Differential Geometry Basics

Daniel Allcock
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allcock@math.utah.edu
web page: *http://www.math.utah.edu/~allcock*

Abstract.
This is a very terse development of differential geometry, written for the author's benefit. The point is to gather all of the basic constructions together with consistent sign and other conventions.

1 Conventions

M is a smooth (C^∞) manifold. All maps, curves, etc. are smooth unless otherwise specified. E, F and G are vector bundles over M. We use Penrose's abstract notation [6], as described in [1], with shape symbols for E, F and G being $\heartsuit$, $\triangle$ and $\bowtie$, respectively. We use capital greek indices for E and capital roman indices for F, the indices appearing as superscripts. We use subscripts for E^* and F^*. We will not need index symbols for G. A section of a tensor product of copies of TM and T^*M will sometimes be called a world-tensor. A metric is a (not necessary positive definite) nondegenerate symmetric element of $\mathfrak{S}_{\diamond\diamond}$. Our notational and sign conventions follow Penrose.

2 The Exterior Derivative of a Function

If $X \in \mathfrak{S}^\diamond$ then X is a derivation on functions, $X : \mathfrak{S} \to \mathfrak{S}$. For given $f \in \mathfrak{S}$, the map $X \mapsto X(f)$ is $\mathfrak{S}$-linear in X, so is induced by an element of $\mathfrak{S}_\diamond$. We call this tensor the differential (or exterior derivative) of f and denote it df. In terms of it we have

$$X(f) = (df)_a X^a$$

for all $f \in \mathfrak{S}$, $X \in \mathfrak{S}^\diamond$.

The map $f \mapsto df$ is a derivation: First, for all $X \in \mathfrak{S}^\diamond$, we have

$$X(f+g) = X(f) + X(g),$$

so

$$(d(f+g))_a X^a = (df)_a X^a + (dg)_a X^a.$$

Since X is arbitrary, we have $(d(f+g))_a = (df)_a + (dg)_a$, so $d(f+g) = df + dg$. Second, for all $X \in \mathfrak{S}^\diamond$ we have

$$X(fg) = X(f)g + fX(g)$$

because X is a derivation. That is,

$$(d(fg))_a X^a = g(df)_a X^a + f(dg)_a X^a.$$

Since X is arbitrary, this implies $d(fg) = (df)g + f(dg)$, so $d : \mathfrak{S} \to \mathfrak{S}_\diamond$ is a derivation. See section 8 for a generalization of the operator d.

3 Connections on Vector Bundles

A connection on E is a map $\nabla : \mathfrak{S}^{\heartsuit} \to \mathfrak{S}_{\diamond}{}^{\heartsuit}$ satisfying the conditions $\nabla(S+T) = \nabla S + \nabla T$ and $\nabla(fS) = df \otimes \nabla S + f\nabla S$ for all $f \in \mathfrak{S}$ and $S, T \in \mathfrak{S}^{\heartsuit}$. ∇ is sometimes called a covariant derivative operator. If $E = TM$ then we sometimes call ∇ an affine connection. We use the conventions of [1] for attaching indices to operators, so that ∇_a, ∇_b, etc. are maps on $\mathfrak{S}^{\Gamma}$, $\mathfrak{S}^{\Delta}$, etc.

If $X \in \mathfrak{S}^{\diamond}$ and $S \in \mathfrak{S}^{\heartsuit}$ then the tensor $X^a \nabla_a S^{\Gamma}$ is called the covariant derivative of S^{Γ} along X. We say that S is (covariantly) constant along X if this vanishes; if this holds for all $X \in \mathfrak{S}^{\diamond}$ (so that $\nabla S = 0$) then we say that S is (covariantly) constant.

Suppose γ is a smooth embedded curve in M with nowhere vanishing tangent vector $\dot\gamma$ and S is a smooth section of E defined over the image of γ (that is, $S \circ \gamma$ is a smooth map). Lemma 5.1of [5] and the discussion following it assures us that there is a smooth vector field X (resp. a section S' of E) whose restriction to the image of γ is $\dot\gamma$ (resp. S). Furthermore, the covariant derivative of S' along X, at points in the image of γ, depends only on the values of X and S' on this image. Thus we may speak unambiguously of the covariant derivative of S along γ even though S is only defined on the image of γ.

If ∇ is an affine connection then a geodesic in M (with respect to ∇) is a smooth curve γ whose tangent vector $\dot\gamma$ is covariantly constant along γ. That is, $\dot\gamma^a \nabla_a \dot\gamma^b = 0$. We will say more about geodesics in section 14.

We note that if ∇ is any connection on E then the identity map $\mathbf{1}_{\heartsuit}{}^{\heartsuit}$ on E is annihilated by ∇. To see this, let e_α be a basis of sections of E and let E^α be the dual basis for E^*. Then $\mathbf{1}_{\Gamma}{}^{\Delta} = \sum_\alpha (E^\alpha)_\Gamma (e_\alpha)^\Delta$. For any β, γ we have

$$
\begin{aligned}
(E^\beta)_\Gamma (e_\gamma)^\Delta \nabla_a \mathbf{1}_{\Delta}{}^{\Gamma} &= \sum_\alpha (E^\beta)_\Gamma (e_\gamma)^\Delta \nabla_a \left[(E^\alpha)_\Delta (e_\alpha)^\Gamma\right] \\
&= \sum_\alpha (E^\beta)_\Gamma (e_\gamma)^\Delta \left[\nabla_a (E^\alpha)_\Delta (e_\alpha)^\Gamma + (E^\alpha)_\Delta \nabla_a (e_\alpha)^\Gamma\right] \\
&= (e_\gamma)^\Delta \nabla_a (E^\beta)_\Delta + (E^\beta)_\Gamma \nabla_a (e_\Delta)^\Gamma \\
&= \nabla_a ((e_\gamma)^\Delta (E^\beta)_\Delta) = 0.
\end{aligned}
$$

Since β and γ are arbitrary, this implies that $\nabla \mathbf{1} = 0$.

Theorem 3.1. *There exists a connection ∇ on E.*

Proof: Suppose $U \subseteq M$ is a coordinate neighborhood in M over which E is trivial. Let $\sigma_1, \ldots, \sigma_{\dim E}$ be a basis of sections of E over U and let $x_1, \ldots, x_{\dim M}$ be coordinates on U. We will define a connection ${}^U\nabla$ on $E|_U$. If S is any section of E over U then there are unique $s_\beta \in \mathfrak{S}$ such that $S = \sum_{\beta=1}^{\dim E} s_\beta \sigma_\beta$. For such S, define

$$
{}^U\nabla S = \sum_{\alpha=1}^{\dim M} \sum_{\beta=1}^{\dim E} \frac{\partial s_\beta}{\partial x_\alpha} (dx_\alpha) \otimes \sigma_\beta .
$$

It follows from the additivity and product rule for partial derivatives that ${}^U\nabla$ is a connection on $E|_U$. Suppose sets U_γ form a locally finite cover of M and that for each γ, h_γ is a smooth function supported in U_γ, with $\sum_\gamma h_\gamma = 1$. Then for $S \in \mathfrak{S}^{\heartsuit}$ we define

$$
\nabla S = \sum_\gamma (h_\gamma) \left({}^{U_\gamma}\nabla \left(S|_{U_\gamma}\right)\right).
$$

It is easy to check that ∇ is a connection on E. $\square$

It will sometimes be useful for computation with bundles over M besides TM to introduce an arbitrary affine connection; at the end of section 5 we introduce a notation for reflecting such a choice.

4 The Torsion of an Affine Connection

Let ∇ be an affine connection. A computation using the axioms for ∇ shows that for all $f, g \in \mathfrak{S}$, we have

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(fg) = g(\nabla_a \nabla_b - \nabla_b \nabla_a)f + f(\nabla_a \nabla_b - \nabla_b \nabla_a)g,$$

showing that for all $X, Y \in \mathfrak{S}^\diamond$, the operator $X^a Y^b(\nabla_a \nabla_b - \nabla_b \nabla_a)$ on $\mathfrak{S}$ is a derivation and thus a vector field. Since the assignment of this vector field to X and Y is obviously $\mathfrak{S}$-linear in X and Y, we see that there is a tensor field $T \in \mathfrak{S}_{\diamond\diamond}{}^\diamond$ such that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)f = T_{ab}{}^c \nabla_c f$$

for all $f \in \mathfrak{S}$. T is called the torsion of ∇. Since the left hand is obviously antisymmetric in a and b, $T_{\diamond\diamond}{}^\diamond$ is antisymmetric in its lower indices.

5 New Connections Derived from Old

Given a connection $\nabla : \mathfrak{S}^\heartsuit \to \mathfrak{S}_\diamond{}^\heartsuit$ on E there are several ways to extend it. First is a trivial extension to $\mathfrak{S}$, by

$$\nabla f = df.$$

As shown in section 2, ∇ is a derivation and thus a connection on $\mathfrak{S}$.

A second sort of extension is less trivial: ∇ induces a natural connection $\widetilde{\nabla} : \mathfrak{S}_\heartsuit \to \mathfrak{S}_{\diamond\heartsuit}$ on E^*, as follows. (After this section, we will regard $\widetilde{\nabla}$ as an extension of ∇ and suppress the tilde.) For any $X \in \mathfrak{S}_\heartsuit$, consider the map $\mathfrak{S}^\heartsuit \to \mathfrak{S}_\diamond$ defined on $Y \in \mathfrak{S}^\heartsuit$ by

$$Y^\Gamma \mapsto \nabla_a(X_\Gamma Y^\Gamma) - X_\Gamma \nabla_a Y^\Gamma.$$

Note that we have used the extension of ∇ to $\mathfrak{S}$ in the first term on the right hand side. Straightforward computation reveals that this map is $\mathfrak{S}$-linear in Y. Therefore it is given by contraction with a unique element of $\mathfrak{S}_{a\Gamma}$. We define $\widetilde{\nabla} : \mathfrak{S}_\heartsuit \to \mathfrak{S}_{\diamond\heartsuit}$ by means of this map. That is, we define $\widetilde{\nabla} X$ by requiring that

$$(\widetilde{\nabla}_a X_\Gamma) Y^\Gamma = \nabla_a(X_\Gamma Y^\Gamma) - X_\Gamma \nabla_a Y^\Gamma \tag{5.1}$$

hold for all $Y \in \mathfrak{S}^\heartsuit$. Now we check that $\widetilde{\nabla}$ is a connection. If $X, Z \in \mathfrak{S}_\heartsuit$ and $f \in \mathfrak{S}$ then it is straightforward to check that for all $Y \in \mathfrak{S}^\heartsuit$ we have

$$Y^\Gamma \widetilde{\nabla}_a(X_\Gamma + Z_\Gamma) = Y^\Gamma(\widetilde{\nabla}_a X_\Gamma) + Y^\Gamma(\widetilde{\nabla}_a Z_\Gamma)$$

and

$$Y^\Gamma \widetilde{\nabla}_a(f X_\Gamma) = (\nabla_a f) Y^\Gamma X_\Gamma + f Y^\Gamma(\widetilde{\nabla}_a X_\Gamma).$$

Since Y was arbitrary, we see that $\widetilde{\nabla}_a(X_\Gamma + Z_\Gamma) = \widetilde{\nabla}_a X_\Gamma + \widetilde{\nabla}_a Z_\Gamma$ and $\widetilde{\nabla}_a(f X_\Gamma) = (df)_a X_\Gamma + f \widetilde{\nabla}_a X_\Gamma$. Therefore $\widetilde{\nabla}(X + Z) = \widetilde{\nabla} X + \widetilde{\nabla} Z$ and $\widetilde{\nabla}(fX) = df \otimes X + f \nabla X$, which is to say that $\widetilde{\nabla}$ is a connection on E^*. As mentioned above, we will henceforth regard $\widetilde{\nabla}$ as an extension of ∇ and suppress the tilde. Observe that (5.1) can be expressed

$$\nabla_a(X_\Gamma Y^\Gamma) = \nabla_a X_\Gamma Y^\Gamma + X_\Gamma \nabla_a Y^\Gamma, \tag{5.2}$$

which is sort of a Leibniz rule.

Now suppose that E and F are vector bundles over M and that ${}^E\nabla : \mathfrak{S}^{\heartsuit} \to \mathfrak{S}_{\diamond}{}^{\heartsuit}$ and ${}^F\nabla : \mathfrak{S}^{\triangle} \to \mathfrak{S}_{\diamond}{}^{\triangle}$ are connections on them. Define a map ${}^{EF}\nabla : \mathfrak{S}^{\heartsuit\triangle} \to \mathfrak{S}_{\diamond}{}^{\heartsuit\triangle}$ by

$$ {}^{EF}\nabla_a(X^\Gamma Y^A) = ({}^E\nabla_a X^\Gamma)Y^A + X^\Gamma({}^F\nabla_a Y^A). $$

The verification that this map is well-defined, that is that the identitities

$$ \begin{aligned} {}^{EF}\nabla(fX \otimes Y) &= {}^{EF}\nabla(X \otimes fY), \\ {}^{EF}((X + X') \otimes Y) &= {}^{EF}(X \otimes Y) + {}^{EF}(X' \otimes Y), \qquad \text{and} \\ {}^{EF}(X \otimes (Y + Y')) &= {}^{EF}(X \otimes Y) + {}^{EF}(X \otimes Y') \end{aligned} $$

hold, is easy. Furthermore, in the process of checking these identities one verifies that ${}^{EF}\nabla$ is a connection, sometimes called the tensor product connection or the composite connection.

Suppose E, F and G are three vector bundles over M, with connections ${}^E\nabla$, ${}^F\nabla$ and ${}^G\nabla$, so that ${}^E\nabla : \mathfrak{S}^{\heartsuit} \to \mathfrak{S}_{\diamond}{}^{\heartsuit}$, ${}^F\nabla : \mathfrak{S}^{\triangle} \to \mathfrak{S}_{\diamond}{}^{\triangle}$ and ${}^G\nabla : \mathfrak{S}^{\bowtie} \to \mathfrak{S}_{\diamond}{}^{\bowtie}$. Then applying the above construction to ${}^E\nabla$ and ${}^F\nabla$ we obtain a connection ${}^{EF}\nabla : \mathfrak{S}^{\heartsuit\triangle} \to \mathfrak{S}_{\diamond}{}^{\heartsuit\triangle}$. We can then use ${}^{EF}\nabla$ and ${}^G\nabla$ to obtain a connection ${}^{(EF)G}\nabla : \mathfrak{S}^{\heartsuit\triangle\bowtie} \to \mathfrak{S}_{\diamond}{}^{\heartsuit\triangle\bowtie}$. Alternately, by first using ${}^F\nabla$ and ${}^G\nabla$ and then using ${}^E\nabla$ we can construct another connection ${}^{E(FG)}\nabla : \mathfrak{S}^{\heartsuit\triangle\bowtie} \to \mathfrak{S}_{\diamond}{}^{\heartsuit\triangle\bowtie}$. It is easy to check that ${}^{(EF)G}\nabla = {}^{E(FG)}\nabla$. Therefore inductive application of this construction yields exactly one connection on each bundle which is expressed as a tensor product of some number of copies (in some order) of E, F and G.

In particular, taking $F = E^*$ and ignoring G, we may take ${}^F\nabla$ to be the connection on E^* induced by ${}^E\nabla$, as above. Thus ${}^E\nabla$ induces a connection on $E \otimes E$, $E \otimes E^* \otimes E$. etc., so we have covariant derivative operators $\mathfrak{S}^{\heartsuit\heartsuit} \to \mathfrak{S}_{\diamond}{}^{\heartsuit\heartsuit}$, $\mathfrak{S}^{\heartsuit}{}_{\heartsuit}{}^{\heartsuit} \to \mathfrak{S}_{\diamond}{}^{\heartsuit}{}_{\heartsuit}{}^{\heartsuit}$, and so on. We will usually denote all of these operators by the same symbol as that of the original operator on $\mathfrak{S}^{\heartsuit}$, in this case ${}^E\nabla$. By definition, these connections satisfy the Leibniz rule

$$ {}^E\nabla_a \left(S^{\Gamma\ldots\Delta}{}_{\Theta\ldots\Lambda} T^{\Xi\ldots\Pi}{}_{\Sigma\ldots\Phi}\right) = \left({}^E\nabla_a S^{\Gamma\ldots\Delta}{}_{\Theta\ldots\Lambda}\right) T^{\Xi\ldots\Pi}{}_{\Sigma\ldots\Phi} + S^{\Gamma\ldots\Delta}{}_{\Theta\ldots\Lambda} \left({}^{E\nabla}{}_a T^{\Xi\ldots\Pi}{}_{\Sigma\ldots\Phi}\right). $$

if no repeated indices are present. However, in light of (5.2) this rule applies even if repeated indices are present.

In section 3 we stated that for some computations we would introduce an arbitrary affine connection; if ∇ is a connection on E then the affine connection together with ∇ define connections on every bundle expressed as a tensor product of TM, T^*M, E and E^*. We will denote these connections by ∇ or similar notation, placing a dot at the center of the symbol for the given connection.

6 The Difference Between Two Connections

We show here that if ∇ and $\widetilde{\nabla}$ are two connections on E then their difference is described by a tensor field $Q \in \mathfrak{S}_{\diamond\heartsuit}{}^{\heartsuit}$. For all $f \in \mathfrak{S}$ and $X \in \mathfrak{S}^{\heartsuit}$ we have

$$ \begin{aligned} (\widetilde{\nabla}_a - \nabla_a)(fX^\Gamma) &= (df)_a X^\Gamma + f\widetilde{\nabla}_a X^\Gamma - (df)_a X^\Gamma - f\nabla_a X^\Gamma \\ &= f(\widetilde{\nabla}_a - \nabla_a)X^\Gamma. \end{aligned} $$

Therefore there is a tensor field $Q_{\diamond\heartsuit}{}^{\heartsuit}$ such that

$$ (\widetilde{\nabla}_a - \nabla_a)X^\Gamma = Q_{a\Delta}{}^\Gamma X^\Delta $$

for all $X \in \mathfrak{S}^{\heartsuit}$. Conversely, if ∇ is any connection on E and $Q \in \mathfrak{S}_{\diamond\heartsuit}{}^{\heartsuit}$ then the map $\widetilde{\nabla} : \mathfrak{S}^{\heartsuit} \to \mathfrak{S}_{\diamond}{}^{\heartsuit}$ defined by

$$\widetilde{\nabla}_a X^{\Gamma} = \nabla_a X^{\Gamma} + Q_{a\Gamma}{}^{\Delta} X^{\Gamma},$$

is also a connection on E. (The computations are easy.) Thus, we know what all of the connections on E are, in terms of any given one.

We now investigate how changing the connection ∇ on E changes the connection induced by ∇ on E^* (see section 5). Suppose $\nabla, \widetilde{\nabla} : \mathfrak{S}^{\heartsuit} \to \mathfrak{S}_{\diamond}{}^{\heartsuit}$ are connections, with $Q_{\diamond\heartsuit}{}^{\heartsuit}$ such that

$$(\widetilde{\nabla}_a - \nabla_a) X^{\Gamma} = Q_{a\Delta}{}^{\Gamma} X^{\Delta}$$

for all $X \in \mathfrak{S}^{\heartsuit}$. The connections induced by ∇ and $\widetilde{\nabla}$ on $\mathfrak{S}_{\heartsuit}$ also have a tensor as their "difference". We compute it as follows. Let $Z \in \mathfrak{S}_{\heartsuit}$ be fixed and let $Y \in \mathfrak{S}^{\heartsuit}$ vary arbitrarily. By definition of the induced connections, we have

$$\begin{aligned} Y^{\Gamma} \widetilde{\nabla}_a Z_{\Gamma} &= \widetilde{\nabla}_a (Z_{\Gamma} Y^{\Gamma}) - Z_{\Gamma} \widetilde{\nabla}_a Y^{\Gamma}. \quad \text{and} \\ Y^{\Gamma} \nabla_a Z_{\Gamma} &= \nabla_a (Z_{\Gamma} Y^{\Gamma}) - Z_{\Gamma} \nabla_a Y^{\Gamma} \end{aligned}$$

Subtracting and using the fact that ∇ and $\widetilde{\nabla}$ agree on $\mathfrak{S}$, we find

$$\begin{aligned} Y^{\Gamma} (\widetilde{\nabla}_a - \nabla_a) Z_{\Gamma} &= -Z_{\Gamma} (\widetilde{\nabla}_a - \nabla_a) Y^{\Gamma} \\ &= -Z_{\Gamma} Q_{a\Delta}{}^{\Gamma} Y^{\Delta} \\ &= -Z_{\Delta} Q_{a\Gamma}{}^{\Delta} Y^{\Gamma}, \end{aligned}$$

so $(\widetilde{\nabla}_a - \nabla_a) Z_{\Gamma} = -Q_{a\Gamma}{}^{\Delta} Z_{\Delta}$. Thus, the action of $\widetilde{\nabla} - \nabla$ on $\mathfrak{S}_{\heartsuit}$ is given by contraction with the negative of the tensor that describes the difference $\widetilde{\nabla} - \nabla$ on $\mathfrak{S}^{\heartsuit}$.

Now we investigate the effect of simultaneous change of connections on E and F on the induced connection (section 5) on $E \otimes F$. Suppose that ∇ and $\widetilde{\nabla}$ are connections on E and ∇' and $\widetilde{\nabla}'$ are connections on F. Suppose

$$(\widetilde{\nabla}_a - \nabla_a) X^{\Gamma} = Q_{a\Delta}{}^{\Gamma} X^{\Delta}$$

and

$$(\widetilde{\nabla}'_a - \nabla'_a) Y^{A} = Q'{}_{aB}{}^{A} Y^{B}$$

for all $X \in \mathfrak{S}^{\heartsuit}$ and $Y \in \mathfrak{S}^{\triangle}$. As in section 5, we build the connections $\widetilde{\Box}$ and $\Box$ on $E \otimes F$ by defining

$$\begin{aligned} \widetilde{\Box}(X^{\Gamma} Y^{A}) &= \widetilde{\nabla} X^{\Gamma} Y^{A} + X^{\Gamma} \widetilde{\nabla}' Y^{A} \qquad \text{and} \\ \Box(X^{\Gamma} Y^{A}) &= \nabla X^{\Gamma} Y^{A} + X^{\Gamma} \nabla' Y^{A}. \end{aligned}$$

Then we have

$$\begin{aligned} (\widetilde{\Box}_a - \Box_a)(X^{\Gamma} Y^{A}) &= Q_{a\Delta}{}^{\Gamma} X^{\Delta} Y^{A} + Q'{}_{aB}{}^{A} X^{\Gamma} Y^{B} \\ &= (Q_{a\Delta}{}^{C} \mathbf{1}_{B}{}^{A} + \mathbf{1}_{\Delta}{}^{\Gamma} Q'{}_{aB}{}^{A}) X^{\Delta} Y^{B}. \end{aligned}$$

Applying this result inductively, we obtain

$$\begin{aligned} (\widetilde{\Box}_a - \Box_a) S^{\Gamma_1 \dots \Gamma_k A_1 \dots A_\ell} = &\sum_{i=1}^{k} Q_{a\Delta}{}^{\Gamma_i} S^{\Gamma_1 \dots \Gamma_{i-1} \Delta \Gamma_{i+1} \dots \Gamma_k A_1 \dots A_\ell} \\ &+ \sum_{i=1}^{\ell} Q'{}_{aB}{}^{A_i} S^{\Gamma_1 \dots \Gamma_k A_1 \dots A_{i-1} B A_{i+1} \dots A_\ell}, \end{aligned} \tag{6.1}$$

which completely describes the difference between the two composite connections $\widetilde{\Box}$ and $\Box$. Taking $F = E^*$, so that indices for F are written as capital greek subscripts, and ∇' (resp. $\widetilde{\nabla}'$) derived from ∇ (resp. $\widetilde{\nabla}$) by duality, we have $Q'{}_{a\Delta}{}^{\Gamma} = -Q_{a\Delta}{}^{\Gamma}$ and so

$$
\begin{aligned}
(\widetilde{\nabla}_a - \nabla_a) S^{\Gamma_1 \ldots \Gamma_k}{}_{\Delta_1 \ldots \Delta_\ell} = & \sum_{i=1}^{k} Q_{a\Phi}{}^{\Gamma_i} S^{\Gamma_1 \ldots \Gamma_{i-1} \Phi \Gamma_{i+1} \ldots \Gamma_k}{}_{\Delta_1 \ldots \Delta_\ell} \\
& - \sum_{i=1}^{\ell} Q_{a\Delta_i}{}^{\Phi} S^{\Gamma_1 \ldots \Gamma_k}{}_{\Delta_1 \ldots \Delta_{i-1} \Phi \Delta_{i+1} \ldots \Delta_\ell}.
\end{aligned} \tag{6.2}
$$

If S has a repeated index, so that Γ_i and Δ_j are the same letter, then the ith term of the first sum cancels with the jth term of the second. A convenient way to express this fact is that each sum need only extend over *nonrepeated* indices of S.

Finally, there is a relation between the difference of two affine connections and the difference of their torsion tensors:

Theorem 6.1. *Suppose ∇ and $\widetilde{\nabla}$ are affine connections on M with torsion tensors T and $\widetilde{T}$ respectively, and $(\widetilde{\nabla}_a - \nabla_a)X^c = Q_{ab}{}^c X^b$ for all $X \in \mathfrak{S}^\diamond$. Then*

$$(\widetilde{T} - T)_{ab}{}^c = -2Q_{[ab]}{}^c.$$

Proof: For any $f \in \mathfrak{S}$,

$$
\begin{aligned}
\widetilde{T}_{ab}{}^c \widetilde{\nabla}_c f &= (\widetilde{\nabla}_a \widetilde{\nabla}_b - \widetilde{\nabla}_b \widetilde{\nabla}_a) f \\
&= \widetilde{\nabla}_a (df)_b - \widetilde{\nabla}_b (df)_a \\
&= \nabla_a (df)_b - Q_{ab}{}^c (df)_c - \nabla_b (df)_a + Q_{ba}{}^c (df)_c \\
&= (\nabla_a \nabla_b - \nabla_b \nabla_a) f - 2Q_{[ab]}{}^c (df)_c \\
&= T_{ab}{}^c \nabla_c f - 2Q_{[ab]}{}^c (df)_c \\
&= T_{ab}{}^c \widetilde{\nabla}_c f - 2Q_{[ab]}{}^c \widetilde{\nabla}_c f.
\end{aligned}
$$

The theorem follows because f was arbitrary. □

Corollary 6.2. *On any manifold M there exists a torsion-free affine connection.*

Proof: Let ∇ be any affine connection on M; such exists by theorem 3.1. Suppose ∇ has torsion T. Let $\widetilde{\nabla}$ be defined by $(\widetilde{\nabla}_a - \nabla_a)X^b = Q_{ab}{}^c X^b$ for all $X \in \mathfrak{S}^\diamond$, where $Q = T/2$. By theorem 6.1, the torsion of $\widetilde{\nabla}$ vanishes. □

7 Lie Brackets and Lie Derivatives

Suppose $X, Y \in \mathfrak{S}^\diamond$. Computation reveals that their commutator $[X, Y] = XY - YX$, considered as an operator on $\mathfrak{S}$, is a derivation and thus a vector field. Taking ∇ to be an affine connection with torsion T, we can find an expression for $[X, Y]$ in terms of ∇.

Theorem 7.1. *With notation as above,*

$$[X, Y]^b = X^a \nabla_a Y^b - Y^a \nabla_a X^b + X^c Y^d T_{cd}{}^b.$$

Proof: For arbitrary $f \in \mathfrak{S}$ we have

$$[X, Y](f) = X(Y(f)) - Y(X(f)),$$

which is to say

$$[X,Y]^b \nabla_b f = X^a \nabla_a (Y^b \nabla_b f) - Y^a \nabla_a (X^b \nabla_b f).$$

After applying the Leibniz rule and the definition of T, and renaming some of repeated indices, we see that

$$[X,Y]^b \nabla_b f = (X^a \nabla_a Y^b - Y^a \nabla_a X^b + X^c Y^d T_{cd}{}^b) \nabla_b f,$$

which proves the theorem. □

The Lie derivative operator with respect to a vector field X, denoted $\mathfrak{L}_X(S)$, is discussed in [2, §§3.6–3.7]. $\mathfrak{L}_X$ maps the tensors of any given shape to each other and has the following properties:

1. $\mathfrak{L}_X(S+T) = \mathfrak{L}_X S + \mathfrak{L}_X T$.
2. $\mathfrak{L}_X(S \otimes T) = (\mathfrak{L}_X S) \otimes T + S \otimes (\mathfrak{L}_X T)$.
3. $\mathfrak{L}_X S = [X,S]$ if $S \in \mathfrak{S}^\diamond$.
4. $\mathfrak{L}_X f = X(f)$ if $f \in \mathfrak{S}$.
5. $\mathfrak{L}_X(S^a T_a) = (\mathfrak{L}_X S)^a T_a + S^a (\mathfrak{L}_X T)_a$ if $S \in \mathfrak{S}^\diamond$ and $T \in \mathfrak{S}_\diamond$.

It follows from property 1 that $\mathfrak{L}_X$ is determined by its action on simple world-tensors, then from property 2 that it is determined by its actions on $\mathfrak{S}^\diamond$ and $\mathfrak{S}_\diamond$, and from properties 4 and 5 that it is determined by its action on $\mathfrak{S}^\diamond$. Finally, condition 3 shows that this action is given in terms of any affine connection ∇ by theorem 7.1. We will use the conventions of [1] for attaching indices to the operator $\mathfrak{L}_X$, which is to say that we will regard each operator $\mathfrak{L}_X$ as a map on the tensor spaces $\mathfrak{S}^{abc}{}_d{}^e$ etc. as well as on the spaces like $\mathfrak{S}^{\diamond\diamond}{}_{\diamond\diamond}{}^\diamond$.

Now we work out explicit formulas for Lie derivatives. If $V_1, \dots, V_A \in \mathfrak{S}^\diamond$ then

$$\begin{aligned}
\mathfrak{L}_X((V_1)^{a_1} \cdots (V_A)^{a_A}) &= \sum_{\alpha=1}^{A} \left(\mathfrak{L}_X (V_\alpha)^{a_\alpha}\right) \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{A} (V_\beta)^{a_\beta} \\
&= \sum_{\alpha=1}^{A} \Big[X^c \nabla_c (V_\alpha)^{a_\alpha} - (V_\alpha)^c \nabla_c X^{a_\alpha} + X^c (V_\alpha)^d T_{cd}{}^{a_\alpha} \Big] \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{A} (V_\beta)^{a_\beta} \\
&= X^c \nabla_c ((V_1)^{a_1} \cdots (V_A)^{a_A}) \\
&\quad - \sum_{\alpha=1}^{A} \nabla_c X^{a_\alpha} (V_1)^{a_1} \cdots (V_{\alpha-1})^{a_{\alpha-1}} (V_\alpha)^c (V_{\alpha+1})^{a_{\alpha+1}} \cdots (V_A)^{a_A} \\
&\quad + X^c \sum_{\alpha=1}^{A} T_{cd}{}^{a_\alpha} (V_1)^{a_1} \cdots (V_{\alpha-1})^{a_{\alpha-1}} (V_\alpha)^d (V_{\alpha+1})^{a_{\alpha+1}} \cdots (V_A)^{a_A}. \quad (7.1)
\end{aligned}$$

For $W \in \mathfrak{S}_\diamond$ we compute $\mathfrak{L}_X W$ as follows: for any $V \in \mathfrak{S}^\diamond$ we have

$$\begin{aligned}
V^b \mathfrak{L}_X W_b &= \mathfrak{L}_X (V^b W_b) - (\mathfrak{L}_X V^b) W_b \\
&= X^c \nabla_c (V^b W_b) - \left(X^c \nabla_c V^b - V^c \nabla_c X^b + X^c V^d T_{cd}{}^b \right) W_b \\
&= X^c V^b \nabla_c W_b + V^c W_b \nabla_c X^b - X^c V^d T_{cd}{}^b W_b.
\end{aligned}$$

After relabelling some of the repeated indices we may use the fact that V was arbitrary to deduce

$$\mathfrak{L}_X W_b = X^c \nabla_c W_b + W_c \nabla_b X^c - X^c T_{cb}{}^d W_d.$$

By a derivation similar to that of (7.1) we deduce

$$\begin{aligned}\mathfrak{L}_X\left((W_1)_{b_1}\cdots(W_B)_{b_B}\right) &= X^c\nabla_c\left((W_1)_{b_1}\cdots(W_B)_{b_B}\right)\\ &\quad+\sum_{\beta=1}^{B}\left(\nabla_{b_\beta}X^c\right)(W_1)_{b_1}\cdots(W_{\beta-1})_{b_{\beta-1}}(W_\beta)_c(W_{\beta+1})_{b_{\beta+1}}\cdots(W_B)_{b_B}\\ &\quad-X^c\sum_{\beta=1}^{B}T_{cb_\beta}{}^{d}(W_1)_{b_1}\cdots(W_{\beta-1})_{b_{\beta-1}}(W_\beta)_d(W_{\beta+1})_{b_{\beta+1}}\cdots(W_B)_{b_B}.\end{aligned}\tag{7.2}$$

Finally, it follows from (7.1), (7.2) and properties 1, 2 and 5 of $\mathfrak{L}$ that the Lie derivitive of a general world-tensor S is given by

$$\begin{aligned}\mathfrak{L}_X S^{a_1\cdots a_A}{}_{b_1\cdots b_B} &= X^c\nabla_c S^{a_1\cdots a_A}{}_{b_1\cdots b_B}\\ &\quad-\nabla_c X^{a_1}S^{c\cdots a_A}{}_{b_1\cdots b_B}-\cdots-\nabla_c X^{a_A}S^{a_1\cdots c}{}_{b_1\cdots b_B}\\ &\quad+\nabla_{b_1}X^c S^{a_1\cdots a_A}{}_{c\cdots b_B}+\cdots+\nabla_{b_B}X^c S^{a_1\cdots a_A}{}_{b_1\cdots c}\\ &\quad+X^c\Big[T_{cd}{}^{a_1}S^{d\cdots a_A}{}_{b_1\cdots b_B}+\cdots T_{cd}{}^{a_A}S^{a_1\cdots d}{}_{b_1\cdots b_B}\\ &\qquad\qquad -T_{cb_1}{}^{d}S^{a_1\cdots a_A}{}_{d\cdots b_B}-\cdots-T_{cb_B}{}^{d}S^{a_1\cdots a_A}{}_{b_1\cdots d}\Big].\end{aligned}\tag{7.3}$$

Of course, this expression simplifies dramatically if ∇ is torsion-free.

8 Differential Forms

A differential p-form on M is a totally antisymmetry tensor $\theta_{a_1\ldots a_p}\in\mathfrak{S}_{\diamond\cdots\diamond}$, which is to say a section of $\wedge^p(T^*M)$. We call p the degree $\deg\theta$ of θ. If θ and ξ are differential forms then we define their exterior (or wedge) product to be

$$(\theta\wedge\xi)_{a_1\ldots a_p b_1\ldots b_q}=\theta_{[a_1\ldots a_q}\xi_{b_1\ldots b_q]}.$$

The exterior product is associative and satisfies the "commutativity" relation

$$\theta\wedge\xi=(-1)^{(\deg\theta)(\deg\xi)}\xi\wedge\theta.$$

We sometimes denote the space of all p-forms on M by $\Omega^p(M)$. Some references, such as [3], use a different definition of $\theta\wedge\xi$—one differeing by a factor of $(\deg\theta)!(\deg\xi)!$. This changes some of the formulas (notably (8.6)), but not the mathematics.

Section 4.3 of [2] describes an operator $d:\Omega^p(M)\to\Omega^{p+1}(M)$ called exterior differentiation. It is characterized by the following properties:

1. d behaves on $\Omega^0(M)=\mathfrak{S}$ as in section 2,
2. $d(\theta\wedge\xi)=d\theta\wedge\xi+(-1)^p\theta\wedge d\xi$ if $\theta\in\Omega^p(M)$.
3. $d(d\theta)=0$ for all forms θ.
4. $d(\theta+\xi)=d\theta+d\xi$.

One checks that with respect to the covariant derivative operator ∂ of a coordinate system, the map

$$\theta_{a_1\ldots a_p}\mapsto\partial_{[a_0}\theta_{a_1\ldots a_p]}\tag{8.1}$$

has these properties and thus provides a way to compute exterior derivatives: $(d\theta)_{a_0\ldots a_p}$ is given by the right hand side. Furthermore, if ∇ is any torsion-free affine connection them in fact we have

$$\nabla_{[a_0}\theta_{a_1\ldots a_p]}=\partial_{[a_0}\theta_{a_1\ldots a_p]}\tag{8.2}$$

for any $\theta \in \mathfrak{S}_{\diamond\cdots\diamond}$ (i.e., θ need not be a form). This holds because with $(\partial_a - \nabla_a)X^c = Q_{ab}{}^c X^b$ for all $X \in \mathfrak{S}^\diamond$, we have $Q_{ab}{}^c = Q_{ba}{}^c$ because ∇ and ∂ have the same torsion. Then by (6.2) we have

$$(\partial_{a_0} - \nabla_{a_0})\theta_{a_1\dots a_p} = -Q_{a_0 a_1}{}^b \theta_{b a_2 \dots a_p} - \cdots - Q_{a_0 a_p}{}^b \theta_{a_1 \dots a_{p-1} a_p}.$$

Antisymmetrizing both sides over $a_0, \dots a_p$, the right side vanishes, proving (8.2). Thus we can compute exterior derivatives as in (8.1), with any torsion-free affine connection in place of ∂.

If $\Phi : M \to N$ is a smooth map of manifolds then we'll denote by Φ^* the pullback operators $(T^*N)^{\otimes p} \to (T^*M)^{\otimes p}$, and we will also write $\Phi^*\theta$ for the section of $(T^*M)^{\otimes p}$ obtained by composing a section θ of $(T^*N)^{\otimes p}$ with Φ^*. A basic functorality property of pullbacks is that if θ and ξ are covariant tensors on M then $\Phi^*(\theta \otimes \xi) = \Phi^*\theta \otimes \Phi^*\xi$. It follows immediately that $\Phi^*(\theta \wedge \xi) = \Phi^*\theta \wedge \Phi^*\xi$. In light of this it follows by induction from the fact that $\Phi^*(df) = d(\Phi^* f)$ for $f \in C^\infty(N)$ that $\Phi^*(d\theta) = d(\Phi^*\theta)$ for any differential form θ on N. That is, the exterior derivative commutes with pullbacks.

A computation reveals that if $X \in \mathfrak{S}^\diamond$ then $\mathfrak{L}_X$ preserves each of the spaces $\Omega^p(M)$. If $\theta \in \Omega^p(M)$ then by (7.3) we have

$$(\mathfrak{L}_X\theta)_{a_1\dots a_p} = X^b \nabla_b \theta_{a_1\dots a_p} + \nabla_{a_1} X^b \theta_{b a_2 \dots a_p} + \cdots + \nabla_{a_p} X^b \theta_{a_1 \dots a_{p-1} b},$$

by (7.3). We study what happens when indices a_i and a_{i+1} are exchanged on the right hand side. Each term except for the $(i+1)$st and $(i+2)$nd remain as they were except with its sign reversed, by the antisymmetry of θ. After the index exchange, the remaining two terms become

$$\nabla_{a_{i+1}} X^b \theta_{a_1 \dots a_{i-1} b a_i \dots a_p} + \nabla_{a_i} X^b \theta_{a_1 \dots a_{i+1} b a_{i+2} \dots a_p};$$

applying the antisymmetry of θ, we see that this is just the negative of sum of the two terms we started with. This proves that $\mathfrak{L}_X\theta$ is a p-form. A less computational way to see this is to define (as in [2]) the Lie derivative $\mathfrak{L}_X$ in terms of the limit of pullback maps along the flow of X. In terms of this definition, the above result and the one below follow from the naturality properties of pullbacks.

Finally, we show that $\mathfrak{L}_X(d\theta) = d(\mathfrak{L}_X\theta)$ for all $\theta \in \Omega^p(M)$. This is easy if θ is a function:

$$\begin{aligned}
(d(\mathfrak{L}_X\theta))_a &= (d(X^b\nabla_b\theta))_a \\
&= \nabla_a(X^b\nabla_b\theta) \\
&= \nabla_a X^b \nabla_b\theta + X^b\nabla_a\nabla_b\theta \\
&= \nabla_a X^b \nabla_b\theta + X^b\nabla_b\nabla_a\theta \\
&= \mathfrak{L}_X(\nabla_a\theta) = (\mathfrak{L}_X(d\theta))_a,
\end{aligned}$$

where we have used the fact that ∇ is torsion-free. The result for general θ follows by induction; given the result for θ and ξ we prove it for $\theta \wedge \xi$ (with θ a p-form):

$$\begin{aligned}
d(\mathfrak{L}_X(\theta \wedge \xi)) &= d(\mathfrak{L}_X\theta \wedge \xi + \theta \wedge \mathfrak{L}_X\xi) \\
&= d\mathfrak{L}_X\theta \wedge \xi + (-1)^p \mathfrak{L}_X\theta \wedge d\xi + d\theta \wedge \mathfrak{L}_X\xi + (-1)^p\theta \wedge d\mathfrak{L}_X\xi \\
&= \mathfrak{L}_X d\theta \wedge \xi + d\theta \wedge \mathfrak{L}_X\xi + (-1)^p \mathfrak{L}_X\theta \wedge d\xi + (-1)^p\theta \wedge d\mathfrak{L}_X\xi \\
&= \mathfrak{L}_X(d\theta \wedge \xi) + (-1)^p \mathfrak{L}_X(\theta \wedge d\xi) \\
&= \mathfrak{L}_X(d(\theta \wedge \xi)).
\end{aligned}$$

A fact useful for work with differential forms is that

$$(p+1)\theta_{[ab_1\dots b_p]} = \theta_{a[b_1\dots b_p]} - \theta_{[b_1|a|b_2\dots b_p]} + - + - \cdots + (-1)^p\theta_{[b_1\dots b_p]a} \tag{8.3}$$

for all $\theta \in \Omega^p(M)$. Given a vector field v, we define the inner derivative $i_v\theta$ of a form $\theta \in \Omega^p(M)$ for $p > 0$ as

$$(i_v(\theta))_{a_2\dots a_p} = v^{a_1}\theta_{a_1\dots a_p}; \tag{8.4}$$

we set $i_v f = 0$ for $f \in \Omega^0(M) = \mathfrak{S}$. Note that $i_v \circ i_v$ vanishes. We call i_v a derivation because when $\theta \in \Omega^p(M)$ and $\xi \in \Omega^q(M)$ we have

$$(p+q)i_v(\theta \wedge \xi) = p\, i_v(\theta) \wedge \xi + (-1)^p q\, \theta \wedge i_v(\xi), \tag{8.5}$$

which follows from (8.3) and some calculations. This allows us to prove Cartan's "magic formula",

$$\mathfrak{L}_v\theta = p\, d i_v\theta + (p+1) i_v d\theta \tag{8.6}$$

for $\theta \in \Omega^p(M)$. Checking it for the cases $p = 0$ and 1 is an easy computation, and for general p the result follows by induction using the product rules for the Lie, exterior and inner derivatives.

The main theorems concerning differential forms are the Poincare lemma and its inverse, the theorem of Stokes and the theory of de Rham cohomology. Frobenius's theorem also has a nice interpretation in terms of differential forms. For these topics we refer to [9].

9 The Curvature Tensor

Given $X \in \mathfrak{S}^\diamond$ and a connection ∇ on E, we define the operator $\underset{X}{\nabla} : \mathfrak{S}^\heartsuit \to \mathfrak{S}^\heartsuit$ by $Y^\Gamma \mapsto X^a\nabla_a Y^\Gamma$. For given $X, Y \in \mathfrak{S}^\diamond$, we define an operator on $\mathfrak{S}^\heartsuit$ by

$$Z \mapsto \left(\underset{X}{\nabla}\underset{Y}{\nabla} - \underset{Y}{\nabla}\underset{X}{\nabla} - \underset{[X,Y]}{\nabla}\right) Z.$$

We claim that this operator is $\mathfrak{S}$-linear in each of X, Y and Z. The computations are facilitated by choosing an arbitrary affine connection on M (with torsion denoted T) and letting ∇ denote the resulting connection on the tensor products of copies of E and TM.

We will find an expression for this operator, which will make obvious the $\mathfrak{S}$-linearity in X and Y. We have

$$\begin{aligned}
\left(\underset{X}{\nabla}\underset{Y}{\nabla} - \underset{Y}{\nabla}\underset{X}{\nabla} - \underset{[X,Y]}{\nabla}\right) Z^\Gamma &= X^a\nabla_a(Y^b\nabla_b Z^\Gamma) - Y^b\nabla_b(X^a\nabla_a Z^\Gamma) - [X,Y]^c\nabla_c Z^\Gamma \\
&= X^a\nabla_a(Y^b\nabla_b Z^\Gamma) - Y^b\nabla_b(X^a\nabla_a Z^\Gamma) - [X,Y]^c\nabla_c Z^\Gamma \\
&= X^a(\nabla_a Y^b)(\nabla_b Z^\Gamma) + X^a Y^b\nabla_a\nabla_b Z^\Gamma - Y^b(\nabla_b X^a)(\nabla_a Z^\Gamma) \\
&\qquad - Y^b X^a\nabla_b\nabla_a Z^\Gamma - \left(X^a\nabla_a Y^c - Y^a\nabla_a X^c + X^a Y^b T_{ab}{}^c\right)\nabla_c Z^\Gamma \\
&= X^a Y^b(\nabla_a\nabla_b - \nabla_b\nabla_a)Z^\Gamma - X^a Y^b T_{ab}{}^c\nabla_c Z^\Gamma,
\end{aligned}$$

proving linearity in X and Y. Now we show linearity in Z; let $Д : \mathfrak{S}^\heartsuit \to \mathfrak{S}_{\diamond\diamond}{}^\heartsuit$ be the operator defined by

$$Д_{ab}Z^\Gamma = (\nabla_a\nabla_b - \nabla_b\nabla_a - T_{ab}{}^c\nabla_c)Z^\Gamma. \tag{9.1}$$

By the discussion above, $Д$ is independent of our choice of affine connection. We have

$$Д_{ab}(fZ^\Gamma) = \nabla_a\nabla_b(fZ^\Gamma) - \nabla_b\nabla_a(fZ^\Gamma) - T_{ab}{}^c\nabla_c(fZ^\Gamma).$$

After expanding using the Leibniz rule and then cancelling some terms and using the definition of T, the right hand side reduces to $f Л_{ab} Z^\Gamma$, as desired. The additivity is trivial, showing $\mathfrak{S}$-linearity in Z.

Therefore there is a tensor $K_{\diamond\diamond\heartsuit}{}^{\heartsuit}$ such that

$$Л_{ab} Z^\Gamma = K_{ab\Delta}{}^{\Gamma} Z^\Delta$$

and

$$\left(\underset{X}{\nabla}\underset{Y}{\nabla} - \underset{Y}{\nabla}\underset{X}{\nabla} - \underset{[X,Y]}{\nabla} \right) Z^\Gamma = X^a Y^b K_{ab\Delta}{}^{\Gamma} Z^\Delta$$

for all $Z \in \mathfrak{S}^\heartsuit$ and $X, Y \in \mathfrak{S}^\diamond$. Since the operator at the left is antisymmetric in X and Y, we have $K_{ab\Delta}{}^{\Gamma} = -K_{ba\Delta}{}^{\Gamma}$. Since $Л$ is independent of the choice of affine connection in the above computations, so is K. K is called the curvature tensor of ∇. We say that ∇ is flat if its curvature vanishes.

Following the argument in section 3.2 of [8], and taking intoa ccount the fact that Wald follows the opposite sign conventions for curvature than we, we can relate K to the failure of parallel propagation around closed paths to preserve tensors. Suppose $p \in M$, that $S, T \in T_p M$ and that s, t are part of a coordinate system centered at p such that the coordinate vector field $\partial/\partial s$ (resp. $\partial/\partial t$) is S (resp. T) at p. For each $\Delta s, \Delta t > 0$ we define γ to be the path beginning at p, lying in the s-t coordinate plane, with its s-coordinate increasing from 0 to Δs, then its t-coordinate increasing from 0 to Δ, then its s-coordinate decreasing to 0,a nd finally its t-coordinate decreasing to 0. (We also assume that only one of the s and t coordinates of γ is changing at any given point along γ.) If $Z \in \mathfrak{S}^\heartsuit$ then we set Z_p to be the value of Z at p, Z'_p to be the element of the fiber of E over p obtained by parallel-propagating Z_p arounf γ, and ΔZ to be $Z'_p - Z_p$. Then to first order in Δs and Δt, we have

$$(\Delta Z)^\Gamma = \Delta s \Delta t S^a T^b K_{ab\Phi}{}^{\Gamma} Z^\Phi.$$

When ∇ is an affine connection then the curvature is usually denoted R rather than K, in honor of Riemann. In this case, we define the Ricci curvature tensor by

$$R_{ac} = R_{abc}{}^{b}.$$

Because both the Riemann and Ricci tensors are denoted by R there is potential for confusion. When necessary we will make explicit which tensor is meant by using $R_{\diamond\diamond}$ or $R_{\diamond\diamond\diamond}{}^{\diamond}$ or somesuch.

Now we investigate the behavior of the operator $Л$ on sections of tensor bundles on which exist connections by virtue of the constructions of section 5. Direct computation using the definition of T shows that $Л_{ab} f = 0$ for all $f \in \mathfrak{S}$. Suppose ${}^E\nabla$ (resp. ${}^F\nabla$) is a connection on E (resp. F), ${}^{EF}\nabla$ denotes the induced connection on $E \otimes F$, and ∇ is defined using ${}^{EF}\nabla$ and the arbitrary affine connection. Then

$$Л_{ab}(S^\Gamma U^A) = (\nabla_a \nabla_b - \nabla_b \nabla_a - T_{ab}{}^{c} \nabla_c)(S^\Gamma U^A);$$

after expanding the derivative operators using the Leibniz rule and cancelling like terms, one obtains

$$Л_{ab}(S^\Gamma U^A) = Л_{ab}(S^\Gamma) U^A + S^\Gamma Л_{ab}(U^A). \tag{9.2}$$

Thus if $K_{\diamond\diamond\heartsuit}{}^{\heartsuit}$ and $L_{\diamond\diamond\triangle}{}^{\triangle}$ are the respective curvatures of ${}^E\nabla$ and ${}^F\nabla$ then

$$\begin{aligned} Л_{ab}(S^\Gamma U^A) &= K_{ab\Delta}{}^{\Gamma} S^\Delta U^A + L_{abB}{}^{A} S^\Gamma U^B \\ &= (K_{ab\Delta}{}^{\Gamma} \mathbf{1}_B{}^{A} + L_{abB}{}^{A} \mathbf{1}_\Delta{}^{\Gamma}) S^\Delta U^B, \end{aligned} \tag{9.3}$$

The expression in parentheses being an expression for the curvature tensor of ${}^{EF}\nabla$.

This allows us to compute the action of Д on $\mathfrak{S}_{\heartsuit}$. For all $S \in \mathfrak{S}^{\heartsuit}$ and $U \in \mathfrak{S}_{\heartsuit}$ we have

$$0 = \text{Д}_{ab}(S^{\Gamma}U_{\Gamma}) = K_{ab\Delta}{}^{\Gamma}S^{\Delta}U_{\Gamma} + S^{\Delta}\text{Д}_{ab}U_{\Delta}.$$

Since S was arbitrary we deduce $\text{Д}_{ab}U_{\Delta} = -K_{ab\Delta}{}^{\Gamma}U_{\Gamma}$. Together with (9.2) this describes the action of Д on any bundle on which we have defined a connection. A special case is for sections of $\mathfrak{S}^{\heartsuit\cdots\heartsuit}{}_{\heartsuit\cdots\heartsuit}$, where we have

$$\begin{aligned}\text{Д}_{ab}S^{\Gamma_1\cdots\Gamma_m}{}_{\Delta_1\cdots\Delta_n} &= K_{ab\Phi}{}^{\Gamma_1}S^{\Phi\Gamma_2\cdots\Gamma_m}{}_{\Delta_1\cdots\Delta_n} + \cdots + K_{ab\Phi}{}^{\Gamma_m}S^{\Gamma_1\cdots\Gamma_{m-1}\Phi}{}_{\Delta_1\cdots\Delta_n} \\ &\quad - K_{ab\Delta_1}{}^{\Phi}S^{\Gamma_1\cdots\Gamma_m}{}_{\Phi\Delta_2\cdots\Delta_n} - \cdots - K_{ab\Delta_n}{}^{\Phi}S^{\Gamma_1\cdots\Gamma_m}{}_{\Delta_1\cdots\Delta_{n-1}\Phi}.\end{aligned}$$

As in (7.3), if there are repeated indices present, so that $a_\alpha = b_\beta$ for some α and β, then the αth term of the second line cancels with the βth term of the third, and the αth term of the fourth line cancels with the βth term of the fifth. That is, one may restrict each sum in this formula to vary over just the nonrepeated indices of S.

10 Change of Curvature Under a Change of Connection

Suppose ∇ and $\widetilde{\nabla}$ are connections on E and that $(\widetilde{\nabla}_a - \nabla_a) = Q_{a\Delta}{}^{\Gamma}S^{\Delta}$ for all $S \in \mathfrak{S}^{\heartsuit}$. Then their respective curvature tensors $\widetilde{K}$ and K are related. For the computation, it helps to choose an arbitrary affine connection (we'll call its torsion T) and denote the resulting connections on $TM \otimes E$ by $\dot{\nabla}$ and $\widetilde{\dot{\nabla}}$.

For any $S \in \mathfrak{S}^{\heartsuit}$ we have

$$\begin{aligned}(\widetilde{K} - K)_{ab\Delta}{}^{\Gamma}S^{\Delta} &= \left(\widetilde{\text{Д}}_{ab} - \text{Д}_{ab}\right)S^{\Gamma} \\ &= \left(2\widetilde{\dot{\nabla}}_{[a}\widetilde{\dot{\nabla}}_{b]} - T_{ab}{}^{c}\widetilde{\dot{\nabla}}_{c} - 2\dot{\nabla}_{[a}\dot{\nabla}_{b]} + T_{ab}{}^{c}\dot{\nabla}_{c}\right)S^{\Gamma} \\ &= 2\widetilde{\dot{\nabla}}_{[a}(\widetilde{\dot{\nabla}}_{b]}S^{\Gamma}) - 2\dot{\nabla}_{[a}(\dot{\nabla}_{b]}S^{\Gamma}) - T_{ab}{}^{c}(\widetilde{\nabla}_c - \nabla_c)S^{\Gamma} \\ &= 2\widetilde{\dot{\nabla}}_{[a}(\dot{\nabla}_{b]}S^{\Gamma} + Q_{b]\Delta}{}^{\Gamma}S^{\Delta}) - 2\dot{\nabla}_{[a}(\dot{\nabla}_{b]}S^{\Gamma}) - T_{ab}{}^{c}Q_{c\Delta}{}^{\Gamma}S^{\Delta} \\ &= 2(\widetilde{\dot{\nabla}}_{[a} - \dot{\nabla}_{[a})(\dot{\nabla}_{b]}S^{\Gamma}) + 2\widetilde{\dot{\nabla}}_{[a}(Q_{b]\Delta}{}^{\Gamma}S^{\Delta}) - T_{ab}{}^{c}Q_{c\Delta}{}^{\Gamma}S^{\Delta} \\ &= 2Q_{[a|\Delta|}{}^{\Gamma}\dot{\nabla}_{b]}S^{\Delta} + 2\dot{\nabla}_{[a}(Q_{b]\Delta}{}^{\Gamma}S^{\Delta}) + 2Q_{[a|\Phi|}{}^{\Gamma}Q_{b]\Delta}{}^{\Phi}S^{\Delta} - T_{ab}{}^{c}Q_{c\Delta}{}^{\Gamma}S^{\Delta} \\ &= 2Q_{[a|\Delta|}{}^{\Gamma}\dot{\nabla}_{b]}S^{\Delta} + 2\dot{\nabla}_{[a}Q_{b]\Delta}{}^{\Gamma}S^{\Delta} + 2\dot{\nabla}_{[a}S^{\Delta}Q_{b]\Delta}{}^{\Gamma} \\ &\qquad + 2Q_{[a|\Phi|}{}^{\Gamma}Q_{b]\Delta}{}^{\Phi}S^{\Delta} - T_{ab}{}^{c}Q_{c\Delta}{}^{\Gamma}S^{\Delta}.\end{aligned}$$

Cancelling the first and third terms and using the fact that S was arbitrary, we deduce

$$(\widetilde{K} - K)_{ab\Delta}{}^{\Gamma} = 2\dot{\nabla}_{[a}Q_{b]\Delta}{}^{\Gamma} + 2Q_{[a|\Phi|}{}^{\Gamma}Q_{b]\Delta}{}^{\Phi} - T_{ab}{}^{c}Q_{c\Delta}{}^{\Gamma}. \tag{10.1}$$

The expression on the right hand side involves ∇; sometimes it is convenient to express it in terms of $\widetilde{\nabla}$ instead. Expressing $\dot{\nabla}$ in terms of $\widetilde{\dot{\nabla}}$, cancelling like terms and exchanging indices a and b in one term, (10.1) becomes

$$(\widetilde{K} - K)_{ab\Delta}{}^{\Gamma} = 2\widetilde{\dot{\nabla}}_{[a}Q_{b]\Delta}{}^{\Gamma} - 2Q_{[a|\Phi|}{}^{\Gamma}Q_{b]\Delta}{}^{\Phi} - T_{ab}{}^{c}Q_{c\Delta}{}^{\Gamma}. \tag{10.2}$$

Finally, we treat the case in which both ∇ and $\widetilde{\nabla}$ are affine connections, with curvatures R and $\widetilde{R}$ and torsions T and $\widetilde{T}$ respectively. Then the choice of an additional arbitrary affine connection is somewhat artificial, since we can choose one of ∇ or $\widetilde{\nabla}$. If we take the arbitrary connection to be ∇ then (10.1) becomes

$$(\widetilde{R}-R)_{abc}{}^{d} = 2\nabla_{[a}Q_{b]c}{}^{d} + 2Q_{[a|e|}{}^{d}Q_{b]c}{}^{e} - T_{ab}{}^{e}Q_{ec}{}^{d}, \tag{10.3}$$

because ∇ is now just ∇ and T refers to the torsion of ∇. If we take the arbitrary affine connection to be $\widetilde{\nabla}$ then (10.2) becomes

$$(\widetilde{R}-R)_{abc}{}^{d} = 2\widetilde{\nabla}_{[a}Q_{b]c}{}^{d} - 2Q_{[a|e|}{}^{d}Q_{b]c}{}^{e} - \widetilde{T}_{ab}{}^{e}Q_{ec}{}^{d}, \tag{10.4}$$

where $\widetilde{\nabla}$ is now just $\widetilde{\nabla}$ and $\widetilde{T}$ is the torsion of $\widetilde{\nabla}$.

11 The Bianchi Identities

Theorem 11.1 (Bianchi's first identity). *If ∇ is an affine connection with curvature R and torsion T then*

$$R_{[abc]}{}^{d} + \nabla_{[a}T_{bc]}{}^{d} + T_{[ab}{}^{e}T_{c]e}{}^{d} = 0.$$

Proof: Assume first that $T = 0$. Then for any $f \in \mathfrak{S}$ we have

$$\nabla_{[[a}\nabla_{b]}\nabla_{c]}f = \nabla_{[a}\nabla_{[b}\nabla_{c]]}f,$$

which is to say $R_{[abc]}{}^{d}\nabla_d f = \nabla_{[a}(T_{bc]}{}^{d}\nabla_d f) = 0$, as desired. Now if ∇ has torsion T then let $\widetilde{\nabla}$ be the connection defined by $(\widetilde{\nabla}_a - \nabla_a)X^b = T_{ac}{}^{b}X^c/2$ for all $X \in \mathfrak{S}^{\diamond}$. By theorem 6.1, $\widetilde{\nabla}$ is torsion-free, so by the above computation we see that its curvature $\widetilde{R}$ satisfies $\widetilde{R}_{[abc]}{}^{d} = 0$. Applying (10.3) we see that

$$(\widetilde{R}-R)_{abc}{}^{d} = \frac{2}{2}\nabla_{[a}T_{b]c}{}^{d} + \frac{1}{2}T_{[a|e|}{}^{d}T_{b]c}{}^{e} - \frac{1}{2}T_{ab}{}^{e}T_{ec}{}^{d}$$

and so

$$R_{[abc]}{}^{d} = -\nabla_{[a}T_{bc]}{}^{d} - \frac{1}{2}T_{[a|e|}{}^{d}T_{bc]}{}^{e} + \frac{1}{2}T_{[ab}{}^{e}T_{|e|c]}{}^{d}.$$

Applying the symmetries

$$T_{[a|e|}{}^{d}T_{bc]}{}^{e} = T_{[bc}{}^{e}T_{a]e}{}^{d} = T_{[ab}{}^{e}T_{c]e}{}^{d}$$

and

$$T_{[ab}{}^{e}T_{|e|c]}{}^{d} = -T_{[ab}{}^{e}T_{c]e}{}^{d}$$

to the last two terms, we see that

$$R_{[abc]}{}^{d} = -\nabla_{[a}T_{bc]}{}^{d} - T_{[ab}{}^{e}T_{c]e}{}^{d},$$

proving the theorem. □

Corrollary 11.2. *The curvature tensor R of a torsion-free affine connection satisfies the identify*

$$R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d} = 0.$$

Proof: Immediate from the symmetries $R_{abc}{}^{d} = -R_{bac}{}^{d}$ and $R_{[abc]}{}^{d} = 0$. □

Theorem 11.3 (Bianchi's second identity). *Suppose ∇ is a connection on E with curvature K, $\square$ is an arbitrary affine connection with torsion T, and ∇ denotes the composite connection derived from ∇ and $\square$. Then*

$$\nabla_{[a}K_{bc]\Delta}{}^{\Gamma} + T_{[ab}{}^{d}K_{c]d\Delta}{}^{\Gamma} = 0.$$

Proof: This is very similar to the proof of theorem 11.1. Suppose first that $\widetilde{\square}$ is an affine connection with curvature $\widetilde{R}$ and vanishing torsion and that $\widetilde{\nabla}$ denotes the composite connection built from $\widetilde{\square}$ and ∇. Then for all $S \in \mathfrak{S}^{\heartsuit}$ we have

$$\widetilde{\nabla}_{[[a}\widetilde{\nabla}_{b]}\widetilde{\nabla}_{c]}S^{\Gamma} = \widetilde{\nabla}_{[a}\widetilde{\nabla}_{[b}\widetilde{\nabla}_{c]]}S^{\Gamma},$$

which is to say

$$-\widetilde{R}_{[abc]}{}^{d}\widetilde{\nabla}_{d}S^{\Gamma} + K_{[ab|\Delta|}{}^{\Gamma}\widetilde{\nabla}_{c]}S^{\Delta} = \widetilde{\nabla}_{[a}(K_{bc]\Delta}{}^{\Gamma}S^{\Delta}).$$

Applying the first Bianchi identity to the left and expanding the right side using the product rule,

$$K_{[ab|\Delta|}{}^{\Gamma}\widetilde{\nabla}_{c]}S^{\Delta} = \widetilde{\nabla}_{[a}K_{bc]\Delta}{}^{\Gamma}S^{\Delta} + \widetilde{\nabla}_{[a}S^{\Delta}K_{bc]\Delta}{}^{\Gamma}.$$

The last terms of each side cancel (after rearranging a, b and c by an even permutation), so we see that $\widetilde{\nabla}_{[a}K_{bc]\Delta}{}^{\Gamma}S^{\Delta} = 0$ for all S. Therefore $\widetilde{\nabla}_{[a}K_{bc]\Delta}{}^{\Gamma} = 0$, as desired.

Now suppose that $\square$ is an affine connection with torsion T and that $\widetilde{\square}$ is defined by $(\widetilde{\square}_a - \square_a)X^b = T_{ac}{}^{b}X^c/2$ for all $X \in \mathfrak{S}^{\diamond}$. Then by theorem 6.1, $\widetilde{\square}$ is torsion-free. Let ∇ be as in the statement of the theorem. Then

$$\widetilde{\nabla}_a K_{bc\Delta}{}^{\Gamma} = \nabla_a K_{bc\Delta}{}^{\Gamma} - \frac{1}{2}T_{ab}{}^{d}K_{dc\Delta}{}^{\Gamma} - \frac{1}{2}T_{ac}{}^{d}K_{bd\Delta}{}^{\Gamma}.$$

Antisymmetrizing over a, b and c and using the known result for the operator $\widetilde{\nabla}$, we find

$$0 = \nabla_{[a}K_{bc]\Delta}{}^{\Gamma} - \frac{1}{2}T_{[ab}{}^{d}K_{|d|c]\Delta}{}^{\Gamma} - \frac{1}{2}T_{[ac}{}^{d}K_{b]d\Delta}{}^{\Gamma}.$$

Applying the relations

$$T_{[ab}{}^{d}K_{|d|c]\Delta}{}^{\Gamma} = -T_{[ab}{}^{d}K_{c]d\Delta}{}^{\Gamma}$$

and

$$T_{[ac}{}^{d}K_{b]d\Delta}{}^{\Gamma} = -T_{[ab}{}^{d}K_{c]d\Delta}{}^{\Gamma}$$

to the last two terms on the left, we obtain the result. $\square$

12 Affine Connections from Fields of Frames

Suppose $e_1, \ldots, e_{\dim M}$ form a basis of vector fields on M, and that $E^1, \ldots, E^{\dim M}$ are the dual 1-forms. We may define an affine connection ∂ on M as follows. Any vector field S may be expressed as a sum, $S = \sum_\beta s^\beta e_\beta$, with each $s^\beta \in \mathfrak{S}$. (This sum and all others of this section runs from 1 through $\dim M$.) We define

$$\partial S = \sum_{\alpha,\beta} e_\alpha(s^\beta)E^\alpha \otimes e_\beta.$$

(This is similar to the connections introduced in the proof of theorem 3.1.) It is easy to see that this defines a connection: the additivity is trivial and we have

$$\begin{aligned}
\partial(fS) &= \sum_{\alpha,\beta} e_a(fs^\beta)E^\alpha \otimes e_\beta \\
&= \sum_{\alpha,\beta} \left(e_\alpha(f)s^\beta + fe_\alpha(s^\beta)\right) E^\alpha \otimes e_\beta \\
&= \left(\sum_\alpha e_\alpha(f)E^\alpha\right) \otimes \left(\sum_\beta s^\beta e_\beta\right) + f\sum_{\alpha,\beta} e_\alpha(s^\beta)E^\alpha \otimes e_\beta \\
&= df \otimes S + f\partial S.
\end{aligned}$$

Note that we have $\partial e_\alpha = 0$ for all α, and thus also $\partial E^\alpha = 0$ for all α. This implies that if $U = \sum_\beta u_\beta E^\beta$ is a one-form (with $u^\beta \in \mathfrak{S}$ for all β) then

$$\partial U = \sum_{\alpha,\beta} e_\alpha(u^\beta)E^\alpha \otimes E^\beta.$$

Now we compute the torsion $\widetilde{T}$ of ∂. For any $f \in \mathfrak{S}$ we have

$$\begin{aligned}
\widetilde{T}_{ab}{}^c\partial_c f &= 2\partial_{[a}\partial_{b]}f \\
&= 2\partial_{[a}(df)_{b]} \\
&= \partial_a \sum_\beta e_\beta(f)(E^\beta)_b - \partial_b \sum_\beta e_\beta(f)(E^\beta)_a \\
&= 2\sum_{\alpha,\beta} e_\alpha(e_\beta(f))(E^\alpha)_{[a}(E^\beta)_{b]} \\
&= 2\sum_{\alpha,\beta} [e_\alpha, e_\beta](f)(E^\alpha)_a(E^\beta)_b \\
&= 2\sum_{\alpha,\beta} (E^\alpha)_a(E^\beta)_b[e_\alpha, e_\beta]^c\partial_c f.
\end{aligned}$$

Since f was arbitrary, we deduce

$$\widetilde{T}_{ab}{}^c = 2\sum_{\alpha,\beta} (E^\alpha)_a(E^\beta)_b[e_\alpha, e_\beta]^c.$$

In particular, if the e_α are a basis of coordinate vector fields then ∂ is torsion-free.

Next we show that the curvature $\widetilde{R}$ of ∂ vanishes. For any $S \in \mathfrak{S}^\diamond$, say $S = \sum_\gamma s^\gamma e_\gamma$, we perform a computation similar to that above:

$$\begin{aligned}
\widetilde{R}_{abc}{}^d S^c &= (2\partial_{[a}\partial_{b]} - \widetilde{T}_{ab}{}^c\partial_c)S^d \\
&= 2\sum_{\alpha,\beta,\gamma} [e_\alpha, e_\beta](s^\gamma)(E^\alpha)_a(E^\beta)_b(e_\gamma)^d - \widetilde{T}_{ab}{}^c\partial_c S^d \\
&= \left(2\sum_{\alpha,\beta}[e_\alpha, e_\beta]^c(E^\alpha)_a(E^\beta)_b\right)\left(\sum_\gamma \partial_c(s^\gamma)(e_\gamma)^d\right) - \widetilde{T}_{ab}{}^c\partial_c S^d \\
&= \widetilde{T}_{ab}{}^c\partial_c S^d - \widetilde{T}_{ab}{}^c\partial_c S^d = 0.
\end{aligned}$$

If ∇ is some fixed affine connection on M then there is a tensor field $Q_{\diamond\diamond}{}^{\diamond}$ such that $(\partial_a - \nabla_a)X^c = Q_{ab}{}^c X^b$ for all $X \in \mathfrak{S}^{\diamond}$. (This is as in section 6, with ∂ in place of $\widetilde{\nabla}$.) In this context Q is usually renamed $-\Gamma$ and called the Christoffel symbol(s), with respect to the given basis of vector fields:

$$(\nabla_a - \partial_a)X^c = \Gamma_{ab}{}^c X^b$$

for all $X \in \mathfrak{S}^{\diamond}$. We can use this formula and (10.4) to compute the curvature tensor R of ∇ from the Christoffel symbols and the torsion of ∂. Replacing $\widetilde{\nabla}$ with ∂ in (10.4), and using $\widetilde{R} = 0$ and $\Gamma = -Q$, we have

$$R_{abc}{}^d = 2\partial_{[a}\Gamma_{b]c}{}^d + 2\Gamma_{[a|e|}{}^d\Gamma_{b]c}{}^e + 2\left(\sum_{\alpha,\beta}(E^\alpha)_a(E^\beta)_b[e_\alpha, e_\beta]^c\right)\Gamma_{ec}{}^d. \tag{12.1}$$

Of course, if ∂ is the connection derived from a basis of coordinate vector fields then the last term vanishes and then in components we have

$$R_{\alpha\beta\gamma}{}^\delta = \frac{\partial\Gamma_{\beta\gamma}{}^\delta}{\partial x_\alpha} - \frac{\partial\Gamma_{\alpha\gamma}{}^\delta}{\partial x_\beta} + \sum_\varepsilon\left(\Gamma_{\alpha\varepsilon}{}^\delta\Gamma_{\beta\gamma}{}^\varepsilon - \Gamma_{\beta\varepsilon}{}^\delta\Gamma_{\alpha\gamma}{}^\varepsilon\right). \tag{12.2}$$

13 The Levi-Civita Connection

Theorem 13.1. *On a (pseudo-)Riemannian manifold M with metric tensor g, there is a unique torsion-free affine connection ∇ such that $\nabla g = 0$.*

Proof: By theorem 6.2, there exists a torsion-free affine connection on M, say $\widetilde{\nabla}$. We seek $Q \in S_{\diamond\diamond}{}^{\diamond}$ such that the operator ∇ defined by $(\widetilde{\nabla}_a - \nabla_a)X^b = Q_{ac}{}^b X^c$ (for all $X \in \mathfrak{S}^{\diamond}$) is torsion-free and annihilates g. Since $\widetilde{\nabla}$ is torsion-free, by theorem 6.1 the requirement that ∇ also be torsion free is the requirement that $Q_{ac}{}^b = Q_{ca}{}^b$. We will solve for Q.

Since $(\widetilde{\nabla}_a - \nabla_a)g_{bc} = -Q_{ab}{}^d g_{dc} - Q_{ac}{}^d g_{bd}$ and we require $\nabla_a g_{bc} = 0$, after lowering indices on the right hand side we find that we need to solve

$$\widetilde{\nabla}_a g_{bc} = -Q_{abc} - Q_{acb}. \tag{13.1}$$

Cyclically permuting the indices a, b and c we find the additional equations

$$\widetilde{\nabla}_b g_{ca} = -Q_{bca} - Q_{bac} \tag{13.2}$$

and

$$\widetilde{\nabla}_c g_{ab} = -Q_{cab} - Q_{cba}. \tag{13.3}$$

Adding equations (13.1) and (13.3), subtracting (13.2), and using the symmetry of Q, we find

$$\widetilde{\nabla}_a g_{bc} + \widetilde{\nabla}_b g_{ca} - \widetilde{\nabla}_c g_{ab} = 2Q_{abc},$$

or in other words,

$$Q_{ab}{}^c = -\frac{(g^{-1})^{cd}}{2}(\widetilde{\nabla}_a g_{bd} + \widetilde{\nabla}_b g_{ad} - \widetilde{\nabla}_d g_{ab}). \tag{13.4}$$

This proves both existence and uniqueness. $\square$

Unless otherwise specified, on a Riemannian manifold ∇ will always denote this connection, called the Levi-Civita connection. Note that for all $X \in \mathfrak{S}^\diamond$ or $\mathfrak{S}_\diamond$ we have

$$\nabla_d X_a = \nabla_d (g_{ab} X^b) = \nabla_d g_{ab} X^b + g_{ab} \nabla_d X^b = g_{ab} \nabla_d X^b,$$

so that we see that ∇ commutes with the raising and lowering of indices.

A special case of (13.3) is when $\widetilde{\nabla}$ is the connection ∂ defined by a coordinate system $x_1, \dots,$ $x_{\dim M}$ (see section 12). Then with $\Gamma = -Q$, we have

$$\Gamma_{ab}{}^c = \frac{(g^{-1})^{cd}}{2} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right), \tag{13.5}$$

or in components

$$\Gamma_{\alpha\beta}{}^\gamma = \frac{1}{2} \sum_\delta (g^{-1})^{\gamma\delta} \left(\frac{\partial g_{\beta\delta}}{\partial x_\alpha} + \frac{\partial g_{\alpha\delta}}{\partial x_\beta} - \frac{\partial g_{\alpha\beta}}{\partial x_\delta} \right). \tag{13.6}$$

This provides a means for computing Γ if g is given in terms of the coordinates. The curvature is given in terms of the Christoffel symbols in (12.1) and (12.2).

The curvature of the Levi-Civita connection is antisymmetric in its last two indices: we have

$$0 = \nabla_{[a} \nabla_{b]} g_{cd} = -R_{abc}{}^e g_{ed} - R_{abd}{}^e g_{ce}$$

and so $R_{abcd} = -R_{abdc}$. Coupled with the first Bianchi identity and the antisymmetry of R_{abcd} in a and b, we also obtain the "interchange" symmetry $R_{abcd} = R_{cdab}$:

$$\begin{aligned}
2R_{abcd} &= R_{abcd} + R_{badc} \\
&= R_{cabd} - R_{bcad} - R_{dbac} - R_{adbc} \\
&= (R_{cadb} + R_{adcb}) + (R_{bcda} + R_{dbca}) \\
&= -R_{dcab} - R_{cdba} \\
&= R_{cdab} + R_{cdab} = 2R_{cdab}.
\end{aligned}$$

This symmetry implies the symmetry of the Ricci tensor:

$$R_{ac} = R_{abc}{}^b = R_{abcd} g^{bd} = R_{cdab} g^{db} = R_{ca}.$$

We define the scalar curvature R by $R = R_a{}^a = R_{ab} g^{ab}$. Another important tensor is the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \tag{13.7}$$

G is obviously symmetric, and by twice contracting the second Bianchi identity we find that its divergence $\nabla^a G_{ab}$ vanishes.

14 Geodesics

In section 3 we defined the notion of a geodesic with repect to an affine connection. Suppose ∇ is the connection, with Christoffel symbols Γ with respect to a connection ∂ derived from a coordinate system $x_1, \dots, x_n$. Then if a curve with coordinates $x_i(t)$ is a geodesic, with tangent vector $v(t) = \sum_{i=1}^n \dot{x}_i(t) (\partial / \partial x_i)$ then we have

$$0 = v^a \nabla_a v^b = v^a \left(\partial_a v^b + \Gamma_{ac}{}^b v^c \right).$$

In coordinates this is the assertion that for all $\beta = 1, \ldots, N$ we have

$$0 = \sum_{\alpha} \dot{x}_\alpha \frac{\partial}{\partial x_\alpha} \dot{x}_\beta + \sum_{\alpha,\gamma} \Gamma_{\alpha\gamma}{}^{\beta} \dot{x}_\alpha \dot{x}_\gamma$$

or alternately

$$0 = \ddot{x}_\beta + \sum_{\alpha,\gamma} \Gamma_{\alpha\gamma}{}^{\beta} \dot{x}_\alpha \dot{x}_\gamma. \tag{14.1}$$

This is known as the geodesic equation. Note that because the sum is symmetric in α and γ, only the symmetric part of the Christoffel symbols enters the geodesic equation. That is, if $\widetilde{\nabla}$ is the torsion-free connection obtained from ∇ as in the proof of theorem 6.2, then ∇ and $\widetilde{\nabla}$ have the same geodesics.

The basic results concerning geodesics are below; the basic tool used in their proofs is the fundamental theorem on the existence and uniqueness of solutions to ordinary differential equations (see [7]), together with the smooth dependence of the solutions on initial conditions. (The ODE theorem is applied to an appropriate flow on TM.)

Theorem 14.1. *For any $p \in M$ there is a nonempty open set U_p of T_pM, starshaped about 0 and maximal with respect to the property that for all $v \in U_p$ there is a unique geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. The map assigning to each $v \in U_p$ the other endpoint $\gamma(1)$ of this geodesic is a smooth map to M.*

We call the map $U_p \to M$ described in the theorem the exponential map at p, and denote it $\exp_p$. (A subset U of a vector space is called starshaped around one of its points of the segment joining that point to any other point of U lies entirely in U.)

Theorem 14.2. *For any $p \in M$ there is an open set $V \subseteq U_p$ containing 0 such that $\exp_p|_V$ is a diffeomorphism onto its image. For any $p \in M$ there is a fiberwise convex neighborhood W of $(p,0)$ in TM such that for all $(q,v) \in W$ we have $v \in U_q$, and that the map $(q,v) \mapsto (q, \exp_q v)$ is a diffeomorphism of W onto a neighborhood of (p,p) in $M \times M$.*

The image in M of a set V as in the theorem and starshaped about 0 is called a normal neighborhood of p. It is obvious that if Y is a normal neighborhood of p then there is a unique geodesic in Y joining p to any other given point of Y. One can do even better:

Theorem 14.3. *For all $p \in M$ there is a normal neighborhood of p which is also a normal neighborhood of each of its points.*

Such a set is called (geodesically) convex.

If ∇ is derived from a metric g then the condition that a path γ be a geodesic is precisely the condition that for each interval $[a,b]$ in the domain of γ, the integral $\int_a^b g_{ab}\dot{\gamma}^a\dot{\gamma}^b dt$ is extremized among nearby (in the C^∞ sense, and probably in weaker senses too) smooth paths. The Euler-Lagrange equations derived from the Lagrangian $g_{ab}\dot{\gamma}^a\dot{\gamma}^b$ are just the equations (14.1). Deriving the Euler-Lagrange equations is sometimes the best way to compute the Christoffel symbols.

15 Geodesic Deviation

Suppose we have a family $\gamma_s(t)$ of geodesics such that the map $(s,t) \mapsto \gamma_s(t)$ is a smooth embedding of its domain onto its image S in M. We denote the image of $\partial/\partial t$ (resp. $\partial/\partial s$) by $\dot{\gamma}$ (resp. γ'). By hypothesis, $\dot{\gamma}^a \nabla_a \dot{\gamma}^b = 0$. After extending $\dot{\gamma}$ and γ' to a neighborhood of S, we can consider the Lie bracket of the extensions. The restriction of this vector field to S depends only on the values of $\dot{\gamma}$ and γ' on S. Since they Lie-commute there, we have $\dot{\gamma}^a \nabla_a \gamma'^b = \gamma'^a \nabla_a \dot{\gamma}^b$ on S.

One should interpret γ' as the separation between neighboring geodesics and $\dot\gamma^a\nabla_a\gamma'$ as the relative velocity of the geodesics. Thus $a = \dot\gamma^a\nabla_a(\dot\gamma^b\nabla_b\gamma')$ represents the relative acceleration of nearby geodesics. We can find an explicit formula for a:

$$\begin{aligned}
a^c &= \dot\gamma^a\nabla_a(\dot\gamma^b\nabla_b{\gamma'}^c) \\
&= \dot\gamma^a\nabla_a({\gamma'}^b\nabla_b\dot\gamma^c) \\
&= \dot\gamma^a\nabla_a{\gamma'}^b\nabla_b\dot\gamma^c + \dot\gamma^a{\gamma'}^b\nabla_a\nabla_b\dot\gamma^c \\
&= {\gamma'}^a\nabla_a\dot\gamma^b\nabla_b\dot\gamma^c + \dot\gamma^a{\gamma'}^b R_{abd}{}^c\dot\gamma^d + \dot\gamma^a{\gamma'}^b\nabla_b\nabla_a\dot\gamma^c \\
&= \dot\gamma^a\dot\gamma^d{\gamma'}^b R_{abd}{}^c + {\gamma'}^a\left[0 - \dot\gamma^b\nabla_a\nabla_b\dot\gamma^c\right] + \dot\gamma^a{\gamma'}^b\nabla_b\nabla_a\dot\gamma^c \\
&= \dot\gamma^a\dot\gamma^d{\gamma'}^b R_{abd}{}^c.
\end{aligned}$$

If ∇ is the Levi-Civita connection for a metric g on M, then we can also conclude that the inner product of $\dot\gamma$ and γ' is constant along the geodesics γ_s. To prove this we first note that

$$\begin{aligned}
\dot\gamma^b\nabla_a\dot\gamma_b &= \nabla_a(\dot\gamma^b\dot\gamma_b) - \dot\gamma_b\nabla_a\dot\gamma^b \\
&= 0 - \dot\gamma^b\nabla_a\dot\gamma_b
\end{aligned}$$

and so $\dot\gamma^b\nabla_a\dot\gamma_b = 0$. Then we have

$$\begin{aligned}
\dot\gamma^a\nabla_a(\dot\gamma^b{\gamma'}_b) &= \dot\gamma^a\nabla_a\dot\gamma^b{\gamma'}_b + \dot\gamma^a\dot\gamma^b\nabla_a{\gamma'}_b \\
&= 0 + {\gamma'}^a\dot\gamma^b\nabla_a\dot\gamma_b \\
&= 0.
\end{aligned}$$

16 Killing Forms and Vector fields

If ∇ is any affine connection then a Killing form is a totally symmetric element of $\mathfrak{S}_{\diamond\ldots\diamond}$ which satisfies Killing's equation

$$\nabla_{(a}K_{bc\ldots d)} = 0.$$

The quantity $K_{a_1\ldots a_k}\dot\gamma^{a_1}\cdots\dot\gamma^{a_k}$ is constant along γ, if γ is a geodesic in M with tangent vector $\dot\gamma$ and K is a Killing form. . The proof is easy:

$$\begin{aligned}
\dot\gamma^b\nabla_b(K_{a_1\ldots a_k}\dot\gamma^{a_1}\cdots\dot\gamma^{a_k}) &= \dot\gamma^{a_1}\cdots\dot\gamma^{a_k}\dot\gamma^b\nabla_b K_{a_1\ldots a_k} + \sum_{i=1}^{k}\dot\gamma^{a_1}\dot\gamma^{a_{i-1}}\dot\gamma^{a_{i+1}}\dot\gamma^{a_k}K_{a_1\ldots a_k}\dot\gamma^b\nabla_b\dot\gamma^{a_i} \\
&= \dot\gamma^{(a_1}\dot\gamma^{a_k}\dot\gamma^{b)}\nabla_b K_{a_1\ldots a_k} \\
&= \dot\gamma^{a_1}\dot\gamma^{a_k}\dot\gamma^b\nabla_{(b}K_{a_1\ldots a_k)} \\
&= 0.
\end{aligned}$$

If ∇ is torsion free and K is a Killing one-form then by repeatedly using the symmetry $\nabla_{(a}K_{b)} = 0$, the definition of the Riemann tensor, and the first Bianchi identity, we find that K satisfies the equation

$$\nabla_a\nabla_b K_c = R_{bca}{}^d K_d.$$

Killing forms of higher degree similar but more complicated equations; more complicated still are the analogues of these equations when ∇ has torsion.

Finally, if M is equipped with a metric then the condition that the flow of a vector field $K^\diamond$ preserve the metric, that is, that $\mathfrak{L}_K g = 0$, is just the condition that $K_\diamond$ be a Killing one-form. The proof is easy:

$$\begin{aligned} 0 &= (\mathfrak{L}_K g)_{bc} \\ &= K^a \nabla_a g_{bc} + \nabla_b K^a g_{ac} + \nabla_c K^a g_{ba} \\ &= \nabla_b K_c + \nabla_c K_b = 2\nabla_{(b} K_{c)}. \end{aligned}$$

In this case, K is called a Killing vector field. It follows from considerations earlier in this section that a geodesic in M has constant inner product with any Killing field.

17 Change of Levi-Civita Connection with Change of Metric

Let g and $\tilde{g}$ be metrics on M, with Levi-Civita connections ∇ and $\widetilde{\nabla}$. Suppose $(\widetilde{\nabla}_a - \nabla_a)X^b = Q_{ab}{}^c X^b$ for all $X \in \mathfrak{S}^\diamond$. Then by (13.4) we have

$$Q_{ab}{}^c = -\frac{(g^{-1})^{cd}}{2}\left(\widetilde{\nabla}_a g_{bd} + \widetilde{\nabla}_b g_{ad} - \widetilde{\nabla}_d g_{ab}\right).$$

Now suppose that $g_{ab} = \tilde{g}_{cd} T_a{}^c T_b{}^d$ for some bundle automorphism T of TM. Let $\bar{T}$ be the inverse of T, so that $T_a{}^b \bar{T}_b{}^c = \bar{T}_a{}^b T_b{}^c = \mathbf{1}_a{}^c$. Then $(g^{-1})^{cd} = (\tilde{g}^{-1})^{ef} \bar{T}_e{}^c \bar{T}_f{}^d$.

We can now compute $Q_{ab}{}^c$; because $\widetilde{\nabla}\tilde{g} = 0$ we have

$$\begin{aligned} Q_{ab}{}^c &= -\frac{1}{2}(g^{-1})^{cd}\tilde{g}_{ef}\left(\widetilde{\nabla}_a(T_b{}^e T_d{}^f) + \widetilde{\nabla}_b(T_a{}^e T_d{}^f - \widetilde{\nabla}_d(T_a{}^e T_b{}^f)\right) \\ &= -\frac{1}{2}(\tilde{g}^{-1})^{ij}\bar{T}_i{}^c \bar{T}_j{}^d \tilde{g}_{ef} T_d{}^f(\widetilde{\nabla}_a T_b{}^e + \widetilde{\nabla}_b T_a{}^e) \\ &\quad - \frac{1}{2}(g^{-1})^{cd}\tilde{g}_{ef}(T_b{}^e \widetilde{\nabla}_a T_d{}^f + T_a{}^e \widetilde{\nabla}_b T_d{}^f) \\ &\quad + \frac{1}{2}(g^{-1})^{cd}\tilde{g}_{ef}(T_b{}^f \widetilde{\nabla}_d T_a{}^e + T_a{}^e \widetilde{\nabla}_d T_b{}^f). \end{aligned}$$

In the first term, the $\tilde{g}$'s and most of the T's cancel out. The term in the last pair of parentheses is obviously invariant under simultaneous interchange of a with e and b with f. Since both terms are contracted over e and f with the symmetric tensor $\tilde{g}$, the last term is actually equal to $(g^{-1})^{cd}\tilde{g}_{ef} T_a{}^e \widetilde{\nabla}_d T_b{}^f$. Thus we have

$$Q_{ab}{}^c = -\bar{T}_e{}^c \widetilde{\nabla}_{(a} T_{b)}^e - (g^{-1})^{cd}\tilde{g}_{ef} T_{(a}{}^e \widetilde{\nabla}_{b)} T_d{}^f + (g^{-1})^{cd}\tilde{g}_{ef} T_{(a}{}^e \widetilde{\nabla}_{d)} T_b{}^f.$$

By virtue of $g_{ab}\bar{T}_c{}^a = \tilde{g}_{ca} T_b{}^a$ this simplifies to

$$\begin{aligned} Q_{ab}{}^c &= -\bar{T}_e{}^c \widetilde{\nabla}_{(a} T_{b)}^e - (g^{-1})^{cd} g_{e(a} \bar{T}_{|f|}{}^e \widetilde{\nabla}_{b)} T_d{}^f + (g^{-1})^{cd} g_{ea} \bar{T}_f{}^e \widetilde{\nabla}_d T_b{}^f \\ &= -\bar{T}_e{}^c \widetilde{\nabla}_{(a} T_{b)}^e - (g^{-1})^{cd} \bar{T}_f{}^e (g_{e(a} \widetilde{\nabla}_{b)} T_d{}^f - g_{ea} \widetilde{\nabla}_d T_b{}^f). \end{aligned} \tag{17.1}$$

18 Conformal Transformations

A special case of two metrics being related as in section 17 is when $T = \Omega\mathbf{1}$, so that $g = \Omega^2\tilde{g}$, where $\Omega \in \mathfrak{S}$. We say that g and $\tilde{g}$ are conformally equivalent metrics. As a special case of (17.1) we find

$$\begin{aligned} Q_{ab}{}^c &= -\Omega^{-1\mathbf{1}}{}_E{}^c \widetilde{\nabla}_{(a} \Omega \mathbf{1}_{b)}^e - (g^{-1})^{cd}\Omega^{-1}\mathbf{1}_d{}^e g_{e(a}\widetilde{\nabla}_{b)}\Omega\mathbf{1}_d{}^f \\ &\quad + (g^{-1})^{cd}\Omega^{-1}\mathbf{1}_f{}^e g_{ea}\widetilde{\nabla}_d \Omega \mathbf{1}_b{}^f \\ &= -\mathbf{1}_{(b}{}^c \widetilde{\nabla}_{a)} \ln\Omega - \mathbf{1}_{(a}{}^c \widetilde{\nabla}_{b)\,\ln\Omega + (g}{}^{-1})^{cd} g_{ab} \widetilde{\nabla}_d \ln\Omega \\ &= -2\ \mathbf{1}_{(a}{}^c \nabla_b \ln\Omega + g_{ab}(g^{-1})^{cd}\nabla_d \ln\Omega, \end{aligned} \tag{18.1}$$

where we have used the facts that ∇ and $\widetilde{\nabla}$ agree on $\mathfrak{S}$ and both annihilate $\mathbf{1}$.

Using (18.1) we can compute the difference of the curvatures of ∇ and $\widetilde{\nabla}$. Lengthy calculation reveals

$$\begin{aligned}(\widetilde{R}-R)_{abc}{}^{d} = {} & \mathbf{1}_{[a}{}^{d}\nabla_{b]}\nabla_c \ln\Omega + (g^{-1})^{df} g_{c[b}\nabla_{a]}\nabla_f \ln\Omega \\ & + \mathbf{1}_{[a}{}^{d}\nabla_{b]}\ln\Omega\nabla_c\ln\Omega - \mathbf{1}_{[a}{}^{d} g_{b]c}(g^{-1})^{ef}\nabla_e\ln\Omega\nabla_f\ln\Omega \\ & + \nabla_{[a}\ln\Omega g_{b]c}(g^{-1})^{de}\nabla_e\ln\Omega.\end{aligned}$$

Contraction on b and d yields

$$\begin{aligned}(\widetilde{R}-R)_{ac} = {} & g_{ac}(g^{-1})^{ef}\nabla_e\nabla_f\ln\Omega \\ & + (n-2)\left(g_{ac}(g^{-1})^{ef}\nabla_e\ln\Omega\nabla_f\ln\Omega - \nabla_a\nabla_c\ln\Omega - \nabla_a\ln\Omega\nabla_a\ln\Omega\right).\end{aligned}$$

This allows one to compute the scalar curvatures $\widetilde{R} = \widetilde{R}_{ab}(\tilde{g}^{-1})^{ab}$ and $R_{ab}(g^{-1})^{ab}$ and compare them. The result is

$$\begin{aligned}\Omega^{-2}\widetilde{R} = {} & R - 2(n-1)(g^{-1})^{ab}\nabla_a\nabla_b\ln\Omega \\ & + (n-2)(n-2)(g^{-1})^{ab}\nabla_a\ln\Omega\nabla_b\ln\Omega.\end{aligned}$$

These computations set the stage for the Weyl conformal tensor $C_{\diamond\diamond\diamond}{}^{\diamond}$, which is defined for $n > 2$ by

$$C_{abc}{}^{d} = R_{abc}{}^{d} - \frac{1}{n-2} g_{[a|[c}R_{e]|b]}(g^{-1})^{de} + \frac{2}{(n-1)(n-2)} R g_{a[c}g_{e]b}(g^{-1})^{de}.$$

Computations reveal that $C_{\diamond\diamond\diamond\diamond}$ has all the symmetries of $R_{\diamond\diamond\diamond\diamond}$, and that in addition $C_{\diamond\diamond\diamond\diamond}$ is trace-free on any pair of its indices. Since in dimension 3 the map $\mathfrak{S}_{[\diamond\diamond]}{}^{[\diamond\diamond]} \to \mathfrak{S}_{\diamond}{}^{\diamond}$ given by $S_{ab}{}^{cd} \mapsto S_{ab}{}^{cb}$ is an isomorphism (proof: count dimensions and check surjectivity), the Weyl tensor vanishes on any 3-dimensional manifold. (We use the fact that $C_{ab}{}^{cd} = 0$ if and only if $C_{ab}{}^{cb} = 0$, which holds because of the trace-free property of C.)

Naturally one makes the analogous definition for $\widetilde{C}$, using $\tilde{g}$ and $\widetilde{R}$ in place of g and the various curvature tensors R. The big surprise is that $\widetilde{C} = C$. That is, the Weyl tensor is an invariant of the conformal class of g. The proof is a very lengthy computation, best performed in the diagrammatic notation described in the appendix of [6].

Finally, there is a differential relation between the Weyl and Ricci tensors. The second Bianchi identity may be written

$$0 = \nabla_e R_{abc}{}^{d} + \nabla_b R_{eac}{}^{d} + \nabla_a R_{bec}{}^{d}.$$

After contracting on d and e one obtains

$$\nabla_d R_{abc}{}^{d} = 2\nabla_{[b}R_{a]c}. \tag{18.2}$$

Raising c and then contracting on c and a yields

$$\nabla_d R_b{}^{d} = \frac{1}{2}\nabla_b R. \tag{18.3}$$

These equations allow us to compute the divergence of the Weyl tensor in terms of the Ricci tensor. The result is

$$\nabla_d C_{abc}{}^{d} = 2\frac{n-3}{n-2}\left(\nabla_{[b}R_{a]c} + \frac{1}{2(n-1)}\nabla_{[a}R g_{b]c}\right) \tag{18.4}$$

19 Volume and the Hodge Star Operator

A volume form on an N-dimensional real vector space V is a nonzero element of $\wedge^N(V^*)$. An orientation is an equivalence class of volume forms, two such forms being equivalent if they differ by multiplication by a positive number. There are two orientations on V. A basis $f_1, \ldots, f_N$ is said to be compatible with an orientation if for some (hence for any) representative ω of that orientation we have

$$\omega_{a_1 \ldots a_N}(f_1)^{a_1} \cdots (f_N)^{a_N} > 0.$$

V is called oriented if one of its orientations is distinguished. In this case we say that a basis is positively oriented (or just oriented) if it is compatible with that orientation.

If V is equipped with a metric g then g induces an inner product on $\wedge^N(V^*)$. If $f_1, \ldots, f_N$ is an orthonormal basis and $F^1, \ldots, F^N$ are a dual basis, then the norm of $F^1 \wedge \cdots \wedge F^N$ equals $(-1)^\sigma$, where σ is the number of the f_i with negative norm. Thus, given g there is a canonical choice e of volume form for each orientation—the unique representative of that orientation with norm ± 1. Furthermore, because we have

$$(F^1 \wedge \cdots \wedge F^N)(f_1, \ldots, f_N) = 1 > 0,$$

we see that $e = F^1 \wedge \cdots \wedge F^N$ if $f_1, \ldots, f_N$ is positively oriented. If $f'_1, \ldots, f'_N$ are another oriented basis for V, with dual basis ${F'}^1, \ldots, {F'}^N$ and $T \in \operatorname{Aut} V$ such that $T(f_i) = f'_i$, then we have

$$\begin{aligned}
(F^1 \wedge \cdots \wedge F^N)(f'_1, \ldots, f'_N) &= (F^1 \wedge \cdots \wedge F^N)(Tf_1, \ldots, Tf_N) \\
&= (T^\dagger F^1 \wedge \cdots \wedge T^\dagger F^N)(f_1, \ldots, f_N) \\
&= T^\dagger (F^1 \wedge \cdots \wedge F^N)(f_1, \ldots, f_N) \\
&= \det T.
\end{aligned}$$

Here $T^\dagger$ stands for the adjoint map on V^* and the map it induces on $\wedge^N(V^*)$. We have also used the definition of the determinant. Since both bases are oriented, we know that $\det T > 0$. Since we have $({F'}^1 \wedge \cdots {F'}^N)(f'_1, \ldots, f'_N) = 1$, we therefore have

$$(\det T)({F'}^1 \wedge \cdots \wedge {F'}^N) = F^1 \wedge \cdots \wedge F^N.$$

Writing elements of V as column vectors with respect to the primed basis, if $x, y \in V$ then their inner product is $x^\dagger G' y$, where G' is the matrix of inner products of the f'_i. Since we have $G' = T^\dagger G T$, where G is the matrix of inner products with respect to the unprimed basis, we have $\det G' = (\det T)^2 (\det G)$. Since $\det G$ and $\det G'$ have the same sign, we know that $|\det G| = 1$ and $\det T > 0$, we conclude that $\det T = +\sqrt{|\det G'|}$. Therefore

$$e = \sqrt{|\det G'|}\, {F'}^1 \wedge \cdots \wedge {F'}^N.$$

This allows one to express the natural volume form conveniently in any basis, given the matrix of inner products.

The Hodge star operator $*$ is a map from $\wedge^p(V^*)$ to $\wedge^{N-p}(V^*)$ for each $p = 0, \ldots, N$. It is given by the map

$$(*\theta)_{a_1 \ldots a_{N-p}} = e_{a_1 \ldots a_{N-p} b_1 \ldots b_p} \theta^{b_1 \ldots b_p}.$$

We now work out the action of $*$ explicitly in terms of a basis. Suppose $f_1, \ldots, f_N$ is an orthonormal bais of V, with the square norm of f_i being $\varepsilon_i = \pm 1$, so that $(F^\alpha)^a = \varepsilon_\alpha (f_\alpha)^a$. Suppose that

$i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$ are two disjoint sets whose union is $\{1, \ldots, N\}$. Let π be the sign of the permutation that carries $(1, 2, \ldots, N)$ to $(i_1, \ldots, i_p, j_1, \ldots, j_q)$. Let $A = F^{i_1} \wedge \cdots \wedge F^{i_p}$ and $B = F^{j_1} \wedge \cdots \wedge F^{j_q}$. Let ε_A (resp. ε_B) be the number of f_{i_k} (resp. f_{j_k}) of norm -1. Then

$$\begin{aligned}
(*B) &= (-1)^{\varepsilon_B}(F^1 \wedge \cdots \wedge F^N)(f_{j_1}, \ldots, f_{j_q}) \\
&= (-1)^{\varepsilon_B}\pi(F^{i_1} \wedge \cdots \wedge F^{i_p} \wedge F^{j_1} \wedge \cdots \wedge F^{j_q})(f_{j_1}, \ldots, f_{j_q}) \\
&= (-1)^{\varepsilon_B}\pi F^{i_1} \wedge \cdots \wedge F^{i_p} \\
&= \pi(-1)^{\varepsilon_B} A
\end{aligned}$$

and

$$\begin{aligned}
(*A) &= (-1)^{\varepsilon_A}(F^1 \wedge \cdots \wedge F^N)(f_{i_1}, \ldots, f_{i_p}) \\
&= (-1)^{\varepsilon_A}\pi(F^{i_1} \wedge \cdots \wedge F^{i_p} \wedge F^{j_1} \wedge \cdots \wedge F^{j_q})(f_{i_1}, \ldots, f_{i_p}) \\
&= (-1)^{\varepsilon_A}(-1)^{p(N-p)}\pi(F^{j_1} \wedge \cdots \wedge F^{j_q} \wedge F^{i_1} \wedge \cdots \wedge F^{i_p})(f_{i_1}, \ldots, f_{i_p}) \\
&= \pi(-1)^{\varepsilon_A}(-1)^{p(N-p)} B.
\end{aligned}$$

So

$$**A = \pi(-1)^{\varepsilon_A}(-1)^{p(N-p)}\pi(-1)_{\varepsilon_B} A = (-1)^{\sigma + p(N-p)} A.$$

The exponent can be simplified because $p^2 \equiv p \pmod 2$, so $**A = (-1)^{\sigma+p(n-1)}A$. These computations also provide an algorithm for computing the dual of a form $F^{i_1} \wedge \cdots \wedge F^{i_p}$, which may be summed up as follows. (i) write down the form $F^1 \wedge \cdots \wedge F^N$. (ii) permute the factors so that $F^{i_1} \wedge \cdots \wedge F^{i_p}$ is "at the right hand end"; this may introdue a sign depending in the parity of the permutation used. (iii) 'cancel' the factor $F^{i_1} \wedge \cdots \wedge F^{i_p}$. (iv) multiply what's left by the product of the norms of the f_{i_k}; this may introduce another sign.

All of these constructions apply to manifolds, by applying them pointwise. A volume form on M is a nonvanishing element of $\Omega^N(M)$. M is said to be orientable if it admits a volume form; henceforth we will assume M to be orientable. An orientation on M is an equivalence class of volume forms, two being equivalent if they differ by multiplication by a positive element of $\mathfrak{S}$. If M is connected the there are exactly two orientations on M. If one of these is distinguished then we say that M is oriented. If M is oriented and equipped with a metric then it admits a natural volume form e, obtained by applying the above constructions pointwise. To check that this yields a volume form, all we have to do is check that e is a smooth section of $\wedge^N(T^*M)$. This is easy because in local coordinates $x_1, \ldots, x_N$ an expression for e is

$$e = \sqrt{|\det g|}\, dx_1 \wedge \cdots \wedge dx_N$$

where $\det g$ is the determinant of the inner product matrix of the coordinate vector fields. The Hodge star operator is then a map $\Omega^p(M) \to \Omega^{N-p}(M)$ for each $p = 0, \ldots, N$. It satisfies the same relation

$$* * \theta = (-1)^{\sigma + p(N-1)}\theta$$

for $\theta \in \Omega^p(M)$ as we met above.

The volume form e is covariantly constant on M: let X be a vector field, let $p \in M$ and let $f_1, \ldots f_N$ be an orthonormal basis parallel-transported along the flow line of X through p. Then $e_{a_1 \ldots a_N} = \pm(f_1)_{[a_1} \cdots (f_{|N|})_{a_N}$, and since $X^a \nabla_a (f_\alpha)_b = 0$ for all α we have $X^a \nabla_a e_{a_1 \ldots a_N} = 0$ along the flow line of X through p. Since p and X were arbitrary, we must have $\nabla e = 0$.

20 The Method of Moving Frames

This is a technique due to Cartan for computing the curvature of the Levi-Civita connection ∇ of a metric g. Suppose that $e_1, \ldots, e_N$ are a basis of vector fields and that $E^1, \ldots, E^N$ are a dual basis. We define the functions $h_{\alpha\beta} = g_{ab}(e_\alpha)^a(e_\beta)^b$; these are the entries in the matrix of inner products of the basis elements with each other. Let $(h^{-1})^{\alpha\beta}$ be the functions such that $\sum_\beta h_{\alpha\beta}(h^{-1})^{\beta\gamma} = 1$ or 0 according as $\alpha = \gamma$ or $\alpha \neq \gamma$. Then we have

$$(E^\mu)_a = \sum_\alpha (h^{-1})^{\alpha\mu}(e_\alpha)_a$$

for each μ. If our basis is orthogonal, then of course the sum simplifies to a single term. We define the Ricci spin coeffiencients $\omega_{a\mu\nu} \in \mathfrak{S}_\diamond$ by

$$\omega_{a\mu\nu} = (e_\mu)^b \nabla_a (e_\nu)_b.$$

Since ∇ annihilates g we have

$$\begin{aligned}
\omega_{a\mu\nu} &= (e_\mu)^b \nabla_a (e_\nu)_b \\
&= \nabla\left((e_\mu)^b (e_\nu)_b\right) - (e_\nu)_b \nabla_a (e_\mu)^b \\
&= \nabla_a (h_{\mu\nu}) - (e_\nu)^b \nabla_a (e_\mu)_b \\
&= (dh_{\mu\nu})_a - \omega_{a\nu\mu}.
\end{aligned} \tag{20.1}$$

In particular, if the functions $h_{\alpha\beta}$ are constant then the we have $\omega_{a\mu\nu} = -\omega_{a\nu\mu}$. At the end of this section we show that if (20.1) holds then ∇ annihilates g.

A characterization of the fact that ∇ is torsion-free is the identity $\nabla_{[a}(e_\nu)_{b]} = \partial_{[a}(e_\nu)_{b]}$, where ∂ is any coordinate connection and we adopt the convention that greek indices are not subject to symmetrization operators. (If this holds for all ν then we have

$$0 = \left(\nabla_{[a} - \partial_{[a}\right)(e_\nu)_{b]} = -\Gamma_{[ab]}{}^c (e_\nu)_c$$

for all ν, so the Christoffel symbols are symmetric, so ∇ is torsion-free.) This provides a way to write down equations for the spin coefficients. We have

$$\nabla_a (e_\nu)_b = \sum_\mu (E^\mu)_b \omega_{a\mu\nu}$$

for all ν, as can be verified by (for each λ) contracting both sides with $(e_\lambda)^b$ and using the definition of the spin coefficients. Thus

$$\begin{aligned}
\partial_{[a}(e_\nu)_{b]} &= \sum_\mu (E^\mu)_{[b}\omega_{a]\mu\nu} \\
&= \sum_{\mu,\alpha} (h^{-1})^{\alpha\mu}(e_\alpha)_{[b}\omega_{a]\mu\nu}
\end{aligned} \tag{20.2}$$

Now we can compute the Riemannian curvature. We have

$$\begin{aligned}
R_{\alpha\beta\gamma\delta} &= R_{abcd}(e_\alpha)^a(e_\beta)^b(e_\gamma)^c(e_\delta)^d \\
&= -(e_\alpha)^a(e_\beta)^b(e_\gamma)^c(\nabla_a\nabla_b - \nabla_b\nabla_a)(e_\delta)_c.
\end{aligned} \tag{20.3}$$

We also have

$$
\begin{aligned}
(e_\gamma)^c\nabla_a\nabla_b(e_\delta)_c. &= \nabla_a\left((e_\gamma)^c\nabla_b(e_\delta)_c\right) - \nabla_a(e_\gamma)^c\nabla_b(e_\delta)_c \\
&= \nabla_a\left((e_\gamma)^c\nabla_b(e_\delta)_c\right) - \nabla_a(e_\gamma)^c\mathbf{1}_c{}^d\nabla_b(e_\delta)_d \\
&= \nabla_a\omega_{b\gamma\delta} - \sum_\mu \nabla_a(e_\gamma)^c(E^\mu)_c(e_\mu)^d\nabla_b(e_\delta)_d \\
&= \nabla_a\omega_{b\gamma\delta} - \sum_\mu \nabla_a(e_\gamma)^c\left(\sum_\nu (h^{-1})^{\mu\nu}(e_\nu)_c\right)\omega_{b\mu\delta} \\
&= \nabla_a\omega_{b\gamma\delta} - \sum_{\mu,\nu}(h^{-1})^{\mu\nu}\omega_{a\nu\gamma}\omega_{b\mu\delta}.
\end{aligned}
$$

Antisymmetrizing and plugging into (20.3) we find

$$
R_{ab\gamma\delta} = -\partial_{[a}\omega_{b]\gamma\delta} + 2\sum_{\mu\nu}(h^{-1})^{\mu\nu}\omega_{[a|\nu\gamma}\omega_{b]\mu\delta}. \tag{20.4}
$$

If we suppress the abstract indices, so that the spin coefficients and the components $R_{ab\gamma\delta}$ of the curvature are considered as differential forms, then we can rewrite this as

$$
R_{\gamma\delta} = -d\omega_{\gamma\delta} + 2\sum_{\mu,\nu}(h^{-1})^{\mu\nu}\omega_{\nu\gamma}\wedge\omega_{\mu\delta}.
$$

We show below that (20.1) implies $\nabla g = 0$. In light of this and the fact that (20.2) characterizes ∇ as torsion-free, we see that together (20.1) and (20.2) specify ∇ the spin coefficients completely. So suppose (20.1) holds. Then we have, for each μ and ν,

$$
\begin{aligned}
\omega_{a\mu\nu} &= (e_\mu)^b\nabla_a(e_\nu)_b \\
&= (e_\mu)^b\nabla_a\left(g_{bc}(e_\nu)^c\right) \\
&= (e_\mu)^b g_{bc}\nabla_a(e_\nu)^c + (e_\mu)^b(e_\nu)^c\nabla_a g_{bc} && (20.5) \\
&= g_{bc}\nabla_a\left[(e_\mu)^b(e_\nu)^c\right] - g_{bc}(e_\nu)^c\nabla_a(e_\mu)^b + (e_\mu)^b(e_\nu)^c\nabla_a g_{bc} && (20.6).
\end{aligned}
$$

Reversing the roles of μ and ν in (20.5) we obtain

$$
\omega_{a\nu\mu} = (e_\nu)^b g_{bc}\nabla_a(e_\mu)^c + (e_\nu)^b(e_\mu)^c\nabla_a g_{bc}.
$$

Exchanging the roles of b and c, we find

$$
\omega_{a\nu\mu} = (e_\nu)^c g_{bc}\nabla_a(e_\mu)^b + (e_\nu)^c(e_\mu)^b\nabla_a g_{bc}.
$$

Adding this to (20.6) we obtain

$$
\omega_{a\mu\nu} + \omega_{a\nu\mu} = g_{bc}\nabla_a\left[(e_\mu)^b(e_\nu)^c\right] + 2(e_\mu)^b(e_\nu)^c\nabla_a g_{bc}.
$$

By (20.1) and the definition of $h_{\mu\nu}$, this implies

$$
\nabla_a\left[g_{bc}(e_\mu)^b(e_\nu)^c\right] = g_{bc}\nabla_a\left[(e_\mu)^b(e_\nu)^c\right] + 2(e_\mu)^b(e_\nu)^c\nabla_a g_{bc}.
$$

Applying the Leibniz rule to the left and cancelling terms ,we are left with

$$
0 = (e_\mu)^b(e_\nu)^c\nabla_a g_{bc}.
$$

Since μ and ν were arbitrary we must have $\nabla g = 0$.

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