Topics in Classical Algebraic Geometry

IGOR V. DOLGACHEV and ALESSANDRO VERRA

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Chapter 9

Apolarity

9.1 Polar polyhedra

9.1.1 Apolar schemes

Let E be a complex vector space of dimension $n+1$. Recall from Chapter 1 that we have a natural polarity pairing

$$S^kE \times S^dE^* \to S^{d-k}E^*, \quad (\Phi, F) \mapsto P_\Phi(F), \quad d \geq k,$$

which extends the canonical pairing $E \times E^* \to \mathbb{C}$. By choosing a basis in E and the dual basis in E^*, we view the ring $\mathrm{Sym}^\bullet E^*$ as the polynomial algebra $\mathbb{C}[T_0, \ldots, T_n]$ and $\mathrm{Sym}^\bullet E$ as the ring of differential operators $\mathbb{C}[\partial_0, \ldots, \partial_n]$. The polarity pairing is induced by the natural action of operators on polynomials.

Definition 9.1. *A homogeneous form $\Phi \in S^kE$ is called* apolar to a homogeneous form $F \in S^dE^*$ if $P_\Phi(F) = 0$.

Lemma 9.1.1. *For any $\Phi \in S^kE, \Phi' \in S^mE$ and $F \in S^dE^*$,*

$$P_{\Phi'}(P_\Phi(F)) = P_{\Phi\Phi'}(F).$$

Proof. By linearity and induction on the degree, it suffices to verify the assertions in the case when $\Phi = \partial_i$ and $\Phi' = \partial_j$. In this case the assertions are obvious. □

Corollary 9.1.2. *Let $F \in S^dE^*$. Let $\mathrm{AP}_k(F)$ be the subspace in S^kE spanned by apolar forms of degree k to F. Then*

$$\mathrm{AP}(F) = \bigoplus_{k=0}^{\infty} AP_k(F)$$

is a homogeneous ideal in the ring $\mathrm{Sym}^\bullet E$.

Definition 9.2. *The quotient ring*

$$A_F = \text{Sym}^\bullet E/\text{AP}(F)$$

is called the apolar ring of F.

The ring A_F inherits the grading of $\text{Sym}^\bullet E$. Since any polynomial $\Phi \in S^r E$ with $r > d$ is apolar to F, we see that A_F is killed by the ideal $\mathfrak{m}_+^{d+1} = (\partial_0, \ldots, \partial_n)^{d+1}$. Thus A_F is an Artinian graded local algebra over $\mathbb{C}$. Since the pairing between $S^d E$ and $S^d E^*$ has values in $S^0 E^* = \mathbb{C}$, we see that $\text{AP}(F)_d$ is of codimension 1 in $S^d E$. Thus $(A_F)_d$ is a vector space of dimension 1 over $\mathbb{C}$ and coincides with the *socle* of A_F, i.e. the ideal of elements of A_F annulated by its maximal ideal.

Note that the latter property characterizes Gorenstein graded local Artinian rings, see [Eisenbud, Iarrobino-Kanev].

Proposition 9.1.3. *(Macaulay) The correspondence $F \mapsto A_F$ is a bijection between $\mathbb{P}(S^d E)$ and graded Artinian quotient algebras* $\text{Sym}^\bullet E/I$ *whose socle is one-dimensional.*

Proof. We have only to show how to reconstruct $\mathbb{C}F$ from $S(V)/I$. The multiplication of d vectors in V composed with the projection to $S^d E/I_d$ defines a linear map $S^d E \to S^d E/I_d$. Since $(\text{Sym}^\bullet E/I)_d$ is one-dimensional. Choosing a basis $(\text{Sym}^\bullet E/I)_d$, we obtain a linear function on $S^d E$. It corresponds to an element of $S^d E^*$. This is our F.

□

Recall that for any closed subscheme $Z \subset \mathbb{P}^n$ is defined by a unique saturated homogeneous ideal I_Z in $\mathbb{C}[T_0, \ldots, T_n]$. Its locus of zeros in the affine space $\mathbb{A}^{n+1}$ is the affine cone C_Z over Z isomorphic to $\text{Spec}(\mathbb{C}[T_0, \ldots, T_n]/I_Z)$.

Definition 9.3. *Let $F \in S^d E^*$. A subscheme $Z \subset \mathbb{P}(E)$ is called* apolar to F if its homogeneous ideal I_Z is contained in $\text{AP}(F)$, or, equivalently, $\text{Spec}(A_F)$ is a closed subscheme of the affine cone C_Z of Z.

This definition agrees with the definition of an apolar homogeneous form Φ. A homogeneous form $\Phi \in S^k E$ is apolar to F if and only if the hypersurface $V(\Phi)$ is apolar to F.

Consider the natural pairing

$$(A_F)_k \times (A_F)_{d-k} \to (A_F)_d \cong \mathbb{C} \tag{9.1}$$

defined by multiplication of polynomials. It is well defined because of Lemma 9.1.1. The left kernel of this pairing consists of $\Phi \in S^kE \mod \mathrm{AP}(F)$ such that $P_{\Phi\Phi'}(F) = 0$ for all $\Phi' \in S^{d-k}E$. By Lemma 9.1.1, $P_{\Phi\Phi'}(F) = P_{\Phi'}(P_\Phi(F)) = 0$ for all $\Phi' \in S^{d-k}E$. This implies $P_\Phi(F) = 0$. Thus $\Phi \in \mathrm{AP}(F)$ is zero in A_F. This shows that the pairing (11.21) is a perfect pairing. This is one of the nice features of a Gorenstein artinian algebra.

It follows that the Hilbert poynomial

$$H_{A_F}(t) = \sum_{i=0}^{d} \dim(A_F))_i t^i = a_d t^d + \ldots + a_0$$

is a reciprocal monic polynomial, i.e. $a_i = a_{d-i}, a_d = 1$. It is an important invariant of a homogenous form F.

Example 9.1.1. Let F be the dth power of a linear form $l \in E^*$. For any $\Phi \in S^kE = (S^kE^*)^*$ we have

$$P_\Phi(l^d) = d(d-1)\ldots(d-k+1)l^{d-k}\Phi(l) = d!l^{[d-k]}\Phi(l),$$

where we set

$$l^{[i]} = \frac{1}{i!}l^i.$$

Here we view $\Phi \in S^dE$ as a homogeneous function on E^*. In coordinates, $l = \sum_{i=0}^n a_iT_i, \Phi = \Phi(\partial_0, \ldots, \partial_n)$ and $\Phi(l) = \Phi(a_0, \ldots, a_n)$. Thus we see that $AP(F)_k, k \leq d$, consists of polynomial of degree k vanishing at l. Assume for simplicity that $l = T_0$. The ideal $AP(F)$ is generated by $\partial_1, \partial_n, \partial_0^{d+1}$. The Hilbert polynomial is equal to $1 + t + \ldots + t^d$.

9.1.2 Sums of powers

Suppose F is equal to a sum of powers of nonzero linear forms

$$F = l_1^d + \ldots + l_s^d.$$

This implies that for any $\Phi \in S^kE$,

$$P_\Phi(F) = P_\Phi(\sum_{i=1}^{s} l_i^d) = d! \sum_{i=1}^{s} \Phi(l_i) l_i^{[d-k]} \tag{9.2}$$

In particular, taking $d = k$, we obtain that

$$\langle l_1^d, \ldots, l_s^d \rangle_{S^dE}^{\perp} = \{\Phi \in S^dE : \Phi(l_i) = 0, i = 1, \ldots, s\} = (I_Z)_d,$$

where Z is the closed subscheme of points $\{[l_1], \ldots, [l_s]\} \subset \mathbb{P}(E^*)$ corresponding to the linear forms l_i.

This implies that the codimension of $\langle l_1^d, \ldots, l_s^d \rangle$ in $S^d E^*$ is equal to the dimension of $(I_Z)_d$, hence the forms $l_1^d, \ldots, l_s^d$ are linearly independent if and only if the points $[l_1], \ldots, [l_s]$ impose independent conditions on hypersurfaces of degree d in $\mathbb{P}(E)$.

Suppose $F \in \langle l_1^d, \ldots, l_s^d \rangle$, then $(I_Z)_d \subset \mathrm{AP}_d(F)$. Conversely, if this is true, we have

$$F \in \mathrm{AP}_d(F)^\perp \subset (I_Z)_d^\perp = \langle l_1^d, \ldots, l_s^d \rangle.$$

If we additionally assume that $(I_{Z'})_d \not\subset \mathrm{AP}_d(F)$ for any proper subset Z' of Z, we obtain, after replacing the forms $l_i's$ by proportional ones, that

$$F = l_1^d + \ldots + l_s^d.$$

Definition 9.4. *A* polar s-polyhedron of F is a set of hyperplanes $H_i = V(l_i), i = 1, \ldots, s$, in $\mathbb{P}(E)$ such that

$$F = l_1^d + \ldots + l_s^d,$$

and, considered as points $[l_i]$ in $\mathbb{P}(E^*)$, the hyperplanes H_i impose independent conditions in the linear system $|\mathcal{O}_{\mathbb{P}(E)}(d)|$.

Note that this definition does not depend on the choice of linear forms defining the hyperplanes. Nor does it depend on the choice of the equation defining the hypersurface $V(F)$.

The following propositions follow from the above discussion.

Proposition 9.1.4. *Let $F \in S^d E^*$. Then $Z = \{[l_1], \ldots, [l_s]\}$ form a polar s-polyhedron of F if and only if the following properties are satisfied*

(i) $I_Z(d) \subset \mathrm{AP}_d(F)$*;*

(ii) $l_1^d, \ldots, l_s^d$ *are linearly independent in $S^d E^*$ or*

(ii') $I'_Z(d) \not\subset \mathrm{AP}_d(F)$ *for any proper subset Z' of Z.*

Proposition 9.1.5. *A set $Z = \{[l_1], \ldots, [l_s]\}$ is a polar s-polyhedron of $F \in S^d E^*$ if and only if Z, considered as a closed subscheme of $\mathbb{P}(E)$, is apolar to F but no proper subscheme of Z is apolar to F.*

9.1.3 Generalized polar polyhedra

Proposition 9.1.5 allows one to generalize the definition of a polar polyhedron. A polar polyhedron can be viewed as a reduced closed subscheme Z of $\mathbb{P}(E^*)$ consisting of s points. Obviously, $h^0(\mathcal{O}_Z) = \dim H^0(\mathbb{P}(E), \mathcal{O}_Z) = s$. More generally, we may consider non-reduced closed subschemes Z of $\mathbb{P}(E)$ of dimension 0 satisfying $h^0(\mathcal{O}_Z) = s$. The set of such subschemes is parametrized by a projective algebraic variety $\mathrm{Hilb}^s(\mathbb{P}(E^*))$ called the *punctual Hilbert scheme* of $\mathbb{P}(E^*)$ of length s.

Any $Z \in \mathrm{Hilb}^s(\mathbb{P}(E^*))$ defines the subspace

$$I_Z(d) = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d)) \subset H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) = S^dE^*.$$

The exact sequence

$$0 \to H^0(\mathbb{P}(E), \mathcal{I}_Z(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_Z) \quad (9.3)$$
$$\to H^1(\mathbb{P}(E), \mathcal{I}_Z(d)) \to 0$$

shows that the dimension of the subspace

$$\langle Z \rangle_d = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d))^{\perp}) \subset \mathbb{P}(S^dE) \quad (9.4)$$

is equal to $h^0(\mathcal{O}_Z) - h^1(\mathcal{I}_Z(d)) - 1 = s - 1 - h^1(\mathcal{I}_Z(d))$. If $Z = Z_{red} = \{p_1, \ldots, p_s\}$, then $\langle Z \rangle_d = \langle v_d(p_1), \ldots, v_d(p_s) \rangle$, where $v_d : \mathbb{P}(E) \to \mathbb{P}(S^dE)$ is the Veronese map. Hence $\dim\langle Z \rangle = s - 1$ if the points $v_d(p_1), \ldots, v_d(p_s)$ are linearly independent. We say that Z is *linearly d-independent* if $\dim\langle Z \rangle_d = s-1$.

Definition 9.5. *A* generalized s-polyhedron *of F is a linearly d-independent subscheme $Z \in$ $\mathrm{Hilb}^s(\mathbb{P}(E))$ which is apolar to F.*

Recall that Z is apolar to F if, for each $k \geq 0$,

$$I_Z(k) = H^0(\mathbb{P}(E^*), \mathcal{I}_Z(k)) \subset \mathrm{AP}_k(F). \quad (9.5)$$

In view of this definition a polar polyhedron is a reduced generalized polyhedron. The following is a generalization of Proposition 9.1.4.

Proposition 9.1.6. *A linear independent subscheme $Z \in$ $\mathrm{Hilb}^s(\mathbb{P}(E))$ is a generalized polar s-polyhedron of $F \in S^dE^*$ if and only if*

$$I_Z(d) \subset \mathrm{AP}_d(F).$$

Proof. We have to show that the inclusion in the assertion implies $I_Z(d) \subset \mathrm{AP}_k(F)$ for any $k \leq d$. For any $\Phi' \in S^{d-k}E$ and any $\Phi \in I(Z)_k$, the product $\Phi\Phi'$ belongs to $I(Z)_d$. Thus $P_{\Phi\Phi'}(F) = 0$. By the duality, $P_{\Phi}(F) = 0$, i.e. $\Phi \in \mathrm{AP}_k(F)$. □

Example 9.1.2. Let $Z = m_1p_1 + \ldots + m_kp_k \in \mathrm{Hilb}^s(\mathbb{P}(E))$ be the union of *fat points* p_k, i.e. at each $p_i \in Z$ the ideal $\mathcal{I}_{Z,p_i}$ is equal to the m_ith power of the maximal ideal. Obviously,

$$s = \sum_{i=1}^{k} \binom{n+m_i-1}{m_i-1}.$$

Then the linear system $|\mathcal{O}_{\mathbb{P}(E)}(d) - Z)|$ consists of hypersurfaces of degree which have singularity at p_i of multiplicity $\geq m_i$ for each $i = 1, \ldots, k$. One can show (see [Iarrobino-Kanev], Theorem 5.3) that Z is apolar to F if and only if

$$F = l_1^{d-m_i+1}A_1 + \ldots + l_k^{d-m_k+1}A_k,$$

where $p_i = [l_i]$ and A_i is a homogeneous polynomial of degree $m_i - 1$.

Remark 9.1.1. It is not known whether the set of generalized s-polyhedra of F is a closed subset of $\mathrm{Hilb}^s(\mathbb{P}(E^*))$. It is known to be true for $s \leq d+1$ since in this case $\dim I(Z)_d = t := \dim S^dE - s$ for all $Z \in \mathrm{Hilb}^s(\mathbb{P}(E^*)$ (see [IK], p.48). This defines a regular map of $\mathrm{Hilb}^s(\mathbb{P}(E^*))$ to the Grassmannian $G(t, S^dE)$ and the set of generalized s-polyhedra is equal to the preimage of a closed subset consisting of subspaces contained in $\mathrm{AP}_d(F)$. Also we see that $h^1(\mathcal{I}_Z(d)) = 0$, hence Z is always linearly d-independent.

9.1.4 Secant varieties

The notion of a polar polyhedron has a simple geometric interpretation. Let

$$v_d : \mathbb{P}(E^*) \to \mathbb{P}(S^dE^*),\ l \mapsto l^d,$$

be the Veronese map. Denote by Ver_d^n its image. Then $F \in S^dE^* \setminus \{0\}$ represents a point $[F]$ in $\mathbb{P}(S^dE^*)$. A set of hyperplanes $H_i = V(l_i), i = 1, \ldots, s$, represents a set of points $[l_i^d]$ in the Veronese variety Ver_d^n. It is a polar s-polyhedron of F if and only if $[F]$ belongs to the linear span $\langle [l_1^d], \ldots, [l_s^d] \rangle$, a $(s-1)$-secant of the Veronese variety.

Recall that for any irreducible nondegenerate projective variety $X \subset \mathbb{P}^r$ of dimension n its *t-secant variety* $\mathrm{Sec}_t(X)$ is defined to be the Zariski closure of the set of points in $\mathbb{P}^r$ which lie in the linear span of dimension t of some set of $t+1$ linear independent points in X.

Counting constants easily gives

$$\dim \mathrm{Sec}_t(X) \leq \min((n+1)(t+1) - 1, r).$$

The subvariety $X \subset \mathbb{P}^r$ is called *t-defective* if the inequality is strict. An example of a 1-defective variety is a Veronese surface in $\mathbb{P}^5$.

A fundamental result about secant varieties is the following lemma whose modern proof can be found, for example in [Dale], [Zak,Prop.1.10].

Lemma 9.1.7. *(A. Terracini) Let $p_1, \ldots, p_{t+1}$ be general $t+1$ points in X and p be a general point in their span. Then*

$$\mathrm{PT}(\mathrm{Sec}_t(X))_p = \langle \mathrm{PT}(X)_{p_1}, \ldots, \mathrm{PT}(X)_{p_{t+1}} \rangle,$$

where $\mathrm{PT}(V)_x$ denotes the embedded Zariski tangent space of a closed subvariety of a projective space at a point x.

The inclusion part

$$\langle \mathrm{PT}(X)_{p_1}, \ldots, \mathrm{PT}(X)_{p_{t+1}} \rangle \subset \mathrm{PT}(\mathrm{Sec}_t(X))_p$$

is easy to prove. We assume for simplicity that $t = 1$. Then $\mathrm{Sec}_1(X)$ contains the cone $C(p_1, X)$ which is sweeped by the lines $\langle p, q \rangle, q \in X$. Therefore $\mathrm{PT}(C(p_1, X)) \subset \mathrm{PT}(\mathrm{Sec}_1(X))_p$. However, it is easy to see that $\mathrm{PT}(C(p_1, X))$ contains $\mathrm{PT}(X)_{p_1}$.

Corollary 9.1.8. $\mathrm{Sec}_t(X) \neq \mathbb{P}^r$ *if and only if for any $t+1$ general points of X there exists a hyperplane section of X singular at these points. In particular, if $r \leq (n+1)(t+1) - 1$, the variety X is t-defective if and only if for any $t+1$ general points of X there exists a hyperplane section of X singular at these points.*

Example 9.1.3. Let $X = \mathrm{Ver}_d^n \subset \mathbb{P}^{\binom{d+n}{n}-1}$ be the image of $\mathbb{P}^n$ under a Veronese map defined by homogeneous polynomials of degree d. Assume $(n+1)(t+1) \geq \binom{d+n}{n} - 1$. A hyperplane section of X is isomorphic to a hypersurface of degree d in $\mathbb{P}^n$. Thus $\mathrm{Sec}_t(\mathrm{Ver}_d^n) \neq \mathbb{P}(S^d E^*)$ if and only if for any $t+1$ general points in $\mathbb{P}^n$ there exists a hypersurface of degree d singular at these points.

Take $n = 1$. Then $r = d$ and $r \leq (n+1)(t+1) - 1 = 2t+1$ for $t \geq (d-1)/2$. Since $t+1 > d/2$ there are no homogeneous forms of degree d which have $t+1$ multiple roots. Thus the Veronese curve $R_d = v_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is not t-degenerate for $t \geq (d-1)/2$.

Take $n = 2$ and $d = 2$. For any two points in $\mathbb{P}^2$ there exists a conic singular at these points, namely the double line through the points. This explains why a Veronese surface V_2^2 is 1-defective.

Another example is $\mathrm{Ver}_4^2 \subset \mathbb{P}^{14}$ and $t = 4$. The expected dimension of $\mathrm{Sec}_4(X)$ is equal to 14. For any 5 points in $\mathbb{P}^2$ there exists a conic passing through these points. Taking it with multiplicity 2 we obtain a quartic which is singular at these points. This shows that Ver_4^2 is 4-defective.

The following corollary of Terracini's Lemma is called the *First main theorem on apolarity* in [Ehrenborg-Rota]. They gave an algebraic proof of this theorem without using (or probably without knowing) Terracini's Lemma.

Corollary 9.1.9. *A general form $F \in S^dE^*$ admits a polar s-polyhedron if and only if there exists linear forms $l_1, \ldots, l_s \in V^*$ such that for any nonzero $\Phi \in S^dE$ the ideal $AP(\Phi) \subset \mathrm{Sym}E^*$ does not contain $\{l_1^{d-1}, \ldots, l_s^{d-1}\}$.*

Proof. A general form $F \in S^dE^*$ admits a polar s-polyhedron if and only if the secant variety $\mathrm{Sec}_s(\mathrm{Ver}_d^n)$ is equal to the whole space. This means that for some points $[l_1^d], \ldots, [l_s^d]$ the span of the tangent spaces at the points $\langle [l_1^d], \ldots, [l_s^d] \rangle$ is equal to the whole space. By Terracini's Lemma, this is equivalent to that the tangent spaces of the Veronese variety at the points $[l_i^d]$ are not contained in a hyperplane defined by some $\Phi \in S^dE = (S^dE^*)^*$. It remains to use that the tangent space of the Veronese variety at $[l_i]^d$ is equal to the projective space of all homogeneous forms of the form $l_i^{d-1}l, l \in E^*$ (see Exercises). Thus, for any nonzero $\Phi \in S^dE$, it is impossible that $P_{l_i^{d-1}l}(\Phi) = 0$ for all l and i. But $P_{l_i^{d-1}l}(\Phi) = 0$ for all l if and only if $P_{l_i^{d-1}}(\Phi) = 0$. This proves the assertion. □

The following fundamental result is due to J. Alexander and A. Hirschowitz [Alexander-Hir].

Theorem 9.1.10. Ver_d^n *is t-defective if and only if*

$$(n, d, t) = (2, 2, 1), (2, 4, 4), (3, 4, 8), (4, 3, 6), (4, 4, 13).$$

In all these cases the secant variety $\mathrm{Sec}_t(\mathrm{Ver}_d^n)$ is a hypersurface.

For the sufficiency of the condition, only the case $(4, 3, 6)$ is not trivial. It asserts that for 7 general points in $\mathbb{P}^3$ there exists a cubic hypersurface which is singular at these points. Other cases are easy. We have seen already the first two cases. The third case follows from the existence of a quadric through 9 general points in $\mathbb{P}^3$. The square of its equation defines a quartic with 9 points. The last case is similar. For any 14 general points there exists a quadric in $\mathbb{P}^4$ containing these points.

Corollary 9.1.11. *Assume $s(n+1) \geq \binom{d+n}{n}$. Then a general homogeneous polynomial $F \in \mathbb{C}[T_0, \ldots, T_n]_d$ can be written as a sum of dth powers of s linear forms unless $(n, d, s) = (2, 2, 2), (2, 4, 5), (3, 4, 9), (4, 3, 7), (4, 4, 14)$.*

9.1.5 The Waring problems

The well-known Waring problem in number theory asks about the smallest number $s(d)$ such that each natural number can be written as a sum of $s(d)$ dth powers of natural numbers. It also asks in how many ways it can be done. Its polynomial analog asks about the smallest number $s(d,n)$ such that a general homogeneous polynomial of degree d in $n+1$ variables can be written as a sum of s dth powers of linear forms.

The Alexander-Hirschowitz Theorem completely solves this problem. We have $s(d,n)$ is equal to the smallest natural number s_0 such that $s_0(n+1) \geq \binom{n+d}{n}$ unless $(n,d) = (2,2),(2,4),(3,4),(4,3),(4,4)$, where $s(d,n) = s_0+1$.

Other versions of the Waring problem ask the following questions:

- (W1) Given a homogeneous form $F \in S^dE^*$, study the subvariety $\mathrm{VSP}(F;s)^o$ of $(\mathbb{P}(E^*)^{(s)}$ (the *variety of power sums* which consists of polar s-polyhedra of F or more general the subvariety $\mathrm{VSP}(F;s)$ of $\mathrm{Hilb}^s(\mathbb{P}(E^*)$ parametrizing generalized s-polyhedra.

- (W2) For given s find the equations of the closure $\mathrm{PS}(s,d;n)$ in S^dE^* of the locus of homogeneous forms of degree d which can be written as a sum of s powers of linear forms.

Note that $\mathrm{PS}(s,d;n)$ is the affine cone over the secant variety $\mathrm{Sec}_{s-1}(\mathrm{Ver}_d^n)$.

In the language of secant varieties, the variety $\mathrm{VSP}(F;s)^o$ is the set of linear independent sets of s points $p_1,\ldots,p_s$ in Ver_d^n such that $[F] \in \langle p_1,\ldots,p_s\rangle$. The variety $\mathrm{VSP}(F,s)$ is the set of linearly independent $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ such that $[F] \in \langle Z\rangle$. Note that we have a natural map

$$\mathrm{VSP}(F,s) \to G(s,S^dE), \quad Z \mapsto \langle Z\rangle_d,$$

where $G(s,S^dE)$ is the Grassmannian of s-dimensional subspaces of S^dE. This map is not injective in general.

Also note that for a general form F the variety $\mathrm{VSP}(F;s)$ is equal to the closure of $\mathrm{VSP}(F,s)^o$ in the Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}(E^*))$ (see [Iarrobino-Kanev], 7.2). It is not true for an arbitrary form F. One can also embed $\mathrm{VSP}(F;s)^o$ in $\mathbb{P}(S^dE^*)$ by assigning to $\{l_1,\ldots,l_s\}$ the product $l_1\cdots l_s$. Thus we can compactify $\mathrm{VSP}(F,s)^o$ by taking its closure in $\mathbb{P}(S^dE^*)$. In general, this closure is not isomorphic to $\mathrm{VSP}(F,s)$.

Proposition 9.1.12. *Assume $n=2$. For general $F \in S^dE^*$ the variety* $\mathrm{VSP}(F;s)$ *is either empty or a smooth irreducible variety of dimension $3s - \binom{2+d}{d}$.*

Proof. We consider $\mathrm{VSP}(F;s)$ as the closure of $\mathrm{VSP}(F;s)^o$ in the Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}(E^*))$. Recall that $Z \in \mathrm{Hilb}^s(\mathbb{P}(E^*))$ is a generalized polar polyhedron of F if and only if $F \in I_Z(d)^\perp$ but this is not true for any proper closed subscheme Z' of Z. Consider the incidence variety

$$X = \{(Z,F) \in \mathrm{Hilb}^s(\mathbb{P}(E^*) \times S^dV^* : Z \in \mathrm{VSP}(F;s)\}.$$

It is known that for any nonsingular surface the punctual Hilbert scheme is nonsingular (see [Fogarty]). Let U be the open subset of the first factor such that for any point $Z \in U$, $\dim I_Z(d) = \dim S^dE - s$. The fibre of the first projection over $Z \in U$ is an open Zariski subset of the linear space $I_Z(d)^\perp$. This shows that X is irreducible and nonsingular. The fibres of the second projection are the varieties $\mathrm{VSP}(F;s)$. Thus for an open Zariski subset of S^dE^* the varieties $\mathrm{VSP}(F;s)$ are empty or irreducible and nonsingular.

□

9.2 Dual homogeneous forms

9.2.1 Catalecticant matrices

Let $F \in S^dE^*$. Consider the linear map (the *apolarity map*)

$$\mathrm{ap}_F^k : S^kE \to S^{d-k}E^*, \quad \Phi \mapsto P_\Phi(F). \tag{9.6}$$

Its kernel is the space $\mathrm{AP}_k(F)$ of forms of degree k which are apolar to F.

By the polarity duality, the dual space of $S^{d-k}E^*$ can be identified with $S^{d-k}E$. Applying Lemma 9.1.1, we obtain

$$^t(\mathrm{ap}_F^k) = \mathrm{ap}_F^{d-k}. \tag{9.7}$$

Assume that $F = \sum_{i=1}^s l_i^d$ for some $l_i \in E^*$. It follows from (9.2) that

$$\mathrm{ap}_F^k(S^kE) \subset \langle l_1^{d-k}, \ldots, l_s^{d-k}\rangle,$$

and hence

$$\mathrm{rank}(\mathrm{ap}_F^k) \le s. \tag{9.8}$$

If we choose a basis in E and a basis in E^*, then ap_F^k is given by a matrix of size $\binom{k+n}{k} \times \binom{n+d-k}{d-k}$ whose entries are linear forms in coefficients of F.

Choose a basis $\xi_0, \ldots, \xi_n$ in E and the dual basis $T_0, \ldots, T_n$ in E^*. Consider a monomial basis in S^kE (resp. in $S^{d-k}E^*$) which is lexigraphically ordered. The matrix of ap_F^k with respect to these bases is called the kth *catalecticant matrix* of

F and is denoted by $\mathrm{Cat}_k(F)$. Its entries $c_{\mathbf{uv}}$ are parametrized by pairs $(\mathbf{u},\mathbf{v}) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$ with $|\mathbf{u}| = d-k$ and $|\mathbf{v}| = k$. If we write

$$F = d! \sum_{|\mathbf{i}|=d} \frac{1}{\mathbf{i}!} a_{\mathbf{i}} T^{\mathbf{i}},$$

then

$$c_{\mathbf{uv}} = a_{\mathbf{u}+\mathbf{v}}.$$

This follows easily from the formula

$$\partial_0^{i_0}\cdots\partial_n^{i_n}(T_0^{j_0}\cdots T_n^{j_n}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!} & \text{if } \mathbf{j}-\mathbf{i} \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Considering $a_{\mathbf{i}}$ as independent variables $T_{\mathbf{i}}$, we obtain the definition of a general catalecticant matrix $\mathrm{Cat}_k(d,n)$.

Example 9.2.1. Let $n = 1$. Write $F = \sum_{i=0}^d \binom{d}{i} a_i T_0^{d-i} T_1^i$. Then

$$\mathrm{Cat}_k(F) = \begin{pmatrix} a_0 & a_1 & \dots & a_k \\ a_1 & a_2 & \dots & a_{k+1} \\ \vdots & \vdots & \dots & \vdots \\ a_{d-k} & a_{d-k+1} & \dots & a_d \end{pmatrix}$$

A matrix of this type is called a *Hankel matrix*. It follows from (9.8) that $F \in \mathrm{PS}(s,d;1)$ implies that all $s+1 \times s+1$ minors of $\mathrm{Cat}_k(F)$ are equal to zero. Thus we obtain that $\mathrm{Sec}_{s-1}(\mathrm{Ver}_d^1)$ is contained in the subvariety of $\mathbb{P}^d$ defined by $(s+1)\times(s+1)$-minors of the matrices

$$\mathrm{Cat}_k(d,1) = \begin{pmatrix} T_0 & T_1 & \dots & T_k \\ T_1 & T_2 & \dots & T_{k+1} \\ \vdots & \vdots & \dots & \vdots \\ T_{d-k} & T_{d-k+1} & \dots & T_d \end{pmatrix}, \quad k = 1,\dots,\min d-s, s$$

For example, if $s = 1$, we obtain that the Veronese curve $\mathrm{Ver}_d^1 \subset \mathbb{P}^d$ satisfies the equations $T_iT_j - T_kT_l = 0$, where $i+j = k+l$. It is well known that these equations generate the homogeneous ideal of the Veronese curve.

Assume $d = 2k$. Then the Hankel matrix is a square matrix of size $k+1$. Its determinant vanishes if and only if F admits a nonzero apolar form of degree k. The set of such F's is a hypersurface in $\mathbb{C}[T_0,T_1]_{2k}$. It contains a Zariski open subset of forms which can be written as a sum of k powers of linear forms (see section 9.3.1).

For example, take $k = 2$. Then the equation

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0 \tag{9.9}$$

describes binary quartics

$$F = a_0T_0 + 4a_1T_0^3T_1 + 6a_2T_0^2T_1^2 + 4a_3T_0T_1^3 + a_4T_1^4$$

which lie in the Zariski closure of the locus of quartics represented in the form $(\alpha_0T_0 + \beta_0T_1)^4 + (\alpha_1T_0 + \beta_1T_1)^4$. Note that a quartic of this form has simple roots unless it has a root of multiplicity 4. Thus any binary quartic with simple roots satisfying equation (9.9) can be represented as a sum of two powers of linear forms.

The cubic hypersurface in $\mathbb{P}^4$ defined by equation (9.9) is equal to the 1-secant variety of a Veronese curve in $\mathbb{P}^4$.

Note that

$$\dim \text{AP}_i(F) = \dim \text{Ker}(\text{ap}_F^i) = \binom{n+k}{i} - \text{rank Cat}_i(F).$$

Therefore,

$$\dim(A_F)_i = \text{rank Cat}_i(F),$$

and

$$H_{A_F}(t) = \sum_{i=0}^{d} \text{rank Cat}_i(F)t^i. \tag{9.10}$$

It follows from (9.7) that

$$\text{rank Cat}_i(F) = \text{rank Cat}_{d-i}(F)$$

confirming that $H_{A_F}(t)$ is a reciprocal monic polynomial.

Suppose $d = 2k$ is even. Then the coefficient at t^k in $H_F(t)$ is equal to rank $\text{Cat}_k(F)$. The matrix $\text{Cat}_k(F)$ is a square matrix of size $\binom{n+k}{k}$. One can show that for a general F, this matrix is nonsingular. A polynomial F is called *degenerate* if $\det \text{Cat}_k(F) = 0$. Thus, the set of degenerate polynomials is a hypersurface (*catalecticant hypersurface*) given by the equation

$$\det \text{Cat}_k(2k, n) = 0. \tag{9.11}$$

The polynomial in variables $T_{\mathbf{i}}$, $|\mathbf{i}| = d$, is called the *catalecticant determinant*.

Example 9.2.2. Let $d = 2$. It is easy to see that the catalecticant polynomial is the discriminant polynomial. Thus a quadratic form is degenerate if and only if it is degenerate in the usual sense. The Hilbert polynomial of a quadratic form F is

$$H_F(t) = 1 + rt + t^2,$$

where r is the rank of the quadratic form.

Example 9.2.3. Suppose $F = T_0^d + \ldots + T_s^d, s \leq n$. Then $T_0^i, \ldots, T_s^i$ are linearly independent for any i, and hence rank $\mathrm{Cat}_i(F) = s$ for $0 < i < d$. This shows that

$$H_F(t) = 1 + s(t + \ldots + t^{d-1}) + t^d.$$

Let $\mathcal{P}$ be the set of reciprocal monic polynomials of degree d. One can stratify the space S^dE^* by setting, for any $P \in \mathcal{P}$,

$$S^dE^*_P = \{F \in S^dE^* : H_F = P\}.$$

If $F \in \mathrm{PS}(s, d; n)$ we know that

$$\text{rank } \mathrm{Cat}_k(F) \leq h(s, d, n)_k = \min(s, \tbinom{n+k}{n}, \tbinom{n+d-k}{n}).$$

One can show that for general enough F, we have the equality (see [Iarrobino-Kanev], Lemma 1.7]). Thus there is a Zariski open subset of $\mathrm{PS}(s, d; n)$ which belongs to the strata $S^dE^*_P$, where $P = \sum_{i=0}^d h(s, d, n)_i t^i$.

9.2.2 Dual forms

In Chapter 1 we introduced the notion of a dual quadric. If $Q = V(F)$, where F is a nondegenerate quadratic form, then the dual variety $\check{Q}$ is a quadric defined by the quadratic form $\check{F}$ whose matrix is the adjugate matrix of F. For any homogeneous form of even degree $F \in S^{2k}E^*$ one can define the dual homogeneous form $\check{F} \in S^{2k}E$ in a similar fashion using the notion of the catalecticant matrix.

Let

$$\mathrm{ap}_F^k : S^kE \to S^kE^* \tag{9.12}$$

be the apolarity map (9.6). We can view this map as a symmetric bilinear form

$$\Omega_F : S^kE \times S^kE \to \mathbb{C}, \quad \Omega_F(\Phi_1, \Phi_2) = \mathrm{ap}_F^k(\Phi_1)(\Phi_2) = \langle \Phi_2, \mathrm{ap}_F^k(\Phi_1) \rangle. \tag{9.13}$$

Its matrix with respect to a monomial basis in S^kE and its dual monomial basis in S^kE^* is the catalecticant matrix $\mathrm{Cat}_k(F)$.

Let us identify Ω_F with the associated quadratic form on S^kE (the restriction of Ω_F to the diagonal). This defines a linear map

$$\Omega : S^{2k}E^* \to S^2S^kE^*, \quad F \mapsto \Omega_F.$$

There is also a natural left inverse map of Ω

$$P : S^2S^kE^* \to S^{2k}E^*$$

defined by multiplication $S^kE^* \times S^kE^* \to S^{2k}E^*$. All these maps are $\mathrm{GL}(E)$-equivariant and realize the linear representation $S^{2k}E^*$ as a direct summand in the representation $S^2S^kE^*$.

Theorem 9.2.1. *Assume that $F \in S^{2k}E^*$ is nondegenerate. There exists a unique homogeneous form $\check{F} \in S^{2k}E$ (the* dual homogeneous form*) such that*

$$\Omega_{\check{F}} = \check{\Omega}_F.$$

Proof. We know that $\check{\Omega}(F)$ is defined by the cofactor matrix $\mathrm{adj}(\mathrm{Cat}_k(F)) = (c^*_{\mathbf{uv}})$ so that

$$\check{\Omega}_F = \sum c^*_{\mathbf{uv}} \xi^{\mathbf{u}} \xi^{\mathbf{v}}.$$

Let

$$\check{F} = \sum_{|\mathbf{u}+\mathbf{v}|=2k} \tfrac{d!}{(\mathbf{u}+\mathbf{v})!} c^*_{\mathbf{uv}} \xi^{\mathbf{u}+\mathbf{v}}.$$

Recall that the entries $c_{\mathbf{uv}}$ of the catalecticant matrix depend only on the sum of the indices. Thus the entries of the cofactor matrix $\mathrm{adj}(\mathrm{Cat}_k(F)) = (c^*_{\mathbf{uv}})$ depend only on the sum of the indices. For any $t^{\mathbf{i}} \in S^kV^*$, we have

$$P_{t^{\mathbf{i}}}(\check{F}) = \sum_{\mathbf{u},\mathbf{v},\mathbf{u}+\mathbf{v}\geq \mathbf{i}} \tfrac{d!}{(\mathbf{u}+\mathbf{v})!} c^*_{\mathbf{uv}} \tfrac{(\mathbf{u}+\mathbf{v})!}{(\mathbf{u}+\mathbf{v}-\mathbf{i})!} \xi^{\mathbf{u}+\mathbf{v}-\mathbf{i}} = \sum_{\mathbf{j},|\mathbf{j}|=k} \tfrac{d!}{\mathbf{j}!} c^*_{\mathbf{ij}} \xi^{\mathbf{j}}$$

This checks that the matrix of the linear map $S^kE^* \to S^kE$ defined by $\Omega_{\check{F}}$ is equal to the matrix $\mathrm{adj}(\mathrm{Cat}_k(F))$. Thus the quadratic form $\Omega_{\check{F}}$ is the dual of the quadratic form Ω_F. □

Recall that the locus of zeros of the quadric $\check{Q}$ in E^* is equal to the set of linear functions of the form $l = b_Q(v)$ such that $\langle v, l\rangle = 0$. The same is true for the dual form $\check{F}$. Its locus of zeros consists of linear forms l such that $\Omega_F^{-1}(l^k) \in S^kE$ vanishes on l. The degree k homogeneous form $\Omega_F^{-1}(l^k)$ is classically known as the *anti-polar* of l (with respect to F).

Definition 9.6. *Two linear forms $l, m \in E^*$ are called* conjugate *with respect to a nondegenerate form $F \in S^{2k}E^*$ if*

$$\Omega_{\check{F}}(l^k, m^k) = \check{F}(l^k m^k) = 0.$$

Proposition 9.2.2. *Suppose F is given by* (9.14)*, where the powers l_i^k are linearly independent in S^kE^*. Then each pair l_i, l_j is conjugate with respect to F.*

Proof. It follows from computation of Ω_F in the proof of Proposition 9.2.3 that it suffices to check the assertion for quadratic forms. Choose a coordinate system such that $l_i = t_0, l_j = t_1$ and $F = t_0^2 + t_2^2 + \ldots + t_n^2$. Then $\check{F} = \xi_0^2 + \ldots + \xi_n^2$, where $\xi_0, \ldots, \xi_n$ are dual coordinates. Now the assertion is easily checked. □

9.2.3 The Waring rank of a homogeneous form

Since any quadratic form Q can be reduced to a sum of squares, one can characterize its rank as the smallest number r such that

$$Q = l_1^2 + \ldots + l_r^2$$

for some linear forms $l_1, \ldots, l_r$.

Definition 9.7. *Let $F \in S^dE^*$. Its* Waring rank $\mathrm{wrk}(F)$ *is the smallest number r such that*

$$F = l_1^d + \ldots + l_r^d \tag{9.14}$$

for some linear forms $l_1, \ldots, l_r$.

Proposition 9.2.3. *Let Ω_F be the quadratic form on S^kV associated to $F \in S^{2k}E^*$. Then the Waring rank of F is greater or equal than the rank of Ω_F.*

Proof. Suppose (9.14) holds with $d = 2k$. Since Ω_F is linear with respect to F, we have $\Omega_F = \sum \Omega_{l_i^{2k}}$. If we choose coordinates such that l_i is a coordinate function t_0, we easily compute the catalecticant matrix of l_i^{2k}. It is equal to the matrix with 1 at the upper left corner and zero elsewhere. The corresponding quadratic form is equal to $(t_0^k)^2$. Thus $\Omega_{l_i^{2k}} = (l_i^k)^2$ and we obtain

$$\Omega_F = \sum_{i=1}^r \Omega_{l_i^{2k}} = \sum_{i=1}^r (l_i^k)^2.$$

Thus the rank of F is greater or equal than the rank of Ω_F. □

Corollary 9.2.4. *Suppose F is a nondegenerate form of even degree $2k$, then*

$$\mathrm{wrk}(F) \geq \binom{k+n}{n}.$$

A naive way to compute the Waring rank is by counting constants. Consider the map

$$s : (E^*)^r \to \mathbb{C}^{\binom{d+n}{n}}, \quad (l_1, \dots, l_r) \mapsto \sum l_i^d. \tag{9.15}$$

If $r(n+1) \geq \binom{d+n}{n}$ one expects that this map is surjective and hence $\mathrm{wrk}(F) \leq r$ for general F. Here "general" means that the coefficients of F belong to an open Zariski subset of the affine space $\mathbb{C}^{\binom{d+n}{n}}$. It follows from Theorem 9.1.10 that the only exceptional cases when it is false and the map s fails to be surjective are the following cases:

- $n = 2, d = 2, r = 2, \mathrm{wrk}(F) = 3$;
- $n = 2, d = 4, r = 5, \mathrm{wrk}(F) = 6$;
- $n = 3, d = 4, r = 9, \mathrm{wrk}(F) = 10$;
- $n = 4, d = 3, r = 7, \mathrm{wrk}(F) = 8$;
- $n = 4, d = 4, r = 14, \mathrm{wrk}(F) = 15$;

Proposition 9.2.5. *Let F be a general homogeneous form of even degree $2k$. Then*

$$\mathrm{wrk}(F) > \mathrm{rank}\,\Omega_F$$

except in the following cases, where the equality takes place,:

- $k = 1$;
- $n = 1$;
- $n = 2, k \leq 4$;
- $n = 3, k = 2$.

Proof. The first case is obvious. It follows from considering the map (9.15) that $\mathrm{wrk}(F) \geq \binom{n+2k}{n}/(n+1)$. On the other hand the rank of Ω_F for general F is equal to $\dim S^k E = \binom{n+k}{n}$.

We know that the case $n = 1$ is not exceptional so that we can compute the Waring rank of F by counting constants and get $\mathrm{wrk}(F) = k + 1 = \mathrm{rank}\,\Omega_F$.

If $n = 2$, we get $\mathrm{wrk}(F) \geq (2k+2)(2k+1)/6 = (k+1)(2k+1)/3$ and rank $\Omega_F = \binom{k+2}{2} = (k+2)(k+1)/2$. We have $(k+1)(2k+1)/3 > (k+2)(k+1)/2$ if $k > 4$. By Theorem 9.1.10,

$$\mathrm{wrk}(F) = \begin{cases} 6 & \text{if k = 2,} \\ 10 & \text{if k = 3,} \\ 15 & \text{if k = 3.} \end{cases}$$

This shows that $\mathrm{wrk}(F) = \mathrm{rank}\ \Omega_F$ in all these cases.

If $n = 3$, we get

$$\mathrm{wrk}(F) \geq (2k+3)(2k+2)(2k+1)/24 > \binom{k+3}{3} = (k+3)(k+2)(k+1)/6$$

unless $k = 2$.

Finally, it is easy to see that for $n > 3$

$$\mathrm{wrk}(F) \geq \tfrac{1}{n+1}\binom{2k+n}{n} > \binom{k+n}{n}$$

for $k > 1$. □

9.2.4 Mukai's skew-symmetric form

Let $\omega \in \Lambda^2 E$ be a skew-symmetric bilinear form on E^*. It admits a unique extension to a Poisson bracket $\{,\}_\omega$ on $S^\bullet E^*$ which restricts to a skew-symmetric bilinear form

$$\{,\}_\omega : S^{k+1}E^* \times S^{k+1}E^* \to S^{2k}E^*. \tag{9.16}$$

Recall that a *Poisson bracket* on a commutative algebra A is a skew-symmetric bilinear map $A \times A \to A, (f,g) \mapsto \{f,g\}$ such that its left and right partial maps $A \to A$ are derivations.

Let $F \in S^{2k}E^*$ be a nondegenerate form and $\check{F} \in S^{2k}E = (S^{2k}E^*)^*$ be its dual form. For each ω as above define $\sigma_{\omega,F} \in \Lambda^2(S^{k+1}E)^*$ by

$$\sigma_{\omega,F}(f,g) = \check{F}(\{f,g\}_\omega).$$

Theorem 9.2.6. *Let F be a nondegenerate form in $S^{2k}E^*$ of Waring rank N. Assume that $N = \mathrm{rank}\ \Omega_F = \binom{n+2k}{n}$. For any $Z = \{[l_1],\ldots,[l_s]\} \in \mathrm{VSP}(F;s)^o$ let $\langle Z\rangle_{k+1}$ be the linear span of the powers l_i^{k+1} in $S^{k+1}E^*$. Then*

(i) $\langle \mathcal{P}\rangle$ *is isotropic with respect to each form $\sigma_{\omega,F}$;*

(ii) $\mathrm{ap}_F^{k-1}(S^{k-1}E) \subset \langle Z\rangle_{k+1}$;

(iii) $\mathrm{ap}_F^{k-1}(S^{k-1}E)$ *is contained in the radical of each* $\sigma_{\omega,F}$.

Proof. To prove the first assertion it is enough to check that $\sigma_{\omega,F}(l_i^{k+1}, l_j^{k+1}) = 0$ for all i, j. We have

$$\sigma_{\omega,F}(l_i^{k+1}, l_j^{k+1}) = \check{F}(\{l_i^k, l_j^k\}_\omega) = \check{F}(l_i^k l_j^k)\omega(l_i, l_j).$$

By Proposition 9.2.2, $\check{F}(l_i^k l_j^k) = \Omega_{\check{F}}(l_i^k, l_j^k) = 0$. This checks the first assertion.

For any $\Phi \in S^{k-1}V$,

$$P_\Phi(F) = P_\Phi(\sum_{i=1}^N l_i^{2k}) = \sum P_\Phi(l_i^{2k}) = \tfrac{(2k)!}{(k+1)!}\sum_{i=1}^N P_\Phi(l^{k-1})l_i^{k+1}.$$

This shows that $\mathrm{ap}_F^{k-1}(S^{k-1}E)$ is contained in $L(\mathcal{P})$. It remains to check that for any $\Phi \in S^{k-1}V, G \in S^{k+1}E^*$ and any $\omega \in \Lambda^2 E$, one has $\sigma_{\omega,F}(P_\Phi(F), G) = 0$. Choose coordinates $t_0, \ldots, t_n$ in E and the dual coordinates $\xi_0, \ldots, \xi_n$ in E^*. The space $\Lambda^2 E$ is spanned by the forms $\omega_{ij} = \xi_i \wedge \xi_j$. We have

$$\{P_\Phi(F), G\}_{\omega_{ij}} = P_{\xi_i}(P_\Phi(F))P_{\xi_j}(G) - P_{\xi_j}(P_\Phi(F))P_{\xi_i}(G)$$

$$= P_{\xi_i\Phi}(F)P_{\xi_j}(G) - P_{\xi_j\Phi}(F)P_{\xi_i}(G) = P_{\Phi\xi_i}(F)P_{\xi_j}(G) - P_{\Phi\xi_j}(F)P_{\xi_i}(G).$$

For any $A, B \in S^k E^*$,

$$\check{F}(AB) = \Omega_{\check{F}}(A, B) = \langle \Omega_F^{-1}(A), B\rangle.$$

Thus

$$\sigma_{\omega_{ij},F}(P_\Phi(F), G) = \check{F}(P_{\Phi\xi_i}(F)P_{\xi_j}(G) - P_{\Phi\xi_j}(F)P_{\xi_i}(G))$$

$$= \langle \Phi\xi_i, P_{\xi_j}(G)\rangle - \langle \Phi\xi_j, P_{\xi_i}(G)\rangle = P_\Phi(P_{\xi_i\xi_j}(G) - P_{\xi_j\xi_i}(G)) = P_\Phi(0) = 0.$$

□

Let $Z = \{[l_1], \ldots, [l_s]\} \in \mathrm{VSP}(F; s)^o$ be a polar s-polyhedron of a nondegenerate form $F \in S^{2k}E^*$ and, as before, let $\langle Z\rangle_{k+1}$ be the linear span of $k+1$th powers of the linear forms l_i. Let

$$L(Z) = \langle Z\rangle_{k+1}/\mathrm{ap}_F^{k-1}(S^{k-1}E). \tag{9.17}$$

It is a subspace of $W = S^{k+1}E^*/\mathrm{ap}_F^{k-1}(S^{k-1}E)$. By (9.7),

$$W^* = \mathrm{ap}_F^{k-1}(S^{k-1}E)^\perp = \mathrm{AP}_{k+1}(F),$$

where we identify the dual space of $S^{k+1}E^*$ with $S^{k+1}E$. Now observe that $\langle\mathcal{P}\rangle_{k+1}^{\perp}$ is equal to $I_{\mathcal{P}}(k+1)$, where we identify $\mathcal{P}$ with the reduced closed subscheme of the dual projective space $\mathbb{P}(E^*)$. This allows one to extend the definition of $L(\mathcal{P})$ to any generalized polar s-polyhedron $Z \in \mathrm{VSP}(F;s)$:

$$L(Z) = I_Z(k+1)^{\perp}/\mathrm{ap}_F^{k-1}(S^{k-1}E) \subset S^kE^*/\mathrm{ap}_F^{k-1}(S^{k-1}E).$$

Proposition 9.2.7. *Let F be a nondegenerate homogeneous form $S^{2k}E^*$ of Waring rank equal to $N_k = \binom{n+k}{k}$. Let $Z, Z' \in \mathrm{VSP}(F;s)$. Then*

$$L(Z) = L(Z') \Longleftrightarrow Z = Z'.$$

Proof. It is enough to show that

$$I_Z(k+1) = I_{Z'}(k+1) \Longrightarrow Z = Z'.$$

Suppose $Z \neq Z'$. Choose a subscheme Z_0 of Z of length $N_k - 1$ which is not a subscheme of Z'. Since $\dim I_{Z_0}(k) \geq \dim S^kE^* - h^0(\mathcal{O}_Z) = \binom{n+k}{k} - N_k + 1 = 1$, we can find a nonzero $\Phi \in I_{Z_0}(k)$. The sheaf $\mathcal{I}_Z/\mathcal{I}_{Z_0}$ is concentrated at one point x and is annihilated by the maximal ideal $\mathfrak{m}_x$. Thus $\mathfrak{m}_x\mathcal{I}_{Z_0} \subset \mathcal{I}_Z$. Let $\xi \in E$ be a linear form on E^* vanishing at x but not vanishing at any subscheme of Z'. This implies that $\xi\Phi \in I_Z(k+1) = I_{Z'}(k+1)$ and hence $\Phi \in I_{Z'}(k) \subset \mathrm{AP}_k(F)$ contradicting the nondegeneracy of F. □

It follows from Theorem 9.2.6 that each $\omega \in \Lambda^2 E$ defines a skew-symmetric 2-form $\sigma_{\omega,F}$ on $S^{k+1}E$ which factors through a skew-symmetric 2-form $\bar{\sigma}_{\omega,F}$ on $W = S^{k+1}E/\mathrm{ap}_F^{k-1}(S^{k-1}E)$. We call this form a *Mukai 2-form*. For each $\mathcal{P} \in \mathrm{VSP}(F;N_k)^o$ the subspace $L(Z) \subset W$ is isotropic with respect to $\bar{\sigma}_{\omega,F}$.

9.3 First examples

9.3.1 Binary forms

This is the case $n = 1$. The zero subscheme of a homogeneous form of degree d in 2 variables $F(T_0,T_1)$ is a positive divisor $D = \sum m_ip_i$ of degree d. Each such divisor is obtained in this way. Thus we can identify $\mathbb{P}(S^dE^*)$ with $|\mathcal{O}_{\mathbb{P}(E)}(d)| \cong \mathbb{P}^d$ and also with the symmetric product $\mathbb{P}(E)^{(d)} = \mathbb{P}(E)^d/S_d$ and the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{P}(E))$. A generalized s-polyhedron of F is a positive divisor $Z = \sum_{i=1}^k m_i[l_i]$ of degree s in $\mathbb{P}(E^*)$ such that $[F] \in \langle Z\rangle = \mathbb{P}(H^0(\mathbb{P}(E),\mathcal{I}_Z(d))^{\perp})$. Note that in our case Z is automatically linearly independent (because $H^1(\mathcal{I}_Z(d)) = 0$). Obviously, $H^0(\mathbb{P}(E),\mathcal{I}_Z(d))$ consists of polynomials of degree d which are divisible by $\Phi = \xi_1^{m_1}\cdots\xi_k^{m_k}$, where $\xi_i \in \mathrm{AP}(l_i)_1$.

In coordinates, if $l_i = a_iT_0 + b_iT_1$, then $\xi_i = b_i\partial_0 - a_i\partial_1$. Thus F is orthogonal to this space if and only if $P_{\Phi\Phi'}(F) = 0$ for all $\Phi' \in S^{d-s}(E)$. By the apolarity duality this implies that $P_\Phi(F) = 0$, hence $\Phi \in \mathrm{AP}(F)_s$. Thus we obtain

Theorem 9.3.1. *A positive divisor $Z = V(l_1^{m_1} \cdots l_k^{m_k})$ of degree is a generalized s-polyhedron of F if and only if $\xi_1^{m_1} \cdots \xi_k^{m_k} \in \mathrm{AP}(F)_s$.*

Corollary 9.3.2. *Assume $n = 1$. Then*

$$\mathrm{VSP}(F; s) = \mathbb{P}(\mathrm{AP}_s(F)).$$

Note that the kernel of the map

$$S^sE \to S^{d-s}E^*, \quad \Phi \mapsto P_\Phi(F)$$

is of dimension $\geq \dim S^sE - \dim S^{d-s}E^* = s + 1 - (d - s + 1) = 2s - d$. Thus $P_\Phi(F) = 0$ for some nonzero $\Phi \in S^sE$, whenever $2s > d$. This shows that a F has always generilized polar s-polyhedron for $s > d/2$. If d is even, a binary form has an apolar $d/2$-form if and only $\det \mathrm{Cat}_{d/2}(F) = 0$. This is a divisor in the space of all binary d-forms.

Example 9.3.1. Take $d = 3$. Assume that F admits a polar 2-polyhedron. Then

$$F = (a_1T_0 + b_1T_1)^3 + (a_2T_0 + b_2T_1)^3.$$

It is clear that F has 3 distinct roots. Thus, if $F = (a_1T_0+b_1T_1)^2(a_2T_0+b_2T-1)^2$ has a double root, it does not admit a polar 2-polyhedron. However, it admits a generalized 2-polyhedron defined by the divisor $2p$, where $p = (b_1, -a_1)$. In the secant variety interpretation, we know that any point in $\mathbb{P}(S^3E^*)$ either lies on a unique secant or on a unique tangent line of the Veronese cubic curve. The space $\mathrm{AP}(F)_2$ is always one-dimensional. It is generated either by a binary quadric $(-b_1\xi_0 + a_1\xi_1)(-b_2\xi_0 + a_2\xi_1)$ or by $(-b_1\xi_0 + a_1\xi_1)^2$.

Thus $\mathrm{VSP}(F; 2)^o$ consists of one point or empty but $\mathrm{VSP}(F; 2)$ always consists of one point. This example shows that $\mathrm{VSP}(F; 2) \neq \overline{\mathrm{VSP}(F; 2)^o}$ in general.

9.3.2 Quadrics

It follows from Example 9.1.1 that $\mathrm{Sec}_t(Ver_2^n) \neq \mathbb{P}(S^2E^*)$ if only if there exists a quadric with $t + 1$ singular points in general position. Since the singular locus of a quadric $V(Q)$ is a linear subspace of dimension equal to $\mathrm{corank}(Q) - 1$, we obtain that $\mathrm{Sec}_{n+1}(\mathrm{Ver}_2^n) = \mathbb{P}(S^2E^*)$, hence any general quadratic form can be written as a sum of $n + 1$ squares of linear forms $l_0, \ldots, l_n$. Of course, linear algebra gives more. Any quadratic form of rank $n + 1$ can be reduced to sum of squares of the coordinate functions. Assume that $Q = T_0^2 + \ldots + T_n^2$. Suppose we also have $Q = l_0^2 + \ldots + l_n^2$. Then the linear transformation $T_i \mapsto l_i$ preserves Q and hence is an orthogonal transformation. Since polar polyhedra of Q and λQ are the same, we see that the projective orthogonal group $\mathrm{PO}(n+1)$ acts transitively on the set $\mathrm{VSP}(F; n+1)^o$ of polar $(n+1)$-polyhedra of Q. The stabilizer group G of the coordinate polar polyhedron is generated by permutations of coordinates and diagonal orthogonal matrices. It is isomorphic to the semi-direct product $2^n \cdot S_{n+1}$ (the Weyl group of roots systems of type B_n, D_n), where we use the standard notation 2^k for the 2-elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^k$. Thus we obtain

Theorem 9.3.3. *Let Q be a quadratic form in $n+1$ variables of rank $n+1$. Then*

$$\mathrm{VSP}(Q; n+1)^o \cong \mathrm{PO}(n+1)/2^n \cdot S_{n+1}.$$

The dimension of $\mathrm{VSP}(Q; n+1)^o$ *is equal to* $\frac{1}{2}n(n+1)$.

Example 9.3.2. Take $n = 1$. Using the Veronese map $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$, we consider a nonsingular quadric $V(Q)$ as a point p in $\mathbb{P}^2$ not lying on the conic $C = V(T_0T_2 - T_1^2)$. A polar 2-gon of Q is a pair of distinct points p_1, p_2 on C such that $p \in \langle p_1, p_2 \rangle$. It can be identified with the pencil of lines through p with the two tangent lines to C deleted. Thus $W(Q, 2)^o = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$. There are two generalized 2-gons $2p_0$ and $2p_\infty$ defined by the tangent lines. Each of them gives the representation of Q as $l_1 l_2$, where $V(l_i)$ are the tangents. We have $\mathrm{VSP}(F; 2) = \overline{\mathrm{VSP}(F; 2)}^o \cong \mathbb{P}^1$.

It is an interesting question to define a good compactification of this space similar to the one we found in Chapter 2 in the case $n = 2$.

Let $Q \in S^2E^*$ be a non-degenerate quadratic form. For each $Z \in \mathrm{VSP}(Q; n+1)$ the linear space $L(Z) = \langle Z \rangle_2 / \mathbb{C}Q \subset S^2E^*/\mathbb{C}Q$ is of dimension n. It is an isotropic subspace of $W = S^2E^*/\mathbb{C}Q$ with respect to any Mukai's 2-form $\bar{\sigma}_{\omega,Q}$. This defines a map

$$\mu : \mathrm{VSP}(Q; n+1) \to G(n, W), \quad Z \mapsto L(Z). \tag{9.18}$$

By Proposition 9.4.3, the map is injective. The image of $\mathrm{VSP}(Q, n+1)^o$ is contained in the locus $G(n, W)^\mu$ of subspaces which are isotropic with respect to any Mukai's 2-form $\bar{\sigma}_{Q,\omega}$. Since for general F the variety $\mathrm{VSP}(Q, n+1)$ is the closure of $\mathrm{VSP}(Q, n+1)^o$ in the Hilbert scheme, the image of $\mathrm{VSP}(Q; n+1)$ is contained $G(n, W)^\mu$. Since all nonsingular quadrics are isomorphic, the assertion is true for any nondegenerate quadratic for F.

Recall that the Grassmann variety $G(n, W)$ carries the natural rank n vector bundle $\mathcal{S}$, the *tautological bundle*. Its fibre over a point $L \in G(n, W)$ is equal to

L. It is a subbundle of the trivial bundle $W_{G(n,W)}$ associated to the vector space W. We have a natural exact sequence

$$0 \to \mathcal{S} \to W_{G(n,V)} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the *universal quotient bundle*, whose fibre over L is equal to W/L. We can consider each element σ of $\Lambda^2 W^*$ as a section of the trivial bundle $\Lambda^2 W^*_{G(n,W)}$. Restricting σ to the subbundle $\mathcal{S}$, we get a section of the vector bundle $\Lambda^2\mathcal{S}^*$. Thus we can view a Mukai's 2-form $\bar{\sigma}_{Q,\omega}$ as section $s_{Q,\omega}$ of $\Lambda^2\mathcal{S}^*$.

It follows from above that the image of the map (9.18) is contained in the set of common zeros of the sections $s_{Q,\omega}$ of $\Lambda^2\mathcal{S}^*$.

The next result has been proven already, by different method, in Chapter 2, Part I.

Corollary 9.3.4. *Let Q be a nondegenerate quadratic form on a three-dimensional vector space E. Then the image of* VSP$(Q;3)$ *in $G(2,W)$, embedded in the Plücker space $\mathbb{P}(\Lambda^2 W)$, is a smooth irreducible 3-fold equal to the intersection X of $G(2,W)$ with a linear space of codimension 3.*

Proof. We have $\dim W = 5$, so $G(2,W) \cong G(2,5)$ is of dimension 6. Hyperplanes in the Plücker space are elements of the space $\mathbb{P}(\Lambda^2 W^*)$. Note that the functions $s_{Q,\omega}$ are linearly independent. In fact, a basis ξ_0,ξ_1,ξ_2 in E gives a basis $\omega_{01} = \xi_0\wedge\xi_1, \omega_{02} = \xi_0\wedge\xi_2, \omega_{12} = \xi_1\wedge\xi_2$ in $\Lambda^2 E$. Thus the space of sections $s_{Q,\omega}$ is spanned by 3 sections s_{01}, s_{02}, s_{12} corresponding to the forms ω_{ij}. Without loss of generality we may assume that $Q = T_0^2 + T_1^2 + T_2^2$. If we take $A = T_0T_1 + T_2^2, B = -T_0^2 + T_1^2 + T_2^2$, we see that $s_{01}(A,B) \neq 0, s_{12}(A,B) = 0, s_{02}(A,B) = 0$. Thus a linear dependence between the functions s_{ij} implies the linear dependence between two of the functions. It is easy to see that no two functions are proportional. So our 3 functions $s_{ij}, 0 \le i < j \le 2$ span a 3-dimensional subspace of $\Lambda^2 W^*$ and hence define a codimension 3 projective subspace L in the Plücker space $\mathbb{P}(\Lambda^2 W)$. The image of VSP$(Q;3)$ under the map (9.18) is contained in the intersection $G(2,E)\cap L$. This is a 3-dimensional subvariety of $G(2,W)$, and hence contains $\mu($VSP$(Q;3))$ as an irreducible component. We skip an argument, based on counting constants, which proves that the subspace L belongs to an open Zariski subspace of codimension 3 subspaces of $\Lambda^2 W$ for which the intersection $L\cap G(2,W)$ is smooth and irreducible (see [Dolgachev]. □

If $n > 2$, the vector bundle $\Lambda^2\mathcal{S}^*$ is of rank $r = \binom{n}{2} > 1$. The zero locus of its nonzero section is of expected codimension equal to r. We have $\binom{n+1}{2}$ sections s_{ij} of $\Lambda^2\mathcal{S}$ and $\dim G(n,E) = n(\binom{n+2}{2} - n - 1)$. For example, when $n = 3$, we have 6 sections s_{ij} each vanishing on a codimension 3 subvariety of 18-dimensional

Grassmannian $G(3,9)$. So there must be some dependence between the functions s_{ij}.

Remark 9.3.1. One can also consider the varieties $\mathrm{VSP}(Q;s)$ for $s > n+1$. For example, we have

$$\begin{aligned} T_0^2 &= \tfrac{1}{2}(T_0+2T_1)^2 + \tfrac{1}{2}(T_0-2T_1)^2 - 4T_1^2, \\ T_0^2 &= \tfrac{1}{2}(T_0+T_1)^2 + \tfrac{1}{2}(T_0-T_1)^2 - \tfrac{1}{2}(T_2+T_1)^2 + \tfrac{1}{2}(T_2-T_1)^2. \end{aligned}$$

This shows that $\mathrm{VSP}(Q;n{+}3)$, $\mathrm{VSP}(Q;n{+}4)$ are not empty for any nondegenerate quadric Q.

9.3.3 Plane cubics

We will start with cubic forms in 3 variables and, in the next chapter discuss the case of cubic forms in 4 variables. Since for any three general points in $\mathbb{P}^2$ there exists a plane cubic singular at these points (the union of three lines), a general ternary cubic form does not admit polar triangles. Of course this is easy to see by counting constants.

A plane cubic curve projectively isomorphic to the cubic $V(T_0^3+T_1^3+T_2^3)$ will be called a *Fermat cubic*. Obviously such a curve admits a polar 3-polyhedron (*polar triangle*).

Proposition 9.3.5. *A plane cubic admits a polar triangle if and only if either it is a Fermat cubic or it is equal to the union of three distinct concurrent lines.*

Proof. Suppose $F = l_1^3+l_2^3+l_3^3$. Without loss of generality we may assume that l_1^3 is not proportional to l_2^3. Thus, after coordinate change $F = T_0^3+T_1^3+l^3$. If $l(T_0,T_1,T_2)$ does not depend on T_2, the curve $V(F)$ is the union of three distinct concurrent lines. Otherwise we can change coordinates to assume that $l = T_2$ and get a Fermat cubic. □

Remark 9.3.2. If F is a Fermat cubic, then its polar triangle is unique. Its sides are the three first polars of F which are double lines.

By counting constants, we see that a general cubic admits a polar 4-polyhedron ((*polar quadrangle*). We call a polar quadrangle $\{l_1,\ldots,l_4\}$ *nondegenerate* if it is defined by 4 points in $\mathbb{P}(E)$ no three of which are collinear. It is clear that a polar quadrangle is non-degenerate if and only if the linear system of conics in $\mathbb{P}(E)$ through the points $[l_1],\ldots,[l_4]$ is an irreducible pencil (i.e. a linear system of dimension 1 whose general member is irreducible). This allows us to define a *nondegenerate generalized polar quadrangle* of F as a generalized polyhedron Z of F such that $|\mathcal{I}_Z(2)|$ is an irreducible pencil.

Lemma 9.3.6. *F admits a degenerate polar quadrangle if and only if $V(F)$ is one of the following curves:*

(i) a Fermat cubic;

(ii) a cuspidal cubic;

(ii) the union of three concurrent lines (not necessary distinct);

Proof. We have

$$T_0^3 + T_1^3 + T_2^3 = \frac{1}{3}(T_0 + T_1)^3 + \frac{1}{3}(T_0 + aT_1)^3 + \frac{1}{3}(T_0 + a^2T_1)^3 + T_2^3,$$

where $a = e^{2\pi i/3}$.

We have

$$T_0T_1(9T_0 + 15T_1) = (T_0 + T_1)^3 + (T_0 + 2T_1)^3 - 2T_0^3 - 5T_1^3.$$

Since the union of three distinct concurrent lines $V(F)$ is projectively equivalent to $V(T_0T_1(9T_0 + 15T_1))$, we see that F admits a degenerate quadrangle.

We also have

$$T_0T_1^2 = (2T_0 + T_1)^3 + (T_0 - 4T_1)^3 - 9T_0^3 + 15T_1^3,$$

$$(2 + c^3)T_0^3 = (T_0 + aT_1)^3 + (T_0 + bT_1)^3 - (cT_0 + dT_1)^3 - (a^3 + b^3 - d^3)T_1^3,$$

where $a^3 + b^3 = c^2d, a + b = cd^2, c^3 + 2 \neq 0$. This shows that case (iii) occurs.

All cuspidal cubics are projectively equivalently. So it is enough to demonstrate a degenerate polar quadrangle for $V(T_0^3 + 6T_1^2T_2)$. We have

$$T_0^3 + 6T_1^2T_2 = (T_1 + T_2)^3 + (T_2 - T_1)^3 - 2T_2^3 + T_0^3.$$

Now let us prove the converse. Suppose

$$F = l_1^3 + l_2^3 + l_3^3 + l_4^3,$$

where l_1, l_2, l_3 vanish at a common point a. Let ξ be a linear form on E corresponding to a. We have

$$\frac{1}{3}P_\xi(F) = l_1^2\xi(l_1) + l_2^2\xi(l_2) + l_3^2\xi(l_3) + l_4^2\xi(l_4) = l_4^2\xi(l_4).$$

This shows that the first polar $P_a(F) = P_\xi(F)$ is either the whole $\mathbb{P}^2$ or a double line $V(l_4^2)$. In the first case F is the union of three concurrent lines. Assume the second case occurs. We can choose coordinates such that $a = (1,0,0)$ and $l = V(T_0)$. Write

$$F = L_0T_0^3 + T_0^2L_1 + T_0L_2 + L_3,$$

where L_i are homogeneous forms of degree i in variables T_1, T_2. Then $P_a(F) = \partial_0(F) = 3T_0^2L_0 + 2T_0L_1 + L_2$. This can be proportional to T_0^2 only if $L_1 = L_2 = 0$. Thus $V(F) = V(T_0^3 + L_3(T_1, T_2))$. If L_3 does not multiple linear factors, we can choose coordinates such that $L_3 = T_1^3 + T_2^3$, and get the cubic. If L_3 has a linear factor with multiplicity 2, we reduce L_3 to the form $T_1^2T_2$. This is the case of a cuspidal cubic. Finally, if L_3 is a cube of a linear form, we reduce the latter to the form T_1^3 and get three concurrent lines.

□

Remark 9.3.3. The set of Fermat cubics is a hypersurface in the space $\mathbb{P}(S^3E^*)$ isomorphic to the homogeneous space $\mathrm{PSL}(3)/3^2.S_3$. Its closure in $\mathbb{P}(S^3E^*)$ consists of curves listed in the assertion of the previous lemma and also reducible cubics equal to the unions of irreducible conics with its tangent line.

Lemma 9.3.7. *The following properties equivalent*

(i) $\mathrm{AP}_1(F) \neq \{0\}$*;*

(ii) $\dim \mathrm{AP}_2(F) > 2$*;*

(iii) $V(F)$ *is equal to the union of three concurrent lines.*

Proof. By the apolarity duality

$$(A_F)_1 \times (A_F)_2 \to (A_F)_3 \cong \mathbb{C},$$

we have

$$\dim(A_F)_1 = 3 - \dim AP_1(F) = \dim(A_F)_2 = 6 - \dim \mathrm{AP}_2(F).$$

Thus $\dim \mathrm{AP}_2(F) = 3 + \dim AP_1(F)$. This proves the equivalence of (i) and (ii). By definition, $\mathrm{AP}(F)_1 \neq \{0\}$ if and only if $P_\Phi(F) = 0$ for some nonzero linear operator $\Phi = \sum a_i\partial_i$. After a linear change of variables, we may assume that $\Phi = \partial_0$, and then $\partial_0(F) = 0$ if and only if F does not depend on T_0, i.e. $V(F)$ is the union of three concurrent lines. □

Lemma 9.3.8. *Let Z be a nondegenerate generalized polar quadrangle of F. Then $|\mathcal{I}_Z(2)|$ is a pencil of conics in $\mathbb{P}(E^*)$ contained in $|\mathrm{AP}_2(F)|$. Conversely, let Z be a 0-dimensional cycle of length 4 in $\mathbb{P}(E)$. Assume that $|\mathcal{I}_Z(2)|$ is an irreducible pencil contained in $|\mathrm{AP}_2(F)|$. Then Z is a nondegenerate generalized polar quadrangle of F.*

Proof. The first assertion follows from the definition of non-degeneracy and Proposition 9.1.6. Let us prove the converse. Let $V(\lambda Q_1 + \mu Q_2)$ be the pencil of conics $|\mathcal{I}_Z(2)|$. Since $\mathrm{AP}(F)$ is an ideal, the linear system L of cubics of the form $V(Q_1L_1+Q_2L_2)$, where L_1, L_2 are linear forms, is contained in $\mathbb{P}(\mathrm{AP}_3(F))$. Obviously it is contained in $|\mathcal{I}_Z(3)|$. Since $|\mathcal{I}_Z(2)|$ has no fixed part we may choose Q_1 and Q_2 with no common factors. Then the map $E^* \oplus E^* \to I_Z(3)$ defined by $(L_1, L_2) \to Q_1L_1+Q_2L_2$ is injective hence $\dim L = 5$. Assume $\dim |\mathcal{I}_Z(3)| \geq 6$. Choose 3 points in general position on an irreducible member C of $|\mathcal{I}_Z(2)|$ and 3 non-collinear points outside C. Then find a cubic K from $|\mathcal{I}_Z(3)|$ which passes through these points. Then K intersects C with total multiplicity $4+3=7$, hence contains C. The other component of K must be a line passing through 3 non-collinear points which is absurd. So, $\dim |\mathcal{I}_Z(3)| = 5$ and we have $L = |\mathcal{I}_Z(3)|$. Thus $|\mathcal{I}_Z(3)| \subset \mathbb{P}(\mathrm{AP}_3(F))$ and, by Proposition 9.1.6, Z is a generalized polar quadrangle of F.

□

Corollary 9.3.9. *Suppose F is not the union of three concurrent lines. Then subset of* $\mathrm{VSP}(F;4)$ *consisting of nondegenerated generalized polar quadrangles is isomorphic to an open subset of the plane* $|\mathrm{AP}_2(F)|^*$.

Example 9.3.3. Let $V(F)$ be the union of an irreducible conic and its tangent line. After a linear change of variables we may assume that $F = T_0(T_0T_1 + T_2^2)$. It is easy to check that $\mathrm{AP}_2(F)$ is spanned by $\xi_1^2, \xi_1\xi_2, \xi_2^2 - \xi_0\xi_1$. It follows from Lemma 9.3.6 that F does not admit degenerate polar quadrangle. Thus any polar quadrangles of F is the base locus of an irreducible pencil in $|\mathrm{AP}_2(F)|$. However, it is easy to see that all nonsingular conics in $|\mathrm{AP}_2(F)|$ are tangent at the point $(0,1,0)$. Thus no pencil has 4 distinct base points. This shows that

$$\mathrm{VSP}(F;4)^o = \emptyset.$$

Of course, $\mathrm{VSP}(F;4) \neq \emptyset$. Any irreducible pencil in $|\mathrm{AP}_2(F)|$ defines a generalized polar quadrangle. It is easy to see that the only reducible pencil is $V(\lambda\partial_1^2 + \mu\partial_1\partial_2)$. Thus $\mathrm{VSP}(F;4)$ contains a subvariety isomorphic to a complement of one point in $\mathbb{P}^2 = |\mathrm{AP}_2(F)|^*$. To compactify it by $\mathbb{P}^2$ we need to find one more generalized polar quadrangle. Consider the subscheme Z of degree 4 concentrated at the point $(1,0,0)$ with ideal at this point generated by (x^2, xy, y^3), where we use inhomogeneous coordinates $x = \xi_1/\xi_0, y = \xi_2/\xi_0$. The linear system $|\mathcal{I}_Z(3)|$ is of dimension 5 and consists of cubics of the form $V(\xi_0\xi_1(a\xi_1 + b\xi_2) + G_3(\xi_1,\xi_2))$. Thus Z is linearly 3-independent. One easily computes $\mathrm{AP}_3(F)$. It is generated by all monomials except $\xi_0^2\xi_1$ and $\xi_0\xi_2^2$ and also the polynomial $\xi_0\xi_2^2 - \xi_0^2\xi_1$. We see that $|\mathcal{I}_Z(3)| \subset \mathbb{P}(\mathrm{AP}_3(F)$. Thus Z is a generalized polar quadrangle of F.

It is non-degenerate since $|\mathcal{I}_Z(2)|$ is the pencil $V(\lambda\xi_1^2 + \mu\xi_1\xi_2)$. So, we see that VSP$(F;4)$ is isomorphic to the plane $|\text{AP}_2(F)|^*$.

Example 9.3.4. Let $V(F)$ be an irreducible nodal cubic. Without loss of generality we may assume that $F = T_2^2T_0 + T_1^3 + T_1^2T_0$. The space of apolar quadratic forms is spanned by $\partial_0^2, \partial_1\partial_2, \partial_2^2 - \partial_1^2$. The net $|\text{AP}_2(F)|$ is base-point-free. It is easy to see that its discriminant curve is the union of three distinct non-concurrent lines. Each line defines a pencil with singular general member but without fixed part. So, VSP$(F;4) = |\text{AP}_2(F)|^*$.

Example 9.3.5. Let $V(F)$ be the union of an irreducible conic and a line which intersects the conic transversally. Without loss of generality we may assume that $F = T_0(T_0^2 + T_1T_2)$. The space of apolar quadratic forms is spanned by $\xi_1^2, \xi_2^2, 6\xi_1\xi_2 - \xi_0^2$. The net $|\text{AP}_2(F)|$ is base-point-free. It is easy to see that its discriminant curve is the union of a conic and a line intersecting the conic transversally. The line defines a pencil with singular general member but without fixed part. So, VSP$(F;4) = |\text{AP}_2(F)|^*$.

Example 9.3.6. Let $V(F)$ be a cuspidal cubic. Without loss of generality we may assume that $F = T_1^2T_0 + T_2^3$. The space of apolar quadratic forms is spanned by $\xi_0^2, \xi_0\xi_2, \xi_2\xi_1$. The net $|\text{AP}_2(F)|$ has 2 base points $(0,1,0)$ and $(0,0,1)$. The point $(0,0,1)$ is a simple base-point. The point $(0,1,0)$ is of multiplicity 2 with the ideal locally defined by (x^2, y). Thus base-point scheme of any irreducible pencil is not reduced. There are no polar 4-polyhedra defined by the base-locus of a pencil of conics in $|\text{AP}_2(F)|$. The discriminant curve is the inion of two lines, each defining a pencil with a fixed line. So $|\text{AP}_2(F)|^*$ minus 2 points parametrizes generalized polar 4-polyhedra. We know that $V(F)$ admits degenerate polar 4-polyhedra. Thus VSP$(F;4)^o$ is not empty and consists of degenerate polar 4-polyhedra.

Example 9.3.7. Let $V(F)$ be a nonsingular cubic curve. We know that its equation can be reduced to a Hesse form $V(T_0^3 + T_1^3 + T_2^3 + 6aT_0T_1T_2)$, where $1 + 8a^3 \neq 0$. The space of apolar quadratic forms is spanned by $a\xi_0\xi_1 - \xi_2^2, a\xi_1\xi_2 - \xi_0^2, a\xi_0\xi_2^2 - \xi_1^2$. The curve $V(F)$ is Fermat if and only if $a(a^3 - 1) = 0$. In this case the net has 3 ordinary base points and the discriminant curve is the union of 3 non-concurrent lines. The net has 3 pencils with fixed part defined by these lines. Thus the set of nondegenerate generalized polyhedrons is equal to the complement of 3 points in $|\text{AP}_2(F)|^*$. We know that a Fermat cubic admits degenerate polar 4-polyhedra.

Suppose $V(F)$ is not a Fermat cubic. Then the net $|\text{AP}_2(F)|$ is base-point-free. Its discriminant curve is a nonsingular cubic. All pencils are irreducible. There are no degenerate generalized polygons. So, VSP$(F;4) = |\text{AP}_2(F)|^*$.

Example 9.3.8. Assume that $V(F) = V(T_0T_1T_2)$ is the union of 3 non-concurrent lines. Then $\text{AP}_2(F)$ is spanned by $\xi_0^2, \xi_1^2, \xi_2^2$. The net $|\text{AP}_2(F)|$ is base-point-free. The discriminant curve is the union of three non-concurrent lines representing pencils without fixed point but with singular general member. Thus $\mathrm{VSP}(F;4) = |\mathrm{AP}_2(F)|^*$.

It follows from the previous examples that $\mathrm{AP}_2(F)|$ is base-point-free net of conics if and only if F does not belong to the closure of the orbit of Fermat cubics.

Theorem 9.3.10. *Assume that F does not belong to the closure of the orbit of Fermat cubics. Then $|\mathrm{AP}_2(F)|$ is base-point-free net of conics and*

$$\mathrm{VSP}(F;4) \cong |\mathrm{AP}_2(F)|^* \cong \mathbb{P}^2.$$

The variety $\mathrm{VSP}(F;4)^o$ is isomorphic to the open subset of $|\mathrm{AP}_2(F)|^*$ whose complement is the curve B of pencils with non-reduced base-locus. The curve B is a plane sextic with 9 cusps if $V(F)$ is a nonsingular curve, the union of three non-concurrent lines if $V(F)$ is an irreducible nodal curve or the union of three lines, and the union of a conic and its two tangent lines if $V(F)$ is the union of a conic and a line.

Proof. The first assertion follows from the Examples 9.3.4-9.3.8. Since the linear system of conics $|\mathrm{AP}_2(F)|$ is base-point-free, it defines a regular map

$$\phi : \mathbb{P}(E) \to |\mathrm{AP}_2(F)|^*.$$

The pre-image of a line is a conic from $\mathcal{N}$. The lines through a point q in $|\mathrm{AP}_2(F)|^*$ define a pencil with base locus $\phi^{-1}(q)$. Thus pencils with non-reduced locus are parametrized by the branch curve B of the map ϕ.

If $V(F)$ is a nonsingular cubic, we know from Example 9.3.7 that the discriminant curve Δ is a nonsingular cubic. A line in $|\mathrm{AP}_2(F)|$ defines a pencil of conics. Its singular members are the intersection points of the line and Δ. It is easy to see that the pencil has exactly 3 singular members if and only its base-point locus consists of 4 distinct points. Thus the curve B is the dual curve of Δ. By the duality, B is dual of Δ. We know from Part 1 that it is a sextic with 9 cusps.

If $V(F)$ is an irreducible nodal curve, we know from Example 9.3.4 that Δ is the union of three non-concurrent lines. The locus of lines intersecting Δ not transversally is the union of three pencils of lines. As above B must be the union of three non-concurrent lines.

If $V(F)$ is the union of a conic and and a line, we know from Example 9.3.5 that Δ is the union of a conic and and a line intersecting the conic transversally. Obviously B must contain an irreducible component dual to the conic. Other irreducible components must be two tangent lines to the conic.

Finally if $V(F)$ is the union of three lines, the map ϕ is given by $(t_0,t_1,t_2) \to (t_0^2,t_1^2,t_2^2)$ and as is easy to see its branch locus is the union of the coordinate lines. □

Let $C \subset \mathbb{P}(S^3E^*) \cong \mathbb{P}^9$ be the locus of three concurrent lines. For each $V(F) \in \mathbb{P}(S^3E^*) \setminus C$, the space $\mathrm{AP}_2(F)$ is 3-dimensional. This defines a regular map $a : \mathbb{P}(S^3E^*) \setminus C \to G(3, S^2E)$. Both the varieties are 9-dimensional. Fix a 3-dimensional subspace L of S^2E and consider the linear map

$$\tilde{a} : S^3E^* \to \mathrm{Hom}(L, V^*), \quad a(F)(\Phi) = P_\Phi(F).$$

Its kernel consists of cubic forms F such that $L \subset \mathrm{AP}_2(F)$. Note that the map $\tilde{a}$ is a linear map from a 10-dimensional space to a 9-dimensional space. One expects that its kernel is 1-dimensional. This shows that, for a general point $L \in G(3, S^2E)$ the pre-image a^{-1} is a one-point. Thus the map a is birational.

9.4 Plane quartics

9.4.1 Clebsch and Lüroth quartics

Since 5 general points in $\mathbb{P}(E^*)$ lie on a singular quartic (a double conic), a general quartic does not admit a polar 5-polyhedron (*polar pentagon*) although the count of constant suggests that this is possible. This remarkable fact was first discovered by J. Lüroth in 1868. Suppose a quartic admits a polar pentagon $\{[l_1], \ldots, [l_5]\}$. Let $C = V(Q)$ be a conic in $\mathbb{P}(E^*)$ passing through the points $[l_1], \ldots, [l_5]$. Then $Q \in \mathrm{AP}_2(F)$. The space $\mathrm{AP}_2(F) \neq \{0\}$ if and only $\det \mathrm{Cat}_2(F) = 0$. Thus the set of quartics admitting a polar pentagon is the locus of the catalecticant invariant on the space $\mathbb{P}(S^4E^*)$. It is a polynomial of degree 6 in the coefficients of a homogeneous form of degree 4.

Definition 9.8. *A plane quartic admitting a polar pentagon is called a* Clebsch quartic.

A Clebsch quartic is called *nondegenerate* if $\dim \mathrm{AP}_2(F) = 1$. Thus the polar pentagon of a nondegenerate Clebsch quartic lies on a unique conic. We call it the *associated conic*. The associated conic is reducible if and only if the corresponding operator is the product of two linear operators. This means that the second polar $P_{ab}(F) = 0$ for some points $a, b \in \mathbb{P}(E)$.

Proposition 9.4.1. *Let $F \in S^4E^*$ be such that the second polar $P_{ab}(F) = 0$ for some $a, b \in \mathbb{P}(E)$. Then, in appropriate coordinate system*

$$F = G_3(T_0, T_1)T_0 + H_4(T_1, T_2), \quad a \neq b,$$

$$F = A_3(T_1, T_2)T_0 + B_4(T_1, T_2), \quad a = b.$$

In particular, $P_{aa}(F) = 0$ if and only if $V(F)$ has a triple point.

Proof. Suppose $a \neq b$. Choose coordinates such that $a = (1,0,0), b = (0,0,1)$ and write

$$F = \sum_{i=0}^{4} L_i(T_1,T_2)T_0^{4-i}.$$

Then $P_{aa} = \frac{\partial^2}{\partial T_0^2}$, $P_{ab}(F) = \frac{\partial^2}{\partial T_2 \partial T_0}(F) = 0$. Now the assertions easily follow. □

We will assume that the apolar conic of a nondegenerate Clebsch quartic is irreducible.

Let $\{[l_1],\ldots,[l_5]\}$ be a polar pentagon of F such that $F = l_1^4 + \ldots + l_5^4$. For any $1 \leq i < j \leq 5$, let $a_{ij} = V(l_i) \cap V(l_j) \in \mathbb{P}(E)$. We can identify a_{ij} with a linear operator $\Phi_{ij} \in E$ (defined up to a constant factor). Obviously, $P_{\Phi_{ij}}(F)$ coincides with the first polar $P_{a_{ij}}(F)$. Applying Φ_{ij} we obtain

$$P_{\Phi_{ij}}(F) = P_{\Phi_{ij}}(l_1^4 + \ldots + l_5^4) = 4\sum_{k\neq i,j} \Phi_{ij}(l_k)l_k^3.$$

Thus $[l_k], k \neq i,j$, form a polar triangle of $P_{a_{ij}}(F)$. Since the associated conic is irreducible no three points among the $[l_{ij}]$'s are linearly dependent. Thus $P_{a_{ij}}(F)$ is a Fermat cubic.

Lemma 9.4.2. *Let $F \in S^4E^*$. Assume that $P_{ab}(F) \neq 0$ for any $a,b \in \mathbb{P}(E)$. Let S be the locus of points $a \in \mathbb{P}(E)$ such that the first polar of $V(F)$ is a Fermat cubic or belongs to the closure of its orbit. Then S is a plane quartic.*

Proof. Let $I_4 : S^3E^* \to \mathbb{C}$ be the Clebsch invariant vanishing on the locus of Fermat cubics. It is a polynomial of degree 4 in coefficients of a cubic. If the cubic is written in a Weierstrass form $F = T_0T_2^2 + T_1^3 + aT_0^2T_1 + bT_0^3 = 0$, then $I_4(F) = \lambda a$, for some nonzero constant λ independent of F.

Compose I_4 with the polarization map $E \times S^4E^* \to S^3E^*$, $(\Phi,F) \mapsto P_\Phi(F)$. We get a bihomogeneous map of degree $(4,4)$ $E \times S^4E^* \to \mathbb{C}$. It defines a degree 4 homogeneous map

$$\text{Sc} : S^4E^* \to S^4E^* \tag{9.19}$$

This map is called the *Clebsch quartic covariant*. It assigns to a quartic form in 3 variables another quartic form in 3-variables. By construction, this map does not depend on the choice of coordinates. Thus it is a *covariant* of quartics, i.e. a $\text{GL}(E)$-equivariant map from S^4E^* to some S^dE^*. By definition, the locus of $a \in E$ such that $\text{Sc}(F)(v) = 0$ is the set of vectors $v \in E$ such that $I_4(P_a(F)) = 0$, i.e., $V(P_a(F))$ belongs to the closure of the Fermat locus. □

Example 9.4.1. Assume that the equation of F is given in the form

$$F = aT_0^4 + bT_1^4 + cT_2^4 + 6fT_1^2T_2^2 + 6gT_0^2T_1^2 + 6hT_0^2T_1^2.$$

Then the explicit formula for the Clebsch covariant gives

$$\mathrm{Sc}(F) = a'T_0^4 + b'T_1^4 + c'T_2^4 + 6f'T_1^2T_2^2 + 6g'T_0^2T_1^2 + 6h'T_0^2T_1^2,$$

where

$$\begin{aligned} a' &= 6g^2h^2 \\ b' &= 6h^2f^2 \\ c' &= 6f^2g^2 \\ f' &= bcgh - f(bg^2 + ch^2) - ghf^2 \\ g' &= acfh - g(ch^2 + af^2) - fhg^2 \\ h' &= abfg - h(af^2 + bg^2) - fgh^2 \end{aligned}$$

For a general F the formula for S is too long.

Note that the map Sc defines a rational map

$$\mathrm{Sc} : \mathbb{P}(S^4E^*) - \to \mathbb{P}(S^4E^*) \tag{9.20}$$

We call it the *Scorza map* in honor of Gaetano Scorza who studied the geometry of this map. Note that the Scorza map is not defined on the closed subset of quartics $V(F)$ such that $V(P_a(F))$ belongs to the closure of the Fermat locus for any $a \in \mathbb{P}(E)$.

Proposition 9.4.3. *The Scorza map is not defined on $V(F)$ if and only if $V(F)$ is a Clebsch quartic admitting a reducible apolar conic.*

We refer for a proof to [Dolgachev-Kanev].

For any quartic curve C satisfying the assumption of the previous proposition, the curve $\mathrm{Sc}(C)$ will be called the *Scorza quartic* associated to C. If C is a nondegenerate Clebsch quartic, then, as we explained in above, the vertices of its polar pentagon must belong to the Scorza quartic $\mathrm{Sc}(C)$. This gives

Proposition 9.4.4. *Let F be a nondegenerate Clebsch quartic. Then each polar pentagon of F is inscribed in the quartic curve $V(\mathrm{Sc}(F))$.*

Lemma 9.4.5. *A quartic curve C which can be circusmscibed about a pentagon defined by 5 lines $V(l_i)$ can be written in the form $C = V(G)$, where*

$$G = l_1 \cdots l_5 \sum_{i=1} \frac{a_i}{l_i}.$$

Proof. Consider the linear system of quartics passing through 10 vertices of a pentagon. The expected dimension of this linear system is equal to 4. Suppose it is larger than 4. Since each side of the pentagon contains 4 vertices, requiring that a quartic vanishes at some additional point on the side forces the quartic contain the side. Since we have 5 sides, we will be able to find a quartic containing the union of 5 lines, obviously a contradiction. Now consider the linear system of quartics whose equation can be wriitten as in the assertion of the lemma. The equations have 5 parameters and it is easy to see that the polynomials $l_1 \cdots l_5/l_i, i = 1, \ldots, 5$, are linearly independent. □

Definition 9.9. *A plane quartic circumscribed about a pentagon is called a* Lüroth *quartic.*

Thus we see that for any Clebsch quartic C the Scorza quartic $\mathrm{Sc}(C)$ is a Lüroth quartic. One can prove that any Lüroth quartic is obtained in this way from a unique Clebsch quartic (see [DK]). Since the locus of Clebsch quartics is a hypersurface (of degree 6) in the space of all quartics, the locus of Lüroth quartics is also a hypersurface. Its degree is equal to 54 and coincides with one of the coefficients of the Donaldson polynomial for the preojective plane (see [Le Poitier, A. Tjurin]).

Let $C = V(F)$ be a general Clebsch quartic. Consider the map

$$c : \mathrm{VSP}(F; 5)^o \to \mathbb{P}(S^2E^*) \tag{9.21}$$

defined by assigning to $\{l_1, \ldots, l_5\}$ the unique conic containing the points $[l_1], \ldots, [l_5]$. The fibres of this map are polar pentagons of F inscribed in the apolar conic. We know that the closure of the set of Clebsch quartics is defined by one polynomial in coefficients of quartic, the catalecticant invariant. Thus the varierty of Clebsch quartics is of dimension 13. Consider the map $(E^*)^5 \to \mathbb{P}(S^4E^*)$ defined by $(l_1, \ldots, l_5) \mapsto V(l_1^4 + \ldots + l_4^4)$. The image of this map is the variety of Clebsch quartics. A general fibre must be of dimension $15 - 13 = 2$. However, scaling the l_i by the same factor, defines the same quartic. Thus the dimesnion of the space of all polar pentagons of a general Clebsch quartic is equal to 1. Over an open subset of the Clebsch locus, the fibres of c are irreducible one-dimensional varietes.

Proposition 9.4.6. *Let $V(F)$ be a nondegenerate Clebsch quartic and C be its apolar conic. Consider any polar pentagon of F as a set of 5 points on C (the dual of its sides). Then* $\mathrm{VSP}(F; 5)^o$ *is an open non-empty subset of a linear pencil on C of degree 5.*

Proof. Consider the correspondence

$$X = \{(x, \{l_1, \ldots, l_5\}) \in C \times \mathrm{VSP}(F; 5)^o : x = [l_i] \text{ for some } i = 1, \ldots, 5\}.$$

Let us look at the fibres of the projection to C. Suppose we have 2 polar pentagons of F with the same side $[l]$. We can write

$$F - l^4 = l_1^4 + \ldots + l_4^2,$$

$$F - \lambda l^4 = m_1^4 + \ldots + m_4^4.$$

For any $\Phi \in S^2E$ such that $\Phi([l_i]) = 0, i = 1, \ldots, 4$, we get $P_\Phi(F) = 12\Phi(l)l^2$. Similarly, for any $\Phi' \in S^2E$ such that $\Phi'([m_i]) = 0, i = 1, \ldots, 4$, we get $P_{\Phi'}(F) = 12\lambda\Phi'(l)l^2$. This implies that $\Phi(l)\Phi' - \Phi'(l)\Phi$ is an apolar conic to F. Since F was a general Clebsch quartic, there is only one apolar conic. The set of $V(\Phi)$'s is a pencil with base points $[l_i]$, the set of $V(\Phi')$ is a pencil with base points $[m_i]$. This gives a contradiction unless the two pencils coincide. But then their base points coincide and the two pentagons are equal. This shows that the projection to C is a one-to-one map. In particular, X is an irreducible curve.

Now it is easy to finish the proof. The set of degree 5 positive divisors on $C \cong \mathbb{P}^1$ is the projective space $|\mathcal{O}_{\mathbb{P}^1}(5)|$. The closure $\mathcal{P}$ of our curve of polar pentagons lies in this space. All divisors containing one fixed point in their support form a hyperplane. Thus the polar pentagons containing one common side $[l]$ correspond to a hyperplane section of $\mathcal{P}$. Since we know that there is only one such pentagon and we take $[l]$ in an open Zariski subset of C, we see that the curve is of degree 1, i.e. a line. So our curve is contained in one-dimensional linear system of divisors of degree 5. □

9.4.2 The Scorza map

Here we assume that $VF)$ is not projectively equivalent to the quartics from Proposition 9.4.1. Let us study the Scorza map (9.20) more closely. Let $C = V(F)$ be any quartic and $S = \mathrm{Sc}(C)$. For any $a \in S$, the first polar $P_a(F)$ defines a Fermat curve (or its degeneration). As we saw in the proof of Lemma 9.3.6, these curves are characterized by the property that there exists a point b such that the first polar is a double line. This defines a correspondence

$$R_C = \{(a, b) \in S \times S : P_b(P_b(F)) \text{ is a double line}\}.$$

Recall that a *correspondence* Z on a variety X is a closed subvariety of $X \times X$ such that the both projections are surjective. A correspondence is called *finite* of degree (d_1, d_2) if the projection to the ith factor is a finite morphism of degree d_i. A correspondence is called *symmetric* if $(x, y) \in R \Leftrightarrow (y, x) \in R$.

We will be dealing with the case when X is a nonsingular projective curve of genus g. For any $x \in X$ we set $R(x) = p_1^{-1}(x)$ considered as a closed subscheme of $\{x\} \times X$ identified with a positive divisor on X. The map $x \mapsto R(x)$ can be extended to a map $c : \mathrm{Div}(X)^0(X) \to \mathrm{Jac}(X)$ by assigning to any divisor $\sum n_i x_i$ of degree 0 the divisor class of $\sum n_i R(x_i)$. Since for any rational function f on X we have $c(\mathrm{div}(f)) = [\mathrm{div}(p_2^*(f))]$, the map c factors to an endomorphism of the Jacobian variety $c : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$. It is known that for a general curve X any endomorphism is equal to a multiplication endomorphism $[m] : [D] \mapsto m[D]$. We say that R is of *valency* v if $c = [v]$ for some integer v. This implies that the divisor class of $R(x) - vx$ does not depend on x (fix a point x_0 and show that $[R(x) - vx] = [R(x_0) - vx_0]$).

Let Δ be the diagonal of $X \times X$. The points in $\Delta \cap R$ are called *fixed points* of R.

Lemma 9.4.7. *(Brill-Cayley formula). Suppose R is a finite equivalence of degree (d_1, d_2) on a nonsingular projective curve X of genus g with valency v. Then*

$$[R] \cap [\Delta] = d_1 + d_2 - 2gv.$$

Proof. Here $[R], [\Delta]$ denote the class of R, Δ in the group $\mathrm{Num}(X)$ of divisor classes on the surface $X \times X$ modulo numerical equivalence (or in $H^2(X \times X, \mathbb{Z})$). Let E_1, E_2 be the classes of fibres of the projections. For any $x \in X$ the restriction of the divisor class of $R - v\Delta$ to each fibre $p_1^{-1}(x)$ is equal to the restriction of a divisor class $p_2^{-1}(D)$, where D does not depend on x. Thus $R - v\Delta - p_2^{-1}(D)$ restricts to the trivial divisor class on each fibre $p_1^{-1}(x)$. This implies that $R - v\Delta - p_2^{-1}(D) = p_1^{-1}(D')$ for some divisor class on X (see [Hartshorne], Ex. 12.4). Thus we obtain the equality

$$[R] = v[\Delta] + aE_1 + bE_2$$

in $\mathrm{Num}(X \times X)$ for some integers a, b. Intersecting with E_1 we get $d_1 = v + b$. Intersecting with E_2, we get $d_2 = v + a$. Intersecting with $[\Delta]$ and using the well-known fact from topology that $[\Delta]^2 = 2 - 2g$, we get $[R] \cap [\Delta] = 2v(1-g) + a + b$. Thus $a = d_2 - v, b = d_1 - v$ and $[R] \cap [\Delta] = 2v(1 - g) + (d_1 + d_2) - 2v = d_1 + d_2 - 2gv$. □

Proposition 9.4.8. *Let $C = V(F)$ be a general plane quartic. Then R_C is a finite symmetric correspondence of degree $(3, 3)$ on $S = \mathrm{Sc}(C)$ without fixed points and valency 1.*

Proof. The symmetry of R_F is obvious. We have a map from S to the closure $\mathcal{F}$ of the Fermat locus defined by $a \mapsto V(P_a(F))$. For any curve in $\mathcal{F}$, except the union of three lines, the set of points such that the first polar is a double line is finite. It is equal to the set of double points of the Hessian curve and consists of 3 points for Fermat curves, one point for cuspidal cubics and 2 points for the unions of a conic and a line. If F is general enough the image of S in $\mathcal{F}$ does not intersect the locus of the unions of three lines (which is of codimension 2). Thus we see that each projection from R_F to S is a finite map of degree 3. Its branch points correspond to the intersection of the image of S with the boundary of the orbit of Fermat curves.

For any general point $x \in S$ the first polar $P_x(C)$ is a Fermat cubic. The divisor $R_C(x)$ consists of the three vertices of its unique polar triangle. For any $y \in R_C(x)$ the side $H = V(l)$ opposite to y is defined by $P_y(P_x(C)) = P_x(P_y(C)) = \lambda l^2$. It is a common side of the polar triangles of $P_x(C)$ and $P_y(C)$. We have $H \cap S = y_1 + y_2 + x_1 + x_2$, where $R_C(x) = \{y, y_1, y_2\}$ and $R_C(y) = \{x, x_1, x_2\}$. This gives

$$y_1 + y_2 + x_1 + x_2 = (R_C(x) - x) + (R_C(y) - y) \in |K_S|.$$

Consider the map $f : S \to \mathrm{Pic}^2(S)$ given by $x \to [R(x) - x]$. Assume f is not constant. If we replace in the previous formula y with y_1 or y_2, we obtain that $f(y) = f(y_1) = f(y_2) = K_S - f(x)$. Thus f is of degree ≥ 3 on its image $\bar{S}$ and factors to a finite map to the normalization W of the image. Since a rational curve does not admit non-constant maps to an abelian variety, we obtain that W is of positive genus. By Hurwitz formula, the genus of W is less or equal than 2. It is easy to see, counting constans for branch points, that genus 3 curves admitting a finite map onto a curve of genus 1 or 2 is of codimension ≥ 2 in the moduli space $\mathcal{M}_3$. Since $\mathbb{P}(S^4E^*)/\mathrm{GL}(E)$ is birationally isomorphic to $\mathcal{M}_3$, we see that the locus of quartics for which the map f is not constant is of codimension ≥ 2. Taking S in an open Zariski subset of the hypersurface of Lüroth quartics (the Scorza curves of Clebsch quartics), we see that for a general Scorza curve S, the map f is constant. Hence R_C has valency $v = 1$. Using the Brill-Cayley formula for $d_1 = d_3 = g = 3$, we obtain that has no fixed points. In particular, P_{aa} is never a double line. □

Let X be a nonsingular projective curve of genus g and θ a theta characteristic with $h^0(\theta) = 0$. For any $x \in X$ the Riemann-Roch gives $h^0(\theta + x) = 1$. Thus $|\theta + x|$ consists of a unique positive divisor $x_1 + \ldots + x_g$. For any x_i we have $h^0(\theta + x - x_i) > 0$.

Consider a correspondence R_θ on X defined by

$$R_\theta = \{(a, b) \in X \times X : h^0(\theta + a - b) > 0\}.$$

Assume that F is general enough such that S is a nonsingular quartic. Consider the *difference map*

$$d : S \times S \to \mathrm{Jac}(S), \quad (x, y) \mapsto [x - y].$$

By Riemann-Roch,

$$0 = h^0(\theta + a - b) - h^1(\theta + a - b) = h^0(\theta + a - b) - h^0(K_S - \theta - a + b)$$

$$= h^0(\theta + a - b) - h^1(\theta + b - a).$$

Thus R_θ is a symmetric correspondence. Its degree is equal to (g, g). It has no fixed points because $h^0(\theta) = 0$. By its definition $[R_\theta(x) - x] = \theta$, so R_θ is of valency 1.

Theorem 9.4.9. *Let X be a non-hyperelliptic nonsingular projective curve of genus $g > 0$. Let R be a finite symmetric correspondence R of degree (g, g) on X without fixed points and of some valency v. Then $R = R_\theta$ for a unique theta characteristic θ with $h^0(\theta) = 0$.*

Proof. It follows from the Brill-Clebsch formula that the valency v of R is equal to 1. Thus $\theta = [R(x) - x]$ does not depend on x. Since R has no fixed points $R(x) - x$ is not effective, i.e., $h^0(\theta) = 0$. Consider the difference map

$$\phi_1 : X \times X \to \text{Jac}(X), \quad (x, y) \mapsto [x - y].$$

Since $\phi_1(x - y) = \phi_1(x; -y')$ implies that $x + y' \sim x' + y$, the assumption that X is not hyperelliptic implies that ϕ_1 is birational onto its image. It obviously blows down the diagonal to the zero point. For any $(x, y) \in R$, the divisor class $[y - x] + \theta$ is effective (use Riemann-Roch) and of degree $g - 1$. Let W_{g-1} be the divisor of effective divisor classes in $\text{Pic}^{g-1}(X)$ (the theta divisor) and $W_{g-1} - \theta$ its translate in $\text{Jac}(X)$. We see that $\phi_1(R) + \theta \subset W_{g-1}$. Let $\sigma : X \times X \to X \times X$ be the switch of the factors. Then

$$\phi(R) = \phi(\sigma(R)) = [-1](\phi(R)) \subset [-1](W_{g-1} - \theta) \subset W_{g-1} + \theta',$$

where $\theta' = K_X - \theta$. Since $R \cap \Delta = \emptyset$, the difference map ϕ_1 is injective on R. Thus

$$R = \phi_1^{-1}(W_{g-1} - \theta) = \phi_1^{-1}(W_{g-1} - \theta').$$

Restricting to $\{x\} \times X$ we see that the divisor classes θ and θ' are equal. Hence θ is a theta characteristic. By assumption, $h^0(\theta) = h^0(R(x) - x) = 0$. □

Note that a nonsingular plane quartic curve X is a nonhyperelliptic curve of genus 3. This implies that, for any theta characteristic θ on X, $h^0(\theta)$ is even (i.e. θ is an even theta characteristic) and $h^0(\theta) = 0$ are equivalent properties. The number of such theta characteristics is equal to 36 (see Part I).

Corollary 9.4.10. *Let C be a general plane quartic curve and R_C be the correspondence on the Scorza curve $S = \mathrm{Sc}(C)$. Then there exists a unique even theta characteristic θ such that $R_C = R_\theta$.*

Recall from Part I that an even theta characteristic θ on a nonsingular plane quartic $X = V(F)$ defines a net $\mathcal{N}$ of quadrics in $|\theta + H|^* = \mathbb{P}^3$, where H is the divisor class of a line. It can be naturally identified with the plane $\mathbb{P}(E)$. The the discriminant variety of singular quadrics is equal to the curve C. The set of such nets up to the natural action of $\mathrm{GL}(4)$ is a cover of degree 36 over the space of nonsingular quartics. Let $\mathbb{P}(S^4E^*)^{\mathrm{ev}}$ denotes the normalization of $\mathbb{P}(S^4E^*)$ in the field of rational functions on this cover. We have a finite morphism

$$\mathbb{P}(S^4E^*)^{\mathrm{ev}} \to \mathbb{P}(S^4E^*)$$

of degree 36 which is not ramified over the open subset of nonsingular quartics. Its fibre over a nonsingular quartic X can be naturally identified with the set of even theta characteristics on X.

Example 9.4.2. (Bert van Geemen) Let C be a Clebsch quartic. Then $\mathrm{Sc}(C)$ is a Lüroth quartic, and by above it comes with a special even theta characteristic θ. We know that θ defines a representation of $\mathrm{Sc}(C)$ as the determinant of 3×3 symmetric matrix with linear forms as its entries. It can be done explicitly.

If $C = l^4 + l_1^4 + l_2^4 + l_3^4 + l_4^4$, then just take the 4×4 matrix M all of whose entries are l and add to this matrix the diagonal matrix with entries l_1, l_2, l_3, l_4. Its determinant is sum of all products of 4 of the 5 linear forms. It defines the Lüroth quartic.

We omit the proof of the following theorem of G. Scorza whose modern proof can be found in [DK]:

Theorem 9.4.11. *Let* $\mathrm{Sc} : \mathbb{P}(S^4E^*) - \to \mathbb{P}(S^4E^*)$ *be the Scorza map. Then the map $C \mapsto (\mathrm{Sc}(C), \theta)$, where $R_C = R_\theta$, is a birational map from $\mathbb{P}(S^4E^*)$ to $\mathbb{P}(S^4E^*)^{\mathrm{ev}}$. In particular, the degree of the Scorza map is equal to 36.*

Remark 9.4.1. The Scorza theorem generalizes to genus 3 the fact that the map from the space of plane cubics to itself defined by the Hessian is a birational map to the cover $\mathbb{P}(S^3E^*)^{\mathrm{ev}}$, formed by pairs (X, ϵ), where ϵ is a non-trivial 2-torsion point (an even characteristic in this case). Note that the Hessian covariant is defined similarly to the Clebsch invariant. We compose the polarization map $V \times S^3E^* \to S^2E^*$ with the discriminant invariant $S^2E^* \to \mathbb{C}$.

Under certain assumptions, which have not been yet verified, Scorza defines a map from the space of canonical curves of genus g in $\mathbb{P}^{g-1}$ to the space of quartic hypersurfaces in $\mathbb{P}^{g-1}$ (see [DK]).

9.4.3 Polar hexagons

A general quartic admits a polar 6-polyhedron (*polar hexagon*). It follows from Proposition 9.1.12 that the variety VSP$(F;6)$ is a smooth irreducible 3-fold.

Let us see how to construct polar hexagons of F explicitly. Let $Z = \{[l_1],\ldots,[l_6]\} \in$ VSP$(F;6)^o$ and

$$F = l_1^4 + \ldots + l_6^4.$$

We know that each pair $l_i, l_j, i \neq j$, is conjugate with respect to F, i.e.,

$$\Omega_{\check{F}}(l_i^2, l_j^2) = \check{F}(l_i^2 l_j^2) = 0.$$

Let Φ_i and $\Phi_j \in S^2E$ be the anti-polars of l_i and l_j with respect to F, i.e.

$$P_{\Phi_i}(F) = l_i^2, \quad P_{\Phi_j}(F) = l_j^2.$$

It follows from (9.2.2) that

$$\Phi_i(l_j) = \Phi_j(l_i) = 0.$$

Assume that Φ_i is irreducible. Then the map

$$S^2E \times S^2E \to S^4E, \quad (A,B) \mapsto A\Phi_i + B\Phi_j$$

has one-dimensional kernel spanned by $(B,-A)$. This easily implies that the dimension of the linear space L of quartic forms $A\Phi_i + B\Phi_j$ is equal to 9. Thus L coincides with $I_Z(4)$. Note that any form from L vanishes on l_i, l_j and common zeroes of A and B. This shows that $Z' = \{[l_i],[l_j]\} \cup V(A) \cap V(B)$ is a polar hexagon of F. By Proposition it must coincide with Z. This shows that the points $[l_k], k \neq i,j$ are reconstructed from the points $[l_i],[l_j]$. It also suggests the following construction of polar hexagons of F.

Start with any $[l] \in \mathbb{P}(E^*)$ such that its anti-polar Φ is irreducible and does not vanish at l. This is an open condition on $[l]$. Note that the latter condition means that $[l]$ does not belong to the quartic $V(\check{F}) \subset \mathbb{P}(E^*)$. Let $[l'] \in V(\Phi)$ and let Φ' be the anti-polar of l'. For any $A, A' \in S^2E$ with $A(l) = A'(l') = 0$, we have

$$P_{A\Phi+B\Phi'}(F) = P_A(P_\Phi(F)) + P_{A'}(P_{\Phi'}(F)) = P_A(l^2) + P_{A'}(l'^2) = 0.$$

This shows that the linear space L of quartic forms $A\Phi + B\Phi'$ is contained in AP$_4(F)$. As before we compute its dimension to find that it is equal to 9. Thus L coincide with $I_Z(4)$, where $Z = \{[l],[l']\} \cup (V(A) \cap V(A'))$. By Proposition 9.1.4, Z is a generalized polar hexagon of F (an ordinary one if $V(A)$ intersects $V(A')$ transversally). Note that this confirms the dimension of VSP$(F;6)$. We can choose $[l]$ in ∞^2 ways, and then choose $[l']$ in ∞^1 ways.

Remark 9.4.2. Consider the variety

$$\widetilde{\mathrm{VSP}}(F;6) = \{([l], Z) \in \mathbb{P}(E^*) \times \mathrm{VSP}(F;6) : \{[l]\} \subset Z\}.$$

The projection to the second factor is a degree 6 map. The general fibre over a point $[l]$ is isomorphic to the anti-polar conic $V(\Phi)$ of $[l]$.

9.4.4 A Fano model of VSP$(F;6)$

Recall that each $Z \in \mathrm{VSP}(F;6)$ defines a subspace $I_Z(3) \subset \mathrm{AP}_3(F)$. Its dual space $I_Z(3)^\perp \subset W = S^3E^*/\mathrm{ap}_F^1(E)$ is an isotropic subspace with respect to Mukai's 2-forms.

Lemma 9.4.12. *Let F be a non-degenerate quartic form and $Z \in \mathrm{VSP}(F;6)$. Then*

$$\dim I_Z(3) = 4.$$

Proof. Counting constants based on the exact sequence (9.3) shows that $\dim I_Z(3) \geq 10 - 6 = 4$. Assume $\dim I_Z(3) > 4$. Let Z_1 be a closed subscheme of Z of length 5. Again counting constant we get $I_{Z_1}(2) \neq \{0\}$. Let C be a conic from the linear system $|I_{Z_1}(2)|$. Obviously $C \notin I_Z(2)$ since otherwise $\mathrm{AP}_2(F) \neq \{0\}$ contradicting the nondegeneracy assumption on F. Choose a 0-dimensional scheme Z_0 of length 2 such that each irreducible component of C contains a subscheme of $Z' = Z_0 \cup Z_1$ of length ≥ 4. It is always possible. Now, counting constants, we see that $\dim I_{Z'}(3) > 4 - 2 = 2$. By Bézout's Theorem, all cubics K from $|I_{Z'}(3)|$ are of the form $C + \ell$, where ℓ is a line. Thus the residual lines form a linear system of lines of dimension ≥ 2, hence each line is realized as a component of some cubic K. However, $|I_{Z'}(3)| \subset |I_Z(3)$ and therefore all lines pass through the point in $Z \setminus Z_1$. This is obviously impossible. □

Applying the previous lemma we obtain a well-defined map

$$\mu : \mathrm{VSP}(F;6) \to G(3,W) \cong G(3,7), \quad Z \mapsto I_Z(3)^\perp.$$

By Proposition 9.4.3, this map is injective. Its image is contained in the locus of subspaces which are isotropic with respect to Mukai's forms.

Theorem 9.4.13. *(S. Mukai) Let $F \in S^4E^*$ be a general quartic form in 3 variables. Then the map*

$$\mu : \mathrm{VSP}(F;6) \to G(3,7)$$

is an isomorphism onto a smooth subvariety X equal to the locus of common zeroes of a 3-dimensional space of sections of the vector bundle $\Lambda^2\mathcal{S}$, where $\mathcal{S}$ is the tautological vector bundle over the Grassmannian. The canonical class of X is equal to $-H$, where H is a hyperplane section of X in the Plücker embedding.

Proof. We refer to the proof to the original paper of Mukai [Mukai], or to [Dolgachev], where some details of Mukai's proof are provided. □

Recall that a *Fano variety* of dimension n is a smooth projective variety X with ample $-K_X$. If $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $-K_X = mH$, where H is an ample generator of the Picard group, then X is said to be of *index* m. The *degree* of X is the self-intersection number H^n . The number $g = \frac{1}{2}H^n + 1$ is called the *genus*.

Remark 9.4.3. The variety X_2 was omitted in the original classification of Fano varieties with the Picard number 1 due to Gino Fano. It was discovered by Vasily Iskovskikh. It has the same Betti numbers as the $\mathbb{P}^3$. It was proven by Mukai that every such variety arises as a smooth projective model of $W(F_4; 6)$ for a unique quartic for F_4.

Remark 9.4.4. Another approach to Mukai's description of $\mathrm{VSP}(F; 6)$ for a general plane quartic $V(F)$ is due to K. Ranestad and F.-O. Schreyer (J. reine angew. Math. 525 (2000)). It allows them also to extend Theorem 9.4.13 to other 2 cases where $n = 2$ and $\mathrm{wrk}(F) = \binom{2+k}{k}$ ($k = 3, 4$).

$$\mathrm{VSP}(F; 10) \subset G(4, 9) \text{ is a K3 surface of degree } 38 \text{ in } \mathbb{P}^{20},\ k = 3$$

$$\mathrm{VSP}(F; 15)^o \subset G(5, 11) \text{ is a set of 16 points},\ k = 4.$$

Although these descriptions were certainly known to Mukai, he did not have a chance to provide the details of his proofs. The approach of Ranestad and Schreyer is based on a well-known result of D. Bucksbaum and D. Eisenbud that any Gorenstein Artinian quotient R/I of the polynomial algebra in 3 variables admits a resolution

$$0 \to R(-d-3) \to \sum_{i=1}^{s} R(-e_i) \to \sum_{i=1}^{s} R(-d_i) \to R \to R/I \to 0,$$

where the map ϕ is given by an alternating matrix whose entries a_{ij} are homogeneous polynomials of degree $e_i - d_j$ and d is the degree of the socle of R/I. It follows easily that $(s-1)(d+3) = 2(d_1 + \ldots + d_s)$. In our case when $R/I = A_F$, the ideal $I = \mathrm{AP}(F)$ is generated by forms of degree $k + 1$. the dimension of this space is $2k + 3$. This gives $s = 2k + 3$ and $d_1 = \ldots = d_s = k + 1$. From this we obtain $e_1 = \ldots = e_{2k+3} = k$. This implies that the map ϕ is given by an alternating $2k + 3 \times 2k + 3$ matrix whose entries are linear forms. The image of $\sum_{i=1}^{s} R(-k-1)_{k+1} = \mathbb{C}^{2k+3}$ in R is equal to $\mathrm{AP}_{k+1}(F)$ and can be identified with the dual of our space W. The map ϕ can be identified with a linear map $V \to \Lambda^2 W^*$. Its image corresponds to the subspace generated by our forms σ_{ij}. Then the authors show that any apolar scheme of lenghth N_k defines a subspace of W of dimension $k + 1$ in which the image of V in $\Lambda^2 W^*$ vanishes.

Remark 9.4.5. We refer to [Melliez-Ranestad] for the beautiful geometry of the variety VSP$(F, 6)$, where $V(F)$ is the Klein quartic.

Exercises

9.1 Let $F \in S^2E^*$. Show that the map $V \to V^*$ defined by $\Phi \mapsto P_\Phi(F)$ corresponds to the symmetric bilinear form $V \times V \to \mathbb{C}$ associated to Q.

9.2 Show that the embedded tangent space of the Veronese variety Ver_d^n at a point represented by the form l^d is equal to the projectivization of the linear space of homogeneous polynomials of degree d of the form $l^{d-1}m$.

9.3 Show using the following steps that Ver_3^4 is 6-defective by proving that for 7 general points p_i in $\mathbb{P}^4$ there is a cubic hypersurface with singular points at the p_i's.

(i) Show that there exists a Veronese curve R_4 of degree 4 through the seven points,

(ii) Show that the secant variety of R_4 is a cubic hypersurface which is singular along R_4.

9.4 Let Q be a nondgenerate quadratic form in $n+1$ variables. Show that VSP$(Q; n+1)^o$ embedded in $G(n, E)$ is contained in the linear subspace of codimension n.

9.5 Compute the catalecticant matrix $\mathrm{Cat}_2(F)$, where F is a homogeneous form of degree 4 in 3 variables.

9.6 Let $F \in S^{2k}E^*$ and Ω_F be the corresponding quadratic form on S^kE. Show that the quadric $V(\Omega_F)$ in $\mathbb{P}(S^kE)$ is characterisezed by the following two properties:

- Its pre-image under the Veronese map $\nu_k : \mathbb{P}(E) \to \mathbb{P}(S^kE)$ is equal to $V(F)$;
- Ω_F is apolar to any quadric in $\mathbb{P}(S^kE^*)$ which contains the image of the Veronese map $\mathbb{P}(E^*) \to \mathbb{P}(S^kE^*)$.

9.7 Let C_k be the locus in $\mathbb{P}(S^{2k}E)$ of hypersurfaces $V(F)$ such that $\det \mathrm{Cat}_k(F) = 0$. Show that C_k is a rational variety. [Hint: Consider the rational map $C_k - \to \mathbb{P}(S^kE)$ which assigns to $V(F)$ the point defined by the subspace $\mathrm{AP}_k(F)$ and study its fibres].

9.8 Give an example of a polar 4-gon of the cubic $T_0T_1T_2$.

9.9 Find all binary forms of degree d for which VSP$(F; 2)^o = \emptyset$.

9.10 Let F be a form of degree d in $n+1$ variables. Show that the variety $\mathrm{VSP}(F;\binom{n+d}{d})^o$ is an irreducible variety of dimension $n\binom{n+d}{d}$.

9.11 Describe the variety $\mathrm{VSP}(F;4)$, where F is a nondegenerate quadric in 3 variables.

9.12 Find all ternary cubics F such that $\mathrm{VSP}(F;4)^o=\emptyset$.

9.13 Show that a plane cubic curve belongs to the closure of the Fermat locus if and only if it admits a first polar equal to a double line or the whole space.

9.14 A plane quartic $C=V(F)$ is called a *Capolari quartic* if $\mathrm{VSP}(F;4)^o\neq\emptyset$.

(i) What is the dimension of the locus of the Capolary quartics?

(ii) Show that the Scorza quartic $S=\mathrm{Sc}(C)$ is reducible.

(iii) Find the intersection of the loci of Fermat quartics and Capolari quartics.

9.15 Let Q be a nondegenerate quadratic form in 3 variables. Show that $W(Q^2;6)^o$ is a homogeneous space for the group $\mathrm{PSL}(2,\mathbb{C})$.

9.16 Let C be a hyperelliptic curve of genus g. Show that the graph of the hyperelliptic involution has valency 2.

9.17 Let $F=T_0^3T_1+T_1^3T_2+T_2^3T_0$. Show that $V(\mathrm{Sc}(F))=V(F)$.

9.18 Prove the assertion from Example 9.4.2.

9.19 Show that the binary form $F=T_0(T_0+2T_1)^2$ does not admit nondegenerate polar 2th polyhedra.

9.20 Show that the locus of lines $\ell=V(l)$ such that the anti-polar of l^2 with respect to a quartic curve $V(F)$ is a plane curve of degree 6 in the dual plane.

9.21 Show that the Scorza quartic of the Fermat quartic C is equal to C.

Chapter 10

Cubic surfaces

10.1 The E_6-lattice

10.1.1 Lattices

A *lattice* is a free abelian group $M \cong \mathbb{Z}^r$ equipped with a symmetric bilinear form $M \times M \to \mathbb{Z}$. A relevant example of a lattice is the second cohomology group modulo torsion of a compact 4-manifold (e.g. a nonsingular projective surface) with respect to the cup-product. Another relevant example is the Picard group modulo numerical equivalence of a nonsingular projective surface equipped with the intersection pairing.

The values of the symmetric bilinear form will be often denoted by (x, y) or $x \cdot y$. We write $x^2 = (x, x)$. The map $x \mapsto x^2$ is an integral valued quadratic form on M. Conversely, such a quadratic form $q : M \to \mathbb{Z}$ defines a symmetric bilinear form by the formula $(x, y) = q(x + y) - q(x) - q(y)$. Note that $x^2 = 2q(x)$.

Let $M^* = \text{Hom}_Z(M, \mathbb{Z})$ and

$$\iota_M : M \to M^*, \quad \iota_M(x)(y) = (x, y)$$

defined by the bilinear form. We say that M is *non-degenerate* if ι_M is injective. In this case the group

$$\text{Disc}(M) = M^*/\iota_M(M)$$

is a finite abelian group. It is called the *discriminant group* of M. If we choose a basis to represent the symmetric bilinear form by a matrix A, then the order of Disc(M) is equal to $|\det(A)|$. The number disc$(M) = \det(A)$ is called the *discriminant* of M. A different choice of a basis changes A to ${}^tCAC$ for some $C \in \text{GL}(n, \mathbb{Z})$, so it does not change $\det(A)$. A lattice is called *unimodular* if $|\text{disc}(M)| = 1$

Tensoring M with reals, we get a real symmetric bilinear form on $M_{\mathbb{R}} \cong \mathbb{R}^r$. Its Sylvester signature (t_+, t_-, t_0) is called the *signature* of M. We write (t_+, t_-) if $t_0 = 0$. For example, the signature of $H^2(X, \mathbb{Z})/\text{Tors} \cong \mathbb{Z}^{b_2}$ for a nonsingular projective surface X is equal to $(2p_g + 1, b_2 - 2p_g - 1)$, where $p_g = \dim H^0(X, \mathcal{O}_X(K_X))$. The signature on $\text{Num}(X) = \text{Pic}(X)/\text{num} \cong \mathbb{Z}^\rho$ is equal to $(1, \rho - 1)$ (this is called the Hodge Index Theorem, see [Hartshorne], Chap. V, Thm. 1.9).

Let $N \subset M$ be a subgroup of M. The restriction of the bilinear form to N defines a structure of a lattice on N. We say that N together with this form is a *sublattice* of M. We say that N is of *finite index* m if M/N is a finite group of order m. Let

$$N^\perp = \{x \in M : (x, y) = 0, \forall y \in N\}.$$

Note that $N \subset (N^\perp)^\perp$ and the equality takes place if and only if N is a *saturated sublattice* (i.e. M/N is torsion-free).

We will need the following lemmas.

Lemma 10.1.1. *Let M be a nondegenerate lattice and N be its nondegenerate sublattice of finite index m. Then*

$$|\text{disc}(N)| = m^2 |\text{disc}(M)|.$$

Proof. Since N is of finite index in M the restriction homomorphism $M^* \to N^*$ is injective. We will identify M^* with its image in N^*. We will also identify M with its image $\iota_M(M)$ in M^*. Consider the chain of subgroups

$$N \subset M \subset M^* \subset N^*.$$

Choose a basis in M, a basis in N, and the dual bases in M^* and N^*. The inclusion homomorphism $N \to M$ is given by a matrix A and the inclusion $N^* \to M^*$ is given by its transpose ${}^t A$. The order m of the quotient M/N is equal to $|\det(A)|$. The order of N^*/M^* is equal to $|\det({}^t A)|$. They are equal. Now the chain from above has the first and the last quotient of order equal to m and the middle quotient is of order $|\text{disc}(M)|$. The total quotient N^*/N is of order $|\text{disc}(N)|$. The assertion follows. □

Lemma 10.1.2. *Let M be a unimodular lattice and N be its nondegenerate saturated sublattice. Then*

$$|\text{disc}(N^\perp)| = |\text{disc}(N)|.$$

Proof. Consider the restriction homomorphism $r : M \to N^*$, where we identify M with M^* by means of ι_M. Its kernel is equal to $N^\perp$. Composing r with the projection $N^*/\iota(N)$ we obtain an injective homomorphism

$$M/(N + N^\perp) \to N^*/N.$$

Notice that $N^\perp \cap N = \{0\}$ because N is a nondegenerate sublattice. Thus $N^\perp + N = N^\perp \oplus N$ is of finite index i in M. Also the sum is orthogonal, so that the matrix representing the symmetric bilinear form on $N \oplus N^\perp$ can be chosen to be a block matrix. This shows that $\mathrm{disc}(N \oplus N^\perp) = \mathrm{disc}(N)\mathrm{disc}(N^\perp)$. Applying Lemma 10.1.1, we get

$$\#(M/N + N^\perp) = \sqrt{|\mathrm{disc}(N^\perp)||\mathrm{disc}(N)|} \leq \#(N^*/N) = |\mathrm{disc}(N)|.$$

This gives $|\mathrm{disc}(N^\perp)| \leq |\mathrm{disc}(N)|$. Since $N = (N^\perp)^\perp$, exchanging the roles of N and $N^\perp$, we get the opposite inequality. □

Lemma 10.1.3. *Let N be a nondegenerate sublattice of a unimodular lattice M. Then*

$$\iota_M(N^\perp) = \mathrm{Ann}(N) = \mathrm{Ker}(r : M^* \to N^*) \cong (M/N)^*.$$

Proof. Under the isomorphism $\iota_M : M \to M^*$ the image of $N^\perp$ is equal to $\mathrm{Ann}(N)$. Since the functor $\mathrm{Hom}_{\mathbb{Z}}(?, \mathbb{Z})$ is left exact, applying it to the exact sequence

$$0 \to N \to M \to M/N \to 0,$$

we obtain an isomorphism $\mathrm{Ann}(N) \cong (M/N)^*$. □

10.1.2 The E_6-lattice

Let $\mathbb{Z}^{1,6} = \mathbb{Z}^7$ equipped with the symmetric bilinear form defined by the diagonal matrix $\mathrm{diag}(1, -1, -1, -1, -1, -1, -1)$ with respect to the standard unit basis

$$\mathbf{e}_0 = (1, 0, \ldots, 0),\ \mathbf{e}_1 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_6 = (0, \ldots, 0, 1)$$

of $\mathbb{Z}^7$. Any basis defining the same matrix will be called an *orthonormal basis*. The lattice $\mathbb{Z}^{1,6}$ is a unimodular lattice of signature $(1, 6)$.

Consider the special vector in $\mathbb{Z}^{1,6}$ defined by

$$\kappa_6 = (-3, 1, 1, 1, 1, 1, 1) = -3\mathbf{e}_0 + \sum_{i=1}^{6} \mathbf{e}_i. \tag{10.1}$$

We define the *E_6-lattice* as a sublattice of $\mathbb{Z}^{1,6}$ given by

$$\mathbf{E}_6 = (\mathbb{Z}\kappa_6)^\perp.$$

Since $\kappa_6^2 = 3$, it follows from Lemma 10.1.2, that $\mathbf{E}_6$ is a negative definite lattice of discriminant 3.

Lemma 10.1.4. *The following vectors form a basis of* $\mathbf{E}_6$

$$\alpha_0 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3,\ \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1},\ i = 1, \ldots, 5.$$

The matrix of the symmetric bilinear form of $\mathbf{E}_6$ *with respect to this basis is equal to the following matrix (called a* Cartan matrix *of type* E_6*):*

$$A = \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

It can be expressed by the Dynkin diagram of type E_6 *(see Part I).*

Proof. By inspection, each α_i is orthogonal to κ_6. Suppose we have a vector $(a_0, a_1, \ldots, a_6)$ orthogonal to κ_6. Then

$$3a_0 + a_1 + \ldots + a_6 = 0. \tag{10.2}$$

We can write this vector as follows

$$(a_0, a_1, \ldots, a_6) = a_0\alpha_0 + (a_0 + a_1)\alpha_1 + (2a_0 + a_1 + a_2)\alpha_2 + (3a_0 + a_1 + a_2 + a_3)\alpha_3$$
$$+ (3a_0 + a_1 + a_2 + a_3 + a_4)\alpha_4 + (3a_0 + a_1 + a_2 + a_3 + a_5)\alpha_5$$

We use here that (10.2) implies that the last coefficient is equal to $-a_6$. We leave the computation of the matrix to the reader. □

Remark 10.1.1. One can similarly define lattices corresponding to Cartan matrix of other types $A_n, D_n (n \geq 4), E_7, E_8$ (see Part I). We denote these lattices by $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_7, \mathbf{E}_8$. they are negative definite lattices of discriminant $(-1)^n(n+1), 4, 2, 1$, respectively.

10.1.3 Roots

A vector $\alpha \in \mathbf{E}_6$ is called a *root* if $\alpha^2 = -2$.

Lemma 10.1.5. *The following is the list of roots:*

$$\begin{aligned} \pm\alpha_{\max} &= \pm(2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_6), \\ \pm\alpha_{ijk} &= \pm(\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k),\ 0 < i < j < k \leq 6, \\ \pm\alpha_{ij} &= \pm(\mathbf{e}_1 - \mathbf{e}_j),\ 0 < i < j \leq 6. \end{aligned}$$

The number of roots is equal to 72.

Proof. Let $\alpha = (a_0, a_1, \ldots, a_6)$. We have equality (10.2) and the equality

$$a_0^2 - a_1^2 - \ldots - a_6^2 = -2. \tag{10.3}$$

Applying the Cauchy-Schwartz inequality for the standard inner product in $\mathbb{R}^6$ we obtain

$$(a_1 + \ldots + a_6)^2 \leq 6(a_1^2 + \ldots + a_6^2). \tag{10.4}$$

This gives

$$9a_0^2 = (a_1 + \ldots + a_6)^2 \leq 6a_0^2 + 12,$$

and hence $|a_0| \leq 2$. Obviously α and $-\alpha$ are both roots or not roots. So, we may assume that $a_0 \geq 0$.

If $a_0 = 2$, we have the equality in (10.4) and hence $(a_1, \ldots, a_6) = (1, 1, 1, 1, 1, 1)$. If $a_0 = 1$, we have $a_1^2 + \ldots + a_6^2 = 3, a_1 + \ldots + a_6 = -3$. This gives 20 vectors of type α_{ijk}. Finally, if $a_0 = 0$, we have $a_1^2 + \ldots + a_6^2 = 2, a_1 + \ldots + a_6 = 0$. This gives 15 vectors of type α_{ij}. □

Let α be a root. Consider the transformation

$$r_\alpha : \mathbb{Z}^{1,6} \to \mathbb{Z}^{1,6},\ x \mapsto x + (x, \alpha)\alpha.$$

It is called a *reflection* with respect to α. Since

$$(r_\alpha(x), r_\beta(y)) = (x + (x, \alpha)\alpha, y + (y, \alpha)\alpha)$$

$$= (x, y) + 2(x, \alpha)(\alpha, y) + (x, \alpha)(\alpha, y)\alpha^2 = (x, y),$$

r_α preserves the symmetric bilinear form on $\mathbb{Z}^{1,6}$. It also preserves the vector κ_6 since $(\alpha, \kappa_6) = 0$.

It is immediately checked that r_α^2 is the identity. It leaves invariant the orthogonal complement of α and sends α to $-\alpha$.

Definition 10.1. *The subgroup $W(\mathbf{E}_6)$ of* $\mathrm{GL}(\mathbb{Z}^{1,6})$ *generated by the transformations r_α is called the* Weyl group *of the lattice $\mathbf{E}_6$.*

As we remarked already in Chapter 7 of Part I, the Weyl group $W(\mathbf{E}_6)$ is of index 2 in the orthogonal group of $\mathbf{E}_6$. The unique non-trivial coset is the coset of the transformation $x \mapsto -x$.

10.1.4 Exceptional vectors

Let $\mathbf{E}_6^* = \mathrm{Hom}(\mathbf{E}_6, \mathbb{Z})$ be the dual abelian group. By Lemma 10.1.3, we have

$$\mathbf{E}_6 = (\mathbb{Z}^{1,6}/\mathbb{Z}\kappa_6)^*, \ \mathbf{E}_6^* = \mathbb{Z}^{1,6}/\mathbb{Z}\kappa_6.$$

Let $\mathbb{R}^{1,6} = (\mathbb{Z}^{1,6})_{\mathbb{R}}$. Consider the affine space

$$\mathbb{R}_1^{1,6} = \{x \in \mathbb{R}^{1,6} : (x, \kappa_6) = -1\}.$$

Since $\kappa_6^2 \neq 0$, each coset in $\mathbb{Z}^{1,6}/\mathbb{Z}\kappa_6$ can be uniquely represented by a vector in $\mathbb{R}_1^{1,6}$ with rational coordinates. We will identify $\mathbf{E}_6^*$ with a subset of $\mathbb{R}_1^{1,6}$.

A vector v in $\mathbb{R}_1^{1,6}$ with integer coordinates satisfying $v^2 = -1$ will be called a *exceptional vector*.

Lemma 10.1.6. *The following is the list of exceptional vectors:*

$$\mathbf{a}_i = \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{10.5}$$

$$\mathbf{b}_i = 2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_6 + \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{10.6}$$

$$\mathbf{c}_{ij} = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, \quad 1 \leq i < j \leq 6. \tag{10.7}$$

The number of exceptional vectors is equal to 27.

Proof. Let $v = (a_0, -a_1, \ldots, -a_6)$ be a exceptional vector. We have two equalities

$$\begin{aligned} 1 &= 3a_0 + a_1 + \ldots + a_6 \\ -1 &= a_0^2 - a_1^2 - \ldots - a_6^2 \end{aligned}$$

Applying the Cauchy-Schwartz inequality, we get as in the proof of Lemma 10.3

$$(3a_0 - 1)^2 \leq 6(a_0^2 + 1).$$

This gives $3a_0^2 - 6a_0 - 5 \leq 0$, hence $\frac{3-\sqrt{24}}{3} \leq a_0 \leq \frac{3+\sqrt{24}}{3}$. Since $a_0 \in \mathbb{Z}$, this gives $0 \leq a_0 \leq 2$. If $a_0 = 2$, we get $a_1 + \ldots + a_6 = -5, a_1^2 + \ldots + a_6^2 = 5$. This gives 6 vectors $\mathbf{b}_i$. If $a_0 = 1$, we get $a_1 + \ldots + a_6 = -2, a_1^2 + \ldots + a_6^2 = 2$. This gives 15 vectors $\mathbf{c}_{ij}$ of the second type. Finally, if $a_0 = 0$ we get 6 vectors $\mathbf{a}_i$. □

Proposition 10.1.7. *For any two different exceptional vectors* v, w

$$0 \leq (v, w) \leq 1.$$

Proof. This can be seen directly from the list however we prefer to give a proof independent of the classification. Since $(v, \kappa_6) = (w, \kappa_6)$, we have $v - w \in \mathbf{E}_6$. Since $\mathbf{E}_6$ is a negative definite lattice we have $(v-w)^2 = -2 - 2(v,w) < 0$. This gives $(v,w) \geq 0$. Assume $(v,w) > 1$. Let $h = -\kappa_6 - v - w$. We have $(v+w)^2 = -2 + 2(v,w) > 1$ and $h^2 = 3 - 4 + (v+w)^2 > 0, (h, -\kappa_6, h) = 1$. This implies that the matrix

$$\begin{pmatrix} h^2 & (h, -\kappa_6) \\ (h, -\kappa_6) & (-\kappa_6)^2 \end{pmatrix}$$

is positive definite. Since the signature of $\mathbb{Z}^{1,6}$ is $(1,6)$ this is a contradiction. □

10.1.5 Sixers

A *sixer* is a set of 6 mutually orthogonal exceptional vectors. An example of a sixer is the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_6\}$.

Lemma 10.1.8. *Let $\{v_1, \ldots, v_6\}$ be a sixer. Then there exists a unique root α such that*

$$(v_i, \alpha) = 1, \quad i = 1, \ldots, 6.$$

Moreover, $(w_1, \ldots, w_6) = (s_\alpha(v_1), \ldots, s_\alpha(v_6))$ is a sixer satisfying

$$(v_i, w_j) = 1 - \delta_{ij}.$$

The root associated to $(w_1, \ldots, w_6)$ is equal to $-\alpha$.

Proof. The uniqueness is obvious since $v_1, \ldots, v_6$ are linear independent, so no vector is orthogonal to all of them. Let

$$v_0 = \frac{1}{3}(-\kappa_6 + v_1 + \ldots + v_6) \in \mathbb{R}^{1,6}.$$

First we show that $v_0 \in \mathbb{Z}^{1,6}$. Since $M^* = M$ it is enough to show that, for any $x \in \mathbb{Z}^{1,6}$, $(v_0, x) \in \mathbb{Z}$. Consider the sublattice N of $\mathbb{Z}^{1,6}$ spanned by $v_1, \ldots, v_6, \kappa_6$. We have $(v_0, v_i) = 0, i > 0$, and $(v_0, \kappa_6) = -3$. Thus $(v_0, M) \subset 3\mathbb{Z}$. By computing the discriminant of N, we find that it is equal to 9. By Lemma 10.1.1 N is a sublattice of index 3 in $\mathbb{Z}^{1,6}$. Hence for any $x \in \mathbb{Z}^{1,6}$ we have $3x \in N$. This shows that

$$(v_0, x) = \frac{1}{3}(v_0, 3x) \in \mathbb{Z}.$$

Now let us set

$$\alpha = 2v_0 - v_1 - \ldots - v_6.$$

We check that α is a root, and $(\alpha, v_i) = 1, i = 1, \ldots, 6$.

It remains to check the second assertion. Since r_α preserves the symmetric bilinear form, $\{w_1, \ldots, w_6\}$ is a sixer. We have

$$(v_i, w_j) = \big(v_i, r_\alpha(v_j)\big) = \big(v_i, v_j + (v_j, \alpha)\alpha\big) = (v_i, v_j) + (v_i, \alpha)(v_j, \alpha) =$$

$$(v_i, v_j) + 1 = 1 - \delta_{ij}.$$

Finally we check that

$$(s_\alpha(v_i), -\alpha) = \big(s_\alpha^2(v_i), -s_\alpha(\alpha)\big) = -(v_i, \alpha) = 1.$$

□

The two sixers with opposite associated roots form a *double-six* of exceptional vectors.

Theorem 10.1.9. *The following is a list of 36 double-sixes with corresponding associated roots:*

1 of type D

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & & \alpha_{\max} \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & & -\alpha_{\max} \end{matrix}$$

15 of type D_{ij}

$$\begin{matrix} a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} & & \alpha_{ij} \\ a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} & & -\alpha_{ij} \end{matrix},$$

20 of type D_{ijk}

$$\begin{matrix} a_i & a_j & a_k & c_{lm} & c_{mn} & c_{ln} & & \alpha_{ijk} \\ c_{jk} & c_{ik} & c_{ij} & b_n & b_l & b_m & & -\alpha_{ijk} \end{matrix}.$$

Proof. We have constructed a map from the set of sixers (resp. double-sixes) to the set of roots (resp. pairs of opposite roots). Let us show that no two sixers $\{v_1, \ldots, v_6\}, \{w_1, \ldots, w_6\}$ can define the same root. Since $w_1, \ldots, w_6, \kappa_6$ span a sublattice of finite index in $\mathbb{Z}^{1,6}$, we can write

$$v_i = \sum_{j=1}^{6} a_j w_j + a_0 \kappa_6 \tag{10.8}$$

with some $a_j \in \mathbb{Q}$. Assume that $v_i \neq w_j$ for all j. Intersecting both sides with α, we get

$$1 = a_0 + \ldots + a_6. \tag{10.9}$$

Intersecting both sides with $-\kappa_6$, we get $1 = a_1 + \ldots + a_6 - 3a_0$, hence $a_0 = 0$. Intersecting both sides with w_j we obtain $-a_j = (v_i, w_j)$. Applying Proposition 10.1.7, we get $a_j \leq -1$. This contradicts (10.9). Thus each v_i is equal to some w_j. □

Theorem 10.1.10. *The group $W(\mathbf{E}_6)$ acts transitively on the sets of roots, exceptional vectors, sixes, double-sixes. The stabilizer subgroup is of respective order* $6!, 2^4 5!, 6!, 2.6!$. *In particular,*

$$\#W(\mathbf{E}_6) = 72.6! = 51840.$$

Proof. Observe that the subgroup of $W(\mathbf{E}_6)$ generated by the reflections with respect to the roots α_{ij} is isomorphic to the permutation group S_6. It acts on the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_6\}$ by permuting its elements and leaves $\mathbf{e}_0$ invariant. This implies that S_6 acts on the roots $\alpha_{ij}, \alpha_{ijk}$ via its action on the set of subsets of $\{1, \ldots, 6\}$. Thus it acts transitively on the set of roots α_{ij} and on the set of roots α_{ijk}. Also applying s_α to α we get $-\alpha$. To show that $W(\mathbf{E}_6)$ acts transitively on the set of roots it suffices to show that $\alpha_{\max}$ can be transformed to one of the roots α_{ij} and one of the roots α_{ijk}. We have

$$\begin{aligned} s_{\alpha_{123}}(\alpha_{\max}) &= \alpha_{\max} + (\alpha_{\max}, \alpha_{123})\alpha_{123} = \alpha_{\max} - \alpha_{123} = \alpha_{456}, \\ s_{\alpha_{345}} \circ s_{\alpha_{123}}(\alpha_{\max}) &= s_{\alpha_{345}}(\alpha_{456}) = \alpha_{36}. \end{aligned}$$

Since sixers correspond to roots, $W(\mathbf{E}_6)$ acts transitively on the set of sixers. Since $w(-\alpha) = -w(\alpha)$ for any $w \in W(\mathbf{E}_6)$, we get also a transitive action on double-sixes.

We use similar argument to prove that $W(\mathbf{E}_6)$ acts transitively on the set of exceptional vectors. It acts transitively on the lines $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_{ij}$'s by permuting the indices. We have

$$s_{\alpha_0}(\mathbf{a}_i) = \mathbf{b}_i, \ s_{\alpha_{123}}(\mathbf{a}_1) = \mathbf{c}_{23}.$$

It is easy to compute the stabilizers. The stabilizer subgroup of the sixer $(\mathbf{a}_1, \ldots, \mathbf{a}_6)$ (and hence of a root) is the group S_6. The stabilizer of the double-six D is the subgroup $\langle S_6, s_{\alpha_0} \rangle$ of order $2.6!$. Now we can compute the order of the Weyl group and hence to compute the order of the stabilizer subgroup of any exceptional vector. □

Definition 10.2. *A* canonical root basis *in $\mathbf{E}_6$ is a basis $(\beta_1, \ldots, \beta_6)$ formed by roots such that the matrix $\big((\beta_i, \beta_j)\big)$ is equal to the Cartan matrix from Lemma 10.1.4.*

Theorem 10.1.11. *Any canonical root basis is obtained from a unique ordered sixer* $(v_1, \ldots, v_6)$ *by the formula*

$$\beta_0 = v_0 - v_1 - v_2 - v_3,\ \beta_i = v_i - v_{i+1},\ i = 1, \ldots, 5. \tag{10.10}$$

where $3v_0 = -\kappa_6 + v_1 + \ldots + v_6$. *The root* α *corresponding to the sixer is equal to*

$$\beta_{\max} = 2\beta_0 + \beta_1 + 2\beta_2 + 3\beta_3 + 2\beta_4 + \beta_5.$$

Proof. It is checked immediately that $(\beta_0, \ldots, \beta_5)$ is a canonical root basis. Conversely, given a canonical root basis $(\beta_0, \ldots, \beta_5)$ we solve for v_i in the system of equations (10.10). We have

$$v_5 = v_6 + \beta_5, v_4 = v_6 + \beta_5 + \beta_4, \ldots, v_1 = v_6 + \beta_5 + \ldots + \beta_1,$$

$$v_0 = \beta_0 + v_1 + v_2 + v_3 = 3v_6 + 3(\beta_5 + \beta_4 + \beta_3) + 2\beta_2 + \beta_1 + \beta_0,$$

$$-\kappa_6 = 3v_0 + v_1 + \ldots + v_6 = 9v_6 + 9(\beta_5 + \beta_4 + \beta_3) + 6\beta_2 + 3\beta_1 + 3\beta_0$$

$$-(6v_6+5\beta_5+4\beta_4+3\beta_3+2\beta_2+\beta_1+3\beta_0) = 3v_6+4\beta_5+5\beta_4+6\beta_3+4\beta_2+2\beta_1+3\beta_0.$$

This gives

$$v_6 = -\frac{1}{3}(\kappa_6 + 4\beta_5 + 5\beta_4 + 6\beta_3 + 4\beta_2 + 2\beta_1 + 3\beta_0).$$

Intersecting both sides with β_i we find $(v_6, \beta_i) = 0, i = 1, \ldots, 5, (v_6, \beta_5) = 1$. Thus $v_6 \in \mathbf{E}_6^*$ and hence all $v_i \in \mathbf{E}_6^*$. Now it is immediately checked that $(v_1, \ldots, v_6)$ is a sixer. The last assertion is also immediate. □

Let $\underline{\beta} = (\beta_1, \ldots, \beta_6)$ be a canonical root basis. The subset

$$\mathcal{C}_{\underline{\beta}} = \{x \in \mathbb{R}^{1,6} : (x, \beta_i) \geq 0, i = 1, \ldots, 6\} \tag{10.11}$$

is called a *Weyl chamber* with respect to $\underline{\beta}$. A subset of a Weyl chamber which consists of vectors such that $(v, \beta_i) = 0$ for some subset $I \subset \{1, \ldots, 6\}$ is called a *face*. A face corresponding to the empty set is equal to the interior of the Weyl chamber. The face corresponding to the subset $\{1, \ldots, 6\}$ is spanned by the vector κ_6.

Corollary 10.1.12. *The Weyl group acts simply transitively on canonical root bases and Weyl chambers.*

Proof. The Weyl group acts transitively on the set of sixers with stabilizer subgroup equal to the permutation of the sixer. Thus it acts simply transitively on the set of ordered sixers, hence on the set of canonical root bases and Weyl chambers. □

It follows from Lemma 10.3 that each root can be written either as a non-negative or as a non-positive linear combination of some canonical root basis. Since the Weyl group acts transitively on the set of canonical root bases, we see that each canonical basis defines the partition of the set of roots

$$\mathcal{R} = \mathcal{R}_+ \coprod \mathcal{R}_-$$

as a sum of *positive roots* (nonegative combinations) and *negative roots* (nonpositive combinations). It is clear that $\mathcal{R}_- = \{-\alpha, \alpha \in \mathcal{R}_+\}$.

The proof of the following result is more involved and we refer for it to any text-book on Lie algebras.

Theorem 10.1.13. *The union of Weyl chambers is equal to $\mathbb{R}^{1,6}$. Two Weyl chambers may intersect only along a common face. Each orbit of $W(\mathbf{E}_6)$ is represented by a unique vector x in a fixed Weyl chamber. Its stabilizer is equal to the subgroup generated by reflection $s_{\beta_i}, i \in I$, where x belongs to a face defined by I but not to a face defined by a subset $J \supsetneq I$.*

By taking the orbit of κ_6, we obtain that $W(\mathbf{E}_6)$ is generated by the reflections $s_{\beta_0}, \ldots, s_{\beta_5}$, where $(\beta_0, \ldots, \beta_5)$ is any canonical root basis.

10.1.6 Steiner triads of double-sixes

It is easy to see that two different double-sixes can share either 4 or 6 exceptional vectors. More precisely, we have

$$\#D \cap D_{ij} = 4, \ \#D \cap D_{ijk} = 6,$$

$$\#D_{ij} \cap D_{kl} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 0 \\ 6 & \text{otherwise} \end{cases}$$

$$\#D_{ij} \cap D_{klm} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l,m\} = 0,2 \\ 6 & \text{otherwise} \end{cases}$$

$$\#D_{ijk} \cap D_{lmn} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 1 \\ 6 & \text{otherwise} \end{cases}$$

A pair of double-sixes is called a *syzygetic duad* (resp. *azygetic duad*) if they have 4 (resp. 6) exceptional vectors in common.

The next lemma is an easy computation.

Lemma 10.1.14. *Two double-sixes with associated roots α, β form a syzygetic duad if and only if $(\alpha, \beta) \in 2\mathbb{Z}$.*

This can be interpreted as follows. Consider the vector space

$$\bar{Q} = Q/2Q \cong \mathbb{F}_2^6 \tag{10.12}$$

equipped with the quadratic form

$$q(x + 2Q) = \tfrac{1}{2}(x, x) \mod 2.$$

Notice that the lattice $\mathbf{E}_6$ is an *even lattice*, i.e. its quadratic form $x \mapsto x^2$ takes only even values. So the definition makes sense. The associated symmetric bilinear form is the symplectic form

$$(x + 2Q, y + 2Q) = (x, y) \mod 2.$$

Each pair of opposite roots $\pm\alpha$ defines a vector v in $\bar{Q}$ with $q(v) = 1$. It is easy to see that the quadric q has Arf invariant (see Chapter 5, Part I) equal to 1 and hence vanishes on 28 vectors. The remaining 36 vectors correspond to 36 pairs of opposite roots or, equivalently, double-sixes.

Note that we have a natural homomorphism of groups

$$W(\mathbf{E}_6) \cong O(6, \mathbb{F}_2)^- \tag{10.13}$$

obtaned from the action of $W(\mathbf{E}_6)$ on $Q/2Q$. It is an isomorphism. This is checked by verifying that the automorphism $v \to -v$ of the lattice Q does not belong to the Weyl group W and then comparing the known orders of the groups.

It follows from above that an syzygetic pair of double-sixes corresponds to orthogonal vectors v, w. Since $q(v+w) = q(v)+q(w)+(v, w) = 0 = 1+1+0 = 0$, we see that each nonzero vector in the isotropic plane spanned by v, w comes from a double-six.

A triple of pairwise syzygetic double-sixes is called a *syzygetic triad* of double-sixes. They span an isotropic plane. Similarly we see that a pair of azygetic double-sixes span a non-isotropic plane in $\bar{Q}$ with three nonzero vectors corresponding to a triple of double-sixes which are pairwise azygetic. It is called an *azygetic triad* of double-sixes.

We say that three azygetic triads form a *Steiner compex of triads of double sixes* if the corresponding planes in $\bar{Q}$ are mutually orthogonal. It is easy to see that an azygetic triad contains 18 exceptional vectors and thus defines a set of 9 exceptional (the omitted ones). The 27 exceptional vectors omitted from three triads in a Steiner complex is equal to the set of 27 exceptional vectors $\mathbf{E}_6^*$. There are 40 Steiner complexes of triads:

10 of type

$$\Gamma_{ijk,lmn} = (D, D_{ijk}, D_{lmn}),\ (D_{ij}, D_{ik}, D_{jk}),\ (D_{lm}, D_{ln}, D_{mn}),$$

30 of type

$$\Gamma_{ij,kl,mn} = (D_{ij}, D_{ikl}, D_{jkl}),\ (D_{kl}, D_{kmn}, D_{lmn}),\ (D_{mn}, D_{mij}, D_{nij}).$$

Theorem 10.1.15. *The Weyl group $W(\mathbf{E}_6)$ acts transitively on the set of triads of azygetic double-sixes with stabilizer subgroup isomorphic to the group $S_3 \times (S_3 \wr S_2)$ of order 432. It also acts transitively on Steiner complexes of triads of double-sixes. A stabilizer subgroup is isomorphic to a maximal subgroup of $W(\mathbf{E}_6)$ of order 1296 isomorphic to the wreath product $S_3 \wr S_3$.*

Proof. We know that a triad of azygetic double-sixes corresponds to a pair of roots (up to replacing the root with its negative) α, β with $(\alpha, \beta) = \pm 1$. This pair spans a root sublattice Q of $\mathbf{E}_6$ of type A_2. Fix a root basis. Since the Weyl group acts transitively on the set of roots, we find $w \in W$ such that $w(\alpha) = \alpha_{\max}$. Since $(w(\beta), \alpha_{\max}) = (\beta, \alpha) = 1$, we see that $w(\beta) = \pm\alpha_{ijk}$ for some i, j, k. Applying elements from S_6, we may assume that $w(\beta) = -\alpha_{123}$. Obviously, the roots $\alpha_{12}, \alpha_{23}, \alpha_{45}, \alpha_{56}$ are orthogonal to $w(\alpha)$ and $w(\beta)$. These roots span a root sublattice isomorphic to $\mathbf{A}_2 \perp \mathbf{A}_2$. Thus we obtain that the orthogonal complement of Q in $\mathbf{E}_6$ contains a sublattice isomorphic to $\mathbf{A}_2 \perp \mathbf{A}_2$. Since $|\text{disc}(\mathbf{A}_2)| = 3$, it follows easily from Lemma 10.1.1 that $Q^\perp \cong \mathbf{A}_2 \perp \mathbf{A}_2$. Obviously any automorphism which leaves the two roots α, β invariant leaves invariant the sublattice Q and its orthogonal complement $Q^\perp$. Thus the stabilizer contains a subgroup isomorphic to $W(\mathbf{A}_2) \times W(\mathbf{A}_2) \times W(\mathbf{A}_2)$ and the permutation of order 2 which switches the two copies of $\mathbf{A}_2$ in $Q^\perp$. Since $W(\mathbf{A}_2) \cong S_3$ we obtain that a stabilizer subgroup contains a subgroup of order $2.6^3 = 432$. Since its index is equal to 120, it must coincide with the stabilizer group.

It follows from above that a Steiner complex corresponds a root sublattice of type $\mathbf{A}_2 \perp \mathbf{A}_2 \perp \mathbf{A}_2$ contained in $\mathbf{E}_6$. The group $W(\mathbf{A}_2) \wr S_3$ of order 3.432 is contained in the stabilizer. Since its index is equal to 40, it coincides with the stabilizer. □

Remark 10.1.2. The notions of syzygetic (azygetic) pairs, triads and a Steiner complex of triads of double-six is analogous to the notions of syzygetic (azygetic) pairs, triads, and a Steiner complex of bitangents of a plane quartic (see Chapter 6, Part I). In both cases we deal with a 6-dimensional quadratic space $\mathbb{F}_2^6$. However, the difference is that the quadratic forms are of different types.

10.1.7 Tritangent trios

A triple of v_1, v_2, v_3 of exceptional vectors is called a *tritangent trio* if

$$v_1 + v_2 + v_3 = -\kappa_6.$$

If we view exceptional vectors as cosets in $\mathbb{Z}^{1,6}/\mathbb{Z}\kappa_6$ this is equivalent to saying that the cosets add up to zero.

It is easy to list all tritangent trios.

Lemma 10.1.16. *There 45 tritangent trios:*

30 of type

$$\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_{ij}, \quad i \neq j,$$

15 of type

$$\mathbf{c}_{ij}, \mathbf{c}_{kl}, \mathbf{c}_{mn}, \quad \{i,j\} \cup \{k,l\} \cup \{m,n\} = \{1,2,3,4,5,6\},$$

Theorem 10.1.17. *The Weyl group acts transitively on the set of tritangent trios.*

Proof. We know that the permutation subgroup S_6 of the Weyl group acts on tritangent trios by permuting the indices. Thus it acts transitively on the set of tritangent trios of the same type. Now consider the reflection with respect to the root α_{123}. We have

$$\begin{aligned} s_{\alpha_{123}}(\mathbf{a}_1) &= \mathbf{e}_1 + \alpha_{123} = \mathbf{e}_0 - \mathbf{e}_3 - \mathbf{e}_4 = \mathbf{c}_{34}, \\ s_{\alpha_{123}}(\mathbf{b}_2) &= (2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6) - \alpha_{123} = \mathbf{e}_0 - \mathbf{e}_5 - \mathbf{e}_6 = \mathbf{c}_{56}, \\ s_{\alpha_{123}}(\mathbf{c}_{12}) &= \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 = \mathbf{c}_{12}. \end{aligned}$$

Thus w transforms the tritangent trio (a_1, b_2, c_{12}) to the tritangent trio (c_{12}, c_{34}, c_{56}). This proves the assertion. □

Remark 10.1.3. The stabilizer subgroup of a tritangent triplet is a maximal subgroup of $W(\mathbf{E}_6)$ of index 45 isomorphic to the Weyl group of the root system of type F_4.

Let $\Pi_1 = \{v_1, v_2, v_3\}$ and $\Pi_2 = \{w_1, w_2, w_3\}$ be two tritangent trios with no common elements. We have

$$(v_i, w_1 + w_2 + w_3) = -(v_i, \kappa_6) = 1.$$

and by Proposition 10.1.7, $(v_i, w_j) \geq 0$. This implies that there exists a unique j such that $(v_i, w_j) = 1$. After reordering, we may assume $j = i$. Let $u_i = -\kappa_6 - v_i - w_i$. Since $u_i^2 = -1, (u_i, \kappa_6) = -1$, the vector u_i is a exceptional vector. Since

$$u_1 + u_2 + u_3 = \sum_{i=1}^{3}(-\kappa_6 - v_i - w_i) = -3\kappa_6 - \sum_{i=1}^{3} v_i - \sum_{i=1}^{3} w_i = -\kappa_6,$$

we get a new tritangent trio $\Pi_3 = (u_1, u_2, u_3)$. The union $\Pi_1 \cup \Pi_2 \cup \Pi_3$ contains 9 lines $v_i, w_i, u_i, i = 1, 2, 3$. There is a unique triple of tritangents trios which consists of the same lines. It is formed by tritangents trios $\Pi'_i = (v_i, w_i, u_i), i = 1, 2, 3$. It is easy to see that any pair of triples of tritangents trios which consist of the same set of 9 lines is obtained in this way. Such a pair of triples of tritangent trios is called a pair of *conjugate triads of tritangent trios*.

We can easily list all conjugate pairs of triads of tritangent trios:

$$(I)\ \begin{matrix} a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j \\ c_{ik} & a_k & b_i \end{matrix}, \quad (II)\ \begin{matrix} c_{ij} & c_{kl} & c_{mn} \\ c_{ln} & c_{im} & c_{jk} \\ c_{km} & c_{jn} & c_{il} \end{matrix}, \quad (III)\ \begin{matrix} a_i & b_j & c_{ij} \\ b_k & a_l & c_{kl} \\ c_{ik} & c_{jl} & c_{mn} \end{matrix}$$

Here the conjugate triad can be read as the columns of the matrix. Altogether we have $20 + 10 + 90 = 120$ different triads.

There is a bijection from the set of pairs of conjugate triads to the set of azygetic triads of double-sixes. The 18 lines contained in the union of the latter is the complementary set of the set of 9 lines defined by a triad in the pair. Here is the explicit bijection.

$$\begin{matrix} a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j \\ c_{ik} & a_k & b_i \end{matrix} \quad \leftrightarrow \quad D_{ij}, D_{ik}, D_{jk}$$

$$\begin{matrix} c_{ij} & c_{kl} & c_{mn} \\ c_{ln} & c_{im} & c_{jk} \\ c_{km} & c_{jn} & c_{il} \end{matrix} \quad \leftrightarrow \quad D, D_{ikn}, D_{jlm}$$

$$\begin{matrix} a_i & b_j & c_{ij} \\ b_k & a_l & c_{kl} \\ c_{ik} & c_{jl} & c_{mn} \end{matrix} \quad \leftrightarrow \quad D_{mn}, D_{jkm}, D_{jkn}$$

Recall that the set of lines omitted from each triad entering in a Steiner complex of triads of azygetic double-six is the set of 27 exceptional vectors. Thus a Steiner complex defines three pairs of conjugate triads of tritangent trios which contains all 27 exceptional vectors. We have 40 such triples of conjugate pairs.

Theorem 10.1.18. *The Weyl group acts transitively on the set of 120 conjugate pairs of triples of tritangent trios. A stabilizer subgroup H is contained in the maximal subgroup of $W(\mathbf{E}_6)$ of index 40 realized as a stabilizer of a Steiner complex. The quotient group is a cyclic group of order 3.*

Proof. This follows from the established bijection between pairs of conjugate triads and triads of azygetic double-sixes and Theorem 10.1.15. In fact it is easy to see directly the transitivity of the action. It is clear that the permutation subgroup S_6 acts transitively on the set of pairs of conjugate triads of the same type. Since the Weyl group acts transitively on the set of tritangent trios, we can send a tritangent trio (c_{ij}, c_{kl}, c_{mn}) to a tritangent trio (a_i, b_j, c_{ij}). As is easy to see from inspection that this sends a conjugate pair of type III to a pair of conjugate triads of type I. Also it sends a conjugate pair of type II to type I or III. Thus all pairs are W-equivalent. □

Remark 10.1.4. Note that each monomial entering into the expression of the determinant of the matrix expressing a conjugate pair of triads represents three orthogonal exceptional vectors. If we take only monomials corresponding to even permutations (resp. odd) we get a partition of the set of 9 exceptional vectors into the union of 3 triples of orthogonal exceptional vectors such that each exceptional vector from one triple intersects has non-zero intersection with two exceptional vectors from any other triple.

10.1.8 Lines on a nonsingular cubic surface

Let S be a surface obtained by blowing up 6 points in $\mathbb{P}^2$ where some of them could be infinitely near. Recall from Chapter 8 of part I that the ordered sequence of the blow-ups

$$\pi : S \xrightarrow{\pi_6} S_5 \xrightarrow{\pi_5} \ldots \xrightarrow{\pi_1} \mathbb{P}^2$$

defines a geometric marking $\phi : \mathbb{Z}^{1,6} \to \text{Pic}(S)$. It is an isometry of lattices which sends κ_6 to the canonical class K_S. We have

$$\phi(\mathbf{e}_0) = \pi^*(l),\ \phi(\mathbf{e}_i) = (\pi_5 \circ \ldots \circ \pi_i)^*(E_i),$$

where E_i is the divisor class of the exceptional curve of π_i, and l is the class of a line in $\mathbb{P}^2$.

Now let S be a cubic surface with at most rational double points as singularities. Let $f : X \to S$ be its minimal resolution. The surface X is a Del Pezzo surface. Recall that X is called a Del Pezzo surface if $-K_X$ is nef and big and is called a Fano surface if $-K_X$ is ample.

Theorem 10.1.19. *The following properties are equivalent*

(i) S is nonsingular;

(ii) $-K_X$ is ample;

(iii) the image $\phi(\alpha)$ of any root α is a non-effective divisor class;

(iv) *the image $\phi(v)$ of any exceptional vector v is the divisor class of an exceptional curve of the first kind E (i.e., $E \cong \mathbb{P}^1, E^2 = -1$).*

Proof. $(i) \Leftrightarrow (ii)$ If S is nonsingular, $X = S$ and $-K_X$ is ample (equal to the class of a plane section of the cubic surface). Conversely, if $-K_X$ is ample, f must be an isomorphism (otherwise $-K_X \cdot C = 0$ for curve contained in the exceptional curve of the resolution).

$(ii) \Leftrightarrow (iii)$ Obvious since $K_X \cdot \phi(\alpha) = (\kappa_6, \alpha) = 0$. Since $-K_X$ is ample, $-K_X \cdot C > 0$ for any effective curve C.

$(iii) \Leftrightarrow (iv)$ Let $D = \phi(v)$. We have $D^2 = -1, D \cdot K_X = -1$. Since $-K_X$ is ample and $-K_X \cdot (K_X - D) = -2 < 0$, we have $h^2(D) = h^0(K_X - D) = 0$. By Riemann-Roch, $h^0(D) \geq \frac{1}{2}(D^2 - D \cdot K_X) = 1$, hence D is effective. If we write an effective representative E of D as a sum of its irreducible components $\sum E_i$, after intersecting both sides with $-K_X$ we obtain that $1 = \sum(-K_X \cdot E_i)$. Since $-K_X$ is ample, only one component is present. Thus E is irreducible. The adjunction formula gives $E \cong \mathbb{P}^1$.

$(iv) \Leftrightarrow (iii)$ Any root α can be written as a difference of two exceptional vectors $v - w$. In fact, it is enough to check it for one root, say $\alpha = \mathbf{e}_1 - \mathbf{e}_2$. Assume $\phi(\alpha)$ is represented by an curve R. Let $E_1 = \phi(v), E_2 = \phi(w)$. These are irreducible curves with self-intersection -1. We have $E_1 \sim E_2 + R$. Intersecting both sides with E_1, we get $R \cdot E_1 = -1 - (E_1 \cdot E_2) < 0$. This shows that E_1 is an irreducible component of R. Subtracting E_1, we obtain $E_1 + (R - E_1) \sim 0$, a contradiction (an effective nonzero divisor is not equivalent to zero).

$(iii) \Leftrightarrow (ii)$ To check that $-K_X$ is ample we have to verify that $-K_X \cdot C > 0$ for any irreducible curve C. Since $-K_X$ is nef, $K_X \cdot C \leq 0$. Suppose $-K_X \cdot C = 0$ for some irreducible curve C. Then $C^2 < 0$ (since the signature on Pic$(X) \cong \mathbb{Z}^{1,6}$ is $(1, \rho - 1)$). By the adjunction formula, $C^2 = -2$. Thus $\alpha = \phi^{-1}([C])$ is a root, and $\phi(\alpha)$ is effective. A contradiction. □

Let S be an nonsingular cubic surface in $\mathbb{P}^3$. It has 27 lines (the images of exceptional vectors under any geometric marking), it has 45 triples of coplanar lines. These are the images of 45 tritangents trios of exceptional vectors. The corresponding plane is called a *tritangent plane*. Under isomorphism $\phi : \mathbb{Z}^{1,6} \cong$ Pic(S), we can transfer all the notions and the statements from the previous sections to the Picard lattice Pic(S).

Two different geometric markings π, π' define a Cremona transformation of $\pi' \circ \pi^{-1} : \mathbb{P}^2 - \to \mathbb{P}^2$. They also define the isometry $\phi' \circ \pi^{-1}$ of Pic(S) which preserves the canonical class. The group of such isometries is isomorphic to the Weyl group $W(\mathbf{E}_6)$ and is generated by reflections in vectors $\phi(\alpha)$.

If $\pi : S \to \mathbb{P}^2$ and $\pi' : S \to \mathbb{P}^2$ be the blowing-down structures corresponding to an ordered double-six, then the characteristic matrix of the Cremona transformation $\pi' \circ \pi^{-1}$ is equal to

$$\begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix} \tag{10.14}$$

10.1.9 Schur's quadrics

Let $Q \in S^2E^*$ be a quadratic form on a finite-dimensional vector space V. Recall that the apolarity map defines a linear map

$$\mathrm{ap}_1^Q : V \to V^*,\ v \mapsto P_v(Q)$$

which we identify with Q. For any linear subspace $L \subset V$, we have the *polar subspace* with respect to Q

$$L_Q^\perp = \{x \in V : b_Q(x,y) = 0, \forall y \in L\} = Q(L)^\perp = \cap_{v\in L} P_v(Q)^\perp. \tag{10.15}$$

Here for any subspace W of V, we denote by $W^\perp$ the subspace of linear functions on V which are identical zero on W (the annulator of W, or the dual of W). If Q is nondegenerate, then

$$L_Q^\perp = Q^{-1}(L^\perp).$$

If M is a linear subspace of E^*, we define its *polar subspace* with respect to Q by

$$M_Q^\perp = Q(M^\perp).$$

If Q is nondegenerate, then $M_Q^\perp$ is the orthogonal complement of M^* with respect to the dual quadratic form $\check{Q}$ on E^* defined by the linear map $Q^{-1} : V^* \to V$.

All of this can be extended to the projective space $\mathbb{P}(E)$ and a quadric hypersurface Q in $\mathbb{P}(E)$. For example, for any linear subspace $L \subset \mathbb{P}(E)$ the dual subspace $L^\perp$ is a linear subspace of $\mathbb{P}(E^*)$ spanned by the hyperplanes (considered as points in $\mathbb{P}(E^*)$) containing L. Thus the dual $H^\perp$ of a hyperplane $H \subset \mathbb{P}(E)$ is the point $a \in \mathbb{P}(E^*)$ corresponding to this hyperplane. Also, if Q is a quadric hypersurface in $\mathbb{P}(E)$ and L is a linear subspace of $\mathbb{P}(E)$, then the polar subspace of L with respect to Q is equal to

$$L_Q^\perp = \cap_{a\in L} P_a(Q). \tag{10.16}$$

Also for any linear subspace W of $\mathbb{P}(E^*)$

$$W_Q^\perp = (\cap_{a\in W^\perp} P_a(Q))^\perp \tag{10.17}$$

Obviously, it is enough to do the intersection for a spanning set of the subspace.

Let $l_1, \ldots, l_6$ be a set of skew lines on a nonsingular cubic surface $S \subset \mathbb{P}^3$. A nonsingular quadric Q in $\mathbb{P}^3$ defines six skew lines $l_1', \ldots, l_6'$, where l_i' is polar to l_i with respect to l_i.

The following beautiful result of Ferdinand Schur (Math. Ann. 1881) shows that there exists a unique nonsingular quadric Q such that the set $l_1', \ldots, l_6'$ is a set of skew lines on S (which together with $l_1, \ldots, l_6$ makes a double six).

Theorem 10.1.20. *Let $(l_1, \ldots, l_6), (l_1', \ldots, l_6')$ be a double-six of lines on a nonsingular cubic surface S. There exists a unique nondegenerate quadric in $\mathbb{P}^3$ such that l_i' is the polar line of l_i with respect to Q for each $i = 1, \ldots, 6$.*

Proof. Fix an ordered double-six $(l_1, \ldots, l_6), (l_1', \ldots, l_6)$ on a nonsingular cubic surface S. Choose a geometric marking $\phi : \mathbb{Z}^{1,6} \to \mathrm{Pic}(S)$ such that $\phi(\mathbf{e}_i) = e_i = [l_i], i = 1, \ldots, l_6$. Then the lineat system $\big|\phi(\mathbf{e}_0)\big|$ defines a birational map $\pi : S \to \mathbb{P}^2$ which blows the lines l_i to the points p_i. The image of the lines l_i' is the conic C_i passing through all p_j except p_i. The pre-image of l_i' with respect to ϕ is the exceptional vector b_i. Let $\phi' : \mathbb{Z}^{1,6} \to \mathrm{Pic}(S)$ be the geometric marking such that $\phi'(\mathbf{e}_i) = l_i'$. It is obtained from ϕ by composing ϕ with the reflection $s_{\alpha_{\max}} \in O(\mathbb{Z}^{1,6})$. We have $\phi'(\mathbf{e}_0) = \phi(s_{\alpha_{\max}}(\mathbf{e}_0)) = \phi(5\mathbf{e}_0 - 2\mathbf{e}_1 - \ldots - 2\mathbf{e}_6)$. Thus the linear system $|e_0'| = |5\mathbf{e}_0 - 2\mathbf{e}_1 - \ldots - 2\mathbf{e}_6|$ defines a birational map $\pi' : S \to \mathbb{P}^2$ which blows down the lines l_i' to points q_i. Note that there is no canonical identification of two $\mathbb{P}^2$'s. One views them as different planes ${}_1\mathbb{P}^2$ and ${}_2\mathbb{P}^2$.

For any line ℓ in ${}_1\mathbb{P}^2$, its full pre-image in S belongs to the linear system $|e_0|$. Since $e_0 \cdot (-K_S) = 3$, the curves in $|e_0|$ are rational curves of degree 3 (maybe reducible). Similarly, the pre-images of lines l' in ${}_2\mathbb{P}^2$ are rational curves of degree 3 on S. Now

$$\pi^*(\ell) + \pi'^*(\ell') \in |e_0 + 5e_0 - 2e_1 - \ldots - 2e_6| = |-2K_S|.$$

Thus the union of two rational curves $\pi^*(\ell)$ and $\pi'^*(\ell')$ is cut out by a quadric $Q_{l,l'}$ in $\mathbb{P}^3 = |-K_S|^*$. Note that the intersection of a quadric and a cubic is a curve of arithmetic genus 4. Our curves are reducible curves of arithmetic genus 4. When we vary ℓ and ℓ', the corresponding quadrics span a hyperplane $\mathcal{H}$ in $|-2K_S|$. The map

$${}_1\check{\mathbb{P}}^2 \times {}_2\check{\mathbb{P}}^2 \to \mathcal{H}, \quad (l, l') \mapsto Q_{l.l'}$$

is isomorphic to the Segre map.

Recall that our surface S lies in $\mathbb{P}^3 \cong |-K_S|^*$. Consider the dual space $\check{\mathbb{P}}^3 = |-K_S|$ and let $\check{Q}$ be the quadric in this space which is apolar to all quadrics in the hyperplane $\mathcal{H}$, i.e. orthogonal to $\mathcal{H}$ with respect to the apolarity map

$$S^2(H^0(S, \mathcal{O}_S(-K_S)) \times S^2(H^0(S, \mathcal{O}_S(-K_S)^*) \to \mathbb{C}.$$

In particular, if a quadric from $\mathcal{H}$ is a pair of planes $\Pi_1 \cup \Pi_2$ corresponding to points $a = \Pi_1^\perp$ and $b = \Pi_2^\perp$ in $\check{\mathbb{P}}^3$, then $P_{a,b}(\check{Q}) = 0$.

Now choose 3 special lines $\langle p_i, p_j \rangle, \langle p_i, p_k \rangle, \langle p_j, p_k \rangle$ in the first plane and similar lines $\langle q_i, q_j \rangle, \langle q_i, q_k \rangle, \langle q_j, q_k \rangle$ in the second plane. Then $R_{ij} = \pi^*(\langle p_i, p_j \rangle) = l_{ij} + l_i + l_j$, where $l_{ij} = \phi(c_{ij})$. Similarly, $R'_{jk} = \pi'^*(\langle q_j, q_k \rangle)) = l'_j + l'_k + l'_{jk}$, where

$$l'_{jk} \sim e'_0 - l'_j - l'_k = (5e_0 - 2\sum_{i=1}^{6} e_i) - (2e_0 - \sum_{i=1}^{6} e_i + l_j) - (2e_0 - \sum_{i=1}^{6} e_i + e_k)$$

$$= e_0 - e_j - e_k \sim l_{ij}.$$

Thus the lines l'_{jk} and l_{ij} coincide.

Now notice that the curve $R_{ij} + R'_{jk}$ is cut out by the reducible quadric $H_{ij} \cup H_{jk}$, where H_{ij} is the tritangent plane containing the lines l_i, l'_j, l_{ij} and H_{jk} is the tritangent plane containing the lines l_j, l'_k, l_{jk}.

Let $a \in \mathbb{P}(E^*)$ and $H_a = a^\perp$ be the corresponding hyperplane in $\mathbb{P}(E)$. If $\langle a, b \rangle_{\check{Q}} = 0$ for $a, b \in \mathbb{P}(E^*)$, then

$$\check{Q}(a) = (H_a)^\perp_{\check{Q}} \in H_b.$$

Let $P_{ij} = (H_{ij})^\perp_{\check{Q}}$. Since each pair of planes H_{ab}, H_{bc}, considered as points in the dual space, are orthogonal with respect to $\check{Q}$, the point P_{ij} belongs to $H_{ki} \cap H_{jk} \cap H_{ji}$. It is easy to see that this point is $a_j \cap b_i$. Since $a_i \cap b_j \in H_{ij}$, the points P_{ij} and P_{ji} are polar to each other with respect to $\check{Q}$. Similarly, we find that the points (P_{ki}, P_{ik}) are polar with respect to $\check{Q}$, hence the lines l_i and l'_i are polar with respect to Q.

Let us show that $\check{Q}$ is a nondegenerate quadric. Suppose $\check{Q}$ is degenerate, then its set of singular points Sing$(\check{Q})$ is a linear space of positive dimension equal to the kernel of the symmetric bilinear form associated to $\check{Q}$. Thus, for any subspace L of $\mathbb{P}^3$, the polar subspace $L^\perp_{\check{Q}}$ with respect to $\check{Q}$ lies in Sing$(\check{Q})^\perp$. Therefore the points P_{ij} lie in a proper subspace of $\mathbb{P}^3$. But this is obvioulsy impossible, since some of these points lie on a pair of skew lines and span $\mathbb{P}^3$. Thus we can define

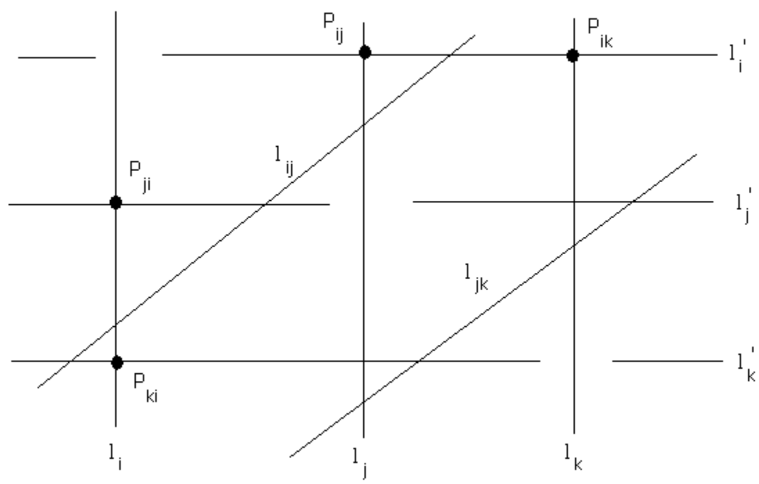

Figure 10.1:

the dual quadric Q of $\check{Q}$ and obtain that the lines l_i and l_i' are polar with respect to Q.

Let us show the uniqueness of Q. Suppose we have two quadrics Q_1 and Q_2 such that $l_i' = (l_i)^{\perp}_{Q_i}, i = 1, \ldots, 6$. Let Q be a singular quadric in the pencil spanned by Q_1 and Q_2. Let K be its space of singular points.

Assume first that $\dim K = 0$. Without loss of generality we may assume that $K \notin l_1 \cup l_2$. Then $\dim (l_1)^{\perp}_{Q} = \dim (l_2)^{\perp}_{Q} = 1$. On the other hand, $l_1' \subset (l_1)^{\perp}_{Q}, l_2' \subset (l_2)^{\perp}_{Q}$. Thus we have the equalities. But now $l_1 \subset K^{\perp}_{Q} = \mathbb{P}^3$, hence $K \subset (l_1)^{\perp}_{Q} = l_1'$ and similarly, $K \subset l_2'$. Since l_1', l_2' are skew, we get a contradiction.

Assume now that $\dim K = 1$. Since K cannot intersect all six lines l_i (otherwise it is contained in S and there are no such lines in S), we may assume that K does not intersect l_1. Then, as above, $l_1' = (l_1)^{\perp}_{Q}$ and $K = l_1'$. Now, K does not intersect l_2'. Repeating the argument, we obtain that $K = l_2$. Thus $l_1' = l_2$, which is a contradiction.

Finally assume that $\dim K = 2$. Then K intersects all lines. Then $(l_i)^{\perp}_{Q}$ are all of dimension ≥ 2 and contain K. Since K may contain at most two lines from the double-six, we may assume that $(l_1)^{\perp}_{Q} = (l_2)^{\perp}_{Q} = K$. Since $l_1' \subset (l_1)^{\perp}_{Q} = K, l_2' \subset (l_2)^{\perp}_{Q} = K$, we see that the lines l_1', l_2' are coplanar and hence intersect. This is a contradiction. □

Definition 10.3. *The quadric Q is called the* Schur quadric *with respect to a given double-six.*

Consider the intersection curve C of the Schur quadric Q with the cubic surface S. Obviously it belongs to the linear system $|-2K_S|$. Let $\pi : S \to \mathbb{P}^2, \pi' : S \to \mathbb{P}^2$ be the birational morphisms defined by the double-six $(l_1, \ldots, l_6), (l_1', \ldots, l_6')$ corresponding to Q. The image of C under π (or π') is a curve of degree 6 with double points at the points $p_i = \pi(l_i)$. We call this curve the *Schur sextic*. It is defined as soon as we choose 6 points on $\mathbb{P}^2$ such that S is isomorphic to the blow-up of these points.

Proposition 10.1.21. *The six double points of the Schur sextic are bi-flexes, i.e. the tangent line to each branch is tangent to the branch with multiplicity* ≥ 3.

Proof. Let $l_i \cap Q = \{a, b\}$ and $l_i' \cap Q = \{a', b'\}$. We know that

$$P_a(Q) \cap Q = \{x \in Q : a \in PT(Q)_x\}.$$

Since $l_i' = (l_i)^{\perp}_Q$, we have

$$l_i' \cap Q = (P_a(Q) \cap P_b(Q)) \cap Q = \{a', b'\}.$$

This implies that $a', b' \in PT(Q)_a$ and hence the lines $\langle a, a' \rangle, \langle a, b' \rangle$ span the tangent space of Q at the point a. The tangent plane $PT(Q)_a$ contains the line l_i' and hence intersects the cubic surface S along l_i' and a conic K_a. We have

$$PT(K_a) = PT(S)_a \cap PT(Q)_a = PT(Q \cap S)_a.$$

Thus the conic K_a and the curve $\mathcal{C} = Q \cap S$ are tangent at the point a. Since the line l_i' is equal to the proper inverse transform of the conic C_i in $\mathbb{P}^2$ passing through the points $p_j, j \neq i$, the conic K_a is the proper inverse transform of some line ℓ in the plane passing through p_i. The point a corresponds to the tangent direction at p_i defined by a branch of the Schur sextic at p_i. The fact that K_a is tangent to $\mathcal{C}$ at a means that the line ℓ is tangent to the branch with multiplicity ≥ 3. Since similar is true, when we replace a with b, we obtain that p_i is a bi-flex of the Schur sextic. □

Remark 10.1.5. A bi-flex is locally given by an equation whose Taylor expansion looks like $xy + xy(ax + by) + f_4(x, y) + \ldots$. This shows that one has impose 5 conditions to get a bi-flex. To get 6 bi-flexes for a curve of degree 6 one has to satisfy 30 linear equations. The space of homogeneous polynomials of degree 6 in 3 variables is of dimension 28. So, the fact that such sextics exist is very surprising.

Also observe that the set of quadrics Q such that $l^{\perp}_Q = l'$ for a fixed pair of skew lines (l, l') is a linear (projective) subspace of codimension 4 of the 9-dimensional space of quadrics. So the existence of the Schur quadric is quite unexpected!

I do not know whether for a given set of 6 points on $\mathbb{P}^2$ defining a nonsingular cubic surface, there exists a unique sextic with bi-flexes at these points.

Example 10.1.1. Let S be the *Clebsch diagonal surface* given by 2 equations in $\mathbb{P}^4$:

$$\sum_{i=0}^{4} T_i = \sum_{i=0}^{4} T_i^3 = 0. \tag{10.18}$$

It exhibits an obvious symmetry defined by permutations of the coordinates. Let $a = \frac{1}{2}(1+\sqrt{5}), a' = a = \frac{1}{2}(1-\sqrt{5})$ be two roots of the equation $x^2 - x - 1 = 0$. One checks that the skew lines

$$l : T_1 + T_3 + aT_2 = aT_3 + T_2 + T_4 = aT_2 + aT_3 - T_5 = 0$$

and

$$l' : T_1 + T_2 + a'T_4 = T_3 + a'T_1 + T_4 = a'T_1 + a'T_4 - T_5 = 0$$

lie on S. Applying to each line even permutations we obtain a double-six. The Schur quadric is $\sum T_i^2 = 0$.

10.2 Singularities

10.2.1 Non-normal cubic surfaces

Let X be an irreducible cubic surface in $\mathbb{P}^3$. Assume that X is not normal. Then its singular locus contains a one-dimensional part C. Let $C_1, \ldots, C_k$ be irreducible components of C and m_i be the multiplicity of a general point η_i of C_i as a point on X. A general section of X is a plane cubic curve H. Its intersection points with C_i are singular points of multiplicity m_i. Their number is equal to $d_i = \deg(C_i)$. By Bertini's theorem, H is irreducible. Since an irreducible plane cubic curve has only one singular point of multiplicity 2, we obtain that C is irreducible and of degree 1.

Let us choose coordinates in such a way that C is given by the equations $T_0 = T_1 = 0$. Then the equation of X must look like

$$L_0 T_0^2 + 2L_1 T_0 T_1 + L_2 T_1^2 = 0,$$

where $L_i, i = 0, 1, 2$, are linear forms in T_0, T_1, T_2. This shows that the left-hand side contains T_2 and T_3 only in degree 1. Thus we can rewrite the equation in the form

$$T_2 A + T_3 B + C = 0. \tag{10.19}$$

where A, B, C are binary forms in T_0, T_1, the first two of degree 2, and the third one of degree 3.

Suppose A, B have no common zeros. Then the map $\mathbb{P}^1 \to \mathbb{P}^1$ defined by A, B is of degree 2, and hence has two ramification points. This implies that $A =$

$aL^2 + bM^2, B = a'L^2 + b'M^2$ for some linear polynomials L, M. After linear change of variables we may assume that $L = T_0, M = T_1$. Thus every monomial in the left-hand side of the equation (10.19) is divisible either by T_0^2 or by T_1^2. Thus we can rewrite it in the form $PT_0^2 + QT_1^2$, where P, Q are linear forms in T_0, T_1, T_2, T_3. Without loss of generality, we may assume that P has a non-zero coefficient at T_3. After a linear change of variables we may assume that $P = T_3$. If Q has zero coefficient at T_2, our surface is a cone over a singular plane cubic. If the coefficient is non-zero, after a linear change of variables we may assume that $Q = T_3$ and the equation becomes

$$T_2T_0^2 + T_3T_1^2 = 0.$$

Suppose A, B has one common non-multiple zero. After a linear change of variables T_0, T_1, we may assume that $A = T_0T_1, B = T_0(T_0+T_1)$ and the equation becomes

$$T_2T_0T_1 + T_3T_0T_1 + T_3T_0^2 + T_0T_1(aT_0 + bT_1) + cT_0^3 + dT_1^3 = 0.$$

After the linear change of variables

$$T_2 \to T_2 + T_3 + aT_0 + bT_1, \quad T_3 \mapsto T_3 + cT_0,$$

we reduce the equation to the form

$$T_2T_0T_1 + T_3T_0^2 + dT_1^3 = 0.$$

Obviously, $d \neq 0$. Multiplying by d^2 and changing $T_0 \to dT_0, T_1 \to dT_1, T_2 \to d^{-1}T_2$, we may assume that $d = 1$.

Finally, if A is proportional to B, say $B = \lambda A$, replacing T_2 with $T_2 + \lambda T_3$, we reduce the equation 10.19 to the form $T_2A + C = 0$. In this case X is again a cone.

Summarizing we get

Theorem 10.2.1. *Let X be an irreducible non-normal cubic surface. Then, either X is a cone over an irreducible singular plane cubic, or it is projectively equivalent to one of the following cubic surfaces singular along a line:*

(i) $T_0^2T_2 + T_1^2T_3 = 0$;

(ii) $T_2T_0T_1 + T_3T_0^2 + T_1^3 = 0$.

The two surfaces are not projectively isomorphic.

The last assertion follows from considering the normalization $\bar{X}$ of the surface X. In case (i) the surface $\bar{X}$ is nonsingular. In case (ii), $\bar{X}$ has one ordinary double point.

10.2.2 Normal cubic surfaces

A normal cubic surface S has only isolated singularities. Let P be a singular point. Choose projective coordinates such that $P = (1,0,0,0)$. Then the equation of the surface can be written in the form

$$T_0Q_2(T_1,T_2,T_3) + Q_3(T_1,T_2,T_3) = 0,$$

where Q_2 and Q_3 are homogeneous polynomials of degree given by the subscripts. If $Q_2 = 0$, the surface is a cone over a nonsingular plane cubic curve. If $Q_2 \neq 0$, then P is a singular point of multiplicity 2. Projecting from P we see that S is birationally isomorphic to $\mathbb{P}^2$. Let $\pi : S' \to S$ be a minimal resolution of singulartities of S. The sheaf $R^1\pi_*\mathcal{O}_{S'}$ has support only at singular points of S. Since S is normal, $\pi_*\mathcal{O}_{S'} = \mathcal{O}_S$. Applying the Leray spectral sequence we obtain an exact sequence

$$0 \to H^1(S,\mathcal{O}_S) \to H^1(S',\mathcal{O}_{S'}) \to H^0(S,R^1\pi_*\mathcal{O}_{S'}) \to H^2(S,\mathcal{O}_S).$$

Since S' is a nonsingular rational surface, we have $H^1(S',\mathcal{O}_{S'}) = 0$. The canonical sheaf of S is $\mathcal{O}_S(-1)$, hence, by Serre's duality, $H^2(S,\mathcal{O}_S) = H^0(S,\omega_S) = 0$. Thus we obtain that $H^0(S,R^1\pi_*\mathcal{O}_{S'}) = 0$. This shows that, for any singular points $s \in S$, we have $(R^1\pi_*\mathcal{O}_{S'})_s = 0$. As is known (see [Reid]) this characterizes canonical singularities (or RDP) of a surface.

This gives

Theorem 10.2.2. *Let S be a normal cubic surface in $\mathbb{P}^3$. Then S is either a cone over a nonsingular plane cubic curve or an anticanonical model of a Del Pezzo surface of degree 3.*

10.2.3 Canonical singularities

From now on we assume that S is a cubic surface with canonical singularities. Let $f : X \to S$ be a minimal resolution of S. As we know X is a Del Pezzo surface and the exceptional divisor E of f is the union of Dynkin curves of type A, D, E. We would like to classify all possibilities for E. This will classify the types of singularities which may occur on S.

Let Q be the sublattice of Pic(X) spanned by the irreducible components of E. We know that Q is a negative definite lattice of rank r equal to the number of irreducible components of E and is the orthogonal sum of root lattices of type A_n, D_n, E_n. Since Pic(X) is an indefinite lattice of rank 7, we obtain that $r \leq 6$. A priori, we have the following possibilities for Q:

$$\begin{array}{ll} (r=6) & E_6,\ D_6,\ A_6,\ D_5+A_1,\ D_4+A_2,\ D_4+2A_1, \\ & \sum_{k=1}^{s} A_{i_k}, i_1+\ldots+i_s=6 \\ (r=5) & D_5,\ D_4+A_1,\ \sum_{k=1}^{s} A_{i_k}, i_1+\ldots+i_s=5 \\ (r=4) & D_4,\ \sum_{k=1}^{s} A_{i_k}, i_1+\ldots+i_s=4 \\ (r=3) & A_3,\ A_2+A_1,\ 3A_1 \\ (r=2) & A_2,\ A_1+A_1 \\ (r=1) & A_1. \end{array}$$

Recall from Part I the local equations of rational double points of type A_n, D_n and E_6:

$$\begin{array}{rcl} A_n & : & xy+z^{n+1}=0 \\ D_n & : & z^2+x(x^2+y^{n-2})=0, n\geq 4 \\ E_6 & : & z^2+x^3+y^4=0 \end{array}$$

Lemma 10.2.3. *The cases $D_6, D_5+A_1, D_4+A_1+A_1$ do not occur.*

Proof. This follows from Lemma 10.1.1 since the discriminants of the lattices $D_6, D_5+A_1, D_4+A_1+A_1$ are not divisible by 3. □

Lemma 10.2.4. *Let $p_0=(1,0,0,0)$ be a singular point of $V(F_3)$. Write*

$$F_3 = T_0G_2(T_1,T_2,T_3)+G_3(T_1,T_2,T_3),$$

where G_2, G_3 are homogeneous polynomials of degree 2,3. Let $p=(t_0,t_1,t_2,t_3)\in V(F_3)$. If the line $\langle p_0,p\rangle$ is contained in $V(F_3)$, then the point $q=(t_1,t_2,t_3)$ is a common point of the conic $V(G_2)$ and the cubic $V(G_3)$. If moreover, p is a singular point of $V(F_3)$, then the conic and the cubic intersect at q with multiplicity >1.

Proof. This is easy to verify and is left to the reader. □

Corollary 10.2.5. *$V(F_3)$ has at most 4 singular points. Moreover, if $V(F_3)$ has 4 singular points, then each point is of type A_1.*

Proof. let p_0 be a singular point. Choose coordinates such that $p_0 = (1,0,0,0)$ and apply Lemma 10.2.4. Suppose we have more than 4 singular points. The conic and the cubic will intersect at at least 4 singular points with multiplicity > 1. Since they do not share an irreducible component (otherwise F_3 is reducible), this contradicts Bézout's Theorem. Suppose we have 4 singular points and p_0 is not of type A_1. Since p_0 is not an ordinary double point, the conic $V(G_2)$ is reducible. Then the cubic $V(G_3)$ intersects it at 3 points with multiplicity > 1 at each point. It is easy to see that this also contradicts Bézout's Theorem. □

Lemma 10.2.6. *The cases, $A_{i_1} + \ldots + A_{i_k}, i_1 + \ldots + i_k = 6$, except the cases $3A_2, A_5 + A_1$ do not occur.*

Proof. Assume $M = A_{i_1} + \ldots + A_{i_k}, i_1 + \ldots + i_k = 6$. Then $d_M = (i_1 + 1) \cdots (i_k + 1)$. Since $3|d_M$, one of the numbers, say $i_1 + 1$, is equal either to 3 or 6. If $i_1 + 1 = 6$, then $M = A_5 + A_1$. If $i_1 + 1 = 3$, then $(i_2 + 1) \ldots (i_k + 1)$ must be a square, and $i_2 + \ldots + i_k = 4$. It is easy to see that the only possibility are $i_2 = i_3 = 2$ and $i_2 = i_3 = i_4 = i_5 = 1$. The last possibility is excluded by applying Corollary 10.2.5. □

Lemma 10.2.7. *The cases $D_4 + A_1$ and $D_4 + A_2$ do not occur*

Proof. Let p_0 be a singular point of S of type D_4. Again, we assume that $p_0 = (1,0,0,0)$ and apply Lemma 10.2.4. As we have already noted, the singularity of type D_4 is analytically (or formally) isomorphic to the singularity $z^2 + xy(x+y) = 0$. This shows that the conic $V(G_2)$ is a double line l. The plane $z = 0$ cuts out a germ of a curve with 3 different branches. Thus there exists a plane section of $S = V(F_3)$ passing through p_0 which is a plane cubic with 3 different braches at P. Obviously, it must be a union of 3 lines with a common point at p_0. Now the cubic $V(G_3)$ intersects the line l at 3 points corresponding to the lines through p_0. Thus S cannot have more singular points. □

Let us show that all remaining cases are realized. We will exhibit the corresponding Del Pezzo surface as a the blow-up of 6 bubble points $p_1, \ldots, p_6$ in $\mathbb{P}^2$.

A_1: 6 points in $\mathbb{P}^2$ on an irreducible conic.

A_2: $p_3 \succ_1 p_1$.

$2A_1$: $p_2 \succ_1 p_1, p_4 \succ_1 p_2$.

A_3: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$.

$A_2 + A_1$:$p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$.

A_4: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$.

$3A_1$:$p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$.

$2A_2$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$.

$A+3+A_1$: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$.

A_5:$p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$.

D_4: $p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$ and p_1, p_3, p_5 are collinear.

A_2+2A_1: $p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$, and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$.

A_4+A_1: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|2\ell - p_1 - \ldots - p_6| \neq \emptyset$.

D_5: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|\ell - p_1 - p_2 - p_6| \neq \emptyset$.

$4A_1$: $p_1, \ldots, p_6$ are the intersection points of 4 lines in a general linear position.

$2A_2+A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$ and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$.

A_3+2A_1: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$ and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$.

A_5+A_1: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|2\ell - p_1 - \ldots - p_6| \neq \emptyset$.

E_6:$p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$.

$3A_2$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$, $|\ell - p_1 - p_2 - p_3| \neq \emptyset$, $|\ell - p_4 - p_5 - p_6| \neq \emptyset$.

Remark 10.2.1. Let R be a subset of roots in a root system of some type A_n, D_n, E_n such that it defines a basis of a root system. The corresponding Dynkin diagram can be obtained by the following procedure due to Borel, de Ziebenthal and Dynkin. Let D be the Coxeter-Dynkin diagram of $\mathcal{R}$. Consider the extended diagram by adding one more vertex which is connected to other edges as shown on the following *extended Dynkin diagrams*.

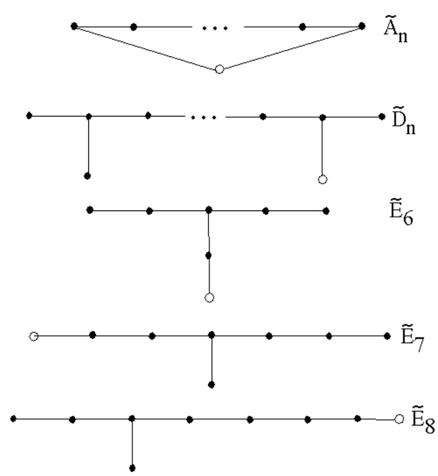

Figure 10.2:

Then $D(\mathcal{R})$ can be obtained as a proper subgraph of the extended Dynkin diagram. In the special case E_6, we get types $E_6, A_1 + A_5, A_2^3$ if we delete 1 vertex, types $A_5, A_1+2A_2, 2A_1+A_3, A_4+A_1, D_5$ if we delete 2 vertices, types $2A_2, A_1+A_3, 2A_1 + A_2, A_4, 4A_1, D_4$ if we delete 3 vertices, types $A_3, A_2 + A_1, 3A_1$ if we delete 4 vertices, types $2A_1, A_2$ if delete 5 and type A_1 if we delete 6. This agrees with our theorem.

Projecting from a singular point and applying Lemma 10.2.4 we see that each singular cubic surface can be given by the following equation.

A_1: $V(T_0G_2(T_1, T_2, T_3) + G_3(T_1, T_2, T_3))$, where $V(G_2)$ is a nonsingular conic which intersects $V(G_3)$ transversally.

A_2: $V(T_0T_1T_2 + G_3(T_1, T_2, T_3))$,where $V(T_1T_2)$ intersects $V(G_3)$ transversally.

$2A_1$: $V(T_0G_2(T_1, T_2, T_3) + G_3(T_1, T_2, T_3))$, where $V(G_2)$ is a nonsingular conic which intersects $V(G_3)$ transversally.

A_3: $V(T_0T_1T_2 + G_3(T_1, T_2, T_3))$, where $V(T_1T_2)$ intersects $V(G_3)$ at the point $(0, 0, 1)$ and at other 4 distinct points.

$A_2 + A_1$: $V(T_0T_1T_2 + G_3(T_1, T_2, T_3))$, where $V(G_3)$ is tangent to $V(T_1T_2)$ at $(1, 0, 0)$.

A_4: $V(T_0T_1T_2+G_3(T_1, T_2, T_3))$, where $V(G_3)$ is tangent to $V(T_1)$ at $(0, 0, 1)$.

$3A_1$: $V(T_0G_2(T_1, T_2, T_3)+G_3(T_1, T_2, T_3))$, where $V(G_2)$ is nonsingular and is tangent to $V(G_3)$ at 2 points.

$2A_2$: $V(T_0T_1T_2+G_3(T_1, T_2, T_3))$, where $V(T_1)$ intersects $V(G_3)$ transverally and $V(G_2)$ is a flex tangent to $V(G_3)$ at $(1, 0, 0)$.

$A_3 + A_1$: $V(T_0T_1T_2 + G_3(T_1, T_2, T_3))$, where $V(G_3)$ passes through $(0, 0, 1)$ and $V(T_1)$ is tangent to $V(G_3)$ at a point $(1, 0, 0)$.

A_5: $V(T_0T_1T_2 + G_3(T_1, T_2, T_3))$, where $V(T_1)$ is a flex tangent of $V(G_3)$ at the point $(0, 0, 1)$.

D_4: $V(T_0T_1^2 + G_3(T_1, T_2, T_3))$, where $V(T_1)$ intersects transversally $V(G_3)$.

$A_2 + 2A_1$: $V(T_0T_1T_2 + G_3(T_1, T_2, T_3))$, where $V(G_3)$ is tangent $V(T_1T_2)$ at two points not equal to $(0, 0, 1)$.

$A_4 + A_1$: $V(T_0T_1T_2 + G_3(T_1, T_2, T_3))$, where $V(G_3)$ is tangent to $V(T_1)$ at $(0, 0, 1)$ and is tangent to $V(T_2)$ at $(1, 0, 0)$.

D_5: $V(T_0T_1^2 + G_3(T_1, T_2, T_3))$, where $V(T_1)$ is tangent to $V(G_3)$ at $(0, 0, 1)$.

$4A_1$: $V(T_0G_2(T_1, T_2, T_3)+G_3(T_1, T_2, T_3))$, where $V(G_2)$ is nonsingular and is tangent to $V(G_3)$ at 3 points.

$2A_2+A_1$: $V(T_0G_2(T_1,T_2,T_3)+G_3(T_1,T_2,T_3))$, where $V(G_2)$ is tangent to $V(G_3)$ at 2 points $(1,0,0)$ with multiplicity 3.

A_3+2A_1: $V(T_0T_1T_2+G_3(T_1,T_2,T_3))$, where $V(G_3)$ passes through $(0,0,1)$ and is tangent to $V(G_1)$ and to $V(G_2)$ at one point not equal to $(0,0,1)$.

A_5+A_1: $V(T_0T_1T_2+G_3(T_1,T_2,T_3))$, where $V(T_1)$ is a flex tangent of $V(G_3)$ at the point $(0,0,1)$ and $V(T_2)$ is tangent to $V(G_3)$.

E_6: $V(T_0T_1^2+G_3(T_1,T_2,T_3))$, where $V(T_1)$ is a flex tangent of $V(G_3)$.

$3A_2$: $V(T_0T_1T_2+G_3(T_1,T_2,T_3))$, where $V(T_1),V(T_2)$ are flex tangents of $V(G_3)$ at points different from $(0,0,1)$.

Remark 10.2.2. Applying a linear change of variables, one can simplify the equations. For example, in the case XXI, we may assume that the inflection points are $(1,0,0)$ and $(0,1,0)$. Then $G_3=T_3^3+T_1T_2L(T_1,T_2,T_3)$. Replacing T_0 with $T_0'=T_0+L(T_1,T_2,T_3)$, we reduce the equation to the form

$$T_0T_1T_2+T_3^3=0 \tag{10.20}$$

Another example is the E_6-singularity (case XX). We may assume that the flex point is $(0,0,1)$. Then $G_3=T_2^3+T_1A_2(T_1,T_2,T_3)$. The coefficient at T_3^2 is not equal to zero, otherwise the equation is reducible. After a linear change of variables we may assume that $A_2=T_3^2+aT_1^2+bT_1T_2+cT_2^2$. Replacing T_0 with $T_0+aT_1+bT_2$, we may assume that $a=b=0$. After scaling the unknowns, we get

$$T_0T_1^2+T_1T_2^2+T_2^3=0. \tag{10.21}$$

10.2.4 4-nodal cubic surface

Let S be given by the equation

$$T_0T_1T_2+T_0T_1T_3+T_0T_2T_3+T_1T_2T_3=0. \tag{10.22}$$

It is immediately verified that S has 4 ordinary nodes as singularities. They are the points $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$. Let us prove that any surface with 4 nodes is projectively isomorphic to the surface given by the above equation.

First we notice that the four singular points are not coplanar. Indeed, otherwise S has a plane section with 4 singular points, however a plane cubic curve has at most 3 singular points unless it contains a multiple line. We have already explained in the previous section that no three nodes lie on a line. Chose a coordinate system such that the singular points are $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$. This implies that a polynomial defining S contains the unknowns T_i only in degree 1. Thus the equation of S can be given in the form

$$aT_0T_1T_2 + bT_0T_1T_3 + cT_0T_2T_3 + dT_1T_2T_3 = 0,$$

where a, b, c, d are nonzero constants. It remains to scale the unknowns to make the constants equal to 1.

The 4-nodal cubic surface exhibits an obvious symmetry defined by the permutation group S_4. It also admits a double cover ramified only over its singular points. This can be seen in many different ways. We give only one, the others can be found in Exercises.

First, note that S can be obtained as the blow-up of the vertices of a complete quadrangle. Let X' be a Del Pezzo surface of degree 2 obtained as a minimal resolution of the double cover of $\mathbb{P}^2$ branched along the union of the four sides of the complete quadrangle. The double cover extends to a double cover $f : X' \to X$ of a minimal resolution X of S branched over the exceptional curves of the singularities. The ramification locus of f consists of the union of 4 disjoint (-1)-curves. Blowing them down we get a Del Pezzo surface Y of degree 6. The cover descends to a double cover of S ramified over the nodes.

10.2.5 The Table

The following table gives the classification of possible canonical singularities of a cubic surface, the number of lines and the class of the surface (i.e., the degree of the dual surface).

Type	Singularity	Lines	Class	Type	Singularity	Lines	Class
I	$\emptyset$	27	12	XII	D_4	6	6
II	A_1	21	10	XIII	$A_2 + 2A_1$	8	5
III	A_2	15	9	XIV	$A_4 + A_1$	4	5
IV	$2A_1$	16	8	XV	D_5	3	5
V	A_3	10	8	XVI	$4A_1$	9	4
VI	$A_2 + A_1$	11	7	XVII	$2A_2 + A_1$	5	4
VII	A_4	6	8	XVIII	$A_3 + 2A_1$	5	4
VIII	$3A_1$	12	6	XIX	$A_5 + A_1$	2	4
IX	$2A_2$	7	6	XX	E_6	1	4
X	$A_3 + A_1$	7	6	XXI	$3A_2$	3	3
XI	A_5	3	6				

Table 10.1: Singular cubic surfaces

Note that the number of lines can be checked directly by using the equations. The map from $\mathbb{P}^2$ to S is given by the linear system of cubics generated by $V(G_3), V(T_1G_2), V(T_2G_2), V(T_3G_2)$. The lines are images of lines or conics which has intersection 1 with a general member of the linear system. We omit the computation of the class of the surface.

10.3 Determinantal equations

10.3.1 Cayley-Salmon equation

Let l_1, l_2, l_3 be three skew lines in $\mathbb{P}^3$. Let $\mathcal{P}_i$ be the pencil of planes through the line l_i. Let us identify $\mathcal{P}_i$ with $\mathbb{P}^1$ and consider the rational map

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 - \rightarrow \mathbb{P}^3$$

which assigns to the triple of planes (Π_1, Π_2, Π_3) the intersection point $\Pi_1 \cap \Pi_2 \cap \Pi_3$. This map is undefined at a triple (Π_1, Π_2, Π_3) such that the line $l_{ij} = \Pi_i \cap \Pi_j$ is contained in Π_k, where $\{i, j, k\} = \{1, 2, 3\}$. The line l_{ij} obviously intersects all three lines. The union of such lines is the nonsingular quadric Q containing l_1, l_2, l_3 (count parameters to convince yourself that any 3 skew lines are contained in a unique nonsingular quadric). A plane from $\mathcal{P}_i$ intersects Q along l_i and a line m_i on Q from another ruling. The triple belongs to the set fo undeterminacy locus I of f if and only if $m_1 = m_2 = m_3$. Consider the map

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow (\mathbb{P}^1)^3, \quad (\Pi_1, \Pi_2, \Pi_3) \mapsto (m_1, m_2, m_3).$$

We see that $I = \phi^{-1}(\Delta)$, where Δ is the small diagonal. Obvioulsy ϕ is an isomorphism, so I is a smooth rational curve. Let Δ_{ij} be one of the three diagonals (the locus of points with equal ith and jth coordinates). Its pre-image $D_i = \phi^{-1}(\Delta_{ij})$ is blown down under f to the line l_k. In fact, if $(\Pi_1, \Pi_2, \Pi_3) \in D_{12}$, then $m_1 = m_2$ and $\Pi_1 \cap \Pi_2 \cap \Pi_3 = m_1 \cap l_3$. Clearly, D_{12}, D_{13}, D_{23} are divisors on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1, 1, 0), (1, 0, 1), (0, 1, 1)$, respectively. The map f can now be resolved by blowing up the curve I, followed by blowing down the proper inverse transforms of the divisors D_{ij} to the lines l_k. One should compare it with the standard birational map from the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ defined by the projection from a point.

Note that in coordinates, $f(\Pi_1, \Pi_2, \Pi_3)$ is the line of solutions of a system of 3 linear equations, thus depends linearly in coefficients of each equations. This shows that the rational map is given by a linear system of divisors of degree $(1, 1, 1)$. Let S be a cubic surface containing the lines l_1, l_2, l_3. Its full pre-image under f is a divisor of degree $(3, 3, 3)$. It contains the divisors D_{12}, D_{13}, D_{23}

whose sum is the divisor of degree $(2,2,2)$. Let R be the residual divisor of degree $(1,1,1)$. It is equal to the proper inverse transform of S. Let

$$R = V\big(\sum_{i,j,k=0,1} a_{i,j,k}\lambda_i\mu_j\gamma_k\big).$$

where $(\lambda_0,\lambda_1),(\mu_0,\mu_1),(\gamma_0,\gamma_1)$ are coordinates in the pencils $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3$.

Thus we obtain

Theorem 10.3.1. *(F. August) Any cubic surface containing 3 skew lines l_1,l_2,l_3 can be generated by 3 pencils of planes in the following sense. There exists a correspondence R of degree $(1,1,1)$ on $\mathcal{P}_1\times\mathcal{P}_2\times\mathcal{P}_3$ such that*

$$S = \{x\in\mathbb{P}^3 : x\in\Pi_1\cap\Pi_2\cap\Pi_3 \textit{ for some } (\Pi_1,\Pi_2,\Pi_3)\in R\}$$

Let us rewrite R in the form

$$R = V(\lambda_0A_0(\mu_0,\mu_1,\gamma_0,\gamma_1)+\lambda_1A_1(\mu_0,\mu_1,\gamma_0,\gamma_1)),$$

where A_0,A_1 are bihomogeneous forms in (λ_0,λ_1) and (γ_0,γ_1). Suppose that S contains two distinct lines l,m which intersect l_2,l_3 but do not intersect l_1. Let $\Pi_2=\langle l,l_2\rangle,\Pi_3=\langle l,l_3\rangle$. Since any plane Π in $\mathcal{P}_1$ intersects l, the point (Π,Π_2,Π_3) is mapped to S but not contained in any divisor D_{12},D_{13},D_{23}. Thus it belongs to R. Since Π is arbitrary, the point (Π_2,Π_3) is the intersection point of the curves $V(A_0),V(A_1)$ in $\mathcal{P}_2\times\mathcal{P}_3$. This shows that the curves $V(A_0),V(A_1)$ of bi-degree $(1,1)$ intersect at two distinct points. Change coordinates μ,γ to assume that these points are $((0,1),(1,0))$ and $((1,0),(0,1))$. Plugging in the equations of $A_0=0,A_1=0$, we see that $V(A_0),V(A_1)$ belong to the pencil spanned by the curves $V(\mu_0\gamma_0),V(\mu_1\gamma_1)$. Changing the coordinates (λ_0,λ_1) we may assume that

$$R = V(\lambda_0\mu_0\gamma_0+\lambda_1\mu_1\gamma_1). \tag{10.23}$$

The surface S is the set of solutions (t_0,t_1,t_2,t_3) of the system of equations

$$\begin{aligned}\lambda_0L_1(T_0,T_1,T_2,T_3) &= \lambda_1M_1(T_0,T_1,T_2,T_3)\\ \mu_0L_2(T_0,T_1,T_2,T_3) &= \mu_1M_2(T_0,T_1,T_2,T_3)\\ \gamma_0L_3(T_0,T_1,T_2,T_3) &= \gamma_1M_3i(T_0,T_1,T_2,T_3).\end{aligned}$$

where L_i,M_i are linear forms and

$$\lambda_0\mu_0\gamma_0+\lambda_1\mu_1\gamma_1=0.$$

Multiplying the left-hand sides and the right-hand-sides, we get

$$S = V(L_1L_2L_3+M_1M_2M_3). \tag{10.24}$$

Corollary 10.3.2. *Assume additionally that S contains 2 distinct lines which intersect two of the lines l_1, l_2, l_3 but not the third one. Then S can be given by the equation*

$$L_1L_2L_3 + M_1M_2M_3 = 0. \tag{10.25}$$

An equation of cubic surface of this type is called a *Cayley-Salmon equation*.

Observe that S contains the lines $l_{ij} = V(L_i, M_j)$. Obviously, $l_{ii} = l_i$ and l_{23}, l_{32} are the two lines which intersect l_2, l_3 but not l_1. The lines l_{12}, l_{21} intersect l_1, l_2 but not l_3 (since otherwise $V(A_0), V(A_1)$ in above have more than 2 intersection points). Similarly, we see that l_{13}, l_{31} intersect l_1, l_3 but not l_2. Thus we have 9 different lines. As is easy to see they form a pair of two conjugate triads of tritangent planes (which can be defined as in the nonsingular case)

$$\begin{matrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23}. \\ l_{31} & l_{32} & l_{33} \end{matrix} \tag{10.26}$$

Thus the condition on S imposed in Corollary 10.3.2 implies that S contains a pair of conjugate triples of tritangent planes. Conversely, such a set of 9 lines gives a Cayley-Salmon equation of S. In fact, each triple of the tritangent planes defines intersect S along the same set of 9 lines. Thus S is contained in the pencil spanned by surfaces $V(L_1L_2L_3), V(M_1M_2M_3)$. It is clear that two Cayley-Salmon equations defining the same set of 9 lines can be transformed to one another by a linear change of variables. Thus the number of essentially different Cayley-Salmon equations is equal to the number of pairs of conjugate triads of tritangent planes.

Theorem 10.3.3. *Let S be a normal cubic surface. The number of different Cayley-Salmon equations for S is equal to 120 (type I), 10 (type II), 1 (type III,IV, VIII), and zero otherwise.*

Proof. We know that the number of conjugate pairs of triads of tritangent trios of exceptional vectors is equal to 120. Thus the number of conjugate triads of triples of tritangent planes on a nonsingular cubic surface is equal to 120. It follows from the proof of Corollary 10.3.2 that a pair of conjugate triples of tritangent planes on a singular surface exists only if we can find 3 skew lines and 2 lines which intersect two of them but not the third. Also we know that the number of lines on S must be at least 9. So we have to check only types $II - VI$ and $VIII$. We leave to the reader to verify the assertion in these cases. □

Corollary 10.3.4. *Let S be a nonsingular cubic surface. Then S is projectively equivalent to a surface*

$$V(T_0T_1T_2 + T_3(T_0 + T_1 + T_2 + T_3)L(T_0, \ldots, T_3)).$$

A general S can be written in this form in exactly 120 ways (up to projective equivalence).

Proof. Consider a Cayley-Salmon equation $L_1L_2L_3 + M_1M_2M_3 = 0$ of S. Let (10.26) be the corresponding 9 lines on S. If L_1, L_2, L_3, M_j are linear independent, we choose a coordinate system such that $L_1 = T_0, L_2 = T_1, L_3 = T_2, M_j = T_3$. If not, the lines l_{1j}, l_{2j}, l_{3j} intersect at one point $p_j = L_1 \cap L_2 \cap L_3$. Assume that this is not the case for all j so that S is projectively equivalent to $V(T_0T_1T_2 + T_3M_2M_3)$. Let $M_2 = \sum a_iT_i$. If one of the a_i's is equal to zero, say $a_3 = 0$, the linear form M_2 is a linear combination of coordinates T_0, T_1, T_2. We have assumed that this does not happen. Thus, after scaling the coordinates we may assume that $M_2 = \sum T_i$. this gives the promised equation. Since we can start with any conjugate pair of triads of tritangent planes, the previous assumption is not satisfied only if any such pair consists of tritangent planes containing three concurrent lines. We will see later that the number of such tritangent planes on a nonsingular surface is at most 18. So we can always start with a conjugate triad of tritangent planes for which each plain does not contain concurrent lines. □

Corollary 10.3.5. *Let S be a normal surface of type $I - IV$ or $VIII$. Then there exists a 3×3 matrix $A(T)$ with linear forms in $T_0, \ldots, T_3$ such that*

$$S = V\big(\det(A(T))\big).$$

Proof. Observe that

$$L_1L_2L_3 + M_1M_2M_3 = \det\begin{pmatrix} L_1 & M_1 & 0 \\ 0 & L_2 & M_2 \\ M_3 & 0 & L_3 \end{pmatrix}.$$

□

10.3.2 Hilbert-Burch Theorem

By other methods we will see that Corollary 10.3.5 can be generalized to any normal cubic surface of type different from XX. We will begin with the approach using the following well-known result from Commutative Algebra (see [Eisenbud]).

Theorem 10.3.6. *(Hilbert-Burch) Let I be an ideal in polynomial ring R such that* $\mathrm{depth}(I) = \mathrm{codim} I = 2$ *(thus R/I is a Cohen-Macaulay ring). Then there exists a projective resolution*

$$0 \longrightarrow R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \longrightarrow R/I \longrightarrow 0.$$

The i-th entry of the vector $(a_1, \ldots, a_n)$ defining ϕ_1 is equal to $(-1)^i c_i$, where c_i is the complementary minor obtained from the matrix A defining ϕ_2 by deleting its ith row.

We apply this theorem to the case when $R = \mathbb{C}[X_0, X_1, X_2]$ and I is the homogeneous ideal of a closed 0-dimensional subscheme Z of $\mathbb{P}^2 = \text{Proj}(R)$ generated by 4 linear independent homogeneous polynomials of degree 3. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then $(I_Z)_m = H^0(\mathbb{P}^2, \mathcal{I}_Z(m))$. By assumption

$$H^0(\mathbb{P}^2, \mathcal{I}_Z(2)) = 0. \tag{10.27}$$

Applying the Hilbert-Burch Theorem, we find a resolution of the graded ring R/I

$$0 \longrightarrow R(-4)^3 \xrightarrow{\phi_2} R(-3)^4 \xrightarrow{\phi_1} R \longrightarrow R/I \to 0,$$

where ϕ_2 is given by a 3×4 matrix $A(X)$ whose entries are linear forms in X_0, X_1, X_2. Passing to the projective spectrum, we get an exact sequence of sheaves

$$0 \longrightarrow W_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-4) \xrightarrow{\phi_2} W_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\phi_1} \mathcal{I}_Z \longrightarrow 0,$$

where W_2, W_1 are vector spaces of dimension 3 and 4. Twisting by $\mathcal{O}_{\mathbb{P}^2}(3)$, we get the exact sequence

$$0 \longrightarrow W_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\tilde{\phi}_2} W_1 \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\tilde{\phi}_1} \mathcal{I}_Z(3) \longrightarrow 0. \tag{10.28}$$

Taking global sections, we obtain $W_1 = H^0(\mathbb{P}^2, \mathcal{I}_Z(3))$. Twisting by $\mathcal{O}_{\mathbb{P}^2}(-2)$, and using a canonical isomorphism $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$, we obtain that $W_2 = H^1(\mathbb{P}^2, \mathcal{I}_Z(1))$. The exact sequence

$$0 \to \mathcal{I}_Z(1) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_Z \to 0$$

shows that

$$W_2 \cong \text{Coker}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_Z)) \cong \text{Coker}\big(\mathbb{C}^3 \to \mathbb{C}^{h^0(\mathcal{O}_Z)}\big).$$

Since $\dim W_2 = 3$, we obtain that $h^0(\mathcal{O}_Z) = 6$. Thus Z is a 0-cycle of length 6.

Now we see that the homomorphism $\tilde{\phi}_2$ of vector bundles is defined by a linear map

$$t : V \to \text{Hom}(W_2, W_1), \tag{10.29}$$

where $\mathbb{P}^2 = \mathbb{P}(E)$. We can identify the linear map t with the tensor $V^* \otimes W_2^* \otimes W_1$. Let us now view this tensor as a linear map

$$u : W_1^* \to \text{Hom}(V, W_2^*). \tag{10.30}$$

In plain language, if t is viewed as a system of 3 linear equations with unknowns t_0, t_1, t_2, t_3 whose coefficients are linear forms in variables x_0, x_1, x_2, then u is the same system rewritten as a system of 3 equations with unknowns x_0, x_1, x_2 whose coefficients are linear forms in variables t_0, t_1, t_2, t_3.

The linear map (10.29) defines a rational map

$$f : \mathbb{P}(E) \to \mathbb{P}(W_1^*) = |\mathcal{I}_Z(3)|^*, \quad [v] \mapsto \mathbb{P}(t(v)(W_2)^\perp).$$

This is the map given by the linear system $|\mathcal{I}_Z(3)|$. In coordinates, it is given by maximal minors of the matrix $A(X)$ defining ϕ_2. For any $\alpha \in t(v)(W_2)^\perp$, we have $u(\alpha)(v) = 0$. This shows that rank$(u(\alpha)) < 3$. Thus S is contained in the locus of $[\alpha]$ such that α belongs to the pre-image of the determinantal locus in Hom(V, W_2^*), i.e. the locus of linear maps of rank < 3. It is a cubic hypersurface in the space Hom(V, W_2^*). Thus the image S' of f is contained in a determinantal cubic surface S. Since the intersection scheme of two general members C_1, C_2 of the linear system $|\mathcal{I}_Z(3)|$ is equal to the 0-cycle Z of degree 6, the image of f is a cubic surface. This gives a determinantal representation of S.

Theorem 10.3.7. *Assume S is a normal cubic surface which does not have a singular point of type E_6. Then S admits a determinantal representation $S = V(\det(A))$, where A is a matrix whose entries are linear forms. A surface with a singular point of type E_6 does not admit such a representation.*

Proof. Assume S has no singular point of type E_6 and let X be a minimal resolution of S. Then the set of (-2)-curves is a proper subset of the set of roots of the lattice $K_X^\perp$ and span a proper sublattice M of Pic(X). Let α be a root in $K_X^\perp$ which does not belong to M. Since the Weyl group $W(X)$ acts transitively on the set of roots, we can choose a marking $\phi : \mathbb{Z}^{1,6} \to$ Pic(X) such that $\alpha = 2e_0 - e_1 - \ldots - e_6$. Let $w \in W(X)^n$ be a an element of the Weyl group generated by reflections with respect to (-2)-curves such that $w \circ \phi$ is a geometric marking defining a geometric basis $e_0', e_1', \ldots, e_6'$. Since w preserves M, $w(\alpha) = 2e_0' - e_1' - \ldots - e_6'$ is not a linear combination of (-2)-curves. However any effective root x is a linear combination of (-2)-curves (use that $x \cdot K_X) = 0$ and for any irreducible component E of x with $E^2 \neq -2$ we have $E \cdot K_X < 0$). Now X is obtained by blowing up a set of 6 points (maybe infinitely near) not lying on a conic. It is easy that this blow-up is isomorphic to the blow-up of a 0-dimensional cycle Z of length 6. Blowing up a sequence of k infinitely near points $p_k \succ_1 \ldots \succ_1 p_1$ is the same as to blow up the ideal (x, y^k). The linear system $|\mathcal{I}_Z(3)|$ is equal to the linear system of cubics through the points $p_1, \ldots, p_6$. The ideal I_Z is generated by a basis of the 3-dimensional linear system $|\mathcal{I}_Z(3)|$ defining a rational map $\mathbb{P}^2 - \to S \subset \mathbb{P}^3$. Thus we can apply the Hilbert-Burch Theorem to obtain a determinant representation of S.

Assume S has a singular point of type E_6 and $A(T) = (A_{ij})_{1\le i,j\le 3}$. Consider the system of linear equations

$$\sum_{j=1}^{3} L_{ij}(t_0,\dots,t_3)x_j = 0, \quad i = 1,2,3. \tag{10.31}$$

For any $x = (x_0,x_1,x_2) \in \mathbb{P}^2$ the set of points $p = (t_0,t_1,t_2,t_3)$ such that $A(t)x = 0$ is a linear space. Consider the rational map $\pi : S \to \mathbb{P}^2$ which assigns to $t \in S$ the solution x of $A(t)x = 0$. Since π is not bijective, there exists a line l on S which is blown down to a point (a_1,a_2,a_3). This means that the equations (10.31) with (x_0,x_1,x_2) substituted with (a_1,a_2,a_3) define three planes intersecting along a line. Thus the three planes are linearly dependent, hence we can write

$$\alpha(a_j\sum_{j=1}^{3} L_{1j}) + \beta(a_j\sum_{j=1}^{3} L_{2j}) + \gamma(\sum_{j=1}^{3} L_{3j})$$
$$= \sum_{j=1}^{3} a_j(\alpha L_{1j} + \beta L_{2j} + \gamma L_{3j}) = 0.$$

for some α,β,γ not all zeros. Choose coordinates for x such that $(a_1,a_2,a_3) = (1,0,0)$. Then we obtain that the entries in the first column of $A(t)$ are linearly dependent. This allows us to assume that $L_{11} = 0$ in the matrix $A(t)$. The equations $L_{21} = L_{31} = 0$ define the line l. The equations $L_{12} = L_{13} = 0$ define a line m. Obviously, $l \neq m$ since otherwise S has equation

$$-L_{12}L_{21}L_{33} + L_{12}L_{31}L_{23} + L_{13}L_{21}L_{32} - L_{13}L_{31}L_{23} = 0,$$

which shows that the line $l = m$ is the double line of S. So, we see that S has at least two lines, but a surface of type XX has only one line. □

We have already seen that each time S is represented as the image of S under a rational map given by the linear system $|\mathcal{I}_Z(3)|$, where Z is a 0-cycle of length 6 satisfying condition (10.27), we can write S by a determinantal equation. A minimal resolution of indeterminacy points defines a blowing down morphism $\pi : X \to \mathbb{P}^2$ of the Del Pezzo surface X isomorphic to a minimal resolution of S. The inverse map is given by assigning to $t \in S$ the nullspace $N(A(t))$. Changing Z to a projectively equivalent set replaces the matrix $A(t)$ by a matrix $A(t)C$, where C is an invertible scalar matrix. This does not change the equation of S. Thus the number of essentially different determinantal representations is equal to the number of linear systems $|e_0|$ on X which define a blowing down morphism $X \to \mathbb{P}^2$ such that the corresponding geometric markings $(e_0,e_1,\dots,e_6)$ of X satisfy $|2e_0 - e_1 - \dots - e_6| = \emptyset$. This gives

Theorem 10.3.8. *The number of essentially different determinantal representations of S is equal to the number of unordered geometric markings $(e_0, e_1, \ldots, e_6)$ of* $\mathrm{Pic}(X)$ *such that $|2e_0 - e_1 - \ldots - e_6| = \emptyset$. It is equal to 72 if S is nonsingular.*

Consider again (10.29) as a tensor $t \in W_1 \otimes V^* \otimes W_2^*$ which defines a linear map $t : W_1^* \to \mathrm{Hom}(W_2, V^*)$. We have the corresponding rational map

$$g : S \to \mathbb{P}(W_2) \cong \mathbb{P}^2, \quad [w] \mapsto \mathrm{Ker}(t^*).$$

Let $\mathbb{P}^3 = \mathbb{P}(W_1^*)$. Consider the projective resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \otimes W_2 \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^3} \otimes V^* \longrightarrow \mathcal{F} \to 0,$$

where ϕ is defined by the linear map viewed as a 3×3 matrix with entries in $W_1 = (W_1^*)^*$. Since the determinant of the matrix is equal to the equation of S, the sheaf $\mathcal{F}$ is locally isomorphic to $\mathcal{O}_S$. We can write it as $\mathcal{F} = \mathcal{O}_S(D)$ for some divisor class D. Taking global sections, we obtain $H^0(S, \mathcal{O}_S(D)) \cong V^*$. Thus the linear system $|D|$ on S defines our rational map $\pi : S \to \mathbb{P}(E)$. Twisting by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and taking cohomology, we obtain an isomorphism $W_2 \cong H^2(S, \mathcal{O}_S(-3H + D))$, where H is a hyperplane section. Since $\mathcal{O}_S(H) \cong \mathcal{O}_S(-K_S)$, we obtain an isomorphism

$$W_2 \cong H^2(S, \mathcal{O}_S(3K_S + D)) \cong H^0(S, \mathcal{O}_S(-2K_S - D))^*.$$

Let $\pi' : X \to \mathbb{P}(W_2)$ be the rational map of a minimal resolution defined by g. The pre-image of D on X is equal to the class e_0 and $-K_S = 3e_0 - e_1 - \ldots - e_6$, where $(e_0, e_1, \ldots, e_6)$ is a geometric marking of X defined by the blowing down morphism π'. Thus π' is given by the linear system $|5e_0 - 2e_1 - \ldots - 2e_6|$. If S is nonsingular, π blows down 6 skew lines $l_1, \ldots, l_6$ to 6 points $p_1, \ldots, p_6$on $\mathbb{P}(E)$ and the map $g = \pi'$ blows down the proper inverse transforms $m_1, \ldots, m_6$ of conics C_j through the points $p_i, i \neq j$ to some points $q_1, \ldots, q_6$ in $\mathbb{P}(W_2)$. The lines $(l_1, \ldots, l_6; m_1, \ldots, m_6)$ form a double-six on S. It follows from above, that the lines $(m_1, \ldots, m_6)$ define a determinantal representation of S corresponding to the transpose of the matrix $A(T)$.

Remark 10.3.1. We can also deduce Theorem 10.3.8 from the theory of determinantal representations from Chapter "Determinantal equations" from Part I. Applying this theory we obtain that S admits a determinantal equation with entries linear forms if it contains a projectively normal curve C such that

$$H^0(S, \mathcal{O}_S(C)(-1)) = H^2(S, \mathcal{O}_S(C)(-2)) = 0. \tag{10.32}$$

Moreover, the set of non-equivalent determinantal representation is equal to the set of divisor classes of such curves. Let $f : X \to S$ be a minimal resolution and $C' = f^*(C)$. Since $f^*(\mathcal{O}_S(-1) = \mathcal{O}_X(K_X)$, the conditions (10.32) are equivalent to

$$H^0(X, \mathcal{O}_X(C' + K_X)) = H^2(X, \mathcal{O}_X(C' + 2K_X)) = 0. \tag{10.33}$$

Since C' is nef, $H^1(X, \mathcal{O}_X(C' + K_X)) = 0$. Also $H^2(X, \mathcal{O}_X(C' + K_X)) = H^0(X, \mathcal{O}_X(-C')) = 0$. By Riemann-Roch,

$$\begin{aligned} 0 = \chi(X, \mathcal{O}_X(C' + K_X)) &= \tfrac{1}{2}((C' + K_X)^2 - (C' + K_X) \cdot K_X) + 1 \\ &= \tfrac{1}{2}(C'^2 + C' \cdot K_X) + 1. \end{aligned}$$

Thus C' is a smooth rational curve, hence C is a smooth rational curve. As we know from Chapter ??, a projectively normal rational curve in $\mathbb{P}^3$ must be of degree 3. Thus $-K_X \cdot C' = 3$, hence $C'^2 = 1$. The linear system $|C'|$ defines a birational map $\pi : X \to \mathbb{P}^2$. Let $e_0 = [C'], e_1, \ldots, e_6$ be the corresponding geometric basis of Pic(X). We have $K_X = -3e_0 + e_1 + \ldots + e_6$ and the condition

$$0 = H^2(X, \mathcal{O}_X(C' + 2K_X)) = H^0(X, \mathcal{O}_X(-C' - K_X)) = 0$$

is equivalent to

$$|2e_0 - e_1 - \ldots - e_6| = \emptyset. \tag{10.34}$$

10.3.3 The cubo-cubic Cremona transformation

Consider a system of linear equations in variables t_0, t_1, t_2, t_3

$$\sum_{j=1}^{3} L_{ij}(Z_0, Z_1, Z_2, Z_3)T_j = 0, i = 1, 2, 3, 4, \tag{10.35}$$

where L_{ij} are linear forms in variables Z_0, Z_1, Z_2, Z_3. It defines a rational map $\Phi : \mathbb{P}^3 - \to \mathbb{P}^3$ by assigning to $(z_0, \ldots, z_3)$ the space of solutions of the system (10.35) with z_i substituted in Z_i. In coordinate-free way, we can consider the system as a linear map $t : W \to \text{Hom}(W_1, W_2)$, where $\dim W = 4, \dim W_1 = 4, \dim W_2 = 3$. The map $\Phi : \mathbb{P}(W) \to \mathbb{P}(W_1)$ is defined by sending $w \in W$ to the linear space Ker$(t(v)) \subset W_1$. The inverse map Φ^{-1} is defined by rewriting the system as a system with unknowns Z_i. Or, in corrdinate-free language, by viewing the tensor $t \in W^* \otimes W_1^* \otimes W_2$ as a linear map $W_1 \to \text{Hom}(W, W_2)$. Let D be the set of linear maps (considered up to proportionality) $W_1 \to W_2$ of rank ≤ 2. The map Φ is not defined at the pre-image of D in $\mathbb{P}(W)$. It is given by the common zeros of the four maximal minors Δ_i of the matrix $(L_{ij}(Z))$. The map Φ is given by

$$(z_0, \ldots, z_3) \mapsto (\Delta_1, -\Delta_2, \Delta_3, -\Delta_4).$$

Lemma 10.3.9. *The scheme-theoretical locus Z of common zeros of the cubic polynomials Δ_i is a connected curve of degree 6 and arithmetic genus 3.*

Proof. We apply the Hilbert-Burch theorem to the ring $R = \mathbb{C}[Z_0, Z_1, Z_2, Z_3]$ and the homogeneous ideal I of Z. We get a resolution

$$0 \longrightarrow R(-4)^3 \xrightarrow{\phi_2} R(-3)^4 \xrightarrow{\phi_1} I \longrightarrow 0. \tag{10.36}$$

Twisting by n and computing the Euler-Poincaré characteristic we obtain the Hilbert polynomial of the scheme Z

$$P(Z; n) = \chi(\mathbb{P}^3, \mathcal{O}_Z(n)) = 6n - 2.$$

This shows that Z is one-dimensional, and comparing with Riemann-Roch, we see that $\deg(Z) = 6$ and $\chi(\mathcal{O}_Z) = -2$. The exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Z \to 0$$

gives $\dim H^0(\mathcal{O}_Z) = 1$ if and only if $H^1(\mathbb{P}^3, \mathcal{I}_Z) = 0$. The latter equality follows from considering the resolution of $\mathcal{I}_Z$. Thus Z is connected and $p_a(Z) = 1 - \chi(\mathcal{O}_Z) = 3$. $\square$

The inverse map is also given by cubic polynomials. This explains the classical name for the transformation T. The pre-images of planes under Φ are cubic surfaces in $\mathbb{P}(W)$ containing the curve Z. The images of planes under Φ are cubic surfaces in $\mathbb{P}(W_1)$ containing the curve Z', defined similarly to Z for the inverse map Φ^{-1}.

Now let V be a 3-dimensional subspace of W. Then restricting $t : W \to \mathrm{Hom}(W_1, W_2)$ to V we obtain a determinantal representation of the cubic surface $S = \Phi(\mathbb{P}(E)) \subset \mathbb{P}(W_1)$. The map $\Phi : \mathbb{P}(E) - \to \mathbb{P}(W_1)$ is not defined at the set $\mathbb{P}(E) \cap Z$. Tensoring (10.36) with $\mathcal{O}_{\mathbb{P}(E)}$ we obtain a projective resolution for the ideal sheaf of $Z \cap \mathbb{P}(E)$ in $\mathbb{P}(E)$ (use that $\mathrm{Tor}_1(R/J, I) = 0$, where J is the ideal generated by the hyperplane in W defining V see [Eisenbud], Exercise A3.16). If $\mathbb{P}(E)$ intersects Z transversally at 6 points, we see that S is a nonsingular cubic surface.

10.3.4 Cubic symmetroids

A *cubic symmetroid* is a cubic surface admitting a representation as a symmetric (3×3)-determinant. An example of a cubic symmetroid is a 4-nodal cubic surface

$$T_0T_1T_2 + T_0T_1T_3 + T_0T_2T_3 + T_1T_2T_3 = \det \begin{pmatrix} T_0 & 0 & T_2 \\ 0 & T_1 & -T_3 \\ -T_3 & T_3 & T_2 + T_3 \end{pmatrix}$$

Note that the condition for a determinantal representation of a cubic surface with canonical singularities to be a symmetric determinantal representation is the existence of an isomorphism

$$\mathcal{O}_S(C) \cong \mathcal{O}_S(C)(2).$$

This is obviously impossible for a nonsingular cubic surface.

Lemma 10.3.10. *Let $L \subset |\mathcal{O}_{\mathbb{P}^2}(2)$ be a pencil of conics. Then it is projectively isomorphic to one of the following pencils:*

(i) $\lambda(T_0T_1 - T_0T_2) + \mu(T_1T_2 - T_0T_2) = 0$*;*

(ii) $\lambda(T_0T_1 + T_0T_2) + \mu T_1T_2 = 0$*;*

(iii) $\lambda T_2^2 + \mu(T_0T_1 + T_0T_2 + T_1T_2) = 0$*;*

(iv) $\lambda T_2(T_2 - T_0) + \mu T_1(T_0 + T_2) = 0$*;*

(v) $\lambda T_0^2 + \mu(T_0T_2 + T_1^2) = 0$*;*

(vi) $\lambda T_0^2 + \mu T_1^2 = 0$*;*

(vii) $\lambda T_0T_1 + \mu T_0T_2 = 0$*;*

(viii) $\lambda T_0T_1 + \mu T_0^2 = 0$*.*

Proof. Let $C_1 = V(F_1), C_2 = V(F_2)$ be two generators of a pencil. If C_1 and C_2 have a common irreducible component we easily reduce it, by a projective transformation to cases (vii) or (viii). Assume now that C_1 do not have a common component. Let $k = \#C_1 \cap C_2$.

Assume $k = 4$. Then, no three of the intersection points lie on a line since otherwise the line is contained in both conics. By a linear transformation we may assume that the intersection points are $(1,0,0), (0,1,0), (0,0,1), (1,1,1)$. The linear system of conics passing through these points is given in (i).

Assume $k = 3$. After a linear change of variables we may assume that C_1, C_2 are tangent at $(1,0,0)$ with tangency direction $T_1 + T_2 = 0$ and intersect transversally at $(0,1,0)$ and $(0,0,1)$. The linear system of conics passing through the three points is $\lambda T_0T_1 + \mu T_0T_2 + \gamma T_1T_2 = 0$. The tangency condition gives $\lambda = \mu$. This gives case (ii).

Assume $k = 2$. Let $(1,0,0)$ and $(0,1,0)$ be the base points. First we assume that C_1 and C_2 are tangent at both points. Obviously, one of the conics from the pencil is the double line $T_2^2 = 0$. We can also fix the tangency directions to be $T_1 + T_2 = 0$ at $(1,0,0)$ and $T_0 + T_2 = 0$ at $(0,1,0)$. The other conic could be $T_0T_1 + T_0T_2 + T_1T_2 = 0$. This gives case (iii).

Now we assume that C_1 and C_2 intersect transversally at $(1,0,0)$ and with multiplicity 3 at $(0,1,0)$ with tangency direction $T_0+T_2=0$. A conic passing through $(1,0,0)$ and tangent to the line $T_0+T_2=0$ at the point $(0,1,0)$ has equation $aT_2^2+b(T_0T_1+T_1T_2)+dT_0T_2=0$. It is easy to check that the condition of triple tangency is $a+d=0$. This gives case (iv).

Finally assume that $k=1$. Obviously the pencil is spanned by a conic and its tangent line taken with multiplicity 2. By a projective transformation it is reduced to form given in case (v) if the conic is irreducible and case (vi) if the conic is a double line . □

Theorem 10.3.11. *Let S be an irreducible cubic symmetroid. Assume that S has only canonical singularities. Then S is projectively isomorphic to one of the following determinantal surfaces:*

(i) $\mathcal{C}_3 = V(T_0T_1T_2+T_0T_1T_3+T_0T_2T_3+T_1T_2T_3)$ *with four RDP of type A_1;*

(ii) $\mathcal{C}_3' = V(T_0T_1T_2+T_1T_3^2-T_2T_3^2)$ *with two RDP of type A_1 and one RDP of type A_3;*

(iii) $\mathcal{C}_3'' = V(T_0T_1T_2-T_3^2(T_0+T_2)-T_1T_2^2)$ *with one RDP of type A_1 and one RDP of type A_5.*

Proof. Let $A=(L_{ij})$ be a symmetric 3×3 matrix with linear entries $L_{ij}(t_0,t_1,t_2,t_3)$ defining the equation of S. It can be written in the form $A(t)=t_0A_0+t_1A_1+t_2A_2+t_3A_3$, where $A_i, i=1,2,3,4,$ are symmetric 3×3 matrices. Let W be a linear system of conics spanned by the conics

$$C_i = (T_0,T_1,T_2)\cdot A\cdot \begin{pmatrix} T_0\\ T_1\\ T_2\end{pmatrix} = 0.$$

The matrices A_i are linearly independent since otherwise $S=V(\det A(t))$ is a cone with vertex (c_0,c_1,c_2,c_3), where $\sum c_iA_i=0$. Thus W is a web of conics. Let $\mathbb{P}^2=\mathbb{P}(E)$ so that $W=\mathbb{P}(\bar{W})$ for a 4-dimensional subspace of S^2E^*. Consider the polarity $S^2E\cong (S^2E^*)^*$. Then the projectivization of the dual of $\bar{W}$ is a pencil L of apolar conics in dual projective space $\mathbb{P}(E^*)$. Since the apolarity is equivariant with respect to the representation of SL(3) in S^2E and in S^2E^*, we see that we may assume that L is given in one of the cases from the previous lemma. Here we have to replace the unknowns T_i with the differential operators ∂_i. We list the corresponding dual 4-dimensional spaces of quadratic forms.

(i) $t_0T_0^2+t_1T_1^2+t_2T_2^2+2t_3(T_0T_2+T_1T_2+T_1T_2)=0;$

(ii) $t_0T_0^2 + t_1T_1^2 + t_2T_2^2 + 2t_3(T_0T_1 - T_0T_2) = 0;$

(iii) $t_0T_0^2 + t_1T_1^2 + 2t_2(T_0T_1 - T_1T_2) + 2t_3(T_0T_2 - T_1T_2) = 0;$

(iv) $t_0T_0^2 + t_1T_1^2 + 2t_2(T_2^2 + T_0T_2) + 2t_3(T_1T_2 - T_0T_1) = 0;$

(v) $t_0(2T_0T_2 - T_1^2) + t_1T_2^2 + 2t_2T_0T_1 + 2t_3T_1T_2 = 0;$

(vi) $t_0T_2^2 + 2t_1T_0T_1 + 2t_2T_1T_2 + 2t_3T_0T_2 = 0;$

(vii) $t_0T_0^2 + t_1T_1^2 + t_2T_2^2 + 2t_3T_0T_1 = 0;$

(viii) $t_0T_1^2 + t_1T_2^2 + 2t_2T_0T_2 + 2t_3T_1T_2 = 0.$

The corresponding determinantal varieties are the following.

(i)

$$\det\begin{pmatrix} t_0 & t_3 & t_3 \\ t_3 & t_1 & t_3 \\ t_3 & t_3 & t_2 \end{pmatrix} = t_0t_1t_2 + t_3^2(-t_0 - t_2 - t_1 + t_3) = 0.$$

It has 4 singular points $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,1,1)$ and hence is isomorphic to the 4-nodal cubic surface from case (i).

(ii)

$$\det\begin{pmatrix} t_0 & t_3 & -t_3 \\ t_3 & t_1 & 0 \\ -t_3 & 0 & t_2 \end{pmatrix} = t_0t_1t_2 + t_1t_3^2 - t_2t_3^2 = 0.$$

It has 2 ordinary nodes $(0,1,0,0), (0,0,1,0)$ and a RDP $(1,0,0,0)$ of type A_3.

(iii)

$$\det\begin{pmatrix} t_0 & t_2 & t_3 \\ t_2 & t_1 & -2(t_2+t_3) \\ t_3 & -2(t_2+t_3) & 0 \end{pmatrix} = 4(t_2+t_3)^2t_0+(t_2+t_3)t_2t_3+t_1t_3^2 = 0.$$

The surface has a double line given by $t_3 = t_2 + t_3 = 0$. This case is excluded.

(iv)

$$\det\begin{pmatrix} t_0 & -t_3 & t_2 \\ -t_3 & t_1 & t_3 \\ t_2 & t_3 & 2t_2 \end{pmatrix} = 2t_0t_1t_2 - t_3^2(t_0 + 4t_2) - t_1t_2^2 = 0.$$

The surface has one ordinary double point $(1,0,0,0)$ and one RDP $(0,1,0,0)$ of type A_5.

(v)

$$\det\begin{pmatrix}0 & t_2 & t_0\\ t_2 & -t_0 & t_3\\ t_0 & t_3 & t_1\end{pmatrix} = t_1t_2^2 + 2t_0t_2t_3 + t_0^3 = 0.$$

The surface has a double line $t_0 = t_2 = 0$. This case has to be excluded.

(vi)

$$\det\begin{pmatrix}0 & t_1 & t_3\\ t_1 & 0 & t_2\\ t_3 & t_2 & t_0\end{pmatrix} = -t_1(t_0t_1 + 2t_2t_3) = 0.$$

This surface is reducible, the union of a plane and a nonsingular quadric.

(vii)

$$\det\begin{pmatrix}t_0 & 0 & 0\\ 0 & t_1 & t_3\\ 0 & t_3 & t_2\end{pmatrix} = t_0(t_1t_2 - t_3^2) = 0.$$

The surface is the union of a plane and a quadratic cone.

(viii)

$$\det\begin{pmatrix}0 & 0 & t_2\\ 0 & t_0 & t_3\\ t_2 & t_3 & t_1\end{pmatrix} = t_0t_2^2 = 0.$$

The surface is a cone.

□

Remark 10.3.2. If S is a cone over a plane cubic curve C. Then S admits a symmetric determinantal representation if and only if C admits such a representation. We refer to Chapter ? for determinantal representations of plane cubics.

If S is irreducible non-normal surface, then S admits a symmetric determinantal representation. This corresponds to cases (iii) and (v) from the proof of the previous theorem. Case (iii) gives a surface isomorphic to the surface from case (i) of Theorem 9.1.2 and case (v) gives a surface isomorphic to the surface isomorphic to the surface from case (ii) of Theorem 9.1.2. We also see that a reducible cubic surface which is not a cone admits a symmetric determinantal representation only if it is the union of an irreducible quadric and a plane which intersects the quadric transversally.

10.4 Representations as sums of cubes

10.4.1 Sylvester's pentahedron

Counting constants we see that it is possible that a general homogeneous cubic form in 4 variables can be written as a sum of 5 cubes of linear forms in finitely many ways. Since there are no cubic surfaces singular at 5 general points, the theory of apolarity tells us that the count of constants gives a correct answer. The following result of J. Sylvester gives more:

Theorem 10.4.1. *A general homogeneous cubic form in 4 variables can be written uniquely as a sum*

$$F = L_1^3 + L_2^3 + L_3^3 + L_4^3 + L_5^3,$$

where L_i are linear forms in 4 variables.

Proof. Suppose

$$F = \sum_{i=1}^{5} L_i^3 = \sum_{i=1}^{5} M_i^3.$$

Let x_i, y_i be the points in $\check{\mathbb{P}}^3$ corresponding to the hyperplanes $V(L_i), V(M_i)$. Consider the linear system of quadrics in $\check{\mathbb{P}}^3$ which pass through the points x_5, $y_1, \ldots, y_5$. If x_5 is not equal to any y_j, this is a linear projective subspace of dimension 3. Applying the corresponding differential operators to F we find 4 linearly independent relations between the linear forms L_1, L_2, L_3, L_4. This shows that the points x_1, x_2, x_3, x_4 are coplanar. It does not happen for general F. Thus we may assume that $x_5 = y_5$, so that we can write $M_5 = \lambda_5 L_5$ for some λ_5. After subtraction, we get

$$\sum_{i=1}^{4} L_i^3 + (1 - \lambda_5^3) L_5 = \sum_{i=1}^{4} M_i^3.$$

Now we consider quadrics through y_1, y_2, y_3, y_4. This is defined by 6-dimensional linear space. Its elements define linear relations between the forms $L_1, \ldots, L_5$. Since the dimension of the linear span of these forms is equal to 4 (the genericity assumption), we obtain that there exists a 4-dimensional linear system of quadrics in $\check{\mathbb{P}}^3$ vanishing at $x_1, \ldots, x_5, y_1, \ldots, y_4$. Assume that all of these points are distinct. Two different quadrics from the linear system intersect along a curve B of degree 4. If this curve is irreducible, a third quadric intersect it at ≤ 8 points. So, the curve B must be reducible. Recall that the linear system of quadrics through an irreducible curve of degree 3 is of dimension 2. Thus, one of the quadrics in our linear system must contain a curve of degree ≤ 2 and so the base locus contains a curve of degree ≤ 2. The dimension of a linear system of quadrics containing an irreducibel conic is of dimension 4 and its base locus is equal to the conic. Since the points x_i's are not coplanar, we see that this case does not occur. Assume that the base locus contains a line. Then 3 linearly independent quadrics intersect along a line l and a cubic curve R. A forth quadric will intersect R at ≤ 4 points outside l. Thus we have at most 4 points among $x_1, \ldots, x_5, y_1, \ldots, y_4$ which do not lie on l. This implies that 5 points lie on the line l. Since no three points among the x_i's and y_j's can lie on a line, we obtain a contradiction.

Thus one of the x_j's coincide with some y_i. We may now assume that $M_4 = \lambda_4 L_4$ and get

$$\sum_{i=1}^{3} L_i^3 + (1-\lambda_4^3)L_4^3 + (1-\lambda_5^3)L_5^3 = \sum_{i=1}^{3} M_i^3.$$

Take a plane through y_1, y_2, y_3 and get a linear dependence

$$a_1 L_1^2 + a_2 L_2^3 + a_3 L_3^2 + a_4(1-\lambda_4^3)L_4^2 + a_5(1-\lambda_5^2)L_5^2.$$

Here $a_4, a_5 \neq 0$ since otherwise the points y_1, y_2, y_3 and $x_4 = y_4$ or $x_4 = y_5$ are coplanar. A linear dependence between squares of linear forms means that the corresponding points in the dual space do not impose independent conditions on quadrics. The subvariety of $(\mathbb{P}^3)^5$ of such 5-tuples is a proper closed subset. By generality assumption we may assume that our set of 5 points $x_1, \ldots, x_5$ is not in this variety. Now we get $\lambda_4^3 = \lambda^3 = 1$ and

$$\sum_{i=1}^{3} L_i^3 = \sum_{i=1}^{3} M_i^3.$$

A plane through y_1, y_2, y_3 gives a linear relation between L_1^2, L_2^2, L_3^2 which as we saw before must be trivial. Thus the points $x_1, x_2, x_3, y_1, y_2, y_3$ lie in the same plane Π. A linear system $\mathcal{Q}$ of quadrics in $\mathbb{P}^3$ through 3 non-collinear points does not have unassigned base points (i.e. its base locus consists of the three points). Since linear relations between L_1, L_2, L_3 form a one-dimensional linear space, $\mathcal{Q}$ contains a hyperplane of quadrics vanishing at additional 3 points x_1, x_2, x_3. Thus the linear system of quadrics through a set of 4 points x_1, x_2, x_3, y_i or y_1, y_2, y_3, x_i has 2 unassigned base points lying in the same plane. By restricting the linear system to the plane we see that this is impossible unless all 6 points are collinear. This is excluded by the generality assumption. This final contradiction shows that $M_i = \lambda_i L_i$ and $\sum_{i=1}^{3}(1-\lambda_i^3)L_i^3 = 0$. Since a general form cannot be written as a sum of 4 cubes, we get $\lambda_i^3 = 1$ and $L_i^3 = M_i^3$ for all $i = 1, \ldots, 6$. $\square$

Assume that $F = \sum_{i=1}^5 L_i^3$, where the linear forms L_i are linearly independent. Let $x_1, \ldots, x_5$ be the corresponding points in the dual $\mathbb{P}^3$. We can apply a projective transformation to assume that

$$x_1 = (1,0,0,0), x_2 = (0,1,0,0), x_3 = (0,0,0,1), x_4 = (0,0,0,1), x_5 = (1,1,1,1).$$

This implies that $V(F)$ is projectively isomorphic to the cubic surface

$$V(\lambda_0 T_0^3 + \lambda_1 T_1^3 + \lambda_2 T_2^3 + \lambda_3 T_3^3 + \lambda_4 (T_0 + T_1 + T_2 + T_3)^3).$$

By Sylvester's Theorem the choice of this equation depends only on the order of the linear forms L_i. This shows that the moduli space of general cubic surfaces (defined as the orbit space of some open Zariski subset of cubic surfaces with respect to the action of the projective linear group of $\mathbb{P}^3$) is birationally isomorphic to $\mathbb{C}^5/S_5 \times \mathbb{C}^*$. Consider the natural map $\mathbb{C}^5 \to \mathbb{C}^5$ given by the elementary symmetric functions $s_1, \ldots, s_5$. It shows that $\mathbb{C}^5/S_5 \cong \mathbb{C}^5$ and

$$\mathbb{C}^5/S_5 \times \mathbb{C}^* \cong \mathbb{P}(1,2,3,4,5). \tag{10.37}$$

This gives

Corollary 10.4.2. *The moduli space of cubic surfaces is a rational variety of dimension 4.*

Remark 10.4.1. One can construct the moduli space $\mathcal{M}$ of cubic surfaces using geometric invariant theory. It is defined as the projective spectrum of the graded ring of invariants

$$R = \bigoplus_{n=0}^{\infty} S^n(S^3(\mathbb{C}^4)^*)^*)^{\mathrm{SL}(4)}.$$

The explicit computation of this ring made by Clebsch and Salmon shows that this ring is generated by invariants of degree $8, 16, 24, 32, 40, 100$, where the square of the last invariant is a polynomial in the the first 5 invariants. This easily shows that the graded subalgebra $R^{[2]}$ of R generated by the elements of even degree is freely generated by the first 5 invariants. This gives an isomorphism

$$\mathcal{M} \cong \mathbb{P}(1,2,3,4,5).$$

One should compare it with isomorphism (10.37), which gives only a birational model of $\mathcal{M}$. I have no explanation for this.

10.4.2 The Hessian surface

The Sylvester theorem gives the equation of the Hessian surface of a general cubic surface.

Definition 10.4. *Let $S = V(F)$ be a general cubic surface and $F = \sum_{i=1}^5 L_i^3$ be its equation. The set of 5 planes $V(L_i)$ is called the* Sylvester pentahedron. *The points $V(L_i, L_j, L_k), 1 \leq i < j < k \leq 5$, are called the* vertices, *and the lines $V(L_i, L_j), 1 \leq i < j \leq 5$, are called the* edges.

Theorem 10.4.3. *Let $S = V(F)$ be a general cubic surface and* $\mathrm{He}(S)$ *be the Hessian surface of S. Assume $F = \sum_{i=1}^5 L_i^3$. Then* $\mathrm{He}(S)$ *contains the edges of the Sylvester pentahedron, and the vertices are its ordinary double points. The equation of* $\mathrm{He}(S)$ *can be written in the form*

$$L_1 L_2 L_3 L_4 L_5 \sum_{i=1}^5 \frac{a_i^2}{L_i} = 0,$$

where $\sum_{i=1}^5 a_i L_i = 0$.

Proof. Recall that

$$\mathrm{He}(S) = \{x \in \mathbb{P}^3 : P_x(F) \text{ is singular }\}.$$

For any point $x \in V(L_i, L_j)$ we have $P_x(F) = \sum_{k \neq i,j} \lambda_k L_k^2$. This is a quadric of rank ≤ 3. Thus each edge is contained in $\mathrm{He}(S)$. Since each vertex lies in 3 non-coplanar edges it must be a singular point. Observe that any edge contains 3 vertices. Any quartic containing 10 vertices and two general points on each edge contains the 10 edges. Thus the linear system of quartics containing 10 edges is of dimension $34 - 30 = 4$. Obviously any quartic with equation $\sum_{i=1}^5 \frac{\lambda_i}{L_i} = 0$ contains the edges. Thus the equation of $\mathrm{He}(S)$ can be written in this form. We derive the same conclusion in another way which will also allow us to compute the coefficients λ_i.

Consider the isomorphism from S to a surface in $\mathbb{P}^4$ given by 2 equations

$$\sum_{i=1}^5 z_i^3 = \sum_{i=1}^5 a_i z_i = 0.$$

This isomorphism is given by the map $\mathbb{P}^3 \to \mathbb{P}^4$ defines by the linear forms L_i. The polar quadric $V(P_x(F))$ is given by 2 equations in $\mathbb{P}^4$

$$\sum L_i(a) z_i^2 = \sum_{i=1}^5 a_i z_i = 0.$$

It is singular if and only if the matrix

$$\begin{pmatrix} L_1(x)z_1 & L_2(x)z_2 & L_3(x)z_3 & zL_4(x)z_4 & L_5(x)z_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}$$

is of rank 1 at some point $z = (z_1, \ldots, z_5) = (L_1(t), \ldots, L_5(t))$. This can be expressed by the equalities $L_i(x) = \lambda a_i / L_i(t), i = 1, \ldots, 5$. Since $\sum_{i=1}^5 a_i L_i(x) = 0$, we obtain

$$0 = \sum_{i=1}^{5} a_i L_i(x) = \sum_{i=1}^{5} a_i^2 / L_i(t).$$

This gives the asserted equation of He(S). □

Remark 10.4.2. Recall that the Hessian of any cubic hypersurface admits a birational automorphism σ which assigns to the polar quadric of corank 1 its singular point. Let X be a minimal nonsingular model of He(S). It is a K3 surface. The birational automorphism σ extends to a biregular automorphism of X. It exchanges the proper inverse transforms of the edges with the exceptional curves of the resolution. One can show that for a general S, the automorphism of X has no fixed points, and hence the quotient is an Enriques surface.

Remark 10.4.3. Not every cubic surface, even a nonsingular one, can be written by a Sylvester equation. For example, consider the Ferma surface $V(\sum_{i=1}^3 T_i^3)$. Its Hessian is $V(T_0T_1T_2T_3)$ and it does not contain the edges of a pentahedron.

10.4.3 Cremona's hexahedral equations

It follows from Sylvester's Theorem that a general cubic surface is isomorphic to a hyperplane section of the Fermat cubic hypersurface in $\mathbb{P}^4$. This defines a dominant rational map from $\mathbb{P}^4$ to the moduli space $\mathcal{M}_{\text{nscub}}$ of nonsingular cubic surfaces (which exists a geometric quotient of the corresponding open subset of cubic surfaces). It follows from the proof of Corollary 10.4.2 that this map is birational. Cremona's hexahedral equations which we discuss here allows one to define a regular map of degree 36 from an open Zariski subset U of $\mathbb{P}^4$ to $\mathcal{M}_{\text{nscub}}$. Its fibres can be viewed as a choice of a double-six on the surface.

Theorem 10.4.4. *(L. Cremona) Assume that a cubic surface $V(F)$ is not a cone and admits a Cayley-Salmon equation (e.g. $V(F)$ is a nonsingular surface). Then $V(F)$ is isomorphic to a cubic surface in $\mathbb{P}^5$ given by the equations*

$$\sum_{i=1}^{6} Z_i^3 = \sum_{i=1}^{6} Z_i = \sum_{i=1}^{6} a_i Z_i = 0. \tag{10.38}$$

Proof. Let $F = PQR + STU$ be a Cayley-Salmon equation of F. Let us try to find some constants such that after scaling the linear forms they add up to zero. Write

$$P' = pP, Q' = qQ, R' = rR, S' = sS, T' = tT, U' = uU.$$

Since $V(F)$ is not a cone, four of the linear forms are linearly independent. After reordering the linear forms, we may assume that the forms P, Q, R, S are linearly independent (we will not use that P, Q, R form one of the two products defining the equation of F). Let

$$T = aP + bQ + cR + dS,\ U = a'P + b'Q + c'R + d'S.$$

The constants p, q, r, s, t, u must satisfy the following system of equations

$$\begin{aligned} p + ta + ua' &= 0 \\ q + tb + ub' &= 0 \\ r + tc + uc' &= 0 \\ s + td + ud' &= 0 \\ pqr + stu &= 0. \end{aligned}$$

The first 4 linear equations allow us to express linearly all unknowns in terms of two, say t, u. Plugging in the last equation, we get a cubic equation in t/u. Solving it, we get a solution. Now set

$$z_1 = Q' + R' - P', \quad z_2 = R' + P' - Q', \quad z_3 = P' + Q' - R',$$

$$z_4 = T' + U' - S', \quad z_5 = U' + S' - T', \quad z_6 = S' + T' - U'.$$

One checks that these six linear forms satisfy the equations from the assertion of the theorem. □

Corollary 10.4.5. *(T. Reye) A general homogeneous cubic form F in 4 variables can be written as a sum of 6 cubes in ∞^4 different ways. In other words,*

$$\dim \mathrm{VSP}(F; 6)^o = 4.$$

Proof. This follows from the proof of the previous theorem. Consider the map

$$(\mathbb{C}^4)^6 \to \mathbb{C}^{20}, \quad (L_1, \ldots, L_6) \mapsto L_1^3 + \ldots + L_6^3.$$

It is enough to show that it is dominant. We show that the image contains the open subset of nonsingular cubic surfaces. In fact, we can use a Clebsch-Salmon equation $PQR+STU$ for $V(F)$ and apply the proof of the theorem to obtain that, up to a constant factor,

$$F = (Q'+R'-P')^3+(R'+P'-Q')^3+(P'+Q'-R')^3+(T'+U'-S')^3$$
$$+(U'+S'-T')^3+(S'+T'-U')^3,$$

where P',Q',R',S',T',U' are appropriately scaled. □

Remark 10.4.4. I do not know how to describe explicitly the variety $\mathrm{VSP}(F;6)^o$. Does it admit a Fano model?

Now let us see in how many ways I can write a surface by a Cremona hexahedral equation.

Suppose a nonsingular S is given by equations (10.38) which one calls *Cremona hexahedral equations*. They allow us to locate 15 lines on S such that the remaining lines form a double-six. The equations of these lines in $\mathbb{P}^5$ are

$$z_i+z_j=0,\ z_k+z_l=0,\ z_m+z_n=0, \sum_{i=1} a_iz_i=0,$$

where $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$. Let us denote the line given by the above equations by $l_{ij,kl,mn}$.

Let us identify a pair a,b of distinct elements in $\{1,2,3,4,5,6\}$ with a transposition (ab) in S_6. We have the product $(ij)(kl)(mn)$ of three commuting transpositions corresponding to each line $l_{ij,kl,mn}$. The group S_6 admits a unique (up to a composition with a conjugation) exterior automorphism which sends each transposition to the product of three commuting transpositions. In this way we can match lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab} of the $\mathbf{E}_6$ lattice. To do it explicitly, one groups together 5 products of three commuting transpositions lines in such a way that they do not contain a common transposition. Such a set is called a *total*. Here is the set of 6 totals

$$\begin{aligned}
A &= (12)(34)(56),\ (13)(25)(46),\ (15)(24)(36),\ (14)(26)(35),\ (16)(23)(45)\\
B &= (12)(34)(56),\ (23)(46)(15),\ (25)(36)(14),\ (24)(16)(35),\ (26)(13)(45)\\
C &= (13)(45)(26),\ (34)(25)(16),\ (36)(24)(15),\ (35)(12)(46),\ (23)(14)(56)\\
D &= (14)(23)(56),\ (45)(36)(12),\ (24)(35)(16),\ (46)(13)(25),\ (34)(15)(26)\\
E &= (15)(34)(26),\ (56)(24)(13),\ (35)(46)(12),\ (25)(14)(36),\ (45)(16)(23)\\
F &= (16)(45)(23),\ (26)(14)(35),\ (46)(13)(25),\ (36)(15)(24),\ (56)(12)(34).
\end{aligned}$$

Two different totals contain one common product $(ij)(kl)(mn)$. This matches the products with subsets of two elements of the set $\{A,B,C,D,E,F\}$ which we can identify with $\{1,2,3,4,5,6\}$. For example, $(12)(34)(56)$ is matched with $(AB) = (12)$ and $(13)(45)(56)$ is matched with $(BC) = (23)$.

After we matched the lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab}, we check that this matching defines an isomorphism of the incidence subgraph of the lines with with the subgraph of the incidence graph of 27 lines on a cubic surface whose vertices correspond to exceptional vectors c_{ab} .

Theorem 10.4.6. *Each Cremona hexahedral equations of a nonsingular cubic surface S defines an ordered double-six of lines. Conversely, a choice of an ordererd double-six defines uniquely Cremona hexahedral equations of S.*

Proof. We have seen already the first assertion of the theorem. If two surfaces given by hexahedral equations define the same double-six, then they have common 15 lines. Obviously this is impossible. Thus the number of different hexahedral equations of S is less or equal than 36. Now consider the identity

$$(z_1+\ldots+z_6)\big((z_1+z_2+z_3)^2+(z_4+z_5+z_6)^2-(z_1+z_2+z_3)(z_4+z_5+z_6)\big)$$
$$=(z_1+z_2+z_3)^3+(z_4+z_5+z_6)^3=z_1^3+\ldots+z_6^3$$
$$+3(z_2+z_3)(z_1+z_3)(z_1+z_2)+3(z_4+z_5)(z_5+z_6)(z_4+z_6).$$

It shows that Cremona hexahedral equations define a Cayley-Salmon equation

$$(z_2+z_3)(z_1+z_3)(z_1+z_2)+(z_4+z_5)(z_5+z_6)(z_4+z_6)=0,$$

where we have to eliminate one unknown with help of the equation $\sum a_iz_i = 0$. Applying permutations of $z_1,\ldots,z_6$, we get 10 Cayley-Salmon equations of S. Each 9 lines formed by the corresponding conjugate pair of triads of tritangent planes are among the 15 lines determined by the hexahedral equation. It follows from the classification of the conjugate pairs that we have 10 such pairs composed of lines c_{ij}'s (type II). Thus a choice of Cremona hexahedral equations defines exactly 10 Cayley-Salmon equations of S. Conversely, it follows from the proof of Theorem that each Cayley-Salmon equation gives 3 Cremona hexahedral equations (unless the cubic equation has a multiple root). Since we have 120 Cayley-Salmon equations for S we get $36 = 360/10$ hexahedral equations for S. They match with 36 double-sixes. □

Remark 10.4.5. One can give another proof of the previous theorem which does not use the assumption that the cubic equation used in the proof of Theorem 10.4.3 has no multiple roots. Choose a double-six which defines an isomorphism from S to the blow-up 6 points $p_1,\ldots,p_6$ in $\mathbb{P}^2$. For any pair (p_i,p_j) consider the line $\langle p_i,p_j\rangle$ and let (ijT) denote a linear form defining this line. The cubic curve defined by the equation $(ijT)(klT)(mnT) = 0$ passes through each point p_s. Here, as always different small letters denote different numbers. Let A be a total. Define the cubic curve X_A by the equation

$$P_A = (12T)(34T)(56T) + (13T)(25T)(46T) + (15T)(24T)(36T)$$

$$+(14T)(26T)(35T) + (16T)(23T)(45T) = 0.$$

In a similar way we define the curves X_B, X_C, X_D, X_E. Thus we get six cubic curves containing 6 points $(p_1, \ldots, p_6)$. Define a map

$$S \to \mathbb{P}^5,\ x \mapsto (P_A(x), \ldots, P_F(x)).$$

One checks that the image is contained in the intersection of the cubic 4-fold

$$\sum_{i=0}^{5} Z_i^3 = 0$$

and a 3-dimensional subspace defined by equations

$$\sum_{i=0}^{5} Z_i = \sum_{i=0}^{5} a_i Z_i = 0.$$

These are the *Sylvester's hexahedral equations*.

One also uses the invariant theory of 6 ordered points in $\mathbb{P}^1$. It defines an isomorphism between the GIT-quotient $(\mathbb{P}^1)^6//\mathrm{SL}(2)$ and the Segre cubic hypersurface in $\mathbb{P}^4$ given by 2 equations in $\mathbb{P}^5$

$$\mathcal{S}_3 : \sum_{i=1}^{5} z_i^3 = \sum_{i=1}^{5} z_i = 0.$$

Its section by a transversal hyperplane $V(\sum_{i=1}^5 z_i)$ defines a nonsingular cubic surface given by Cremona hexaedral equations. Now fix a general line l in $\mathbb{P}^2$ and consider the projection of the points p_i to l from a general point x. Taking the orbit of the correspondinbg set in $l \cong \mathbb{P}^1$ we get a point on $\mathcal{S}_3$. When x varies we get a surface in $\mathcal{S}_3$. One can show that its closure is a hyperplane section of $\mathcal{S}_3$ isomorphic to S.

10.5 Automorphisms of a nonsingular cubic surface

10.5.1 Eckardt points

A point of intersection of three lines in a tritangent plane is called an *Eckardt point*. As we will see later the locus of nonsingular cubic surfaces with an Eckardt point is of codimension 1 in the moduli space of cubic surfaces.

Proposition 10.5.1. *There is a bijective correspondence between Eckardt points on a nonsingular cubic surface S and automorphisms of order 2 with one isolated fixed point.*

Proof. Let $p \in S$ be an Eckardt point and let $\pi : S' \to S$ be the blow-up of p. This is a Del Pezzo surface of degree 2. The pre-image of the linear system $|-K_S - p|$ is the linear system $|-K_{S'}|$. It defines a degree 2 regular map $f : S' \to \mathbb{P}^2$ whose restriction to $S \setminus \pi^{-1}(p) \cong S \setminus \{p\}$ is the linear projection of S with center at p. Let R_1, R_2, R_3 be the proper inverse transforms of the lines in S from the tritangent plane defined by p. These are (-2)-curves on S'. Their image in $\mathbb{P}^2$ is a singular point of the branch curve B of degree 4. The image of $E = \pi^{-1}(p)$ is a line passing through 3 singular points of a quartic curve. It must be an irreducible component of B. Thus B is the union of a line ℓ and a cubic curve C which intersect at three distinct points x_1, x_2, x_3. Let X be the double cover of the blow-up of $\mathbb{P}^2$ at the points x_1, x_2, x_3 ramified along the proper transform of the curve B. We have a birational map of $f : S' - \to X$ which is a regular map outside the union of curves R_i. It is easy to see that it extends to the whole S' by mapping the curves R_i isomorphically to the pre-images of the points x_i under the map $X \to \mathbb{P}^2$.

Thus $f : S' \to X$ is a finite map of degree 2, and hence is a Galois cover of degree 2. The corresponding automorphism of S' leaves the curve E pointwisely invariant, and hence descends to an automorphism g of the cubic surface S. Since it must leave $|-K_S|$ invariant, it is induced by a linear projective transformation $\bar{g}$ of $\mathbb{P}^3$. Its set of fixed points in $\mathbb{P}^3$ is the point p and a plane which intersects S along a curve C'. The linear projection from p maps C' isomorphically to the plane cubic C. Thus g has one isolated fixed point on S.

Conversely, assume S admits an automorphism g of order 2 with one isolated fixed point p. As above we see that g is induced by a projective transformation $\bar{g}$. Diagonalizing the corresponding linear map of $\mathbb{C}^4$, we see that $\bar{g}$ has one eigenspace of dimension 1 and one eigensubspace of dimension 3. Thus in $\mathbb{P}^3$ it fixes a point and a plane Π. The fixed locus of g is the point p and a plane section C' not passing through p. Let P be the tangent plane of S at p. It is obviously invariant and its intersection with S is a cubic plane curve Z with a singular point at p. Its intersection with C' gives 3 fixed nonsingular points on Z. If Z is irreducible, its normalization is isomorphic to $\mathbb{P}^1$ which has only 2 fixed point of any non-trivial automorphism of order 2. Thus Z is reducible. If it consists of a line and a conic, then one of the components has 3 fixed points including the point p. Again this is impossible. So we conclude that Z consists of three concurrent lines and hence a tritangent plane. It is clear that the automorphism associated to this tritangent plane coincides with g. □

Example 10.5.1. Consider a cubic surface given by equation

$$F_3(T_0, T_1, T_2) + T_3^3 = 0,$$

where $C = V(F_3(T_0, T_1, T_2)$ is a nonsingulat plane cubic. Let L be a flex tangent of C. It is easy to see that the pre-image of C under the projection $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2)$ splits in the union of three lines passing through a coomon point (the pre-image of the inflection point). Thus the surface contains 9 Eckardt points. Note that the corresponding 9 tritangent planes contain all 27 lines.

Example 10.5.2. Consider a cubic surface given by an equation in $\mathbb{P}^4$ of the form

$$\sum_{i=0}^{4} a_i T_i = \sum_{i=0}^{4} T_i = 0,$$

where $a_i \neq 0$. Assume $a_1 = a_2$. Then the point $p = (1, -1, 0, 0, 0)$ is an Eckardt point. In fact, the tangent plane at this point is $T_0 = T_1 = 0$. It cuts out the surface along the union of three lines intersecting at the point p. Similarly, we have an Eckardt point whenever $a_i = a_j$ for some $i \neq j$. Thus we may have 1,3,6 or 10 Eckardt points dependent on whether we have two,three,four or five equal a_i's.

For the future need let us prove the following.

Proposition 10.5.2. *Let p_1 and p_2 be two Eckardt points on S such that the line $l = \langle p_1, p_2 \rangle$ is not contained in S. Then l intersects S in a third Eckardt point.*

Proof. Let g be an automorphism of S defined by the projection from the point p_1. Then l intersects S at the point $p_3 = g(p_2)$. Note the projection of any line ℓ passing through p_2 must contain a singular point of the branch locus since otherwise ℓ intersects the line component of the branch locus at a nonsingular point and hence passes through p_1. Thus ℓ intersects one of the lines passing through p_1 and the plane spanned by these two lines cuts out S in an additional line passing through p_3. In this way we find three lines through p_3. □

Proposition 10.5.3. *No more than two Eckardt points lie on a line contained in the surface.*

Proof. Consider the linear projection from one of the Eckardt points p_1. Its branch curve is the union of a plane cubic C and a line intersecting at three points. The second Eckardt point p_2 is projected to one of the intersection points, say q. The plane spanned by the lines $\langle p_1, p_2\rangle$ and one of the other 2 lines passing through p_2 is a tritangent plane with Eckardt point p_2. Since it is invariant with respect to the involution g defined by p_1, the point p_2 is a fixed point and hence lies on the curve of fixed points of g. The projection of the tritangent plane is a line which intersects C only at the point q. Hence q is an inflection point. Clearly this shows that if there is a third Eckardt point p_3, it must coincide with p_2. $\square$

10.5.2 The Weyl representation

Let S be a nonsingular cubic surface and $\mathrm{Aut}(S)$ be its group of biregular automorphims. By functoriality $\mathrm{Aut}(S)$ acts on $Pic(S)$ leaving the canonical class K_S invariant. Thus $\mathrm{Aut}(S)$ acts on the the lattice $Q = (\mathbb{Z}K_S)^\perp$ preserving the intersection form. Let

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(Q), \quad g \mapsto g^*,$$

be the corresponding homomorphism.

Proposition 10.5.4. *The image of ρ is contained in the Weyl group $W(Q)$. The kernel of ρ is trivial.*

Proof. Clearly, any automorphism induces a Cremona isometry of $\mathrm{Pic}(S)$. We know that it is contained in the Weyl group. We know that $\mathrm{O}(Q) = W(Q) \times \{-\pm\}$. We have to show that -1 is not realized by an automorhism. But this is obvious because g^* sends an effective divisor to an effective divisor. An element in the kernel does not change any geometric basis of $\mathrm{Pic}(S)$. Thus it descends to an automorphism of $\mathbb{P}^2$ which fixes an ordered set of 6 points in general linear position. Obviously it is the identity transformation. $\square$

Corollary 10.5.5. $\mathrm{Aut}(S)$ *is isomorphic to a subgroup of the Weyl group $W(S)$.*

We will need some known information about the structure of the Weyl group $W(\mathbf{E}_6)$.

Lemma 10.5.6. *Let H be a maximal subgroup of $W(\mathbf{E}_6)$. Then one of the following cases occurs:*

(i) $H \cong 2^4 : S_5$ *of order* $2^4 5!$ *and index 27;*

(ii) $H \cong S_6 \times 2$ *of order* $2.6!$ *and index 36;*

(iii) $H \cong 3_+^{1+2} : 2S_4$ *of order* 1296 *and index 40;*

(iv) $H \cong 3^3 : (S_4 \times 2)$ *of order* 1296 *and index 40;*

(v) $H \cong \big(2.(A_4 \times A_4).2\big).2$ *of order is* 1152 *and index 45.*

Here we use the ATLAS notations for cyclic groups: $\mathbb{Z}/n\mathbb{Z} = n$ and semi-direct products: $H \ltimes G = H : G$, 3_+^{1+2} denotes the group of order 3^3 of exponent p, $A.B$ is a group with normal subgroup isomorphic to A and quotient isomorphic to B.

We recognize a group from (i) as the stabilizer subgroup of a exceptional vector (or a line on a cubic surface). If we choose a simple root basis $(\alpha_0, \ldots, \alpha_5)$ such that the exceptional vector is equal to α_5^*, then H is generated by the reflections $s_i = s_{\alpha_i}, i \neq 5$. It is naturally isomorphic to the Weyl group $W(D_5)$.

A group H of type (ii) is the stabilizer subgroup of a double-six. The subgroup S_6 permutes the lines, the subgroup 2 switches the two sixers. In the geometric root basis $\alpha_0 = e_0 - e_1 - e_2 - e_3, \alpha_i = e_i - e_{i+1}$, the stabilizer subgroup of the double-sixdouble-six $(e_1, \ldots, e_6; e_1', \ldots, e_6')$, where $e_i' = 2e_0 - e_1 - \ldots - e_6 + e_i$, generated by permutations of e_i's and the reflection with respect to the maximal root $2e_0 - e_1 - \ldots - e_6$.

A group of type (iv) is the stabilizer subgroup of a Steiner triad of a double-sixes.

A group of type (v) is the stabilizer subgroup of a tritangent plane (or a triple of exceptional vectors added up to 0).

Proposition 10.5.7. $W(\mathbf{E}_6)$ *contains a unique normal subgroup* $W(\mathbf{E}_6)'$. *It is a simple group and its index is equal to 2.*

Proof. Choose a root basis $(\alpha_0, \ldots, \alpha_5)$ in the root lattice Q of type E_6. Let $s_0, \ldots, s_5$ be the corresponding simple reflections. Each element $w \in W(\mathbf{E}_6)$ can be written as a product of the simple reflections. Let $\ell(w)$ is the minimal length of the word needed to write w as such a product. for example, $\ell(1) = 0, \ell(s_i) = 1$. one shows that the function $\ell : W(\mathbf{E}_6) \to \mathbb{Z}/2\mathbb{Z},\ w \mapsto \ell(w) \mod 2$ is a homomorphism of groups. Its kernel $W(\mathbf{E}_6)'$ is a subgroup of index 2. The restriction of the function ℓ to the subgroup $H \cong S_6$ generated by the reflections $s_1, \ldots, s_5$ is the sign function. Suppose K is a normal subgroup of $W(\mathbf{E}_6)'$. Then $K \cap H$ is either trivial or equal to the alternating subgroup A_6 of index 2. It remains to use that $H \times (r)$ is a maximal subgroup of $W(\mathbf{E}_6)$ and r is a reflection which does not belong to $W(\mathbf{E}_6)'$. □

Remark 10.5.1. Recall that we have an isomorphism (10.13) of groups

$$W(\mathbf{E}_6) \cong \mathrm{O}(6, \mathbb{F}_2)^-.$$

The subgroup $W(\mathbf{E}_6)'$ is isomorphic to the commutator subgroup of $\mathrm{O}(6,\mathbb{F}_2)^-$.

Let us mention other realizations of the Weyl group $W(\mathbf{E}_6)$.

Proposition 10.5.8.

$$W(\mathbf{E}_6)' \cong \mathrm{SU}_4(2),$$

where $\mathrm{U}_4(2)$ *is the group of linear transformations with determinant 1 of* $\mathbb{F}_4^4$ *preserving a non-degenerate Hermitian product with respect to the Frobenius automorphism of* $\mathbb{F}_4$.

Proof. Let $\mathbf{F} : x \mapsto x^2$ be the Frobenius automorphism of $\mathbb{F}_4$. We view the expression

$$\sum_{i=0}^{3} x_i^3 = \sum_{i=0}^{3} x_i \mathbf{F}(x_i)$$

as a nondegenerate hermitian form in $\mathbb{F}_4^4$. Thus $\mathrm{SU}_4(2)$ is isomorphic to the subgroup of the automorphism group of the cubic surface S defined by the equation

$$T_0^3 + T_1^3 + T_2^3 + T_3^3 = 0$$

over the field $\bar{\mathbb{F}}_2$. The Weyl representation (which is defined for nonsingular cubic surfaces over fields of arbitrary characteristic) of $\mathrm{Aut}(S)$ defines a homomorphism $\mathrm{SU}_4(2) \to W(\mathbf{E}_6)$. The group $\mathrm{SU}_4(2)$ is known to be simple and of order equal to $\frac{1}{2}|W(\mathbf{E}_6)|$. This defines an isomorphism $\mathrm{SU}_4(2) \cong W(\mathbf{E}_6)'$. □

Proposition 10.5.9.

$$W(\mathbf{E}_6) \cong \mathrm{SO}(5,\mathbb{F}_3), \quad W(\mathbf{E}_6)' \cong \mathrm{SO}(5,\mathbb{F}_3)^+,$$

where $\mathrm{SO}(5,\mathbb{F}_3)^+$ *is the subgroup of elements of spinor norm 1.*

Proof. Let $\bar{Q} = Q/3Q$. Since the determinant of the Cartan matrix of type E_6 is equal to 3, the symmetric bilinear form defined by

$$\langle v + 3Q, w + 3Q\rangle = -(v,w) \mod 3$$

is degenerate. It has one-dimensional radical spanned by the vector

$$v_0 = 2\alpha_1 + \alpha_1 + 2\alpha_4 + \alpha_5 \mod 3Q$$

The quadratic form $q(v) = (v,v) \mod 3$ defines a non-degenerate quadratic form on $V = \bar{Q}/\mathbb{F}_3 v_0 \cong \mathbb{F}_3^5$. We have a natural injective homomorphism $W(\mathbf{E}_6) \to \mathrm{O}(5,\mathbb{F}_2)$. Comparing the orders, we find that the image is a subgroup of index 2. It must coincide with $\mathrm{SO}(5,\mathbb{F}_3)$. Its unique normal subgroup of index 2 is $\mathrm{SO}(5,\mathbb{F}_3)^+$. □

Remark 10.5.2. Let V be a vector space of odd dimension $2k+1$ over a finite field $\mathbb{F}_q$ equipped with a non-degenerate symmetric bilinear form. An element $v \in V$ is called a *plus vector* (resp. *minus vector*) if (v,v) is a square in $\mathbb{F}_q^*$ (resp. is not a square $\in \mathbb{F}_q^*$). The orthogonal group $\mathrm{O}(V)$ has three orbits in $\mathbb{P}(E)$: the set of isotropic lines, the set of lines spanned by a plus vector and the set of lines spanned by a minus vector. The isotropic subgroup of a non-isotropic vector v is isomorphic to the orthogonal group of the subspace $v^\perp$. The restriction of the quadratic form to $v^\perp$ is of Witt index k if v is a plus vector and of Witt index $k-1$ if v is a minus vector. Thus the the stabilizer group is isomorphic to $\mathrm{O}(2k,\mathbb{F}_q)^\pm$. In our case when $k=2, q=3$, we obtain that minus vectors correspond to cosets of roots in $\bar{Q}$, hence the stabilizer of a minus vector is isomorphic to the stabilizer of a douible six, i.e. a maximal subgroup of $W(\mathbf{E}_6)$ of index 36. The stabilizer subgroup of a plus vector is a group of index 45 and isomorphic to the stabilizer of a tritangent plane. The stabilizer of an isotropic plane is a maximal subgroup of type (iii), and the stabilizer subgroup of an isotropic line is a maximal subgroup of type (iv).

10.5.3 Automorphisms of finite order

Since any automorphism of a nonsingular cubic surface S preserves $|-K_S|$, it is induced by a projective transformation. After diagonalization we may assume that any automorphism is represented by a diagonal matrix with roots of unity as its entries.

Lemma 10.5.10. *Let $S=V(F)$ be a nonsingular cubic surface which is invariant with respect to a projective transformation g of order $n>1$. Then, after a linear change of variables, F is given in the following list. Also, a generator of the group (g) can be defined by $(x_0,x_1,x_2,x_3)\to(x_0,\zeta_n^a,\zeta_n^b x_2,\zeta_n^c x_3)$, where ζ_n is a primitive nth root of unity.*

(i) $(n=2)$, $(a,b,c)=(0,0,1)$,

$$F=T_3^2L_1(T_0,T_1,T_2)+L_3(T_0,T_1,T_2).$$

(ii) $(n=2)$, $(a,b,c)=(0,1,1)$,

$$F=T_0L_2(T_2,T_3)+T_1M_2(T_2,T_3)+L_3(T_0,T_1).$$

(iii) $(n=3)$, $(a,b,c)=(0,0,1)$,

$$F=T_3^3+L_3(T_0,T_1,T_2).$$

(iv) $(n = 3)$, $(a, b, c) = (0, 1, 1)$,

$$F = L_3(T_0, T_1) + M_3(T_2, T_3).$$

(v) $(n = 3)$, $(a, b, c) = (0, 1, 2)$,

$$F = L_3(T_0, T_1) + T_2T_3L_1(T_0, T_1) + M_3(T_2, T_3).$$

(vi) $(n = 4)$, $(a, b, c) = (0, 2, 1)$,

$$F = T_3^2T_2 + L_3(T_0, T_1) + T_2^2M_1(T_0, T_1).$$

(vii) $(n = 4)$, $(a, b, c) = (2, 3, 1)$,

$$F = T_0^3 + T_0T_1^2 + T_1T_3^2 + T_1T_2^2.$$

(viii) $(n = 5)$, $(a, b, c) = (4, 1, 2)$,

$$F = T_0^2T_1 + T_1^2T_2 + T_2^2T_3 + T_3^2T_0,$$

(ix) $(n = 6)$, $(a, b, c) = (0, 3, 2)$,

$$F = L_3(T_0, T_1) + T_3^3 + T_2^2L_1(T_0, T_1).$$

(x) $(n = 6)$, $(a, b, c) = (0, 2, 5)$,

$$F = L_3(T_0, T_1) + T_3^2T_2 + T_2^3.$$

(xi) $(n = 6)$, $(a, b, c) = (4, 2, 1)$,

$$F = T_3^2T_1 + T_0^3 + T_1^3 + T_2^3 + \lambda T_0T_1T_2.$$

(xii) $(n = 6)$, $(a, b, c) = (4, 1, 3)$,

$$F = T_0^3 + bT_0T_3^2 + T_2^2T_1 + T_1^3.$$

(xiii) $(n = 8)$, $(a, b, c) = (4, 3, 2)$,

$$F = T_3^2T_1 + T_2^2T_3 + T_0T_1^2 + T_0^3.$$

(xiv) $(n = 9)$, $(a, b, c) = (4, 1, 7)$,

$$F = T_3^2T_1 + T_1^2T_2 + T_2^2T_3 + T_0^3.$$

(xv) $(n = 12)$, $(a, b, c) = (4, 1, 10)$,

$$F = T_3^2 T_1 + T_2^2 T_3 + T_0^3 + T_1^3.$$

Here the subscripts in L_i, M_i indicates the degree of the polynomial.

Proof. Choose a coordinate system where g diagonalizes as in the statement of the lemma. Let $P_1 = (1, 0, 0, 0), \ldots, P_4 = (0, 0, 0, 1)$ be the reference points. They are fixed under the action of g in $\mathbb{P}^3$. We will use frequently that F is of degree ≥ 2 in each variable. This follows from the assumption that the surface is nonsingular. We will also feel free to scale the variables to simplify the equation.

Case 1: Two of a, b, c, say a, b, are equal to zero. Write F as a polynomial in T_3. Assume $P_3 \notin V(F)$. Then

$$F = T_3^3 + T_3^2 L_1(T_0, T_1, T_2) + T_3 L_2(T_0, T_1, T_2) + L_3(T_0, T_1, T_2).$$

Since F is an eigenvector with the eigenvalue equal to ζ_n^{3c} and $L_3 \neq 0$, we must have $n = 3$ and $L_1 = L_2 = 0$. This is case (iii). Assume $P_3 \in V(F)$. Then

$$F = T_3^2 L_1(T_0, T_1, T_2) + T_3 L_2(T_0, T_1, T_2) + L_3(T_0, T_1, T_2).$$

As above this gives $n = 2, L_2 = 0$. This is case (i).

Case 2: One of (a, b, c), say a, is equal to zero. Write F as a polynomial in the form

$$F = L_3(T_0, T_1) + T_0 L_2(T_2, T_3) + T_1 M_2(T_2, T_3) + M_3(T_2, T_3).$$

Assume that $L_2 = M_2 = 0$. If M_3 is of degree 3 in T_3 or T_2, say T_2, then $3b = 0 \mod n$. If L_3 is of degree in T_3 too, we get $3c = 0 \mod n$, hence $n = 3$. Without loss of generality we may assume that $(b, c) = (1, 1)$ or $(2, 1)$. In the first case M_3 is any polynomial in T_2, T_3 of degree ≥ 2 in T_2, T_3. This is case (iv). In the second case $M_3 = T_3^3 + T_2^3$. This is a special case of case (v).

If M_3 is of degree 2 in T_3, then L_3 contains $T_3^2 T_2$, hence $2c + b = 0 \mod n$. This gives $n = 6, (a, b, c) = (0, 2, 5)$. This is case (x).

Assume now that L_2 or M_2 is not equal to zero. If T_2^2, T_3^2 do not enter in L_2 and M_2, then $T_0 T_1$ must enter in one of them. This gives $b + c = 0 \mod n$. If T_2^3 or T_2^3 enters in M_3, then $3b = 0 \mod n$ or $3c = 0 \mod n$. This gives $n = 3, (a, b, c) = (0, 1, 2)$ or $(0, 2, 1)$. This is case (v). If T_2^3, T_3^3 do not enter in M_3, then $T_2^2 T_3$ and $T_2 T_3^2$ both enter and we get $2b + c = b + 2c = 0 \mod n$. This again implies $n = 3$ and we are in case (v).

Now we may assume that T_2^2 enters in L_2 or M_2, then $2b = 0 \mod n$. If T_3^2 also enters in L_2 or M_2, then $2c = 0 \mod n$. This implies $n = 2$ and $M_3 = 0$. This is case (ii).

If T_3^2 does not enter in L_2 and M_2, then M_3 is of degree ≥ 2 in T_3. If T_3^3 enters in M_3, then $3c = 0 \mod n$, hence $n = 6$ and $(a, b, c) = (0, 3, 2)$. Thus

$$F = L_3(T_0, T_1) + T_2^2 M_1(T_0, T_1) + T_3^3.$$

This gives case (ix).

If T_3^2 and T_2^3 do not enter in L_2 and in M_2 but T_2T_3 enters in one of these polynomials, then we get $b + c = 0 \mod n$. If $T_3^2T_2$ enters in L_3, then $b + 2c = 0 \mod n$, hence $4c = 0$ and $n = 4, (a, b, c) = (0, 2, 1)$ or $(0, 2, 3)$. This is case (vi).

Case 3: $0, a, b, c$ are all distinct. Note that If two of (a, b, c) are equal, then, by scaling and permuting coordinates we will be in the previous Cases. This obviously implies that $n > 3$. Also monomials $T_i^2T_j$ and $T_iT_j^2$ cannot both enter in F.

Case 3a: All the reference points P_i belong to the surface.

In this case F does not contain cubes of the variables T_i and we can write

$$F = T_0^2 A_1(T_1, T_2, T_3) + T_1^2 B_1(T_0, T_2, T_3) + T_2^2 C_1(T_0, T_1, T_3) + T_3^2 D_1(T_0, T_1, T_2),$$

where A_1, B_1, C_1, D_1 are nonzero linear polynomials. Since all $0, a, b, c$ are distinct, each of these linear polynomials contains only one variable. If the coefficients at T_i and T_j contain the same variable T_k, then the plane $V(T_k)$ is tangent to the surface along a line. It is easy to see that this does not happen for a nonsingular surface. Thus without loss of generarily we may assume that

$$F = T_0^2 T_1 + T_1^2 T_2 + T_2^2 T_3 + T_3^2 T_0.$$

Then $a + b = 2b + c - a = 2c - a = 0 \mod n$. This implies $n = 5, (a, b, c) = (4, 1, 2)$. This is case (viii).

Case 3b. Three reference point belong to the surface.

By scaling and permuting variables we may assume that P_1 does not belong to $V(F)$. The equation contains T_0^3 but does not contain the cubes of other variables. Since F is g-invariant, T_0^2 does not enter in F. We can write

$$F = T_0^3 + T_0 L_2(T_1, T_2, T_3) + L_3(T_1, T_2, T_3). \tag{10.39}$$

Each line $\ell_i = \langle P_1, P_i \rangle$ does not belong to the surface and contains 2 fixed point of g. Suppose each line ℓ_i intersect $V(F)$ only at one point P_i. Then L_2 does not contain squares of the variables and L_3 contains squares of each variable but not cubes. Without loss of generality we may assume that L_3 contains $T_2^2T_3$. Then $2b + c = 0 \mod n$. Since $T_2T_3^2$ does not enter in L_3, the monomial $T_3^2T_1$ must enter. This gives $2c + a = 0 \mod n$. Solving for (a, b, c) we find that $n =$

$9, (a, b, c) = (4, 1, 7)$. The polynomial L_2 cannot contain T_iT_j and hence is equal to zero. This gives us case (xiv).

Now we are in the situation when one of the lines ℓ_i intersects $V(F)$ at a point P different from P_i. If there is no other point in the intersection, then P is a third fixed point of g on the line. This is impossible, and therefore ℓ_i intersects the surface at three distinct points P_i, P, Q. Since g permutes P and Q, we see that the restriction of g^2 to ℓ_i is the identity. Without loss of generality we may assume that $i = 2$ and, hence $2a = 0 \mod n$. Obviously, $T_1^2T_2, T_1^2T_3$ do not enter in L_3 and one of T_2^2 or T_3^2 does not enter in L_2. Assume T_2^2 does not enter in L_2. Then $T_2^2T_1$ or $T_2^2T_3$ enters in L_3. In the first case $2b + a = 0 \mod n$. This gives $n = 4, (a, b, c) = (2, 1, 3)$ or $(2, 3, 1)$. This leads to case (vii).

In the second case, we get $2b + c = 0 \mod n$. Since $T_3^2T_2$ does not enter in L_3, $T_3^2T_1$ must enter giving $2c + a = 0 \mod n$. This easily gives $n = 8$ and $(a, b, c) = (4, 3, 2)$. This leads to case (xiii).

Case 3c. Two reference points do not belong to the surface. We may assume that P_1, P_2 are not in the surface. Thus T_0^3, T_1^3 enter in F, hence $3a = 0 \mod n$. We may assume that F is as in (10.39), where T_1 enters in L_3. Clearly, T_1^2 does not enter in L_2. If T_3^2 (or T_2^2) enters in L_2, then $2c = 0 \mod n$, and we get $n = 6, (a, b, c) = (2, b, 3)$ or $(4, b, 3)$. Since $b \neq 3$, T_2^2 does not enter in L_2. Thus $T_2^2T_3$ or $T_2^2T_1$ enter in L_3. In the first case $2b + c = 0 \mod 6$, hence $2b = 3 \mod 6$ which is impossible. Thus $T_2^2T_1$ enters giving $2b + a = 0 \mod 6$. This gives case $(a, b, c) = (4, 1, 3)$ or $(2, 5, 3)$. This is case (xii).

Now we may assume that T_3^2 and T_2^2 do not enter in L_2. If $T_2^2T_1$ enters, we are led to the previous case (xii). So we may assume that $T_2^2T_3$ enters giving $2b + c = 0$. This implies that $T_3^2T_1$ enters, hence $2c + a = 0 \mod n$. This easily gives $n = 12, (a, b, c) = (4, 1, 10)$. This is case (xv).

Case 3d. Three reference points do not belong to the surface.

We may assume that P_1, P_2, P_3 are not in the surface. Thus T_0^3, T_1^3, T_2^3 enter in F, hence $3a = 3b = 0 \mod n$. We may assume that F is as in (10.39), where T_1, T_2 enter in L_3. Clearly, T_1^2, T_2^2 do not enter in L_2. If T_3^2 enters in L_2, then $2c = 0 \mod n$, and we get $n = 6, (a, b, c) = (2, 4, 3)$ or $(4, 2, 3)$. This gives case (xi).

Assume T_3^2 does not enter in L_2. Without loss of generality we may assume that $T_3^2T_1$ enters in L_3. This gives $2c + a = 0 \mod n$. From this follows that $n = 6$ and $(a, b, c) = (4, 2, 1)$. This case is isomorphic to case (xi).

Case 3e. No reference point belongs to the surface.

In this case each T_i^3 enters in F, hence $3a = 3b = 3c = 0 \mod n$. This is impossible for $n > 3$.

□

In the natural representation of Aut(S) in $W(\mathbf{E}_6)$ each nontrivial automorphism g defines a conjugacy class in $W(E_6)$. The following table gives the list of the conjugacy classes. This can be found in [ATLAS, Carter, Manin]. It is given in the following table.

	Atlas	Carter	Manin	Ord	$\|C(w)\|$	Tr	Char
x	1A	$\emptyset$	c_{25}	6	51840	7	$(t-1)^7$
x	2A	$4A_1$	c_3	2	1152	-1	$P_1^4(t-1)^3$
x	2B	$2A_1$	c_2	2	192	3	$P_1^2(t-1)^5$
	2C	A_1	c_{16}	2	1440	5	$P_1(t-1)^6$
	2D	$3A_1$	c_{17}	2	96	1	$P_1^3(t-1)^4$
x	3A	$3A_2$	c_{11}	3	648	-2	$P_2^3(t-1)$
x	3C	A_2	c_6	3	216	4	$P_2(t-1)^5$
x	3D	$2A_2$	c_9	3	108	1	$P_2^2(t-1)^3$
x	4A	$D_4(a_1)$	c_4	4	96	3	$(t^2+1)^2(t-1)^3$
x	4B	A_1+A_3	c_5	4	16	1	$P_1P_3(t-1)^3$
	4C	$2A_1+A_3$	c_{19}	4	96	-1	$P_1^2P_3(t-1)^3$
	4D	A_3	c_{18}	4	32	3	$P_3(t-1)^4$
x	5A	A_4	c_{15}	5	10	2	$P_4(t-1)^3$
x	6A	$E_6(a_2)$	c_{12}	6	72	2	$P_2(t^2-t+1)^2(t-1)$
x	6C	D_4	c_{21}	6	36	2	$P_1^2(t^2-t+1)(t-1)^3$
x	6E	A_1+A_5	c_{10}	6	36	-1	$P_1P_5(t-1)$
x	6F	$2A_1+A_2$	c_8	6	24	0	$P_1^2P_2(t-1)^3$
	6G	A_1+A_2	c_7	6	36	2	$P_1P_2(t-1)^4$
	6H	A_1+2A_2	c_{10}	6	36	-1	$P_1P_2^2(t-1)^2$
	6I	A_5	c_{23}	6	12	1	$P_5(t-1)^2$
x	8A	D_5	c_{20}	8	8	1	$P_1(t^4+1)(t-1)^2$
x	9A	$E_6(a_1)$	c_{14}	9	9	1	$t^6+t^3+1)(t-1)$
	10A	A_1+A_4	c_{25}	10	36	0	$P_1P_4(t-1)^2$
x	12A	E_6	c_{13}	12	12	0	$P_2(t^4-t^2+1)(t-1)$
	12C	$D_5(a_1)$	c_{24}	12	12	2	$(t^3+1)(t^2+1)(t-1)^2$

Here we mark with the cross the conjugacy classes realized by automorphisms of nonsingular cubic surfaces. Also $|C(w)|$ denotes the cardinality of the centralizer of an element w from the conjugacy class, $\mathrm{Tr}_{\mathrm{Pic}}$ denotes the trace in the Picard lattice (equal to the trace in the root lattice plus 1), Char denotes the characteristic polynomial in Pic(S) and $P_e = t^e + t^{e-1} + \ldots + 1$.

To determine to which conjugacy class our g corresponds under the Weyl representation we use the following well-known formula which is a special case of the *topological Lefschetz fixed-point formula.*

Lemma 10.5.11. *Let g be an automorphism of prime order $p > 1$ of a nonsingular algebraic surface. Let S^g consists of curves $R_1, \dots, R_t$ and isolated points $p_1, \dots, p_s$. Denote by Tr_i the trace of g in its natural action in the cohomology space $H^i(S, \mathbb{C})$. Then*

$$2 - 2\mathrm{Tr}_1(g) + \mathrm{Tr}_2(g) = \sum_{j=1}^{t}(2 - 2g(R_j) + s.$$

The next theorem rewrites the list from Lemma 10.5.10 in the same order, renaming the cases with indication to which conjugacy class they correspond. Also, we simplify the formulae for F by scaling, and reducing a cubic ternary form to the Hesse form, and a cubic binary form to sum of cubes, and a quadratic binary forms to the product of the variables. Each time we use that the forms are non-degenerate because the surface is nonsingular.

Theorem 10.5.12. *Let S be a nonsingular cubic surface admitting a non-trivial automorphism g of order n. Then S is equivariantly isomorphic to one of the following surfaces $V(F)$ with diagonal action defined by a generator of (g) via the formula*

$$(x_0, x_1, x_2, x_3) = (x_0, \zeta_n^a x_1, \zeta_n^b x_2, \zeta_n^c x_3).$$

(2A) $(n = 2)$, $(a, b, c) = (0, 0, 1)$,

$$F = T_3^2 L_1(T_0, T_1, T_2) + T_1^3 + T_2^3 + T_3^3 + \lambda T_1 T_2 T_3.$$

(2B) $(n = 2)$, $(a, b, c) = (0, 1, 1)$,

$$F = T_0 T_2 T_3 + T_1(T_2^2 + T_3^2 + a T_2 T_3) + T_0^3 + T_1^3.$$

(3A) $(n = 3)$, $(a, b, c) = (0, 0, 1)$,

$$F = T_3^3 + T_1^3 + T_2^3 + T_3^3 + \lambda T_1 T_2 T_3.$$

(3C) $(n = 3)$, $(a, b, c) = (0, 1, 1)$,

$$F = T_0^3 + T_1^3 + T_2^3 + T_3^3.$$

(3D) $(n = 3)$, $(a, b, c) = (0, 1, 2)$,

$$F = L_3(T_0, T_1) + T_2 T_3 L_1(T_0, T_1) + M_3(T_2, T_3).$$

(4A) $(n = 4)$, $(a, b, c) = (0, 2, 1)$,

$$F = T_3^2 T_2 + L_3(T_0, T_1) + T_2^2 M_1(T_0, T_1).$$

(4B) $(n = 4)$, $(a, b, c) = (2, 1, 3)$,

$$F = T_0^3 + T_0 T_1^2 + T_1 T_3^2 + T_1 T_2^2.$$

(5A) $(n = 5)$, $(a, b, c) = (4, 1, 2)$,

$$F = T_0^2 T_1 + T_1^2 T_2 + T_2^2 T_3 + T_3^2 T_0,$$

(6A) $(n = 6)$, $(a, b, c) = (0, 3, 2)$,

$$F = T_0^3 + T_1^3 + T_3^3 + T_2^2 L_1(T_0, T_1).$$

(6C) $(n = 6)$, $(a, b, c) = (0, 2, 5)$,

$$F = L_3(T_0, T_1) + T_3^2 T_2 + T_2^3.$$

(6E) $(n = 6)$, $(a, b, c) = (4, 2, 1)$,

$$F = T_3^2 T_1 + T_0^3 + T_1^3 + T_2^3 + \lambda T_0 T_1 T_2.$$

(6F) $(n = 6)$, $(a, b, c) = (4, 1, 3)$,

$$F = T_0^3 + b T_0 T_3^2 + T_2^2 T_1 + T_1^3.$$

(8A) $(n = 8)$, $(a, b, c) = (4, 3, 2)$,

$$F = T_3^2 T_1 + T_2^2 T_3 + T_0 T_1^2 + T_0^3.$$

(9A) $(n = 9)$, $(a, b, c) = (4, 1, 7)$,

$$F = T_3^2 T_1 + T_1^2 T_2 + T_2^2 T_3 + T_0^3.$$

(12A) $(n = 12)$, $(a, b, c) = (4, 1, 10)$,

$$F = T_3^2 T_1 + T_2^2 T_3 + T_0^3 + T_1^3.$$

Proof. We will be computing the trace of g^* by using Lemma 10.5.11. We use the classification from Lemma 10.5.10.

Order 2.

In case (i), the fixed locus is the nonsingular elliptic curve given by equations $T_3 = L_3 = 0$ and isolated point $(0,0,0,1)$. The Euler-Poincaré characteristic of the fixed locus is equal to 1. Hence the trace in Pic(S) is equal to -1. This gives the conjugacy class $2A$. In case (ii), the fixed locus is the line $T_0 = T_1 = 0$ and three isolated points lying on the line $T_2 = T_3$ (not contained in the surface). The Euler-Poincaré characteristic of the fixed locus is equal to 5. Hence the trace in Pic(S) is equal to 3. This gives the conjugacy class 2B.

Order 3.

In case (iii), the fixed locus is a nonsingular elliptic curve given by equations $T_3 = L_3 = 0$. The Euler-Poincaré characteristic of the fixed locus is equal to 0. Hence the trace in Pic(S) is equal to -2. This gives the conjugacy class 3A.

In case (iv), the fixed locus is the set of 6 points lying on the lines $T_0 = T_1 = 0$ and $T_2 = T_3 = 0$. Here we use that the polynomials L_3, M_3 do not have multiple roots since otherwise S is singular. The Euler-Poincaré characteristic of the fixed locus is equal to 6. Hence the trace in Pic(S) is equal to 4. This gives the conjugacy class 3C.

In case $(iv)'$, the fixed locus consists of 3 points lying on the line $T_2 = T_3 = 0$. Hence the trace in Pic(S) is equal to 1. This gives the conjugacy class 3D.

In case (v), the fixed locus is the set of 3 points lying on the line $T_2 = T_3 = 0$. Again we use that L_3 does not have multiple roots. The Euler-Poincaré characteristic of the fixed locus is equal to 3. Hence the trace in Pic(S) is equal to 1. This gives the conjugacy class 3D.

Order 4.

In case (vi), the fixed locus is the set of 5 points lying on the lines $T_0 = T_1 = 0$ and two reference points $P_3 = (0,0,1,0)$ and $P_4 = (0,0,0,1)$. The Euler-Poincaré characteristic of the fixed locus is equal to 5. Hence the trace in Pic(S) is equal to 3. This gives the conjugacy class $4A$ or $4D$. To distinguish the two classes, we notice that g^2 acts as in case (i). This implies that g^2 belongs to the conjugacy class $2A$. On the other hand, the characteristic polynomial of $4D$ shows that $4D^2$ is the conjucacy class $2B$. Thus we have the conjugacy class 4A.

In case (vii), we have three isolated fixed points $(1,0,0,0), (0,1,0,0), (0,0,1,0)$. Thus the trace is equal to 1. This gives the conjugacy class 4B.

Order 5.

This is the unique conjugacy class of order 5. It is realized in case (viii). We have 4 isolated fixed points confirming that the trace is equal to 2.

Order 6.

In case (ix) we have 4 isolated fixed points so that the trace is equal to 2. This gives possible conjugacy classes 6A,6C,6G. We know that our surface is g^2-equivariantly isomorphic to a surface from case (iii). Thus g^2 belongs to the conjugacy class $3A$. Using characteristic polynomials we check that only $6A^2$ is equal to $3A$.

In case (x) we have 4 isolated fixed points. This gives that the trace is equal to 2. It is clear that g^3 acts as in case (i), thus g^3 belongs to $2A$. Also g^2 acts as in case (iv). This shows that g^2 belongs to $3C$. Comparing the characteristic polynomials, this leaves only the possibility that g belongs to $6C$.

In case (xi) we have only one isolated fixed point $(0,0,0,1)$. This gives that the trace is equal to -1 and hence g belongs to $6E$.

In case (xii) we have 2 isolated fixed points so that the trace is equal to 2. The only conjugacy class with trace zero is $6F$.

Order 8.

$8A$ is the unique conjugacy class of order 8. Its trace is 1. This agrees with case (xiii), where we have 3 fixed points.

Order 9.

$9A$ is the unique conjugacy class of order 9. Its trace is 1. This agrees with case (xiv), where we have 3 fixed points.

Order 12.

We have 2 fixed points giving the trace of g equal to 0. This chooses the conjugacy class $12A$. □

Remark 10.5.3. Some of the conjugacy classes (maybe all ?) are realized by automorphisms of minimal resolutions of singular surfaces. Also two non-conjugate elements from Aut(S) may define the same conjugacy class in $W(\mathbf{E}_6)$. An example is an automorphism g from case (3A) and its square.

10.5.4 Automorphisms groups

Theorem 10.5.13. *The following is the list of all possible groups of automorphisms of nonsingular cubic surfaces.*

Proof. Let S be a nonsingular cubic surface and G be a subgroup of Aut(S). Suppose G contains an element of order 3 from the conjugacy class 3C. Applying Theorem 10.5.12, we see that S is isomorphic to the Fermat surface $V(T_0^3+T_1^3+T_2^3+T_3^3)$. It has 27 lines given by the equations $T_0+\epsilon T_1=0, T_2+\eta T_3=0, \epsilon^3=\eta^3=-1$, or their transforms under permuting the variables. It is clear that any automorphism of S permutes the planes $T_i+\epsilon T_j=0$ and hence Aut(S) consists of

Type	Order	Structure	Eckardt	Equation	Parameters
I	648	$3^3 : S_4$	18	$T_0^3 + T_1^3 + T_2^3 + T_3^3$	
II	120	S_5	10	$T_0^2T_1 + T_0T_2^2 + T_2T_3^2 + T_3T_1^2$	
III	108	$(3^2 : 4) \times 3$	9	$T_0^3 + T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$	$20a^3 + 8a^6 = 1$
IV	54	$(3^2 : 2) \times 3$	9	$T_0^3 + T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$	$a - a^4 \neq 0,$ $8a^3 \neq -1,$ $20a^3 + 8a^6 \neq 1$
V	24	S_4	6	$T_0^3 + T_0(T_1^2 + T_2^2 + T_3^2) + aT_1T_2T_3$	$9a^3 \neq 8a$
VI	12	$S_3 \times 2$	4	$T_2^3 + T_3^3 + aT_2T_3(T_0 + T_1) + T_0^3 + T_1^3$	$a \neq 0$
VII	8	8	1	$T_3^2T_2 + T_2^2T_1 + T_0^3 + T_0T_1^2$	
VIII	6	S_3	3	$T_2^3 + T_3^3 + T_2T_3(T_0 + aT_1) + T_0^3 + T_1^3$	$a^3 \neq -1$
IX	4	4	1	$T_3^2T_2 + T_2^2T_1 + T_0^3 + T_0T_1^2 + aT_1^3$	$a \neq 0$
X	4	2^2	2	$T_0^2(T_1 + T_2 + aT_3) + T_1^3 + T_2^3 + T_3^3 + 6bT_1T_2T_3$	$8b^3 \neq -1$
XI	2	2	1	$T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$ $+T_0^2(T_1 + bT_2 + cT_3)$	$b^3, c^3 \neq 1$ $b^3 \neq c^3$ $8a^3 \neq -1,$

permutations of the variables and multiplying the variables by cube roots of unity. This gives case I. It is easy to see that each plane $T_i + \epsilon T_j = 0$ is a tritangent plane with an Eckardt point. Thus we have 18 Eckardt points, maximal possible.

Assume that G contains an element of order 5. Applying Theorem 10.5.12, we see that S is isomorphic to the *Clebsch diagonal surface*

$$T_0^2T_1 + T_0T_2^2 + T_2T_3^2 + T_3T_1^2 = 0. \tag{10.40}$$

Consider the embedding of S in $\mathbb{P}^4$ given by the linear functions

$$\begin{aligned} Z_0 &= T_0 + T_1 + T_2 + T_3 \qquad (10.41)\\ Z_1 &= \zeta_5^3T_0 + \zeta_5^4T_1 + \zeta_5^2T_2 + \zeta_5T_3 \\ Z_2 &= \zeta_5T_0 + \zeta_5^3T_1 + \zeta_5^4T_2 + \zeta_5^2T_3 \\ Z_3 &= \zeta_5^4T_0 + \zeta_5^2T_1 + \zeta_5T_2 + \zeta_5^3T_3 \\ Z_4 &= \zeta_5^2T_0 + \zeta_5T_1 + \zeta_5^3T_2 + \zeta_5^4T_3 \end{aligned}$$

Then one easily checks that $\sum_{i=0}^4 Z_i = 0$ and (10.40) implies that also $\sum_{i=0}^4 Z_i^3 = 0$. This shows that S is isomorphic the following surface in $\mathbb{P}^4$:

$$\sum_{i=0}^4 T_i^3 = \sum_{i=0}^4 T_i = 0 \tag{10.42}$$

These equations exhibit an obvious symmetry which is the group S_5.

Observe that S has 10 Eckardt points $(1,-1,0,0,0)$ and other one obtain by permutations of coordinates. Also notice that any point, say $(1,-1,0,0)$ is joined by a line in the surface to three other points $(0,0,1,-1,0),(0,0,0,1,-1)$, $(0,0,1,0,-1)$. The graph whose vertices are Eckardt points and edges the lines is a famous tri-valent *Peterson graph* whose group of symmetry is isomorphic to S_5.

Assume G is larger than S_5. Consider the representation of G in the symmetry group of the graph of Eckardt points. Its image is equal to S_5, hence its kernel is non-trivial. Let H be a maximal subgroup of $W(\mathbf{E}_6)$ which contains G. It follows from Lemma 10.5.6 that G must contain S_6 or an involution. The restriction of the representation to S_6 must be trivial, since the kernel is non-trivial and is not equal to A_6. This is impossible since S_6 contains our S_5. If the kernel contains an involution, then the involution fixes 10 points. Since no involutions in $W(\mathbf{E}_6)$ has trace equal to 8, we get a contradiction. Thus Aut$(S)\cong S_5$.

Assume that G contains an element g from the conjugacy class 3A. Then we are in case (iii) of Theorem 10.5.12. The plane cubic curve $C=V(T_1^3+T_2^3+T_3^3+aT_1T_2T_3)$ has the projective group of automorphisms isomorphic to $3^2:2$. Together with g this generates a group G_1 of order 54 isomorphic to $(3^2:2)\times 3$. Note that for a special value a, C may acquire an additional isomorphism of order 4 or 6. In the latter case C is projectively isomorphic to the Fermat cubic $T_1^3+T_2^3+T_3^3$, hence we are in case I. In the former case the group of automorphisms contains a group G' isomorphic to $(3^2:4)\times 3$. According to example 10.5.1 S contains 9 tritangent planes with Eckardt points. Each plane is the preimage of a line under the projection to the plane Π containing the curve F of fixed points.

Suppose there is a symmetry g' not belonging to G_1. Since G_1 acts transitively on the set of Eckardt points, we may assume that g' fixes a tritangent plane containing an Eckardt point. Thus g' fixes the plane Π and hence is an automorphism of the plane cubic F. This proves that $G=G_1$ if F has no extra automorphim, $G=G'$ if F has an automorphism of order 4, and S is of type I if F has an automorphism of order 6.

Assume that S contains an element g of order 8. Then S is isomorphic to the surface from case (xiii) of Theorem 10.5.12. The only maximal subgroup of $W(\mathbf{E}_6)$ which contains an element of order 8 is a subgroup H of order 1152. As we know it stabilizes a tritangent plane. In our case the tritangent plane is $T_2=0$. It has the Eckardt point $x=(0,0,0,1)$. Thus $G=$ Aut(S) is a subgroup of the linear tangent space TS_x. If any element of G acts identically on the set of lines in the tritangent plane, then it acts identically on the projectivized tangent space, and hence G is a cyclic group. Obvioulsy this implies that G is of order 8. Assume that there is an element σ which permutes cyclically the lines. Let G' be the subgroup generated by σ and g. Obviously, $\sigma^3=g^k$. Since G does not contain elements of order 24, we may assume that $k = 2$ or 4. Obviously, σ normalizes (g) since otherwise we have two distinct cyclic groups of order 8 acting on a line with a common fixed point. It is easy to see that this is impossible. Since $\text{Aut}(\mathbb{Z}/8\mathbb{Z})) \cong (\mathbb{Z}/2\mathbb{Z})^2$ this implies that σ and g commute. Thus $g\sigma'$ is of order 24 which is impossible. This shows that $\text{Aut}(S) \cong \mathbb{Z}/8\mathbb{Z}$.

It is easy to see that the square of the conjugacy classes $6C, 6E$ is equal to 3C and the square of $6F$ is equal to 3A. Also the cube of the conjugacy class $9A$ is equal to 3C. Since surfaces with automorphism of these types and also with an automorphisms of order 5 and 8 have been already classified we may assume that $\text{Aut}(S)$ does not contain elements of order 5,8,9,12. By the previous analysis we may assume that any element of order 3 belongs to the conjugacy class 3D, and elements of order 6 to the conjugacy class 6A.

Assume $\text{Aut}(S)$ contains an element g from conjugacy class 3D. Then the surface is g-equivariantly isomorphic to the surface from case (v) of Theorem 10.5.12.

$$T_2^3 + T_3^3 + T_2T_3T_0 + L_3(T_0, T_1) = 0.$$

We can reduce it to the form

$$T_2^3 + T_3^3 + T_2T_3(T_0 + aT_1) + T_0^3 + T_1^3 = 0. \tag{10.43}$$

The fixed points of g are the points $q_i = (a_i, b_i, 0, 0)$, where $L_3(a_i, b_i) = 0$. Observe that we have 3 involutions $g_i, i = 0, 1, 2$, defined by $(x_0, x_1, x_2, x_3) \to (x_0, x_1, \zeta_3^i x_3, \zeta_3^{2i} x_2)$. The set of fixed points of g_i is the nonsingular plane section $T_3 = \zeta_i^i T_2$ and an isolated fixed point $p_i = (0, 0, 1, -\zeta_3^i)$. Thus each g_i belongs to the conjugacy class $2A$. The point p_i is an Eckardt point in the tritangent plane $T_0 - \zeta_3^i T_2 - \zeta_3^{2i} T_3 = 0$. Notice that they lie on the line $T_0 = T_1 = 0$. This line is uniquely determined by g, it is spanned by isolated fixed point of g in $\mathbb{P}^3$. It is immediately checked that $g_i g_j = (g_j g_i)^{-1} = g^{i+2j}$ for $i \neq j$. This implies that the group G_1 generated by g, g_0, g_1, g_2 is isomorphic to S_3. The element g belongs to a unique such group determined by the line l. The elements of order 2 in G_1 correspond to the Eckardt points on l.

Suppose one of the fixed point of g, say q_1, is an Eckardt point. A straightforward computation shows that this happens only if in equation (10.43) $a^3 = -1$. Also it shows that only one of the fixed points could be an Eckardt point.

Let P_3 be the 3-Sylow subgroup of $G = \text{Aut}(S)$. It is a cyclic group of order 3. Since $|\text{Aut}(S)| = 2^a 3^b$, the Sylow Theorems gives that the number of 3-Sylow subgroups divides $2^a 3^b$ and $\equiv 1 \mod 3$. This shows that this number is equal to 2^{2k}. If $k > 1$, then G is contained in a maximal subgroups of $W(\mathbf{E}_6)$ of order divisible by 2^4. This could be either a group isomorphic to $2^4 : S_5$ or $S_6 \times 2$. In the first case G stabilizes a line in S, and then any element of order 3 has 2 fixed points on this line. But, as we saw in above the fixed points of an element of order 3 do not lie on a line contained in the surface. In the second case, we use that any subgroup of S_6 containing 16 elements subgroups of order 3 must coincide with S_6. This certainly impossible. Thus $k = 0$ or 2.

If $k = 0$, then G has a unique subgroup (g) of order 3. So, either Aut$(S) = G_1 \cong S_3$, or G contains an involution $\tau \notin (g)$. It must commute with g, hence it leaves invariant the set of 3 collinear fixed points of g. Thus it fixes $a = 1$ or 3 fixed point of g. If τ is of type 2B, then it has 5 isolated fixed points. The group (g) leaves this set invariant and has 2 or 5 fixed points in this set. This shows that τ must be of type 2A and hence its isolated fixed point is one of the fixed points of g which is an Eckardt point. As we have observed earlier, there could be only one such point. Hence there is only one additional element of order 2. The line joining this point with an Eckardt point p_i must be contained in S, since otherwise, by Proposition 10.5.2 we have a third Eckardt point on this line. Thus τ commutes with any involution g_i in G_1. Hence Aut$(S) \cong S_3 \times 2$.

If $k = 1$, G is isomorphic to a transitive subgroup of S_4 which contains an S_3. It must be isomorphic to S_4. Each subgroup of Aut(S) isomorphic to S_3 defines a line with 3 Eckardt points. Since any two such subgroups have a common element of order 2, each line intersects other 3 lines at one point. This shows that the four lines are coplanar and form a complete quadrangle in this plane. Also, since each of the three diagonals d_i has only two Eckardt points on it, we see that each diagonal is contained in the surface. Now choose coordinates such that the plane of the quadrangle has equation $T_0 = 0$ and the diagonals have the equations $T_0 = T_i = 0$. The equation of the surface must now look as follows.

$$aT_0^3 + T_0^2 L_1(T_1, T_2, T_3) + T_0 L_2(T_1, T_2, T_3) + cT_1T_2T_3 = 0.$$

The group Aut(S) leaves the quadrangle invariant and hence acts by permuting the coordinates T_i's and multiplying them by ± 1.. This easily implies that the equation can be reduced to the form of type IV.

Assume that Aut(S) contains an element g from conjugacy class 2B. Then the equation of the surface looks like

$$aT_0T_2T_3 + T_1(T_2^2 + T_3^2 + bT_2T_3) + T_0^3 + T_1^3 = 0.$$

It exhibits an obvious symmetry of order 3 defined by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, \zeta_2 x_2, \zeta_2^2 x_3).$$

Thus we are in one of the above cases.

Suppose $\mathrm{Aut}(S)$ contains an element g of order 4. If g belongs to the conjugacy class 4B, then g^2 belongs to 2B and hence this case has been already considered. If g belongs to $2A$ then the equation of the surface looks like

$$T_2T_3^2 + T_0^3 + aT_1^3 + T_2^2(T_0 + T_1) = 0.$$

Here we have to assume that the surface is not isomorphic to the surface of type VII. It follows from the proof of the next corollary that in all previous cases, except type VII, the automorphism group is generated by involutions of type 2A. Thus our surface cannot be reduced to one of the previous cases.

Finally it remains to consider the case when only involutions of type 2A are present. Suppose we have 2 such involution. They define two Eckardt points p_1 and p_2. In order the involution commute the line joining the two points is contained in S. Suppose we have a third involution defining a third Eckardt point p_3. Then we have a tritangent plane formed by the lines $\langle p_i, p_j\rangle$. Obviously it must coincide with each tritangent plane corresponding to the Eckardt points p_i. This contradiction shows that we can have at most 2 commuting involutions. This gives the last two cases of our theorem. The condition that there is only one involution of type 2A is that the line $L_1(T_0, T_1, T_2) = 0$ does not pass through a flex point of $L_3(T_0, T_1, T_2) = 0$. □

Corollary 10.5.14. *Let* $\mathrm{Aut}(S)^o$ *be the subgroup of* $\mathrm{Aut}(S)$ *generated by involutions of type 2A. Then* $\mathrm{Aut}(S)^o$ *is a normal subgroup of* $\mathrm{Aut}(S)$ *such that the quotient group is either trivial or a cyclic group of order 2 or 4. The order is 4 could occur only for the surface of type VII. The order is 2 could occur only for surfaces of type X.*

Proof. We do it case by case. For surfaces of type I, the group $\mathrm{Aut}(S)$ is generated by transformations of type

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, \epsilon x_3, x_2),$$

where $\epsilon^3 = 1$. It is easy to see that it is an involution of type 2A corresponding to the Eckardt point $(0, 0, 1, -\epsilon)$.

For surfaces of type II given by equation (10.42), the group $\mathrm{Aut}(S)$ is generated by transpositions of coordinates. They correspond to involutions of type 2A associated with Eckardt points of type $(1, -1, 0, 0, 0)$.

In the case of surfaces of type III, we use that a line in $\mathbb{P}^3$ joining 2 Eckardt points contains the third Eckardt point. Thus any such line generate a subgroup isomorphic to S_3. We have 12 lines which contain 9 flex points. They are the projections of these line in $\mathbb{P}^2$ from the center of projection $(1, 0, 0, 0)$. One can show that the group generated by these 12 subgroups must coincide with the whole group.

The remaining cases follow from the proof of the theorem.

□

Exercises

10.1 Let $Y \subset \mathbb{P}^4$ be the image of $\mathbb{P}^2$ under a rational map given by the linear system $|3\ell - 2p_1 - p_2 - p_3|$, where the points p_1, p_2, p_3 are not on a line, and p_2, p_3 is not infinitely near to p_1.

(i) Show that Y is a surface of degree 4 and it is nonsingular if and only if the points p_2, p_3 are in $\mathbb{P}^2$.

(ii) Show that the projection of Y from a point not on the surface is a non-normal cubic surface in $\mathbb{P}^3$.

(iii) Show that any non-normal cubic surface in $\mathbb{P}^3$ can be obtained in this way.

10.2 Show that the dual of the 4-nodal cubic surfrace is isomorphic to the quartic surface given by the equation

$$\sqrt{T_0} + \sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3} = 0.$$

10.3 Let $T : (x, y) \to (x^{-1}, y^{-1})$ be the standard Cremona transformation. Show that T extends to a biregular automorphism σ of a Del Pezzo surface X of degree 6 and the orbit space $X/(\sigma)$ is isomorphic to a 4-nodal cubic surface.

10.4 Show that a cubic surface can be obtained as the blow-up of 5 points on $\mathbb{P}^1 \times \mathbb{P}^1$. Find the conditions on the 5 points such that the blow-up is isomorphic to a nonsingular cubic surface.

10.5 Compute the number of m-tuples of skew lines on a nonsingular surface for $m = 2, 3, 4, 5$.

10.6 Suppose a quadric intersects a cubic along the union of three conics. Show that the three planes defined by the conics pass through three lines in a tritangent plane

10.7 Let Γ and Γ' be two twisted cibics in $\mathbb{P}^3$ containing a common point P. For a general plane Π through P let $\Pi \cap \Gamma = \{P, p_1, p_2\}, \Pi \cap \Gamma' = \{p, p_1', p_2'\}$ and $f(P) = \langle p_1, p_1 \rangle \cap \langle p_1', p_2' \rangle$. Consider the set of planes through P as a hyperplane H in the dual space $\check{\mathbb{P}}^3$. Show that the the image of the rational map $H - \to \mathbb{P}^3, \Pi \mapsto$

$f(\Pi)$ is a nonsingular cubic surface and every such cubic surface can be obtained in this way

10.8 Show that the linear system of quadrics in $\mathbb{P}^3$ spanned by quadrics which contain a degree 3 rational curve on a nonsingular cubic surface S can be spanned by the quadrics defined by the minors of a matrix defining a determinantal reprfesentation of S.

10.9 Sow that the linear system of cubic surfaces in $\mathbb{P}^3$ containing 3 skew lines defines a birational map from $\mathbb{P}^3$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

10.10 Show that non-normal cubic surfaces are scrolls, i.e. contain a one-dimensional family of lines.

10.11 Show that all singular surfaces of type $VII, X, XI, XIII - XXI$ are isomorphic and there are two non-isomorphic surfaces of type XII.

10.12 Prove that the linear system of cubic surfaces in $\mathbb{P}^3$ containing three skew lines defines a birational map from $\mathbb{P}^3$ to $\mathbb{P}^7$ whose image is equal to the Segre variety $(\mathbb{P}^1)^3$.

10.13 Compute the number of determinantal representations of a singular cubic surface.

10.14 Find determinantal representations of an irreducible non-normal cubic surface.

10.15 Let l be a line on a cubic surface with canonical singularities and E be its proper inverse transform on the corresponding Del Pezzo surface X. Let $\mathcal{N}$ be the sublattice of Pic(X) spanned by irreducible components of exceptional divisors of $\pi : X \to S$. Define the multiplicity of l by

$$m(l) = \frac{\#\{\sigma \in O(\text{Pic}(X)) : \sigma(E) - E \in \mathcal{N}\}}{\#\{\sigma \in O(\text{Pic}(X)) : \sigma(E) = E\}}.$$

Show that the sum of the multiplicities is always equal to 27.

10.16 Show that every line on a nonsingular cubic surface S is tangent to the Hessian surface $H(S)$ at two points. Also show that the 24 points obtained in this way from a double-six of lines are the intersection points of S and the Schur quadric associated to the double-six.

Chapter 11

Geometry of Lines

11.1 Grassmanians of lines

Let V be a complex vector space of dimension $n+1$, in $V \otimes V$ we have the subset

$$I = \{v \otimes w \in V \otimes V,\ v, w \in V\}$$

of indecomposable vectors as well as

$$I^+ = \{v \otimes w + v \otimes w,\ v, w \in V\}$$

and

$$I^- = \{v \otimes w - w \otimes v,\ v, w \in V\}.$$

The elements $v \otimes w + v \otimes w$ of I^+ are denoted by vw, the elements $v \otimes w - v \otimes w$ of I^- are denoted by $v \wedge w$. The subsets I^+ and I^- respectively generate the subspaces

$$S^2V \text{ and } \bigwedge^2 V$$

of symmetric and antisymmetric tensors. Both I^+ and I^- are affine cones with vertex at the origin in the affine space $V \otimes V$ andthe same is true for I. The corresponding projectivizations define some well known projective varieties:

$$\big|I\big| \subset \mathbb{P}(V \otimes V)$$

is the *Segre variety* $s_2(\mathbb{P}^n \times \mathbb{P}^2)$.

$$|I^+| \subset \mathbb{P}(S^2V)$$

is the quotient of $\mathbb{P}^n \times \mathbb{P}^n$ under the natural involution sending (x, y) to (y, x). It is equal to the image of the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$ under the Segre map. Finally

$$|I^-| \subset \mathbb{P}(\bigwedge^2 V)$$

is the Grassmann variety $G(2, V)$ of 2-dimensional linear subspaces of V, or, equivalently, lines in $\mathbb{P}^n = \mathbb{P}(E)$. Recall that this description is obtained by assigning to any nonzero element $v \wedge w \in I^-$ the linear subspace spanned by the vectors v, w. Conversely any spanning set v, w of a 2-dimensional subspace of V defines a unique element $v \wedge w$ in $|I^-|$.

Note that

$$V \otimes V = S^2 V \oplus \bigwedge^2 V$$

and hence $G(2, V)$ is equal to the image of the projection of the Segre variety to $\mathbb{P}(\bigwedge^2 V)$ with center at $|I^+|$. Similarly, $|I^+|$ is the projected Segre variety with center at $G(2, V)$.

To obtain a system of equations defining $G(2, V)$ observe that $\bigwedge^2 V$ is canonically isomorphic to the space ofalternating bilinear forms $\omega : V^* \times V^* \to \mathbb{C}$ and that

$$I^- \setminus \{\bar{0}\} = \{\omega \in \bigwedge^2 V : \text{rank}(\omega) = 2\}.$$

Therefore, after fixing a basis in V, we can identify $\bigwedge^2 V$ with the space of $(n+1) \times (n+1)$ skew-symmetric matrices$A = (p_{ij})$and $G(2, V)$ with the locus of matrices of rank 2 (there are no nonzero skew-symmetric matrices of rank ≤ 1). The entries $p_{ij}, i < j$, define projective coordinateson $\mathbb{P}(\bigwedge^2 V)$, the so called *Plücker coordinates*. In particular $G(2, V)$ is the zero set of the 4×4 pfaffians of A. In fact, a stronger assertion is true.

Proposition 11.1.1. *The homogeneous ideal of $G(2, V) \subset \mathbb{P}(\bigwedge^2 V)$ is generated by 4×4 pffafians of a general skew-symmetric matrix of size $n + 1$.*

Remark 11.1.1. Another way to look at a point L of $G(2, V)$ as a $2 \times (n+1)$ matrix X whose rows are the coordinate vectors of a basis of L. A different choice of a basis changes X to CX, where C is an invertible matrix of size 2. Thus $G(2, V)$ can be viewed as the orbit space $\text{Mat}'_{2,n+1}/\text{GL}(2)$ of the set of rank 2 matrices of size $2 \times n + 1$ with respect to the action of $\text{GL}(2)$ by left multiplications. The Plücker coordinates can be identified with $\binom{n+1}{2}$ maximal minors of X. They are $\text{SL}(2)$ invariant polynomials on the space $\text{Mat}_{2,n+1}$. The First Fundamental Theorem of invariant theory asserts that they generate the algebra of $\text{SL}(2)$-invariant polynomials. The Second Fundamental Theorem asserts that the relations between the maximal minors are generated by the pfaffians.

Proposition 11.1.2. *$G(2,V)$ is a smooth, irreducible variety of dimension $2(n-1)$.*

Proof. This follows from the fact that $G(2,V)$ can be covered by open affine subsets isomorphic to affine space $\mathbb{A}^{2(n-1)}$. If we choose to represent points of $G(2,V)$ as skew-symmetric matrices, then the open subsets are matrices with one entry p_{ij} not equal to zero. All other $2(n-1)$ entries above the diagonal could be arbitrary. If we choose to represent points of $G(2,V)$ as $2\times(n+1)$ matrices, the the open subset corresponds to matrices with one minor $\det M_{ij}$ not equal to zero. Multiplying the matrix by M_{ij}^{-1} on the left we may assume that $M_{ij} = I_2$ is the identity matrix. Then the remaining $2(n-1)$ entries could be arbitrary. □

Remark 11.1.2. A reader familiar with the general notion of a Hilbert scheme should observe that $G(2,V)$ is the Hilbert scheme of subschemes of $\mathbb{P}(E)$ with Hilbert polynomial $P(t) = t+1$. Since the normal bundle N_l of any line l is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{n-1}$, we see that $H^1(l,N_l) = 0, H^0(l,N_l) = 2(n-1)$. The first equality implies that the Hilbert scheme is smooth at l, the second one gives the dimension of the Zariski tangent space.

11.1.1 Tangent and secant varieties

By 11.1.1 the ideal of $G(2,V)$ is generated by $\binom{n+1}{4}$ quadratic forms of rank 6:

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = \mathrm{Pf}\begin{pmatrix} 0 & p_{ij} & p_{ik} & p_{il} \\ -p_{ij} & 0 & p_{jk} & p_{jl} \\ -p_{ik} & -p_{jk} & 0 & p_{kl} \\ -p_{il} & -p_{jl} & -p_{kl} & 0 \end{pmatrix}$$

with $1 \le i < j < k < l \le n+1$. If $n = 3$ then G is the *Klein quadric*

$$V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) \subset \mathbb{P}^5$$

defining the Grassmannian of lines in $\mathbb{P}^3$. A point $p \in \mathbb{P}(\bigwedge^2 V)$ is a1-dimensional vector space generated by an alternating bilinear form

$$w_p : V^* \times V^* \to \mathbb{C}.$$

Since w_p is alternating, the rank of w_p is always even.

Definition 11.1. *The rank of p is the rank of w_p.*

It is clear that w_p has rank $\le 2k$ iff thereexist matrices $s_1,\ldots,s_k$ of rank 2 such that

$$w_p = s_1 + \ldots + s_k.$$

In other words p has rank $\le 2k$ if and only if$p \in S$,where S is a space of dimension $k-1$ which is at least k-secant to G. This gives the following.

Proposition 11.1.3. *The variety*

$$G_k =: \{p \in \mathbb{P}(\bigwedge^2 V) : \ p \text{ has rank} \leq 2k+2\}$$

is the k-secant variety $\mathrm{Sec}_k(G)$ *of* $G = G(2, V)$.

Let

$$t = [\frac{n-3}{2}],$$

then t is the maximal number k such that $\mathrm{Sec}_k G \neq \mathbb{P}(\bigwedge^2 V)$. So the Plücker space is stratified by the rankof its points and the strata are the following:

$$\mathbb{P}^n \setminus \mathrm{Sec}_t(G),\ \mathrm{Sec}_t(G) \setminus \mathrm{Sec}_{t-1},\ \ldots, \mathrm{Sec}_1 \setminus G,\ G. \tag{11.1}$$

It follows from the previous remarks that $\mathrm{Sec}_k(G) \setminus \mathrm{Sec}_{k-1}(G)$ is the orbit of a matrix of rank $2k+2$ and size $n+1$ under the action of $\mathrm{GL}(n+1)$. Therefore

$$\dim \mathrm{Sec}_k(G) = \dim \mathrm{GL}(n+1)/H_k$$

where H_k is the stabilizer of a skew symmetric matrix of rank $2k+2$ (e.g. with the standard symplectic matrix J_{2k+2} in the left upper corner and zero elsewhere). An easy computation gives the following.

Proposition 11.1.4. *Let* $0 \leq k \leq t$*, then*

$$d_k = \dim \mathrm{Sec}_k(G) = (k+1)(2n-2k-1) - 1.$$

Let $X \subset \mathbb{P}^r$ be a reduced and non degenerate variety: the *k-th defect* of X can be defined as

$$\delta_k(X) = \min\big((k+1)\dim X + k, r\big) - \dim \mathrm{Sec}_k(X),$$

which is the difference between the expected dimension of the k-secant variety of X and the effective one. We say that X is *k-defective* if $\mathrm{Sec}_k(X)$ is a proper subvariety and $\delta_k(X) > 0$.

Example 11.1.1. Let$n = 2t+3$, then$\mathrm{Sec}_t(G) \subset \mathbb{P}(\bigwedge^2 V)$is the pfaffian hypersurface of degree $t+2$ in $\mathbb{P}(\bigwedge^2 V)$ parametrizing singular skew-symmetric matrices (p_{ij}) of size $2t+4$. The expected dimension of $\mathrm{Sec}_t(G)$ is equal to $4t^2+8t+5$ which is larger than $\dim \mathbb{P}(\bigwedge^2 V) = \binom{2t+4}{2} - 1$. Thus $d_t(G) = \dim \mathrm{Sec}_t(G) + 1$ and $\delta_t(G) = 1$.

In the special case $n = 5$, we have $t = 2$ and $\dim G = 8$. Recall that a nondegenerate subvariety $X \subset \mathbb{P}^r$ with $\dim X = [\frac{2r}{3}] - 1$ is called a *Severi-Zak variety* if $\mathrm{Sec}_1(X) \neq \mathbb{P}^r$. There are four non-isomorphic Severi-Zak varieties and $G(2,6)$ is one of them. The other three are the Veronese surface in $\mathbb{P}^5$, the Segre variety $s_2(\mathbb{P}^2 \times \mathbb{P}^2)$ in $\mathbb{P}^8$ and the E_6-variety of dimension 16 in $\mathbb{P}^{26}$.

11.1.2 The incidence variety

Consider the incidence correspondence:

$$Z = \{(x, l) \in \mathbb{P}^n \times G : x \in l\}$$

and the corresponding projections:

$$p : Z \to \mathbb{P}^3, \quad q : Z \to G.$$

The fibre of p over a point x is isomorphic to the set of lines containing the point x. If $x = [v]$ for some $v \in V$, then a line containing p corresponds to a point in $\mathbb{P}(V/\mathbb{C}v)$. Thus the fibres of p are isomorphic to $\mathbb{P}^{n-1}$. The fibre of q over a line $l \in G$ is isomorphic to $l \cong \mathbb{P}^1$.

Let us see that both projections are the structure projections of the corresponding projective bundles. Recall that we identify a vector space V over a field K with K-points of the affine space Spec$(S^\bullet V^*)$, where $S^\bullet$ denotes the graded symmetric algebra of V. The projectivization $\mathbb{P}(E)$ is the set of points of the projective scheme Proj$(S^\bullet V^*)$. Similarly, any locally free sheaf $\mathcal{E}$ over a scheme X defines the vector bundle $E = \mathbb{V}(\mathcal{E}) =$ Spec$(S^\bullet \mathcal{E}^*)$ and the projective bundle $\mathbb{P}(E) =$ Proj$(S^\bullet \mathcal{E}^*)$. The sheaf of local sections of E is isomorphic to $\mathcal{E}$ and the set of rational points of the fibre E_x over a point $x \in X$ with residue field $k(x)$ can be identified with the linear space $\mathcal{E} \otimes_{\mathcal{O}_X} k(x)$.

We will identify locally free sheaves with the corresponding vector bundles. Thus an exact sequence of vector bundles means an exact sequence of the corresponding sheaves. Note that our notation is dual to one used in [Hartshorne], where $\mathbb{P}(E) =$ Proj$(S^\bullet \mathcal{E})$.

Let $\pi : \mathbb{P}(E) \to X$ be the canonical structure morphism of a X-scheme. There exists a unique invertible sheaf (line bundle) $\mathcal{L}$ on $\mathbb{P}(E)$ together with a surjective morphism of sheaves $\pi^*(\mathcal{E}^*) \to \mathcal{L}$ such that the corresponding morphism

$$\mathcal{E}^* \cong \pi_*(\pi^*(\mathcal{E}^*)) \to \pi_*(\mathcal{L})$$

is an isomorphism (see [Hartshorne]). This sheaf is denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$. Note that, for any invertibel sheaf $\mathcal{M}$, the projective bundles $\mathbb{P}(E)$ and $\mathbb{P}(E \otimes \mathcal{M})$ are isomorphic, however the corresponding $\mathcal{O}(1)$ sheaves are not.

Let us return to our Grassmannians. Consider the sheaf $\mathcal{F} = q_* p^* \mathcal{O}_{\mathbb{P}^n}(1)$ on G. It is clear that the restriction of $\mathcal{F}$ to each fibre l of q is equal to the restriction of $\mathcal{O}_{\mathbb{P}^n}(1)$ to the corresponding line and hence is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$. Since $H^i(\mathcal{O}_{\mathbb{P}^1}(1)) = 0, i > 0$, one can use Corollary 9.4 from [Hartshorne, Chapter 3] to deduce that $\mathcal{F}$ is locally free of rank 2. Note that

$$H^0(G, \mathcal{F}) = H^0(G, q_* p^* \mathcal{O}_{\mathbb{P}^n}(1)) \cong H^0(Z, p^* \mathcal{O}_{\mathbb{P}^n}(1)) \cong H^0(\mathbb{P}^n, p_* p^* \mathcal{O}_{\mathbb{P}^n}(1)).$$

By the projection formula, $p_*p^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^1} \otimes p_*(\mathcal{O}_Z) \cong \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^n}$. Thus we get a canonical isomorphism between the spaces $H^0(G, \mathcal{F})$ and E^*. Let V_G^* be the trivial bundle associated with vector space E^*. Consider the map $e : V_G^* \to \mathcal{F}$ defined by taking the germs of global sections. Since the fibre $\mathcal{F}_l$ over a line $l \in G$ is isomorphic to $H^0(l, \mathcal{O}_l(1))$, the restriction of the map e to fibres corresponds to the restriction map $V^* \to H^0(l, \mathcal{O}_l(1))$ which is obviously surjective. Thus e is surjective map. Its kernel $\mathcal{G}$ is a locally free sheaf on G of rank $n-1$.

Definition 11.2. *The sheaf* $\mathcal{F}^* = (q_*p^*\mathcal{O}_{\mathbb{P}^n}(1))^*$ *is denoted by* $\mathcal{S}_G$ *and is called* tautological (universal) subbundle *on* G. *The sheaf* $G^* = \mathrm{Ker}(e)^*$ *is denoted by* $\mathcal{Q}_G$ *and is called the* universal quotient bundle. *By definition we have the following exact sequence*

$$0 \to \mathcal{S}_G \to V_G \to \mathcal{Q}_G \to 0. \tag{11.2}$$

This is called the tautological exact sequence *on* G. *We will also use the dual exact sequence*

$$0 \to \mathcal{Q}_G^* \to V_G^* \to \mathcal{S}_G^* \to 0. \tag{11.3}$$

Remark 11.1.3. The Grassmann variety $G(r, V)$ of m-dimensional subspaces in V represents a functor $\mathbf{G}$ which assigns to a scheme S the set of rank r vector bundles $\mathcal{E}$ together with an injective map $\sigma : V^* \to \Gamma(\mathcal{E})$ such that the image generates $\mathcal{E}$. Given such a pair $(\mathcal{E}, \sigma)$, one defines a morphism $S \to G(r, V)$ by assigning to a point $s \in S$, the vector space $\mathrm{Ker}(r_x \circ \sigma)^\perp \subset V$, where $r_x : \Gamma(\mathcal{E}) \to \mathcal{E} \otimes k(x)$ is the natural restriction map to the fibre (not the stalk) of $\mathcal{E}$. The sheaf $\mathcal{S}_G^*$ is the universal sheaf in the sense of representable functors. For any $(\mathcal{E}, \phi) \in \mathbf{G}(S)$ let $\phi : S \to G(r, V)$ be the corresponding morphism. Then $\mathcal{E} = \phi^*(\mathcal{S}_G^*), \sigma = \phi^*(r)$, where $s : V^* \to \Gamma(\mathcal{S}_G^*)$ is the natural map defined by (11.3).

Consider the Plücker embedding $i : G \to \mathbb{P}(\bigwedge^2 V)$. It is given by assignment to the line $l = \mathbb{P}(L)$ the one-dimensional subspace $\bigwedge^2 L \subset \bigwedge^2 V$. The space $\bigwedge^2 L$ corresponds to the point $i(l)$. This shows that $\mathcal{O}_G(-1) = i^*\mathcal{O}_{\mathbb{P}(\bigwedge^2 V)}(-1) = \bigwedge^2(\mathcal{S}_G)$, hence

$$c_1(\mathcal{Q}_G) = -c_1(\mathcal{S}_G^*) = c_1(\mathcal{O}_G(1)), \tag{11.4}$$

or equivalently

$$\bigwedge^{n-1} \mathcal{Q}_g \cong \mathcal{O}_G(1).$$

Lemma 11.1.5. *(i) The projection* $p \colon Z \to \mathbb{P}^n$ *is isomorphic to the projective tangent bundle*

$$\mathbb{P}(T_{\mathbb{P}^n}(-1)) \to \mathbb{P}^n.$$

(ii) The projection $q \colon Z \to G$ *is isomorphic to the the projective bundle associated to the universal subbundle* $\mathcal{S}_G$ *on* G.

Proof. (i) Recall the Euler sequence describing the tangent bundle $T_{\mathbb{P}^n} = (\Omega^1_{\mathbb{P}^n})^*$ of $\mathbb{P}^n$:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to T_{\mathbb{P}^n} \to 0$$

Twisting it by $\mathcal{O}_{\mathbb{P}^n}(-1)$, we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \otimes V \to T_{\mathbb{P}^n}(-1) \to 0.$$

Localizing the sequence at any point $x \in \mathbb{P}^n$ we deduce that the fibre of $T_{\mathbb{P}^n}(-1)$ at a point $x = [v] \in \mathbb{P}^n$ is equal to the quotient space $V/\mathbb{C}v$. Thus the projection p is the projective bundle $\mathbb{P}(T_{\mathbb{P}^n}(-1)) \cong \mathbb{P}(T_{\mathbb{P}^n})$, the projectivization of the tangent bundle. Moreover the associated long exact sequence yields the identification

$$H^0(T_{\mathbb{P}^n}(-1)) = V. \tag{11.5}$$

(ii) Again applying Corollary 9.4 from [Hartshorne, Chapter 3], we get an isomorphism $S^k(q_*p^*\mathcal{O}_{\mathbb{P}^1}(1)) \cong q_*(\mathcal{O}_{\mathbb{P}^1}(k))$ for any $k \geq 0$. Consider the complete linear system on Z defined by the invertible sheaf $\mathcal{L} = p^*\mathcal{O}_{\mathbb{P}^1}(1)$. It defines a map $Z \to \text{Proj}(S^\bullet\mathcal{S}_G^*) = \mathbb{P}(\mathcal{S}_G)$ which as is easy to see (by restriction to fibres) must be an isomorphism. Note for the future use that

$$\mathcal{O}_{\mathbb{P}(\mathcal{S}_G)}(1) \cong p^*\mathcal{O}_{\mathbb{P}^1}(1). \tag{11.6}$$

□

Remark 11.1.4. The equality (11.5) implies that sections $s \in H^0(T_{\mathbb{P}^n}(-1))$ can be described very geometrically as follows.Let $s \neq 0$ and let $x_s \in \mathbb{P}^n$ be the point defined by s, then s vanishes exactly at x_s. Moreover we have the rational section

$$f_s : \mathbb{P}^n \to Z, \quad x \mapsto (x_s, \langle x_s, x\rangle).$$

Let us compute the canonical sheaf ω_G of G.

Lemma 11.1.6. *Let T_G be the tangent bundle of G. There is a natural isomorphism of sheaves*

$$\begin{aligned} T_G &\cong \mathcal{S}_G^* \otimes \mathcal{Q}_G, \\ \omega_G &\cong \mathcal{O}_G(-n-1), \end{aligned}$$

where $\mathcal{O}_G(1)$ is taken with respect to the Plïcker embedding.

Proof. This was an Exercise from Chapter 2 of Part I, so we give its solution. It is easy to see (same as for the projective space) that the tangent space TG_l is canonically isomorphic to $\mathrm{Hom}(L, V/L) \cong L^* \otimes V/L = (\mathcal{S}_G)_l^* \otimes (\mathcal{Q}_G)_l$, where $l = \mathbb{P}(L)$. One can show that this isomorphism can be extended to the isomorphism of sheaves (see the details in [Altman-Kleiman, Lect. Notes in Math. 146]). Globalizing we easily get the first isomorphism. To get the second isomorphism we have to compute $\bigwedge^{2n-2}(\mathcal{S}_G^* \otimes \mathcal{Q}_G)^*$. Recall that for any locally free sheaf $\mathcal{E}$ of rank r, its first Chern class is defined by $c_1(\mathcal{E}) = c_1(\bigwedge^r(\mathcal{E})$, where the first Chern class of an invertible sheaf is the corresponding divisor class. We use the splitting principle, according to which we may assume that $\mathcal{E}$ is the direct sum of line bundles L_i. In this case $c_1(\mathcal{E}) = \oplus c_1(L_i)$. This easily implies that, for any locally sheaves $\mathcal{A}$ and $\mathcal{B}$ of ranks r_1, r_2, respectively we have

$$c_1(\mathcal{A} \otimes \mathcal{B}) = r_2 c_1(\mathcal{A}) + r_1 c_1(\mathcal{B}). \tag{11.7}$$

Indeed, if we write $\mathcal{A} = \oplus_{i=1}^{r_1} \mathcal{A}_i, \mathcal{B} = \oplus_{i=1}^{r_2} \mathcal{B}_i$, we get, using that the determinant of the direct sum is equal to the tensor product of the determinants of each summand,

$$c_1(\mathcal{A} \otimes \mathcal{B}) = c_1(\bigoplus_{i,j} \mathcal{A}_i \otimes \mathcal{B}_j) = \sum_{i,j} c_1(\mathcal{A}_i) + c_1(\mathcal{B}_j)$$

$$= r_2(\sum c_1(\mathcal{A}_i)) + r_1(\sum c_1(\mathcal{B}_j)) = r_2 c_1(\mathcal{A}) + r_2 c_1(\mathcal{B}).$$

Applying (11.7) to our situation, we get

$$c_1(\omega_G^*) = c_1(\mathcal{S}_G^* \otimes \mathcal{Q}_G) = (n-1)c_1(\mathcal{S}_G^*) + 2c_1(\mathcal{Q}_G). \tag{11.8}$$

Now we use the tautological exact sequence (11.2) to get $\bigwedge^2 \mathcal{S}_G \otimes \bigwedge^{n-1} \mathcal{Q}_G = \mathcal{O}_G$. This gives(11.7), $c_1(\mathcal{S}_G) = -c_1(\mathcal{Q}_G)$. It follows from the splitting principle that $c_1(\mathcal{E}^*) = -c_1(\mathcal{E})$. Now formula (11.8) gives

$$c_1(\omega_G^*) = (n-1)c_1(\mathcal{Q}_G) + 2c_1(\mathcal{Q}_G) = (n+1)c_1(\mathcal{Q}_G)).$$

Applying formula (11.6) we get

$$c_1(\omega_G) = -(n+1)c_1(\mathcal{Q}_G) = -(n+1)c_1(\mathcal{O}_G(1).$$

□

Next lemma computes the canonical sheaf of Z as well as the relative canonical sheaves with respect to the projections p and q.

Lemma 11.1.7. *Let $\mathcal{O}_Z(1)$ be the tautological line bundle of the projective bundle $Z = \mathbb{P}(T_{\mathbb{P}^n}(-1))$ (i.e. $p_*(\mathcal{O}_Z(1)) = T_{\mathbb{P}^n}(-1)$). Then*

$$\begin{aligned}
\omega_{Z/\mathbb{P}^n} &\cong p^*\mathcal{O}_{\mathbb{P}^n}(n-1) \otimes q^*\mathcal{O}_G(-n),\\
\omega_{Z/G} &\cong p^*\mathcal{O}_{\mathbb{P}^n}(-2) \otimes q^*\mathcal{O}_G(1),\\
\omega_Z &\cong p^*\mathcal{O}_{\mathbb{P}^n}(-2) \otimes q^*\mathcal{O}_G(-n),\\
\mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^n}(-1))}(1) &\cong p^*(\mathcal{O}_{\mathbb{P}^n}(-1)) \otimes q^*(\mathcal{O}_G(1)),
\end{aligned}$$

where $\mathcal{O}_G(1)$ is the line bundle associated to a hyperplane section of G in its Plücker embedding.

Proof. We use the following relative Euler sequence which computes the relative tangent bundle of the projective bundle $\pi : \mathbb{P}(E) \to X$.

$$0 \to \mathcal{O}_{\mathbb{P}(E)} \to \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*(E) \to T_{\mathbb{P}(E)/X} \to 0 \tag{11.9}$$

We apply this formula to both projections p and q. First use p to get the exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^n}(-1))}(1) \otimes p^*(T_{\mathbb{P}^n}(-1)) \to T_{Z/\mathbb{P}^n} \to 0.$$

Taking the dual exact sequence and then the exterior powers we get

$$\omega_{Z/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^n}(-1))}(-n) \otimes p^*\omega_{\mathbb{P}^n}(n) \cong \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^n}(-1))}(-n) \otimes p^*\mathcal{O}_{\mathbb{P}^n}(-1). \tag{11.10}$$

Now we use the Euler exact sequence for the projection q

$$0 \to \mathcal{O}_Z \to \mathcal{O}_{\mathbb{P}(\mathcal{S}_G)}(1) \otimes q^*(\mathcal{S}_G) \to T_{Z/G} \to 0.$$

As before, we get

$$\omega_{Z/G} \cong \mathcal{O}_{\mathbb{P}(\mathcal{S}_G)}(-2) \otimes q^*(\bigwedge^2 \mathcal{S}_G^*) \cong p^*\mathcal{O}_{\mathbb{P}^n}(-2) \otimes q^*\mathcal{O}_G(1). \tag{11.11}$$

For the last isomorphism we used formula (11.6).

Now we compute the canonical sheaf of Z in two ways by using the two projections.

$$\omega_Z \cong p^*\omega_{\mathbb{P}^n} \otimes \omega_{Z/\mathbb{P}^n} \cong q^*\omega_G \otimes \omega_{Z/G}.$$

This gives

$$\begin{aligned}
\omega_{Z/\mathbb{P}^n} &\cong q^*\omega_G \otimes \omega_{Z/G} \otimes p^*\omega_{\mathbb{P}^n}^{-1}\\
&\cong \omega_{Z/G} \otimes q^*\mathcal{O}_G(-n-1) \otimes p^*\mathcal{O}_{\mathbb{P}^n}(n+1)
\end{aligned}$$

$$\cong p^*\mathcal{O}_{\mathbb{P}^n}(-2) \otimes q^*\mathcal{O}_G(1) \otimes q^*\mathcal{O}_G(-n-1) \otimes p^*\mathcal{O}_{\mathbb{P}^n}(n+1)$$

$$\cong q^*\mathcal{O}_G(-n) \otimes p^*\mathcal{O}_{\mathbb{P}^n}(n-1).$$

Now we use (11.10) and (11.11) to obtain

$$\mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^n}(-1))}(-n) \cong p^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \omega_{Z/\mathbb{P}^n}$$

$$\cong p^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes q^*\mathcal{O}_G(-n) \otimes p^*\mathcal{O}_{\mathbb{P}^n}(n-1) \cong q^*\mathcal{O}_G(-n) \otimes p^*\mathcal{O}_{\mathbb{P}^n}(n)$$

It remains to use that $\mathrm{Pic}(G)$ has no torsion. Even more is true.

$$\mathrm{Pic}(G) \cong \mathbb{Z}\mathcal{O}_G(1). \tag{11.12}$$

In fact a section of $\mathcal{O}_G(1)$ defined by a Plücker coordinate vanishes along a hyperplane section of G and the complement set is isomorphoc to affine space $\mathbb{A}^{2n-2}$. □

Let X be a subvariety of G, and Z_X be the pre-image of X under the projection $p : Z \to G$. The image of Z_X in $\mathbb{P}^n$ is the union of lines $l \in X$. We will need the description of its set of nonsingular points. The next lemma follows from the definitions of tangent spaces.

Proposition 11.1.8. *The projection $p_X : Z_X \to \mathbb{P}^n$ is smooth at (x, l) if and only if*

$$\dim_l \Omega(x) \cap \mathrm{PT}(X)_l = \dim_{(x,l)} p_K^{-1}(x)$$

Proof. Let $(x, l) \in Z_X$ and let F be the fibre of $p_X : Z_X \to \mathbb{P}^n$ passing through the point (x, l) identified with the subset $\Omega(x) \cap X$ under the projection $q : Z_X \to G$. Then

$$\mathrm{PT}(F)_{x,l} = \mathrm{PT}(\Omega(x))_l \cap \mathrm{PT}(X)_l = \Omega(x) \cap \mathrm{PT}(X)_l. \tag{11.13}$$

This proves the assertion. □

Corollary 11.1.9. *Let $Y = p_X(Z_X) \subset \mathbb{P}^n$ be the union of lines $l \in X$. Assume X is nonsingular and $p_X^{-1}(x)$ is a finite set. Suppose $\dim_l \Omega(x) \cap \mathrm{PT}(X)_l = 0$ for some $l \in X$ containing x. Then x is nonsingular as a point of Y.*

11.1.3 Schubert varieties

These varieties parametrize lines with special linear condition with respect to fixed linear subspaces. Fix a flag

$$A_0 \subsetneq A_1 \subset \mathbb{P}^n$$

of subspaces of dimension a_0, a_1 and define the *Schuber variety*

$$\Omega(A_0, A_1) = \{l \in G : \dim L \cap A_i \geq i\}.$$

This is a closed subvariety of G of dimension $a_0 + a_1 - 1$. It admits a rational map to A_0 (sending l to $l \cap A_0$) with fibres isomorphic to projective space $\mathbb{P}^{a_1-1}$. The varieties

$$\Omega(A_0, \mathbb{P}^n) = \{l \in G : l \cap A_0 \neq \emptyset\}$$

are called *special Schuber varieties*.

The *Schubert cycles* are the classes of the Schubert varieties in the Chow ring $\mathrm{CH}^\bullet(G)$ of algebraic cycles on G modulo rational equivalence (see [Fulton]). A reader not familiar with these notions may substitute $\mathrm{CH}^\bullet(G)$ with the cohomology ring $H^*(G, \mathbb{Z})$. The Schubert cycle of $\Omega(A_0, A_1)$ depends only on a_0, a_1 and is denoted by $\sigma_{n-1-a_0, n-a_1}$ or just σ_{n-1-a_0} if $a_1 = n$ (in classical notation $\sigma_{\lambda_0,\lambda_1}$ is just (λ_0, λ_1)). Its codimension is equal to the sum of the numbers. Observe that $\Omega(A_0, A_1)$ is equal to the Grassmannian $G_1(A_1) \cong G(2, a_1 + 1)$ of lines in A_1 if $a_0 = a_1 - 1$ since any line in A_1 intersects A_0. Thus

$$(\lambda_0, \lambda_0) = [G_1(A_1)], \ \dim A_1 = n - \lambda_0.$$

A proof of the following result can be found in [Fulton] or [Hodge-Pedoe].

Proposition 11.1.10. *The Chow ring* $\mathrm{CH}^\bullet(G)$ *is generated by the classes of the Schubert varieties. The Schubert cycles of codimension* r *form a free basis of the group* $\mathrm{CH}^r(G)$ *of classes of algebraic cycles of codimension* r*.*

Recall that the Chern classes $c_m(\mathcal{E})$ of a vector bundle $\mathcal{E}$ are defined as the coefficients c_m in the relation in $\mathrm{CH}^\bullet(\mathbb{P}(\mathcal{E}))$

$$\sum_{m=0}^{r} p^* c_m(\mathcal{E}) \cdot \xi^{r-m} = 0, \tag{11.14}$$

where $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

The special Schubert cycles

$$\sigma_m = [\Omega(A_0, \mathbb{P}^n)], \quad m = n - 1 - a_0$$

can be realized as Chern classes of the vector bundle $\mathcal{Q}_G$ (see [Fulton]).

$$c_m(\mathcal{Q}_G) = \sigma_m. \tag{11.15}$$

This also computes the Chern classes of $\mathcal{S}_G$ by using the relation

$$(1 + c_1(\mathcal{S}_G) + c_2(\mathcal{S}_G))(1 + c_1(Q_G) + \ldots + c_{n-1}(\mathcal{Q}_G)) = 1.$$

For example,

$$c_1(\mathcal{S}_G) = -\sigma_1, \quad c_2(\mathcal{S}_G) = -\sigma_2 + \sigma_1^2. \tag{11.16}$$

Codimension one Schubert cycle is the class

$$\sigma_1 = \text{lines meeting a fixed codimension two subspace}$$

Thus, we obtain

$$\text{Pic}(G) = \mathbb{Z}\sigma_1.$$

Comparing with (11.12) we see that

$$\sigma_1 = c_1(\mathcal{O}_G(1)) = c_1(\mathcal{Q}_G)). \tag{11.17}$$

Codimension two Schubert cycles are the classes

$$\begin{aligned} \sigma_{1,1} &= \text{Grassmannian of lines in a fixed hyperplane of } \mathbb{P}^n \\ \sigma_2 &= \text{lines intersecting a codimension 3 subspace.} \end{aligned}$$

Codimension three Schubert cycles are the classes

$$\begin{aligned} \sigma_{2,1} &= \text{Grassmannian of lines in a fixed hyperplane } H \text{ of } \mathbb{P}^n \\ &\quad \text{intersecting a fixed subspace of } H \text{ of codimension 2.} \\ \sigma_3 &= \text{Grassmannian of lines in a fixed codimension 2 subspace of } \mathbb{P}^n \end{aligned}$$

The Schubert calculus allows one to write explicitly the intersection product in CH(G) in terms of a basis given by Schubert cycles. In particular, one can compute the degree of the Grassmanian (equal to σ_1^{2n-2}) and of Schubert varieties in their Plücker embedding. We refer for this to [Fulton], Example 14.7.11.

Proposition 11.1.11.

$$\begin{aligned} \deg\big(G(2, n+1)\big) &= \frac{(2n-2)!}{(n-1)!n!} \\ \deg \Omega(a_0, a_1) &= \frac{(a_0 + a_1 - 1)!}{a_0!a_1!}(a_1 - a_0)! \end{aligned}$$

Example 11.1.2. Let us look at the Grassmanian $G(2,4)$ of lines in $\mathbb{P}^3$. We know that this is a nonsingular quadric in $\mathbb{P}^5$. The Schubert cycle of codimension 1 is represented by the special Schubert variety $\Omega(l)$ of lines intersecting a given line l. We have two codimension 2 Schubert cycles σ_2 and $\sigma_{1,1}$ represented by the Schubert varieties $\Omega(x)$ of lines containing a given point and Ω_π of lines containing in a given plane π. Each of these varieties is isomorphic to $\mathbb{P}^2$. We have one-dimensional Schubert cycle $\sigma_{2,1}$ represented by the Schubert variety $\Omega(x,\pi)$ of lines in a plane π containing a given point $x \in \pi$. It is isomorphic to $\mathbb{P}^1$. Finally we have a 0-dimensional Schubert variety $\{l\}$ representing lines contained in a given line l. Thus

$$\mathrm{CH}(G(2,4)) = \mathbb{Z}[G] \oplus \mathbb{Z}\sigma_1 \oplus (\mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_{1,1}) \oplus \mathbb{Z}\sigma_{2,1} \oplus \mathbb{Z}[\text{point}].$$

Note that the two classes in codimension 2 represent two different rulings of the Klein quadric by planes.

Let us confirm that the degree of $G(2,4)$ is equal to 2 by doing the intersection theory in $G(2,4)$. We have

$$\sigma_1^2 = m\sigma_2 + n\sigma_{1,1}. \tag{11.18}$$

Representing σ_2 by Schubert varieties $\Omega(x_1)$ and $\Omega(x_2)$ for different points x_1, x_2 we find that $\#\Omega(x_1)\cap\Omega(x_2) = 1$. One can check that the intersection is transversal so $\sigma_2^2 = 1$. Similarly we find that $\sigma_{1,1}^2 = 1$ and $\sigma_1 \cdot \sigma_{1,1} = 0$. Intersecting both sides of (11.18) with σ_2 we get $m = \sigma_1^2 \cdot \sigma_2$. Since $\Omega(l) \cap \Omega_x = \Omega_{x,\pi}$, where π is the plane spanned by l and $x \notin l$, we obtain $m = \#\Omega_{l'} \cap \Omega_\pi = 1$ (there is a unique line in π containing x and the intersection point of l and π). Similarly, we see that $n = 1$. Now squaring both sides of (11.18), we get

$$\sigma_1^4 = (\sigma_2 + \sigma_{1,1})^2 = \sigma_2^2 + \sigma_{1,1}^2 = 1 + 1 = 2.$$

Similar computation gives the intersection numbers of two Schubert cycles of complementary codimensions

$$\sigma_{\lambda_0,\lambda_1} \cdot \sigma_{\lambda_0',\lambda_1'} = \begin{cases} 1 & \text{if } \lambda_i + \lambda_{1-i}' = n-1,\ i = 0,1 \\ 0 & \text{otherwise.} \end{cases} \tag{11.19}$$

Using Schubert varieties one can describe the projective tangent space of $\mathrm{Sec}_k(G)$ at a given point $p \neq \mathrm{Sec}_{k-1}(G)$. Consider p as linear map $V^* \to V$ and let K be its kernel. The rank of p is equal to $2k+2$. Thus orthogonal subspace $K^\perp \subset V$ defines a linear susbpace $\Lambda_p = \mathbb{P}(K^\perp)$ of $\mathbb{P}(E)$ of dimension $2k+1$. Let $\Omega(\Lambda_p)$ be the corresponding special Schubert variety and $\langle\Omega(\Lambda_p)\rangle$ be its linear span in the Plücker space.

Proposition 11.1.12.

$$\mathrm{PT}(\mathrm{Sec}_k(G))_p = \langle \Omega(\Lambda_p)\rangle.$$

Proof. Since $\mathrm{Sec}_k(G) \setminus \mathrm{Sec}_{k-1}(G)$ is a homogeneous space for $\mathrm{GL}(n+1)$ we may assume that the point p is represented by a bi-vector $\omega = \sum_{i=1}^{k+1} e_i \wedge e_{i+1}$. The corresponding subspace $K^\perp$ is spanned by $e_1, \ldots, e_{2k+2}$. A line l intersects $\mathbb{P}(\Lambda_p)$ if and only if it can be represented by a bi-vector $v\wedge w$, where $v \in K^\perp$. Thus $W = \langle \Omega(\Lambda_p)\rangle$ is the span of bi-vectors $e_i \wedge e_j$, where either i or j is less or equal than $2k$. In other words W is given by vanishing $\binom{n-2k-1}{2}$ Plücker coordinates p_{ab}, where $a, b > 2k+2$. It is easy to see that this agrees with formula (11.1.4) for $\dim \mathrm{Sec}_k(G)$. So, it is enough to show that W is contained in the tangent space. We know that the equations of $\mathrm{Sec}_k(G)$ are given by Pffafians of size $4k+4$. Recall the formula for the Pffafians from Part I, Chapter 2, Exercise 2.1

$$\mathrm{Pf}(A) = \sum_{S\in\mathcal{S}} \pm \prod_{(ij)\in S} a_{ij},$$

where S is a set of pairs $(i_1 j_1), \ldots, (i_{2k+2}, j_{2k+2})$ such that $1 \le i_s < j_s \le 4k+4$, $s = 1, \ldots 2k+2$, $\{i_1, \ldots, i_{2k+2}, j_1, \ldots, j_{2k+2}\} = \{1, \ldots, 4k+4\}$. Consider the Jacobian matrix of $\mathrm{Sec}_k(G)$ at the point p. Each equation of $\mathrm{Sec}_k(G)$ is obtained by a choice of a subset I of $\{1, \ldots, n+1\}$ of cardinality $4k+4$ and writing the Pffafian of the submatrix of (p_{ij}) formed by the columns and rows with indices in I. The corresponding row of the Jacobian matrix is obtained by taking the partials of this equation with respect to all p_{ij} evaluated at the point p. If $a, b \le 2k+2$, then one of the factors in the product $\prod_{(ij)\in S} p_{ij}$ corresponds to a pair (i, j), where $i, j > 2k+2$. When we differentiate with respect to p_{ab} its value at p is equal to zero. Thus the corresponding entry in the Jacobian matrix is equal to zero. So, all nonzero entries in a row of the Jacobian matrix correspond to the coordinates of vectors from W which are equal to zero. Thus W is contained in the space of solutions. □

Taking $k = 0$, we obtain

Corollary 11.1.13.

$$\mathrm{PT}(G)_l = \langle \Omega(l)\rangle.$$

Let Λ be any subspace of $\mathbb{P}^n$ of dimension $2k+1$. Consider the set of points

$$P_\Lambda = \{p \in \mathbb{P}(\bigwedge^2 V) : \Lambda = \Lambda(p)\}.$$

This is the projectivization of the linear space of skew-symmetric matrices of rank $2k+2$ with the given nullspace of dimension $2k+2$. An easy computation using the formula (11.1.4) for $d_k = \dim \mathrm{Sec}_k(G)$ shows that its dimension is equal to $(2k+1)(k+1)-1$.

Let

$$\gamma_k : \mathrm{Sec}_k(G) \setminus \mathrm{Sec}_{k-1}(G) \to G(d_k+1, \Lambda^2 V), \quad d_k = \dim \mathrm{Sec}_k(G),$$

be the *Gauss map* which assigns to a point its projective tangent space. Applying Proposition 11.1.12, we obtain

Corollary 11.1.14.

$$\gamma_k^{-1}(\langle \Omega(\Lambda) \rangle) = P_\Lambda.$$

In particular, any hyperplane in the Plücker space containing $\Omega(\Lambda)$ *is tangent to* $\mathrm{Sec}_k(G)$ *along the subvariety* P_Λ *of dimension* $(2k+1)(k+1)-1$.

Example 11.1.3. Let $n = 5$. The secant variety $\mathrm{Sec}_1(G)$ is a cubic hypersurface in $\mathbb{P}^{14}$ defined by the pffafian of 6×6 skew-symmetric matrix whose entries are Plücker coordinates p_{ij}. The Gauss map is the restriction to $\mathrm{Sec}_1(G)$ of the polar map $\mathcal{P} : \mathbb{P}^{14} - \to \check{\mathbb{P}}^{14}$ given by the partials of the cubic. The singular locus of $\mathrm{Sec}_1(G)$ is G. The Plücker equations of G are the partials of the pffafian cubic. The map $\mathcal{P}$ is a Cremona transformation in $\mathbb{P}^{14}$ defined by the linear system of quadrics defining the Plücker equations of G. It can be resolved by blowing up G and then blowing down the proper inverse transform of $\mathrm{Sec}_1(G)$ to a subvariety isomorphic to G^*, where $G^* = G(2, V^*)$. The image of the exceptional locus of the blow-up is equal to $\mathrm{Sec}_1(G^*)$. Three other Severi-Zak varieties define a similar Cremona transformation (of $\mathbb{P}^5$, $\mathbb{P}^8$ and $\mathbb{P}^{26}$). It is given by the partials of the cubic form defining the first secant variety.

11.2 Linear complexes of lines

An effective divisor$D \subset G$is called a *complex of lines*. Since we know that $\mathrm{Pic}(G)$ is generated by $\mathcal{O}_G(1)$ we see that

$$D \in \mid \mathcal{O}_G(d) \mid$$

for some $d \geq 1$.The *degree* of D is d.

An example of a complex of degree d in $G(2, n+1)$ is the *Chow variety* of a subvariety $X \subset \mathbb{P}^n$ of codimension 2. It parametrizes lines which have non-empty intersection with X. Its degree is equal to the degree of X. When X is linear, this is of course the Schubert variety $\Omega(X)$.

A *linear complex* is a complex of degree one, that is a hyperplane section $H$of G. If no confusion arises we will sometimes identify Ω with the corresponding hyperplane $\langle H \rangle$.

In coordinates, a linear complex (or, more precisely, the corresponding hyperplane) can be written as

$$\sum_{1\le i<j\le n+1} a_{ij}p_{ij} = 0.$$

For example, the complex $p_{ij} = 0$ parametrizes the lines intersecting the coordinate $(n-2)$-plane $x_k = 0, k \neq i, j$, in $\mathbb{P}^n$.

Since any hyperplane in $\mathbb{P}(\bigwedge^2 V)$ is a point in $\mathbb{P}(\bigwedge^2 V^*)$ we can use the stratification (11.1) to classify the orbits of linear complexes up to the natural action of $\mathrm{GL}(V)$ on $H^0(\mathcal{O}_{\mathcal{G}}(1)) = V^*$. So, let

$$G^* = G(2, V^*) \cong G(n-1, V).$$

The $\mathrm{GL}(V)$-orbit of a linear complex Ω is uniquely determined by the *rank* $2k$ of $\langle\Omega\rangle$ considered as a bilinear form ω on V defined by the corresponding element of $\bigwedge^2 V^*$ or as a linear map $\alpha_\omega : V \to V^*$. Let $\mathrm{Ker}(\omega)$ be the radical of the bilinear form ω (or the kernel of the corresponding linear map $\alpha_\omega : V \to V^*$) and

$$C_H = \mathbb{P}(\mathrm{Ker}(\omega)). \tag{11.20}$$

It is called the *center* of a linear complex H. We have encountered with this in Chapter 2 of Part I. This is a linear subspace of $\mathbb{P}(E)$ of dimension $n-2k$, where $2k$ is the rank of H.

Proposition 11.2.1. *Let H be a linear complex and Λ_H be the linear subspace* (11.20) *associated to it. Then*

$$\Omega(C_H) \subset H,$$

$$G_1(C_H) = \mathrm{Sing}(H).$$

Proof. Since $\mathrm{GL}(V)$ acts transitively on the set of linear complexes of equal rank, we may assume that H is given by $\omega = \sum_{i=1}^k e_i^* \wedge e_{k+i}^*$, where $e_1^*, \ldots, e_{n+1}^*$ is a basis of E^* dual to a basis $e_1, \ldots, e_{n+1}$ of V. The linear space $\mathrm{Ker}(\omega)$ is spanned by $e_i, i > 2k$. A line l intersects C_H if and only it can be represented by a bivector $v \wedge w \in \bigwedge^2 V$, where $v \in C_H$. The linear span of $\Omega(\Lambda_H)$ is spanned by bivectors $e_i \wedge e_j$, where $i < 2k$. It is obvious that it is contained in the hyperplane $\langle H\rangle \subset \bigwedge^2 V$ defined by $\omega \in (\bigwedge^2 V)^*$. This checks the first assertion.

It follows from Corollary 11.1.13 that

$$l \in \mathrm{Sing}(H) \Longleftrightarrow \mathrm{PT}(G)_l \subset H \Longleftrightarrow \Omega(l) \subset H.$$

Suppose $\Omega(l) \subset H$ but l does not belong to C_H. We can find a point in l represented by a vector $v = \sum a_i e_i$, where $a_i \neq 0$ for some $i \leq 2k$. Then the line represented by a 2-vector $v \wedge e_{k+i}$ intersects l but does not belong to H (since $w_H(v \wedge e_{k+i}) = a_i \neq 0$). Thus $\Omega(l) \subset H$ implies $l \subset C_H$. Conversely, this inclusion implies $\Omega(l) \subset \Omega(C_H) \subset H$. This proves the second assertion. □

It follows from this proposition that any linear complex is singular unless its rank is equal to $2[\frac{n+1}{2}]$, maximal possible. Thus the set of hyperplanes in the Plücker space which are tangent to G can be identified with the set of linear complexes of rank $\leq 2[\frac{n-1}{2}]$. This gives

Corollary 11.2.2. *Let $t = [\frac{n-3}{2}]$, then $Sec_t(G^*)$ is equal the dual variety $\check{G} \subset \mathbb{P}(\bigwedge^2 V^*)$ of G.*

When $n = 3, 4$ we see that $G^* = \check{G}$, when $n = 5$ we obtain that $\check{G} = \mathrm{Sec}_1(G^*)$. This agrees with Example 11.1.3.

11.2.1 Linear complexes and apolarity

A smooth linear complex H considered as a linear map $\alpha_\omega : V \to V^*$ defined by a skew-symmetric bilinear form ω on V defines a polarity between linear subspaces in V by

$$E \mapsto E_\omega^\perp = \alpha_\omega(\Lambda)^\perp = \{w \in V : \omega(v, w) = 0, \forall v \in E\}.$$

Since ω is skew-symmetric, we have

$$\mathrm{Ker}(\omega) \subset E_\omega^\perp.$$

For any subspace $\Lambda = \mathbb{P}(E) \subset \mathbb{P}^n$ let

$$i_H(\Lambda) = \mathbb{P}(E_\omega^\perp).$$

Clearly $i_H(\Lambda)$ contains the center $C_H = \mathbb{P}(\mathrm{Ker}(\omega))$ of H. Its dimension is equal to $n + 1 - \dim \Lambda + \dim \Lambda \cap C_H$.

Since ω is skew-symmetric, for any point $x \in \mathbb{P}(E)$,

$$x \in i_H(x).$$

A correspondence between points and hyperplanes satisfying this property is classically known as a *null-system*.

In the special case when $n = 3$ and H is nonsingular, we have the *polar duality* between points and planes. The plane $\Pi(x)$ corresponding to a point x is called the *null-plane* of x. The point x_Π corresponding to a plane Π is called the *null-point* of Π. Note that $x \in \Pi(x)$ and $x_\Pi \in \Pi$.

We also have a correspondence between lines in $\mathbb{P}^3$

$$i_H : G \to G, \quad l = \mathbb{P}(L) \mapsto \mathbb{P}(L_H^{\perp}).$$

Note that the lines l and $i_H(l)$ are always skew or coincide. The set of fixed points of i_H on G is equal to H. It is easy to see that i_H corresponds to the projection of the quadric G in $\mathbb{P}^5$ whose center is the point c dual to H with respect to G. Thus

$$G/(i_H) \cong \mathbb{P}^4.$$

The hyperplane $\langle H \rangle$ is the polar hyperplane $P_c(G)$. The ramification divisor of the projection $G \to \mathbb{P}^4$ is the linear compex $H = P_c(G) \cap G$. The branch divisor is a quadric in $\mathbb{P}^4$.

Proposition 11.2.3. *Let l be a line in $\mathbb{P}^3$ and $l' = i_H(l)$ be its polar line with respect to a nonsingular linear complex H. Then any line $m \in H$ intersecting l intersects l'. The linear complex K consists of lines intersecting a pair of polar lines.*

Proof. Let $x = l \cap m$. Since $x \in m$, we have $m = m' \subset i_H(p)$. Also $l' \subset i_H(p)$. Thus the null-plane of x contains l' and m. Thus m and l' lie in the same plane and hence intersect. Conversely, suppose m intersects l at a point x and l' at a point x'. Then $i_H(m) = \Pi_x \cap \Pi_{x'}$ and hence contains x and x'. Thus it coincides with m. But we have noticed already that

$$H = \{l : i_H(l) = l\}.$$

□

Definition 11.3. *A linear complex H in $\mathbb{P}(\bigwedge^2 V)$ is called* apolar *to a linear complex H^* in $\mathbb{P}(\bigwedge^2 V^*)$ if $\omega_{H^*}(\omega_H) = 0$.*

In the case $n = 3$, we can identify $\mathbb{P}(\bigwedge^2 V)$ with $\mathbb{P}(\bigwedge^2 V^*)$ by means of the wedge-product pairing

$$\bigwedge^2 V \times \bigwedge^2 V \to \bigwedge^4 V \cong \mathbb{C} \tag{11.21}$$

defined by the Klein quadric. Thus we can speak about apolar linear complexes in $\mathbb{P}^3$. Their corresponding elements ω_H wedge to zero. In other words, H_1, H_2 are apolar if and only if, considered as point in $\mathbb{P}(\bigwedge^2 V)$, one has $P_{H_1 H_2}(Q) = 0$, where Q is the Klein quadric. In Plücker coordinates, this gives the relation

$$a_{12}b_{34} + a_{13}b_{24} - a_{14}b_{23} + a_{23}b_{14} - a_{24}b_{13} + a_{34}b_{12} = 0. \tag{11.22}$$

Lemma 11.2.4. *Let H and H' be two nonsingular linear compexes in $\mathbb{P}^3$ and $A : V \to V^*, B : V \to V^*$ be the corresponding linear maps. Then H and H' are apolar to each other if and only if $g = B^{-1} \circ A$, considered as a transformation of $\mathbb{P}(E)$, satisfies $g^2 = 1$.*

Proof. Take 2 skew lines l, l' in the intersection $H \cap H'$ of two complexes. Choose coordinates in V such that l and l are two opposite edges of the coordinate tetrahedron, say $l = V(x_1, x_3), l' = V(x_2, x_4)$. Then the linear complexes have the following equations in Plücker coordinates

$$H : ap_{12} + bp_{34} = 0; \quad H' : cp_{12} + dp_{34}.$$

The condition that H and H' are apolar is $ad + bc = 0$. Now we have

$$A = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & c & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{pmatrix},$$

$$B^{-1} = \begin{pmatrix} 0 & -c^{-1} & 0 & 0 \\ c^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -d^{-1} \\ 0 & 0 & d^{-1} & 0 \end{pmatrix}$$

This gives

$$AB^{-1} = \begin{pmatrix} a/c & 0 & 0 & 0 \\ 0 & a/c & 0 & 0 \\ 0 & 0 & b/d & - \\ 0 & 0 & 0 & b/d \end{pmatrix} = \frac{a}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This shows that $(AB^{-1})^2$ defines the identical transformation of $\mathbb{P}(E)$. It is easy to see that conversely, this implies that $ad + bc = 0$. □

In particular, a pair of apolar linear complexes defines an involution of $\mathbb{P}(E)$. Any pair of linear complexes defines a projective transformation of $\mathbb{P}(E)$ as follows. Take a point x, define its null-plane Π_H, and then take its null-point y with respect to H'. For apolar complexes we must get an involution. That is, the null-plane of y with respect to H must coincide with the null-plane of x with respect to H'.

Since any set of mutually apolar linear complexes is linearly independent, we see that the maximal number of mutually apolar linear complexes is equal to 6. If we choose these complexes as coordinates t_i in $\bigwedge^2 V^*$ we can write the equation of the Klein quadric as the sum

$$Q = \sum_{i=1}^{6} t_i^2.$$

Since each pair of apolar linear complexes defines an involution in $\mathbb{P}(\bigwedge^2 V)$ we obtain 15 involutions. They form an elementary abelian group of order 2^4 of projective transformations in $\mathbb{P}^3$. This group is called the *Heisenberg group*.

An example of six mutually apolar linear complexes is the set

$$(t_1, \ldots, t_6) = (i(p_{14}-p_{23}), p_{14}+p_{23}, i(p_{24}-p_{13}), p_{24}+p_{13}, i(p_{34}-p_{12}), p_{34}+p_{12}),$$

where $i = \sqrt{-1}$. These coordinates in the Plücker space are called the *Klein coordinates*.

It is easy to compute the corresponding Heisenberg group. Note that the skew-symmetric matrix A_i defining the coordinate t_1, t_3, t_5 satisfy $A_i^2 = -1$, the rest satisify $A_i^2 = 1$. Denote by (ab) the transformation defined by a pair (t_i, t_j) of complexes. We have

$$\begin{aligned}(12) : (x_0, x_1, x_2, x_3) &\mapsto (x_0, -x_1, -x_2, x_3),\\ (13) : (x_0, x_1, x_2, x_3) &\mapsto (x_1, x_0, -x_3, -x_2),\end{aligned}$$

We leave to the reader to finish the computation and convince yourself that the group is generated by the following transformations

$$(x_0, x_1, x_2, x_3) \mapsto (\epsilon_0 x_0, \epsilon_1 x_1, \epsilon_2 x_2, \epsilon_2 x_3),\ \epsilon_i = \pm 1, \quad \sum_{i=1} \epsilon_i = 0 \mod 2,$$

$$(x_0, x_1, x_2, x_3) \mapsto (x_1, x_0, x_2, x_3),$$

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_3, x_2),$$

11.2.2 6 lines

We know that any 5 lines in $\mathbb{P}^3$ are contained in a linear complex. In fact, in a unique linear complex when the lines are linear independent as vectors in $\bigwedge^2 V$. A set of 6 lines is contained in a linear complex only if they are linear dependent. The 6×6 matrix of its polar coordinates must have a nonzero determinant. An example of 6 dependent lines is the set of lines intersecting a given line l. They are contained in the linear span of the Schubert linear complex $\Omega(l)$. We will give a geometric characterization of a set of 6 linearly dependent lines which contains a subset of 5 linearly independent lines.

Lemma 11.2.5. *Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be an involution (i.e. an automorphism of order 2). Then its graph is an irreducible curve $\Gamma_g \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 1)$ such that $\iota(\Gamma_g) = \Gamma_g$, where ι is the automorphism $(x, y) \mapsto (y, x)$. Conversely, any curve on $\mathbb{P}^1 \times \mathbb{P}^1$ with these properties is equal to the graph of some involution.*

Proof. This is easy and left to the reader. □

Corollary 11.2.6. *Let g, g' be two different involutions of $\mathbb{P}^1$. Then there exists a unique common orbit $\{x, y\}$ with respect to g and g'.*

We will need the following result of M. Chasles (1849).

Theorem 11.2.7. *Let Q be a nondegenerate quadric in $\mathbb{P}^3$ and σ be a automorphism of order 2 of Q which is the identity on one of the rulings. Then the set of lines in $\mathbb{P}^3$ which are either contained in this ruling or intersect an orbit of lines in the second ruling form a linear complex. Conversely, any linear complex is obtained in this way from some pair (Q, σ).*

Proof. Consider the set H of lines defined as in the first assertion of the theorem. Take a general plane Π and a point $x \in \Pi$. Consider the Schubert variety $\Omega(x, \Pi)$. It is a line in the Plücker space. The plane interesects Q along a conic C. Each line from $\Omega(x, \Pi)$ intersects C at two points. This defines an involution on C. Each line from the second ruling intersects C at one point. Hence σ defines another involution on C. By Corollary 11.2.6 there is a unique common orbit. Thus there is a unique line from $\Omega(x, \Pi)$ which belongs to H. Thus H is a linear complex.

Let l_1, l_2, l_3 be any three skew lines in H. Let Q be a quadric containing these lines. It is obviously nonsingular. The lines belong to some ruling of Q, Take any line l from the other ruling. Its polar line $l' = l_H^\perp$ intersect l_1, l_2, l_3 (because $(l_i)_K^\perp = l_i, i = 1, 2, 3$). Hence l' lies on Q. Now we have an involution on the second ruling defined by the polarity with respect to H. If $m \in H$ and is not contained in the first ruling, then m intersect a line l from the second ruling, by Proposition 11.2.3, it also intersects l'. This is the description of H from the assertion of the theorem.

□

Remark 11.2.1. Let C be the curve in $G(2, 4)$ parametrizing lines in a ruling of the quadric Q. Take a general line l in $\mathbb{P}^3$. Then $\Omega(l)$ contains two lines from each ruling, the ones which pass through the points $Q \cap l$. This implies that C is a conic in the Plücker embedding. A linear complex H either intersects each conic at two points and contains two or one line from the ruling or contains C and hence contains all lines from the ruling.

Lemma 11.2.8. *Let l be a line intersecting a nonsingular quadric Q in $\mathbb{P}^3$ at two different points x, y. Let $\mathrm{PT}(Q)_x \cap Q = l_1 \cup l_2$ and $PT(Q)_y \cap Q = m_1 \cup m_2$, where l_1, m_1 and l_2, m_2 belong to the same ruling. Then the polar line $l_Q^\perp$ intersects Q at the points $x' = l_1 \cap m_2$ and $y' = l_2 \cap m_1$.*

Proof. Each line on Q is self-polar to itself. Thus $P_x(Q)$ is the tangent plane $\mathrm{PT}(Q)_x$ and similarly $P_y(Q) = \mathrm{PT}(Q)_y$. This shows that $l_Q^\perp = \mathrm{PT}(Q)_x \cap \mathrm{PT}(Q)_y = \langle x', y' \rangle$. □

Lemma 11.2.9. *let l_1, l_2, l_3, l_4 be four skew lines in $\mathbb{P}^3$. Suppose not all of them are contained in a quadric. Then there are exactly 2 lines which intersect all of them. These lines may coincide.*

Proof. This is of course well-known. It can be checked by using the Schubert calculus since $\sigma_1^4 = \# \cap_{i=1}^4 \Omega(l_i) = 2$. A better geometric proof is as follows. Let Q be the quadric containing the first 3 lines. Then l_4 intersects Q at two points p, q which may coincide. The lines through these points belonging to the ruling not containing l_1, l_2, l_3 intersect $l_1, \ldots, l_4$. Conversely, any line intersecting $l_1, \ldots, l_4$ is contained in this ruling (because it intersects Q at 3 points) and passes through the points $l_4 \cap Q$.

□

Theorem 11.2.10. *Let $(l_1, \ldots, l_6)$ be a set of 6 lines and let $(l'_1, \ldots, l'_6)$ be the set of polar lines with respect to some nonsingular quadric Q. Assume that the first five lines are linearly independent in the Plücker space. Then $(l_1, \ldots, l_6)$ belong to a nonsingular linear complex if and only if there exists a projective transformation T such that $T(l_i) = l'_i$. This condition does not depend on the choice of Q.*

Proof. First let us check that this condition does not depend on a choice of Q. For each line l let $l_Q^\perp$ denote the polar line with respect to Q. Suppose $A(l) = l_Q^\perp$ for some projective transformation A. Let Q' be another nonsingular quadric. We have to show that $l_{Q'}^\perp = B(l)$ for some other projective transformation B depending only on A but not on l. Let us identify V with $\mathbb{C}^{n+1}$ and a quadric Q with a nonsingular symmetric matrix. Then $A(l) = l_Q^\perp$ means that $xQAy = 0$ for any vectors x, y in l. We have to find a matrix B such that $xQ'By = 0$. We have

$$xQAy = xQ'(Q'^{-1}QA)y = xQ'By,$$

where $B = Q'^{-1}QA$. This checks the claim.

Suppose the set $(l_1, \ldots, l_6)$ is projectively equivalent to $(l'_1, \ldots, l'_6)$ where l'_i are polar lines with respect to some quadric Q. Replacing Q with a quadric containing the first 3 lines l_1, l_2, l_3 we may assume that $l'_i = l_i, i = 1, 2, 3$. We identify Q

with $\mathbb{P}^1 \times \mathbb{P}^1$. If $l_j \cap Q = (a_j, b_j), (a'_j, b'_j)$ for $j = 4, 5, 6$, then, by Lemma 11.2.8, $l'_j \cap Q = (a_j, b'_j), (a'_j, b_j)$. Suppose $l'_i = A(l_i)$. Then A fixes 3 lines in the first ruling hence sends Q to itself. It is also identical on the first ruling. It acts on the second ruling by switching the coordinates $(b_i, b'_j), j = 4, 5, 6$. Thus A^2 has 3 fixed points on $\mathbb{P}^1$, hence A^2 is the identity. This shows that $A = \sigma$ as in the Chasles theorem. Hence the lines $l_i, l'_i,\ i = 1, \ldots, 6$, belong to the linear complex.

Conversely, assume $l_1, \ldots, l_6$ belong to a nonsingular linear complex H. Applying Lemma 11.2.9 , we find two lines l, m intersecting l_1, l_2, l_3, l_4 (two transversals). By Proposition 11.2.3, the polar line $l' = i_H(l)$ intersects l_1, l_2, l_3, l_4. Hence it must coincide with either l or m. The first case is impossible. In fact, if $l = l'$, then $l \in H$. The pencil of lines through $l \cap l_1$ in the plane $\langle l, l_1\rangle$ is contained in H. Similarly, the line $\Omega(l \cap l_2, \langle l, l_2\rangle)$ is contained in H. Let π be the plane of lines spanned by these two lines in G. It is contained in H. Thus π cuts out in G a pair of lines. Thus H is singular at the point of intersections of these two lines. A contradiction.

Thus we see that $l, l' = m$ is a pair of polar lines. Now the pair of transversals $n, n' = n_K^\perp$ of l_1, l_2, l_3, l_5 is also a pair of polar lines. Consider the quadric Q spanned by l_1, l_2, l_3. The four transversals are the four lines from the second ruling of Q. We can always find an involution σ on Q which preserves the first ruling and such that $\sigma(l) = l', \sigma(n) = n'$. Consider the linear complex H' defined by the pair (Q, σ). Since $l_1, \ldots, l_5$ belong to H, and any complex is determined by 5 linearly independent lines, we have the equality $H = H'$. Thus l_6 intersects Q at a pair of lines in the second ruling which are in the involution σ. But σ is defined by the polarity with respect to H (since $l_1, l_2, l_3 \in H$ and the two involutions share two orbits corresponding to the pairs $(l, l'), (n, n')$). This implies $(l_1, \ldots, l_6) = \sigma(l'_1, \ldots, l'_6)$, where $l'_i = l_i^\perp$.

□

Corollary 11.2.11. *Let $l_1, \ldots, l_6$ be 6 skew lines on a nonsingular cubic surface S. Then they are linear independent in the Plücker space.*

Proof. We first check that any 5 lines among the six lines are linearly independent. Assume that $l_1, \ldots, l_5$ are linearly dependent. Then one of them, ay l_5 lies in the span of l_1, l_2, l_3, l_4. Let $(l'_1, \ldots, l'_6)$ is the set of six skew lines which together with $(l_1, \ldots, l_6)$ form a double-six. Then l_1, l_2, l_3, l_4 lie in the linear complex $\Omega(l'_5)$, hence l_5 lies in it too. But this is impossible because l_5 is skew to l'_5.

We know that there exists the unique quadric Q such that l'_i are polar to Q with respect to Q (the Schur quadric). But $(l'_1, \ldots, l'_6)$ is not projectively equivalent to $(l_1, \ldots, l_6)$. Otherwise S and its image S' under the projective transformation T will have 6 common skew lines. It will also have common transversals of each subset of 4. Thus the degree of the intersection curve is larger than 9. This shows that the cubic surfaces S and S' coincide and T is an automorphism of S. Its action on Pic(S) is a reflection with respect to the root corresponding to the double-six. It follows from Theorem 2.5.15 that S does not admit such an automorphism. □

Remark 11.2.2. The group $\mathrm{SL}(4)$ acts diagonally on the Cartesian product G^6. Consider the sheaf $\mathcal{L}$ on G^6 defines as the tensor product of the sheaves $p_i^*\mathcal{O}_G(1)$, where $p_i : G^6 \to G$ is the ith projection. The group $\mathrm{SL}(4)$ acts naturally in the space of global section of $\mathcal{L}$ and its tensor powers. Let

$$R = \bigoplus_{i=0}^{\infty} H^0(G^6, \mathcal{L}^i)^{\mathrm{SL}(4)}.$$

This is a graded algebra of finite type and its projective spectrum Proj(R) is denoted by $G^6//\mathrm{SL}(4)$. This is an example of geometric invariant theory quotient. The variety G^6 has an open invariant Zariski subset U which is mapped to $G^6//\mathrm{SL}(4)$ with fibres equal to $\mathrm{SL}(4)$-orbits. This implies that $G^6//\mathrm{SL}(4)$ is an irreducible variety of dimension 9. Given 6 ordered general lines in $\mathbb{P}^3$ their Plücker coordinates make a 6×6 matrix. Its determinant can be considered as a section from R_1. The locus of zeros of this section is a closed subvariety of G^6 whose general point is a 6-tuple of lines contained in a linear complex. The image of this locus in $U/\mathrm{SL}(4)$ is a hypersurface F. Now the duality of lines by means of a non-degenerate quadric defines an involution on G^6. Since it does not depend on the choice of a quadric up to projective equivalence, the involution descends to an involution of $U/\mathrm{SL}(4)$. The fixed points of this involution is the hypersurface F. One can show that the quotient by the duality involution is an open subset of a certain explicitly described 9-dimensional toric variety X (see [Dolgachev, Lectures on Invariant Theory]).

Finally observe that a nonsingular cubic surface together with a choice of its geometric marking defines a double-six, which is an orbit of the duality involution in $U/\mathrm{SL}(4)$ and hence a unique point in X which does not belong to the branch locus of the double cover $U/\mathrm{SL}(4) \to X$. This embeds the 4-dimensional moduli space of marked nonsingular cubic surfaces in a 9-dimensional toric variety.

11.2.3 Linear systems of linear complexes

Let $W \subset \bigwedge^2 V^*$ be a linear subspace of dimension $d+1$. After projectivization and restriction to $G(2, V)$ it defines a d-dimensional linear system of linear complexes (pencil, if $d = 1$, net, if $d = 2$, web if $d = 3$). We have encountered already a net of linear complexes in $G(2, 5)$ in Part I, Chapter 2. Its base locus was a compactification of the space of polar triangles of a conic.

A linear system $|W|$ of linear complexes defines three varieties. The first one is the base-locus Bs$(|W|)$ of $|W|$, i.e. the intersection of all $H \in |W|$. It is a subvariety of $G(2, V)$ of dimension $2n - 3 - d$. Its canonical class is given by the formula

$$\omega_{\mathrm{Bs}(|W|)} \cong \mathcal{O}_{\mathrm{Bs}(|W|)}(d - n). \tag{11.23}$$

In particular, it is a Fano variety if $d < n$, a Calabi-Yau variety if $d = n$ and $n \geq 3$ and a variety of general type if $2n - 3 \geq d > n$.

The second variety is the *center variety* $C(|W|)$

$$C(|W|) = \bigcup_{H \in |W|} C_H.$$

The third variety is the *singular variety*

$$\mathrm{Sing}(|W|) = \{x \in \mathbb{P}(E) : \dim \Omega(x) \cap \mathrm{Bs}(|W|) > n - d - 2\}.$$

Notice that each H from $|W|$ cuts out a codimension 1 subspace in $\Omega(x) \cong \mathbb{P}^{n-1}$ so we expect that $\dim \Omega(x) \cap \mathrm{Bs}(|W| = n - d - 2$.

Observe, that for any $x \in C_H$ any line in $\Omega(x)$ is contained in H. Thus

$$C(|W|) \subset \mathrm{Sing}(|W|) \tag{11.24}$$

For any linear subspace Λ in $\mathbb{P}(E)$ we can define the *polar subspace* with respect to $|W|$ by

$$i_{|W|}(\Lambda) = \bigcap_{H \in |W|} i_H(\Lambda).$$

Since $x \in i_H(x)$ for any linear complex H, we obtain that, for any $x \in \mathbb{P}(E)$,

$$x \in i_{|W|}(x).$$

It is easy to see that

$$\dim i_H(x) = n - d + \dim\{H \in |W| : x \in C_H\}. \tag{11.25}$$

Now we are ready to give examples.

Example 11.2.1. Assume $d = 1$ so we have a pencil of linear complexes. Assume $n + 1 = 2k + 1$ and $|W|$ does not intersect the set of linear complexes with corank > 1 (it is of codimension 3 in $\mathbb{P}(\bigwedge^2 V^*)$). Then we have a map $|W| \cong \mathbb{P}^1 \to \mathbb{P}^n$ which assigns to $H \in |W|$ its center C_H. The map is given by the Pfaffians of the principal minors of a skew-symmetric matrix of size $n + 1$, so the center variety of $|W|$ is a rational curve R of degree k in $\mathbb{P}^n$. Let $\mathrm{Sec}_1(R)$ be the 1-secant variety of R. By (11.24) any secant line of R_k must be contained in Bs$(|W|)$. Thus we obtain

Proposition 11.2.12. *Let $|W|$ be a general pencil of linear complexes in $G(2,2k+1)$. Then its base locus contains the secant variety of a rational curve of degree k in $\mathbb{P}^{2k}$.*

Example 11.2.2. Let $d = 2$ so we have a net of linear complexes. Assume $n+1 = 2k+1$ and $|W|$ does not intersect the set of linear complexes with corank > 1 (it is of codimension 3 in $\mathbb{P}(\bigwedge^2 V^*)$). Similarly to the previous example we obtain that $C(|W|)$ is a projection of the Veronese surface $v_k(\mathbb{P}^2)$ and the variety of trisecant lines is contained in Bs$(|W|)$.

Proposition 11.2.13. *Let $|W|$ be a general net of linear complexes in $G(2,2k+1)$. Then its singular variety contains a projected Veronese surface $v_k(\mathbb{P}^2)$ and its base locus contains the variety of trisecant lines of the surface.*

We have seen it already in the case $k = 2$ (see Chapter 2 of Part I).

Example 11.2.3. Let $d = 3$ and $n = 4$ so we have a web $|W|$ of linear complexes in $\mathbb{P}^9 = \mathbb{P}(\bigwedge^2 V^*)$. We assume that $|W|$ is general enough. It intersects the Grassmann variety $G^* = G(2,V^*)$ in finitely many points. We know that the degree of $G(2,5)$ is equal to 5, thus $|W|$ intersects G^* at 5 points. Consider the rational map $|W| = \mathbb{P}^3 - \to C(|W|) \subset \mathbb{P}^4$ which assigns to $H \in |W|$ the center of H. As in the previous examples, it is given by Pffaffians of skew-symmetric matrices of size 4. They all vanish at the set of 5 points $p_1,\ldots,p_5$. We will see later (see Chapter ?) that the image of such a map is the Segre cubic hypersurface with 10 double points (the double points are the images of the lines $\langle p_i,p_j\rangle$). Observe now that the singular surface of $|W|$ is equal to the projection of the incidence variety $\{(x,l) \in \mathbb{P}^4 \times \text{Bs}(|W|) : x \in l\}$ to $\mathbb{P}^4$. It coincides with the center variety $C(|W|)$.

Proposition 11.2.14. *Let $|W|$ be a general web of linear complexes in $G(2,5)$. Its base locus $X = |\text{Bs}(|W|)$ is a Del Pezzo surface of degree 5. The image of the incidence variety Z_K in $\mathbb{P}^4$ is isomorphic to the Segre cubic.*

11.3 Quadratic complexes

11.3.1 Generalities

Recall that a quadratic complex is the intersection K of the Grassmanian $G(2,V) \subset \mathbb{P}(\Lambda^2 V)$ with a quadric hypersurface Q. We shall assume that K is nonsingular, i.e. the intersection is transversal. Thus K is a nonsingular variety of dimension $2n-3$. Since $\omega_G \cong \mathcal{O}_G(-n-1)$, by the adjunction formula

$$\omega_K \cong \mathcal{O}_K(-n+1).$$

Thus K is a Fano variety of index $n-1$.

Consider the incidence variety $Z \subset \mathbb{P}(\mathcal{Q}_G)$ and let Z_K be its restriction over K. We denote by $p_K : Z_K \to \mathbb{P}^n$ and $q_K : Z_K \to K$ the natural projections. For each point $x \in \mathbb{P}^n$ the fibre of p_K is isomorphic to the intersection of the Schubert variety $\Omega(x)$ with Q. We know that $\Omega(x)$ is isomorphic to $\mathbb{P}^{n-1}$ embedded in $\mathbb{P}(\Lambda^2 V)$ as a linear subspace. Thus the fibre is isomorphic to a quadric in $\mathbb{P}^{n-1}$. This shows that K admits a structure of a *quadric bundle*, i.e. a fibration with fibres isomorphic to a quadric hypersurface. The important invariant of a quadric bundle is its *discriminant*. This is the set of points of the base of the fibration over which the fibre is a singular quadric or the whole space. In our case we have

Definition 11.4. *The discriminant Δ of a quadratic complex is the set of points $x \in \mathbb{P}^n$ such that $\Omega(x) \subset Q$ is a singular quadric in $\mathbb{P}^{n-1}$ or $\Omega(x) \subset Q$.*

Proposition 11.3.1. *Δ is a hypersurface of degree $2(n-1)$.*

Proof. Consider the map

$$i : \mathbb{P}(E) \to \mathbb{P}(\bigwedge^2 V),\ x \mapsto \Omega(x). \tag{11.26}$$

If $x = \mathbb{C}v_0$, the linear subspace of $\bigwedge^2 V$ corresponding to $\Omega(x)$ is the image of V in $\bigwedge^2 V$ under the map $v \mapsto v \wedge v_0$. This is a n-dimensional subspace $\Lambda(x)$ of $\bigwedge^2 V$ and hence defines a point in the Grassmann variety $G(n, \bigwedge^2 V)$. If we write $v_0 = \sum a_i e_i$, where we assume that $a_{n+1} \neq 0$, then $\Lambda(x)$ is spanned by the vectors $e_i \wedge v_0 = \sum_{j \neq i} a_j e_j \wedge e_i, i = 1, \ldots, n$. Thus the row of the matrix of Plücker coordinates of the basis are linear functions in coordinates of v_0. Its maximal minors are polynomials of order n. Observe now that each $(in+1)$th columns contains a_{n+1} in the ith row and has zero elsewhere. This easily implies that all maximal minors are divisible by a_{n+1}. Thus the Plücker coordinates of $\Lambda(x)$ are polynomials of degree $n-1$ in coordinates of v_0. We see now that the map i is given by a linear system of divisors of degree $n-1$. Fix a quadric Q in $\mathbb{P}(\bigwedge^2 V)$ which does not vanish on G. For any $n-1$-dimensional linear subspace L of $\mathbb{P}(\bigwedge^2 V)$ the intersection of Q with L is either a quadric or the whole L. Let us consider the locus D of L's such that this intersection is not a nonsingular quadric. We claim that this is a hypersurface of degree 2.

Let $b : E \times E$ be a nondegenerate symmetric bilinear form on a vector space E of dimension r. The restriction of b to a linear subspace $W \subset E$ with a basis $(w_1, \ldots, w_k)$ is a degenerate bilinear form if and only if the determinant of the matrix $(b(w_i, w_j))$ is equal to zero. If we write $w_i = \sum a_{ij} e_j$ in terms of a basis in E, we see that this condition is polynomial of degree $2k$ in coefficients a_{ij}. A well-known theorem from linear algebra asserts that this polynomial can be written as a quadratic polynomial in maximal minors of the matrix (a_{ij}) (in the case $E = \mathbb{R}^3$ it is the statement that the area of the parallelogram formed by vectors v and w is equal to $|v \times w|$). The quadratic polynomial is the quadratic form associated to the bilinear form on $\bigwedge^k E$) obtained by a natural extension of b to the exterior product. Returning back to our situation we interpret the maximal minors as the Plücker coordinates of L and obtain that D is a quadric hypersurface.

It remains to use that $\Delta = i^{-1}(Z)$, where i is given by polynomials of degree $n-1$. □

Let
$$\Delta_k = \{x \in \Delta : \mathrm{corank} Q \cap \Omega(x) \geq k\}.$$
This are closed subvarieties of Δ_k.

Let
$$\tilde{\Delta} = \{(x,l) \in Z_K : \mathrm{rank} d_{p_K}(x,l) < n\}. \tag{11.27}$$
In another words $\tilde{\Delta}$ is the locus of points in Z_K where the projection $p_K : Z_K \to \mathbb{P}^n$ is not smooth. This set admits a structure of a closed subscheme of Z_K defined locally by vanishing of the maximal minors of the jacobian matrix of the map p_K. Globally we have the exact sequence of the sheaves of differentials
$$0 \to p_K^*\Omega^1_{\mathbb{P}^n} \longrightarrow \Omega_{Z_K} \to \Omega^1_{Z_K/\mathbb{P}^n} \to 0, \tag{11.28}$$
and the support of $\tilde{\Delta}$ is equal to the set of pointsJ(A,B) where $\Omega^1_{Z_K/\mathbb{P}^n}$ is not locally free. Locally the map d is given by a $n \times 2n-2$ matrix. Thus $\tilde{\Delta}$ is given locally by $n \times n$ minors of this matrix and is of dimension $n-1$.

Tensoring (11.28) with the residue field $\kappa(p)$ at a point $p = (x,l) \in Z_K$, we see that $\tilde{\Delta}$ is equal to the degeneracy locus of points where the map $d_p : (p_K^*\Omega^1_{\mathbb{P}^n})_p \to (\Omega^1_{Z_K})_p$ is not injective. Using *Thom-Porteous formula* (see [Fulton]), we can express the class of $\tilde{\Delta}$ in $H^*(Z_K, \mathbb{Z})$.

11.3.2 Intersection of 2 quadrics

Let Q_1, Q_2 be two quadrics in $\mathbb{P}^n$ and $X = Q_1 \cap Q_2$. We assume that X is nonsingular and moreover that the pencil $\mathcal{P} = (Q_t)_{t \in \mathbb{P}^1}$ of quadrics spanned by Q_1, Q_2 has exactly $n+1$ singular quadrics of corank 1. This set can be identified with a set of $n+1$ point $p_1, \ldots, p_{n+1}$ in the base $\mathbb{P}^1$ of the pencil.

If $n = 2g+1$ is odd, we get the associated nonsingular hyperelliptic curve C of genus g, the double cover of $\mathbb{P}^1$ branched at $p_1, \ldots, p_{2g+2}$.

The variety X is a Fano variety of degree 4 in $\mathbb{P}^n, n \geq 3$, of dimension $n-2$. Its canonical class is equal to $-(n-3)H$, where H is a hyperplane section. When $n = 4$ it is a Del Pezzo surface.

Theorem 11.3.2. *(A. Weil) Assume $n = 2g + 1$. Let $F(X)$ be the variety of $g-1$-dimensional linear subspaces X. Then $F(X)$ is isomorphic to the Jacobian variety of the curve C and also to the intermediate Jacobian of X.*

Proof. We will restrict ourselves only with the case $g = 2$ leaving to the reader the general case. For each $l \in F(X)$ consider the projection map $p_l : X' = X \setminus l \to \mathbb{P}^3$. For any point $x \in X$ not on l, the fibre over $p_l(x)$ is equal to the intersection of the plane $l_x = \langle l, x\rangle$ with X'. The intersection of this plane with a quadric Q from the pencil $\mathcal{P}$ is a conic containing l and another line l'. If we take two nonsingular generators of $\mathcal{P}$ we see that the fibre is the intersection of two lines or the whole $l' \in F(X)$ intersecting l. In the latter case, all points on $l' \setminus l$ belong to the same fibre. Since all quadrics from the pencil intersect the plane $\langle l, l'\rangle$ along the same quadric there exists a unique quadric $Q_{l'}$ from the pencil which contains $\langle l, l'\rangle$. It belongs to one of the two rulings of planes on $Q_{l'}$ (or a unique family if the quadric is singular). Note that each quadric from the pencil contains at most one plane in each ruling which contains l (two members of the same ruling intersect along a subspace of even codimension). Thus we can identify the following sets:

$$\text{pairs } (Q, r), \text{ where } Q \in \mathcal{P},\ r \text{ is a ruling of planes in } Q$$

$$B = \{l' \in F(X) : l \cap l' \neq \emptyset\}.$$

If we identify $\mathbb{P}^3$ with the set of planes of $\mathbb{P}^3$ containing l, then the latter set is a subset of $\mathbb{P}^3$. Let D be the union of l''s from B. The projection map p_l maps D to B with fibres isomorphic to $\langle l, l'\rangle \setminus \{l\}$.

Extending p_l to a morphism $f : \bar{X} \to \mathbb{P}^3$, where $\bar{X}$ is the blow-up of X with center at l, we obtain that f is an isomorphism outside B and the fibres over points in B are isomorphic to $\mathbb{P}^1$. Observe that $\bar{X}$ is contained in the blow-up $\bar{\mathbb{P}}^3$ along l. The projection f is the restriction of the projection $\bar{\mathbb{P}}^5 \to \mathbb{P}^3$ which is a projective bundle of relative dimension 2. It is known how the intermidiate Jacobian behaves when we blow-up a smooth subvariety. This easily gives that Jac$(X) \cong$ Jac(B).

The crucial observation now is that B is isomorphic to our hyperelliptic curve C. In fact, consider the incidence variety

$$\mathcal{X} = \{(Q, l) \in \mathcal{P} \times G(2, 6) : l \subset Q\}.$$

Its projection to $\mathcal{P}$ has fibre over Q isomorphic to the rulings of planes in Q. It consists of two connected components outside the set of singular quadrics and one connected component over the set of singular quadrics. Taking the Stein factorization we get a double cover of $\mathcal{P} = \mathbb{P}^1$ branched along 6 points. It is isomorphic to C.

A general plane in $\mathbb{P}^3$ intersects C at $d = \deg B$ points. Its pre-image under the projection $X \to \mathbb{P}^3$ is isomorphic to the complete intersection of 2 quadrics in $\mathbb{P}^4$. It is a Del Pezzo surface of degree 4 and hence is obtained by blowing up 5 points in $\mathbb{P}^2$. Thus $d = 5$. An easy argument using Riemann-Roch shows that B lies on a unique quadric $\mathcal{Q} \subset \mathbb{P}^3$. Its pre-image under the projection $\bar{X} \to \mathbb{P}^3$ is the exceptional divisor E of the blow-up $\bar{X} \to X$. One can show that the normal bundle of l in X is trivial so $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and hence $\mathcal{Q}$ is a nonsingular quadric. Thus (X, l) defines a biregular model $B \subset \mathbb{P}^3$ of C such that B is of degree 5 and lies on a unique nonsingular quadric. One can show that the latter condition is equivalent to that the invertible sheaf $\mathcal{O}_B(1) \otimes \omega_B^{-2}$ is not effective. It is easy to see that B is of bidegree $(2, 3)$.

Let us construct an isomorphism between Jac(C) and $F(X)$. Recall that Jac(C) is birationally isomorphic to the symmetric square $C^{(2)}$ of the curve C. The canonical map $C^{(2)} \to \text{Pic}^2(C)$ defined by $x + y \mapsto [x + y]$ is an isomorphism over the complement of one point represented by the canonical class of C. Its fibre over K_C is the linear system $|K_C|$. Note also that Pic$^2(C)$ is canonically identified with Jac(X) by sending a divisor class ξ of degree 2 to the class $\xi - K_C$.

Each line l' skew to l is projected to a secant line of B. In fact, $\langle l, l' \rangle \cap X$ is a quartic curve in $\mathbb{P}^3 \cong \langle l, l' \rangle$ containing two skew line components. The residual part is the union of two skew lines m, m' intersecting both l and l'. Thus l' is projected to the secant line joining two points on C which are the projections of the lines m, m'. If $m = m'$, then l' is projected to a tangent line of B. Thus the open subset of lines in X skew to l is mapped bijectively to an open subset of $C^{(2)}$ represented by "honest" secants of C, i.e. secants which are not 3-secants. Each line $l' \in F(X) \setminus \{l\}$ intersecting l is projected to a point x of B. The line f of the ruling of $\mathcal{Q}$ intersecting B with multiplicity 3 and passing through x defines a divisor D of degree 2 such that $f \cap B = x + D$. The divisor class $[D] \in \text{Pic}^2(C)$ is assigned to l'. So we see that each trisecant line (they are necessary lie on $\mathcal{Q}$) defines 3 lines passing through the same point of l. By taking a section of X by a hyperplane tangent to X at a point x one can see that through any point X is contained in 4 lines (taken with some multiplicity). Finally, the line l itself corresponds to K_C. This establishes an isomorphism between Pic$^2(C)$ and $F(X)$. $\square$

11.3.3 Kummer surface

We consider the case $n = 3$. The quadratic complex is the intersection of two quadrics $G \cap Q$. We shall assume that K is *nondegenerate* in the sense that the pencil spanned by G and Q contains 6 different singular quadrics of corank 1. Let C be the associated hyperelliptic curve of genus 2.

First let us look at the discriminant surface Δ of K. By Proposition 11.3.1 it is a quartic surface. For any point $x \in \Delta$ the conic $C_x = K \cap \Omega(x)$ is the union of 2 lines. A line in G is always equal to a one-dimensional Schubert variety. In fact, G is a nonsingular quadric of dimension 4, and hence contains two 3-dimensional families of planes. These are the families realized by the Schubert planes $\Omega(x)$ and Ω_π. Hence a line must be a pencil in one of these planes, which shows that $C_x = \Omega(x, A_1) \cup \Omega(x, A_2)$ for some planes A_1, A_2 in $\mathbb{P}^3$. Any line in K is equal to some $\Omega(x, A)$ and hence is equal to an irreducible component of the conic C_x. Thus we see that any line in K is realized as an irreducible component of a conic $C_x, x \in K$. It follows from Theorem 11.3.2 that the variety of lines $F(K)$ in K is isomorhic to the Jacobian variety of C.

Proposition 11.3.3. *The variety A of lines in K is a double cover of the quartic surface Δ. The cover ramifies over the set Δ_1 of points such that the conic $C_x = p_K^{-1}(x)$ is a double line.*

Let $x \in \Delta$ and $C_x = \Omega(x, A_1) \cup \Omega(x, A_2)$. A singular point of C_x representing a line in K is called a *singular line* of K. If $x \notin \Delta_1$, then C_x has only one singular point equal to $\Omega(x, A_1) \cap \Omega(x, A_2)$. Otherwise it has the whole line of them.

A point $x \in \Delta$ is classically known as a *focal point* of K. The surface Δ is called the *focal surface* of K. The two pencils of lines defining C_x are called the *confocal pencils*. Thus a singular line is a line common to two confocal pencils.

Let $S \subset K$ be the locus of singular lines.

Theorem 11.3.4. *The set of pairs (x, l), where l is a singular line containing x is isomorphic to the variety $\tilde{\Delta} \subset Z_K$, the locus of points where the morphism $p_K : Z_K \to \mathbb{P}^3$ is not smooth. It is a nonsingular surface with trivial canonical class. The projection $p_K : \tilde{\Delta} \to \Delta$ is a resolution of singularities. The projection $q_K : \tilde{\Delta} \to S$ is an isomorphism. The surface S is equal to $K \cap R$, where R is a quadric in $\mathbb{P}^5$.*

Proof. The first assertion is obvious since the fibres of $p_K : Z_K \to \mathbb{P}^3$ are isomorphic to the conics C_x. To see that $q_k :$ is one-to-one we have to check that a singular line l cannot be a singular point of two different fibres C_x and C_y. The planes $\Omega(x)$ and $\Omega(y)$ intersect at one point $l = \langle x, y \rangle$ and hence span $\mathbb{P}^4$. If Q is tangent to both planes at the same point l, then the two planes are contained in PT$(Q)_l \cap$ PT$(G)_l$, hence $K = Q \cap G$ is singular at l. This contradict our assumption on K. Thus the projection $\tilde{\Delta} \to S$ is one-to-one. Since the fibres of $q_K : Z_K \to K$ are projective lines, this easily implies that the restriction of q_K to $\tilde{\Delta}$ is an isomorphism onto S.

Let us see now that S is cut out by a quadric. Consider the Gauss map $Q \to \check{\mathbb{P}}^5$ defined by assigning to a point $p \in Q$ its tangent hyperplane. A line l passing though a point $x \in \mathbb{P}^3$ is a singular line if and only if Q is tangent to the plane $\Omega(x)$ at l. This happens if and only if the quadric 3-fold $\mathrm{PT}(Q)_l \cap G$ contains the plane $\Omega(x)$. A 3-dimensional quadric contains a plane only if it is singular. Hence $\mathrm{PT}(Q)_l \cap G$ must be singular at some point l', i.e. $\mathrm{PT}(Q)_l$ is a tangent hyperplane of G at some point.

Thus S is equal to the pre-image of the dual quadric $\check{G}$ under the Gauss map. Since the Gauss map of a quadric is linear we obtain that S is cut out by a quadric.

By adjunction formula we obtain

$$\omega_S \cong \mathcal{O}_S.$$

The assertion that S is nonsingular follows from its explicit equations (11.29) given below. □

Theorem 11.3.5. *The set Δ_1 consists of 16 points, each point is an ordinary double point of the focal surface Δ.*

Proof. Let A be the variety of lines in K. We know that it is a double cover of Δ ramified over the set Δ_1. Since Δ is isomorphic to S outside Δ_1, we see that A admits an involution with a finite set F of isolated fixed points such that the quotient is birationally isomorphic to a K3 surface. The open set $A \setminus F)$ is an unramied double cover of the complement of $s = \#F$ projective lines in the K3 surface S. For any variety Z we denote by $e_c(Z)$ the topological Euler-charactaeristic with compact support. By the additivity property of e_c, we get $e_c(A - S) = e(A) - s = 2(e_s(S) - 2s) = 48 - 4s$. Thus $e(A) = 48 - 3s$. Since $A \cong \mathrm{Jac}(C)$, we have $e(A) = 0$. This gives $s = 16$. Thus Δ has 16 singular points. Each point is resolved by a (-2)-curve on S. This implies that each singular point is a rational double point of type A_1, i.e. an ordinary double point. □

Definition 11.5. *For any abelian variety A of dimension g the quotient of A by the involution $a \mapsto -a$ is denoted by* $\mathrm{Kum}(A)$ *and is called the* Kummer variety *of A.*

Note that $\mathrm{Kum}(A)$ has 2^{2g} singular point locally isomorphic to the cone over the Veronese variety $v_g(\mathbb{P}^{g-1})$. In the case $g = 2$ we have 16 ordinary double points. It is easy to see that any involution with this property must coincide with the negation involution (look at its action in the tangent space, and use that A is a complex torus). This gives

Corollary 11.3.6. *The focal surface of K is isomorphic to the Kummer surface of the Jacobian variety of the hyperelliptic curve C of genus 2.*

Proposition 11.3.7. *The surface S contains two sets of 16 disjoint lines.*

Proof. The first set is formed by the lines $q_K(p_K^{-1}(z_i))$, where $z_1, \ldots, z_{16}$ are the singular points of the focal surface. The other set comes from the dual picture. We can consider the dual incidence variety

$$\check{Z}_K = \{(\Pi, l) \in \check{\mathbb{P}}^3 \times K : l \subset \Pi\}.$$

The fibres of the projection to $\check{\mathbb{P}}^3$ are conics. Again we define the focal surface $\check{\Delta}$ as the locus of planes such that the fibre is the union of lines. A line in the fibre is a pencil of lines in the plane. These pencils form the set of lines in K. The lines common to two confocal pencils are singular lines of K. Thus we see that the surface S can be defined in two ways using the incidence Z_K or $\check{Z}_K$. As before we prove that $\check{\Delta}$ is the quotient of the abelian surface A and is isomorphic to the Kummer surface of C. The lines in S corresponding to singular points of $\check{\Delta}$ is the second set of 16 lines. □

Choosing six mutually apolar linear complexes we write the equation of the Klein quadric as a sum of squares. The condition of non-degeneracy allows one to reduce the quadric Q to the diagonal form in these coordinates. Thus the equation of the quadratic complex can be written in the form

$$\sum_{i=1}^{t} T_i^2 = 0, \quad \sum_{i=1}^{t} \lambda_i T_i^2 = 0. \tag{11.29}$$

Since K is nonsingular $\lambda_i \neq \lambda_j,\ i \neq j$. The parameters in the pencil corresponding to 6 singular quadrics are $(t_0, t_1) = (-\lambda_1, 1), i = 1, \ldots, 6$. Thus the hyperelliptic curve C has the equation

$$y^2 = (t_1 + \lambda_1 t_0) \cdots (t_1 + \lambda_6 t_0),$$

which has to be considered as an equation of degree 6 in $\mathbb{P}(3, 1, 1)$. Since the dual of the quadric Q has the equation $\sum \lambda_i^{-1} U_i$, and the dual of G has the equation $\sum U_i = 0$, the pre-image of $\check{G}$ under the Gauss map defined by Q is the quadric $\sum \lambda_i^{-1} T_i = 0$. This show that the surface S, a nonsingular model, of the Kummer surface, is given by the equations

$$\sum_{i=1}^{t} T_i^2 = \sum_{i=1}^{t} \lambda_i T_i^2 = \sum_{i=1}^{t} \lambda_i^2 T_i^2 = 0 \tag{11.30}$$

Consider 6 lines ℓ_i in $\mathbb{P}^2$ given by the equations

$$X_0 + \lambda_i X_1 + \lambda^2 X_2 = 0, \quad i = 1, \ldots, 6 \tag{11.31}$$

Since the points $(1, \lambda_i, \lambda_i^2)$ lie on the conic $X_0 X_2 - X_1^2 = 0$, the lines ℓ_i are tangent to the conic.

Theorem 11.3.8. *The surface S is birationally isomorphic to the double cover of $\mathbb{P}^2$ branched along the six lines ℓ_i.*

Proof. Consider the net $\mathcal{N}$ of quadrics spanned by the three quadrics defining S. Its discriminant curve parametrizing singular quadrics is the union of 6 lines $u_0 + \lambda_i u_1 + \lambda_i^2 u_2 = 0$. For any point t in $\mathcal{N}$ the corresponding quadric Q_t from the net has two rulings of planes. They coincide if the quadric is of corank 1. As in the proof of Theorem 11.3.2 (this was the case of a pencil of lines), this defines a double cover Y of the plane branched along the union of 6 lines. We know that S contains a line ℓ. Take a point $x \in S$ and consider the plane $\langle \ell, x\rangle$. It is contained in a unique quadric Q from $\mathcal{N}$ and belongs to one of its rulings. This defines a rational map from S to Y. For any pair $(Q,r) \in Y$, we find a unique plane from r which contains ℓ. The restriction of N to this plane is a pencil of conics with a fixed component ℓ. The residual pencil of lines has a base point x. This defines a rational map from Y to S. The two maps are inverse to each other. □

Remark 11.3.1. Consider the double cover F of $\mathbb{P}^2$ branched over 6 lines $\ell_1,\ldots,\ell_6$ tangent to an irreducible conic C. It is isomorphic to a hypersurface in $\mathbb{P}(3,1,1,1)$ given by the equation $z^2 - f_6(x_0,x_1,x_2)$, where $V(f_6)$ is the union of 6 lines. The restriction of f_6 to the conic C is the divisor $2D$, where D is the set of points where the lines are tangent to C. Since $C \cong \mathbb{P}^1$ we can find a cubic polynomial $g(x_0,x_1,x_2)$ which cuts out D in C. Then the pre-image of C in F is defined by the equation $z^2 - g_3^2 = 0$ and hence splits into the union of two curves $C_1 = V(z-g_3)$ and $C_2 = V(z+g_3)$ each isomorphic to C. These curves intersect at 6 points. The surface F has 15 ordinary double points over the points $p_{ij} = \ell_i \cap \ell_j$. Let $\bar{F}$ be a minimal resolution of F. It follows from the adjunction formula for a hypersurface in a weighted projective space that the canonical class of F is trivial. Thus $\bar{F}$ is a K3 surface. Since C does not pass through the points p_{ij} we may identify C_1, C_2 with their pre-images in $\bar{F}$. Since $C_1 \cong C_2 \cong \mathbb{P}^1$, we have $C_1^2 = -2$. Consider the divisor class H on $\bar{F}$ equal to $C_1 + L$, where L is the pre-image of a line in $\mathbb{P}^2$. We have

$$H^2 = C_1^2 + 2C_1\cdot L + L^2 = C_1^2 + (C_1+C_2)\cdot L + L^2 = -2+4+2 = 4.$$

We leave to the reader to check that the linear system $|H|$ maps $\bar{F}$ to a quartic surface in $\mathbb{P}^3$. It blows down all 15 exceptional divisors of $\bar{F} \to F$ to double points and blows down C_1 to the sixteenth double point.

Conversely, let Y be a quartic surfse in $\mathbb{P}^3$ with 16 ordinary double points $p_1,\ldots,p_{16}$. Projecting from one of them, say p_1, we get a double cover of $\mathbb{P}^2$ branched along a curve of degree 6. It is the image of the intersection R of Y with the polar cubic $P_{p_1}(Y)$. Obvioulsy, R the singular points of Y are projected to 15 singular points of the branch curve. A plane curve of degree 6 with 15 singular points must be the union of 6 lines $\ell_1, \ldots, \ell_6$. The projection of the tangent cone at p_1 is a conic everywhere tangent to these lines.

11.4 Ruled surfaces

11.4.1 Generalities

Consider an irreducible curve C in $G(2, V)$. The union of the lines parametrized by C is a surface S_C in $\mathbb{P}^n = \mathbb{P}(E)$, the image of the correspondence $Z_C = \{(x, l) \in \mathbb{P}^n \times C : x \in l\}$ in $\mathbb{P}^n$. A surface in $\mathbb{P}^n$ sweeped by lines is called a *ruled surface*, or a 2-dimensional *scroll*. The surface Z_C is a minimal ruled surface isomorphic to $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{S}_G \otimes \mathcal{O}_C$. We know that

$$\bigwedge^2 \mathcal{E} = \mathcal{O}_G(-1) \otimes \mathcal{O}_C = \mathcal{O}_C(-1) = -\deg(C),$$

where the degree of C is its degree in the Plücker embedding. i.e.

$$\deg(C) = \#\{\text{lines from } C \text{ intersecting a general subspace in } \mathbb{P}^n \text{ of codimension } 2\}.$$

We know that $\mathcal{O}_{\mathbb{P}(\mathcal{S}_G)}(1) \cong p^*\mathcal{O}_{\mathbb{P}^n}(1)$. Thus $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong p^*\mathcal{O}_{S_C}(1)$. Let D be the divisor on Z_C corresponding to the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Recall that $(q_C)_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}^* \cong \mathcal{S}_G^* \otimes \mathcal{O}_C$. The tautological exact sequence of vector bundles on G restricted to C gives an inclusion $V^* \subset H^0(\mathcal{E}_C^*)$. The projection map $p_C : Z_C \to S_C$ is given by the linear system $|V^*| \subset |\mathcal{O}_{Z_C}(D)$. In particular, $D^2 = \deg S_C \cdot \deg(p_C)$. Let us compute D^2. Since S_C is irreducible surface, its general hyperplane section is irreducible. Thus a general member of the linear system $|V^*|$ on Z_C is irreducible. This shows that we may assume that D is irreducible. Since it intersects each fibre of $q_C : Z_C \to C$ with multiplicity 1, we see that D is isomorphic to C. By formula (11.2.3), we have

$$\begin{aligned}\omega_{Z_C/C} &\cong \omega_{Z/G} \otimes \mathcal{O}_{Z_C} \cong p^*\mathcal{O}_{\mathbb{P}^n}(-2) \otimes q^*\mathcal{O}_G(1) \\ &\cong \mathcal{O}_{Z_C} \cong \mathcal{O}_{Z_C}(-2D) \otimes q^*\mathcal{O}_C(1).\end{aligned}$$

Thus

$$\omega_{Z_C} \cong \omega_{Z_C/C} \otimes q_C^*\omega_C = \mathcal{O}_{Z_C}(-2D) \otimes q^*\omega_C(1).$$

Since $q_C : D \to C$ is an isomorpphism, by adjunction formula,

$$\omega_D \cong \omega_{Z_C} \otimes \mathcal{O}_{Z_C}(D) \otimes \mathcal{O}_D \cong \mathcal{O}_{Z_C}(-D) \otimes q^*\mathcal{O}_C(1) \otimes \mathcal{O}_D.$$

Cancelling by ω_D, we obtain an isomorphism

$$\mathcal{O}_{Z_C}(D) \otimes \mathcal{O}_D = q^*\mathcal{O}_C(1).$$

This gives

$$D^2 = \deg S_C \cdot \deg p_C = \deg C. \tag{11.32}$$

Example 11.4.1. Suppose $\deg C = 1$, i.e. C is a line in $G(2, V)$. Then (11.32) shows that S_C is a plane and $\deg p_C = 1$. A minimal ruled surface Z_C is mapped birationally to $\mathbb{P}^2$. This could happen only if $Z_C \cong \mathbb{F}_1$ and the morphism p_C blows down the exceptionals ection. Thus S_C is a plane sweeped by lines in a pencil. The vector bundle $\mathcal{E}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Suppose $\deg(C) = 2$, i.e. C is a conic in $G(2, V)$. Assume S_C is a surface. Then (11.32) shows that S_C is either a quadric surface in $\mathbb{P}^n$ and p_C is a birational isomorphism, or S_C is a plane and p_C is of degree 2. In the first case, S_C is contained in $\Lambda \cong \mathbb{P}^3$ and the curve C is contained in the Schubert subvariety $\Omega_\Lambda \cong G_1(\Lambda)$. It parametrizes one rulings of the quadric. In the second case, C is a conic in a plane $\Omega_\Pi \cong \Pi^*$, where $\Pi = S_C$.

From now on we will assume that $d = \deg p_C = 1$, i.e. through a general point of S_C passes only one line from C. Thus S_C is of degree equal to the degree of C. This is of course easy to see. A special hyperplane H corresponding to the Schubert variety $\Omega(\Lambda)$, where codim$\Lambda = 2$ intersects C at the points m which correspond to lines on S_C intersecting one of $\deg S_C$ points in $\Omega \cap S_C$.

We will assume that C is a nonsingular curve of genus g. We say that g is the *genus* of S_C.

The projection map $p_C : Z_C \to S_C$ is a normalization map of the surface S_C. Let Δ be the set of singular points of S_C. Following the classical terminology we call it the *double curve* of a ruled surface S_C. Note that it does not mean that any point of Δ is a double point of S_C.

Assume $n = 3$. Let l be a general line from C. A general plane Π in $\mathbb{P}^n$ containing l intersects S_C along a curve of degree d which has l as an irreducible component. Suppose that the residual curve R is irreducible. Since Π intersects any line from C at one point or contains it, the curve R intersects any line at one point. Thus through each point $x_i, i = 1, \ldots, s$, of $R \cap l$ passes at least one line l_i from C and all the lines are different. One of them, say x_s corresponds to l. So, l must meet $s - 1$ other lines at points $x_1, \ldots, x_{s-1}$. This shows that l intersects the double curve Δ at $s - 1$ points. Assume that $p_C^{-1}(x_i) = k + 1 > 1$ for all $i = 1, \ldots, s - 1$. Then l meets $(s - 1)k$ other lines from C. In general case we expect that $k = 1$ and R intersects l at $d - 2$ points. Then $s = d - 1$ and l meets $d - 2$ other lines. So each line from C is a $(d - 2)$-secant line of Δ. In general, the plane section passing through the points $x_1, \ldots, x_{s-1}$ has an ordinary singular point of multiplicity k, so that $d-1 = l \cdot R = 1+(s-1)k$. Thus $(s-1)k = d-2$.

Now take a general plane section $A = H \cap S_C$ of S_C. Its pre-image A' in Z_C under the projection p_C is a curve from $|D|$ isomorphic to C. Thus the projection $A' \to A$ is the normalization map. We assume that H intersects the double curve transversally at nonsingular points. Thus the degree of the double curve Δ is equal to the number of ordinary points multiplicity $k+1$ of A. Since A is a plane curve of genus g and degree d we get the second assertion.

Proposition 11.4.1. *Let Δ be the double curve of S_C. Assume that for a general point $x \in \Delta$ the number of lines in C passing through x is equal to $k+1$. Then Δ intersects a general line from the ruling at $(d-2)/k$ points. The degree of the double curve is equal to $(\frac{1}{2}(d-1)(d-2)-g)/k$, where g is the genus of C.*

The projection $p_C : Z_C \to S$ is the normalization of the surface S. The double curve is a 1-dimensional component of the singular locus of S.

Example 11.4.2. Let X be a nonsingular curve of genus 3 and degree 6 embedded in $\mathbb{P}^3$ by the linear system $|K_X + D|$, where D is a non-effective divisor of degree 2. For each point $x \in X$ the linear system $|K_X - x|$ is of degree 2 and dimension 0. By Riemann-Roch, for any point $x \in X$, we have $h^0(D+x) = 1$. Let $p+q+r \in |D+x|$. We have $h^0(D+K_X-p-q-r) = 3+h^0(p+q+r-D)-2 = 1+h^0(x) = 2$. Thus the linear system $|D+K_X-p-q-r|$ of planes through the points p, q, r is a pencil. This means that the points p, q, r are on a line. This gives a 1-dimensional family of trisecant lines of X parametrized by the curve X itself. Let S be the union of the trisecants. Suppose $x \in \mathbb{P}^3 \setminus X$ belongs to two trisecants t, t' corresponding to points p, q, r and p', q', r'. Then $p+q+r+p'+q'+r' \sim 2D+x+x'$. Since the plane spanned by t and t' intersects X at the points p, q, r, p', q', r', we obtain that $2D+x+x' \sim D+K_X$, hence $D \sim K_X - x - x'$. By Riemann-Roch, $h^0(K_X - x - x') = 1$. Thus D is an effective divisor contradicting the assumption. We see now that S is a ruled surface. Projecting X from any point x on it, we get a curve of degree 5 of genus 3. It must have 3 double points. Thus there are 3 trisecants passing through x. This shows that X is the double curve of S. Applying Proposition 11.4.1 we see that $\deg(S) = 8$.

11.5 Congruences of lines

11.5.1 Class and Order

A *congruence of lines* in $\mathbb{P}^n$ is a surface in the Grassmannian $G = G(2, n+1)$.

If $n = 3$, congruencesof lines are codimension two subvarieties of G, so the study of congruences of lines in $G(2,4)$ is the next step after the classification of complexes. The intersection productdefines a perfect pairing

$$CH^2(G) \times CH_2(G) \to \mathbb{Z},$$

where $\mathrm{CH}_k(G)$ denotes $\mathrm{CH}^{2n-2-k}(G)$.

We know that the Chow group $CH^2(G)$ is freely generated by the Schubert cycles$\sigma_{1,1}$ and σ_2. The Chow group $\mathrm{CH}_2(G)$ is freely generated by the Schubert cycles $\sigma_{n-2,n-2}$ representing the Schubert variety of lines in a plane, and $\sigma_{n-1,n-3}$ representing the special Schubert variety of lines in a 3-dimensional subspace containing a fixed point. We have

$$\sigma_{1,1} \cdot \sigma_{n-2,n-2} = 1, \sigma_{1,1} \cdot \sigma_{n-1,n-3} = 0,$$

$$\sigma_2 \cdot \sigma_{n-2,n-2} = 0, \sigma_2 \cdot \sigma_{n-1,n-3} = 1,$$

We set

$$\sigma_{n-2,n-2} = \sigma_{1,1}^*, \quad \sigma_{n-1,n-3} = \sigma_2^*.$$

Let S be a congruence of lines, then its class in $\mathrm{CH}_2(G)$ is given by

$$[S] = a\sigma_{1,1}^* + b\sigma_2^*,$$

where

$$a = [S] \cdot \sigma_2, \quad b = [S] \cdot \sigma_{1,1}, \quad .$$

The number a is called the *order* of S, the number b is called the *class* of S.

The following proposition follows from the definition.

Proposition 11.5.1. *The order of S is the number of lines from S intersecting a general codimension 3 subspace of $\mathbb{P}^n$. The class of S is the number of lines in S which are contained in a general hyperplane.*

Let Z_S be the restriction of the incidency variety over S and let $p_S : Z_S \to \mathbb{P}^n$ and $q_S : Z_S \to S$ be the corresponding projections. Let $\xi \in \mathrm{Pic}(Z_S)$ be the divisor class of the line bundle $\mathcal{O}_{Z_S}(1)$, where we view Z_S as $\mathbb{P}(S_G \otimes \mathcal{O}_S)$. Recall from (11.14) that we have the relation in $\mathrm{CH}^*(Z)$

$$\xi^2 + q_S^* c_1(\mathcal{S}_G)\xi + q_S^* c_2(\mathcal{S}_G) = 0. \tag{11.33}$$

Multiplying by ξ and applying (11.16), we get

$$\xi^3 = q_S^*\sigma_1\xi^2 + q_S^*(-\sigma_1^2 + \sigma_2)\xi.$$

Expressing ξ^2 from (11.33) and using that $q_S^*\sigma_1 \cdot q_S^* c_2(\mathcal{S}_G)) = 0$ because $\dim S = 2$, we finally get

$$\xi^3 = \sigma_2 \xi = a. \tag{11.34}$$

Since we know that $\xi = p_S^*\mathcal{O}_{\mathbb{P}^n}(1)$, we obtain the following

Proposition 11.5.2. *Let a be the order of a congruence S. Then*

$$a = \deg p_S(Z_S) \cdot \deg p_S.$$

In particular, when $n = 3$, we see that $p_S(Z_S)$ is equal to $\mathbb{P}^3$ and a is equal to the degree of the map $Z_S \to \mathbb{P}^3$. If $n > 3$ and $\deg p_S = 1$, the image of Z_S is a 3-dimensional scroll of degree equal to the order of S.

Applying (11.2.3), we get

$$\omega_{Z_S} = \omega_{Z_S/S} \otimes q_S^*\omega_S \cong \omega_{Z/G} \otimes \mathcal{O}_{Z_S} \otimes q_S^*\omega_S$$

$$\cong p^*\mathcal{O}_{\mathbb{P}^n}(-2) \otimes q^*\mathcal{O}_G(1) \otimes \mathcal{O}_{Z_S} \otimes q_S^*\omega_S$$

$$\cong p_S^*\mathcal{O}_{\mathbb{P}^n}(-2) \otimes \omega_S(1).$$

Assume $n = 3$ and S is smooth. Let $R(S)$ be the ramification divisor of $p_S : Z_S \to \mathbb{P}^3$, its support is equal to the set of points $(x, l) \in Z_S$ such that p_S is not smooth at (x, l). Recall that for any morphism $f : X \to Y$ of finite degree of nonsingular varieties, the divisor of points $x \in X$ where the map is not smooth is a section of the relative canonical sheaf $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$. Applying this to the map $p_S : Z_S \to \mathbb{P}^3$, we see obtain

$$\mathcal{O}_{Z_S}(R(S)) \cong \omega_{Z_S} \otimes p_S^*(\omega_{\mathbb{P}^3}^{-1} \cong p_S^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes q_S^*\omega_S(1). \tag{11.35}$$

Definition 11.6. *Let S be an irreducible congruence of lines in $\mathbb{P}^3$. The* focal surface *is the branch locus $F(S)$ of the map $p_S : Z_S \to \mathbb{P}^3$, i.e. the image of the ramification divisor $R(S)$. By definition, the focal surface $F(S)$ is the locus of points $x \in \mathbb{P}^3$ such that $S \cap \Omega(x)$ is either of positive dimension (equal to 1 unless $S = \Omega(x)$) or consists of less than a points. A point $x \in F(S)$ is called a* singular point *if $\dim \Omega(x) \cap S > 0$. A one-dimensional component of the locus of singular points (if not empty) is called the* fundamental curve *of S.*

It follows from (11.35) that a general fibre of q_S intersects $R(S)$ at 2 points. The image of these points in $\mathbb{P}^3$ belong to the focal surface. The line l corresponding to the fibre is tangent to $F(S)$ at these points (called the *focal points* of l). Thus the surface S can be realized as an irreducible component of the variety of bitangent lines of the focal surface.

Applying the adjunction formula to $R(S)$, we obtain

$$\omega_{R_S} \cong \omega_{Z_S} \otimes \mathcal{O}_{Z_S}(R(S)) \otimes \mathcal{O}_{R(S)} \cong q_S^*\omega_S(1)^{\otimes 2} \otimes \mathcal{O}_{R(S)}. \tag{11.36}$$

Thus the restriction map $q_S : R(S) \to S$ is of degree 2. Its ramification locus is the divisor $R(S)_1$ such that

$$\mathcal{O}_{R(S)}(R(S)_1) \cong \omega_{R(S)/S} \cong \omega_{R_S} \otimes q_S^*\omega_S^{-1} \cong q_S^*\omega_S(2) \otimes \mathcal{O}_{R(S)} \tag{11.37}$$

Another important invariant of a congruence is the rank.

Definition 11.7. *The* rank *of a congruence S is $r = (a-1)(b-1) - p_a(H)$, where H is a hyperplane section of S.*

The geometric interpretation of the rank in the case of $G(2,4)$ is easy: take a general Schubert variety $H = \Omega(l)$, where $l \notin S$. It follows from Proposition 11.2.1 that $\Omega(l)$ is a quadric singular along the line l. Projecting from l we get a map from $H \setminus l \to Q \subset \mathbb{P}^3$, where Q is a nonsingular quadric. Then the image of the curve $H \cap S$ is a curve H' of bidegree (a,b). Computing the genus of this curve we find that the rank is the expected number of double points of H'.

11.5.2 Congruences of order 1: examples

It follows from formula (11.5.2) that the lines from a congruence of degree 1 sweep a 3-dimensional subspace of $\mathbb{P}^n$. For this reason we may restrict ourselves with the case $n = 3$.

The following examples describe all possible congruences of order one in $\mathbb{P}^3$

Example 11.5.1. In a smooth quadric surface $Q \subset \mathbb{P}^3$ fix the curve $F \cup L$, where F is a smooth rational curve of type $(m-1,1)$ and L is a general line of type $(1,0)$. In particular l intersects F transversally in $m-1$ distinct points:

$$L \cap F = \{o_1 \ldots o_{m-1}\}.$$

Then we consider the linear complex

$$\Omega(L) = \{l \in G :\ l \cap L \neq \emptyset\},$$

and the complex of degree m equal to the Chow variety of F

$$D_F = \{l \in G :\ l \cap F \neq \emptyset\}.$$

Their intersection $\Omega(L) \cap D_F$ is a congruence of bidegree (m,m). It contains the Schubert planes $\Omega(o_i), i = 1,\ldots,m-1$. The residual component is a congruence of order 1 and class m. It is easy to see that an irreducible congruence of order 0 or class 0 must be a Schubert variety $\Omega(x)$ or Ω_Π. Since S does not contain any Schubert variety Ω_Π (any plane intersects F at finitely many points), we see that S is reduced and irreducible.

It is easy to describe S as a surface embedded in G. We have

$$S \subset \Omega(L) \subset \langle \Omega(L) \rangle \cong \mathbb{P}^4.$$

Recall that $\Omega(L)$ is a quadric of rank 4 and that $\mathrm{Sing}(\Omega(L)) = \{L\} \in G$. In particular L is the unique base-point for the two pencils of planes of $\Omega(L)$.

Proposition 11.5.3. *S is a rational ruled surface of degree $m+1$. If $m \geq 3$ then L is its unique singular point. The locus of singular points of S in $\mathbb{P}^3$ is the line L.*

Proof. Consider the pencil $\mathcal{P}$ of planes through the line L. Each plane Π from $\mathcal{P}$ intersects the quadric along L and some other line from another ruling. This line intersects F at one point x. This shows that Π contains one pencil of lines $\Omega(x,\Pi)$ contained in S. Varying Π we see that S has a ruling by lines. It is clear that each point $l \in S$ is contained in one pencil $\Omega(x,\Pi)$, namely in the one where $x = F \cap l$ and $\Pi = \langle l, L \rangle$. This shows that S is a rational ruled surface of degree $m+1$ in $\mathbb{P}^5$. The corresponding curve in $G(2, \bigwedge^2 V)$ is of course F embedded by the map $x \mapsto \Omega(x, \langle x, L \rangle)$. Note that each point o_i is mapped to $P_i = \Omega(o_i, \langle \mathrm{PT}(F)_{o_i}, L \rangle)$. The line L belongs to each pencil $P_i, i = 1, \ldots, m-1$, and hence belongs to the double curve Δ of the ruled surface S if $m \geq 3$. In fact, in this case Δ is just one point L. This follows from Proposition 11.1.8. A point $l \in S$ is nonsingular if there is a unique pencil P of lines from S containing l and this pencil is not the tangent space of F at P. Clearly, any pencil P containing $l \neq L$ is is defined by the plane $\langle x, L \rangle$, where $l \cap F = x$. If $F \cap l = y$ for some $y \neq x$, then l intersects $F \cup L$ at ≥ 3 points and hence is contained in the quadric. It must be in the ruling which does not contain L. But lines from this ruling intersect F at one point. It is easy to see that the pencil P is tangent to F embedded in $\mathbb{P}^5$ if and only if l is tangent to F at x. But again this will imply that l belongs to the quadric and hence cannot be tangent to F.

For any point $x \in \mathbb{P}^3$ not on L, there exists a unique line from S. It is the line in the plane $\langle x, L \rangle$ containing a unique point from F. On the other hand, any point on L is contained in a pencil of lines. □

Example 11.5.2. Consider again the previous example. For each point $x \in F$ we have the plane $\Pi_x = \langle x, L \rangle$ containing L (or $\langle \mathrm{PT}(F)_x.L \rangle$ if $x \in F \cap L$). This defines a regular map f of degree m from $F \cong \mathbb{P}^1$ to the pencil of planes through L and

$$S = \{l \in G : l \subset \Pi_x, x \in F\}.$$

Our second example is a specialization of Example 11.5.1. Fix a finite morphism of degree m
$$f : L \to P_L,$$
where L is a fixed line and P_L is the pencil of planes through L. Denote by Π_x the plane $f(x)$. Let
$$S = \{l \in G : \ x \in l \subset \Pi_x, \ x \in L\}$$
Clearly S is contained in the Schubert variety $\Omega(L)$. It is ruled by pencils $\Omega(x, \Pi_x)$. Obviously, $L \in S$ and belongs to all pencils in S. In other words, S is a cone with the vertex L. For each point $z \in \mathbb{P}^3$ not on L there is only one $l \in S$ containing z, namely the line in $\langle z, L\rangle$ passing through z. Thus S is a congruence of degree 1. Its degree is computed as in the previous example.

Proposition 11.5.4. *S is a cone over a smooth rational curve of degree $m + 1$. It is a congruence of order 1. Its locus of singular points in $\mathbb{P}^3$ is the line L.*

A uniform construction of both examples 11.5.1 and 11.5.2 can be obtained as follows. Consider the projection map from $\mathbb{P}^3$ to $\mathbb{P}^2$ with center at the line L. It extends to the projection map of a $\mathbb{P}^2$-bundle
$$\pi : B_L \to \mathbb{P}^1,$$
where $\sigma : B_L \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ along L. We can identify fibres P_x of π with planes containing L. Consider a section $f : \mathbb{P}^1 \to B_L$ of the projective bundle of degree m with respect to $\mathcal{O}_{B_L}(1)$, where we represent B_L as $\mathbb{P}((\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. Then f defines the congruence of lines
$$S_f = \{l \in G : \ f(x) \in l \subset P_x, x \in \mathbb{P}^1\}.$$
The general example 11.5.1 is recovered when the image of f is not in the exceptional divisor E of σ.If $f(P_L) \subset E$ then we obtain Example 11.5.2.

Example 11.5.3. Let
$$R \subset \mathbb{P}^3 = \mathbb{P}(E)$$
be a rational normal cubic, projecting R in $\mathbb{P}^2$ from a general point we obtain a cubicwith exactly one node. This shows that the secants of R are parametrized by a congruence S of order 1. Its set of singular points in $\mathbb{P}^3$ is the curve R. The surface S is of course the projective plane $\mathbb{P}^2$ isomorphic to the symmetric square $R^{(2)}$ of R. Under this isomorphism $\mathbb{P}^2$ is identified with $\mathbb{P}(W^*)$, where $W = H^0(\mathcal{I}_R(2))$ (any two points x, y on R define a pencil of quadrics containing

R and the the secant $\langle x, y\rangle$). Let us identify $\mathbb{P}(S^2W^*)$ with the space of quartic surfaces containing R. It defines a rational map

$$\mathbb{P}(E) - \to \mathbb{P}(S^2W)$$

which blows down each bisecant of R to a point. This embeds S in $\mathbb{P}^5 = \mathbb{P}(S^2W)$. This corresponds to the Veronese mapping of $\mathbb{P}^2 = \mathbb{P}(W^*)$. Also it allows one to define a natural linear isomorphism

$$\mathbb{P}(S^2W) \cong \mathbb{P}(\Lambda^2)$$

as the unique linear extension from $S \subset \mathbb{P}(\Lambda^2)$ embedded by Plücker and $S \subset \mathbb{P}(S^2W)$ embedded by the Veronese map.

Since $S \cong \mathbb{P}^2$ is embedded by a Veronese map, we see that S is a congruence of degree 4, hence of class 3.

Remark 11.5.1. A congruence of order one defines a rational map

$$\phi_S : \mathbb{P}^3 \to \mathbb{P}^5.$$

Indeed, since S has order one, there is a unque line $l_x \in S$ through a general point $x \in \mathbb{P}^3$. This defines a rational map sending x to l_x.

For example, in the case when S is the congruence of secants of a rational normal curve R, ϕ_S is defined by the linear system $|\mathcal{I}_R(4)|$. If S is from Example 1, then one can show that ϕ_S is defined by the linear system $|\mathcal{J}(m+1)|$, where $\mathcal{J}$ is the sheaf of ideals $\mathcal{I}_F \cap \mathcal{I}_L^n$.

TO BE CONTINUED

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