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New Solutions for Some Important Partial Differential Equations

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Abstract: New solutions of some important partial differential equations are obtained using the first integral method. The efficiency of the method is demonstrated by applying it for the linear Klein-Gordan equation, MKdV equation, Burgers' equation in two and three dimensions.

Key words: First integral method, Klein-Gordan equation; MKdV; Burgers' equation

1 Introduction

In various fields of science and engineering, many problems can be described by non-linear partial differential equations (PDEs). The study of numerical methods for the solution of partial differential equations has enjoyed an intense period of activity over the last 40 years from both theoretical and practical points of view. Improvements in numerical techniques, together with the rapid advance in computer technology, have meant that many of the PDEs arising from engineering and scientific applications, which were previously intractable, can now be routinely solved [1]. In finite difference methods differential operators are approximated and difference equations are solved. In the finite element method the continuous domain is represented as a collection of a finite number N of subdomains known as elements. The collection of elements is called the finite element mesh. The differential equations for time dependent problems are approximated by the finite element method to obtain a set of ordinary differential equations (ODEs) in time. These differential equations are solved approximately by finite difference methods. In all finite difference and finite elements it is necessary to have boundary and initial conditions. However, the Adomian decomposition method, which has been developed by George Adomian [2], depends only on the initial conditions and obtains a solution in series which converges to the exact solution of the problem. In recent years, other ansatz methods have been developed, such as the tanh method [3–5], extended tanh function method [6, 7], the modified extended tanh function method [8], the generalized hyperbolic function [8–10], the variable separation method [11, 12], and the first integral method[13–17]. In this paper, we use the first integral method to find the new exact solutions of the linear Klein-Gordan equation, the MKdV equation, the Burgers' equation in two and three dimensions which will be useful in the theoretical numerical studies.

2 The first integral method

Consider the nonlinear PDE:

$$F(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0 \tag{1}$$

where $u\ (x, t)$ is the solution of the equation (1). We use the transformations

$$u(x, t) = f(\xi), \xi = x - ct \tag{2}$$

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where c is constant. Based on this we obtain

$$\frac{\partial}{\partial t}(\cdot) = -c\frac{d}{d\xi}(\cdot), \frac{\partial}{\partial x}(\cdot) = \frac{d}{d\xi}(\cdot), \frac{\partial^2}{\partial x^2}(\cdot) = \frac{d^2}{d\xi^2}(\cdot), ... \tag{3}$$

We use (3) to transfer the PDE (1) to ODE:

$$G(f, f_\xi, f_{\xi\xi}, ...) = 0 \tag{4}$$

Next, we introduce new independent variables

$$X(\xi) = f(\xi), Y = f_\xi(\xi) \tag{5}$$

which leads to a system of ODEs

$$\begin{cases} X_\xi(\xi) = Y(\xi) \\ Y_\xi(\xi) = F_1(X(\xi), Y(\xi)) \end{cases} \tag{6}$$

By the qualitative theory of ordinary differential equations [18], if we can find the integrals to (6) under the same conditions, then the general solutions to (6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to (6) which reduces (4) to a first order integrable ordinary differential equation. An exact solution to (1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem: Suppose that $P(\omega, z), Q(\omega, z)$ are polynomials in $C[\omega, z]$ and $P(\omega, z)$ is irreducible in $C[\omega, z]$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $C[\omega, z]$ such that

$$Q[\omega, z] = P[\omega, z]G[\omega, z]$$

3 Applications

In this section, we discuss the problems which involve the linear and the nonlinear PDEs using the first integral method described in section 2.

Example 1: We first start with the linear equation of the Klein-Gordan type

$$u_{tt} = u_{xx} + u \tag{7}$$

Using (2) and (3) equation (7) becomes

$$c^2\frac{d^2 f(\xi)}{d\xi^2} = \frac{d^2 f(\xi)}{d\xi^2} + f(\xi) \tag{8}$$

Using (5) we get

$$\dot{X}(\xi) = Y(\xi) \tag{9a}$$

$$\dot{Y}(\xi) = \frac{X(\xi)}{c^2 - 1} \tag{9b}$$

According to the first integral method, suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of (9), and $q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0 \tag{10}$$

where $a_i(X)(i = 0, 1..., m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (10) is called the first integral to (9). Assuming that $m = 2$ in equation (10). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y)\sum_{i=0}^{2} a_i(X)Y^i \tag{11}$$

By equating the coefficients of $Y^i(i = 3, 2, 1, 0)$ on both sides of equation (11), we have

$$\dot{a}_2(X) = h(X)a_2(X) \tag{12a}$$

$$\dot{a}_1(X) = g(X)a_2(X) \tag{12b}$$

$$\dot{a}_0(X) = -2(\frac{X}{c^2 - 1}) + g(X)a_1(X) \tag{12c}$$

$$a_1(X)(\frac{X}{c^2 - 1}) = g(X)a_0(X) \tag{12d}$$

Since, $a_2(X)$ is a polynomial of X , from (12a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$, and balancing the degrees of $g(X), a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 0$ only. Now we discuss this case: if deg $g(X) = 0$, suppose that $g(X) = A_1$, then we find $a_1(X)$, and $a_0(X)$.

$$a_1(X) = A_1X + A_0 \tag{13}$$

$$a_0(X) = d + A_0A_1X + \frac{(-2 - A_1^2 + A_1^2c^2)X^2}{2(-1 + c^2)} \tag{14}$$

where A_0, d are arbitrary integration constants. Substituting $a_0(X), a_1(X)$, and $g(X)$ in (12d) and setting all the coefficients of powers X to be zero. Then, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = 0, A_1 = 0 \tag{15a}$$

$$d = 0, A_0 = 0, A_1 = -\sqrt{2}\sqrt{-\frac{1}{1-c} - \frac{1}{1+c}} \tag{15b}$$

$$d = 0, A_0 = 0, A_1 = \sqrt{2}\sqrt{-\frac{1}{1-c} - \frac{1}{1+c}} \tag{15c}$$

Using (15a) in (10), we obtain

$$Y = -\sqrt{-d + \frac{X^2}{c^2 - 1}}, Y = \sqrt{-d + \frac{X^2}{c^2 - 1}} \tag{16}$$

Combining (16) with (9), we obtain the exact solution to equation (8) as follows:

$$f(\xi) = \frac{ie^{\frac{-i(\xi+\xi_0)}{\sqrt{1-c^2}}}(1 + 4(c^2 - 1)^2 d + e^{\frac{2i(\xi+\xi_0)}{\sqrt{1-c^2}}})}{4\sqrt{(1 - c^2}} \tag{17a}$$

$$f(\xi) = \frac{ie^{\frac{-i(\xi+\xi_0)}{\sqrt{1-c^2}}}(4(c^2 - 1)^2 d + e^{\frac{2i(\xi+\xi_0)}{\sqrt{1-c^2}}})}{4\sqrt{(1 - c^2}} \tag{17b}$$

where ξ_0 is an arbitrary integration constant. Then the exact solution to (7) can be written as

$$u(x, t) = \frac{ie^{\frac{-i(x-ct+\xi_0)}{\sqrt{1-c^2}}}(1 + 4(c^2 - 1)^2 d + e^{\frac{2i(x-ct+\xi_0)}{\sqrt{1-c^2}}})}{4\sqrt{1 - c^2}} \tag{18a}$$

$$u(x, t) = \frac{ie^{\frac{-i(x-ct+\xi_0)}{\sqrt{1-c^2}}}(4(c^2 - 1)^2 d + e^{\frac{2i(x-ct+\xi_0)}{\sqrt{1-c^2}}})}{4\sqrt{1 - c^2}} \tag{18b}$$

Similarly, for the cases of (15b)-(15c), the exact solutions are given respectively by

$$u(x,t) = e^{\sqrt{\frac{1}{c^2-1}}(x-ct+\xi_0)} \tag{18c}$$

$$u(x,t) = e^{-\sqrt{\frac{1}{c^2-1}}(x-ct+\xi_0)} \tag{18d}$$

These solutions are all new exact solutions.

Example 2: We next consider the nonlinear equation of the MKdV [5]

$$u_t = -u^2 u_x - u_{xxx} \tag{19}$$

Using (2) and (3) equation (19) becomes

$$-c\frac{df(\xi)}{d\xi} = -(f(\xi))^2\frac{df(\xi)}{d\xi} - \frac{d^3 f(\xi)}{d\xi^3} \tag{20}$$

Integrating (20) gives

$$-cf(\xi) = -\frac{(f(\xi))^3}{3} - \frac{d^2 f(\xi)}{d\xi^2} \tag{21}$$

The constant of integration equal zero since the solitary wave solution and its derivatives equal zero as $\xi \to \pm\infty$. Using (5) we get

$$\dot{X(\xi)} = Y(\xi) \tag{22a}$$

$$\dot{Y(\xi)} = cX(\xi) - \frac{X(\xi)^3}{3} \tag{22b}$$

According to the first integral method, suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of (22) and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0 \tag{23}$$

where $a_i(X)(i = 0, 1, ..., m)$ are polynomials of X and $a_m(X) \neq 0$. Assuming that $m = 2$ in equation (23). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y)\sum_{i=0}^{2} a_i(X)Y^i \tag{24}$$

By equating the coefficients of $Y^i(i = 3, 2, 1, 0)$ on both sides of equation (24), we have

$$\dot{a_2}(X) = h(X)a_2(X) \tag{25a}$$

$$\dot{a_1}(X) = g(X)a_2(X) \tag{25b}$$

$$\dot{a_0}(X) = -2(cX - \frac{X^3}{3}) + g(X)a_1(X) \tag{25c}$$

$$a_1(X)(cX - \frac{X^3}{3}) = g(X)a_0(X) \tag{25d}$$

Since, $a_2(X)$ is a polynomial of X , from (25a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$, and balancing the degrees of $g(X)$, $a_1(x)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Now we discuss this case. Suppose that $g(X) = A_1X + B_0$ then we find $a_1(X)$ and $a_0(X)$.

$$a_1(X) = \frac{A_1}{2}X^2 + B_0X + A_0 \quad (26)$$

$$a_0(X) = d + A_0B_0X + \frac{1}{2}(A_0A_1 + B_0^2 - 2c)X^2 + \frac{1}{2}A_1B_0X^3 + \frac{1}{24}(4 + 3A_1^2)X^4 \quad (27)$$

where A_0 , d are arbitrary integration constants. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in the equation (25d) and setting all the coefficients of powers X to be zero. Then, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$d = \frac{A_0^2}{4}, c = \frac{-iA_0}{\sqrt{6}}, B_0 = 0, A_1 = -2i\sqrt{\frac{2}{3}} \quad (28a)$$

$$d = \frac{A_0^2}{4}, c = \frac{iA_0}{\sqrt{6}}, B_0 = 0, A_1 = 2i\sqrt{\frac{2}{3}} \quad (28b)$$

$$d = 0, c = 0, A_0 = 0, B_0 = 0, A_1 = -2i\sqrt{\frac{2}{3}} \quad (28c)$$

$$d = 0, c = 0, A_0 = 0, B_0 = 0, A_1 = 2i\sqrt{\frac{2}{3}} \quad (28d)$$

Similarly, as in the last example, the exact solutions are given respectively by

$$u(x,t) = i(\frac{-3}{2})^{\frac{1}{4}}\sqrt{A_0}tanh(\frac{(\frac{1}{2} + \frac{i}{2})\sqrt{A_0}(x - ct + \xi_0)}{6^{\frac{1}{4}}}) \quad (29a)$$

$$u(x,t) = (-1)^{\frac{3}{4}}(\frac{3}{2})^{\frac{1}{4}}\sqrt{A_0}tanh(\frac{(\frac{1}{2} + \frac{i}{2})\sqrt{A_0}(x - ct + \xi_0)}{6^{\frac{1}{4}}}) \quad (29b)$$

$$u(x,t) = \frac{i\sqrt{6}}{x + \xi_0} \quad (29c)$$

$$u(x,t) = \frac{-i\sqrt{6}}{x + \xi_0} \quad (29d)$$

where ξ_0 is an arbitrary integration constant. Now, we take $m = 1$ in equation (23) to solve the MKdV equation then by the similar procedure we get the new exact solutions

$$u(x,t) = (-6)^{\frac{1}{4}}\sqrt{A_0}tan[\frac{(-1)^{\frac{3}{4}}\sqrt{A_0}(x - ct + \xi_0)}{6^{\frac{1}{4}}}] \quad (30a)$$

$$u(x,t) = (-1)^{\frac{3}{4}}(6)^{\frac{1}{4}}\sqrt{A_0}tan[(-\frac{1}{6})^{\frac{1}{4}}\sqrt{A_0}(x - ct + \xi_0)] \quad (30b)$$

$$u(x,t) = \frac{i\sqrt{6}}{x + \xi_0} \quad (30c)$$

$$u(x,t) = \frac{-i\sqrt{6}}{x + \xi_0} \quad (30d)$$

Example 3: In this example, we solve the Burgers' equation in two dimensions which can be written as

$$u_t + \varepsilon(uu_x + uu_y) - \nu(u_{xx} + u_{yy}) \quad (31)$$

where $u(x, y, t)$ is the solution of the equation (31). We use the transformations

$$u(x, y, t) = f(\xi), \xi = x + \beta y - ct \quad (32)$$

where β, c are constants. Then the equation (31) becomes

$$\frac{d^2 f(\xi)}{d\xi^2} = \frac{-c}{\nu(1+\beta^2)}\frac{df(\xi)}{d\xi} + \frac{\varepsilon(1+\beta)}{\nu(1+\beta^2)} f(\xi)\frac{df(\xi)}{d\xi} \tag{33}$$

Using (3) we get

$$\dot{X}(\xi) = Y(\xi) \tag{34a}$$

$$\dot{Y}(\xi) = \frac{1}{\nu(1+\beta^2)}\{-c + \varepsilon(1+\beta)X(\xi)\}Y(\xi) \tag{34b}$$

According to the first integral method, suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of (34), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0 \tag{35}$$

where $a_i(X)(i = 0, 1, ..., m)$ are polynomials of X and $a_m(X) \neq 0$. Due to the Division Theorem, there exists a polynomial $g(x) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y)\sum_{i=0}^{2} a_i(X)Y^i \tag{36}$$

By equating the coefficients of $Y^i(i = 3, 2, 1, 0)$ on both sides of equation (36), we have

$$\dot{a_2}(X) = h(X)a_2(X) \tag{37a}$$

$$\dot{a_1}(X) = -\frac{2a_2(X)}{\nu(1+\beta^2)}\{-c + \varepsilon(1+\beta)X(\xi)\} + g(X)a_2(X) + h(X)a_1(X) \tag{37b}$$

$$\dot{a_0}(X) = -\frac{a_1(X)}{\nu(1+\beta^2)}\{-c + \varepsilon(1+\beta)X(\xi)\} + g(X)a_1(X) + h(X)a_0(X) \tag{37c}$$

$$g(X)a_0(X) = 0 \tag{37d}$$

Since, $a_2(X)$ is a polynomial of X , from (37a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 0$ or deg $g(X) = 1$ only. If deg $g(X) = 0$, suppose that $g(X) = A_1$ then we find $a_1(X)$ and $a_0(X)$.

$$a_1(X) = A_0 - \frac{X^2(1+\beta)\varepsilon}{(1+\beta^2)\nu} + \frac{X(2c + A_1\nu + A_1\beta^2\nu)}{(1+\beta^2)\nu} \tag{38}$$

$$a_0(X) = d + \alpha_4 X^4 + \alpha_3 X^3 + \alpha_2 X^2 + \alpha_1 X \tag{39}$$

where A_0 , d are arbitrary integration constants.

$$\alpha_4 = \frac{(1+\beta)^2\varepsilon^2}{4s\nu}, \alpha_1 = \frac{A_0(c + A_1 s)}{s}$$

$$\alpha_3 = \frac{(-1-\beta)\varepsilon(3c + 2A_1\nu + 2A_1\beta^2\nu)}{3s^2}, s = \nu(1+\beta^2)$$

$$\alpha_2 = \{2c^2 + 3A_1 c\nu + 3A_1 c\beta^2\nu - A_0\varepsilon\nu - A_0\beta\varepsilon\nu - A_0\beta^2\varepsilon\nu - A_0\beta^3\varepsilon\nu + A_1^2 s^2\}/2s^2$$

Substituting $a_0(X)$, and $g(X)$ in (37d) and setting all the coefficients of powers X to be zero. Then, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = 0 \tag{40}$$

Then, by the similar procedure explained above we get the exact solution to (31) which can be expressed as

$$u(x,y,t)=\frac{1}{(1+\beta)\varepsilon}(c+s_1)tan[(x+\beta y-ct+\xi_0)s_1/2s] \quad (41a)$$

$$u(x,y,t)=\frac{1}{(1+\beta)\varepsilon}(c+s_2)tan[(x+\beta y-ct+\xi_0)s_2/2s] \quad (41b)$$

where ξ_0 is an arbitrary integration constant,

$$s_1=\sqrt{(-c^2-(A_0+\sqrt{A_0^2-4d})(1+\beta+\beta^2+\beta^3)\varepsilon\nu)}$$

$$s_2=\sqrt{(-c^2+(-A_0+\sqrt{A_0^2-4d})(1+\beta+\beta^2+\beta^3)\varepsilon\nu)}$$

These solutions are all new exact solutions.

Example 4: In this example, we solve the Burgers' equation in three dimensions which can be written as

$$u_t+\varepsilon(uu_x+uu_y+uu_z)-\nu(u_{xx}+u_{yy}+u_{zz})=0 \quad (42)$$

where $u(x,y,z,t)$ is the solution of the equation (42). We use the transformations

$$u(x,y,z,t)=f(\xi),\xi=x+\beta y+\gamma z-ct \quad (43)$$

where β,γ,c are real constants. Then equation (42) becomes

$$\frac{d^2f(\xi)}{d\xi^2}=\frac{-c}{\nu(1+\beta^2+\gamma^2)}\frac{df(\xi)}{d\xi}+\frac{\varepsilon(1+\beta+\gamma)}{\nu(1+\beta^2+\gamma^2)}f(\xi)\frac{df(\xi)}{d\xi} \quad (44)$$

Using (5) we get

$$\dot{X}(\xi)=Y(\xi) \quad (45a)$$

$$\dot{Y}(\xi)=\frac{-c}{\nu(1+\beta^2+\gamma^2)}Y(\xi)+\frac{\varepsilon(1+\beta+\gamma)}{\nu(1+\beta^2+\gamma^2)}X(\xi)Y(\xi) \quad (45b)$$

According to the first integral method, suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of (45), and $q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i=0$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(\xi),Y(\xi)]=\sum_{i=0}^{m}a_i(X)Y^i=0 \quad (46)$$

where $a_i(X)(i=0,1,...,m)$ are polynomials of X and $a_m(X)\neq 0$. Due to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dq}{d\xi}=\frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi}+\frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi}=(g(X)+h(X)Y)\sum_{i=0}^{2}a_i(X)Y^i \quad (47)$$

By equating the coefficients of $Y^i(i=3,2,1,0)$ on both sides of equation (47), we have

$$\dot{a_2}(X)=h(X)a_2(X) \quad (48a)$$

$$\dot{a_1}(X)=-\frac{2a_2(X)}{r}(-c+\varepsilon(1+\beta+\gamma)X(\xi))+g(X)a_2(X)+h(X)a_1(X) \quad (48b)$$

$$\dot{a_0}(X)=-\frac{a_1(X)}{r}(-c+\varepsilon(1+\beta+\gamma)X(\xi))+g(X)a_1(X)+h(X)a_0(X) \quad (48c)$$

$$g(X)a_0(X)=0 \quad (48d)$$

where $r=\nu(1+\beta^2+\gamma^2)$. Since, $a_2(X)$ is a polynomial of X , from (48a), we deduce that $a_2(X)$ is a constant and $h(X)=0$. For simplicity, we take $a_2(X)=1$, and balancing the degrees of $g(X)$, $a_1(X)$

and $a_0(X)$, we conclude that deg $g(X) = 0$ or deg $g(X) = 1$ only. If deg $g(X) = 0$, suppose that $g(X) = A_1$ then we find $a_1(X)$ and $a_0(X)$.

$$a_1(X) = A_0 - \frac{X^2}{r}(1+\beta+\gamma)\varepsilon + \frac{X}{r}(2c + A_1 r) \tag{49}$$

$$a_0(X) = d + r_4 X^4 + r_3 X^3 + r_2 X^2 + r_1 X \tag{50}$$

where A_0 , d are arbitrary integration constants.

$$r_1 = A_0(c + A_1 r)/r, r_2 = (2c^2 + 3A_1 cr - r_0 + A_1^2\nu^2(1 + 2(\beta^2+\gamma^2) + \beta^4 + \gamma^4))/2r^2$$
$$r_4 = (1+\beta+\gamma)^2\varepsilon^2/4\nu r, r_3 = (-1-\beta-\gamma)\varepsilon(3c + 2A_1 r)/3r^2$$
$$r_0 = A_0\varepsilon\nu(1+\beta+\gamma+\beta^2+\gamma^2+\beta^3+\gamma^3)$$

Substituting $a_0(X)$, and $g(X)$ in (48d) and setting all the coefficients of powers X to be zero. Then, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = 0 \tag{51}$$

Then, by the similar procedure explained above we get the exact solution to (42) which can be written as

$$u(x,y,z,t) = \frac{c + s_3}{(1+\beta+\gamma)\varepsilon} tan[s_3(x + \beta y + \gamma z - ct + \xi_0)/2r] \tag{52a}$$

$$u(x,y,z,t) = \frac{c + s_4}{(1+\beta+\gamma)\varepsilon} tan[s_4(x + \beta y + \gamma z - ct + \xi_0)/2r] \tag{52b}$$

where ξ_0 is an arbitrary integration constant.

$$s_3 = \sqrt{(-c^2 - (A_0 + \sqrt{A_0^2 - 4d})(1 + \beta^3 + \gamma + \gamma^2 + \gamma^3 + \beta^2(1+\gamma) + \beta(1+\gamma))\varepsilon\nu)}$$

$$s_4 = \sqrt{(-c^2 + (-A_0 + \sqrt{A_0^2 - 4d})(1 + \beta^3 + \gamma + \gamma^2 + \gamma^3 + \beta^2(1+\gamma) + \beta(1+\gamma))\varepsilon\nu)}$$

These solutions are all new exact solutions.

4 Conclusion

In this work, the first integral method was applied successfully for solving linear and nonlinear partial differential equations in one, two and three dimensions. Four partial differential equations which are the linear Klein-Gordan equation, the MKdV equation, and the Burgers' equation in two and three dimensions have been solved exactly. The first integral method described herein is not only efficient but also has the merit of being widely applicable. Thus, we deduce that the proposed method can be extended to solve many nonlinear partial differential equations problems which are arising in the theory of solitons and other areas.

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