-1)(d+1) = 3m(d-1)$. The case $d = 1$ corresponds to a projective transformation. Assume $d > 1$. Then we get $d = 3m - 1$ and hence $3(3m-1) - Nm = 3$. Finally, we obtain

$$(9 - N)m = 6, \quad d = 3m - 1.$$

This gives us 4 cases.

(i) $m = 1, N = 3, d = 2$;

(ii) $m = 2, N = 6, d = 5$;

(ii) $m = 3, N = 7, d = 8$;

(iii) $m = 6, N = 8, d = 17$.

The first case is obviously realized by a quadratic transformation with 3 fundamental points. The corresponding bubble cycle is $\eta = x_1 + x_2 + x_3$, where either $\eta = \eta_0$, or $x_2 \succ x_1$, or $x_3 \succ x_2 \succ x_1$.

The second case is realized by the linear system of plane curves of degree 5 with 6 double points. Take 6 points not lying on a conic and no three lie on a line. The space of plane quintics is of dimension 20. The number of conditions for passing through a point with multiplicity ≥ 2 is equal to 3. Thus we have at least ∞^2 plane quintics with given set of 6 double points. One can easily see that the reducible quintics with forms families of dimension less than 2. For example, if the quintic is the union of an irreducible cubic and and a conic. Then the conic passes through at mots 5 points. The cubic must have a double point at the remaining point and pass simply through the five points. The space of such cubics depend on only one parameter.

The third (resp. the forth) case correspond to the *Geiser transformation* (resp. the *Bertini transformation*) which we will discuss later.

7.2 De Jonquières transformations

Assume that there exists a point in the support of η with multiplicity $d-1$. We have

$$d^2-(d-1)^2-\sum_{i=2}^{N}=1,\quad 3d-(d-1)-\sum_{i=2}^{N}=3.$$

This easily implies $\sum_{i=2}^{N} m_i(m_i-1)=0$, hence

$$m_2=\ldots=m_N=1,\quad N=2d-1.$$

The homaloidal system must consist of curves of degree d with singular point p_1 of multiplicity $d-1$ passing simply through $2d-2$ points $p_2,\ldots,p_{2d-1}$. The corresponding Cremona transformation is called *De Jonquières transformation*.

Let us assume that $x_1=(0,0,1)$. Then the equation of a curve from the homaloidal linear system must look like

$$T_2F_{d-1}(T_0,T_1)+F(T_0,T_1)=0. \tag{7.16}$$

Let Γ be a curve of degree $d-1$ which passes through p_1 with multiplicity $d-2$ and simply through the remaining points $p_2,\ldots,p_{2d-1}$. Counting constants, we see that curves of degree $d-1$ depend on $d(d+1)/2$ parameters and the number of conditions is equal to $\frac{1}{2}(d-1)(d-2)+2d-2=d(d+1)/2-1$. We choose point in general position to assume that we can find an irreducible curve Γ. Also, applying the Bézout Theorem we see that such a curve is unique if it is irreducible. Thus the linear system contains a pencil consisting of reducible curves $\Gamma+\ell$, where ℓ is a line through p_1. Since the dimension of our linear system is at least 2, we can find a curve Γ' with equation (7.16) which does not belong to this pencil. Thus a general linear combination of curves $\Gamma+\ell$ and Γ' is an irreducible curve. So, our linear system is homaloidal, and we can choose a basis of the linear system in the form

$$T_0(T_2G_{d-2}+G_{d-1}),T_1(T_2G_{d-2}+G_{d-1}),T_2F_{d-1}+F.$$

Thus the image (y_0,y_1,y_2) of a point (x_0,x_1,x_2) under the Cremona transformation satisfies

$$\begin{aligned}\lambda y_0 &= x_0\big(x_2G_{d-2}(x_0,x_1)+G_{d-1}(x_0,x_1)\big)\\ \lambda y_1 &= x_1\big(x_2G_{d-2}(x_0,x_1)+G_{d-1}(x_0,x_1)\big)\\ \lambda y_2 &= x_2F_{d-1}(x_0,x_1)+F(x_0,x_1),\end{aligned} \tag{7.17}$$

where λ is a nonzero constant.

Let us invert this transformation. We have $y_1/y_0 = x_1/x_0$ and hence $x_0 = cy_0, x_1 = cy_1$ for some constant c. Plugging in the first equation from (7.21), we obtain

$$\lambda y_0 = c^{d-1} y_0(x_2 G_{d-2}(y_0, y_1) + G_{d-1}(y_0, y_1)).$$

Hence

$$\lambda = c^{d-1}(x_2 G_{d-2}(y_0, y_1) + G_{d-1}(y_0, y_1)).$$

From the third equation

$$\lambda y_2 = c^{d-1}(x_2 F_{d-1}(y_0, y_1) + cF(y_0, y_1)).$$

Hence

$$y_2 = \frac{x_2 F_{d-1}(y_0, y_1) + cF(y_0, y_1)}{x_2 G_{d-2}(y_0, y_1) + cG_{d-1}(y_0, y_1)}.$$

Resolving with respect to x_2, we find

$$x_2 = \frac{\lambda(y_2 G_{d-1}(y_0, y_1) - F(y_0, y_1))}{y_2 G_{d-2}(y_0, y_1) - F_{d-1}(y_0, y_1)}.$$

$$\lambda = -\frac{x_2(y_2 G_{d-2}(y_0, y_1) - F_{d-1}(y_0, y_1))}{y_2 G_{d-1}(y_0, y_1) - F(y_0, y_1))}.$$

Thus, the inverse map is given by the formulas

$$\lambda x_0 = y_0(y_2 G_{d-2}(y_0, y_1) - F_{d-1}(y_0, y_1)) \tag{7.18}$$
$$\lambda x_1 = y_1(y_2 G_{d-2}(y_0, y_1) - F_{d-1}(y_0, y_1)) \tag{7.19}$$
$$\lambda x_2 = -y_2 G_{d-1}(y_0, y_1) + F(y_0, y_1), \tag{7.20}$$

If we can find a curve (7.16) such that $F_{d-1} = -G_{d-1}$ we obtain that $T^{-1} = T$.

Note the following properties of a De Jonquirères transformation. The lines $\langle p_1, p_i\rangle$ are blown down to $2d-2$ points $q_2, \ldots, q_{2d-1}$. The curve Γ is blown down to a point y_1. If we resolve the map by $\pi : X \to \mathbb{P}^2$, then the exceptional curve $\pi^{-1}(x_1)$ is mapped to a curve Γ' of order $d-1$ with $(d-2)$-multiple point y_1. The exceptional curves $\pi^{-1}(x_i)$ are mapped to lines $\langle y_1, y_i\rangle$.

Let us find the fixed points of T. They satisfy

$$\begin{aligned} \lambda x_0 &= x_0\big(x_2 G_{d-2}(x_0, x_1) + G_{d-1}(x_0, x_1)\big) \\ \lambda x_1 &= x_1\big(x_2 G_{d-2}(x_0, x_1) + G_{d-1}(x_0, x_1)\big) \\ \lambda x_2 &= x_2 F_{d-1}(x_0, x_1) + F(x_0, x_1), \end{aligned} \tag{7.21}$$

where λ is a nonzero scalar. This gives

$$\lambda = x_2 G_{d-2}(x_0, x_1) + G_{d-1}(x_0, x_1)$$

and

$$x_2^2 G_{d-2}(x_0, x_1) + x_2 G_{d-1}(x_0, x_1) = x_2 F_{d-1}(x_0, x_1) + F(x_0, x_1).$$

Thus the points in the fixed locus (outside of the base points) satisfy the equation

$$x_2^2 G_{d-2}(x_0, x_1) + x_2(G_{d-1}(x_0, x_1) - F_{d-1}(x_0, x_1)) - F(x_0, x_1) = 0. \quad (7.22)$$

The closure of this set is a plane curve X of degree d with a $(d-2)$-multiple point at x_1. It is birationally isomorphic to a hyperelliptic curve of genus $g = d-2$. The corresponding double cover $f : X \to \mathbb{P}^1$ is defined by the projection $(x_0, x_1, x_2) \mapsto (x_0, x_1)$. Its branch points are given by the discriminant of the quadratic equation (in the variable x_2):

$$D = (G_{d-1} - F_{d-1})^2 + 4FG_{d-2}.$$

We have $2d - 2 = 2g + 2$ points as is expected.

7.2.1 Hyperelliptic curves

Let C be a hyperelliptic curve of genus g and g_2^1 its linear system defining a degree 2 map to $\mathbb{P}^1$. Consider the linear system $|D| = |g_2^1 + a_1 + \ldots + a_g|$, where $a_1, \ldots, a_g \in C$. We assume that the divisor $D_1 = a_1 + \ldots + a_g$ is not contained in the linear system $|(g-2)g_2^1|$, or equivalently, $|K_C - D| = \emptyset$. By Riemann-Roch, $\dim |D| = 2$ and defines a map $f : C \to \mathbb{P}^2$. The image of f is a plane curve H_{g+2} of degree $g+2$ with a g-multiple point p_1, the image of the divisor D_1.

By choosing projective coordinates such that $p_1 = (0, 0, 1)$, we can write H_{g+2} by an equation

$$T_2^2 P_g(T_0, T_1) + 2T_2 P_{g+1}(T_0, T_1) + P_{g+2}(T_0, T_1) = 0. \quad (7.23)$$

Let l be a general line through p_1. It intersects H_{g+2} at two points p, q not equal to p_1. For any point $x \in l$ let y be the fourth point such that the the pairs (p, q) and (x, y) are harmonic conjugate (see section 2.1.2). We say that y is *harmonic conjugate* to x with respect to the pair (p, q)).

We would like to define a birational map $T : \mathbb{P}^2 - \to \mathbb{P}^2$ whose restriction to a general line through p_1 takes a point to its harmonic conjugate. Notice that such map is not defined at the points where a line l through p_1 is tangent to H_{g+2}. it is also undefined at the point p_1. Let $p_2, \ldots, p_{2g+3}$ be the tangency points. They correspond to the ramification points of the double map $C \to \mathbb{P}^1$ defined by the hyperelliptic g^1_2. It is a fair guess that the transformation T must be a De Jonquières transformation defined by the linear system

$$|(g+2)\ell - (g+1)p_1 - p_2 - \ldots - p_{2g+3}| \tag{7.24}$$

and the curve H_{g+2} must be the curve of fixed points.

Consider the first polar of H_{g+2} with respect to the point p_1. Its equation is

$$x_2 P_g(T_0, T_1) + P_{g+1}(T_0, T_1) = 0.$$

We know that it passes through the tangency points $p_2, \ldots, p_{2g+3}$. Also it follows from the equation that it has a g-multiple point at p_1. It suggests that the first polar is the curve Γ which was used to define a De Jonquières transformation. Thus we take $d = g+2$ and

$$G_g(T_0, T_1) = P_g(T_0, T_1), \quad G_{g+1}(T_0, T_1) = P_{g+1}(T_0, T_1).$$

As we have already noticed before, to get an involutorial transformation, we need to find a curve $V(T_2 F_{g+1}(T_0, T_1) + F_{g+2}(T_0, T_1))$ from the linear system (7.24) such that $F_{g+1} = -G_{g+1}$. The points $p_i = (1, a_i, b_i), i \geq 2$ belong to the intersection of curves Γ and H_{g+2}. In appropriate coordinate system, we may assume that $b_i \neq 0$. Plugging $P_g(1, a_i) = -P_{g+1}(1, a_i)/b_i$ in the equation of H_{g+2}, we obtain

$$b_i^2 (\frac{-P_{g+1}(1, a_i)}{b_i}) + 2b_i P_{g+1}(1, a_i) + P_{g+2}(1, a_i)$$

$$= b_i P_{g+1}(1, a_i) + P_{g+2}(1, a_i) = 0.$$

Thus the curve given by the equation

$$T_2 P_{g+1}(T_0, T_1) + P_{g+2}(T_0, T_1) = 0$$

belongs to the linear system (7.24). So, we can define the De Jonquières transformation by the formula

$$\begin{aligned} \lambda y_0 &= x_0 \big(x_2 P_g(x_0, x_1) + P_{g+1}(x_0, x_1)\big) \\ \lambda y_1 &= x_1 \big(x_2 P_g(x_0, x_1) + P_{g+1}(x_0, x_1)\big) \\ \lambda y_2 &= -x_2 P_{g+1}(x_0, x_1) - P_{g+2}(x_0, x_1). \end{aligned} \tag{7.25}$$

This transformation satisfies T^2 is the identity. It follows from (7.22) that the curve of fixed points is the curve H_{g+2}. Its restriction to a line $l = V(T_1 - tT_0)$ is given by the formula

$$\begin{aligned}\lambda y_0 &= x_2 P_g(1,t) + x_0 P_{g+1}(1,t)\\ \lambda y_1 &= t\big(x_2 P_g(1,t) + x_0 P_{g+1}(1,t)\big)\\ \lambda y_2 &= -x_2 P_{g+1}(1,t) - x_0 P_{g+2}(1,t).\end{aligned}$$

In affine coordinates x_2/x_0 on the line $T_1 - tT_0 = 0$, the transformation is

$$x \mapsto y = \frac{xP_g(1,t) + P_{g+1}(1,t)}{-xP_{g+1}(1,t) - P_{g+2}(1,t)}.$$

This gives

$$xyP_g(1,t) + (x+y)P_{g+1}(1,t) + P_{g+2}(1,t) = 0. \tag{7.26}$$

The pair (x,y) satisfies the quadratic equation $Z^2 - Z(x+y) + xy$ and the the pair (a,b), where a,b are the points of intersection of the line l with H_{g+2} satisfies the quadratic equation $Z^2P_g(1,t)+2ZP_{g+1}(1,t)+P_{g+2}$. It follows from the definition (2.2) of harmonic conjugates that the equation (7.26) expresses the condition that the pairs (x,y) and (a,b) are harmonic.

Definition 7.6. *The Cremona transformation defined by the formula* (7.25) *is called the De Jonquières involution defined by the hyperelliptic curve* H_{g+2} (7.23)*. It is denoted by* IH_{g+2}.

7.3 Elementary transformations

7.3.1 Segre-Hirzebruch minimal ruled surfaces

First let recall the definition of a minimal ruled surface $\mathbf{F}_n$. If $n = 0$ this is the surface $\mathbb{P}^1 \times \mathbb{P}^1$. If $n = 1$ it is isomorphic to the blow-up of one point in $\mathbb{P}^2$ with the ruling $\pi : \mathbf{F}_1 \to \mathbb{P}^1$ defined by the pencil of lines through the point. If $n > 1$, we consider the cone in $\mathbb{P}^{n+1}$ over a Veronese curve $v_n(\mathbb{P}^1) \subset \mathbb{P}^n$, i.e., we identify $\mathbb{P}^{n-1}$ with a hyperplane in $\mathbb{P}^n$ and consider the union of lines joining a point not on the hyperplane with all points in $v_n(\mathbb{P}^1)$. The surface $\mathbf{F}_n$ is a minimal resolution of its vertex. The exceptional curve of the resolution is a smooth rational curve E_n with $E_n^2 = -n$. The projection from the vertex of the cone, extends to a morphism $p : \mathbf{F}_n \to \mathbb{P}^1$ which defines a ruling (a $\mathbb{P}^1$-bundle). The curve E_n is its section, called the *exceptional section*. In the case $n = 1$, the exceptional curve E_1 of the blow-up $\mathbf{F}_1 \to \mathbb{P}^2$ is also a section of the corresponding ruling $p : \mathbf{F}_1 \to \mathbb{P}^1$. It is also called the exceptional section.

The ruling $p : \mathbf{F}_n \to \mathbb{P}^1$ is a projective vector bundle isomorphic to the projectivization of the vector bundle $\mathbb{V}(\mathcal{E}_n)$, where $\mathcal{E}_n = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. Recall that for any locally free sheaf $\mathcal{E}$ of rank $r+1$ over a scheme S one defines the vector bundle $\mathbb{V}(\mathcal{E})$ as the scheme $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$. Locally, when we choose a trivialization $\mathcal{E}|U \cong \mathcal{O}_U^{r+1}$ over an open affine set $U \subset S$, we get

$$\mathrm{Sym}(\mathcal{E})|U \cong \mathrm{Sym}(\mathcal{O}_U^r) \cong \mathcal{O}(U)[T_0, \ldots, T_r]$$

and $\mathrm{Spec}(\mathcal{E})|U \cong \mathbb{A}_U^{r+1}$. A local section $U \to \mathbb{V}(\mathcal{E})$ is defined by a homomorphism $\mathrm{Sym}(\mathcal{E}) \to \mathcal{O}(U)$ of $\mathcal{O}(U)$-algebras, and hence by a linear map $\mathcal{E}|U \to \mathcal{O}(U)$. Thus the sheaf of local sections of the vector bundle $\mathbb{V}(\mathcal{E})$ is isomorphic to the dual sheaf $\mathcal{E}^*$.

The projectivization of $\mathbb{V}(\mathcal{E})$ is the scheme $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$. It comes with the natural morphism $p : \mathbb{P}(\mathcal{E}) \to S$. In the same notation as above,

$$\mathbb{P}(\mathcal{E})|U \cong \mathrm{Proj}(\mathrm{Sym}(\mathcal{O}_U^{r+1})) \cong \mathrm{Proj}(\mathcal{O}(U)[T_0, \ldots, T_r]) \cong \mathbb{P}_U^r.$$

By definition of the projective spectrum, we have an invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Its sections over $p^{-1}(U)$ are homogeneous elements of degree 1 in $\mathrm{Sym}(\mathcal{O}_U^{r+1})$. This gives for any $k \geq 0$,

$$p_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)) \cong \mathrm{Sym}^k(\mathcal{E}).$$

Note that for any invertible sheaf $\mathcal{L}$ over S, we have $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \cong \mathbb{P}(\mathcal{E})$ as schemes, however the sheaves $\mathcal{O}(1)$ are different.

For any scheme $\pi : X \to S$ over S a morphism of S-schemes $f : X \to \mathbb{P}(\mathcal{E})$ is defined by an invertible sheaf $\mathcal{L}$ over X and a surjection $\phi : \pi^*(\mathcal{E}) \to \mathcal{L}$. Then we trivialize $\mathbb{P}(\mathcal{E})$ over U, the surjection ϕ defines $r+1$ sections of $\mathcal{L}|\pi^{-1}(U)$. These define a local map $x \to (s_0(x), \ldots, s_r(x))$ from $\pi^{-1}(U)$ to $p^{-1}(U) = \mathbb{P}_U^r$. These maps are glued together to define a global map. We have $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

In particular, taking $X = \mathbb{P}(\mathcal{E})$ and f the identity morphism, we obtain a surjection $p^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ when we push it down, we get the identity map $p_*(p^*(\mathcal{E})) = \mathcal{E} \to p_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

Example 7.3.1. Let us take X = S. Then an S-morphism $S \to \mathbb{P}(\mathcal{E})$ is a section $s : S \to \mathbb{P}(\mathcal{E})$. It is defined by an invertible sheaf $\mathcal{L}$ on S and a surjection $\phi : \mathcal{E} \to \mathcal{L}$. We have $\mathcal{L} = s^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Let $\mathcal{N} = \mathrm{Ker}(\phi)$. This is a locally free sheaf of rank r.

Example 7.3.2. Take $x = \mathrm{Spec}(K)$ to be a point in S, and $i : x \to S$ be its inclusion in S. Then an invertible sheaf on a point is the constant sheaf K_x and $i^*(\mathcal{E}) = \mathcal{E}_x = \mathcal{E}/\mathfrak{m}_x\mathcal{E} = \mathcal{E}_x$ is the fibre of the sheaf . The inclusion of x in S is defined by a surjection $\mathcal{E}_x \to K_x$, i.e. by a point in the projective space $\mathbb{P}(\mathcal{E}_x^*)$ (if we prefer to define a projective space as the set of lines). Thus we see that the fibres of the projective bundle $\mathbb{P}(\mathcal{E})$ can be identified with the projective spaces $\mathbb{P}((\mathcal{E}_x)^*)$.

Lemma 7.3.1. *Let* $s : S \to \mathbb{P}(\mathcal{E})$ *be a section and* $\mathcal{N} = \mathrm{Ker}(\mathcal{E} \to s^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. *Let us identify* S *with* $s(S)$. *Then* $\mathcal{N} \otimes \mathcal{L}^{-1}$ *is isomorphic to the conormal sheaf of* $s(S)$ *in* $\mathbb{P}(\mathcal{E})$.

Proof. Recall (see [Hartshorne], Proposition 8.12) that for any closed embedding $i : Y \hookrightarrow X$ of a S-scheme defined by the ideal sheaf $\mathcal{I}$ we have an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to i^*(\Omega^1_{X/S}) \to \Omega^1_{Y/S} \to 0, \tag{7.27}$$

where the first homomorphism is injective if i is a regular embedding (e.g. X, Y are regular schemes). The sheaf $\mathcal{I}/\mathcal{I}^2$ is called the *conormal sheaf* of Y in X and is denoted by $\mathcal{N}^*_{Y/X}$. Its dual sheaf is called the normal sheaf of Y in X and is denoted by $\mathcal{N}_{Y/X}$.

Also recall that the sheaf $\Omega^1_{\mathbb{P}^n}$ of regular 1-forms on projective space can be defined by the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

It is generalized to any projective bundle

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/S} \to p^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to 0. \tag{7.28}$$

Here the homomorphism $p^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ is equal to the homomorphism $p^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ after twisting by -1. Thus

$$\Omega^1_{\mathbb{P}(\mathcal{E})/S}(1) \cong \mathrm{Ker}(p^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)). \tag{7.29}$$

Applying s^* to both sides we get

$$s^*(\Omega^1_{\mathbb{P}(\mathcal{E})/S}(1)) \cong \mathcal{N}. \tag{7.30}$$

Consider the morphism $s : S \to \mathbb{P}(\mathcal{E})\mathbb{P}(\mathcal{E})$ as a morphism of S-schemes. It is equal to the composition of an isomorphism $s : S \to s(S)$ and the closed embedding $s(S) \hookrightarrow \mathbb{P}(\mathcal{E})$ of S-schemes. Since $\Omega^1_{s(S)/S} = \{0\}$, we get from (7.27)

$$\mathcal{N}^*_{s(S)/\mathbb{P}(\mathcal{E})} \cong i^*(\Omega^1_{\mathbb{P}(\mathcal{E})/S}).$$

Applying to both sides s^*, we obtain from (7.30)

$$s^*(\mathcal{N}_{s(S)/\mathbb{P}(\mathcal{E})}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \cong \mathcal{N} \otimes \mathcal{L}^{-1} \cong \mathcal{N}^*_{s(S)/\mathbb{P}(\mathcal{E}example}.$$

This proves the assertion. □

Let us apply this to minimal ruled surfaces $\mathbf{F}_n$. It is known that any locally free sheaf over $\mathbb{P}^1$ is isomorphic to the direct sum of invertible sheaves. Suppose $\mathcal{E}$ is of rank 2. Then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some integers a, b. Since the projective bundle $\mathbb{P}(\mathcal{E})$ does not change if we twist $\mathcal{E}$, we may assume that $a = 0$ and $b = n \geq 0$.

Proposition 7.3.2. *Let $p : S \to \mathbb{P}^1$ be a morphism of a nonsingular surface such that all fibres are isomorphic to $\mathbb{P}^1$. Suppose S has a section E with $E^2 = -n$ for some $n \geq 0$, then $S \cong \mathbf{F}_n$.*

Proof. Let f be the divisor class of a fibre of p and s be the divisor class of the section E. For any divisor class d on S such that $d \cdot f = a$, we obtain $(d - as) \cdot f = 0$. If d represents an irreducible curve C, this implies that $p(C)$ is a point, and hence C is a fibre. Writing every divisor as a linear combination of irreducible curves, we obtain that any divisor class is equal to $af + bs$ for some integers a, b. Let us write $K_{\mathbb{P}(\mathcal{E})} = af + bs$. By adjunction formula, applied to a fibre and the section s, we get

$$-2 = (af + bs) \cdot f, \quad -2 + n = (af + bs) \cdot s = a - 2nb.$$

This gives

$$K_S = (-2 - n)f - 2s. \tag{7.31}$$

Assume $n \neq 0$. Consider the linear system $|nf + s|$. We have

$$(nf + s)^2 = n, \ (nf + s) \cdot ((-2 - n)f - 2s) = -2 - n.$$

By Riemann-Roch, $\dim |nf + s| \geq n + 1$. The linear system $|nf + s|$ has no base points because it contains the linear system $|nf|$ with no base points. Thus it defines a regular map $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$. Since $(nf + s) \cdot s = 0$, it blows down the section s to a point p. Since$(nf + s) \cdot f = a$, it maps fibres to lines passing through p. The degree of the image is $(nf + s)^2 = n$. Thus the image of the map is a surface of degree n equal to the union of lines through a point. It must be a cone over the Veronese curve $v_n(\mathbb{P}^1)$ if $n > 1$ and $\mathbb{P}^2$ if $n = 1$. The map is its minimal resolution of singularities. This proves the assertion in this case.

Assume $n = 0$. We leave to the reader to check that the linear system $|f + s|$ maps S isomorphically to a quadric surface in $\mathbb{P}^3$. □

Corollary 7.3.3.

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \cong \mathbf{F}_n.$$

Proof. The assertion is obvious if $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Assume $n > 0$. Consider the section of $\mathbb{P}(\mathcal{E})$ defined by the surjection

$$\phi : \mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}, \tag{7.32}$$

corresponding to the projection to the first factor. Obviously $\mathcal{N} = \mathrm{Ker}(\phi) \cong \mathcal{O}_{\mathbb{P}^1}(n)$. Applying the lemma, we get

$$\mathcal{N}^*_{s(\mathbb{P}^1)/\mathbb{P}(\mathcal{E})} \cong \mathcal{O}_{\mathbb{P}^1}(n).$$

Now, if C is any curve on a surface X, its ideal sheaf is isomorphic to $\mathcal{O}_X(-C)$ and hence the conormal sheaf is isomorphic to $\mathcal{O}_X(-C)/\mathcal{O}_X(-2C)$. Consider the exact sequence

$$0 \to \mathcal{O}_X(-2C) \to \mathcal{O}_X(-C) \to \mathcal{N}^*_{C/X} \to 0.$$

Tensor it with $\mathcal{O}_X(C)$ we obtain that $\mathcal{N}^*_{C/X} \otimes \mathcal{O}_X(C) \cong \mathcal{O}_C$. This implies

$$\mathcal{N}^*_{C/X} = \mathcal{O}_X(-C) \otimes \mathcal{O}_C,\ \mathcal{N}_{C/X} = \mathcal{O}_X(C) \otimes \mathcal{O}_C. \tag{7.33}$$

In particular, we see that the degree of the invertible sheaf $\mathcal{N}_{C/X}$ on the curve C is equal to the self-intersection C^2.

Thus we obtain that the self-intersection of the section s defined by the surjection (7.32) is equal to $-n$. It remains to apply the previous proposition. □

7.3.2 Elementary transformations

Let $p : \mathbf{F}_n \to \mathbb{P}^1$ be a ruling of $\mathbf{F}_n$ (the unique one if $n \neq 0$). Let $x \in \mathbf{F}_n$ and F_x be the fibre of the ruling containing x. If we blow up x, the proper inverse transform of F_x is an exceptional curve of the first kind. We can blow it down to obtain a nonsingular surface S'. The projection p induces a morphism $p' : S' \to \mathbb{P}^1$ with any fibre isomorphic to $\mathbb{P}^1$. Let E_n be the exceptional section or any section with self-intersection 0 if $n = 0$ (such a section is of course equal to a fibre of the second ruling of $\mathbf{F}_0$). Assume that $x \notin E_n$. The proper inverse transform of E_n on the blow-up has self-intersection equal to $-n$, and its image in S' has the self-intersection equal to $-n+1$. Applying Proposition 7.3.2, we obtain that $S' \cong \mathbf{F}_{n-1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n - \to \mathbf{F}_{n-1}.$$

Assume that $x \in E_n$. Then the proper inverse transform of E_n on the blow-up has self-intersection $-n-1$ and its image in S' has the self-intersection equal to

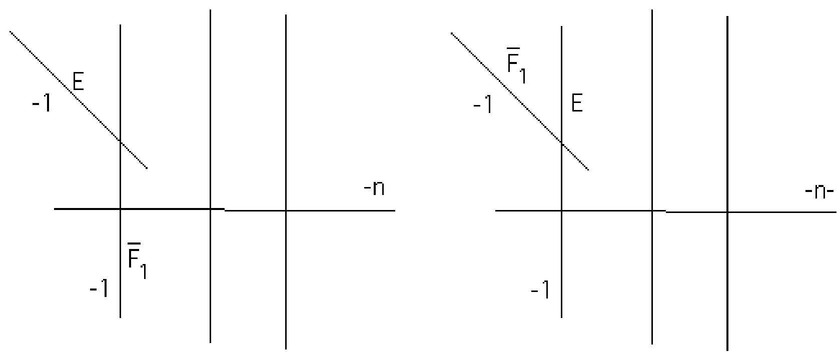

Figure 7.1:

$-n-1$. Applying Proposition 7.3.2, we obtain that $S' \cong \mathbf{F}_{n+1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n - \to \mathbf{F}_{n+1}.$$

A birational map elm_x is called an *elementary transformation*.

Remark 7.3.1. Let $\mathcal{E}$ be a locally free sheaf over a nonsingular curve B. As we explained in Example 7.3.2, a point $x \in \mathbb{P}(\mathcal{E})$ is defined by a surjection $\mathcal{E}_x \to K_x$, where K_x is the constant sheaf on x. Composing this surjection with the natural surjection $\mathcal{E} \to \mathcal{E}_x$, we get a surjective morphism of sheaves $\phi_x : \mathcal{E} \to K_x$. Its sheaf $\mathcal{E}_x = \mathrm{Ker}(\phi_x)$ is a subsheaf of $\mathcal{E}$ which has no torsion. Since the base is a regular one-dimensional scheme, the sheaf $\mathcal{E}_x$ is locally free. Thus we have defined an operation on locally free sheaves. It is also called an elementary transformation.

Consider the special case when $B = \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. We have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n) \xrightarrow{\phi_x} K_x \to 0.$$

The point x belongs to the exceptional section E_n if and only if ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1} \to K_x$. Then $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ and

$$\mathbb{P}(\mathcal{E}') \cong \mathbb{P}(\mathcal{E}'(1)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n+1)) \cong \mathbf{F}_{n+1}.$$

This agrees with our definition of elm_x. If $x \notin E_n$, then ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1}(n)$, and we obtain $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-1)$. In this case $\mathbb{P}(\mathcal{E}') \cong \mathbf{F}_{n-1}$ and again agrees with our definition of elm_x.

Let $x, y \in \mathbf{F}_n$. Assume that $x \in E_n$, $y \notin E_n$ and $p(x) \neq p(y)$. Then the composition

$$t_{x,y} = \mathrm{elm}_y \circ \mathrm{elm}_x : \mathbf{F}_n - \to \mathbf{F}_n$$

is a birational automorphism of $\mathbf{F}_n$. Here we identify the point y with its image in $\mathrm{elm}_x(\mathbf{F}_n)$. Similarly we get a birational automorphism $t_{y,x} = \mathrm{elm}_y \circ \mathrm{elm}_x$ of $\mathbf{F}_n$. We can also extend this definition to the case when $y \succ_1 x$, where y does not correspond to the tangent direction defined by the fibre passing through x or the exceptional section (or any section with self-intersection 0). We blow up x, then y, and then blow down the proper transform of the fibre through x and the the proper inverse transform of the exceptional curve blown up from x.

7.3.3 Birational automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

We will often identify $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with a nonsingular quadric Q in $\mathbb{P}^3$. Let us fix a point $x_0 \in Q$. The linear projection $p_{x_0} : Q \setminus \{x_0\} \to \mathbb{P}^2$ defines a birational map. Let l_1, l_2 be two lines on Q passing through x_0 and q_1, q_2 be their projections. The inverse map $p_{x_0}^{-1}$ blows up the points q_1, q_2 and blows down the proper transform of the line $\langle q_1, q_2\rangle$. For any birational automorphism T of Q the composition $p_{x_0} \circ T \circ p_{x_0}^{-1}$ is a birational transformation of $\mathbb{P}^2$. This defines an isomorphism of groups

$$\Phi_{x_0} : \mathrm{Bir}(Q) \cong \mathrm{Bir}(\mathbb{P}^2).$$

Remark 7.3.2. Let $z_1, \ldots, z_n \in Q$ be F-points of T different from x_0. Let $T^{-1}(x_0)$ be a point if T^{-1} is defined at x_0 or the principal curve of T corresponding to x_0 with x_0 deleted if it contains it. The Cremona transformation $\Phi_{x_0}(T)$ is defined outside the set $q_1, q_2, p_{x_0}(z_1), \ldots, p_{x_0}(z_n), p_{x_0}(T^{-1}(x_0))$. Here, we also include the case of infinitely near fundamental points of T. If some of z_i's lie on a line l_i or infinitely near to points on l_i, their image under p_{x_0} is considered to be an infinitely near point to q_i.

Let $\mathrm{Aut}(Q) \subset \mathrm{Bir}(Q$ be the subgroup of biregular automorphisms of Q. It acts naturally on $\mathrm{Pic}(Q) = \mathbb{Z}f + \mathbb{Z}g$, where $f = [l_1], g = [l_2]$. The kernel $\mathrm{Aut}(Q)^o$ of this action is isomorphic to $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2 \times \mathrm{PGL}_2$. The quotient group is of order 2, and its nontrivial coset can be represented by the automorphism τ of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ defined by $(a, b) \mapsto (b, a)$. Obviously, $\Phi_{x_0}(\tau)$ is a projective automorphism of $\mathbb{P}^2$ (if we use (x, y) as affine coordinates in $\mathbb{P}^2$ and in $\mathbb{P}^1 \times \mathbb{P}^1$, then τ is given by the formula $(x, y) \mapsto (x, y)$).

Proposition 7.3.4. *Let $g \in \mathrm{Aut}(Q)^o$. If $g(x_0) \neq x_0$, then $\Phi_{x_0}(g)$ is a quadratic transformation with fundamental points $q_1, q_2, p_{x_0}(g^{-1}(x_0))$. If $g(x_0) = x_0$, then $\Phi_{x_0}(g)$ is a projective transformation.*

Proof. It follows from Remark 7.3.2 that $\Phi_{x_0}(g)$ has at most 3 fundamental points if $g(x_0) \neq x_0$ and at most 2 fundamental points if $g(x_0) = x_0$. Since any birational map with less than 3 fundamental points (including infinitely near) is regular, we see that in the second case $\Phi_{x_0}(g)$ is a projective automorphism. In the first case, the image of the line $\langle q_1, q_2 \rangle$ is equal to the point $p_{x_0}(g(x_0))$. Thus $\Phi_{x_0}(g)$ is not projective. Since it has at most 3 fundamental points, it must be a quadratic transformation. □

Remark 7.3.3. In general, the product of two quadratic transformations is not a quadratic transformations. However in our case all quadratic transformations coming from $\mathrm{Aut}(Q)$ have a common pair of fundamental points and hence their product is a quadratic transformation. The subgroup $\Phi_{x_0}(\mathrm{Aut}(Q))$ of $\mathrm{Cr}(2) = \mathrm{Bir}(\mathbb{P}^2)$ is an example of a subgroup of the Cremona group $\mathrm{Cr}(2)$ which is isomorphic to an algebraic linear group. According to a theorem of Enriques-Fano, any subgroup of $\mathrm{Cr}(2)$ which is isomorphic to a linear algebraic group, is contained in a subgroup isomorphic to $\mathrm{Aut}(\mathbf{F}_n)$ for some n. There is even a generalization of this result to the group $\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{P}^n)$. Instead of minimal ruled surfaces one considers smooth toric varieties of dimension n.

Take two points x, y no two on the same fibre of each projection $p_1 : \mathbf{F}_0 \to \mathbb{P}^1, p_2 : \mathbf{F}_0 \to \mathbb{P}^1$. Let $x = F_1 \cap F_2$, $y = F_1' \cap F_2'$, where F_1, F_1' are two fibres of p_1 and F_2, F_2' are two fibres of p_2. Then $t_{x,y}$ is a birational automorphism of $\mathbf{F}_0$.

Proposition 7.3.5. *$\Phi_{x_0}(t_{x,y})$ is a product of quadratic transformations. If $x_0 \in \{x, y\}$, then $\Phi_{x_0}(t_{x,y})$ is a quadratic transformation. Otherwise $\Phi_{x_0}(t_{x,y})$ is the product of two quadratic transformation.*

Proof. Assume first that y is not infinitely near to x. Suppose x_0 coincides with one of the points x, y, say $x_0 = x$. It follows from Remark 7.3.2 that $\Phi_{x_0}(T)$ is defined outside $q_1, q_2, p_{x_0}(y)$. On the other hand, the image of the line $\langle q_1, p_{x_0}(y) \rangle$ is a point. Here we assume that the projection $\mathbf{F}_0 \to \mathbb{P}^1$ is chosen in such a way that its fibres are the proper transforms of lines through q_1 under $p_{x_0}^{-1}$. Thus $\Phi_{x_0}(T)$ is not regular with at most three F-points, hence is a quadratic transformation.

If $x_0 \neq x, y$, we compose $t_{x,y}$ with an automorphism g of Q such that $g(x_0) = x$. Then

$$\Phi_{x_0}(t_{x,y} \circ g) = \Phi_{x_0}(t_{x_0, g^{-1}(y)}) = \Phi_{x_0}(t_{x,y}) \circ \Phi_{x_0}(g).$$

By the previous lemma, $\Phi_{x_0}(g)$ is a quadratic transformation. By the previous argument, $\Phi_{x_0}(t_{x_0, g^{-1}(y)})$ is a quadratic transformation. Also the inverse of a quadratic transformation is a quadratic transformation. Thus $\Phi_{x_0}(t_{x,y})$ is a product of two quadratic transformations.

Now assume that $y \succ_1 x$. Take any point $z \neq x$. Then one can easily checks that $t_{x,y} = t_{z,y} \circ t_{x,z}$. Here we view y as an ordinary point on $t_{x,z}(\mathbf{F}_0)$. □

Proposition 7.3.6. *Let $T : \mathbf{F}_n - \to \mathbf{F}_m$ be a birational map. Assume that T commute with the projections of the minimal ruled surfaces to $\mathbb{P}^1$. Then T is a composition of biregular maps and elementary transformations.*

Proof. Let (X, π, f) be a resolution of T. Let $p_1 : \mathbf{F}_n \to \mathbb{P}^1$ and $p_2 : \mathbf{F}_m \to \mathbb{P}^1$ be the projections. We have

$$\phi = p_1 \circ \pi = p_2 \circ f : X \to \mathbb{P}^1.$$

Let $a_1, \ldots, a_k$ be points in $\mathbb{P}^1$ such that $C_i = \phi^{-1}(a_i) = \pi^*(p_1^1(a_i))$ is a reducible curve. We have $\pi_*(C_i) = p_1^{-1}(a_i)$ and $f_*(C_i) = p_2^{-1}(a_i)$. Let E_i be the unique component of C_i which is mapped surjectively to $p_1^{-1}(a_i)$ and E_i' be the unique component of C_i which is mapped surjectively to $p_2^{-1}(a_i)$. Let π be a composition of blow-ups of points $x_1, \ldots, x_N$ and let f be a composition of blow-ups of points $y_1, \ldots, y_N$. The pre-images in X of the maximal points (with respect to the partial order defined by $\succ$) are irreducible curves with self-intersection -1. Let E be a component of C_i with $E^2 = -1$ which is different from E_i, E_i'. We can reorder the order of the blow-ups to assume that $\pi(E) = x_N$ and $f(E) = y_N$. Let $\pi_N : X \to X_{N-1}$ be the blow-up x_N and $f_N : X \to Y_{N-1}$ be the blow-up y_N. Since π_N and f_N are given by the same linear system, there exists an isomorphism $t : X_{N-1} \cong Y_{N-1}$. Thus, we can replace the resolution (X, π, f) with

$$(X_{N-1}, \pi_1 \circ \ldots \circ \pi_{N-1}, f_1 \circ \ldots \circ f_{N-1} \circ t).$$

Continuing in this way, we may assume that x_N and y_N are the only maximal points of π and f such that $p_1(x_N) = p_2(y_N) = a_i$. Let $E = \pi^{-1}(x_N)$ and $E' = f^{-1}(y_N)$. Let $R \neq E'$ be a component of $\phi^{-1}(a_i)$ which intersects E. Let $x = \pi(R)$. Since $x_N \succ x$, and no other points is infinitely near to x, we get $R^2 = -2$. Blowing down E, we get that the image of R has self-intersection -1. Continuing in this way we obtain two possibilities

$$C_i = E_i + E_i', \quad E_i^2 = E_i'^2 = -1, E_i \cdot E_i' = 1,$$

$$C_i = E_i + R_1 + \ldots + R_k + E_i', \quad E_i^2 = E_i'^2 = -1,$$

$$R_i^2 = -1, E_i \cdot R_1 = \ldots = R_i \cdot R_{i+1} = R_k \cdot E_i' = 1$$

and all other intersections are equal to zero.

In the first case, $T = \mathrm{elm}_{x_N}$. In the second case, let $g : X \to X'$ be the blow-down E_i, let $x = \pi(R_1 \cap E_i)$. Then $T = T' \circ \mathrm{elm}_x$, where T' satisfies the assumption of the proposition. Continuing in this way we write T as the composition of elementary transformations. □

7.3.4 De Jonquières transformations again

Let T be a De Jonquières transformation of degree d with fundamental points $p_1, \ldots, p_{2d-1}$. Consider the pencil of lines through p_1. The restriction of the linear system $|d\ell - \eta|$ to a general line from this pencil is of degree 1, and hence maps this line to a line. Since each such line l intersects X at 2 points different from p_1, the image of l is equal to l. Thus T leaves any line from the pencil invariant or blows down it to a point. Let us blow up p_1 to get a birational map $\pi_1 : S_1 \to \mathbb{P}^2$. The surface S_1 is isomorphic to $\mathbf{F}_1$. Its exceptional section is $E_1 = \pi_1^{-1}(p_1)$. The proper transform of the curve Γ is a nonsingular curve $\bar{\Gamma}$. It intersects E_1 at $d-1$ points $z_1, \ldots, z_{2d-2}$ corresponding to the branches of Γ at p_1. Let $l_1, \ldots, l_{d-1}$ be the fibres of the projection $\phi : S_1 \to \mathbb{P}^1$ corresponding to the lines $\langle p_1, p_i\rangle$, where $i = 2, \ldots, 2d-1$. The curve $\bar{C}$ passes through the points $\bar{p}_i = \pi_1^{-1}(p_i) \in l_i$. Let $\pi : X \to \mathbb{P}^2, f : X \to \mathbb{P}^2$ be the resolution of T obtained by blowing up the cycle η. The map factors through $\pi' : X \to S$ which is the blow-up with center at the points $\bar{p}_i$. The proper transform of $\Gamma' = \bar{\Gamma}$ in X is an exceptional curve of the first kind. The map f blows down the proper inverse transforms of the fibres l_i and the curve Γ'. If we stop before blowing down Γ' we get a surface isomorphic to S_1. Thus T can be viewed also as a birational automorphism of $\mathbf{F}_1$ which is the composition of $2d-2$ elementary transformations

$$\mathbf{F}_1 \overset{\mathrm{elm}_{\bar{p}_2}}{- \to} \mathbf{F}_0 - \to \mathbf{F}_1 - \to \ldots - \to \mathbf{F}_0 - \to \mathbf{F}_1.$$

If we take x_0 to be the image of l_1 under $\mathrm{elm}_{\bar{p}_2}$, to define an isomorphism $\Phi_{x_0} : \mathrm{Bir}(\mathbf{F}_0) \to \mathrm{Bir}(\mathbb{P}^2)$, then we obtain that $T = \Phi_{x_0}(T')$, where T' is the composition of transformations $t_{\bar{p}_i, \bar{p}_{i+1}} \in \mathrm{Bir}(\mathbf{F}_0)$, where $i = 3, 5, \ldots, 2d-3$. Applying Proposition 7.3.5, we obtain the following.

Theorem 7.3.7. *Any De Jonquières transformation is a composition of quadratic transformations.*

7.4 Characteristic matrices

Consider a resolution

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow f \\ \mathbb{P}^2 & \overset{T}{- \to} & \mathbb{P}^2 \end{array} \tag{7.34}$$

of a Cremona transformation. Obviously, it gives a resolution of the inverse transformation T^{-1}. The roles of π and f are interchanged. Let

$$f : X = X_M \xrightarrow{f_M} X_{M-1} \xrightarrow{f_{M-1}} \ldots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = \mathbb{P}^2 \tag{7.35}$$

be the decomposition in blow-ups similar to the one we had for π. It defines a bubble cycle ξ and the homaloidal net $|d'h - \xi|$ defining T^{-1}. Let $\mathcal{E}'_1, \ldots, \mathcal{E}'_M$ be the corresponding exceptional configurations.

Lemma 7.4.1. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_N$ be the exceptional configurations for π and $\mathcal{E}'_1, \ldots, \mathcal{E}'_M$ be the exceptional configurations for f. Then*

$$N = M.$$

Proof. Let S be a nonsingular projective surface and $\phi : S' \to S$ be a blow-up map. Then the Picard group Pic(S') is generated by the pre-image $\phi^*(\text{Pic}(S))$ and the divisor class $[E]$ of the exceptional curve. Also we know that $[E]$ is orthogonal to any divisor class from $\phi^*(\text{Pic}(S)$ and this implies that

$$\text{Pic}(S') = \mathbb{Z}[E] \oplus \phi^*(\text{Pic}(S)).$$

In particular, taking $S = \mathbb{P}^2$, we obtain, by induction that

$$\text{Pic}(X) = \pi^*(\text{Pic}(\mathbb{P}^2)) \bigoplus \bigoplus_{i=1}^{N} [\mathcal{E}_i].$$

This implies that Pic(X) is a free abelian group of rank $N + 1$. Replacing π with f, we obtain that the rank is equal to $1 + M$. Thus $N = M$.

□

Remark 7.4.1. It could happen that all exceptional configurations of π are irreducible (i.e. no infinitely points are used to define π) but some of the exceptional configurations of f are reducible. This happens in the case of the transformation given in Exercise 8.2.

Definition 7.7. *A marked resolution of a Cremona transformation is the diagram* (7.34) *together with an order of a sequence of the exceptional curves for f and π.*

Any marked resolution of T defines two bases in Pic(X). The first basis is

$$\underline{e}' : e'_0 = f^*(\ell),\ e'_1 = [\mathcal{E}'_1], \ldots, e'_N = [\mathcal{E}'_N]$$

The second basis is

$$\underline{e} : e_0 = \pi^*(\ell),\ e_1 = [\mathcal{E}_1], \ldots, e_N = [\mathcal{E}_N].$$

The transition matrix A is called the *characteristic matrix* of T with respect to a marked resolution.

The first column of A is the vector $(d, -m_1, \ldots, -m_N)$, where $(d, m_1, \ldots, m_N)$ is the characteristic of T. We write other columns in the form $(d_j, -m_{1j}, \ldots, -m_{Nj})$. They describe the *principal curves* or *P-curves* of T, the curves which are blown down to base points of T^{-1} under T. These are the curves $P_j = \pi(\mathcal{E}'_j)$. They belong to the linear system $|d_j\ell - \sum_{i=1}^N m_{ij}x_i|$. Of course, it makes sense only if $\mathcal{E}'_j$ are irreducible and do not coincide with any $\mathcal{E}_i$. If $\mathcal{E}'_j = \mathcal{E}_i$, then the corresponding column is a unit vector.

Recall that for any rational map $T : X' - \to X$ of irreducible algebraic varieties, one can define the image $T(Z')$ of an irreducible subvariety of X' and the pre-image $T^{-1}(Z)$ of an irreducible subvariety of X. We choose an open subset U' where T is defined, and define $T(Z)$ to be the closure of $T(U' \cap Z')$ in X. Similarly, we choose an open subset U of X, where T^{-1} is defined and define $T^{-1}(Z)$ to be equal to the closure of $T^{-1}(U \cap Z)$ in X.

Proposition 7.4.2. *Let $T : \mathbb{P}^2 - \to \mathbb{P}^2$ be a Cremona transformation with F-points $x_1, \ldots, x_N$ and F-points of T^{-1} equal to $y_1, \ldots, y_N$ and characteristic matrix A. Let C be an irreducible curve on $\mathbb{P}^2$ of degree n which passes through the points y_i of T^{-1} with multiplicity n_i. Let n' be the degree of $T^{-1}(C)$ and let n'_i be the multiplicity of $T^{-1}(C)$ at x_i. Then the vector $v = (n', -n'_1, \ldots, -n'_N)$ is equal to $A \cdot v$, where $v = (n, -n_1, \ldots, -n_N)$.*

Proof. Let (X, π, f) be a minimal resolution of T. The divisor class of the proper inverse transform $f^{-1}(C)$ in X is equal to $v = ne'_0 - \sum n_i e'_i$. If we rewrite it in terms of the basis $(e_0, e_1, \ldots, e_N)$ we obtain that it is equal to $v' = n'e'_0 - \sum n'_i e'_i$, where $v' = Av$. Now the image of $f^{-1}(C)$ under π coincides with $T^{-1}(C)$. By definition of the curves $\mathcal{E}_i$, the curve $T^{-1}(C)$ is a curve of degree n' passing through the fundamental points x_i of T with multiplicities n'_i. □

Example 7.4.1. The following matrix is a characteristic matrix of any standard quadratic transformation T_i.

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}. \tag{7.36}$$

Assume $T = T_1$. Since $T = T^{-1}$, the fundamental points p_1, p_2, p_3 of T and T^{-1} are the same and we choose the same order on them. Let E_1, E_2, E_3 be the exceptional curves of π and E'_1, E'_2, E'_3 be the exceptional curves of f. We know that T blows down the line $\ell_{ij} = \langle p_i, p_j \rangle$ to the point p_k, where $\{i, j, k\} =$

$\{1,2,3\}$. The linear system $|f^*(e_0')|$ is $|2e_0 - e_1 - e_2 - e_3|$, the proper inverse transform of ℓ_{ij} in X has the divisor class $e_0 - e_i - e_j$. Thus $e_k' = e_0 - e_i - e_j$. This gives us the matrix (7.36).

Now assume that $T = T_2$. The resolution $\pi : X \to \mathbb{P}^2$ is the composition of the blow-up of the point $p_1 = (0,0,1)$, followed by blowing up an infinitely near point p_2 corresponding to the tangent direction $x_0 = 0$, and followed by the blowing up the point $p_3 = (1,0,0)$. Let $\mathcal{E}_1 = E_1 + E_2, \mathcal{E}_2 = E_2, \mathcal{E}_3 = E_3$. Here $E_1^2 = -2, E_2^2 = E_3^2 = -1$. Since $T_2 = T_2^{-1}$, we may assume that the fundamental points of T^{-1} are the same p_1, p_2, p_3. It is easy to see that under the map f, the proper transform of the line $x_0 = 0$ is blown down to the point p_3, the proper transform of the line x_1 together with the curve E_1 is blown down to the point p_1. Thus $e_1' = (e_0 - e_1 - e_3) + (e_1 - e_2) = e_0 - e_2 - e_3$, $e_2' = e_0 - e_1 - e_3, e_3' = e_0 - e_1 - e_2$. We get the same matrix. Note that the second column describes the P-curve as a curve from the linear system $|\ell - p_2 - p_3|$. Here p_2 is infinitely near point to p_1. By definition, $p_2 + p_3$ is not a bubble cycle since p_1 is absent. So, $|\ell - p_2 - p_3|$ is not representing a curve on $\mathbb{P}^2$. In fact, $\mathcal{E}_1'$ is reducible and contains a component which is blown down to a point under π.

Now assume $T = T_3$. The resolution π is the composition of the blow-up of $p = (0,0,1)$, followed by the blow-up the infinitely near point corresponding to the direction $x_0 = 0$, and then followed by the blow-up the intersection point of the proper transform of the line $l = V(x_0)$ with the exceptional curve of the first blow-up. We have $\mathcal{E}_1 = E_1 + E_2 + E_3, \mathcal{E}_2 = E_2 + E_3, \mathcal{E}_3 = E_3$. Here $E_1^2 = E_2^2 = -2, E_3^2 = -1$. The blowing down $f : X \to \mathbb{P}^2$ consists of blowing down the proper inverse transform of the line l equal to $e_0 - e_1 - e_2$, followed by the blowing down the image of E_2 and then blowing down the image of E_1. We have $e_1' = (e_0 - e_1 - e_2) + (e_2 - e_3) + (e_1 - e_2) = e_0 - e_2 - e_3$, $e_2' = (e_0 - e_1 - e_2) + (e_2 - e_3) = e_0 - e_1 - e_3$, $e_3' = e_0 - e_1 - e_2$. Again we get the same matrix.

The characteristic matrix defines a homomorphism of free abelian groups

$$\phi_A : \mathbb{Z}^{1+N} \to \mathbb{Z}^{1+N}.$$

We equip $\mathbb{Z}^{1+N}$ with the standard hyperbolic inner product with the norm defined by

$$(a_0, a_1, \ldots, a_N)^2 = a_0^2 - a_1^2 - \ldots - a_N^2.$$

The group $\mathbb{Z}^{1+N}$ equipped with this integral quadratic form is the standard unimodular hyperbolic lattice of odd type. It is customary denoted by $I^{1,N}$. Since both bases $\underline{\mathcal{E}}$ and $\underline{\mathcal{E}'}$ are orthonormal with respect to the inner product, we obtain that the characteristic matrix is orthogonal, i.e. belongs to the group $\mathrm{O}(I^{1,N}) \subset \mathrm{O}(1,N)$.

Recall that the orthogonal group $\mathrm{O}(1,N)$ consists of $N+1\times N+1$ matrices M such that

$$M^{-1} = J_{N+1}\cdot {}^tM\cdot J_{N+1},$$

where J_{N+1} is the diagonal matrix diag$[1,-1,\ldots,-1]$.

In particular, if

$$A = \begin{pmatrix} d & d_1 & \ldots & d_N \\ -m_1 & -m_{11} & \ldots & -m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ -m_N & -m_{N1} & \ldots & -m_{NN} \end{pmatrix}$$

is the characteristic matrix of a Cremona transformation T, then

$$A^{-1} = \begin{pmatrix} d & m_1 & \ldots & m_N \\ -d_1 & -m_{11} & \ldots & -m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ -d_N & -m_{1N} & \ldots & -m_{NN} \end{pmatrix}$$

is the characteristic matrix of T^{-1}. In particular, we see that the linear system defining T^{-1} has characteristic $(d;d_1,\ldots,d_N)$. In the case when T^{-1} has no infinitely near base points, the numbers d_i are the degrees of P-curves of T.

A further observation is that the canonical class K_X is an element of Pic(X) which can be written in both bases as

$$K_X = -3e_0 + \sum_{i=1}^N e_i = -3e_0' + \sum_{i=1}^n e_i'.$$

This shows that the matrix A considered as an orthogonal transformation of $I^{1,N}$ leaves the vector

$$k_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \ldots + \mathbf{e}_N = (-3,1,\ldots,1)$$

invariant. Here, $\mathbf{e}_i$ denotes the unit vector in $\mathbb{Z}^{1+N}$ with $(i+1)$th coordinate equal to 1 and other coordinates equal to zero.

The matrix A defines an orthogonal transformation of $(\mathbb{Z}k_N)^\perp$.

Lemma 7.4.3. *The following vectors form a basis of* $(\mathbb{Z}k_N)^\perp$.

$$\begin{aligned} N\geq 3: \alpha_0 &= \mathbf{e}_0-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3, \quad \alpha_i = \mathbf{e}_i-\mathbf{e}_{i+1},\ i=1,\ldots,N,\\ N=2: \alpha_0 &= \mathbf{e}_0-3\mathbf{e}_1, \quad \alpha_1=\mathbf{e}_1-\mathbf{e}_2\\ N=1: \alpha_0 &= \mathbf{e}_0-3\mathbf{e}_1. \end{aligned}$$

Proof. Obviously the vectors α_i are orthogonal to the vector k_N. Suppose a vector $v = (a_0, a_1, \dots, a_N) \in (\mathbb{Z}k_N)^\perp$. Thus $3a_0 + \sum_{i=1}^N a_i = 0$ (recall that $e_i^2 = -1, i > 0$), hence $-a_N = 3a_0 + \sum_{i=1}^{N-1} a_i$. Assume $N \geq 3$. We can write

$$v = a_0(\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) + (a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2) + (2a_0 + a_1 + a_2)(\mathbf{e}_2 - \mathbf{e}_3)$$
$$+ \sum_{i=3}^{N-1}(3a_0 + a_1 + \cdots + a_i)(\mathbf{e}_i - \mathbf{e}_{i+1}).$$

If $N = 2$, we write $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1) + (3a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2)$. If $N = 1$, $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1)$. □

It is easy to compute the matrix $Q_N = (a_{ij})$ of the restriction of the inner product to $(\mathbb{Z}k_N)^\perp$ with respect to the basis (α_0, α_{N-1}). We have

$$(8), \quad \text{if } N = 1, \quad \begin{pmatrix} -8 & 3 \\ 3 & -2 \end{pmatrix}, \quad \text{if } N = 2.$$

If $N \geq 3$, we have

$$(a_{ij}) = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i - j| = 1 \text{ and } i, j \geq 1 \\ 1 & \text{if } i = 0, j = 3 \\ 0 & \text{otherwise.} \end{cases}$$

For $N \geq 3$, the matrix $A + 2I_N$ is the incidence matrix of the following graph (the *Coxeter-Dynkin diagram* of type $T_{2,3,N-3}$).

For $3 \leq N \leq 8$ this is the Coxeter-Dynkin diagram of the root system of the semi-simple Lie algebra $\mathfrak{sl}_3 + \mathfrak{sl}_2$ of type $A_2 + A_1$ if $N = 3$, of $\mathfrak{sl}_5$ of type A_4 if $N = 4$, of $\mathfrak{so}_{10}$ of type D_5 if $N = 5$ and of the exceptional simple Lie algebra of type E_N if $N = 6, 7, 8$.

We have

$$k_N^2 = 9 - N.$$

This shows that the matrix Q_N is negative definite if $N < 9$, semi-negative definite with one-dimensional null-space for $N = 9$, and of signature $(1, N-1)$ for $N \geq 10$. By a direct computation one checks that its determinant is equal to $N - 9$.

Proposition 7.4.4. *Assume $N \leq 8$. There are only finitely many posssible characteristic matrices. In particular, there are only finitely many possible characteristics of a homaloidal net with ≤ 8 base points.*

Proof. Let G be the group of real matrices $M \in \mathrm{GL}(N)$ such that ${}^tMQ_NM = Q_N$. Since Q_N is negative definite for $N \leq 8$, the group G is isomorphic to the orthogonal group $\mathrm{O}(N)$. The latter group is a compact Lie group. A characteristic matrix belongs to the subgroup $\mathrm{O}(Q_N) = G \cap \mathrm{GL}(N, \mathbb{Z})$. Since the latter is discrete, it must be finite. □

7.4.1 Composition of characteristic matrices

Suppose we have two birational maps $T : \mathbb{P}^2 - \to \mathbb{P}^2, T' : \mathbb{P}^2 - \to \mathbb{P}^2$. We would like to compute the characteristic matrix of the composition $T' \circ T$. Let

$$\begin{array}{ccccccccc} & X & & & & Y & \\ \pi \swarrow & & \searrow f & , & \pi' \swarrow & & \searrow f' \\ \mathbb{P}^2 & \overset{T}{- \to} & \mathbb{P}^2 & & \mathbb{P}^2 & \overset{T'}{- \to} & \mathbb{P}^2 \end{array} \tag{7.37}$$

be resolutions of T and T'. We want to construct a resolution of $T' \circ T$. Let

$$f : X = X_N \xrightarrow{f_N} X_{N-1} \xrightarrow{f_{N-1}} \ldots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = \mathbb{P}^2$$

be a composition of blow-ups of f and

$$\pi' : Y = Y_M \xrightarrow{\pi'_M} Y_{M-1} \xrightarrow{\pi'_{M-1}} \ldots \xrightarrow{\pi'_2} Y_1 \xrightarrow{\pi'_1} Y_0 = \mathbb{P}^2$$

be a composition of blow-ups of π'. Let $x_1, \ldots, x_N$ be the fundamental points of T and $y_1, \ldots, y_N$ be the fundamental points of T^{-1}. Let $x'_1, \ldots, x'_M$ be the fundamental points of T' and $y'_1, \ldots, y'_M$ be the fundamental points of T'^{-1}. For simplicity we will assume that no infinitely near points occur as fundamental points of T, T', T^{-1}, T'^{-1}. We refer to the general case to [M. Alberich-Carraminana, Geometry of the Plane Cremona Maps, Lect. Notes. Math., vol. 1769].

First some of the fundamental points of T^{-1} may coincide with fundamental points of T'. This happens when a P-curve of T contains a fundamental point of T'. Let us assume that

$$y_i = x'_i, \quad i = 1, \ldots, r.$$

In this case the fibred product of $X \xrightarrow{f} \mathbb{P}^2$ and $Y \xrightarrow{\pi'} \mathbb{P}^2$ contains $E_i'^{(1)} \times E_i^{(2)},\ i = 1, \ldots, r$, as irreducible components. When we throw them away, we obtain a marked resolution $(Z, \pi \circ g, f' \circ h)$ of $T' \circ T$, where $g : Z \to X$ is a composition of blow-ups $x'_{r+1}, \ldots, x'_M$ and $h : Z \to Y$ is the composition of the blow-ups of $y_{r+1}, \ldots, y_N$. Consider the following bases of Pic(Z).

$$\underline{e}_1 = \big(g^*(e_0^{(1)}), g^*(e_1^{(1)}), \ldots, g^*(e_N^{(1)}), h^*(e_{r+1}^{(2)}), \ldots, h^*(e_M^{(2)})\big),$$

$$\underline{e}_2 = \big(g^*(e_0'^{(1)}), g^*(e_1'^{(1)}), \ldots, g^*(e_N'^{(1)}), h^*(e_{r+1}^{(2)}), \ldots, h^*(e_M^{(2)})\big),$$

$$\underline{e}_2' = \big(h^*(e_0'^{(2)}), h^*(e_1{}^{(2)}), \ldots, h^*(e_M{}^{(2)}), g^*(e_{r+1}'{}^{(1)}), \ldots, g^*(e_N{}'^{(1)})\big),$$

$$\underline{e}_3 = \big(h^*(e_0^{(2)}), h^*(e_1^{(2)}), \ldots, h^*(e_M^{(2)}), g^*(e_{r+1}'{}^{(1)}), \ldots, g^*(e_N'{}^{(1)})\big),$$

Note that

$$g^*(e_0^{(1)}) = h^*(e_0'^{(2)}).$$

The transition matrix from basis $\underline{e}_1$ to basis $\underline{e}_2$ is

$$\tilde{A}_1 = \begin{pmatrix} A_1 & 0_{N,M-r} \\ 0_{M-r,N} & I_{M-r} \end{pmatrix},$$

where A_1 is the characteristic matrix of T_1. The transition matrix from basis $\underline{e}_2$ to basis $\underline{e}_2'$ is

$$P = \begin{pmatrix} I_{r+1} & 0_{r+1,N-r} & 0_{r+1,N-r} \\ 0_{N-r,r+1} & 0_{N-r,N-r} & I_{N-r} \\ 0_{M-r,r+1} & I_{M-r} & 0_{M-r,M-r} \end{pmatrix}.$$

The transition matrix from basis $\underline{e}_2'$ to basis $\underline{e}_3$ is

$$\tilde{A}_2 = \begin{pmatrix} A_2 & 0_{M,N-r} \\ 0_{N-r,M} & I_{N-r} \end{pmatrix},$$

where A_2 is the characteristic matrix of T_2. The characteristic matrix of $T_2 \circ T_1$ is equal to the product

$$A = \tilde{A}_1 \circ P \circ \tilde{A}_2.$$

In the special case, when $r = N$, i.e., all fundamental points of T_1^{-1} are fundamental points of T_2, we obtain that the characteristic matrix of $T_2 \circ T_1$ is equal to

$$\begin{pmatrix} A_1 & 0_{N,M-N} \\ 0_{M-N,N} & I_{M-N} \end{pmatrix} \cdot A_2. \tag{7.38}$$

Example 7.4.2. Assume that $r = 0$, i.e. no F-point of T^{-1} coincide with a F-point of T'. Then the characteristic matrix of $T' \circ T$ is equal to

$$\begin{pmatrix} dd' & dd'_1 & \ldots & dd'_M & d_1 & \ldots & d_N \\ -d'm_1 & -d'_1m_1 & \ldots & -d'_Mm_1 & -m_{11} & \ldots & -m_{N1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -d'm_N & -d'_1m_N & \ldots & -d'_Mm_N & -m_{1N} & \ldots & -m_{NN} \\ -m'_1 & m'_{11} & \ldots & m'_{1M} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -m'_M & m'_{1M} & \ldots & m'_{MM} & 0 & \ldots & 0 \end{pmatrix}$$

with the obvious meanings of $d, m_i, m_{ij}, d', m'_j, m'_{ij}$. In particular we see that the degree of the composition is equal to the product of the degrees of the factors.

Example 7.4.3. Let T_1 be a quadratic transformation with base points x_1, x_2, x_3. Let y_1, y_2, y_3 be the base points of T_1^{-1}. Let T_2 be a Cremona transformation with base points $x_1, \ldots, x'_M$ and base points $y'_1, \ldots, y'_M$ of T_2^{-1}. Assume that $y_i = x'_i$ for $i \leq r$. Let A be the characteristic matrix of the composition $T_2 \circ T_1$. If $r = 3$, we obtain from (7.38)

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & \ldots & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \cdot \begin{pmatrix} d & d_1 & \ldots & d_M \\ -m_1 & -m_{11} & \ldots & -m_{1M} \\ \vdots & \vdots & \vdots & \vdots \\ -m_M & -m_{M1} & \ldots & -m_{MM} \end{pmatrix}.$$

Here we choose some order on the points y_1, y_2, y_3 which affects the matrix A_1.

For example, we obtain that the characteristic of the composition map is equal to

$$(2d-m_1-m_2-m_3, d-m_2-m_3, d-m_1-m_3, d-m_1-m_2, m_4, \ldots, m_M) \quad (7.39)$$

Assume $r < 3$. We leave to the reader to check that the characteristic of the composition map is equal to

$$(2d - m_1 - m_2, d - m_2, d - m_1, d - m_1 - m_2, m_3, \ldots, m_M), \quad r = 2, \quad (7.40)$$

$$(2d - m_1, d, d - m_1, d - m_1, m_2, \ldots, m_M), \quad r = 1 \quad (7.41)$$

It is not difficult to see that the same formulae are true in the case when some of the points $y_i = x_i$ are infinitely near.

7.4.2 Weyl groups

Let $\mathbf{E}_N = (\mathbb{Z}k_N)^\perp \cong \mathbb{Z}^N$ equipped with the quadratic form obtained by the restriction of the inner product in $I^{1,N}$. Let $\mathrm{O}(E_N)$ be the orthogonal group of $\mathbf{E}_N$. It is isomorphic to the group $\mathrm{O}(Q_N)$ introduced in the proof of the previous corollary. Assume $N \geq 3$. For any vector $\alpha \in \mathbf{E}_N$ with $\alpha^2 = -2$, we define the following element in $\mathrm{O}(E_N)$:

$$r_\alpha : v \mapsto v + (v, \alpha)\alpha.$$

It is called a *reflection* with respect to α. It leaves the orthogonal complement to α pointwisely fixed, and maps α to $-\alpha$.

Definition 7.8. *The subgroup $W(\mathbf{E}_N)$ of $\mathrm{O}(\mathbf{E}_N)$ generated by reflections r_{α_i} is called the* Weyl group *of L_N.*

The following proposition is stated without proof.

Proposition 7.4.5. *The Weyl group $W(\mathbf{E}_N)$ is of infinite index in $\mathrm{O}(\mathbf{E}_N)$ for $N > 10$. For $N \leq 10$,*

$$\mathrm{O}(\mathbf{E}_N) = W(\mathbf{E}_N) \rtimes (\tau),$$

where $\tau^2 = 1$ and $\tau = 1$ if $N = 7, 8$, $\tau = -1$ if $N = 9, 10$ and τ is induced by the symmetry of the Coxeter-Dynkin diagram for $N = 4, 5, 6$.

Note that any reflection can be extended to an orthogonal transformation of the lattice $I^{1,N}$ (use the same formula). The subgroup generated by reflections $r_{\alpha_i}, i \neq 0$, acts as the permutation group S_N of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$.

7.4.3 Noether's inequality

We know that the characteristic vector $c = (d, m_1, \ldots, m_N)$ of a homaloidal net is a vector in $I^{1,N}$ satisfying

$$c^2 = 1, \quad (c, k_N) = -3.$$

We denote the set of such vectors by H_N. It is the set of integer points in a ball of radius 1 in an affine hyperplane of the hyperbolic space $\mathbb{R}^{1,N} = I^{1,N} \otimes \mathbb{R}$.

Lemma 7.4.6. *(Noether's inequality) Let $v = (d, m_1, \ldots, m_N) \in H_N$. Assume $d > 1, m_1 \geq \ldots \geq m_N \geq 0$. Then*

$$m_1 + m_2 + m_3 \geq d + 1,$$

and the equality holds if and only if either $m_1 = \ldots = m_N$ or $m_1 = n - 1, m_2 = \ldots = m_N$.

Proof. We have

$$m_1^2 + \ldots + m_N^2 = d^2 - 1, \quad m_1 + \ldots + m_N = 3d - 3.$$

Multiplying the second equality by m_3 and subtracting from the first one, we get

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) - \sum_{i \geq 4} m_i(m_3 - m_i) = d^2 - 1 - 3m_3(d - 1).$$

From this we get

$$(d-1)(m_1+m_2+m_3-d-1) = (m_1-m_3)(d-1-m_1)+(m_2-m_3)(d-1-m_2)+$$
$$+\sum_{i \geq 4} m_i(m_3 - m_i).$$

Since $d - 1 - m_i \geq 0$, this obviously proves the assertion. □

Corollary 7.4.7.

$$m_1 > d/3.$$

Theorem 7.4.8. *Let A be a characteristic matrix of a homaloidal net. Then A belongs to the Weyl group $W(\mathbf{E}_N)$.*

Proof. Let $c = (d, m_1, \ldots, m_N)$ be a characteristic of the net. Let

$$\mathbf{v} = (d, -m_1, \ldots, -m_N) \in \mathbb{Z}^{N+1}.$$

It is the first column of the characteristic matrix. We order the points such that $\mathbf{v}$ satisfies the conditions of Lemma 7.4.6. Apply the reflection r_{α_0} to $\mathbf{v}$ to obtain

$$r_{\alpha_0}(\mathbf{v}) = \mathbf{v} + (\mathbf{v}, \alpha_0)\alpha_0 = \mathbf{v} + (d - m_1 - m_2 - m_3)(\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)$$

$$= (2d - m_1 - m_2 - m_3, m_2 + m_3 - d, m_1 + m_3 - d, m_1 + m_2 - d, m_4, \ldots, m_N).$$

By Noether's inequality, we have $2d - m_1 - m_2 - m_3 < d$. Since any reflection is an orthogonal transformation and leaves k_N fixed, we see that it leaves invariant the affine space H_N. Thus the vector $r_{\alpha_0}(\mathbf{v}) \in H_N$. We reorder the coordinates, except the first one, to satisfy the assumption of the Noether inequality. This corresponds to applying reflections $r_{\alpha_i}, i \neq 0$. Applying again the reflection r_{α_0} we decrease the first coefficient d'. Continuing in this way we find a composition σ of reflections such that $\sigma(\mathbf{v}) = (1, m'_1, \ldots, m'_N)$. This obviously implies $m'_1 = \ldots = m'_N = 0$. Thus the matrix $A' = \sigma \cdot A$ has the first column equal to the vector $(1, 0, \ldots, 0)$. Since A' is orthogonal matrix (with respect to the hyperbolic inner product), for each column $(d_j, -m_{j1}, \ldots, -m_{jN})$ of the characteristic matrix A', we get $d_j = 0, \sum_{i=1}^N m_{ji}^2 = 1$. This implies $m_{ji} = \pm 1$ for some i and all other m_{ji} are equal to zero. Since two columns are orthogonal, we also get that A' is diagonal with ± 1 at the diagonal. Finally, we use that A defines an endomorphism of $\mathbb{Z}^{1+N}$ which leaves the vector $k_N = (-3, 1, \ldots, 1)$ invariant. This could happen only if $A' = I_N$. This proves the assertion. □

7.5 Cremona group

7.5.1 Noether's factorization theorem

Let T be a Cremona transformation of $\mathbb{P}^2$ defined by a linear system $|d\ell - \sum m_i x_i|$. We order the multiplicities $m_1 \geq \ldots \geq m_N$. Obvioulsy, we may assume that $x_1 \in \mathbb{P}^2$. Assume that one of the points x_2 and x_3 is not infinitely near to x_1 of the first order. Then replacing T with $T \circ Q$, where Q is a quadratic transformation such that the fundamental points of Q^{-1} are equal to x_1, x_2, x_3, we obtain that $T \circ Q$ is given by a linear system of degree $2d - m_1 - m_2 - m_3 < d$ (see (7.39)). Continuing in this way we obtain that $Q_k \circ \cdots Q_1 \circ T$ is given by a linear system of degree 1, i.e. a projective transformation. Unfortunately, this proof is wrong (as was the original proof of M. Noether). The reason is that at a certain step, maybe even at the first one, a quadratic transformation cannot be applied because of infinitely near points $x_2 \succ x_1, x_3 \succ x_1$. We will give a modified version of this proof due to V. Iskovskikh.

7.5.2 Noether-Fano inequality

First we generalize Corollary 7.4.7 to birational maps of any rational surfaces. The same idea works even for higher-dimensional varieties. Let $T : S \to S'$ be a birational map of surfaces. Let $\pi : X \to S, f : X \to S'$ be its resolution. Let $|H'|$ be a linear system on X' without base points. Let $f^*(H') \sim \pi^*(H) - \sum_i m_i \mathcal{E}_i$ for some divisor H on S and exceptional configurations $\mathcal{E}_i$ of the map π. Since $|H'|$ has no base points, $|f^*(H')|$ has no base points. Thus $f^*(H')$ intersects nonnegatively any curve on X. In particular,

$$f^*(H') \cdot \mathcal{E}_i = -m_i \mathcal{E}_i^2 = m_i \geq 0. \tag{7.42}$$

This can be interpreted by saying that $T^{-1}(H')$ belongs to the linear system $|H - \eta|$, where $\eta = \sum m_i x_i$ is a bubble cycle on S (defined in the same was as a bubble cycle on $\mathbb{P}^2$).

Theorem 7.5.1. *(Noether-Fano inequality) Assume that there exists some integer $m_0 \geq 0$ such that $|H' + mK_{S'}| = \emptyset$ for $m \geq m_0$. For any $m \geq m_0$ such that*

$|H+mK_S| \neq \emptyset$ *there exists* i *such that*

$$m_i > m.$$

Proof. We know that $K_X = \pi^*(K_S)+\sum_i \mathcal{E}_i$. Thus we have the equality in $\mathrm{Pic}(X)$

$$f^*(H') + mK_X = (\pi^*(H+mK_S)) + \sum (m-m_i)\mathcal{E}_i.$$

Applying f_* to the left-hand side we get the divisor class $H' + mK_{S'}$ which, by assumption cannot be effective. Since $|\pi^*(H+mK_S)| \neq \emptyset$, applying f_* to the right-hand side, we get the sum of an effective divisor and the image of the divisor $\sum_i (m-m_i)\mathcal{E}_i$. If all $m-m_i$ are nonnegative, it is also an effective divisor, and we get a contradiction. Thus there exists i such that $m-m_i < 0$. □

Example 7.5.1. Assume $S = S' = \mathbb{P}^2$, $H = d\ell$ and $H' = \ell$. We have $|H+K_{S'}| = |-2\ell| = \emptyset$. Thus we can take $m_0 = 1$. If $d \geq 3$, we have for any $1 \leq a \leq d/3$, $|H' + aK_S| = |(d-3a)\ell| \neq \emptyset$. This gives $m_i > d/3$ for some i. This is Corollary 7.4.7.

Example 7.5.2. Let $S = \mathbf{F}_n$ and $S' = \mathbf{F}_r$ be the minimal Segre-Hirzebruch ruled surfaces. Let $|H'| = |f'|$ be the linear system defined by the ruling on S'. It has no base points, so we can write $[f^*(H')] = \pi^*(af+bs) - \sum m_i e_i$, where f, s the divisors classes of a fibre and the exceptional section on S, and $m_i \geq 0$. Here (X, π, f) is a resolution of T (sorry for the using f in two different meanings). Thus $H = af + bs$.

Recall that $K_S = -2s-(2+n)f$, $K_{S'} = -2s'-(2+r)f'$. Thus $|H'+K_{S'}| = |(-1-n)f - 2s| = \emptyset$. We take $m_0 = 1$. We have

$$|af+bs+mK_S| = |(a-m(2+n))f + (b-2m)s|.$$

Assume that

$$1 < b \leq \frac{2a}{2+n}.$$

If $m = [b/2]$, then $m \geq m_0$ and both coefficients $a - m(2+n)$ and $b-2m$ are nonnegative. Thus we can apply Theorem 7.5.1 to find an index i such that $m_i > m \geq b/2$.

In the special case, when $n = 0$, i.e. $S = \mathbb{P}^1 \times \mathbb{P}^1$, the inequality $b \leq a$ implies that there exists i such that $m_i > b/2$.

Similar argument can be also applied to the case $S = \mathbb{P}^2, S' = \mathbb{F}_r$. In this case, $|H| = |a\ell$ and $|aH + mK_S = (a-3m)\ell|$. Thus, we can take $m = [a/3]$, and find i such that $m_i > a/3$.

7.5.3 Noether's Theorem

We shall prove the following.

Theorem 7.5.2. *The group* $\mathrm{Bir}(\mathbf{F}_0)$ *is generated by biregular automorphisms and a birational automorphism* $t_{x,y}$ *for some pair of points* x, y.

Applying Propositions 9.2.1 and 9.2.2 from Chapter 9, we obtain the following Noether's Factorization Theorem.

Corollary 7.5.3. $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by projective automorphisms and quadratic transformations*

Now let us prove Theorem 7.5.2.

Let $T : \mathbf{F}_n - \rightarrow \mathbf{F}_m$ be a birational map. Let

$$\mathrm{Pic}(\mathbf{F}_n) = \mathbb{Z}f + \mathbb{Z}s, \quad \mathrm{Pic}(\mathbb{F}_m) = \mathbb{Z}f' + \mathbb{Z}s',$$

where f, f' are the divisor classes of fibres, and s, s' are the divisor classes of exceptional sections. Similar to the case of birational maps of projective plane, we can define a marked resolution (X, π, f) of T and its characteristic matrix A. We have two bases in $\mathrm{Pic}(X)$

$$\underline{e} : \pi^*(f), s = \pi^*(s), e_i = [\mathcal{E}_i], i = 1, \ldots, N,$$

$$\underline{e'} : \pi^*(f'), \pi^*(s'), e'_i = [\mathcal{E}'_i], i = 1, \ldots, N.$$

for simplicity of notation, let us identify f, s, f', s' with their inverse transforms in $\mathrm{Pic}(X)$. As in the case of Cremona transformations, one can define the characteristic matrix of T. For example, its first column $(a, b; m_1, \ldots, m_N)$ expresses that the pre-image of the linear system $|f'|$ on $\mathbf{F}_m$ is the linear system $|af + bs - \eta|$, where $\eta = \sum m_i x_i$ is a bubble cycle over $\mathbf{F}_n$. The first column of the inverse matrix defines preimage of $|f|$ under T^{-1} (the same as the image under T).

Example 7.5.3. Let $T = \mathrm{elm}_x : \mathbf{F}_n - \rightarrow \mathbf{F}_{n\pm 1}$. Let f, s, e be the classes of a fibre, the exceptional section, and the exceptional curve E on the blow-up $\pi : X \rightarrow \mathbf{F}_n$ of x. Suppose $|s - x| = \emptyset$, i.e., x does not lie on the exceptional divisor. Let $f : X \rightarrow \mathbf{F}_{n-1}$ be the blow-down the proper transform $\bar{F}$ of F. Then

$$f' = f, \quad s' = s + f - e, \quad e' = f - e.$$

If $|s - x| \neq \emptyset$, we have

$$f' = f, \quad s' = s - e, \quad e' = f - e.$$

It is easy to see that these transformations are inverse to each other, as it should be. Thus we get

$$f = f', \quad s = s' - e', \quad e = f' - e', \quad \text{if } |s - x| \neq \emptyset,$$
$$f = f', \quad s = s' + f' - e', \quad e = f' - e', \quad \text{otherwise}.$$

Let $T : \mathbf{F}_n - \to \mathbf{F}_m$. Composing T with elm_x, we get a map $\text{elm}_x \circ T : \mathbf{F}_n - \to \mathbf{F}_{m\pm 1}$. The image of $|f|$ on $\mathbf{F}_{m\pm 1}$ is equal to

$$|(a - m_i)f + bs - (b - m_x)x' - \sum_{y\neq x} m_y y, \quad \text{if } |s - x| = \emptyset, \tag{7.43}$$
$$|(a + b - m_i)f + bs - (b - m_x)x' - \sum_{y\neq x} m_y y|, \quad \text{if } |s - x| \neq \emptyset,$$

where x' is the image of the proper transform of the fibre passing through x.

Lemma 7.5.4. *Let $T : \mathbf{F}_0 - \to \mathbf{F}_0$ be a birational automorphism equal to a composition of elementary transformations. Then T is equal to a composition of biregular automorphisms of $\mathbf{F}_0$ and a transformation $t_{x,y}$ for a fixed pair of points x, y, where y is not infinitely near to x.*

Proof. It follows from Proposition 9.2.2 from Chapter 9 that $t_{x,y}$, where $y \succ_1 x$ can be written as a composition of two transformations of type $t_{x',y'}$ with no infinitely near points. Now notice that the transformations $t_{x,y}$ and $t_{x',y'}$ for different pairs of points differ by an automorphism of $\mathbf{F}_0$ which sends x to x' and y to y'. Suppose we have a composition T of elementary transformations.

$$\mathbf{F}_0 \overset{\text{elm}_{x_1}}{--\to} \mathbf{F}_1 \overset{\text{elm}_{x_2}}{--\to} \ldots \overset{\text{elm}_{x_{k-1}}}{--\to} \mathbf{F}_1 \overset{\text{elm}_{x_k}}{--\to} \mathbf{F}_0$$

If no $\mathbf{F}_0$ occurs among the surfaces $\mathbf{F}_n$ here, then T is a composition of even number k of elementary transformations preserving the projections to $\mathbb{P}^1$. It is clear that not all points x_i are images of points in $\mathbf{F}_0$ lying on the same exceptional section as x_1. Let x_i be such a point (maybe infinitely near to x_1). Then we compose T with t_{x_i,x_1} to obtain a birational map $T' : \mathbf{F}_0 - \to \mathbf{F}_0$ which is a composition of $k-2$ elementary transformations. Continuing in this way we write T as a composition of transformations $t_{x',y'}$.

If $\mathbf{F}_1 \overset{\text{elm}_{x_{i-1}}}{--\to} \mathbf{F}_0 \overset{\text{elm}_{x_i}}{--\to} \mathbf{F}_1$ occurs, then, elm_{x_i} may be defined with respect to another projection to $\mathbb{P}^1$. Then we write as a composition of the switch automorphism τ and the elementary transformation with respect to the first projection. Then we repeat this if such $(\mathbb{F}_0, \text{elm}_j)$ occurs again. □

Let $T : \mathbf{F}_0 - \to \mathbf{F}_0$ be a birational transformation. Assume the image of $|f|$ is equal to $|af + bs - \sum m_x x|$. Applying the automorphism τ, if needed, we may assume that $b \leq a$. Thus, using Example 7.5.2, we can find a point x with $m_x > b/2$. Composing T with elm_x, we obtain that the image of $|f|$ in $\mathbf{F}_1$ is the linear system $|a'f' + bs' - m_{x'}x' - \sum_{y \neq x'} m_y y|$, where $m_{x'} = b - m_x < m_x$. Continuing in this way, using formula (8.1.7) we get a map $T' : \mathbf{F}_0 - \to \mathbf{F}_q$ such that the image of $|f|$ is the linear system $|a'f' + bs' - \sum m_x x|$, where all $m_x \leq b/2$. If $b = 1$, we get all $m_i = 0$. Thus T' is everywhere defined and hence $q = 0$. The assertion of the theorem is verified.

Assume $b \geq 2$. Since all $m_i \leq b/2$, we must have, by Example 7.5.2,

$$b > \frac{2a'}{2+q}.$$

Since the linear system $|a'f' + bs'|$ has no fixed components, we get

$$(a'f' + bs') \cdot s' = a' - bq \geq 0.$$

Thus $q \leq a'/b < (2+q)/2$, and hence $q \leq 1$. If $q = 0$, we get $b > a'$. Applying τ, we will decrease b and will start our algorithm again until we either arrive at the case $b = 1$, and we are done, or arrive at the case $q = 1$, and $b > 2a'/3$ and all $m_{x'} \leq b/2$.

Let $\pi : \mathbf{F}_1 \to \mathbb{P}^2$ be the blowing down the exceptional section s' to a point q. Then the image of a fibre $|f|$ on $\mathbf{F}_1$ under π is equal to $|\ell - q|$. Hence the image of our linear system in $\mathbb{P}^2$ is equal to $|a'\ell - (a'-b)q - \sum_{p \neq q} m'_p p|$. Obviously, we may assume that $a' \geq b$, hence the coefficient at q is non-negative. Since $b > 2a'/3$, we get $a' - b < a'/3$. By Example 7.5.2, there exists a point $p \neq q$ such that $m'_p > a'/3$. Let $\pi(x) = p$ and $\mathcal{E}_1$ be the exceptional curve corresponding to x and S be the exceptional section in $\mathbf{F}_1$. If $x \in S$, the divisor class $s - e_1$ is effective and is represented by the proper inverse transform of S in the blow-up of x. Then

$$(a'f + bs - m'_x e_1 - \sum_{i>1} m'_i e_i) \cdot (s - e_1) \leq a' - b - m'_x < 0.$$

This is impossible because the linear system $|a'f + bs - m_x x - \sum_{y \neq x} y|$ on $\mathbf{F}_1$has no fixed part. Thus x does not lie on S. If we apply elm_x, we arrive at $\mathbf{F}_0$ and may assume that the new coefficient at f' is equal to $a' - m'_x$. Since $m'_x > a'/3$ and $a' < 3b/2$, we see that $a' - m'_x < b$. Now we apply τ to decrease b. Continuing in this way we obtain that T is equal to a product of elementary transformations and automorphisms of $\mathbf{F}_0$. We finish the proof of Theorem 7.5.2 by applying Lemma 7.5.4.

Corollary 7.5.5. *The group* $\mathrm{Cr}(2)$ *of Cremona transformations of* $\mathbb{P}^2$ *is generated by projective automorphisms and the standard Cremona transformation* T_0.

Proof. It is enough to show that the standard quadratic transformations T_2 and T_3 are generated by T_0 and projective transformations. Let T_2 has fundamental points at p_1, p_2 and an infinitely near point $p_3 \succ_1 p_1$. Choose a point q different from p_1, p_2, p_3 and not lying on the line $\langle p_1, p_2 \rangle$. Let T be a quadratic transformation with F-points at p_1, p_2, q. It is easy to check that $T \circ T_2$ is a quadratic transformation with F-points $(p_1, p_2, T_2(q))$. Composing it with projective automorphisms we get the standard quadratic transformation T_1.

Now let us consider the standard quadratic transformation T_3 with F-points $p_3 \succ p_2 \succ p_1$. Take a point q which is not on the line in the linear system $|\ell - p_1 - p_2|$. Consider a quadratic transformation T with F-points p_1, p_2, q. It is easy to see that $T \circ T_3$ is a quadratic transformation with F-points $p_1, p_2, T_3(q)$. Composing it with projective transformations we get the standard quadratic transformation T_2. Then we write T_2 as a composition of T_1 and projective transformations. □

Exercises

7.1 Consider a minimal resolution X of the standard quadratic transformation T_1. Show that T_1 lifts to an automorphism σ of X. Show that σ has 4 fixed points and the orbit space $X/(\sigma)$ is isomorphic to the cubic surface with 4 nodes given by the equation $T_0T_1T_2 + T_0T_1T_3 + T_1T_2T_3 + T_0T_2T_3 = 0$.

7.2 Consider the rational map

$$T : (t_0, t_1, t_2) \mapsto (t_1t_2(t_0-t_2)(t_0-2t_1), t_0t_2(t_1-t_2)(t_0-2t_1), t_0t_1(t_1-t_2)(t_0-t_2)).$$

Show that it is a Cremona transformation and find the Enriques diagram of the corresponding bubble cycle.

7.3 Let C be a plane curve of degree d with a singular point p. Let $\pi : X \to \mathbb{P}^2$ be a sequence of blow-ups which resolves the singularity. Define the bubble cycle $\eta(C, p) = \sum m_i x_i$ as follows: $x_1 = p$ and $m_1 = \mathrm{mult}_p C$, $x_2, \ldots, x_k$ are infinitely near points to p of order 1 such that the proper transform C' of C under the blow-up at p contains these points, $m_i = \mathrm{mult}_{x_i} C', i = 2, \ldots, k$, and so on.

(i) Show that the arithmetic genus of the proper transform of C in X is equal to $\frac{1}{2}(d-1)(d-2) - \frac{1}{2}\sum_i m_i(m_i - 1)$.

(ii) Describe the Enriques diagram of $\eta(C, p)$, where $C = V(T_0^{b-a}T_1^a + T_2^b), p = (1, 0, 0)$, and $a \leq b$ are positive integers.

7.4 Let $f : S' \to S$ be a birational morphism of nonsingular projective surfaces.

(i) Show that it defines a map of the bubble spaces $f^{\text{bb}} : S'^{\text{bb}} \to S^{\text{bb}}$ which coincide with f on S'.

(ii) Consider the action of the group Aut($\mathbb{P}^2$) on the bubble space of $\mathbb{P}^2$. Determine the number of orbits on the sets of less than 4 elements.

7.5 Show that any planar Cremona transformation which leaves invariant a general line from a pencil of lines through some point must be a De Jonquières transformation.

7.6 Let $T : \mathbb{P}^2 - \to \mathbb{P}^2$ be a rational map. Consider the rational map $\Phi_T : \mathbb{P}^2 - \to \check{\mathbb{P}}^2$ defined by sending a point p to the line $\langle p, T(p)\rangle$. Let (u_0, u_1, u_2) be the coordinates in the dual plane such that the line $V(aT_0 + bT_1 + cT_2)$ has coordinates (a, b, c).

(i) Let F_0, F_1, F_2 be a basis of the linear system defining T. Show that the curves $T_2F_1 - T_1F_2, T_2F_0 - T_2F_2, T_0F_1 - T_1F_0$ form a basis of the linear system defining Φ_T.

(ii) Show that the dimension of the image of ϕ_T is equal to 2 unless T defines a De Jonquières transformation.

(iii) Show that for any line l_p in $\check{\mathbb{P}}^2$ corresponding to lines in $\mathbb{P}^2$ through the point p the curve $\omega_p = \Phi_T^{-1}(l_p)$ contains p. Find the equation of the curve ω_p.

(iv) Show that any fixed point of T in the set of definition of T is a base point of Φ_T.

7.7. Let $ABCD$ be a quadrangle in $\mathbb{P}^2$, and P, Q be the intersection points of two pairs of opposite sides $\langle AB, CD\rangle$ and $\langle BC, AD\rangle$. Let P', Q' be the intersection points of the line $\langle P, Q\rangle$ with the diagonals $\langle A, C\rangle$ and $\langle B, D\rangle$. Show that the pairs (P, Q) and (P', Q') are harmonic pairs.

7.8 Let $f : E \to \mathbb{P}^1$ be a double cover of $\mathbb{P}^1$ ramified over four points. Show that E admits an automorphism of order 4 if and only if one can order the branch points such that their cross-ratio is equal to -1.

7.9 Show that the cross-ratio of 4 ordered points in $\mathbb{P}^2$ is invariant with respect to projective transformations.

7.10 Show that two hyperelliptic plane curves H_{g+2} and H'_{g+2} are birationally isomorphic if and only if there exists a De Jonquières transformation which transforms one curve to another.

7.11 Let H_{g+2} be a hyperelliptic curve given by the equation (7.23). Consider the linear system of hyperelliptic curves $H_{q+2} = V(T_2^2 G_q(T_0,T_1) + 2T_2 G_{q+1}(T_0,T_1) + G_{q+2}(T_0,T_1))$ such that $F_g G_{q+2} - 2F_{g+1}G_{q+1} + F_{g+2}G_q = 0$. Show that

(i) the curves H_q exist if $q \geq (g-2)/2$;

(ii) the branch points of H_{g+2} belong to H_q and vice versa;

(iii) the curve H_q is invariant with respect to the De Jonquières involution IH_{g+2} defined by the curve H_{g+2} and the curve H_{g+2} is invariant with respect to the the De Jonquières involution IH_{q+2} defined by the curve H_q;

(iv) the involutions IH_{g+2} and IH_{q+2} commute with each other;

(v) the fixed locus of the composition $H_{g+2} \circ H_{q+2}$ is given by the equation $V(F_{g+q+3})$, where

$$F_{g+q+3} = \det \begin{pmatrix} F_g & F_{g+1} & F_{g+2} \\ G_q & G_{q+1} & G_{q+2} \\ 1 & -T_2 & T_2^2. \end{pmatrix}$$

(vi) the De Jonquières transformations which leave the curve H_{g+2} invariant form a group. It contains an abelian subgroup of index 2 which consist of transformations which leave H_{g+2} pointwisely fixed.

7.12 Consider the linear system $L_{a,b} = |af + bs|$ on $\mathbf{F}_n$, where s is the divisor class of the exceptional section, and f is the divisor class of a fibre. Assume $a, b \geq 0$. Show that

(i) $L_{a,b}$ has no fixed part if and only if $a \geq nb$;

(ii) $L_{a,b}$ has no base points if and only if $a \geq nb$,

(ii) Assume $b = 1$ and $a \geq n$. Show that the linear system $L_{a,1}$ maps $\mathbf{F}_n$ in $\mathbb{P}^{2a-n+1}$ onto a surface $X_{a,n}$ of degree $2a-n$;

(iii) Show that the surface $X_{a,n}$ is isomorphic to the union of lines $\langle v_a(x), v_{a-n}(x)\rangle$, where $v_a : \mathbb{P}^1 \to \mathbb{P}^a, v_{2a-n} : \mathbb{P}^1 \to \mathbb{P}^{a-n}$ are the Veronese maps, and $\mathbb{P}^a$ and $\mathbb{P}^{a-n}$ are identified with two disjoint projective subspaces of $\mathbb{P}^{2a-n+1}$.

7.13 Show that the weighted projective plane $\mathbb{P}(1,1,n)$ is isomorphic to the surface $X_{n,n}$.

7.14 Show that the surface $X_{a,n} \subset \mathbb{P}^{2a-n+1}$ contains a nonsingular curve C of genus $g = 2a - n + 2$ which is embedded in $\mathbb{P}^{2a-n+1}$ by the canonical linear system $|K_C|$.

7.15 Find the automorphism group of the surface $\mathbf{F}_n$.

7.16 Show that a projective automorphism T of $\mathbb{P}^2$ which fixes two points is equal to $\Phi_{x_0}(g)$ for some automorphism of $\mathbf{F}_0$ and a point $x_0 \in \mathbf{F}_0$.

7.17 Compute a characteristic matrix of a De Jonquières transformation.

10.2 Compute a characteristic matrix of symmetric Cremona transformations from Example 8.3.2.

7.18 Let C be an irreducible plane curve of degree $d > 1$ passing through the points $p_1, \ldots, p_n$ with multiplicities $m_1 \geq \ldots \geq m_n$. Assume that its proper inverse transform under the blowing up the points $p_1, \ldots, p_n$ is a smooth rational curve $\bar{C}$ with $\bar{C}^2 = -1$. Show that $m_1 + m_2 + m_3 > d$.

7.19 Compute a characteristic matrix of the composition $T \circ T'$ of a De Jonquières transformation T with F-points $p_1, p_2, \ldots, p_{2d-1}$ and characteristic vector $(d, d-1, 1, \ldots, 1)$ and a quadratic transformation with F-points p_1, p_2, p_3.

7.20 Let $g : \mathbb{A}^2 \to \mathbb{A}^2$ be an automorphism of the affine plane given by a formula $(x, y) \to (x + P(y), y)$, where P is a polynomial of degree d in one variable. Consider g as a Cremona transformation. Compute its characteristic matrix. In the case $d = 3$ write as a composition of projective transformations and quadratic transformations.

7.21 Show that every Cremona transformation is a composition of the following maps ("links"):

(i) the switch involution $\tau : \mathbf{F}_0 \to \mathbf{F}_0$;

(ii) the blowup $\sigma : \mathbf{F}_1 \to \mathbb{P}^2$;

(iii) the inverse $\sigma^1 : \mathbb{P}^2 - \to \mathbf{F}_1$;

(iv) an elementary transformation $\mathrm{elm}_x : \mathbb{F}_q - \to \mathbf{F}_{q\pm 1}$.

7.22 Show that any Cremona transformation is a composition of De Jonquières transformations and projective automorphisms.

7.23 Let $x_0 = (0, 1) \times (1, 0), \times(1, 0) \in \mathbb{P}^1 \times \mathbb{P}^1$, $y_0 = \tau(x_0)$. Show that t_{y_0,x_0} is given by the formula $(u_0, u_1) \times (v_0, v_1) \mapsto (u_0, u_1) \times (u_0v_1, u_2v_0)$. Check that the composition $T = \tau \circ t_{y_0,x_0}$ satisfies $T^3 = \mathrm{id}$.

Chapter 8

Del Pezzo surfaces

8.1 First properties

8.1.1 Varieties of minimal degree

Recall that a subvariety $X \subset \mathbb{P}^n$ is called *nondegenerate* if it is not contained in a proper linear subspace. Let $d = \deg(X)$. We have the following well-known (i.e., can be found in modern text-books, e.g. [Harris]) result.

Theorem 8.1.1. *Let X be an irreducible nondegenerate subvariety of $\mathbb{P}^n$ of dimension k and degree d. Then $d \geq n - k + 1$, and the equality holds only in one of the following cases:*

(i) X is an irreducible quadric hypersurface;

(ii) a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$

(iii) a cone over a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(iv) a rational normal scroll.

Recall that a *rational normal scroll* is defined as follows. Choose k disjoint linear subspaces $L_1, \ldots, L_k$ in $\mathbb{P}^n$ which span the space. Let $a_i = \dim L_i$. We have $\sum_{i=1}^k a_i = n-k+1$. Consider Veronese maps $v_{a_i} : \mathbb{P}^1 \to L_i$ and define $X_{a_1,\ldots,a_k;n}$ to be the union of linear subspaces spanned by the points $v_{a_1}(x), \ldots, v_{a_k}(x)$, where $x \in \mathbb{P}^1$. It is clear that $\dim X_{a_1,\ldots,a_k;n} = k$ and it is easy to see that $\deg X_{a_1,\ldots,a_k;n} = a_1 + \ldots + a_k$.

Corollary 8.1.2. *Let S be an irreducible nondegenerate surface in $\mathbb{P}^n$ of degree d. Then $d \geq n - 1$ and the equality holds only in one of the following cases:*

(i) *X is a nonsingular quadric in $\mathbb{P}^3$;*

(ii) *X is an irreducible quadric cone in $\mathbb{P}^3$;*

(ii) *a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;*

(iii) *a rational normal scroll $X_{a,n} \subset \mathbb{P}^{2a-n+1}$.*

Del Pezzo surfaces come next. We will see that nonsingular surfaces in $\mathbb{P}^n$ of degree n are Del Pezzo surfaces. But first, let us define them.

Definition 8.1. *A Del Pezzo surface is a nonsingular surface with $-K_S$ nef and big. A Del Pezzo surface is called a Fano surface if $-K_S$ is ample.*

Recall that a divisor D is called *nef* if for any irreducible curve C the intersection number $C \cdot D$ is non-negative. It is called *big* if $D^2 > 0$. Note that if we require $C \cdot D > 0$ instead of $C \cdot D \geq 0$, then D is an ample divisor. This follows from the Moishezon-Nakai criterion of ampleness.

8.1.2 A blow-up model

Lemma 8.1.3. *Let S be a Del Pezzo surface. Then, any irreducible curve C on S with negative self-intersection is a smooth rational curve with $C^2 = -1$ or -2.*

Proof. By adjunction

$$C^2 + C \cdot K_S = \deg \omega_C = 2 \dim H^1(C, \mathcal{O}_C) - 2$$

By definition of a Del Pezzo surface, we have $C \cdot K_S \leq 0$. Thus $0 > C^2 > -2$ and $H^1(C, \mathcal{O}_C) = 0$. It is easy to show that the latter equality implies that $C \cong \mathbb{P}^1$ (the genus of the normalization of an irreducible curve is less or equal to the arithmetic genus defined as $\dim H^1(C, \mathcal{O}_C)$ and the difference is positive if the curve is singular). □

Lemma 8.1.4. *Let S be a Del Pezzo surface. Then*

$$H^i(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0.$$

Proof. We write $0 = -K_S + K_S$ and apply the following Ramanujam's Vanishing theorem: for any nef and big divisor D on a nonsingular projective variety X

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0, \ i > 0.$$

□

Theorem 8.1.5. *Let S be a Del Pezzo surface. Then, either $X \cong \mathbf{F}_n, n = 0, 2$, or X is obtained from $\mathbb{P}^2$ by blowing up $k \leq 8$ points in the bubble space.*

Proof. Since K_S is not nef, a minimal model for S is either a minimal ruled surface V (over some base curve B) or $\mathbb{P}^2$. Since $H^1(S, \mathcal{O}_S) = 0$, we must have $B \cong \mathbb{P}^1$ (use that the projection $p : V \to B$ satisfies $p_*(\mathcal{O}_V) \cong \mathcal{O}_B$ and this defines a canonical injective map $H^1(B, \mathcal{O}_B) \to H^1(V, \mathcal{O}_V)$). Thus $V = \mathbf{F}_n$ or $\mathbb{P}^2$. Assume $V = \mathbf{F}_n$. If $n > 2$, the exceptional section of V has self-intersection $r < -2$. Its proper inverse transform on S has self-intersection $\leq r$. This contradicts Lemma 8.1.3. Thus $n \leq 2$. If $n = 1$, then composing the map $S \to \mathbf{F}_1$ with p, we get a birational morphism $S \to \mathbb{P}^2$. Assume $X \ncong \mathbf{F}_n$, where $n = 0, 2$. Then the birational morphism $f : X \to \mathbf{F}_n$ is equal to the composition of $\phi : S \to V'$ and a blow-up $b : V' \to \mathbf{F}_n$ of a point $p \in \mathbf{F}_n$. Assume $n = 0$, and let l_1, l_2 be two lines containing p. Let $V' \to \mathbb{P}^2$ be the blow-down of the proper transforms of the lines. Then the composition $S \to V' \to \mathbb{P}^2$ is a birational morphism to $\mathbb{P}^2$. Assume $n = 2$. The point p does not belong to the exceptional section since otherwise its proper inverse transform in S has self-intersection < -2. Let l be the fibre of $p : \mathbf{F}_2 \to \mathbb{P}^1$ which passes through p. Then elm_p maps $\mathbf{F}_2$ to $\mathbf{F}_1$ and hence blowing down the proper inverse transform of l defines a birational morphism $S \to V' \to \mathbf{F}_1$. Composing it with the birational morphism $\mathbf{F}_1 \to \mathbb{P}^2$, we get a birational morphism $\pi : S \to \mathbb{P}^2$.

The last assertion follows from the known behavior of the canonical class of S under a blow-up. If $\pi : S \to \mathbb{P}^2$ is a birational morphism which is a composition of k blow-ups, then

$$K_S^2 = K_{\mathbb{P}^2}^2 - k = 9 - k. \tag{8.1}$$

By definition, $K_S^2 > 0$, so $k < 9$.

□

Definition 8.2. *The number $d = K_S^2$ is called the* degree *of a Del Pezzo surface.*

We know that S does not contain curves with self-intersection < -2. In particular, any exceptional cycle $\mathcal{E}_i$ of the birational morphism $\pi : S \to \mathbb{P}^2$ contains only smooth rational curves E with $E^2 = -1$ or -2. This easily implies that the bubble points corresponding to $\mathcal{E}_i$ represent a totally ordered chain $x_t \succ_1 x_{t-1} \succ_1 \ldots\ldots \succ_1 x_1$. Thus $\mathcal{E}_i = E_1 + \ldots + E_t$, where $E_i^2 = -2,\ i \neq t, E_t^2 = -1$ and $E_i \cdot E_{i+1} = 1, i = 1, \ldots, t-1$, with all other intersections $E_i \cdot E_j$ equal to zero.

Lemma 8.1.6. *Let X be a nonsingular projective surface with $H^1(X, \mathcal{O}_X) = 0$. Let C be an irreducible curve on X such that $|-K_X - C| \neq \emptyset$ and $C \notin |-K_X|$. Then $C \cong \mathbb{P}^1$.*

Proof. We have $-K_X \sim C + D$ for some nonzero effective divisor D, and hence $K_X + C \sim -D \not\sim 0$. This shows that $|K_X + C| = \emptyset$. By Riemann-Roch,

$$0 = \dim H^0(X, \mathcal{O}_X(K_X + C)) = \tfrac{1}{2}((K_X + C)^2 - (K_X + C) \cdot K_X) + 1$$

$$- \dim H^1(X, \mathcal{O}_X) + \dim H^2(X, \mathcal{O}_X) \geq 1 + \tfrac{1}{2}(C^2 + K_X \cdot C) = \dim H^1(C, \mathcal{O}_C).$$

This $H^1(C, \mathcal{O}_C) = 0$, and as we noted earlier, this implies that $C \cong \mathbb{P}^1$. □

Proposition 8.1.7. *Let S be a Del Pezzo surface.*

(i) Let $f : S \to \bar{S}$ be a blowing down a (-1)-curve E. Then $\bar{S}$ is a Del Pezzo surface.

(ii) Let $\pi : S' \to S$ be the blowing-up with center at a point x not lying on any (-2)-curve. Assume $K_S^2 > 1$. Then S' is a Del Pezzo surface.

Proof. (i) We have $K_S = f^*(K_{\bar{S}}) + E$, and hence, for any curve C on $\bar{C}S$, we have

$$K_{\bar{S}} \cdot C = f^*(K_{\bar{S}}) \cdot f^*(C) = (K_S - E) \cdot f^*(C) = K_S \cdot f^*(C) \leq 0.$$

Also $K_{\bar{S}}^2 = K_S^2 + 1 > 0$. Thus $\bar{S}$ is a Del Pezzo surface.

(ii) Since $K_S^2 > 2$, we have $K_{S'}^2 = K_S^2 - 1 > 0$. By Riemann-Roch,

$$\dim |-K_{S'}| \geq \tfrac{1}{2}((-K_{S'})^2 - (-K_{S'} \cdot K_{S'})) = K_{S'}^2 \geq 0.$$

Thus $|-K_{S'}| \neq \emptyset$, and hence, any irreducible curve C with $-K_{S'} \cdot C < 0$ must be a proper component of some divisor from $|-K_{S'}|$ (it cannot be linearly equivalent to $-K_{S'}$ because $(-K_{S'})^2 > 0$. Let $E = \pi^{-1}(x)$. We have $-K_{S'} \cdot E = 1 > 0$. So we may assume that $C \neq E$. Let $\bar{C} = f(C)$. We have

$$-K_{S'} \cdot C = \pi^*(-K_S) \cdot C - E \cdot C = -K_S \cdot \bar{C} - \text{mult}_x(\bar{C}).$$

Since $f_*(K_{S'}) = K_S$ and $C \neq E$, the curve $\bar{C}$ is a proper irreducible component of some divisor from $|-K_S|$. By Lemma 8.1.6, $\bar{C} \cong \mathbb{P}^1$. Thus $\text{mult}_x\bar{C} \leq 1$ and hence $0 > -K_{S'} \cdot C \geq -K_S \cdot \bar{C} - 1$. This gives $-K_S \cdot \bar{C} = 0$ and $x \in \bar{C}$ and hence $\bar{C}$ is a (-2)-curve. Since x does not lie on any (-2)-curve we get a contradiction. □

8.1.3 (-2)-curves

We call a smooth rational curve with negative self-intersection $-n$ a $(-n)$-*curve*.

Recall that a *symmetric Cartan matrix* is a symmetric positive definite matrix $C = (a_{ij})$ of size n with $a_{ii} = 2$ and $a_{ij} = 0$ or 1 when $i \neq j$. All such matrices can be classified. Each Cartan matrix is a block-sum of irreducible Cartan matrices. There are two infinite series of irreducible matrices of types A_n and D_n and three exceptional irreducible matrices of type E_n, where $n = 6, 7, 8$. The matrix $C - 2I_n$, where C is an irreducible Cartan matrix, is the incidence matrix of the Coxeter-Dynkin diagram of type A_n, D_n, E_n.

Proposition 8.1.8. *Let S be a Del Pezzo surface of degree $d = 9 - k$. The number of (-2)-curves on S is less or equal than k. Let $\mathcal{N}(S)$ be the union of such curves. Then $\mathcal{N}$ is a disjoint sum of connected curves (not necessary irreducible) $\mathcal{N}_i = \sum R_i^{(j)}, j = 1, \ldots, n_i$, such that the matrix $(R_i^{(j)} \cdot R_i^{(j')})$ of intersection numbers is equal to the opposite of an irreducible Cartan matrix.*

Proof. If $S = \mathbf{F}_0$ no such curves exist. If $S = \mathbf{F}_2$, there is only one such curve, the exceptional section. In this case the $\mathcal{N} = \mathcal{N}_1$, and the intersection matrix is (-2), the opposite of the Cartan matrix of the root system of type A_1. Now assume that S is obtained from $\mathbb{P}^2$ by blowing up $k \leq 8$ points. Let R be an irreducible curve with self-intersection -2. We know that $R \cong \mathbb{P}^1$ and $R \cdot K_S = 0$. Thus R belongs to the orthogonal complement $(\mathbb{Z}K_S)^\perp$ in Pic(S). We know from Chapter 9 that the orthogonal complement is a negative definite lattice. I claim that the divisor classes of (-2)-curves are linearly independent. Indeed, suppose that this is not true. Then we can find two disjoint sets of curves $R_i, i \in I$, and $R_j, j \in J$, such that

$$\sum_{i \in I} n_i R_i \sim \sum_{j \in J} m_j R_j,$$

where n_i, m_j are some non-negative integers. Taking intersection of both sides with R_i we obtain that

$$R_i \cdot (\sum_{i \in I} n_i R_i) = R_i \cdot (\sum_{j \in J} m_j R_j) \geq 0.$$

This implies that

$$(\sum_{i \in I} n_i R_i)^2 = \sum_{i \in I} n_i R_i \cdot (\sum_{i \in I} n_i R_i) \geq 0.$$

Since $(\mathbb{Z}K_S)^\perp$ is negative definite, this could happen only if $\sum_{i \in I} n_i R_i \sim 0$. Since all coefficients are non-negative, this happens only if all $n_i = 0$. For the same reason each m_i is equal to 0.

Let N be the subgroup of $(\mathbb{Z}K_S^\perp) \subset \mathrm{Pic}(S)$ generated by the divisor classes of (-2)-curves. We know that $\mathrm{Pic}(S) \cong \mathbb{Z}^{1+k}$ (each blow-up increases the rank of the Picard group by 1). Thus the rank of N is less or equal then k, and the rank is equal to the number of (-2)-curves.

Let A be the matrix of the intersection form in $\mathrm{Pic}(S)$ restricted to N computed in the basis formed by (-2)-curves. Then it is a negative definite symmetric matrix with -2 at the diagonal. For any two distinct (-2)-curves R_i, R_j we have $R_i \cdot R_j \leq 1$. Indded, if $R_i \cdot R_j > 1$, then $(R_i + R_j)^2 = -2 + 2R_i \cdot R_j \geq 0$ contradicting the negative definiteness of the matrix. Thus the matrix $-(R_i \cdot R_j)_{ij}$ is a symmetric Cartan matrix. A connected component of $\mathcal{N}$ corresponds to an irreducible Cartan matrix. □

Definition 8.3. *A* Dynkin curve *is a reduced connected curve R on a projective nonsingular surface X such that its irreducible components R_i are smooth rational curves with self-intersection -2 and the matrix $(R_i \cdot R_j)_{ij}$ is the opposite of a Cartan matrix. The type of a Dynkin curve is the type of the corresponding root system.*

Theorem 8.1.9. *Let R be a Dynkin curve on a projective nonsingular surface X. There is a birational morphism $f : X \to Y$, where Y is a normal surface satisfying the following properties*

(i) *$f(R)$ is a point;*

(ii) *the restriction of f to : $X \setminus R$ is an isomorphism;*

(iii) *$f^*(\omega_Y) \cong \omega_X$.*

Proof. Let H be a very ample divisor on X. Since the intersection matrix of components of $R = \sum_{i=1}^n R_i$ has non-zero determinant, we can find rational numbers r_i such that

$$(\sum_{i=1}^n r_i R_i) \cdot R_j = -H \cdot R_j, \quad j = 1, \ldots, n.$$

It is easy to see that the entries of the inverse of a Cartan matrix are nonpositive. Thus all r_i's are nonnegative numbers. Replacing H by some multiple mH, we may assume that all r_i are nonnegative integers. Let $D = \sum r_i R_i$. Since $H + D$ is effective divisor and $(H + D) \cdot R_i = 0$ for each i, we have $\mathcal{O}_X(H + D) \otimes \mathcal{O}_{R_i} = \mathcal{O}_{R_i}$. Consider the standard exact sequence

$$0 \to \mathcal{O}_X(H) \to \mathcal{O}_X(H + D) \to \mathcal{O}_D \to 0.$$

Replacing H by mH, we may assume, by Serre's Theorem, that $H^1(X, \mathcal{O}_X(H)) = 0$ and $\mathcal{O}_X(H)$ is generated by global sections. Let $s_0, \ldots, s_{N-1}$ be sections of

$\mathcal{O}_X(H)$ which define an embedding in $\mathbb{P}^{N-1}$. Consider them as sections of $\mathcal{O}_X(H+D)$. Let s_{N+1} be a section of $\mathcal{O}_X(H+D))$ which maps to $1 \in H^0(X, \mathcal{O}_D)$. Consider the map $f' : X \to \mathbb{P}^N$ defined by the sections $(s_0, \ldots, s_N)$. Then $f'(D) = (0, \ldots, 0, 1)$ and $f'|X \subset D$ is an embedding. So we obtain a map $f : X \to \mathbb{P}^N$ satisfying properties (i) and (ii). Since X is normal, f' factors through a map $f : X \to Y$, where Y is normal. Let ω_Y be the canonical sheaf of Y (it is defined as $j_*(\omega_{Y\setminus f'(R)})$, where $j : Y \setminus f'(R) \to Y$ is the natural open embedding). We have

$$\omega_X = f^*(\omega_Y) \otimes \mathcal{O}_X(A)$$

for some divisor A. Since $K_X \cdot R_i = 0$ for each i, and $f^*(\omega_Y) \otimes \mathcal{O}_{R_i} = \mathcal{O}_{R_i}$ we get $A \cdot R_i = 0$. Since the intersection matrix of R is negative definite we obtain $A = 0$. □

Definition 8.4. *A point $y \in Y$ of a normal variety Y is called a* canonical singularity *if there exists a resolution $\pi : X \to Y$ such that $\pi^*(\omega_Y) \cong \omega_X$. In the case $\dim Y = 2$, a canonical singularity is called a RDP (*rational double point*).*

We state the next well-known theorem without proof.

Theorem 8.1.10. *Let $y \in Y$ be a RDP and $\pi : X \to Y$ be a resolution such that $\pi^*(\omega_Y) \cong \omega_X$. Then $\pi^{-1}(y)$ is a Dynkin curve. Moreover (Y, y) is analytically equivalent to one of the following singularities*

$$\begin{aligned} A_n &: \quad z^2 + x^2 + y^{n+1} = 0, \quad n \geq 1 \\ D_n &: \quad z^2 + y(x^2 + y^{n-2} = 0, \quad n \geq 4 \\ E_6 &: \quad z^2 + x^3 + y^4 = 0 \\ E_7 &: \quad z^2 + x^3 + xy^3 = 0 \\ E_8 &: \quad z^2 + x^3 + y^5 = 0 \end{aligned} \tag{8.2}$$

The corresponding Dynkin curve is of respective type A_n, D_n, E_n.

8.2 Anti-canonical models

Lemma 8.2.1. *Let S be a Del Pezzo surface with $K_S^2 = d$. Then*

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = 1 + \tfrac{1}{2}r(r+1)d.$$

Proof. By Kodaira-Ramanujam's Vanishing theorem, for any $r \geq 0$ and $i > 0$,

$$H^i(S, \mathcal{O}_S(-rK_S)) = H^i(S, \mathcal{O}_S(K_S + (-r-1)K_S)) = 0. \tag{8.3}$$

The Riemann-Roch Theorem gives

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = \tfrac{1}{2}(-rK_S - K_S)\cdot(-rK_S)) + 1 = 1 + \tfrac{1}{2}r(r+1)d.$$

□

Theorem 8.2.2. *Let S be a Del Pezzo surface of degree d and $\mathcal{N}$ be the union of (-2)-curves on S. Then*

(i) $|-K_S|$ has no fixed part.

(ii) If $d > 1$, then $|-K_S|$ has no base-points.

(iii) If $d > 2$, $|-K_S|$ defines a regular map ϕ to $\mathbb{P}^d$ which is an isomorphism outside $\mathcal{N}$. The image surface $\bar{S}$ is a normal nondegenerate surface of degree d. The image of each connected component of $\mathcal{N}$ is a RDP of $\phi(S)$.

(iv) If $d = 2$, $|-K_S|$ defines a regular map $\phi : S \to \mathbb{P}^2$. It factors as a birational morphism $f : S \to \bar{S}$ onto a normal surface and a finite map $\pi : \bar{S} \to \mathbb{P}^2$ of degree 2 branched along a curve of degree 4. The image of each connected component of $\mathcal{N}$ is a RDP of $\bar{S}$.

(v) If $d = 1$, $|-2K_S|$ defines a regular map $\phi : S \to \mathbb{P}^3$. It factors as a birational morphism $f : S \to \bar{S}$ onto a normal surface and a finite map $\pi : \bar{S} \to Q \subset \mathbb{P}^3$ of degree 2, where Q is a quadric cone. The morphism π is branched along a curve of degree 6 cut out on Q by a cubic surface. The image of each connected component of $\mathcal{N}$ under f is a RDP of $\bar{S}$.

Proof. The assertions are easily verified if $S = \mathbf{F}_0$ or $\mathbf{F}_2$. So we assume that S is obtained from $\mathbb{P}^2$ by blowing up $k = 9 - d$ points x_i.

(i) Assume there is a fixed part F of $|-K_S|$. Write $|-K_S| = F + |M|$, where $|M|$ is the mobile part. If $F^2 > 0$, by Riemann-Roch,

$$\dim |F| \geq \tfrac{1}{2}(F^2 - F\cdot K_S) \geq \tfrac{1}{2}(F^2) > 0,$$

and hence F moves. Thus $F^2 \leq 0$. If $F^2 = 0$, we must also have $F \cdot K_S = 0$. Thus $F = \sum n_i R_i$, where R_i are (-2)-curves. Hence $[F] \in (\mathbb{Z}K_S)^{\perp}$ and hence $F^2 \leq -2$ (the intersection form on $(\mathbb{Z}K_S)^{\perp}$ is negative definite and even). Thus $F^2 \leq -2$. Now

$$\begin{aligned} M^2 &= (-K_S - F)^2 = K_S^2 + 2K_S\cdot F + F^2 \leq K_S^2 + F^2 \leq d - 2, \\ -K_S \cdot M &= K_S^2 + K_S \cdot F \leq d. \end{aligned}$$

Suppose $|M|$ is irreducible. Since $\dim|M| = \dim|-K_S| = d$, the linear system $|M|$ defines a rational map to $\mathbb{P}^d$ whose image is a non-degenerate irreducible surface of degree $\leq d-3$ (strictly less if $|M|$ has base points). This contradicts Theorem 8.1.1.

Now assume that $|M|$ is reducible, i.e. defines a rational map to a non-degenerate curve $W \subset \mathbb{P}^d$ of some degree t. By Theorem 8.1.1, we have $t \geq d$. Since S is rational, W is a rational curve, and then the pre-image of a general hyperplane section is equal to the disjoint sum of t linearly equivalent curves. Thus $M \sim tM_1$ and

$$d \geq -K_S \cdot M = -tK_S \cdot M_1 \geq d(-K_S \cdot M_1).$$

Since $-K_S \cdot M = 0$ implies $M^2 < 0$ and a curve with negative self-intersection does not move, this gives $-K_S \cdot M_1 = 1, d = t$. But then $M^2 = d^2M_1^2 \leq d-2$ gives a contradiction.

(ii) Assume $d > 1$. We have proved that $|-K_S|$ is irreducible. A general member of $|-K_S|$ is an irreducible curve C with $\omega_C = \mathcal{O}_C(C+K_S) = \mathcal{O}_C$. If C is smooth, then it is an elliptic curve and the linear system $|\mathcal{O}_C(C)|$ is of degree $d > 1$ and has no base points. The same is true for a singular irreducible curve of arithmetic genus 1. This is proved in the same way as in the case of a smooth curve. Consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

Applying the exact sequence of cohomology, we see that the restriction of the linear system $|C| = |-K_S|$ to C is surjective. Thus we have an exact sequence of groups

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(S, \mathcal{O}_C(C)) \to 0.$$

Since $|\mathcal{O}_C(C)|$ has no base-points, we have a surjection

$$H^0(S, \mathcal{O}_C(C)) \otimes \mathcal{O}_C \to \mathcal{O}_C(C).$$

This easily implies that the homomorphism

$$H^0(S, \mathcal{O}_S(C)) \otimes \mathcal{O}_C \to \mathcal{O}_S(C)$$

is surjective. Hence $|C| = |-K_S|$ has no base-points.

(iii) Assume $d > 2$. Let $x, y \in S$ be two points outside E. Let $f : S' \to S$ be the blowing up of x and y. By Lemma 8.1.7, blowing them up, we obtain a Del Pezzo surface S' of degree $d-2$. We know that the linear system $|-K_{S'}|$ has no fixed components. Thus

$$\dim|-K_S - x - y| = \dim|-K_{S'} - E_x - E_y| \geq 1.$$

This shows that $|-K_S|$ separates points. Also, the same is true if $y \succ_1 x$ and x does not belong to any (-1)-curve E on S or $x \in E$ and y does not correspond to the tangent direction defined by E. Since $-K_S \cdot E = 1$ and $x \in E$, the latter case does not happen.

Since $\phi : S \to \bar{S}$ is a birational map given by a complete linear system $|-K_S|$, its image is a nondegenerate surface of degree $d = (-K_S)^2$. Since $-K_S \cdot R = 0$ for any (-2)-curve, we see that ϕ blows down R to a point p. If $d = 3$. $\bar{S}$ is a cubic surface with isolated singularities (the images of connected components of $\mathcal{N}$). It is well-known that a hypersurface with no singularities in codimension 1 is a normal variety. Thus $\bar{S}$ is a normal surface. If $d = 4$, then S is obtained by a blow-up one point on a Del Pezzo surface S' of degree 3. This point does not lie on a (-2)-curve. Thus, $\bar{S}'$ is obtained from $\bar{S}$ by a linear projection from a nonsingular point. Since $\bar{S}'$ is normal, $\bar{S}$ must be normal too (its local rings are integral extensions of local rings of $\bar{S}'$, and their fields of fractions coincide). Continuing in this way we see that $\bar{S}$ is normal for any $d > 2$.

The fact that singular points of $\bar{S}$ are RDP is proven in the same way as we have proved assertion (iii) of Theorem 8.1.9.

(iv) Assume $d = 2$. By (ii), the linear system $|-K_S|$ defines a regular map $\phi : S \to \mathbb{P}^2$. Since $K_S^2 = 2$, the map is of degree 2. Using Stein's factorization [Hartshorne], it factors through a birational morphism onto a normal surface $f : S \to \bar{S}$ and a finite degree 2 map $\pi : \bar{S} \to \mathbb{P}^2$. Also we know that $f_*(\mathcal{O}_S) = \mathcal{O}_{\bar{S}}$. A standard Hurwitz's formula gives

$$\omega_{\bar{S}} \cong \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L}), \tag{8.4}$$

where $s \in H^0(\mathbb{P}^2, \mathcal{L}^{\otimes 2})$ vanishes along the branch curve W of π. We have

$$\mathcal{O}_S(K_S) = \omega_S = (\pi \circ f)^*(\mathcal{O}_{\mathbb{P}^2}(-1)) = f^*(\pi^*(\mathcal{O}_{\mathbb{P}^2}(-1))).$$

It follows from the proof of Theorem 8.1.9 (iii) that singular points of $\bar{S}$ are RDP. Thus $f^*(\omega_{\bar{S}}) = \omega_S$, and hence

$$f^*(\omega_{\bar{S}}) \cong f^*(\pi^*(\mathcal{O}_{\mathbb{P}^2}(-1))).$$

Applying f_* and using the projection formula and the fact that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, we get $\omega_{\bar{S}} \cong \pi^*(\mathcal{O}_{\mathbb{P}^2}(-1))$. It follows from (8.4) that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and hence $\deg W = 4$.

Proof of (v). Let $\pi : X \to \mathbb{P}^2$ be the blow-up of 8 points $x_1, \ldots, x_8$. Then $|-K_S|$ is the proper inverse transform of the pencil $|3\ell - x_1 - \ldots - x_8|$ of plane cubics passing through the points $x_1, \ldots, x_8$. Let x_9 be the ninth intersection point of two cubics generating the pencil. The point $x_9' = \pi^{-1}(x_9)$ is the base point of $|-K_S|$. By Bertini's Theorem, all fibres except finitely many, are nonsingular curves (the assumption that the characteristic is zero is important here). Let F be a nonsingular member from $|-K_S|$. Consider the exact sequence

$$0 \to \mathcal{O}_S(-K_S) \to \mathcal{O}_S(-2K_S) \to \mathcal{O}_F(-2K_S) \to 0. \tag{8.5}$$

The linear system $|\mathcal{O}_F(-2K_S)|$ on F is of degree 2. It has no base-points. We know from (8.3) that $H^1(S, \mathcal{O}_S(-K_S)) = 0$. Thus the restriction map

$$H^0(S, \mathcal{O}_S(-2K_S)) \to H^0(F, \mathcal{O}_F(-2K_S))$$

is surjective. By the same argument as we used in the proof of (ii), we obtain that $|-2K_S|$ has no base points. By Lemma 8.2.1, $\dim|-2K_S| = 3$. Let $\phi : S \to \mathbb{P}^3$ be a regular map defined by $|-2K_S|$. Its restriction to any nonsingular member F of $|-K_S|$ is given by the linear system of degree 2 and hence is of degree 2. Therefore the map f is of degree $t > 1$. The image of ϕ is a surface of some degree k. Since $(-2K_S)^2 = 4 = kt$, we conclude that $k = t = 2$. Thus the image of ϕ is a quadric surface Q in $\mathbb{P}^3$ and the images of members F of $-K_S|$ are lines l_F on Q. I claim that Q is a quadric cone. Indeed, all lines l_F intersect at the base point $\phi(x_9')$ of $|-K_S|$. This is possible only of Q is a cone.

Let $S \xrightarrow{\pi} S' \xrightarrow{\phi'} Q$ be the Stein factorization. Note that a (-2)-curve R does not pass through the base point x_9 of $|-K_S|$ (because $-K_S \cdot R = 0$). Thus $\pi(x_9)$ is a nonsingular point q' of S'. Its image in Q is the vertex q of Q. Since ϕ' is a finite map, the local ring $\mathcal{O}_{S',q'}$ is a finite algebra over $\mathcal{O}_{Q,q}$ of degree 2. After completion, we may assume that $\mathcal{O}_{S',q'} \cong \mathbb{C}[[u,v]]$. If $u \in \mathcal{O}_{Q,q}$, then v satisfies a monic equation $v^2 + av + b$ with coefficients in $\mathcal{O}_{Q,q}$, where after changing v to $v + \frac{1}{2}a$ we may assume that $a = 0$. Then $\mathcal{O}_{Q,q}$ is equal to the ring of invariants in $\mathbb{C}[[u,v]]$ under the automorphism $u \mapsto u, v \mapsto -v$ which is easily to see isomorphic to $\mathbb{C}[[u,v^2]]$. However, we know that q is a singular point so the ring $\mathcal{O}_{Q,q}$ is not regular. Thus we may assume that $u^2 = a, v^2 = b$ and then $\mathcal{O}_{Q,q}$ is the ring of invariants for the action $(u,v) \mapsto (-u,-v)$. This action is free outside the maximal ideal (u,v). This shows that the finite map ϕ' is unramified in a neighborhood of q' with q' deleted. In particular, the branch curve Q of ϕ' does not pass through q. We leave to the reader to repeat the argument from the proof of (iv) to show that the branch curve W of ϕ belongs to the linear system $|\mathcal{O}_Q(3)|$. □

Remark 8.2.1. Let $d = 1$ and $a : X \to S$ be the blow-up the base point x_9 of the pencil $-|K_S|$. The pre-image of $|-K_S|$ on X is a base-free pencil which defines a morphism $f : X \to \mathbb{P}^1$ whose general fibres are elliptic curves. The exceptional curve $E = a^{-1}(x_9)$ is a section. The linear system $|a^*(-2K_S) + 2E|$ defines a degree 2 map to the minimal ruled surface $\mathbf{F}_2$. The image of E is the exceptional section s. Let $X \xrightarrow{\pi'} X' \xrightarrow{\varphi'} \mathbf{F}_2$ be the Stein factorization. The morphism π' is a resolution of singularities of rational double points. The morphism φ' is a finite map of degree 2 with branch curve W equal to the union of the exceptional section and the curve B from the linear system $|6f + 3s|$. When S is a Fano surface, the map π' is the identity and B is a nonsingular curve of genus 4.

Remark 8.2.2. The singularities of the branch curves of the double cover $S \to \mathbb{P}^2$ ($d = 2$) and $S \to Q$ (($d = 1$) are *simple singularities*. This means that in appropriate analytic (or formal) coordinates they are given by one of the following equations:

$$\begin{aligned} A_n &: x^2 + y^{n+1} = 0, \quad n \geq 1 \\ D_n &: y(x^2 + y^{n-2} = 0, \quad n \geq 4 \\ E_6 &: x^3 + y^4 = 0 \\ E_7 &: x^3 + xy^3 = 0 \\ E_8 &: x^3 + y^5 = 0 \end{aligned} \tag{8.6}$$

This easily follows from Theorem 8.1.10.

Remark 8.2.3. A double cover of $\mathbb{P}^2$ branched along a plane curve of degree $2k$ can be viewed as a hypersurface of degree $2k$ in the weighted projective space $\mathbb{P}(1, 1, 1, k)$. Thus a Del Pezzo surface of degree 2 is birationally isomorphic to a quartic hypersurface in $\mathbb{P}(1, 1, 1, 2)$.

Similarly, a double cover of the quadratic cone branched along a curve cut out by a hypersurface of degree $2k + 1$ not passing through the vertex can be viewed as a hypersurface of degree $4k + 2$ in $\mathbb{P}(1, 1, 2, 2k + 1)$. Thus a Del Pezzo surface of degree 1 is birationally isomorphic to a sextic hypersurface in $\mathbb{P}(1, 1, 2, 3)$.

8.2.1 Surfaces of degree d in $\mathbb{P}^d$

We saw that the image of a Del Pezzo surface of degree $d > 2$ under the map given by the linear system $| - K_S|$ is a nondegenerate surface of degree d in $\mathbb{P}^d$. We call this surface an *anticanonical model* of S. It is a normal surface with canonical singularities.

Let us prove the converse. First we need the following.

Lemma 8.2.3. *Let C be a nondegenerate nonsingular irreducible curve of degree $d > 2$ in $\mathbb{P}^{d-1}$. If C is projectively normal, then C is an elliptic curve. Otherwise C is a rational curve projected from $\mathbb{P}^d$.*

Proof. Let H be a hyperplane section of C. We have

$$\dim H^0(C, \mathcal{O}_C(H)) = d + 1 - g + \dim H^1(C, \mathcal{O}_C(H)) \geq d, \qquad (8.7)$$

and the equality takes place if and only if C is projectively normal. Thus we obtain

$$g \leq 1 + \dim H^1(C, \mathcal{O}_C(H)) = 1 + \dim H^0(C, \mathcal{O}_C(K_C - H)). \qquad (8.8)$$

If $|K_C - H| = \emptyset$, then $g \leq 1$ and the equality takes place if and only if C is projectively normal. Assume $|K_C - H| \neq \emptyset$. By Clifford's Theorem [Hartshorne],

$$\dim H^0(C, \mathcal{O}_C(K_C - H)) \leq 1 + \tfrac{1}{2}\deg(K_C - H) = 1 + (g - 1 - \tfrac{1}{2}d) = g - \tfrac{1}{2}d,$$

unless $K_C = H$ or C is a hyperellptic curve and $K_S - H = kg_2^1$ for some $k > 0$. If we are not in one of the exceptional cases, we obtain $g \leq 1 + g - \frac{1}{2}d$ which is a contradiction. If $K_C = H$, we get $d = 2g - 2$ and (8.7) gives $g = 2g - 2 + 1 - g + 1 \geq 2g - 2$, hence $g = 2$ and $d = 2g - 2 = 2$, a contradiction. If $K_C - H = kg_2^1$, then $H = (g-1)g_2^1 - kg_2^1 = (g - 1 - k)g_2^1$. Since $\deg H > 2$, we get $k < g - 2$. Thus $\dim H^0(C, \mathcal{O}_C(K_C - H)) = k + 1$, and (8.8) gives $g \leq 1 + k + 1 < 2 + g - 2 = g$, a contradiction again. Thus $|K_C - H| = \emptyset$ and we are done. □

Theorem 8.2.4. *Let V be a nondegenerate normal surface of degree d in $\mathbb{P}^d$. Assume that V has at most canonical singularities. Its minimal resolution of singularities is a Del Pezzo surface S of degree d and V is an anticanonical model of S.*

Proof. Let C be a general hyperplane section of V. Since V has only finitely many singularities, by Bertini's Theorem it is a nonsingular nondegenerate curve of degree d. By the previous lemma, C is of genus ≤ 1. Assume C is a rational curve. Choosing a pencil of hyperplane sections, we find that V admits a rational map to $\mathbb{P}^1$ whose general fibre is a rational curve. It is well known (Tsen's Theorem, see [Hartshorne]) that this implies that the field of rational functions on V is a purely transcendental extension of $\mathbb{C}$. Thus V is a rational surface. Let $\pi : S \to V$ be a minimal resolution of singularities. Since V has only canonical singularities, $\pi^*(\omega_V) \cong \omega_S$. Since V is normal, $\pi_*(\mathcal{O}_S) = \mathcal{O}_V$, and by the projection formula,

$$\pi_*(\omega_S) \cong \omega_V.$$

This implies that the canonical homomorphism $H^1(V, \omega_V) \to H^1(S, \omega_S)$ is injective. Since S is a nonsingular rational surface, we get $H^1(S, \omega_S) \cong H^1(S, \mathcal{O}_S) = 0$. Thus

$$H^1(V, \omega_V) \cong H^1(V, \mathcal{O}_V) = 0,$$

and the exact sequence

$$0 \to \mathcal{O}_V \to \mathcal{O}_V(1) \to \mathcal{O}_C(1) \to 0.$$

shows that the restriction homomorphism $H^0(V, \mathcal{O}_V(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective. Thus the linear system $|\mathcal{O}_C(1)$ is complete which is contradictory to Lemma 8.2.3.

Therefore C is an elliptic curve. Let us identify it with its pre-image under π. By the adjunction formula,

$$\mathcal{O}_C = \omega_C = \mathcal{O}_S(K_S + C) \otimes \mathcal{O}_C.$$

By Riemann-Roch,

$$\dim H^0(S, \mathcal{O}_S(K_S + C)) = \tfrac{1}{2}(K_S \cdot C + C^2) + 1 = 1.$$

Thus $|K_S + C|$ consists of an isolated curve D. Since $D \cdot \pi^*(H) = 0$ for any hyperplane section H of V not passing through singularities, we obtain that each irreducible component R of D is contained in the exceptional curve of the resolution π. Since V has only canonical singularities, R is a (-2)-curve. Since $(K_S + C) \cdot R = K_S \cdot R + C \cdot R = 0$ for any irreducible component of a resolution, we get $D^2 = 0$. Since the sublattice of $\mathrm{Pic}(S)$ generated by the components of a Dynkin curve is negative definite, we get $D = 0$. Thus $K_S + C \sim 0$ and $-K_S = \pi^*(\mathcal{O}_V(1))$ is nef and big. So, S is a Del Pezzo surface of degree $K_S^2 = d$. Clearly, S is its anti-canonical model. □

Corollary 8.2.5. *Let V be a nondegenerate normal surface of degree d in $\mathbb{P}^d$. Assume that V has at most canonical singularities. Then $d \leq 9$. Moreover, V is either surface of degree 8 in $\mathbb{P}^8$ isomorphic to the image of $\mathbf{F}_n$ ($n = 0, 2$) under the map defined by the linear system $|-2K_{\mathbf{F}_n}|$, or a projection of the Veronese surface $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$.*

Proof. Use that a Del Pezzo surface of degree ≥ 3 not isomorphic to $\mathbf{F}_0$ or $\mathbf{F}_2$ is the blow-up of $k \leq 8$ bubble points $x_1, \ldots, x_k$ in $\mathbb{P}^2$ and the linear system $|-K_S| = |3\ell - x_1 - \ldots - x_k|$. It is a subsystem of the complete linear system $|3\ell|$ defining a Veronese map $v_3 : \mathbb{P}^2 \to \mathbb{P}^9$. □

8.3 Lines on Del Pezzo surfaces

8.3.1 Cremona isometries

Definition 8.5. *An orthogonal transformation σ of* $\mathrm{Pic}(S)$ *is called a* Cremona isometry *if $\sigma(K_S) = K_S$ and σ sends any effective class to an effective class. The group of Cremona isometries will be denoted by* $\mathrm{Cr}(S)$.

Proposition 8.3.1. *An isometry σ of* $\mathrm{Pic}(S)$ *is a Cremona isometry if and only if preserves the canonical class and sends a (-2)-curve to a (-2)-curve.*

Proof. Clearly, any Cremona isometry sends the class of irreducible curve to the class of an irreducible curve. Since it also preserves the intersection form, its sends (-2)-curve to an (-2)-curve.

Let us prove the converse. Let D be an effective class in $\mathrm{Pic}(S)$ with $D^2 \geq 0$. Then $-K_S \cdot D > 0$ and $(K_S - D) \cdot D < 0$. This gives $-K_S \cdot \sigma(D) > 0,\ \sigma(D)^2 \geq 0$. Since $(K_S - \sigma(D)) \cdot (-K_S) = -K_S^2 + \sigma(D) \cdot K_S < 0$, we have $|K_S - \sigma(D)| = \emptyset$. By Riemann-Roch, $|\sigma(D)| \neq \emptyset$.

So it remains to show that σ sends any (-1)-curve to an effective divisor class. This follows from the next lemma. □

Lemma 8.3.2. *Let D be a divisor class with $D^2 = D \cdot K_S = -1$. Then $D = E + R$, where R is a nonnegative sum of (-2)-curves, and E is either a (-1)-curve or $K_S^2 = 1$ and $E \in | - K_S|$. Moreover D is a (-1)-curve if and only if for each (-2)-curve R_i on S we have $D \cdot R_i \geq 0$.*

Proof. Let $e_0 = \pi^*(\ell)$, where $\pi : S \to \mathbb{P}^2$ is a birational morphism and ℓ is a line. We know that $e_0^2 = 1, e_0 \cdot K_S = -3$. Thus $((D \cdot e_0)K_S + 3D) \cdot e_0 = 0$ and hence

$$\big((D \cdot e_0)K_S + 3D\big)^2 = -6D \cdot e_0 - 9 + (D \cdot e_0)^2 K_S^2 < 0.$$

Thus $-6D \cdot e_0 - 9 < 0$ and hence $D \cdot e_0 > -9/6 > -2$. This shows that $(K_S - D) \cdot e_0 = -3 - D \cdot e_0 < 0$, and since e_0 is nef, we obtain that $|K_S - D| = \emptyset$. Applying Riemann-Roch we get $\dim |D| \geq 0$. Write an effective representative of D as a sum of irreducible components and use that $D \cdot (-K_S) = 1$. Since $-K_S$ is nef, there is only one component E entering with coefficient 1 satisfying $E \cdot K_S = -1$, all other components are (-2)-curves. If $D \sim E$, then $D^2 = E^2 = -1$ and E is a (-1)-curve. Let $\pi : S' \to S$ be a birational morphism of a Del Pezzo surface of degree 1 (obtained by blowing up $8-k$ points on S in general position not lying on E). We identify E with its pre-image in S'. Then $(E + K_{S'}) \cdot K_{S'} = -1 + 1 = 0$. hence, either $S' = S$ and $E \in | - K_S|$, or

$$(E + K_{S'})^2 = E^2 + 2E \cdot K_{S'} + K_{S'}^2 = E^2 + 2E \cdot K_S + 1 = E^2 - 1 < 0.$$

Since $E \cdot K_S = -1$, E^2 is odd. Thus, the only possibility is $E^2 = -1$.

Assume $R \neq 0$. Since $-1 = E^2 + 2E \cdot R + R^2$ and $E^2 \leq 1$, $R^2 \leq -2$, we get $E \cdot R \geq 0$, where the equality take place only if $E^2 = 1$. In both cases, we get

$$-1 = (E + R)^2 = (E + R) \cdot R + (E + R) \cdot E \geq (E + R) \cdot R.$$

Thus if $D \neq E$, we get $D \cdot R_i < 0$ for some irreducible component of R. This proves the assertion. □

Definition 8.6. *A* blowing down structure *on a Del Pezzo surface S is a composition of birational morphisms*

$$\pi : S = S_k \xrightarrow{\pi_k} S_{k-1} \xrightarrow{\pi_{k-1}} \ldots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2,$$

where each $\pi : S_i \to S_{i-1}$ is the blow-up a point x_i in the bubble space of $\mathbb{P}^2$

Recall from Chapter 9 that a blowing-down structure of a Del Pezzo surface defines a basis in Pic(S) formed by the classes $e_0 = \pi^*(\ell), e_i = (\pi_k \circ \cdots \circ \pi_1)^*(E_i)$, where ℓ is a line on $\mathbb{P}^2$ and $E_i = \pi_i^{-1}(x_i)$. We call it *geometric basis*. A geometric basis defines a root basis $e_0 - e_1 - e_2 - e_3, e_1 - e_2, \ldots, e_{k-1} - e_k$ in the sublattice $(\mathbb{Z}K_S)^\perp$. We fix a standard orthonormal basis $\mathbf{e}_0, \ldots, \mathbf{e}_k$ in $\mathbb{Z}^{1,k} = \mathbb{Z}^{k+1}$ formed by the unit vectors and a basis of simple roots $\alpha_0 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ in the root lattice $Q_k = (\mathbb{Z}\kappa_k)^\perp \subset \mathbb{Z}^{1,k}$, where

$$\kappa_k = -3\mathbf{e}_0 + \mathbf{e}_1 + \ldots + \mathbf{e}_k.$$

Then a blowing-down structure defines an isometry of lattices

$$\phi : \mathbb{Z}^{1,k} \to \text{Pic}(S) \quad \text{such that } \phi(\kappa_k) = K_S.$$

We call such an isometry a *lattice marking* of S. A lattice marking defined by a blowing-down structure is called a *geometric marking*.

Definition 8.7. *A pair (S, ϕ), where S is a Del Pezzo surface and ϕ is a marking (resp. geometric marking) $\phi : \mathbb{Z}^{1,k} \to$ Pic(S) is called a* marked Del Pezzo surface *(resp.* geometrically marked Del Pezzo surface*).*

Recall from Chapter 9 that the group of orthogonal transformations σ of Pic(S) such that $\sigma(K_S) = K_S$ is equal to the Weyl group $W(S)$ of Pic(S) with respect to any geometric basis. A geometric marking defines an isomorphism between $W(S)$ and the Weyl group $W(Q_k)$ of the root lattice Q_k.

Definition 8.8. *An element $\alpha \in Q_k$ is called a* root *if $\alpha^2 = -2$. A root α is called* effective *with respect to a geometric marking of S is $\phi(\alpha)$ is the class of a (-2)-curve on S.*

Corollary 8.3.3. *Let $\mathcal{R}$ be the set of effective roots of a marked Del Pezzo surface (S, ϕ). Then the Cremona group of S is isomorphic to the subgroup of the Weyl group of Q_k which leaves the subset $\mathcal{R}$ invariant.*

Let $W(S)^n$ be the subgroup of $W(S)$ generated by reflections with respect to (-2)-curves. It acts on a marking $\varphi : \mathbb{Z}^{1,k} \to$ Pic(S) by composing on the left.

Lemma 8.3.4. *Let*

$$C^n = \{D \in \text{Pic}(S) : D \cdot R \geq 0 \textit{ for any } (-2)\textit{-curve } R\}$$

For any $D \in \text{Pic}(S)$ there exists $w \in W(S)^n$ such that $w(D) \in C^n$. If $D \in C^n$ and $w(D) \in C^n$ for some $w \in W(S)^n$, then $w(D) = D$. In other words, C^n is a fundamental domain for the action of $W(S)^n$ in $\text{Pic}(S)$.

Proof. The group $G = W(S)^n$ is a finite subgroup of the orthogonal group of the real vector space $V = \text{Pic}(S) \otimes \mathbb{R}$ equipped with the positive definite inner product defined by the intersection form on $\text{Pic}(S)$ multiplied by -1. It is generated by reflections with respect to the set Δ^+ of vectors corresponding to (-2)-curves. Let Δ be the G-orbit of the set Δ^+. For each $v \in \Delta$ let H_v be the hyperplane in V orthogonal to the vector v. A *chamber* is a connected component of the set

$$V \setminus \bigcup_{v \in \Delta} H_v.$$

The group G acts simply transitively on the set of chambers and the closure of each chamber can be taken as a fundamental domain for the action of G in V. Each fundamental domain is a finite convex polyhedron bounded by a set of hyperplanes $H_{w_1}, \ldots, H_{w_t}$. The vectors $w_1, \ldots, w_t$ form a basis in Δ in the following sense. Each $v \in \Delta$ can be written either as a linear combination of vectors $w_1, \ldots, w_t$ with either all nonnegative coefficients or all nonpositive coefficients. Also no proper subset of $w_1, \ldots, w_t$ has this property. Conversely, every basis of Δ defines a chamber by the inequalities $x \cdot w_i > 0, i = 1, \ldots, t$. In our case we can take as a basis the set Δ^+. This gives the assertion of the lemma. We refer to the theory of groups generated by reflections to [Bourbaki]. □

Theorem 8.3.5. *For any marked Del Pezzo surface (S, φ), there exists $w \in W(S)^n$ such that $(S, w \circ \varphi)$ is geometrically marked Del Pezzo surface.*

Proof. We use induction on $k = 9 - K_S^2$. Let $e_i = \phi(\mathbf{e}_i), i = 0, \ldots, k$. It follows from the proof of Lemma 8.3.1, that each e_i is an effective class. Assume e_1 is the class of a (-1)-curve E_1. Let $\pi_k : S \to S_{k-1}$ be the blowing down of E_1. Then $e_0, e_2, \ldots, e_k$ are equal to the pre-images of the divisor classes $e_0', e_2', \ldots, e_k'$ which define a marking of S_{k-1}. By induction, there exists an element $w \in W(S_1)^n$ such that $w(e_0'), w(e_2'), \ldots, w(e_k')$ define a geometric marking. Since $\pi_k(E_1)$ does not lie on any (-2)-curve (otherwise S is not Del Pezzo), we see that for any (-2)-curve R on S_{k-1}, $\pi_k^*(R)$ is a (-2) curve on S. Thus, under the canonical isomorphism $\text{Pic}(S) \cong \pi_k^*(\text{Pic}(S_{k-1}) \perp \mathbb{Z}e_1$, we can identify $W(S_{k-1})^n$ with a subgroup of $W(S)^n$. Applying w to $(e_0, \ldots, e_k)$ we get a geometric marking of S.

If e_1 is not a (-1)-curve, then we apply an element $w \in W(S)^n$ such that $w(e_1) \in C^n$. By Lemma 8.3.2, $w(e_1)$ is a (-1)-curve. Now we have a basis $w(e_0), \ldots, w(e_k)$ satisfying the previous assumption. □

Let $\varphi : \mathbb{Z}^{1,k} \to \text{Pic}(S)$ and $\varphi : \mathbb{Z}^{1,k} \to \text{Pic}(S)$ be two geometric markings corresponding to two blowing-down structures $\pi = \pi_1 \circ \ldots \circ \pi_k$ and $\pi' = \pi_1' \circ \ldots \circ \pi_k'$. Then $T = \pi' \circ \pi^{-1}$ is a Cremona transformation of $\mathbb{P}^2$ and $w = \varphi \circ \varphi'^{-1} \in W(Q_k)$ is its characteristic matrix. Conversely, if T is a Cremona transformation with F-points $x_1, \ldots, x_k$ such that their blow-up is a Del Pezzo surface S, a characteristic matrix of T defines a pair of geometric markings φ, φ' of S and an element $w \in W(Q_k)$ such that

$$\varphi = \varphi' \circ w.$$

Corollary 8.3.6. *Assume S is a Fano surface. Then any marking $\varphi : \mathbb{Z}^{1,k} \to$ $\text{Pic}(S)$ is a geometric marking. The Weyl group of Q_k acts simply transitively on the set of markings by composing on the right.*

Corollary 8.3.7. *There is a bijection from the set of geometric markings on S and the set of left cosets $W(S)/W(S)^n$.*

Proof. The group $W(S)$ acts simply transitively on the set of markings. By Theorem 8.3.5, each orbit of $W(S)^n$ contains a unique geometric marking. □

Corollary 8.3.8. *The group* $\text{Cr}(S)$ *acts on the set of geometric markings of S.*

Proof. Let $(e_0, \ldots, e_k)$ defines a geometric marking, and $\sigma \in \text{Cr}(S)$. Then there exists $w \in W(S)^n$ such that $\omega(\sigma(e_0)), \ldots, \omega(\sigma(e_k))$ defines a geometric marking. Since $\sigma(e_1)$ is a (-1)-curve E_1, it belongs to C^n. Hence, by Lemma 8.3.4, we get $w(\sigma(e_1)) = \sigma(e_1)$. This shows that $w \in W^n(\bar{S})$, where $S \to \bar{S}$ is the blow-down $\sigma(E_1)$. Continuing in this way, we see that $w \in W(\mathbb{P}^2)^n = \{1\}$. Thus $w = 1$ and we obtain that σ sends a geometric marking to a geometric marking. □

Example 8.3.1. The action of $\text{Cr}(S)$ on geometric markings is not transitive in general. For example, consider 6 distinct points $x_1, \ldots, x_6$ in $\mathbb{P}^2$ lying on an irreducible conic C. Let S be their blow-up and ϕ be the corresponding geometric marking. This is a Del Pezzo surface with a (-2)-curve R equal to the proper inverse transform of the conic. Let T be the quadratic transformation with F-points at x_1, x_2, x_3. Then $C \in |2\ell - x_1 - x_2 - x_3|$ and hence is equal to $T^{-1}(l)$, where l is a line in $\mathbb{P}^2$. This line contains the points $q_i = T(p_i), i = 4, 5, 6$. Let $q_1 = T(\langle p_2, p_3 \rangle), q_2 = T(\langle p_1, p_3 \rangle), q_3 = T(\langle p_1, p_2 \rangle)$. Then the blow-up of the points $q_1, \ldots, q_6$ is isomorphic to S and defines a geometric marking ϕ'. Let w be the corresponding element of the Weyl group $W(Q_6) = W(E_6)$. We have

$$R = 2e_0 - e_1 - \ldots - e_6 = e_0' - e_4' - e_5' - e_6'.$$

However, the element $w \in W(S)$ defined by the two bases sends e_i to e_i'. Thus $w(R) \neq R$ and hence $w \notin \mathrm{Cr}(S)$. Note that w is the reflection with respect to the root $\alpha = e_0 - e_1 - e_2 - e_3$ and $\alpha \cdot R = -1$, so that

$$r_\alpha(R) = R - \alpha = (2e_0 - e_1 - \ldots - e_6) - (e_0 - e_1 - e_2 - e_3) = e_0 - e_4 - e_5 - e_6.$$

Since the points p_4, p_5, p_6 are not collinear, this is not an effective class. The group $\mathrm{Cr}(S)$ in this case consists of permutations of the vectors $e_1, \ldots, e_6$ and is isomorphic to S_6. Its index in $W(E_6)$ is equal to 72. The group $W(S)^n$ is generated by the reflection r_α. Thus we get $\frac{1}{2}\#W(E_6) = 36 \cdot 6!$ geometric markings and the group $\mathrm{Cr}(S)$ has 36 orbits on this set.

8.3.2 Lines on Del Pezzo surfaces

Let E be a (-1)-curve on a Del Pezzo surface S. Then $-K_S \cdot E = 1$. If $d > 2$, the image of E in an anti-canonical model $\bar{S}$ of S is a line. Conversely, a line ℓ on $\bar{S}$ satisfies $H \cdot \ell = 1$ for a hyperplane section H and hence its full preimage in S is an effective divisor D such that $-K_S \cdot D = 1$. It follows from Lemma 8.3.2 that $D = E + R$, where R is the sum of (-2)-curves and E is a (-1)-curve.

For this reason, we call any (-1)-curve on S a *line*.

Since an irreducible curve with negative self-intersection does not move, we will identify it with its divisor class in $\mathrm{Pic}(S)$. Since $\mathbb{P}^2, bfF_0$ or $\mathbf{F}_2$ has no lines (in our definition!), we will assume that S is a blow-up of $0 < k \leq 8$ points $x_1, \ldots, x_k$ in $\mathbb{P}^2$. Also, $k = 1$, S contains only one line, the exceptional curve of the blow-up. If $k = 2$, S contains either 3 lines if x_2 is not infinitely near to x_1 and contains 2 lines otherwise. So, we may assume that $k \geq 3$.

Theorem 8.3.9. *A Del Pezzo surface has only finitely many lines.*

Proof. Let $d = K_S^2$. Each line defines a vector $v = dE + K_S$ satisfying $v \cdot K_S = 0, v^2 = -d(d-1) < 0$. We know that the lattice $K_S^\perp$ is negative definite. A negative definite lattice contains only finitely many vectors of given norm (a sphere in $\mathbb{R}^n$ intersects $\mathbb{Z}^n$ in finitely many points). □

We will be able to say more.

Lemma 8.3.10. *Let $e = (d, -m_1, \ldots, -m_k) \in \mathbb{Z}^{1,k}$ be a vector such that $e^2 = -1$ and $e \cdot \kappa_k = -1$. Assume $k \leq 8$ and $d > 1$. Then all $m_i \geq 0$ and there exist i, j, s such that $m_i + m_j + m_s > d$.*

Proof. Let (S, φ) be a marked Fano surface with $K_S^2 = 9 - k$. Snce S has no (-2)-curves, the marking is geometric. By Lemma 8.3.2, $\varphi(e)$ is the divisor class of a (-1)-curve E. Let $e_0, e_1, \ldots, e_k$ be the geometric marking defined by φ. Each e_i is the class of a (-1)-curve. If e is not one of the e_i's, then $m_i = E \cdot e_i \geq 0$. If $E = e_i$ for some $i > 0$, then $d = 0$ and $m_i = -1$.

Assume $d > 0$ and $m_1 + m_2 + m_3 \leq d$. Then

$$-1 = e \cdot \kappa_k = -3d + \sum_{i=1}^{k} m_i < \sum_{i=1}^{k} (m_i - \frac{d}{3}) \leq$$

$$\leq (m_1 + m_2 + m_3 - d) + 6(m_3 - \frac{d}{3}) \leq 2(3m_3 - d) \leq 0.$$

This shows that the last inequality must be the equality, hence $m_1 = m_2 = m_3 = d/3$ and $\sum_i (m_i - \frac{d}{3}) = 0$. Therefore $m_i = d/3$ for all i and $e = (d, -\frac{d}{3}, \ldots, \frac{d}{3})$. But then $-1 = e^2 = d^2 - \frac{kd^2}{9} = d^2(1 - \frac{k}{9})$ is contradictory. □

Corollary 8.3.11. *The Weyl group $W(Q_k)$ acts transitively on the set of vectors e as in the statement of the previous lemma. The stabilizer subgroup $W(Q_k)_e$ is conjugate to the subgroup $W(Q_{k-1})$ of $W(Q_k)$.*

Proof. Let $e = (d, -m_1, \ldots, -m_k)$ with $d > 0$. Applying the reflections r_{α_i}, where $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, i > 0$, we may assume that $m_1 \geq \ldots \geq m_k$. By the previous lemma, $m_1 + m_2 + m_3 > d$. Let $\alpha_0 = e_0 - e_1 - e_2 - e_3$. We have

$$r_{\alpha_0}(e) = e + (e \cdot \alpha_0)\alpha_0 = (d', m_1', m_2', m_3', m_4', \ldots, m_k'),$$

where $d' = 2d - m_1 - m_2 - m_3, m_1' = m - m_2 - m_3, m_2 = m - m_1 - m_3, m_3 = m - m_1 - m_2, m_i' = m_i, i > 3$. We have $d' < d$, and if $d \neq 0$, we continue applying the elements of $W(Q_k)$ until we get a vector with $d = 0$. Finally, applying $r_{\alpha_i}, i > 0,$, we reduce this vector to the vector e_k. Since $e_k \cdot \alpha_i = 0$ for $i = 0, \ldots, k-2$, we see that the subgroup $W(Q_{k-1})$ stabilizes e_k. It is known that $W(Q_{k-1})$ is a maximal subgroup of $W(Q_k)$. □

Corollary 8.3.12. *Let S be a Fano surface of degree $9 - k$. Then the set of lines on S is bijective to the set of left cosets $W(Q_k)/W(Q_{k-1}$. The group $W(Q_k)$ acts transitively on the set of lines preserving the matrix of intersection numbers.*

Consulting the tables in [Bourbaki] containing the orders of the Weyl groups we obtain that the following table giving the number of lines on a Fano surface of degree $d = 9 - k$.

k	1	2	3	4	5	6	7	8
# lines	1	3	6	10	16	27	56	240

If we exhibit S as a blow-up of k points $x_1, \ldots, x_k$ in $\mathbb{P}^2$, then the lines can be easily found. If $k \leq 4$ the lines are the pre-images of the points and the proper transforms of the lines $\langle x_i, x_j \rangle$. If $k = 5$, we have to add also the proper inverse transform of the conic through the 5 points. If $k = 6$, we have to add 6 conics passing through all points except one. If $k = 7$, we add $\binom{7}{2}$ conics passing through all points except two, as well as 7 cubics with one node at some point x_i and passing simply through the remaining points. If $k = 8$, we add $\binom{8}{3}$ conics, $2\binom{8}{2}$ cubics with a node at x_i and passing simply through all points except some x_j, $\binom{8}{3}$ quartics with nodes at three of the points x_i and passing simply through the remaining points, and $\binom{8}{2}$ quintics with nodes at six of the points x_i and passing simply through the remaining two points.

Example 8.3.2. The case $k = 6$ corresponds to nonsingular cubic surfaces. We have 27 lines and the group $W(Q_6) \cong W(E_6)$ acts as the group of incidence preserving symmetries of the set of lines. A blowing-down structure $\pi : S \to \mathbb{P}^2$ defines a set of 6 skew lines $E_i = \pi^{-1}(x_i)$. The proper inverse transforms of conics C_i passing through all points except x_i define another set of 6 skew lines E_i'. We have $E_i \cdot E_j' = 1$ if $i \neq j$ and $E_i \cdot E_i' = 0$. The 12 lines $(E_1, \ldots, E_6, E_1', \ldots, E_6')$ form a *double-six*. Blowing down the lines $E_1', \ldots, E_6'$ defines another blowing-down structure $\pi' : S \to \mathbb{P}^2$. The corresponding Cremona transformation $T = \pi' \circ \pi^{-1}$ is given by the linear system $|5\ell - 2x_1 - \ldots - 2x_6|$. We have $\#W(E_6) = 72 \cdot 6!$ geometric markings. The subgroup S_6 acts by changing the order of points x_i. Thus, we see that the number of (unordered) double-sixes is equal to 36.

Example 8.3.3. Let S be a Fano surface of degree 2 and $\phi : S \to \mathbb{P}^2$ be its degree 2 cover defined by $|-K_S|$. Each line E on S is mapped to a line ℓ in $\mathbb{P}^2$. The pre-image of this line is a member of $|-K_S|$ which splits in the union of E and a curve E' from $|-K_S - E|$. Since $(-K_S - E) \cdot E = 1$ and $(-K_S - E)^2 = -1$, E' is a line on S. Also $E' \cdot E = 2$ shows that ℓ intersect the branch curve at two points. It is easy to see that $\phi^{-1}(\ell)$ splits only if ℓ is a bitangent. Thus all lines came in pairs. Each pair corresponds to a bitangent. We have 28 bitangents and 56 lines on S. Everything agrees.

The deck transformation g of the cover ϕ is a biregular automorphism of S. If $\pi : S \to \mathbb{P}^2$ is a blow-down structure of S, then $\pi \circ g$ is another blowing-down structure. The corresponding birational transformation $\pi' \circ \pi^{-1}$ of $\mathbb{P}^2$ is called a *Geiser transformation*.

We know that, for any (-1)-curve E, we have $g^*(E) \sim -K_S - E$. Let $e_0, \ldots, e_7$ be a geometric basis defined by the blowing-down structure π. Then

$$g^*(e_i) = (3e_0 - e_1 - \ldots - e_7) - e_i, \quad i = 1, \ldots, 7,$$

$$g^*(e_0) = g^*(\frac{1}{3}(-K_S + \sum_{i=1}^{7} e_i)) = \frac{1}{3}g^*(-K_S) + \frac{1}{3}(\sum_{i=1}^{7} g^*(e_i))$$
$$= -\frac{1}{3}K_S + \frac{1}{3}(21e_0 - 8\sum_{i=1}^{7} e_i) = 8e_0 - 3\sum_{i=1}^{7} e_i.$$

Thus we see that the characteristic matrix of a Geiser transformation with respect to the bases $e_0, \ldots, e_7$ and $g^*(e_0), \ldots, g^*(e_7)$ is the following.

$$\begin{pmatrix} 8 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix} \tag{8.9}$$

In particular we see that a Geiser transformation is one of the symmetric Cremona transformations listed in Chapter 8.

Note that in the Weyl group $Q(E_7)$ this matrix corresponds to the element ω_0 of order 2 which has the maximal length as a word in the reflections r_{α_i}.

Example 8.3.4. Let S be a Fano surface of degree 1 and $\phi : S \to Q \subset \mathbb{P}^3$ be its degree 2 cover defined by $|-2K_S|$. The restriction of $|-K_S|$ to a line E is a linear system of degree 2. This shows that $\phi(E)$ is a conic C on Q. Let C be cut out by some plane H. Since $\phi^*(H) \in |-2K_S|$ we see that $\phi^*(H) = E + E'$, where $E' \in |-2K_S - E|$. Since $E' \cdot K_S = 1$ and $E'^2 = -1$, we obtain that E' is a line. Also $E' \cdot E = 3$ shows that H intersects the branch curve at 3 points. It is easy that $\phi^{-1}(C)$ splits only if H is tangent to the branch curve everywhere. It is called a *tritangent plane*. The branch curve is a curve B of degree 6 and genus 4 canonically embedded in $\mathbb{P}^3$. Thus a tritangent plane cuts out a divisor D of degree 3 such that $2D \in |-K_B|$. The divisor class $[D]$ is a theta characteristic. Since D is effective it is an odd theta characteristic. We know that a curve of genus 4 has $2^3(2^4-1) = 120$ odd theta characteristics. Thus we get 120 tritangent planes and hence 120 pairs of lines on S. This agrees with the number 240 equal to the index of $W(E_7)$ in $W(E_8)$.

The deck transformation g of the cover ϕ is a biregular automorphism of S. If $\pi : S \to \mathbb{P}^2$ is a blow-structure of S, then $\pi' = \pi \circ g$ is another blowing-down structure. The corresponding birational transformation $\pi' \circ \pi^{-1}$ of $\mathbb{P}^2$ is called a *Bertini transformation*.

We know that, for any (-1)-curve E, we have $g^*(E) \sim -2K_S - E$. Let $e_0, \ldots, e_8$ be a geometric basis defined by the blowing-down structure π. Then

$$g^*(e_i) = (6e_0 - 2e_1 - \ldots - 2e_7) - e_i, \quad i = 1, \ldots, 8.$$

$$g^*(e_0) = g^*(\frac{1}{3}(-K_S + \sum_{i=1}^{8} e_i)) = \frac{1}{3}g^*(-K_S) + \frac{1}{3}(\sum_{i=1}^{8} g^*(e_i))$$

$$= -\frac{1}{3}K_S + \frac{1}{3}(48e_0 - 17\sum_{i=1}^{8} e_i) = 17e_0 - 6\sum_{i=1}^{8} e_i.$$

Thus we see that the characteristic matrix of a Bertini transformation with respect to the bases $e_0, \ldots, e_8$ and $g^*(e_0), \ldots, g^*(e_8)$ is the following.

$$\begin{pmatrix} 17 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ -6 & -3 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -3 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -3 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -3 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -3 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -3 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -23 & -3 \end{pmatrix} \tag{8.10}$$

In particular we see that a Bertini transformation is one of the symmetric Cremona transformations listed in Chapter 8.

Note that in the Weyl group $Q(E_8)$ this matrix corresponds to the element ω_0 of order 2 which has the maximal length as a word in the reflections r_{α_i}.

The situation with Del Pezzo but not Fano surfaces is more complicated. The number of lines depends on the structure of the set of (-2)-curves.

Theorem 8.3.13. *Fix a line E on a Del Pezzo surface. There is a natural bijection*

$$\textit{Lines on } S \longleftrightarrow W(S)^n \backslash W(S)/W(S)_E.$$

Proof. Let E be a line and $w \in W(S)$. By Lemma 8.3.2 there exists $g \in W(S)^n$ such that $g(w(E))$ is a line. This line is the unique line $l(E, w)$ in the orbit of $w(E)$ with respect to the action of $W(S)^n$. By Corollary 8.3.11, for any line E' there exists $w \in W(S)$ such that $w(E) = E'$. This shows that the map

$$\Phi_E : W(S) \to \text{set of lines}, \quad w \mapsto l(E, w),$$

is surjective. Suppose that $l(E,w) = l(w',E)$. Then $gw(E) = w'(E)$ for some $g \in W(S)^n$. Thus, $w^{-1}gw'(E) = E$, and hence $w'^{-1}gw \in W(S)_E$ and $w' \in W(S)^n wW(S)$. Conversely, each w' in the double coset $W(S)^n wW(S)$ defines the same line $l(E,w)$. □

Example 8.3.5. Let S be a Del Pezzo surface of degree 3 with a unique (-2)-curve. It can be obtained by blowing-up 6 distinct points on a conic. The proper inverse transform of the conic is the (-2)-curve. The anti-canonical model of S is a cubic surface with one ordinary double points (a node). We have 15 lines not passing through the node. They are the images of the proper inverse transforms of the lines $\langle x_i, x_j\rangle$. We also have 6 lines containing the node. They are the images of the exceptional curves of the blow-up. Thus alltogether we have 21 lines. The group $W(S)^n$ is a group of order 2. It acts on the set $W(S)/W(S)_E$ with 15 orbits of cardinality 1 and 6 orbits of cardinality 2.

Exercises

9.1 Prove that $H^1(S,\mathcal{O}_S) = 0$ for a Del Pezzo surface S without using the Kodaira-Ramanujam's Vanishing Theorem.

9.2 Let $f : X' \to X$ be a resolution of a surface with canonical singularities. Show that $R^1 f_*(\mathcal{O}_{X'}) = 0$.

9.3 Describe all possible types of simple singularities which may occur on a plane curve of degree 4.

9.4 Show that an anticanonical model of a Del Pezzo surface of degree 4 is a complete intersection of two quadrics.

9.5 Let $G(2,5)$ be the Grassmannian of lines in $\mathbb{P}^4$ embedded in $\mathbb{P}^9$ by the Plücker embedding. Show that the intersection of $G(2,5)$ with a general linear subspace of codimension 4 is an anticanonical model of a Del Pezzo surface of degree 5.

9.6 Let S be a Del Pezzo surface of degree 6. Show that its anticanonical model is isomorphic to a section of the Segre variety $s(\mathbb{P}^2 \times \mathbb{P}^2)$ in $\mathbb{P}^8$ by a linear subspace of codimension 2.

9.7 Let S be a Del Pezzo surface of degree 6. Show that its anticanonical model is isomorphic to a hyperplane section of the Segre variety $s(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in $\mathbb{P}^7$.

9.8 Describe all Del Pezzo surfaces which are toric varieties (i.e. contain an open Zariski subset isomorphic to the torus $(\mathbb{C}^*)^2$ such that each translation of the torus extends to an automorphism of the surface).

9.9 Describe all possible singularities on a Del Pezzo surface of degree $d \geq 5$.

9.10 Show that a normal cubic surface has at most 4 canonical singularities of type A_1 and surfaces with 4 such singularities are all isomorphic to the surfgace given by the equation $T_0T_1T_2 + T_0T_1T_3 + T_0T_2T_3 + T_1T_2T_3 = 0$.

9.11 Let Q be a quadric intersecting a nonsingular surface S along the union of three conics C_1, C_2, C_3. Let l_1, l_2, l_3 be the lines residual to the conics with respect to a plane section. Show that the lines lie in the same plane. Conversely, the union of three conics residual to three coplanar lines on S are cut out by a quadrics. Prove that, for a fixed tritangent plane, the set of such quadrics span a web of quadrics.

9.12 Show that the first polar of a cubic surface with respect to a point p lying on two lines intersects the surface along the union of a quartic curve C_p of genus 1 and two lines. Let p_1, p_2, p_3 be threevertices of a triangle of coplanar lines and C_1, C_2, C_3 be the corresponding quartic curves. Show that the three pencils of quadrics containing each C_i span a web of quadrics from the previous problem.

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