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PROJECTIVE TRANSFORMATIONS OF PSEUDO-RIEMANNIAN MANIFOLDS

A. V. Aminova

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Dedicated to Professor A. S. Solodovnikov,
whose papers on projective transformations
of Riemannian spaces were very valuable to the author

Introduction

A projective transformation of a pseudo-Riemannian manifold M^n is an automorphism of the induced Riemannian connection of the projective structure that takes geodesics in M^n to geodesics again.

For the first time, the problem of determining Riemannian spaces admitting continuous transformation groups preserving geodesics was considered by Sophus Lie for the case of two-dimensional surfaces (see [306]). However, as Fubini wrote in the preface to [262], "the famous mathematician did not succeed in solving this problem" (which Fubini called the "Lie problem").

Having criticized Lie's method, which, to Fubini's opinion, could not be used for the general solution of the problem, Fubini developed his own approach based on the infinitesimal calculus, later named the "Lie differentiation."

The subsequent development of the theory of projective transformations on linear connection spaces is connected with the names of E. Cartan, Eisenhart, Thomas, Knebelman, Schouten, Yano, Egorov, Vranceanu, Kobayashi, and others.

It is known that in the spaces of constant curvature S^n, when considered in the small, the complete projective group coincides with the projective group of pseudo-Euclidean space, i.e., with the group of bilinear substitutions, and depends on $n(n+2)$ parameters.

In the spaces V^n of nonconstant curvature, the order of the complete projective group does not exceed $n(n-2)-5$ (Egorov [91]), and, moreover, in the majority of cases, this group consists of similarity transformations (homotheties) or isometries. In 1903, in "Turin Academy Notes," Fubini's paper "Groups of geodesic transformations" was published [262], which as mentioned above, laid the foundations for a systematic definition and study of Riemannian spaces admitting infinitesimal projective transformations. Later on, Solodovnikov [156–158] continued Fubini's research and completely solved the problem posed; the works of Fubini and Solodovnikov contain a classification of the Riemannian spaces V^n, $n \geq 3$, in terms of (local) groups of projective transformations which are larger than homothety groups.

The conclusions of Fubini and Solodovnikov are based on the assumption that the metrics considered are positive definite. Taking a given signature as a condition considerably complicates the problem and requires a basically new approach for its solution. It was proposed in the papers of the author [5–11], where the problem of defining all pseudo-Riemannian manifolds with Lorentz signature $(+-\ldots-)$ (Lorentz manifolds) of dimension $n \geq 3$ admitting nonhomothetic infinitesimal projective and affine transformations was solved, and for each of them, the maximal projective and affine Lie algebras, together with the homothetic and isometric subalgebras, were defined.

This paper includes a survey of the results in the theory of projective transformations of pseudo-Riemannian manifolds, in particular, the solution of the classical geometrical problem of determining the pseudo-Riemannian metrics with corresponding geodesics (Sec. 5) and the Lie problem (Sec. 6). By using the technique of skew-normal frames developed by the author in [31] (see Sec. 1), all two-dimensional

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pseudo-Riemannian manifolds that admit Lie algebras of infinitesimal projective transformations are determined. For each of these manifolds, the maximal projective Lie algebra, together with the homothetic and isometric subalgebras, is defined. The metric is constructed for these manifolds, and also basis vector fields and the structure equations for the projective Lie algebras admitted are given in appropriate local coordinates (Theorem 6.7). In Sec. 6.5, properties of connected projective Lie groups of two-dimensional pseudo-Riemannian manifolds of nonconstant curvature are considered.

Special attention is given to concircular geometry, which is closely related to projective transformations (Sec. 4), as well as to applications in physics, mechanics, and the theory of differential equations.

Projective transformations occur systematically in studying symmetries of equations in mathematical physics. It is enough to mention that the Lie algebra of infinitesimal point symmetries of the Korteweg–de Vries equation is a subalgebra of the projective (more precisely, affine) Lie algebra, while the Riccati equation can be considered as a "singular realization" of the projective transformation group on a straight line ([97], p. 36).

The explanation of this becomes clear when we discover that the largest group of point symmetries of the equations in Newtonian dynamics is the 24-dimensional projective group acting on flat 4-dimensional space-time (Aminova [48,54]). This result was obtained by the geometric approach proposed by the author and is based on the ideas of Lie and E. Cartan. It generalizes the well-known Lie theorem stating that the dimension r of the group of point symmetries of the equation

$$\frac{d^2y}{dx^2} = f\big(x, y, \frac{dy}{dx}\big)$$

is at most 8, and the maximum value $r = 8$ is attained for the equation $\frac{d^2y}{dx^2} = 0$ or (as shown by Ovsyannikov [124]) for

$$\frac{d^2y}{dx^2} = a(x)\frac{dy}{dx} + b(x)y;$$

the corresponding group is the projective group on the plane.

In creating the theory of spaces with projective connection, Cartan continually emphasized its importance for studying differential equations [221]. Differential-geometric methods and, in particular, the methods of Cartan's theory allow us to develop a systematic approach for defining local and nonlocal symmetries of a wide class of ordinary and partial differential equations and for finding their solutions ([48, 54]).

Section 7, the concluding of the paper, is devoted to projective geometry of differential systems. In this section, we present the basic concepts and tools and give classifications of second-order differential equations with third-degree polynomials by the derivatives of the unknown function in the right-hand sides and systems of second-order differential equations in two unknowns and quadratic right-hand sides (geodesic equations) with respect to the Lie algebras of infinitesimal symmetries. We also indicate conditions under which the integral curves of the above equations and systems are straight lines.

Among the projectively mobile two-dimensional spaces defined in the paper, there are 1- and 2-parameter sets of surfaces, which makes them particularly attractive from the point of view of geometric research and its applications.

The volume images of some sets of revolution surfaces of revolution admitting projective transformations are given at the end of Sec. 6 (the visualization is carried out by Korneev).

This paper is appropriate for specialists in various fields, since it includes the necessary information about the theory of transformation groups of pseudo-Riemannian manifolds (Secs. 1, 2).

1. Pseudo-Riemannian Geometry in a Skew-Normal Frame

1.1. Let M be a differentiable manifold of dimension n. A linear frame or, briefly, a frame $\xi_{(p)}$ at a point $p \in M$ is an ordered basis of the tangent space T_pM. A moving frame Y over an open set $V \subseteq M$ is an ordered set $(Y_1, \dots, Y_n)$ of n vector fields on V such that their values at each point $p \in V$

form a frame. If (x, U) is a local coordinate system, then the coordinate (or natural) frame over U is $(X_1, \dots, X_n) \equiv (\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$.

Let LM denote the set of all frames at all points of M. The projection $\pi : LM \to M$ is defined as $\pi(\xi_{(p)}) = p$. The general linear group $\mathrm{GL}(n, \mathbb{R})$ acts freely on LM to the right; an element $A = (A^i_j) \in \mathrm{GL}(n, \mathbb{R})$ maps a frame $\xi_{(p)} = (\underset{1}{\xi}{}_{(p)}, \dots, \underset{n}{\xi}{}_{(p)})$ into the frame $\xi_{(p)}{}' = \xi_{(p)} A = (\underset{i}{\xi}{}_{(p)} A^i_1, \dots, \underset{i}{\xi}{}_{(p)} A^i_n)$, and $\pi(\xi_{(p)}) = \pi(\xi_{(p)}{}')$ if and only if $\xi_{(p)}{}' = \xi_{(p)} A$ for some $A \in \mathrm{GL}(n, \mathbb{R})$. Every frame $\xi_{(p)}$ at $p \in U$ can be represented in a unique way as

$$\xi_{(p)} = (\underset{1}{\xi}{}_{(p)}, \dots, \underset{n}{\xi}{}_{(p)})$$

with

$$\underset{i}{\xi}{}_{(p)} = \underset{i}{\xi}{}^k_{(p)} \frac{\partial}{\partial x^k},$$

where $(\underset{i}{\xi}{}^k_{(p)})$ is a nonsingular matrix. We take x^i and $\underset{i}{\xi}{}^k_{(p)}$ as local coordinates in $\pi^{-1}(U)$; this turns LM into a differentiable manifold. The set LM, together with the differentiable structure thus defined on it, is the principal bundle $LM(M, \mathrm{GL}(n, \mathbb{R}))$, which is called the *bundle of (linear) frames over M*. A connection ∇ in the bundle LM of linear frames over M is called a *linear* or *affine connection on* M [110].

In the language of bundles, a moving frame Y on $V \subseteq M$ is a smooth local cross section $\sigma : V \to LM$ of the bundle of frames that associates the linear frame $(Y_1(p), \dots, Y_n(p))$ with each $p \in V$. Every frame ξ with $\pi(\xi) \in V$ can be uniquely expressed in terms of the moving frame Y by setting $Y : \xi = (Y_1, \dots, Y_n)A = (Y_i A^i_1, \dots, Y_i A^i_n)$. On the other hand, for all p in a coordinate neighborhood $U \subseteq V$, we have

$$(Y_1, \dots, Y_n) = \left(\underset{1}{\xi}{}^i \frac{\partial}{\partial x^i}, \dots, \underset{n}{\xi}{}^i \frac{\partial}{\partial x^i} \right).$$

The canonical form θ on LM is an $\mathbb{R}^n$-valued 1-form on LM defined by $\theta(X) = \xi^{-1}(\pi(X))$ for any $X \in T_\xi LM$, where $\xi \in LM$ is considered as a linear mapping from $\mathbb{R}^n$ with the natural basis (E_i) onto $T_{\pi(\xi)}M$. In terms of a local coordinate system, the canonical form $\theta = \theta^i E_i$ is given by the relation

$$\theta^i = \theta^i_j dx^j \quad (dx^k = \underset{i}{\xi}{}^k \theta^i), \tag{1.1}$$

where (θ^i_j) is inverse to the matrix $(\underset{j}{\xi}{}^k)$:

$$\underset{k}{\xi}{}^i \theta^k{}_j = \delta^i_j. \tag{1.2}$$

The ordered set of linearly independent forms θ^i gives at every point $p \in M$ the *coframe* $(\theta^1, \dots, \theta^n)$, i.e., a basis of the space $T^*_p M$ that is dual to the basis ξ of $T_p M$.

Let ∇ be a linear connection on M, and let $\omega = \omega^i{}_j E^j{}_i$ be the connection form, where $(E^j{}_i)$ is a basis of $\mathrm{gl}(n, \mathbb{R})$ such that the matrix E^j_i has entry 1 at the intersection of the ith row and the jth column and is zero elsewhere. In terms of a moving frame Y on U, the components $\omega^i{}_j$ of the connection form ω are defined as follows (see [110, 304], Vol. I, p. 141):

$$\omega^i{}_j = \theta^i_k (d\xi^k_j + \Gamma^k_{ml} \xi^l_j dx^m) \equiv \gamma^i_{jl} \theta^l.$$

Here, γ^i_{jl} are components of the connection in the moving frame Y and Γ^k_{ml} are components of the connection in the natural frame defined by the formula

$$\nabla_{X_j} X_k = \Gamma^i_{jk} X_i \quad (i, j, k = 1, \dots, n), \tag{1.3}$$

where ∇_{X_j} denotes the covariant derivative with respect to the coordinate vector field X_j.

1.2. For any vector field X on a differentiable manifold M, the linear mapping $X : qf \to Xf$ of the algebra $\mathcal{F}M$ of real-valued functions on M into itself is defined by

$$(Xf)(p) = X_p f \quad (p \in M),$$

where $X_p f$ is the derivative of the function f with respect to the vector $X_p \in T_pM$. If ξ^i, $i = 1, \dots, n$, are components of the vector field X in a chart (x, U), then

$$Xf\mid_U = \xi^i \frac{\partial f}{\partial x^i} \equiv \xi^i \partial_i f;$$

in particular,

$$Xx^i = \xi^i \quad (i = 1, \dots, n). \tag{1.4}$$

The set $\mathcal{X}M$ of all (differentiable) vector fields on the differentiable manifold M is an infinite-dimensional real Lie algebra with respect to the *Lie bracket* defined by

$$[X, Y]f = X(Yf) - Y(Xf)$$

for all functions f on M and $X, Y \in \mathcal{X}M$. If $X = \xi^j \partial_j$ and $Y = \eta^j \partial_j$ in a chart (x, U), then by virtue of (1.4), we have

$$[X, Y] = (\xi^j \partial_j \eta^i - \eta^j \partial_j \xi^i)\partial_i. \tag{1.5}$$

In particular, for the coordinate vector fields $X_1 = \partial_1, \dots, X_n = \partial_n$, we obtain

$$[X_i, X_j] = 0 \quad (i, j = 1, \dots, n). \tag{1.6}$$

1.3. Let ∇ be a linear connection on M. We define the mapping $T : \mathcal{X}M \times \mathcal{X}M \to \mathcal{X}M$ by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

and the mapping $R : \mathcal{X}M \times \mathcal{X}M \times \mathcal{X}M \to \mathcal{X}M$ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.7}$$

T is a tensor field of type (1.2), which is called the *torsion tensor field* (or simply, *torsion*), and R is a tensor field of type (1.3), called the *curvature tensor field* (or simply, *curvature*) of the connection ∇. In what follows, it is always assumed that a linear connection has zero torsion. Such a connection is said to be a *torsion-free connection.*

The components R^i_{jkl} of the curvature tensor R in a chart (x, U) are defined by

$$R(X_k, X_e)X_j = R^i_{jkl} X_i$$

and are given by the formula

$$R^i_{jkl} = -R^i_{jlk} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^h_{jl}\Gamma^i_{hk} - \Gamma^h_{jk}\Gamma^i_{hl} \quad (i, j, h, k, l = 1, \dots, n). \tag{1.8}$$

For any tensor field B on M, we have the Ricci identity

$$(\nabla_{X_l}\nabla_{X_k} - \nabla_{X_k}\nabla_{X_l}) B^{j_1 \dots j_s}_{i_1 \dots i_r} \equiv 2B^{j_1 \dots j_s}_{i_1 \dots i_r, [kl]}$$
$$= \sum_{q=1}^{r} B^{j_1 \dots j_s}_{i_1 \dots i_{q-1} h i_{q+1} \dots i_r} R^h_{i_q kl} - \sum_{t=1}^{s} B^{j_1 \dots j_{t-1} h j_{t+1} \dots j_s}_{i_1 \dots i_r} R^{j_t}_{hkl}, \tag{1.9}$$

where R is the curvature tensor of the connection ∇ satisfying the *Bianchi identity*

$$R^i_{jkl,m} + R^i_{jlm,k} + R^i_{jmk,l} = 0 \quad (i, j, k, l, m = 1, \dots, n).$$

1.4. Let M^n and N^k be differentiable manifolds, and let $\varphi : M^n \to N^k$ be a differentiable mapping. For each point $p \in M^n$, the mapping φ induces the linear mapping φ_{*p} (or $(d\varphi)_p$) of the tangent space T_pM^n into the tangent space $T_{\varphi(p)}N^k$ by the formula

$$(\varphi_{*p}X_p)f \circ \varphi = X_p(f \circ \varphi) \quad (X_p \in T_pM^n,\ f \in \mathcal{F}N^k).$$

If (x, U) is a chart around a point p and (y, V), where $V \supset f(U)$, is a chart around the image $\varphi(p) \in N^k$ of the point p, then the matrix of the linear mapping $\varphi_{*p} : T_pM^n \to T_{\varphi(p)}N^k$ is the Jacobi matrix $(\frac{\partial y^\alpha}{\partial x^i})$, $\alpha = 1, \dots, k$, $i = 1, \dots, n$, at the point p in terms of the coordinate basis $(\frac{\partial}{\partial x^i} \mid_p)$ in T_pM^n and the coordinate basis $(\frac{\partial}{\partial y^\alpha} \mid_{\varphi(p)})$ in $T_{\varphi(p)}N^k$.

Thus, for any vector field X on M^n, the vector field $\varphi_{*p}X$ on N^k is defined and has the property

$$X(f \circ \varphi) = (\varphi_{*p}X)f \circ \varphi \quad (f \in \mathcal{F}N^k).$$

The vector fields X and φ_*X are said to be *φ-connected*, and φ_* is called the *differential of the mapping* φ.

The identity mapping $\mathrm{id} : M^n \to M^n$ induces the identity mapping $\mathrm{id}_{*p} : T_pM^n \to T_pM^n :\ X = X$ for all $p \in M^n$.

Manifolds M^n and N^k are said to be *diffeomorphic* if there exists a *diffeomorphism* of one manifold onto another, i.e., a bijection $\varphi :\ M^n \to N^k$ which is differentiable, together with its inverse φ^{-1}. This implies that $\varphi_{*p} :\ T_pM^n \to T_{\varphi(p)}N^k$ is a bijection (linear isomorphism) and $(\varphi_{*p})^{-1} = (\varphi^{-1})_{*\varphi(p)}$, i.e., $\dim T_pM^n = \dim T_{\varphi(p)}N^k$, and hence $n = k$.

1.5. Let M be a differentiable manifold. A *pseudo-Riemannian metric* on M is a differentiable tensor field g of nondegenerate (i.e., for each point $p \in M$, the relation $g(X, Y) = 0$ for all vectors $Y \in T_pM$ implies $X = 0$) symmetric bilinear forms $g(p)$ on the tangent spaces T_pM of the manifold M; g is called the *metric* or *fundamental tensor* on M.

In a local chart (x, U), we have

$$g \mid_U = g_{ij}dx^i \otimes dx^j \equiv g_{ij}dx^i dx^j, \tag{1.10}$$

where the components $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ji}$ are functions of the local coordinates $x^1, \dots, x^n$ and $\det(g_{ij}) \neq 0$.

In the tangent space T_pM of every point $p \in M$, the tensor $g(p)$ defines the *pseudo-Euclidean metric*, which is the inner product

$$(X, Y) \to g(X, Y) \equiv \langle X, Y \rangle = \langle Y, X \rangle \quad (X, Y \in T_pM),$$

and defines the lengths of vectors and angles between them as well as operations of raising and lowering of indices. A metric g is called a *Riemannian metric* if each form $g(p)$ is positive definite. If all the forms $g(p)$ have the same signature $(s, n-s)$ for all $p \in M$, then we say that $M \equiv M^n_{(s)}$ and $g \equiv \overset{(n)}{g}_{(s)}$ both have the *signature* $(s, n-s)$. If $s = 1$ or $n - s = 1$, then g is called a *Lorentz metric* and M is called a *Lorentz manifold*. We also say that they have Lorentz signature. In the case s = 0, the metric g and the manifold M are Riemannian.

The non-degeneracy of a metric tensor g implies that for each 1-form ω on M, there exists one and only one vector field $Y \in \mathcal{X}M$ such that $\omega(X) = \langle Y, X \rangle$ for all $X \in \mathcal{X}M$. This field is said to be *dual* to the form ω with respect to g.

In terms of local coordinates, a metric g is written as the quadratic form

$$ds^2 = g_{ij}dx^i dx^j,$$

which is called the *fundamental* (*quadratic*) *form* in M. This notation shows that along a differentiable curve, g is equal to the square of the differential of the arc length ds. A pair (M, g) (or simply M) is called a *pseudo-Riemannian manifold* or a *pseudo-Riemannian space*.

1.6. Let (M^n, g) and (N^k, a) be pseudo-Riemannian manifolds. A differentiable mapping $\varphi:\ M^n \to N^k$ is said to be *isometric* if it preserves the inner products, i.e., if $\langle X, Y\rangle_{M^n} = \langle \varphi_* X, \varphi_* Y\rangle_{N^k}$ or $g(X, Y) = a(\varphi_* X, \varphi_* Y)$ for each point $p \in M^n$ and all vectors $X, Y \in T_p M^n$. This implies that an isometric mapping has a maximal rank defined at every point by the rank of the Jacobi matrix of the mapping, and hence it is an immersion.

An isometric diffeomorphism is called an *isometry.*

If $M^n \subset N^k$ and the inclusion $i: M^n \to N^k$, $p \to p$, is an isometric immersion, then M^n is called a *pseudo-Riemannian submanifold* of N^k.

If (N^k, a) is a pseudo-Riemannian manifold and $\varphi:\ M^n \to N^k$ is an immersion, then on M^n, there exists the *induced pseudo-Riemannian metric* $g = \varphi^* a$ with the components

$$g_{ij} = (\varphi^* a)_{ij} = a_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} \quad (i, j = 1, \dots, n,\ \ \alpha, \beta = 1, \dots, k). \tag{1.11}$$

By the definition of the induced metric, the mapping $\varphi: (M^n, \varphi^* a) \to (N^k, a)$ is isometric. Conversely, if a mapping $\varphi: (M^n, g) \to (N^k, a)$ is isometric, then $g = \varphi^* a$.

Let M^n be an (immersed) submanifold of the pseudo-Riemannian manifold (N^k, a), and let φ be an immersion. The induced metric $g = \varphi^* a$ turns M^n into a pseudo-Riemannian submanifold of N^k. If the immersion φ is given by the equation

$$y^\alpha = \varphi^\alpha(x^1, \dots, x^n) \quad (\alpha = 1, \dots, k)$$

in local coordinates (x^i) in M^n and (y^α) in N^k, where the rank of the Jacobi matrix $(\frac{\partial y^\alpha}{\partial x^i})$ is equal to n, then the fundamental form in M^n is defined by

$$ds^2 = g_{ij}dx^i dx^j = a_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}dx^i dx^j = a_{\alpha\beta}dy^\alpha dy^\beta|_{y=\varphi(x)}. \tag{1.12}$$

If, in particular, $n = 2$ and M^2 is a surface in the three-dimensional Euclidean space with the linear element

$$ds^2 = dx^2 + dy^2 + dz^2,$$

which is given by the equation $\bar r = \bar r(u, v)$, where $\bar r = (x, y, z)$, then (1.12) is the first quadratic form of the surface:

$$ds^2 = d\bar r^2 = |\bar r_u|^2 du^2 + 2\langle \bar r_u, \bar r_v\rangle\, du\, dv + |\bar r_v|^2 dv^2 \equiv E\, du^2 + 2F\, du\, dv + G dv^2$$

where E, F, and G are the Gaussian coefficients of the surface and $\bar r_u = \frac{\partial \bar r}{\partial u}$ and $\bar r_v = \frac{\partial \bar r}{\partial v}$.

1.7. There is a unique linear connection ∇ in the bundle of linear frames on a pseudo-Riemannian manifold M satisfying the following conditions (see [304], Vol. I, p. 119):

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, \tag{1.13}$$

$$\nabla_X Y = \nabla_Y X + [X, Y] \tag{1.14}$$

for all differentiable vector fields X, Y, and Z on M. The first condition is equivalent to the requirement that the metric tensor be covariantly constant (or, which is the same, (absolute) parallel): $\nabla g = 0$ (*Ricci's lemma*); the second condition means that the torsion tensor vanishes. The connection ∇ defined by (1.13) and (1.14) is called the *Levi-Civita connection* or the *Riemannian connection*, and the parallel translation defined by it is called the *Levi-Civita parallelism.*

Let (x, U) be a chart on M, and let $X_i = \frac{\partial}{\partial x^i}$. If we take into account (1.6), then (1.3), (1.13), and (1.14) imply

$$\frac{1}{2}\big(X_i\langle X_j, X_k\rangle + X_j\langle X_i, X_k\rangle - X_k\langle X_i, X_j\rangle\big)$$
$$= \langle \nabla_{X_i} X_j, X_k\rangle = \Gamma^e_{ij}\langle X_e, X_k\rangle.$$

Substituting $\langle X_i, X_k \rangle = g(X_i, X_k) = g_{ik}$ and introducing the notation

$$\Gamma_{k,ij} = \Gamma^e_{ij} g_{ek}, \tag{1.15}$$

we obtain

$$\Gamma_{k,ij} = \Gamma_{k,ji} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \tag{1.16}$$

If we multiply both sides of (1.15) by g^{hk} and sum the result over k, then, using (1.16), we obtain

$$\Gamma^h_{ij} = \Gamma^h_{ji} = \frac{1}{2} g^{hk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \tag{1.17}$$

The coefficients $\Gamma_{ij,k}$ are called the *Christoffel symbols of the first kind*, and Γ^h_{ij} are called the *Christoffel symbols of the second kind* of the metric g.

The Levi-Civita connection ∇, which is determined in terms of local coordinates by the Christoffel symbols Γ^k_{ij}, is a unique linear connection in the bundle of linear frames over a pseudo-Riemannian manifold (M, g) such that the parallel translation of tangent vectors along any curve in M preserves the inner products.

The curvature tensor R^i_{jkl} of a Levi-Civita connection is called the Riemannian curvature tensor. Contracting the curvature tensor, we obtain the symmetric *Ricci tensor*

$$R_{jl} = R^i_{jil}. \tag{1.18}$$

Contracting it with g^{jl}, we obtain the *scalar curvature* of a pseudo-Riemannian manifold M:

$$\mathcal{R} = g^{jl} R_{jl}. \tag{1.19}$$

1.8. For each plane E in the tangent space T_pM of an n-dimensional pseudo-Riemannian manifold M, the *sectional curvature* $K(p, E)$ at the point $p \in M$ is defined by the formula

$$K(p, E) = \frac{g(R(X,Y)Y, X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{1.20}$$

where R is the curvature tensor of the manifold M and (X, Y) is a basis in E.

If $K(p, E)$ is a constant K for all planes E in T_pM and all $p \in M$, then M is called a *space of constant curvature* K and is denoted by $\mathbb{S}^n(K)$ or $\mathbb{S}^n$. Relation (1.20) implies that (M, g) has a constant curvature K if and only if for all X, Y, $Z \in TM$

$$R(X,Y)Z = K\big(g(Z,Y)X - g(Z,X)Y\big)$$

or, in the local coordinates,

$$R^i_{jkl} = K(\delta^i_k g_{jl} - \delta^i_l g_{jk}) \quad (K = \frac{R}{n(n-1)}), \tag{1.21}$$

where δ^i_j is the Kronecker symbol. This holds iff there exist local coordinates x^i in a neighborhood of every point $p \in M$ in which the metric g is written as the Riemannian form

$$ds^2 = \sigma^{-2}(e_1 dx^{1^2} + \ldots + e_n dx^{n^2}) \quad \left(\sigma = 1 + \frac{K}{4}\sum_{i=1}^{n} e_i dx^{i^2}, \quad e_i \pm 1\right). \tag{1.22}$$

The last condition is equivalent to the requirement that each point $p \in M$ has a neighborhood isometric to an open subset of one of the following spaces: the pseudo-Euclidean space $\mathbb{R}^n_{(s)}$ if $K = 0$, the pseudo-Riemannian spherical space

$$\mathbb{S}^n_{(s)} = \left\{ x \in \mathbb{R}^{n+1}_{(s)} \mid b^{n+1}_{(s)}(x,x) \equiv -\sum_{i=1}^{s} x^{i^2} + \sum_{j=s+1}^{n+1} x^{j^2} = r^2 \right\}$$

if $K = 1/r^2 > 0$, and the pseudo-Riemannian hyperbolic space

$$\mathbb{H}^n_{(s)} = \left\{x \in \mathbb{R}^{n+1}_{(s+1)} \mid b^{n+1}_{(s+1)}(x,x) = -r^2\right\}$$

if $K = -1/r^2 < 0$ with the pseudo-Riemannian metrics induced by pseudo-Euclidean metrics in $\mathbb{R}^{n+1}_{(\cdot)}$ on the corresponding quadrics. It follows that $\mathbb{S}^n_{(0)} = \mathbb{S}^n$ is the ordinary sphere and $\mathbb{H}^n_{(0)} = \mathbb{H}^n$ is the hyperbolic space, if $s = 0$ [407].

1.9. Let M be a pseudo-Riemannian manifold with metric g. Suppose we are given a partition

$$I = \bigcup_{s=1}^{\tau} I_s, \ \ I_s = \{h \mid n_s + 1 \le h \le n_s + m_s\}, \tag{1.23}$$

$$n_1 = 0, \quad n_t = \sum_{l=1}^{t-1} m_l, \quad m_1 + \dots + m_\tau = n, \quad (t = 2, \dots, \tau)$$

of the set of indices $I = \{1, \dots, n\}$ and the bijection

$$I_s \ni h \to \tilde{h} = 2n_s + m_s + 1 - h \in I_s \quad (s = 1, \dots, \tau) \tag{1.24}$$

of this set onto itself. A *skew-normal* (*moving*) *frame* or, briefly, a *skew-frame* over an open set $V \subseteq M$ is an ordered set $Y = (Y_1, \dots, Y_n)$ composed of τ ordered sets $(Y_t \mid t \in I_s)$, $s = 1, \dots, \tau$, of n_t real- or complex-valued vector fields on V satisfying the conditions

$$g(Y_h, Y_l) = e_h \delta^l_{\tilde{h}} \quad (h, l = 1, \dots, n). \tag{1.25}$$

With each collection $\{Y_h \mid h \in I_s\}$ of n_s complex vector fields, one associates the set $\{Y_l \mid l \in I_t\}$ of $n_t = n_s$ complex-conjugate vector fields $Y_{n_t+\sigma} = \overset{*}{Y}_{n_s+\sigma}$, where $\sigma = 1, \dots, n_t$, $e_t = e_s = 1$, and $t = s + 1$ or $t = s - 1$ ([31, 52]).

If, in particular, $\tau = n$ and all Y_i, $i \in I$, are real, then the skew-normal frame is an orthonormal frame over V, i.e., a moving frame orthonormal at every point $P \in V$. Obviously, each skew-frame can be transformed into an orthonormal frame by a nonsingular (complex) linear transformation and vice versa. Since an orthonormal moving frame exists in a neighborhood of each point $p \in M$ ([407], 1st ed., p. 50), then a skew-normal moving frame also exists in a neighborhood of each point $p \in M$.

Let $U \subset V$ be a coordinate neighborhood, (X_i) be a natural frame, and $g_{ij} = g(X_i, X_j)$. Let the 1-form θ_h be dual to a vector field Y_h with respect to $g\,|_V$:

$$\langle \theta_h, Y_l \rangle \equiv \theta_h(Y_l) = g(Y_h, Y_l) \quad (l = 1, \dots, n).$$

Solving the equation $g_{ij} \underset{h}{\xi}{}^i = \underset{h}{\xi}{}_j$ with respect to g_{ij}, we obtain

$$g_{ij} = \sum_{h=1}^{n} e_h \underset{h}{\xi}{}_i \underset{\tilde{h}}{\xi}{}_j \quad \text{or} \quad g\,|_U = \sum_{h=1}^{n} e_h \theta_h \otimes \theta_{\tilde{h}}. \tag{1.26}$$

From these relations, we obtain a formula for passing from the skew-normal frame $(Y_1, \dots, Y_n)$ over U to the coordinate frame $(X_1, \dots, X_n)$:

$$X_j = \sum_{h=1}^{n} e_h \underset{\tilde{h}}{\xi}{}_j Y_h, \tag{1.27}$$

and also a formula for the contravariant components of the metric tensor:

$$\sum_{h=1}^{n} e_h \underset{h}{\xi}{}^i \underset{\tilde{h}}{\xi}{}^j = g^{ij}, \quad \text{or} \quad g_c\,|_U = \sum_{h=1}^{n} e_h Y_h \otimes Y_{\tilde{h}}.$$

By (1.27), for any covariant tensor field $b \in \mathbb{T}_m V$, we have

$$b_{i_1 \dots i_m} = b(X_{i_1}, \dots , X_{i_m}) = \sum_{h_1, \dots , h_m = 1}^{n} e_{h_1} \dots e_{h_m} b(Y_{h_1}, \dots , Y_{h_m}) \underset{\tilde{h}_1}{\xi}{}^{i_1} \dots \underset{\tilde{h}_m}{\xi}{}^{i_m}.$$

Hence, introducing the notation $\bar{b}_{h_1 \dots h_m} = b(Y_{h_1}, \dots , Y_{h_m})$, we obtain

$$b = \sum_{h_1, \dots , h_m = 1}^{n} e_{h_1} \dots e_{h_m} \bar{b}_{h_1, \dots , h_m} \theta_{\tilde{h}_1} \otimes \dots \otimes \theta_{\tilde{h}_m}. \tag{1.28}$$

This formula and also (1.26) are satisfied in any chart (x, U), $U \subset V$ and, therefore, are is valid over the whole V.

We define the set of invariants γ_{lpk} in V by

$$\nabla_{Y_k} Y_l \equiv \sum_{p=1}^{n} e_p \gamma_{lpk} Y_{\tilde{p}} = \gamma^p_{lk} Y_p, \tag{1.29}$$

where $\gamma^p_{lk} = e_p \gamma_{l\tilde{p}k}$ are the connection components in the skew-frame Y:

$$\omega^i_j = \gamma^i_{jl} \theta^l = e_i \gamma_{j\tilde{i}l} \theta^l = \sum_h e_h e_i \gamma_{j\tilde{i}h} \theta_{\tilde{h}};$$

then, using (1.25), we obtain

$$\gamma_{lhk} = \langle \theta_h, \nabla_{Y_k} Y_l \rangle = \langle \nabla_{Y_k} \theta_l, Y_h \rangle \quad \text{and} \quad \gamma_{lhk} = \underset{l}{\xi}{}_{i,j} \underset{h}{\xi}{}^i \underset{k}{\xi}{}^j,$$

where $\underset{l}{\xi}{}_{i,j} = \nabla \theta_l (X_i, X_j)$. Differentiating (1.25), we find that

$$\langle \nabla_{Y_k} \theta_h, Y_l \rangle + \langle \theta_h, \nabla_{Y_k} Y_l \rangle = 0, \quad \text{i.e.,}\ \gamma_{hlk} + \gamma_{lhk} = 0,$$

in particular, $\gamma_{hhl} = 0$.

Differentiating (1.28), we obtain

$$\nabla b(Y_{p_1}, \dots , Y_{p_m}, Y_l) \equiv (\nabla_{Y_l} b)(Y_{p_1}, \dots , Y_{p_m}) = Y_l \bar{b}_{p_1 \dots p_m}$$
$$- \sum_{h=1}^{n} e_h (\gamma_{p_1 \tilde{h} l} \bar{b}_{h p_2 \dots p_m} + \dots + \gamma_{p_m \tilde{h} l} \bar{h}_{p_1 \dots p_{m-1} h}).$$

Using (1.29), we easily verify that

$$[Y_k, Y_h] \equiv \nabla_{Y_k} Y_h - \nabla_{Y_h} Y_k = \sum_{l=1}^{n} e_l (\gamma_{lkh} - \gamma_{lhk}) Y_{\tilde{l}},$$

where $[Y_k, Y_h]$ is the Lie bracket of the vector fields Y_k and Y_h:

$$[Y_k, Y_h] \mid_U = (\underset{k}{\xi}{}^i \partial_i \underset{h}{\xi}{}^j - \underset{h}{\xi}{}^i \partial_i \underset{k}{\xi}{}^j) X_j.$$

Let R be the curvature tensor of the connection ∇. The quantities

$$\gamma_{lpqr} \equiv \langle \theta_l, R(Y_q, Y_r) Y_p \rangle$$

define the components

$$\Omega^i{}_j = \frac{1}{2} \sum_{p,q=1}^{n} e_i e_p e_q \times \gamma_{\tilde{i} jpq} \theta_{\tilde{p}} \wedge \theta_{\tilde{q}}$$

of the curvature form $\Omega = \Omega^i{}_j E_i{}^j$ of the connection ∇. Relations (1.7) and (1.29) imply the Cartan structure equations

$$d\theta^i = -\omega^i{}_j \wedge \theta^j, \quad d\omega^i{}_j = -\omega^i{}_l \wedge \omega^l{}_j + \Omega^i{}_j \tag{1.30}$$

or

$$d\theta = -\omega \wedge \theta, \quad d\omega = -\omega \wedge \omega + \Omega,$$

from which the first and second Bianchi identities can be deduced by using the exterior differentiation [110]:

$$\Omega^i{}_j \wedge \theta^j = 0 \quad \text{and} \quad d\Omega^i{}_j = \Omega^i{}_l \wedge \omega^l{}_j - \omega^i{}_l \wedge \Omega^l{}_j,$$

or, briefly,

$$\Omega \wedge \theta = 0 \quad \text{and} \quad d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

The next theorem generalizes the theorem on structure equations in the orthonormal frame ([407], 1st ed., pp. 50–51).

Theorem 1.1 (structure equations in a skew-normal frame, Aminova [52])**.** *Let M be a pseudo-Rie--mannian manifold with metric g and Levi-Civita connection ∇. Then in the neighborhood of each point $p \in M$, there is a skew-normal (moving) frame. Let $Y = (Y_1, \dots, Y_n)$ be a skew-normal moving frame over a connected open set $V \subseteq M$ defined by partitioning* (1.23) *of the set of indices $I = \{1, \dots, \tau\}$ and bijection* (1.24)*. Let $\omega^i{}_j$ and θ^i be components of the form of the connection ∇ and of the coframe dual to Y (the canonical form). If $g(Y_i, Y_{\tilde{i}}) = e_i$, then*

$$\omega_{ij} + \omega_{ji} = 0 \quad \textit{and} \quad d\theta_i = -\sum_j e_j \omega_{ij} \wedge \theta_{\tilde{j}}. \tag{1.31}$$

In addition,

$$\{\omega_{ij}\} = \{\gamma_{jik}\theta^k\} = \left\{\sum_k e_k \gamma_{jik} \theta_{\tilde{k}}\right\},$$

where the coefficients γ_{jik} are defined by formula (1.29)*, is a unique set of linear differential forms on V that satisfies conditions* (1.31).

2. Projective Structure

2.1. Let M be an n-dimensional differentiable manifold, and let X be a differentiable vector field on M. A curve $\gamma : \mathbb{R} \supset (a, b) \to M : \ t \to x(t)$ is called an *integral curve* (or *trajectory*) of the vector field X if

$$\dot{x}(t) = X(x(t)) \quad (a < t < b). \tag{2.1}$$

In a chart (x, U) on M for which $X|_U = \xi^i(x)\partial_i$, condition (2.1) is written in the form of the set of n first-order differential equations

$$\frac{dx^i}{dt} = \xi^i(x^1(t), \dots, x^n(t)) \quad (i = 1, \dots, n). \tag{2.2}$$

That is why (2.1) is called a *differential equation on the manifold M*. Any vector field X is called an (*autonomous*) *dynamic system* on M, and (2.2) is called its *local notation.*

An integral curve $x(t)$ of a vector field X is called *maximal* if it is not the restriction of an integral curve of this field defined on a larger interval $I \supset (a, b)$. For any vector field X on M and a point $p \in M$, there exists a unique maximal integral curve passing through the point p.

A smooth curve $\gamma = x_t$, $a < t < b$, where $-\infty \le a < b \le \infty$, in a manifold M with linear connection ∇ is called a *geodesic* if the field X of vectors $\dot{x}_t$ tangent to γ at x_t is parallel along γ, i.e., if $\nabla_X X = 0$ for all t, where t is an affine parameter for γ and ∇_X denotes the covariant derivative along X. A geodesic is said to be maximal if it is not an interval of a larger geodesic, and it is said to be complete if the domain of its definition is $-\infty < t < \infty$.

Let $x^i = x^i(t)$ be the equation of a curve $\gamma = x_t$ of class C^2 in M, and let Γ^i_{jk} be the components of a connection ∇ in local coordinates x^i. Then γ is a geodesic if and only if

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \tag{2.3}$$

provided t is an affine parameter ([90], Vol. I, p. 138, [304], Vol. 1, p. 138).

2.2. Two n-dimensional pseudo-Riemannian manifolds (M, g) and (M', g') are called *manifolds with corresponding* (or *common*) *geodesics* if there is a diffeomorphism f from M into M', which is called a *geodesic mapping* or *projective mapping*, such that f maps every geodesic γ in M into a geodesic $f(\gamma)$ in M' and if the preimage $f^{-1}(\gamma')$ of every geodesic γ' in M' in M' is a geodesic in M.

This holds iff in the corresponding local coordinate systems $x^i\ |_{p\in M} = x^i\ |_{f(p)\in M'}$, the following condition holds [191]:

$$\Gamma'^i_{jk} - \Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j \quad \text{(Weyl equation)}, \tag{2.4}$$

where $\Gamma^i_{jk} (\Gamma'^i_{jk})$ are the Christoffel symbols of the connection ∇ (∇') in the local coordinates x^i and $p = p_i dx^i$ is a differential 1-form.

To prove this, we write the geodesic equations in M and M' that are referred to an arbitrary parameter τ:

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \lambda(\tau)\dot{x}^i \quad \text{and} \quad \ddot{x}^i + \Gamma'^i_{jk}\dot{x}^j\dot{x}^k = \lambda'(\tau)\dot{x}^i \qquad \left(\dot{x}^i \equiv \frac{dx^i}{d\tau}\right);$$

hence, denoting

$$\Gamma'^i_{jk} - \Gamma^i_{jk} = p^i_{jk} \equiv p^i_{kj} \quad \text{and} \quad \lambda'(\tau) - \lambda(\tau) = \varphi(\tau),$$

we find from this that

$$p^i_{jk}\dot{x}^j\dot{x}^k = \varphi(\tau)\dot{x}^i \quad \text{and} \quad p^l_{jk}\dot{x}^j\dot{x}^k = \varphi(\tau)\dot{x}^l.$$

Excluding $\varphi(\tau)$, we obtain

$$(p^i_{jk}\delta^l_m - p^l_{jk}\delta^i_m)\dot{x}^j\dot{x}^k\dot{x}^m = 0.$$

Since these relations are identical and only one geodesic passes through each (non-singular) point in any direction, we obtain the Weyl equation (2.4):

$$p^i_{jk} = \Gamma'^i_{jk} - \Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j.$$

It is easy to verify that the condition (2.4) is sufficient.

2.3. If we introduce the *projective Thomas parameters* [388] in M and M':

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{2}{n+1}\delta^i_{(j}\Gamma^l_{k)l}, \tag{2.5}$$

then condition (2.4) becomes

$$\Pi'^i_{jk} = \Pi^i_{jk}. \tag{2.6}$$

This means that a projective mapping preserves the projective Thomas parameters, which implies

$$W'^i_{jkl} = W^i_{jkl},$$

where W^i_{jkl} is the *Weyl projective curvature tensor*:

$$W^i_{jkl} = \Pi^i_{jkl} - \frac{1}{n-1}(\delta^i_k\Pi_{jl} - \delta^i_l\Pi_{jk}), \tag{2.7}$$

$$\Pi^i_{jkl} = \partial_k\Pi^i_{jl} - \partial_l\Pi^i_{jk} + \Pi^h_{jl}\Pi^i_{hk} - \Pi^h_{jk}\Pi^i_{hl}, \qquad \Pi_{jk} = \Pi^h_{jhk}.$$

The projective Thomas parameters transform under a change of coordinates by

$$\Pi^{i'}_{j'k'} = \Pi^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^l}{\partial x^{j'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^l} - \frac{1}{n+1}\left(\delta^{i'}_{j'}\frac{\partial \ln|\Delta|}{\partial x^{k'}} + \delta^{i'}_{k'}\frac{\partial \ln|\Delta}{|}\partial x^{j'}\right),$$

where $\Delta = \det\left(\frac{\partial x^{i'}}{\partial x^i}\right)$, hence they define a geometric object called an *object of projective connection.* [1]

[1]See [161], Vol. I, and also [357], p. 67, for the definition of a geometric object. In what follows, it is sufficient to consider a geometric object as a tensor field or an object of affine (Γ^i_{jk}) or projective (Π^i_{jk}) connection.

If M and M' are pseudo-Riemannian manifolds with metrics g and g', then contracting the Weyl equation over indices i and k and using the *Foss–Weyl formula*

$$\Gamma^i_{il} = \frac{1}{2}\partial_l \ln|\det(g_{ij})|, \tag{2.8}$$

we obtain

$$p = d\psi = d\frac{1}{2(n+1)} \ln \frac{|\det(g_{ij})|}{|\det(g'_{ij})|}.$$

2.4. The affine connection space A^n of dimension n is said to be *projectively flat* or *projectively Euclidean* if its geodesic lines can be mapped into straight lines of the flat space $\mathbb{R}^n$, and hence, in appropriate coordinates, they are expressed by $n-1$ linear equations with constant coefficients or n linear parameter equations. In these coordinates, which are called *projective coordinates*, the connection coefficients Γ^i_{jk} of the flat space vanish, and from the Weyl equation (2.4) and the formulas (2.5) and (2.6), we have

$$\Gamma'^i_{jk} = \delta^i_j p_k + \delta^i_k p_j, \tag{2.9}$$

$$\Pi'^i_{jk} = 0$$

for the coefficients Γ'^i_{jk} and Π'^i_{jk} of affine and projective connections, respectively, in projectively flat space A^n. These connections are called *projectively flat connections*.

From this, it is easy to deduce that *for $n > 2$, the space A^n is projectively flat if and only if its Weyl projective curvature tensor vanishes. The two-dimensional space A^2 is projectively flat if and only if the tensor*

$$p_{ij} = -p_{j,i} + p_i p_j$$

satisfies the equation

$$p_{ij,k} - p_{kj,i} = 0,$$

where the vector p_i is defined by (2.9) ([406]; see also [161], Vol. II).

From formulas (1.8), (1.18), (2.5), and (2.7), it follows that the Weyl projective curvature tensor of a pseudo-Riemannian manifold (M^n, g) has the form

$$W^i_{jkl} = R^i_{jkl} - \frac{1}{n-1}(\delta^i_k R_{jl} - \delta^i_l R_{jk}).$$

From the relation $W_{ijkl} = 0$, we obtain

$$R_{ijkl} = \frac{1}{n-1}(g_{ik} R_{jl} - g_{il} R_{jk}),$$

and hence we find for $n > 2$,

$$R_{ij} = \rho g_{ij},$$

where, according to Schür's theorem, $\rho = \text{const}$ [191].

By Schür's terminology, a two-dimensional projectively flat (pseudo-)Riemannian space is called a *projective surface*. Integrating Eq. (2.9) (see, e.g., [101], Part II, p. 73), it is easy to prove that any projective surface has a constant curvature.

Consequently, a *projectively flat pseudo-Riemannian space is a space of constant curvature* [191].

2.5. The projective Thomas parameters (2.5) give a projective connection locally. To define a projective connection globally, it is necessary to consider a set of 2-jets and to introduce the notion of a projective structure (see [303] and also [30, 52]).

Let f be a mapping from a neighborhood of the origin O in $\mathbb{R}^n$ into a differentiable manifold M^n. If we "glue" all mappings f that have the same partial derivatives of any order $l \leq k$ at O, then we obtain the *k-jet* $J_O^k(f)$ at O. If f is a diffeomorphism from a neighborhood of the origin O onto an open subset in M, then the k-jet $J_O^k(f)$ at O is called the *k-th order frame* or *k-frame* at the point $x = f(O)$.

If h is a diffeomorphism of a neighborhood of the point O in $\mathbb{R}^n$ onto a neighborhood of the point O in $\mathbb{R}^n$, then the set of k-frames $J_O^k(h)$ at the point $O \in \mathbb{R}^n$ forms a group $G^k(n)$ with multiplication $J_O^k(h) \cdot J_O^k(h') = J_O^k(h \cdot h')$.

The set $P^k(M)$ of k-frames of the manifold M is a principal bundle over M with natural projection $\pi : \pi(J_O^k(f)) = f(O)$ and structure group $G^k(n)$ acting on $P^k(M)$ to the right. A first-order frame is an ordinary linear frame; therefore, $G^1(n) = \mathrm{GL}(n, \mathbb{R})$, and $P^1(M)$ is the bundle LM of linear frames over M.

Each 2-frame u in $\mathbb{R}^n$ can be uniquely written as a polynomial $f(x) = (u^i + u^i_j x^j + (1/2)u^i_{jk}x^jx^k)e_i$, where $x = x^ie_i \in \mathbb{R}^n$ and $u^i_{jk} = u^i_{kj}$. The set (u^i, u^i_j, u^i_{jk}) determines a natural coordinate system in $P^2(\mathbb{R}^n)$, and its restriction $(u^i_j, u^i_{jk}) \equiv (s^i_j, s^i_{jk})$ to $G^2(n)$ is a natural coordinate system in $G^2(n)$. The natural coordinate systems (u^i, u^i_j) in $P^1(\mathbf{R}^n)$ and (s^i_j) in $G^1(n)$ are introduced similarly.

A *projective structure on* M is a principal subbundle Π of the bundle $P^2(M)$ with the structure group $L_0 \subset G^2(n)$, where L_0 is the group of matrices in $\mathrm{SL}(n+1, \mathbb{R})$ of the form $\{(A, 0), (\xi, a)\}$, where $A \in \mathrm{GL}(n, \mathbb{R})$ and ξ is an n-dimensional column vector factorized by its center.

Considering $G^1(n) = \mathrm{GL}(n, \mathbb{R})$ and L_0 as subgroups in $G^2(n) : G^1(n) \subset L_0 \subset G^2(n)$, we see that there is a bijection between the cross sections $M \to P^2(M)/G^1(n)$ and the torsion free affine (i.e., linear) connections on the manifold M, and there is a bijection between the cross-sections $M \to P^2(M)/L_0$ and the projective structures on M.

Every affine torsion-free connection $\nabla : M \to P^2(M)/G^1(n)$ composed with the natural mapping $P^2(M)/G^1(n) \to P^2(M)/L_0$ gives the projective structure $\Pi : \to P^2(M)/L_0$. We say that a torsion-free connection ∇ belongs to a projective structure Π if it induces Π in the way described above.

Two affine torsion-free connections ∇ and ∇' on M inducing the same structure Π are said to be *projectively equivalent*.

Since the kernel of the homomorphism $L_0 \to G^1(n)$ consists of elements (s^i_{jk}) of the form $s^i_{jk} = \delta^i_j p_k + \delta^i_k p_j$, two affine torsion-free connections ∇ and ∇' defined on the bundle $P^1(M)$ of linear frames over M induce the same cross section $M \to P^2(M)/L_0$ if and only if the Weyl equation is satisfied ([303], p. 191):

$$\Gamma'^i_{jk} - \Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j;$$

as was shown above, this is equivalent to the condition

$$\Pi'^i_{jk} = \Pi^i_{jk},$$

where Γ'^i_{jk} and Γ^i_{jk} are the components of the connections ∇' and ∇ in the local coordinate system $x^1, \dots, x^n$ and Π'^i_{jk} and Π^i_{jk} are the corresponding projective Thomas parameters.

Consequently, two connections ∇ and ∇' on M belong to the same projective structure if and only if their projective parameters coincide [89]. In view of this, the object of the projective connection defined by the projective Thomas parameters defines a projective structure on M.

3. Infinitesimal Transformations

3.1. Infinitesimal isometries.

3.1.1. Let M be an arbitrary set. A bijection $\varphi : M \to M$ is called a *transformation* of the set M. The mapping id_M, or $\mathrm{id} : x \to x$, $x \in M$, is called the *identity transformation* of M. For any two transformations ψ and φ of the set M, the transformation $\varphi \circ \psi : x \to \varphi \circ \psi(x) = \varphi(\psi(x))$, $x \in M$, is defined and is called their *composition*. The inverse bijection $\varphi^{-1} :\ \varphi^{-1} \circ \varphi = \varphi \circ \varphi^{-1} = \mathrm{id}_M$ is called the *transformation* inverse to φ.

By virtue of the associativity of the composition, every non-empty set Γ of transformations of M such that $\varphi, \psi \in \Gamma$ implies that $\varphi \circ \psi \in \Gamma$ and $\varphi \in \Gamma$ implies that $\varphi^{-1} \in \Gamma$ is a group with the multiplication $(\varphi, \psi) \to \varphi\psi \equiv \varphi \circ \psi$, which is called the *transformation group* of the set M. The identity element e of this group is id_M and the element inverse to φ is the inverse transformation φ^{-1}.

A transformation group Γ of the set M is said to be *transitive* if for any $x, y \in M$, there exists a transformation $\varphi \in \Gamma$ such that $\varphi(x) = y$.

Let M be a differentiable manifold of dimension n.

A 1-*parameter group of* (*differentiable*) *transformations on* M is a mapping $\mathbb{R} \times M \ni (t, p) \to \varphi_t(p) \in M$ such that $\varphi_t :\ p \to \varphi_t(p)$ is a transformation on M for any $t \in \mathbb{R}$ and $\varphi_{t_1+t_2}(p) = \varphi_{t_1}\varphi_{t_2}(p) \equiv \varphi_{t_1}(\varphi_{t_2}(p))$ for all $t_1, t_2 \in \mathbb{R}$.

A mapping $I_\epsilon \times U \ni (t, p) \to \varphi_t(p) \in \varphi_t(U)$ such that $\varphi_t : p \to \varphi_t(p)$ is a diffeomorphism of U onto the open set $\varphi_t(U)$ in M, $\varphi_{t_1+t_2}(p) = \varphi_{t_1}\varphi_{t_2}(p)$ if $t_1, t_2,\ t_1 + t_2 \in I_\epsilon$, and $\varphi_{t_2}(p) \in U$ is called a *local* 1-*parameter group of local transformations* defined on $I_\epsilon \times U$, where U is an open set in M and $I_\epsilon = (-\epsilon, \epsilon)$, $\epsilon > 0$, is an open interval.

3.1.2. Let X be a vector field on M, (x, U) be a chart in M, and $X\,|_U = \xi^i(x)\partial_i$. We consider a set of n ordinary differential equations

$$\frac{dx^{i'}}{dt} = \xi^i(x') \quad (i = 1, \dots, n) \tag{3.1}$$

with an independent variable t and unknown functions $x^{i'}(t)$ satisfying the initial conditions $x^{i'}(0) = x^i$. We write a solution of this set (the restriction to U of an integral curve of the vector field X) in the form of the Taylor series

$$x^{i'}(t) = x^i + \frac{dx^{i'}}{dt}\Big|_{t=0} t + \frac{1}{2!}\frac{d^2x^{i'}}{dt^2}\Big|_{t=0} t^2 + \dots .$$

Differentiating (3.1) with respect to t, we find that

$$\frac{d^2x^{i'}}{dt^2}\Big|_{t=0} = \frac{\partial \xi^i}{\partial x^{j'}}\frac{dx^{j'}}{dt}\Big|_{t=0} = \xi^j\partial_j\xi^i(x).$$

The other derivatives are computed in the same way. As a result, we have

$$x^{i'}(t) = x^i + t\xi^i(x) + \frac{t^2}{2!}\xi^j\partial_j\xi^i(x) + \dots = \left(1 + t\xi^j\partial_j + \frac{t^2}{2!}(\xi^j\partial_j)^2 + \dots\right)x^i, \tag{3.2}$$

or, symbolically,

$$x^{i'} = e^{tX}x^i.$$

This solution defines the transformation $\varphi_t : (x^i) \to (x^{i'}(t))$. Since we obviously have $\varphi_{t_1}\varphi_{t_2} = \varphi_{t_1+t_2}$ and ${\varphi_t}^{-1} = \varphi_{-t}$, the transformations φ_t form a (local) 1-parameter group or (*local*) *flow* $F_X(t)$ *of the vector field* X. We say that X *generates* a (*local*) 1-*parameter group* $F_X(t)$ *of* (*local*) *transformations* in a neighborhood of the point (x^i).

If there exists a global 1-parameter group of transformations φ_t such that for all $p \in M$, the *orbit* $x_t = \varphi_t(p)$ *of the point* p is an integral curve of the vector field X, then X is called a *complete vector field.*

Neglecting in higher powers of t in (3.2), we obtain the *infinitesimal transformation*

$$x^i \to x^{i'} = x^i + \xi^i \delta t,$$

where δt is written instead of t. We say that it *generates a group* $F_X(t)$, and the symbol $X = \xi^i \partial_i$ is called the *generator* or *operator* of this group or *infinitesimal transformations* on M. In the comprehensive literature meaning, the distinctions between the notions of infinitesimal transformation and the vector field were erased; therefore the above terms can be considered as equivalent.

3.1.3. Let $\mathcal{T}M$ be an algebra of tensor fields on M, and let $\Omega \in \mathcal{T}M$. Let $F_X(t)$ be the flow of a vector field X in a neighborhood of a point p. The differential φ_{t^*} of a transformation $\varphi_t \in F_X(t)$ maps each tangent vector $Y \in T_{\varphi_t^{-1}(p)}M$ into the tangent vector $\varphi_{t^*}Y \in T_pM$, which is defined as follows. If Y is tangent to a curve γ at the point $\varphi_t^{-1}(p)$, then $\varphi_{t^*}Y$ is tangent to the curve $\varphi_t(\gamma)$ at the point $\varphi_t(\varphi_t^{-1}(p)) = p$. Let x^i be the coordinates of the point p and let y^i be the coordinates of the point $\varphi_t^{-1}(p)$. If $Y = \eta^i \frac{\partial}{\partial y^i}$ and $\varphi_{t^*}Y = \xi^i \frac{\partial}{\partial x^i}$, then $\xi^i = \eta^l \frac{\partial x^i}{\partial y^l}\big|_{\varphi_t^{-1}(p)}$.

The mapping φ_{t^*} defines a linear isomorphism of the tangent spaces $T_{\varphi_t^{-1}(p)}M$ and T_pM. This isomorphism can be extended to an isomorphism $\tilde{\varphi}_t$ of the tensor algebra $\mathcal{T}(\varphi_t^{-1}(p))$ at the point $\varphi_t^{-1}(p)$ onto the tensor algebra $\mathcal{T}(p)$ at the point p which preserves the tensor type and commutes with contractions.

The tensor field $\tilde{\Omega}(t) \equiv \tilde{\varphi}_t\Omega$ defined by

$$(\tilde{\varphi}_t\Omega)_p = \tilde{\varphi}(\Omega_{\varphi_t^{-1}(p)})$$

is called the *dragged tensor field*, and the mapping L_X of the tensor algebra $\mathcal{T}M$ onto itself transforming Ω into $L_X\Omega$ by

$$(L_X\Omega)_p = \lim_{t\to 0} \frac{1}{t}(\Omega_p - \tilde{\Omega}_p(t)) = \lim_{t\to 0} \frac{1}{t}[\Omega_p - (\tilde{\varphi}_t\Omega)_p] \tag{3.3}$$

is called the *Lie differentiation* along (respectively, with respect to) X. $L_X\Omega$ is called the *Lie derivative* of a tensor field Ω, and $L_X\Omega\delta t$ is called the *Lie differential* of the field Ω with respect to the infinitesimal transformation $x^{i'} = x^i + \xi^i\delta t$.

If a field Ω is dragged with $\xi^i\delta t$, then the Lie differential of this field at a point p taken with the opposite sign is equal up to terms linear in δt to a variation of Ω at this point, i.e., the *variation form* of Ω at the point p. Therefore, one may replace $L_X(\partial_i\Omega)$ by $\partial_i(L_X\Omega)$:

$$L_X(\partial_i\Omega) = \partial_i L_X\Omega. \tag{3.4}$$

We emphasize that one cannot consider $L_X\underbrace{(\partial_i\Omega)}$ as the Lie derivative of $\partial_i\Omega$, since $\partial_i\Omega$ is not a tensor field in general. Since we obviously have $\overbrace{\Phi \otimes \Omega} = \tilde{\Phi} \otimes \tilde{\Omega}$, the Leibnitz rule is valid: $L_X(\Phi \otimes \Omega) = (L_X\Phi) \otimes \Omega + \Phi \otimes L_X\Omega$.

3.1.4. The Lie differentiation preserves the type of tensor field and commutes with contractions, since the isomorphism $\tilde{\varphi}_t$ preserves the type and commutes with contractions. If f is a function on M, then by $\varphi_t^{-1} = \varphi_{-t}$ we have

$$(L_Xf)(p) = (Xf)(p) = (\xi^l\partial_l f)(p).$$

If η_i are coordinates of a covector, then applying the Taylor formula and considering the first-order terms in t, we find that

$$L_X\eta_i = \xi^l\partial_l\eta_i + \eta_l\partial_i\xi^l. \tag{3.5}$$

Similarly, we obtain

$$L_X\eta^i = \xi^l\partial_l\eta^i - \eta^l\partial_l\xi^i, \tag{3.6}$$

and from (1.5), we see that

$$L_XY = [X, Y],$$

where $Y = \eta^i \partial_i$; hence $L_X Y$ is the Lie bracket of the vector fields X and Y.

The Lie derivative for tensors of higher valency is computed for every index individually by (3.5) or (3.6) depending on the nature of the index. In particular, for the curvature tensor, we have

$$L_X R^i_{jkl} = \xi^h \partial_h R^i_{jkl} - R^h_{jkl} \partial_h \xi^i + R^i_{hkl} \partial_j \xi^h + R^i_{jhl} \partial_k \xi^h + R^i_{jkh} \partial_l \xi^h$$
$$\equiv \xi^h R^i_{jkl,h} - R^h_{jkl} \xi^i_{,h} + R^i_{hkl} \xi^h_{,j} + R^i_{jhl} \xi^h_{,k} + R^i_{jkh} \xi^h_{,l}$$

(the comma denotes the covariant differentiation), and the Lie derivative of the Kronecker symbol δ^i_j is zero:

$$L_X \delta^i_j = \xi^l \partial_l \delta^i_j - \delta^h_j \partial_h \xi^i + \delta^i_h \partial_j \xi^h = 0.$$

The notions of the dragged field and the Lie derivative for a geometric object are defined similarly.

3.1.5. Let (M, g) be a pseudo-Riemannian manifold, and let $X = \xi^i \partial_i$ be a vector field on M. We denote by δt an infinitesimal constant and consider an infinitesimal transformation:

$$g_{ij} dx^i dx^j + \delta t X(g_{ij} dx^i dx^j) \equiv g_{ij} dx^i dx^j + \delta t (L_X g_{ij}) dx^i dx^j.$$

Computing

$$X(g_{ij} dx^i dx^j) = (X g_{ij}) dx^i dx^j + g_{ij} d(X x^i) dx^j + g_{ij} dx^i d(X x^j),$$

we find that

$$L_X g_{ij} = \xi^l \partial_l g_{ij} + g_{il} \partial_j \xi^l + g_{lj} \partial_i \xi^l \equiv \xi_{i,j} + \xi_{j,i}, \tag{3.7}$$

where the comma denotes the covariant differentiation with respect to g (i.e., with respect to the Levi-Civita connection of g).

This is the shortest derivation of the formula for the Lie derivative of a metric tensor, which can also be obtained by using (3.5) and (3.6) or immediately from the definition (3.3).

We find the Lie derivative of the Christoffel symbols. Differentiating the equation $g_{lk} g^{li} = \delta^i_k$ and taking into account that the covariant derivative and the Lie derivative of the Kronecker symbol δ^i_k are zero, we obtain

$$(\partial_h g_{lk}) g^{li} + g_{lk} \partial_h g^{li} = 0 \quad \text{and} \quad (L_X g_{lk}) g^{li} + g_{lk} L_X g^{li} = 0.$$

Contracting this with g^{kj} and using (3.7), we have

$$L_X g^{ij} = \xi^l \partial_l g^{ij} - g^{lj} \partial_l \xi^i - g^{il} \partial_l \xi^j. \tag{3.8}$$

Then (1.16), (1.17), and (3.4) imply the formulas

$$2 L_X \Gamma_{l,jk} = \partial_k L_X g_{jl} + \partial_j L_X g_{kl} - \partial_l L_X g_{jk},$$
$$L_X \Gamma^i_{jk} = (L_X g^{il}) \Gamma_{l,jk} + g^{il} L_X \Gamma_{l,jk}.$$

From here, using (3.7) and (3.8), we find that

$$L_X \Gamma^i_{jk} = \xi^l \partial_l \Gamma^i_{jk} - \Gamma^l_{jk} \partial_l \xi^i + \Gamma^i_{lk} \partial_j \xi^l + \Gamma^i_{jl} \partial_k \xi^l + \partial_{jk} \xi^i = \xi^i_{,jk} + \xi^l R^i_{jlk}, \tag{3.9}$$

where R is the curvature tensor of the connection $\nabla(\Gamma^i_{jk})$. We note that the Lie derivative $L_X \Gamma^i_{jk}$ is a tensor unlike the symbols Γ^i_{jk}.

Differentiating (1.8) and taking into account (3.4), we obtain

$$L_X R^i_{jkl} = (L_X \Gamma^i_{jl})_{,k} - (L_X \Gamma^i_{jk})_{,l}. \tag{3.10}$$

3.1.6. If M is a manifold equipped with a linear connection ∇, then (3.5) and (3.6) can be rewritten in terms of covariant derivatives:

$$L_X\eta_i = \xi^l\partial_l\eta_i + \eta_l\partial_i\xi^l = \xi^l\eta_{i,l} + \eta_l\xi^l_{,i},$$

$$L_X\eta^i = \xi^l\partial_l\eta^i - \eta^l\partial_l\xi^i = \xi^l\eta^i_{,l} - \eta^l\xi^i_{,l},$$

It is seen from this that the Lie differentiation preserves the type of a tensor field.

We write the covariant derivative

$$u_{i,j} = \partial_j u_i - \Gamma^l_{ij}u_l$$

and find the Lie derivative:

$$L_X u_{i,j} = \partial_j(L_X u_i) - \Gamma^l_{ij}L_X u_l - (L_X\Gamma^l_{ij})u_l = (L_X u_i)_{,j} - (L_X\Gamma^l_{ij})u_l.$$

In the same way, we obtain

$$L_X u^i_{,j} - (L_X u^i)_{,j} = (L_X\Gamma^i_{jl})u^l.$$

The following commutation rule of the Lie derivative and the covariant derivative of an arbitrary tensor field B is obtained similarly:

$$L_X B^{jk}_{l\ldots,i} - (L_X B^{jk}_{l\ldots})_{,i} = (L_X\Gamma^j_{ih})B^{hk}_{l\ldots} + (L_X\Gamma^k_{ih})B^{jh}_{l\ldots} - (L_X\Gamma^h_{il})B^{jk}_{h\ldots} - \ldots.$$

Applying it to the metric tensor, we find that

$$L_X g_{jk,i} - (L_X g_{jk})_{,i} \equiv -(L_X g_{jk})_{,i} = -(L_X\Gamma^h_{ij})g_{hk} - (L_X\Gamma^h_{ik})g_{jh},$$

which implies

$$L_X\Gamma^i_{jk} = \frac{1}{2}g^{ih}[(L_X g_{kh})_{,j} + (L_X g_{jh})_{,k} - (L_X g_{jk})_{,h}]. \tag{3.11}$$

The next formulas follow from the definitions of the Lie derivative and the exterior differentiation:

$$L_X(\omega_1+\omega_2) = L_X\omega_1 + L_X\omega_2, \quad L_{X+Y}\omega = L_X\omega + L_Y\omega, \quad L_X(\omega_1\wedge\omega_2) = (L_X\omega_1)\wedge\omega^2 + \omega^1\wedge L_X\omega_2,$$

$$L_X d\omega = dL_X\omega, \quad L_X\omega = i_X d\omega + di_X\omega, \quad L_{fX}\omega = fL_X\omega + df\wedge i_X\omega, \quad L_X i_Y\omega = i_{[X,Y]}\omega + i_Y L_X\omega,$$

where ω, ω_1 and ω_2 are differential forms, f is a function, X and Y are vector fields, and $i_X\omega = \mathcal{C}(X\otimes\omega)$ is the *inner product with respect to* X, i.e., the combination of a tensor product and a contraction $\mathcal{C}$ over the first indices, in particular, $i_X\omega = \omega(X)$ for any 1-form ω. For a function f, we set $i_X f = 0$.

For any vector fields X and Y on M, we have $L_{[X,Y]} = [L_X, L_Y]$.

Formulas (2.5) and (3.9) imply

$$L_X\Pi^i_{jk} = \xi^l\partial_l\Pi^i_{jk} - \Pi^l_{jk}\partial_l\xi^i + \Pi^i_{lk}\partial_j\xi^l + \Pi^i_{jl}\partial_k\xi^l + \partial_{jk}\xi^i - \frac{1}{n+1}\left(\delta^i_j\partial_{kl}\xi^l + \delta^i_k\partial_{jl}\xi^l\right). \tag{3.12}$$

The expressions for the Lie derivatives of a tensor field and the Christoffel symbols were obtained in the paper of Fubini [262]. The term “Lie differential” was proposed by Van Danzig.

3.1.7. We consider a linear geometric object[2] Ω on $V\subset M$. Let X be an infinitesimal transformation for which the Lie derivative of Ω vanishes: $L_X\Omega = 0$. By the linearity of the object Ω, its Lie derivative is transformed linearly and uniformly. Therefore, the above relation is valid in every coordinate system.

We choose a local coordinate system (x, U) in a neighborhood of an arbitrary point $p\in V$ so that the condition $X\,|_U = \xi^i\partial_i = \partial_1$ holds. In this chart, the finite transformation φ_t generated by the vector field X has the form $x^{i'}(t) = x^i + t\delta^i_1$ and $L_X\Omega = \partial_1\Omega$ ([357], p. 108); therefore, the dragged field can be represented by the series

$$\tilde{\Omega}(t) = e^{tL_X}\Omega = \Omega + tL_X\Omega + \frac{t^2}{2!}L^2_X\Omega + \ldots$$

[2]That is, a geometric object with a linear (not necessarily uniform) transformation law. In particular, tensor fields and objects of affine and projective connections have this property.

provided that the components of the object Ω are analytic functions. Since $L_X\Omega = 0$, we have $\tilde{\Omega}(t) = \Omega$. This implies the following theorem.

Theorem 3.1. [3] *The field of a linear geometric object Ω is invariant under the action of the (local) 1-parameter group of (local) transformations generated by a vector field X if and only if the Lie derivative of this object vanishes: $L_X\Omega = 0$.*

3.1.8. Let M be a pseudo-Riemannian manifold with metric g and Levi-Civita connection ∇. A diffeomorphism f of the manifold M onto itself is called an isometry if it preserves the metric tensor, i.e., if $f^*g = g$. The group $\widehat{I}(M^n)$ of isometries of a connected pseudo-Riemannian manifold M^n is a Lie group of dimension at most $n(n+1)/2$. If $\dim \widehat{I}(M^n) = n(n+1)/2$, then M^n is a space of constant curvature ([303], Chap. 2, §3).

A vector field X on M is called an *infinitesimal isometry* or *Killing vector field* or (*isometric*) *motion* (*i.m.*) if the local 1-parameter group of local transformations generated by the field X in a neighborhood of each point $p \in M$ consists of local isometries. By Theorem 3.1, X is an infinitesimal isometry if and only if

$$L_X g = 0 \quad (\textit{Killing equation}), \tag{3.13}$$

or, in a local coordinate system (x, U),

$$L_X g_{ij} \equiv \xi^l \partial_l g_{ij} + g_{il}\partial_j \xi^l + g_{jl}\partial_i \xi^l = \xi_{i,j} + \xi_{j,i} = 0, \tag{3.14}$$

where L_X is the Lie derivative along X and $\xi^i\partial_i = X\,|_U$.

The set $I(M^n)$ of all infinitesimal isometries in M^n forms a Lie algebra; for a connected pseudo-Riemannian manifold M^n, its dimension is at most $n(n+1)/2$. If $\dim I(M^n) = n(n+1)/2$, then M^n is a space of constant curvature [191].

If a field X generates a (global) 1-parameter group of isometries it is said to be complete. The set of all complete Killing vector fields forms the Lie algebra of the group $\widehat{I}(M)$ of isometries in M.

In a complete Riemannian manifold M (i.e., in a Riemannian manifold with a complete Riemannian connection for which every maximal geodesic in M is complete), every Killing vector field is complete. Therefore, if M is complete, then the Lie algebra $I(M)$ of all isometric motions in M is isomorphic to the Lie algebra of the group $\widehat{I}(M)$ of isometries in M.

3.2. Infinitesimal affine transformations.

3.2.1. Let M and M' be manifolds with linear (in particular, Riemannian) connections ∇ and ∇', respectively. A differentiable mapping $f : M \to M'$ is called an *affine mapping* if it maps every parallel vector field along any curve τ in M into a parallel vector field along the curve $f(\tau)$ (i.e., if the induced mapping of the tangent bundle TM into TM' maps every horizontal curve into a horizontal curve). Clearly, f maps every geodesic in M into a geodesic in M'.

An affine mapping f of a manifold M onto itself is called an *affine transformation* of M. The set of affine transformations in M forms a group denoted by $\widehat{A}(M)$ or $\widehat{A}(\nabla)$. The group $\widehat{A}(M)$ of affine transformations of a manifold M with a finite number of connected components is a Lie group ([304], Vol. I, p. 229). Since every isometry of a pseudo-Riemannian manifold M is an affine transformation, the group of isometries $\widehat{I}(M) \subset \widehat{A}(M)$.

3.2.2. A vector field X on M is called an *infinitesimal affine transformation* or *affine* (*Killing*) *vector field* or *affine motion* (*a.m.*) if the local 1-parameter group of local transformations φ_t generated by this field in a neighborhood U of each point $p \in M$ preserves the connection ∇, i.e., if $\varphi_t : U \to M$ is an affine

[3] See [416]. For the proof of this theorem for tensor fields, see [304], Vol. 1.

transformation with respect to the restriction $\nabla|_U$ of the connection ∇ to U. By Theorem 3.1, X is an affine motion in M if and only if

$$\nabla_Y(L_X - \nabla_X) = R(X, Y) \tag{3.15}$$

for all the vector fields Y on M or

$$L_X\Gamma^i_{jk} \equiv \partial_{jk}\xi^i + \xi^l\partial_l\Gamma^i_{jk} - \Gamma^l_{jk}\partial_l\xi^i + \Gamma^i_{lk}\partial_j\xi^l + \Gamma^i_{jl}\partial_k\xi^l \equiv \xi^i{}_{,jl} + R^i_{jkl}\xi^k = 0; \tag{3.16}$$

for a pseudo-Riemannian manifold (M, g), this is equivalent to the condition

$$\nabla(L_X g) = 0, \tag{3.17}$$

or

$$L_X g_{ij} = h_{ij}, \qquad h_{ij,k} = 0, \tag{3.18}$$

where ξ^i, Γ^i_{jk}, and R^i_{jkl} are the components of the vector field X, the connection ∇, and the curvature tensor field R, in terms of the coordinate frame $\{\partial_i\}$, respectively.

If M is a connected manifold with a linear connection ∇, then the Lie algebra $A(M)$ of infinitesimal affine transformations of M has dimension at most $n^2 + n$, where $n = \dim M$. If $\dim A(M) = n^2 + n$, then M is flat, i.e., the curvature of manifold M vanishes identically ([304], Vol. 1).

The set of all complete affine vector fields in M forms the Lie algebra of the group $\widehat{A}(M)$ of affine transformations in M.

3.3. Infinitesimal projective transformations.

3.3.1. A projective transformation of a manifold M with an affine (i.e., linear) connection maps geodesic lines to geodesics again and preserves the projective connection just as an affine transformation preserves the affine connection.

Let Π and Π' be two projective structures on manifolds M and M', respectively. A diffeomorphism f from M into M' induces the local isomorphism : $f_* : P^2(M) \to P^2(M')$. If f_* maps Π into Π', then f is called a (local) projective isomorphism from M onto M'. If $M = M'$ and $\Pi = \Pi'$, then the isomorphism f is called a *projective transformation* of M or an *automorphism of the projective structure* Π.

A vector field X on a manifold M with a projective structure Π is called an *infinitesimal projective transformation* or *projective motion* (*p.m.*) if the local 1-parameter group of local transformations, which is generated by this field in a neighborhood of each point $x \in M$, consists of (local) projective transformations, i.e., automorphisms of the projective structure Π.

An infinitesimal projective transformation is also called a *projective* (*Killing*) *vector field*.

As stated above (see Sec. 2.5), two connections ∇ and ∇' on M belong to the same projective structure if and only if their projective Thomas parameters coincide. Therefore, a diffeomorphism f of a pseudo-Riemannian manifold M onto itself is a projective transformation with respect to the projective structure induced on M by a connection ∇ if and only if it preserves the corresponding object of the projective connection that is defined on M by the projective Thomas parameters.

This implies that a vector field X on a manifold M with an affine connection ∇ is an infinitesimal projective transformation if and only if all transformations φ_t from the flow $F_X(t)$ of the vector field X preserve the projective connection, i.e., if the field of the object of the projective connection is invariant under the action of the (local) 1-parameter group of (local) transformations that are generated by the vector field X. By Theorem 3.1, the necessary and sufficient condition of this is that the Lie derivative of the object of the projective connection vanishe:

$$L_X\Pi^i_{jk} = 0,$$

or

$$L_X\Gamma^i_{jk} = \delta^i_j\phi_k + \delta^i_k\phi_j, \quad \phi_k \equiv \frac{1}{n+1}L_X\Gamma^l_{kl},$$

where $\phi = \phi_k dx^k$ is a differential 1-form; this is equivalent to the condition (see (3.9))

$$\xi^i{}_{,jk} + \xi^l R^i_{jlk} = \delta^i_j \phi_k + \delta^i_k \phi_j$$

provided that $X = \xi^i \partial_i$. Introducing the *Kobayashi operator* $A_X \equiv L_X - \nabla_X$, we rewrite the last equation in the invariant form:

$$\nabla_Y (L_X - \nabla_X) = R(X, Y) - \phi(Y) \cdot \mathrm{id} - Y\phi \quad (Y \in \mathcal{X}M),$$

where R is the curvature tensor.

3.3.2. If M is a pseudo-Riemannian manifold with metric g and Riemannian connection ∇, then, using the Foss–Weyl formula (2.8), we obtain

$$\phi_k = L_X \Gamma^l_{kl} = (\xi^l \partial_l \ln \sqrt{|\det(g_{ij})|})_{,k} \equiv \varphi_{,k}\,.$$

Consequently, $X = \xi^i \partial_i$ is a *projective motion in a pseudo-Riemannian manifold* (M, g) *iff*

$$L_X \Gamma^i_{jk} = \delta^i_j \varphi_{,k} + \delta^i_k \varphi_{,j}\,, \tag{3.19}$$

where

$$\varphi = \frac{1}{n+1} L_X \ln \sqrt{|\det(g_{ij})|} = \frac{1}{n+1} \xi^l \partial_l \ln \sqrt{|\det(g_{ij})|}$$

is a function of x^i, which is called the *defining function* of the projective motion X.

We write the condition of parallelism of the metric tensor g_{ij} with respect to the Levi-Civita connection in local coordinates:

$$\partial_k g_{ij} - \Gamma^l_{ki} g_{jl} - \Gamma^l_{kj} g_{il} = 0.$$

Taking the Lie derivative of both sides of this equation and using (3.4) and (3.19), we find that

$$(L_X g_{ij})_{,k} = 2 g_{ij} \varphi_{,k} + g_{ik} \varphi_{,j} + g_{jk} \varphi_{,i}. \tag{3.20}$$

Conversely, if the last equation holds, then (3.11) implies (3.19).

Therefore, *condition* (3.20) *is necessary and sufficient for* X *to be a projective motion in* (M, g).

Equation (3.20) can be written in the form of two relations:

$$L_X g_{ij} \equiv \xi_{i,j} + \xi_{j,i} = h_{ij} \tag{3.21}$$

(*generalized Killing equation*) and

$$h_{ij,k} = 2 g_{ij} \varphi_{,k} + g_{ik} \varphi_{,j} + g_{jk} \varphi_{,i} \tag{3.22}$$

(*Eisenhart equation*) or in the invariant form:

$$L_X g = h, \tag{3.23}$$

$$\nabla h(Y, Z, W) = 2g(Y, Z) W\varphi + g(Y, W) Z\varphi + g(Z, W) Y\varphi \quad (Y, Z, W \in \mathcal{X}M). \tag{3.24}$$

Contracting Eisenhart's equation over indices i, j, we obtain

$$\varphi_{,k} = \frac{1}{2(n+1)} g^{ij} h_{ij,k} = \frac{1}{n+1} g^{ij} \xi_{i,jk} \equiv \frac{1}{n+1} (\mathrm{div}\ X)_{,k}\,.$$

If $\varphi = \mathrm{const}$, i.e., $\mathrm{div} X = \mathrm{const}$, then the vector field X preserves the affine connection: $L_X \Gamma^i_{jk} = 0$, and hence it is an infinitesimal affine transformation.

An affine motion X is an infinitesimal homothety if $h_{ij} = \mathrm{const} \cdot g_{ij}$, and it is an infinitesimal isometry if $h_{ij} = 0$.

Carrying out the symmetrization $\mathcal{S}$ of both sides of Eq. (3.24), we obtain $\mathcal{S}(\nabla q) = 0$, where $q = h - 4\varphi g$. If γ is a geodesic in M, then the field of tangent vectors $\dot\gamma$ is parallel along γ. In view of this,

$q(\dot\gamma,\dot\gamma)$ remains constant along each geodesic γ in M, i.e., it is a first integral of the geodesic equations. Consequently, *with each solution of* (3.24) *in* M*, we can associate a quadratic first integral*

$$(4\varphi g - h)(\dot\gamma,\dot\gamma) = \mathrm{const} \tag{3.25}$$

of the geodesic equations.

For any two projective motions X_1 and X_2 in M, their Lie bracket $X = [X_1, X_2] \equiv L_{X_1} X_2$ is also a projective motion, and, moreover, $\mathrm{div} X = 2X_{[1}\mathrm{div} X_{2]}$. Therefore, the set $P(M)$ of all projective motions in M forms a Lie algebra, which is called the *projective Lie algebra* in M.

Let us prove that projective transformations of a pseudo-Riemannian manifold M^n form a Lie group.

We recall Palais' theorem [347]. Let $\widehat{G}$ be a group of differentiable transformations of a differentiable manifold M, and let G' be the set of all vector fields X on M generating a (global) 1-parameter group of transformations belonging to $\widehat{G}$. Let G be the Lie subalgebra generated by the set G' in the Lie algebra of all vector fields on M. If G is finite-dimensional, then $\widehat{G}$ admits a Lie group structure (such that the mapping $\widehat{G} \times M \to M$ is differentiable) and $G = G'$. The Lie algebra of $\widehat{G}$ is naturally isomorphic to G.

If $\widehat{G}$ is the group $\widehat{P}(M^n)$ of projective transformations on M^n, then G' is the set of all (complete) projective vector fields on M^n and G is the projective Lie algebra $P(M^n)$ on M^n. Consequently, the proof of the assertion is reduced to the proof that the projective Lie algebra $P(M^n)$ is finite-dimensional.

If X is a projective motion of M^n, then Eqs. (3.19) are satisfied in every chart. The integrability conditions for these equations are reduced by using (3.10) and have the form

$$L_X R^i_{jkl} = \delta^i_e \varphi_{,jk} - \delta^i_k \varphi_{,jl}. \tag{3.26}$$

Let $X = \xi^i \partial_i$. We introduce new functions $u^i_j = \xi^i_{,j}$. Then by virtue of (3.9), Eq. (3.19) can be written as of the following set of first-order partial differential equations:

$$\xi^i_{,j} = u^i_j, \quad u^i_{j,k} = \xi^e R^i_{jkl} + 2\delta^i_{(j}\varphi_{,k)}, \quad \varphi_{,ij} = -\frac{1}{n-1}(\xi^e R_{ij,e} + R_{ej} u^e_i + R_{ie} u^e_j)$$

with $n(n+2)$ unknown functions ξ^i, u^i_j and $\varphi_{,i}$ [91]. If the integrability conditions are satisfied identically, then its solution depends on n^2+2n arbitrary parameters. In this case, the space admits a projective Lie algebra of maximal dimension n^2+2n. Hence the dimension of the projective Lie algebra on M^n does not exceed the number n^2+2n. By Palais' theorem, the group $\widehat{P}(M^n)$ of projective transformations of M^n is a Lie group with the Lie algebra $P(M^n)$.

It is known that every Lie algebra is the Lie algebra of some Lie group, and with every Lie group, one associates the Lie algebra. Among the Lie groups with the Lie algebra g, there is only one connected 1-connected Lie group G. Any connected Lie group with the Lie algebra g has the form G/D, where D is a discrete normal subgroup of G contained in its center [135].

Therefore, all problems concerning Lie groups (more precisely, their connected of the components identity) are reduced to the corresponding problems for Lie algebras. Therefore, in studying groups of projective transformations, which are Lie groups as was shown above, we focus our attention on Lie algebras of these groups that are realized in the form of Lie algebras of infinitesimal projective transformations.

Clearly, the following inclusions hold:

$$\widehat{I}(M^n) \subseteq \widehat{H}(M^n) \subseteq \widehat{A}(M^n) \subseteq \widehat{P}(M^n),$$

where $\widehat{I}(^n)$, $\widehat{H}(^n)$, and $\widehat{A}(^n)$ are groups of isometries, homotheties, and affine and projective transformations of a pseudo-Riemannian manifold M^n, respectively. This implies the following inclusions for isometric (I), homothetic (H), and affine (A) Lie algebras in M^n:

$$I(M^n) \subseteq H(M^n) \subseteq A(M^n) \subseteq P(M^n).$$

The dimension of the projective Lie algebra $P(M^n)$ of an n-dimensional connected pseudo-Riemannian manifold M^n is equal to n^2+2n if and only if M^n is a space of constant curvature $\mathbb{S}^n$

[303]; otherwise, dim $P(M^n) \leq n^2 - 2n + 5$ (I. P. Egorov [91]). Since a connection on $\mathbb{S}^n$ is projectively equivalent to a flat connection, what was said above implies that the projective group of a space of constant curvature coincides with the projective group of the Euclidean space $\mathbb{R}^n$, i.e., with the group of bilinear substitutions.

The group $\widehat{P}$ of projective transformations of an n-dimensional manifold M with projective structure Π is a Lie transformation group of dimension $r \leq \dim\ \Pi = n^2 + 2n$. If $\dim\ \widehat{P} = \dim\ \Pi$, then Π is either the natural projective structure on the projective space $\mathbb{P}^n(\mathbb{R})$ or the natural projective structure on its universal covering space $\mathbb{S}^n$ [303].

Let $\nabla(\Gamma)$ and $\nabla'(\Gamma')$ be projectively equivalent connections on M, i.e., connections for which (2.4) is satisfied. If X is an infinitesimal projective transformation for the connection ∇, then it is also an infinitesimal projective transformation for the connection ∇', and, moreover, $\mathrm{div}_{\nabla'} X = (n+1)p(X) + \mathrm{div}_{\nabla} X$ in the case of Riemannian connections ∇ and ∇'. Therefore, the projective Lie algebras of projectively equivalent connections are isomorphic (they coincide).

3.4. Infinitesimal conformal transformations.

3.4.1. A diffeomorphism f of a pseudo-Riemannian manifold (M, g) onto itself is called a conformal transformation if $f^*g = \rho g$, where ρ is a positive function on M. If ρ is constant, then f is called a similarity transformation or *homothety.* If $\rho = 1$, then this homothety is an isometry.

A vector field X on M is called an *infinitesimal conformal transformation* or *conformal* (*Killing*) *vector field* or *conformal motion* (*c.m.*) if it generates a local 1-parameter group of conformal transformations in a neighborhood U of each point $p \in M$. Similarly, a vector field X on M is called an *infinitesimal homothety* or *homothetic motion* (*h.m.*) if it generates a local 1-parameter group of homotheties in the neighborhood of each point $p \in M$. X is a conformal motion on M if and only if

$$L_X g = \sigma g, \tag{3.27}$$

where the function σ is constant for a homothetic motion.

The group $\widehat{C}(M)$ of conformal transformations of a connected n-dimensional pseudo-Riemannian manifold M is a Lie transformation group, and its dimension is $\dim \widehat{C}(M) \leq (n+1)(n+2)/2$ provided that $n \geq 3$. Similarly, the group $\widehat{H}(M)$ of homotheties of a connected n-dimensional pseudo-Riemannian manifold M is a Lie transformation group, and its dimension is $\dim \widehat{C}(M) \leq (n+1)(n+2)/2$.

This assertion is proved in the following way. The integrability conditions for the equation $L_X g = \sigma g$ imply that the dimension of the Lie algebra of conformal motions does not exceed the number $(n+1)(n+2)/2$ [191]. Since any r-dimensional Lie algebra of homothetic motions contains the Lie subalgebra of isometric motions of dimension $r_I \geq r - 1$ [262] and $r_I \leq n(n+1)/2$, we have $r \leq r_I + 1 \leq n(n+1)/2 + 1$. By Palais' theorem (see Sec. 3.3.2), the group of conformal transformations and the group of homotheties are Lie transformation groups.

An infinitesimal transformation of M is homothetic it is simultaneously conformal and affine ([416], p. 167). Every projective conformal motion is a similarity transformation [262].

3.4.2. A vector field X on an n-dimensional connected pseudo-Riemannian manifold (M, g) is called a closed conformal vector field if $\nabla X = f \cdot \mathrm{id}$, where

$$f = \frac{1}{n}\mathrm{div} X \equiv \frac{1}{n}\xi^i_{,i}.$$

Let F be the space of all such vector fields. Let H be the space of all infinitesimal homotheties. If $\dim F \geq n$ and $F \cap H = \{0\}$, then the manifold (M, g) is an Einstein space: $R_{ij} = \kappa g_{ij}$, where $R_{ij} = R^l_{ilj}$ are components of the Ricci tensor in the coordinate frame [396].

We say that a vector field X on a pseudo-Riemannian manifold (M, g) is a *concircular* (respectively, *recurrent*) vector field if $\nabla X = \rho \cdot \mathrm{id} + X \cdot d\Phi$ (respectively, $\nabla X = X \cdot d\Phi$), where ρ and Φ are scalar fields. If $\nabla_Y X = \rho Y + a(Y)X$ for a field of 1-forms a and every vector field Y on M, then X is called a torse-forming vector field [357].

Concircular vector fields occur in studying conformal transformations of pseudo-Riemannian manifolds mapping geodesic circles into geodesic circles (see Sec. 4). The field of converging directions, which is also called a concurrent vector field (see below) and was investigated in detail by Shirokov [185] in 1931, may serve as an example of a concircular vector field. Many authors (see [105, 107, 145, 346, 397] and others) have been engaged in the further research into concircular vector fields.

Takeno considered concircular vector fields with in the framework of general relativity. He investigated in detail concircular vector fields in spherically symmetric space-times [385].

Deszcz [239] found condition for a curvature under which a connected analytic Riemannian manifold V^n $(n > 2)$ admitting a nonzero concircular vector field is an Einstein space. Ferrand [253] established a canonical form of a metric of conformal Euclidean space admitting a concircular vector field.

A scalar field Φ on (M, g) is called a special concircular scalar field if $\nabla^2\Phi = \rho\Phi g$, where $\rho = \mathrm{const} \neq 0$. If the curvature tensor (respectively, the Ricci tensor) of a Riemannian space (M, g) admitting a special concircular scalar field satisfies the condition $R_{hijk,[lm]} = 0$ (respectively $R_{ij,[kl]} = 0$), then M is a space of constant curvature (respectively an Einstein space) [307].

A connected n-dimensional Riemannian manifold M of class C^∞ admits $n+1$ independent solutions of the equation $\nabla^2\rho + c^2\rho g = 0$ if and only if M is isometric to the sphere $S^n(1/c)$ in $\mathbb{E}^{n+1}$ ([386]; see also [264]).

A vector field X on an n-dimensional Riemannian manifold (M, g) of class C^∞ is called a special concircular vector field if $\nabla_Y X = \varphi Y$ for all vector fields Y and some function φ on M. If there exist more than two linearly independent special concircular vector fields $X_{(i)}$ on a manifold M, then $\nabla_Y \varphi_{(i)} = -Kg(Y, X_{(i)})$ for all Y and $\varphi_{(i)} = -K\rho_{(i)} + b_{(i)}$, where $b_{(i)}$ and K are constants [300].

If $\nabla_Y X = Y$ for all Y, then X is called a concurrent vector field ([185, 357]). A concurrent vector field is, obviously, an infinitesimal homothety. If a complete connected Riemannian manifold M admits a concurrent vector field X, then it is isomorphic to the Euclidean space, and, moreover, the corresponding isomorphism maps X into the "radial" vector field $x^i \frac{\partial}{\partial x^i}$ [215].

The papers of Shandra ([174]–[179]) are devoted to concircular fields and geodesic mappings. In [175], it is shown that the set of solutions of Sinyukov's equation

$$(\nabla_Z A)(X, Y) = \lambda(X)\langle Y, Z\rangle > +\lambda(Y)\langle X, Z\rangle$$

in the space $V(K)$ (see Sec. 5) forms a Jordan algebra Q with respect to multiplication

$$(\overset{1}{A} * \overset{2}{A})(X, Y) = K(\langle \overset{1}{a}X, \overset{2}{a}Y\rangle + \langle \overset{2}{a}X, \overset{1}{a}Y\rangle) - (\overset{1}{\lambda} \otimes \overset{2}{\lambda} + \overset{2}{\lambda} \otimes \overset{1}{\lambda})(X, Y);$$

where a is the linear operator defined by $\langle aX, Y\rangle = A(X, Y)$. The set of solutions of Sinyukov's equation that are generated by concircular vector fields is an ideal in the algebra Q.

If a pseudo-Riemannian manifold (M, g) in which there exists a concircular vector field of basic type (i.e., other than an absolutely parallel vector field) admits a geodesic mapping onto a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, then $(\bar{M}, \bar{g})$ also admits a concircular vector field of basic type [176].

In [177], it is shown that the Jordan algebra Q of an n-dimensional space $V(K)$ $(K \neq 0)$ is isomorphic to the Jordan algebra Q_0 of absolutely parallel quadratic forms in some $(n+1)$-dimensional Shirokov space $(\tilde{M}, G)$ (see Sec. 4.3.1). Shandra proved that the degree of geodesic mobility (i.e., the dimension of the solution space of Sinyukov's equation) of an n-dimensional Riemannian manifold (M, g) of non-constant curvature can only be

$$p = \frac{m(m+1)}{2} + l,$$

where m is the number of linearly independent concircular covector fields on M and l runs from 1 to $\left[\frac{n+1-m}{3}\right]$ (the square brackets denote the integral part of a number) [177]. Therefore, the problem of finding all lacunae in the distribution of degrees of geodesic mobility is completely solved for the spaces with indefinite metrics.

In [179], Shandra considered n-dimensional pseudo-Riemannian spaces $(M, g = \langle .,. \rangle)$ admitting a tensor field of the type $(0, q)$ that satisfies the conditions

$$(\nabla_Z T)(X_1, \ldots, X_q) = \sum_{\alpha=1}^{q} \underset{\alpha}{L}(X_1, \ldots, \bar{X}_\alpha, \ldots, X_q)\langle X_\alpha, Z\rangle$$

for all $Z, X_1, \ldots, X_q \in X()$, where $\underset{1}{L} \ldots \underset{q}{L}$ are tensor fields of type $(0, q-1)$, $X()$ is the Lie algebra of vector fields on M, and ∇ is the Riemannian connection of the metric g (the bar over an argument denotes its absence in the corresponding expression). This tensor field is a generalization of a concircular covector field that corresponds to the case of $q = 1$ and is called concircular. Shandra showed that the class of pseudo-Riemannian spaces admitting concircular tensor fields is closed under geodesic mappings.

In [178], an analogue of the notion of concircular field for semi-Riemannian spaces (manifolds with degenerate metrics) was constructed. Special types of concircular fields that have no analogues for pseudo-Riemannian manifolds were defined and studied. For geodesic mappings of fiber bundles admitting generalized concircular vector fields, see [174].

4. Concircular Geometry

4.1. Concircular motions.

4.1.1. A *geodesic circle* is a curve whose first and second curvatures are constant and zero, respectively, and it is given by the equation

$$\frac{\delta^3 u^k}{\delta s^3} + g_{ij}\frac{\delta^2 u^i}{\delta s^2}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta u^k}{\delta s} = 0.$$

In 1940, a number of papers were published by Kentaro Yano [415] in which special conformal mappings of Riemannian spaces preserving geodesic circles were studied. Yano called a conformal mapping

$$\tilde{g}_{ij} = \rho^2 g_{ij}$$

of a Riemannian space V^n into $\tilde{V}^n$ that maps each geodesic circle from V^n into a geodesic circle in $\tilde{V}^n$ a *concircular transformation*. Mapping (4.4) is a concircular transformation if the following conditions [415] hold:

$$\rho_{ij} \equiv \rho_{i,j} - \rho_i\rho_j + \frac{1}{2}g^{kl}\rho_k\rho_l g_{ij} = \Phi g_{ij} \quad (\rho_i \equiv \partial_i \ln \rho),$$

where a comma denotes the covariant differentiation with respect to g_{ij}. Yano defined the *concircular curvature tensor*

$$Z^i_{jkl} = R^i_{jkl} - \frac{R}{n(n-1)}(g_{jl}\delta^i_k - g_{jk}\delta^i_l) \quad (R \equiv R^i_i, R_{ij} \equiv R^k_{ikj})$$

and showed that this tensor remains unchangeable under concircular transformations.

4.1.2. In the author's paper [12], *concircular motions*, i.e., infinitesimal concircular transformations that are given by the equation

$$L_X g_{ij} = \psi g_{ij}, \tag{4.1}$$

$$\psi_{,ij} = \varphi g_{ij} \tag{4.2}$$

were defined and studied. Along with being of purely geometric interest, concircular motions are interesting from the point of view of general relativity, since geodesic circles are trajectories of a uniformly accelerated motion in Einstein's theory. We also refer to Singatullin's attempt to give an invariant definition of axial symmetry in general relativity by using similar transformations [142].

We list briefly some properties of concircular motions stated in [12].

A conformal motion X in a pseudo-Riemannian manifold M^n $(n > 2)$ is concircular iff the concircular curvature tensor is preserved:

$$L_X Z^i_{jkl} = 0.$$

Hence Theorem 3.1 implies that if M^n admits an infinitesimal concircular transformation X, then it also admits a 1-parameter group G_1 of concircular transformations generated by the infinitesimal transformation.

If X_a, $a = 1, \dots, r$, are concircular motions in M^n, then $[X_a, X_b]$ are also concircular motions in M^n if $a, b = 1, \dots, r$, $a \neq b$.

Consequently, the set of all concircular motions in M^n forms a Lie algebra $CC(M^n)$, which is a subalgebra of the conformal Lie algebra $C(M^n)$ and has dimension $\dim CC(M^n) \leq \dim C(M^n) \leq (n+1)(n+2)/2$. By Palais' theorem (see Sec. 3.3.2) the set of all concircular transformations in M^n forms a Lie group $\widehat{CC}(M^n) \subset \widehat{C}(M^n)$, which is called the concircular group in M^n.

The conformal group of an Einstein space G^n, $n > 2$, and, consequently, of a space of constant curvature S^n $(n > 2)$ is a concircular group.

Equation (4.2) was repeatedly studied (see [104, 107, 133, 145, 184, 185, 217, 397]). From our point of view, the most interesting case (4.2) admits $r > 1$ independent solutions. In this case, $\varphi = -K\psi + L$, where K and L are constants and, if ψ^i is a nonzero vector (or else $\varphi = 0$), then M^n is a particular case of the spaces $V(K)$ (Solodovnikov, see Sec. 5), namely, $V_0(K)$ with the r-dimensional principal part ds_o^2 and the metric

$$ds^2 = ds_o^2(x^{i_0}) + \sigma(x^{i_0})ds_1^2(x^{l_1}) \quad (i_0 = 1, \dots, r, \ l_1 = r+1, \dots, n),$$

for which an associated metric $ds_o^2 + \sigma dy^2$ has a constant curvature K.

For any conformal motion (4.1) in $V_0(K)$, the following condition holds: (see [12]);

$$\psi_{,ij} = -K\psi g_{ij} \quad (i, j = 1, \dots, n, \quad K = \mathrm{const}). \tag{4.3}$$

This implies that if M^n admits $r > 1$ independent solutions $\psi_1, \dots, \psi_r$ of the equation $\psi_{,ij} = \varphi g_{ij}$ $(\psi_{,i}\psi_,{}^i \neq 0)$, then all conformal motions in it are concircular.

Therefore, the group of conformal transformations of the space $V_0(K)$ with the principal part of dimension $r > 1$ is a concircular group.

Any conformal motion X in a space $V_0(K)$ with principal part of dimension $r > 1$ when $K = 0$ is a curvature collineation: $L_X R^i_{jkl} = 0$.

The space $V_0(K)$ with principal part of dimension $r > 1$ when $K \neq 0$ does not admit non-isometric homothetic motions.[4]

We denote by $C^n(K)$ the space M^n for which (4.1) implies (4.3). For $K \neq 0$, this property allows us to draw important conclusions about the structure of the conformal group of $C^n(K)$.

Theorem 4.1 (Aminova [15])**.** *If the space $C^n(K)$ with $K \neq 0$ in which there exist τ independent solutions $\underset{\alpha}{\psi}$, $(\alpha = 1, \dots, \tau)$ of Eq.* (4.3) *admits an r-dimensional maximal conformal* (*i.e., concircular*) *Lie algebra CC_r, then this Lie algebra has a homothetic Lie subalgebra of dimension $r - \tau$ consisting of infinitesimal isometries. To obtain a basis for the Lie algebra CC_r, it is sufficient to add τ concircular motions*

$$\underset{c}{X}{}_{\alpha} = D\underset{\alpha}{\psi} \quad (\alpha = 1, \dots, \tau)$$

to a basis of the isometric subalgebra.

Any conformal motion in the space $C^n(K)$ with $K \neq 0$ can be represented in the form

$$X = \tilde{X} - \frac{1}{2K}D\psi,$$

[4] This result is a consequence of a more general result: the spaces $V(K \neq 0)$ admit no affine motions other than isometric motions (see [12, 29]).

where $\tilde{X}$ is an infinitesimal isometry and ψ is a solution of Eq. (4.3).

4.2. Conformal and projective Lie algebras determined by concircular vector fields.

4.2.1. Let a pseudo-Riemannian manifold (M^n, g) with Levi-Civita connection ∇ admit r independent solutions $\underset{a}{\psi}$, $a = 1, \dots, r$, of the equation

$$\nabla^2 \psi + K\psi g = 0 \quad (K = \text{const} \neq 0). \tag{4.4}$$

Then it has r independent concircular motions

$$X_a = D\underset{a}{\psi} \quad (a = 1, \dots, r), \tag{4.5}$$

$r(r+1)/2$ nonaffine projective motions

$$Y_{ab} \equiv Y_{ba} = \underset{a}{\psi}\, D\underset{b}{\psi} + \underset{b}{\psi}\, D\underset{a}{\psi} \quad (a, b = 1, \dots, r) \tag{4.6}$$

and $r(r-1)/2$ isometric motions

$$Z_{ab} \equiv -Z_{ba} = \underset{a}{\psi}\, D\underset{b}{\psi} - \underset{b}{\psi}\, D\underset{a}{\psi} \quad (a, b = 1, \dots, r). \tag{4.7}$$

(see [12]). At each point $p \in M^n$, there are $r(r-1)/2$ parallelograms constructed on the concircular vectors $\underset{a}{\psi}\, D\underset{b}{\psi}$, $a \neq b$. The sides of each of the parallelograms and one of its diagonals give nonaffine projective motions, while the second diagonal gives an infinitesimal isometry. Each of the congruences of geodesics, the trajectories of the concircular motions X_b, is also a congruence of the trajectories of r projective motions $(Y_{ab} + Z_{ab})/2$, $a = 1, \dots, r$, and is the principal congruence of the Ricci tensor with constant principal invariant equal to $(n-l)K$. (Later on, we will call such trajectories concircular.)

4.2.2. If we use the identity

$$\underset{a}{\psi}_{,i}\, \underset{b}{\psi}{}_{,}{}^{i} + K\, \underset{a}{\psi}\, \underset{b}{\psi} \equiv s_{ab} \equiv s_{ba} \equiv \text{const} \tag{4.8}$$

for calculation of the commutators

$$\begin{aligned}
&[Y_{ab}, Y_{cd}] = s_{bc} Z_{ad} + s_{ad} Z_{bc} + s_{bd} Z_{ac} + s_{ac} Z_{bd}, \\
&[Z_{ab}, Z_{cd}] = s_{bc} Z_{ad} + s_{ad} Z_{bc} - s_{bd} Z_{ac} - s_{ac} Z_{bd}, \\
&[Y_{ab}, Z_{cd}] = s_{bc} Y_{ad} - s_{ad} Y_{cb} - s_{bd} Y_{ac} + s_{ac} Y_{db}, \\
&[X_a, Z_{bc}] = 2 s_{a[b} X_{c]}, \quad [X_a, X_b] = K Z_{ab},
\end{aligned} \tag{4.9}$$

then we confirm that the spans $\{\{Z\}\}$, $\{\{X, Z\}\}$, and $\{\{Y, Z\}\}$ of the vector fields (4.5)–(4.7) form Lie algebras. Since these vector fields are linearly independent, the following theorem holds.

Theorem 4.2 (Aminova [15]). *If a pseudo-Riemannian manifold (M^n, g) admits r independent solutions $\underset{a}{\psi}$, $a = 1, \dots, r$, of* (4.4)*, then there exist the actions on M^n of:*

(1) *an isometric Lie algebra $I_{r(r-1)/2}$ which has r subalgebras of dimension $(r-1)(r-2)/2$, each of which has $r-1$ subalgebras of dimension $(r-2)(r-3)/2$, and so on:*

$$I_{r(r-1)/2} \supseteq I_{(r-1)(r-2)/2} \supseteq \cdots \supseteq I_1;$$

$I_{r(r-1)/2}$ is contained in all $\binom{r}{r-s}$ distinct subalgebras I_l of dimension $(r-s)(r-s-1)/2$, where $s = 1, \dots, r-2$;

(2) *a conformal* (*concircular*) *Lie algebra $CC_{r(r+1)/2}$ with the isometric subalgebra $I_{r(r-1)/2}$ having r subalgebras of dimension $(r-1)r/2$, each of which has $r-1$ subalgebras of dimension $(r-2)(r-1)/2$, and so on:*

$$CC_{r(r+1)/2} \supseteq CC_{(r-1)r/2} \supseteq \cdots \supseteq CC_1;$$

the Lie algebra $CC_{r(r+1)/2}$ *is contained in all* $\binom{r}{r-s}$ *distinct subalgebras* CC_l *of dimension* $(r-s)(r-s+1)/2$*, where* $s = 1, \dots, r-2$;[5] *it is intransitive if* $r < n$ *and transitive otherwise*;

(3) *a projective (nonaffine) Lie algebra* P_{r^2} *with the isometric subalgebra* $I_{(r-1)r/2}$ *having* r *subalgebras of dimension* $(r-1)^2$*, each of which contains* $r-1$ *subalgebras of dimension* $(r-2)^2$*, and so on*:

$$P_{r^2} \supseteq P_{(r-1)^2} \supseteq \cdots \supseteq P_1;$$

P_{r^2} *is contained in all* $\binom{r}{r-s}$ *distinct (nonaffine projective) subalgebras* P_l *of dimension* $(r-s)^2$*,* $s = 1, \dots, r-2$*. If* $r < n$*,* P_{r^2} *is intransitive, and it is transitive if* $r = n$*.*

The generators of the Lie algebras I_l, CC_l, and P_l are, respectively, the sets $\{Z\}$, $\{X, Z\}$, and $\{Y, Z\}$ of vector fields from (4.5)–(4.7), and their structural relations are determined by (4.9), where the constants s_{ab} are given by (4.8).

In particular, if $r = 5$, then we have the actions of M^n, a Lie algebra of isometric motions I_{10}, which has five subalgebras of dimension 6 and ten subalgebras of dimension 3:

$$I_{10} \supseteq I_6 \supseteq I_3 \supseteq I_1,$$

of a Lie algebra of conformal motions CC_{15} with five 10-dimensional, ten 6-dimensional, and ten 3-dimensional subalgebras:

$$CC_{15} \supseteq CC_{10} \supseteq CC_6 \supseteq CC_3 \supseteq CC_1,$$

and finally, of a Lie algebra of projective motions P_{25} having five 16-dimensional, ten 9-dimensional, and ten 4-dimensional subalgebras:

$$P_{25} \supseteq P_{16} \supseteq P_9 \supseteq P_4 \supseteq P_1.$$

The characteristic "chain" structure of the Lie algebras under consideration is illustrated by the following diagram.

P_{25}	CC_{15}	I_{10}
P_{16} P_{16} P_{16} P_{16} P_{16}	CC_{10} CC_{10} CC_{10} CC_{10} CC_{10}	I_6 I_6 I_6 I_6 I_6
P_9 P_9 P_9 P_9	CC_6 CC_6 CC_6 CC_6	I_3 I_3 I_3 I_3
P_4 P_4 P_4	CC_3 CC_3 CC_3	I_1 I_1 I_1
P_1 P_1 P_1 P_1	CC_1 CC_1 CC_1	

We note that an M^n in which there are r independent solutions to (4.4) is the space $V(K)$ with r-dimensional principal part [105]. Consequently, the Lie algebra $I_{r(r-1)/2}$ of isometric motions, the Lie algebra $CC_{r(r+1)/2}$ of conformal motions, and the Lie algebra P_{r^2} of projective motions act on each such space. Are these Lie algebras maximal for M^n? Later on, we will ensure that the latter holds automatically for the spaces of constant curvature S^n ([15]; see Sec. 4.4).

4.3. Projective Lie algebras determined by concurrent and parallel vector fields.

[5]Each of these subalgebras is nontrivial, i.e., does not reduce to isometries or homotheties.

4.3.1. In many respects, the following theorem determines the character and structure of the projective Lie algebra in those spaces $V(0)$ which admit a concurrent vector field X, where $X : \nabla X = \mathrm{id}$, i.e., in the Shirokov spaces (see Sec. 3.4.2).

Theorem 4.3 (Aminova [28])**.** *A pseudo-Riemannian manifold (M^n, g) in which there is a concurrent vector field $D\psi$, where $\nabla^2\psi = g$, and $\tau < n$ linearly independent parallel vector fields $D\varphi_a$, where $\nabla^2\varphi_a = 0$, $a = 1, \dots, \tau$, admits a projective Lie algebra P of dimension $(\tau+1)^2$ with maximal isometric (I), homothetic (H), and affine (A) subalgebras of dimension $\tau(\tau+1)/2$, $\tau(\tau+1)/2+1$, and $\tau^2+\tau+1$ respectively*:

$$P_{(\tau+1)^2} \supseteq A_{\tau^2+\tau+1} \supseteq H_{\tau(\tau+1)/2+1} \supseteq I_{\tau(\tau+1)/2}.$$

P is spanned by the basis vector fields

$$Z = D\psi \equiv \psi^i_{,}\partial_i, \quad E_a = D\varphi_a, \quad E^{\pm}_{ab} = \varphi_a E_b \pm \varphi_b E_a, \quad \text{and} \quad Z_a = \varphi_a Z, \tag{4.10}$$

which satisfy the structure relations

$$\begin{aligned}
&[E_a, E_b] = 0, \quad [E_a, E^{\pm}_{cd}] = \alpha_{ac}E_d \pm \alpha_{ad}E_c, \quad [E_a, Z] = E_a, \quad [Z, Z_a] = Z_a + \alpha_a Z,\\
&[E^{\pm}_{ab}, E^{\pm}_{cd}] = \alpha_{bc}E^{-}_{ad} \pm \alpha_{ac}E^{-}_{bd} + \alpha_{ad}E^{-}_{bc} \pm \alpha_{bd}E^{-}_{ac},\\
&[E^{+}_{ab}, E^{-}_{cd}] = \alpha_{bc}E^{+}_{ad} + \alpha_{ac}E^{+}_{bd} - \alpha_{ad}E^{+}_{bc} - \alpha_{bd}E^{+}_{ac},\\
&[E_b, Z_a] = \alpha_{ab}Z + (E^{+}_{ab} + E^{-}_{ab})/2, \ [Z_a, Z_b] = \alpha_b Z_a - \alpha_a Z_b, \ [Z, E^{\pm}_{ab}] = \alpha_a E_b \pm \alpha_b E_a,\\
&[Z_a, E^{\pm}_{bc}] = \alpha_b(E^{+}_{ac} + E^{-}_{ac})/2 \pm \alpha_c(E^{+}_{ab} + E^{-}_{ab})/2 - \alpha_{ac}Z_b \mp \alpha_{ab}Z_c,
\end{aligned}$$

where $\alpha_{ab} \equiv E_a\varphi_b \equiv \alpha_{ba}$ and $\alpha_a \equiv Z\varphi_a - \varphi_a$ are constants,

$$I = \{\{E_a, E^{-}_{ab}\}\}, \ a < b, \quad H = \{\{I, Z\}\}, \ A = \{\{H, E^{+}_{ab}\}\}, \ a \le b,$$

$$P = \{\{A, Z_a\}\}, \quad (a, b, c, d = 1, \dots, \tau).$$

In the case $\tau = n$, i.e., in the case of the flat space $\mathbb{E}^n$, the vector fields (4.10) *generate the maximal projective Lie algebra in E^n*:

$$P_{n^2+2n} \supseteq A_{n^2+n} \supseteq H_{n(n+1)/2+1} \supseteq I_{n(n+1)/2}.$$

In the Cartesian coordinates $ds^2 = e_1 {dx^1}^2 + \ldots + e_n {dx^n}^2$, we have $2\psi = e_1 {x^1}^2 + \ldots + e_n {x^n}^2$, $\varphi_a = x^a$, $\alpha_{ab} = e_a\delta^b_a$, $\alpha_a = 0$, and $a, b = 1, \dots, n$.

4.3.2. The assertions of Theorem 4.3 are true for any number r, $1 \le r \le \tau$, of parallel vector fields existing in M^n. This forces the "chain" structure of the Lie algebras I, H, A, and P, which have τ different subalgebras of dimensions $(\tau-1)\tau/2$, $(\tau-1)\tau/2+1$, $\tau^2-\tau+1$, and τ^2, respectively, each of which has $\tau-1$ subalgebras of dimensions $(\tau-1)(\tau-2)/2$, $(\tau-1)(\tau-2)/2+1$, $\tau^2-3\tau+3$, and $(\tau-1)^2$ respectively, and so on:

$$\begin{gathered}
I \supseteq I_{\tau(\tau-1)/2} \supseteq I_{(\tau-1)(\tau-2)/2} \supseteq \cdots \supseteq I_1, \\
H \supseteq H_{\tau(\tau-1)/2+1} \supseteq H_{(\tau-1)(\tau-2)/2+1} \supseteq \cdots \supseteq H_1, \\
A \supseteq A_{\tau^2-\tau+1} \supseteq A_{\tau^2-3\tau+3} \supseteq \cdots \supseteq A_1, \\
P \supseteq P_{\tau^2} \supseteq P_{(\tau-1)^2} \supseteq \ \ldots \ \supseteq P_1.
\end{gathered} \tag{4.11}$$

Each of the Lie algebras I, H, A, and P are contained in all $\binom{\tau}{s}$ distinct subalgebras I_l, H_l, A_l, and P_l of dimensions $(\tau-s)(\tau-s+1)/2$, $(\tau-s)(\tau-s+1)/2+1$, $(\tau-s)^2+(\tau-s)+1$, and $(\tau-s+1)^2$, respectively ($s = 1, \dots, r-1$).[6] The following diagram represents chains (4.11) in the case $\tau = 4 = n$:

P_{24}	A_{20}	H_{11}	I_{10}
P_{16} P_{16} P_{16} P_{16}	A_{13} A_{13} A_{13} A_{13}	H_7 H_7 H_7 H_7	I_6 I_6 I_6 I_6
P_9 P_9 P_9	A_7 A_7 A_7	H_4 H_4 H_4	I_3 I_3 I_3
P_4 P_4	A_3 A_3	H_2 H_2	I_1 I_1 I_1
P_1	A_1	H_1	

At every point $p \in M^n$, there are $\tau(\tau-1)/2$ parallelograms constructed on the recurrent vectors $\varphi_a D\varphi_b$. The sides of each of the parallelograms and one of its diagonals give non-homothetic affine motions, whereas the second diagonal is an infinitesimal isometry. The geodesics of each of the congruences of trajectories of τ 1-parameter groups of parallel translations are also trajectories of the 1-parameter affine groups, while the geodesics, the trajectories of the 1-parameter group of homotheties, are the trajectories of τ 1-parameter projective groups.

4.4. Lie algebras of conformal and projective motions in spaces of constant curvature. De Sitter universe.

4.4.1. We write the metric of the space $S^n(K)$ of constant curvature $K \neq 0$ in the Riemannian form ([407], p. 88):

$$ds^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n} e_i dx^{i^2}, \quad \sigma = 1 + \frac{K}{4}\sum_{i=1}^{n} e_i x^{i^2} \quad (e_i = \pm 1). \tag{4.12}$$

The general solution of Eq. (4.4) in this space is

$$\psi = \sum_{a=1}^{n+1} c_a \underset{a}{\psi},$$

where c_a are constants of integration,

$$\underset{i}{\psi} = -\frac{2}{K}\frac{x^i}{\sigma}, \quad \underset{n+1}{\psi} = \frac{1}{K}\left(1 - \frac{2}{\sigma}\right) \quad (i = 1, \dots, n)$$

and the following condition is satisfied:

$$\underset{n+1}{\psi}{}^2 = \frac{1}{K^2} - \frac{K}{4}\sum_{i=1}^{n} e_i \underset{i}{\psi}{}^2. \tag{4.13}$$

This implies that $S^n(K)$ admits $n+1$ linearly independent (special) concircular vector fields

$$\underset{i}{\psi}{}_{,}{}^{l} = x^i x^l - \frac{2}{K} e_i \sigma \delta_i^l, \quad \underset{n+1}{\psi}{}_{,}{}^{l} = x^l, \quad (i = 1, \dots, n),$$

[6] Each of the subalgebras H_l, A_l, and P_l is nontrivial, i.e., they are properly homothetic, affine, and projective, respectively.

defining $n+1$ concircular motions of the form (4.5):

$$X_i = x^i x^l \partial_l - \frac{2}{K} e_i \sigma \partial_i, \quad X_{n+1} = x^l \partial_l \quad (i = 1, \dots, n),$$

$\frac{n(n+1)}{2}$ isometric motions of the form (4.7):

$$Z_{ij} = \frac{4}{K^2}(e_j x^i \partial_j - e_i x^j \partial_i),$$

$$Z_{i\ n+1} = -\frac{1}{K}\left(x^i x^l \partial_l + \frac{2}{K} e_i (2-\sigma)\partial_i\right),$$

and $\frac{n(n+1)}{2} + n$ projective motions of the form (4.6):

$$\tilde{Y}_{ij} = -\frac{2}{K}\frac{x^i}{\sigma}\left(x^j x^l - \frac{2}{K} e_j \sigma \delta^l_j\right)\partial_l = \underset{i}{\psi}\ \underset{j}{\psi}{,}^l\, \partial_l,$$

$$\tilde{Y}_{i\ n+1} = -\frac{2}{K}\frac{x^i x^l}{\sigma}\partial_l = \underset{i}{\psi}\ \underset{n+1}{\psi}{,}^l\, \partial_l \quad (i, j = 1, \dots, n,\ i \le j).$$

If we set

$$z^i = -\frac{K}{2}\underset{i}{\psi} = \frac{x^i}{\sigma} \quad \text{and} \quad z^{n+1} = \sqrt{|K|}\underset{n+1}{\psi} = \frac{\operatorname{sgn} K}{\sqrt{|K|}}\left(1 - \frac{2}{\sigma}\right),$$

then (4.12) and (4.13) become

$$ds^2 = \sum_{a=1}^{n+1} e_a dz^{a\,2}, \tag{4.14}$$

$$\sum_{a=1}^{n+1} e_a z^{a\,2} = \frac{1}{K} \quad (e_{n+1} = \operatorname{sgn} K).$$

Consequently z^a are the Weierstrass coordinates, which can be considered as the Cartesian coordinates of the flat space $\mathbb{E}^{n+1}$ with the fundamental form (4.14) in which S^n is embedded, which are defined by the equation

$$e_1 {z^1}^2 + \ldots + e_{n+1} {z^{n+1}}^2 = \frac{1}{K}.$$

The coordinates z^a are defined up to the transformations

$$\tilde{z}^a = A^a_b z^b,$$

where A^a_b are constants satisfying the following conditions (see [191], p. 247):

$$\sum_a e_a (A^a_b)^2 = e_b, \quad \sum_a e_a A^a_b A^a_c = 0 \quad (a, b, c = 1, \dots, n+1,\ b \ne c).$$

The components η^a of the vector field $\lambda^i \partial_i \in TS^n$ in $\mathbb{E}^{n+1}$ are defined by the formula $\eta^a = \lambda^i Z^a_{,i}$ [191]. Hence, using the relations

$$z^a_{,i} z^b{,}^i = -K z^a z^b + e_a \delta^b_a \quad (a, b = 1, \dots, n+1)$$

(see (4.8)), we obtain

$$Z_{ab} = (e_b z^a \delta^c_b - e_a z^b \delta^c_a) q_c, \tag{4.15}$$

$$X_a = (-K z^a z^b + e_a \delta^c_a) q_c, \tag{4.16}$$

$$Y_{ab} = (-2K z^a z^b z^c + e_b z^a \delta^c_b + e_a z^b \delta^c_a) q_c \quad (q_c \equiv \frac{\partial}{\partial z^c}).$$

Consequently, the isometries in $\mathbb{S}^n$ are generated by the infinitesimal rotations (4.15), while the concircular transformations are generated by the projective motions (4.16) in $\mathbb{E}^{n+1}$. Therefore, the conformal group of S^n is generated by a subgroup of the conformal group in $\mathbb{E}^{n+1}$. What was done above implies the following theorem.

Theorem 4.4 (Aminova [15]). *The maximal isometric $I_{n(n+1)/2} = \{\{Z\}\}$, conformal $CC_{(n+1)(n+2)/2} = \{\{X, Z\}\}$, and projective $P_{n^2+2n} = \{\{Y, Z\}\}$ Lie algebras of the space S^n of constant non-zero curvature K are generated by the concircular vector fields (in the sense indicated in Secs. 4.2 and 4.3). The generators of these Lie algebras are completely defined by prescribing $n + l$ functions z^a of an immersion of S^n in the flat space $\mathbb{E}^{n+1}$ and can be obtained by a simple differentiation of these functions with respect to the coordinates in S^n:*

$$X_a = Dz^a, \quad Y_{ab} = 2z^{(a}Dz^{b)}, \quad Z_{ab} = 2z^{[a}Dz^{b]}.$$

The corresponding Lie algebras are defined by the commutation relations

$$\begin{aligned}
&[Z_{ab}, Z_{bd}] = e_b Z_{ad}, \quad (a \neq b \neq d \neq a),\\
&[X_a, Z_{ac}] = e_a X_c, \quad (a \neq c), \quad [X_a, X_b] = KZ_{ab},\\
&[Y_{ab}, Z_{bd}] = e_b Y_{ad}, \quad (a \neq b \neq d \neq a),\\
&[Y_{bb}, Z_{bd}] = 2e_b Y_{bd}, \quad (b \neq d),\\
&[Y_{ab}, Z_{ab}] = e_a Y_{bb} - e_b Y_{aa},\\
&[Y_{ab}, Y_{bd}] = e_b Z_{ad}, \quad (a \neq b \neq d \neq a),\\
&[Y_{bb}, Y_{bd}] = 2e_b Z_{bd},
\end{aligned}$$

and the remaining commutators are zero.

4.4.2. As an illustration, we consider the de Sitter model of a static uniform universe S with the interval ([162, Sec. 136]):

$$ds^2 = -\frac{dr^2}{1 - \frac{r^2}{R^2}} - r^2 d\vartheta^2 - r^2 \sin^2\vartheta \, d\varphi^2 + \left(1 - \frac{r^2}{R^2}\right) dt^2.$$

The general solution of Eq. (4.4) in this space is

$$\psi = \sum_{a=1}^{5} c_a \underset{a}{\psi}, \quad \underset{\alpha}{\psi} = R\sqrt{1 - \frac{r^2}{R^2}} \exp\left(e_\alpha \frac{t}{R}\right), \quad (\alpha = 1, 2, \quad e_1 = -e_2 = 1),$$

$$\underset{3}{\psi} = r \sin\vartheta \cos\varphi, \quad \underset{4}{\psi} = r \sin\vartheta \sin\varphi, \quad \underset{5}{\psi} = r\cos\vartheta, \quad (c_a = \text{const})$$

(see [15]). It defines five independent concircular vector fields, three of which ($\underset{3}{\psi}{,}^i$, $\underset{4}{\psi}{,}^i$, and $\underset{5}{\psi}{,}^i$) are spacelike and others of which are timelike (in the interval $r \leq R$):

$$\{\underset{\alpha}{\psi}{,}^i\} = \left\{\frac{r}{R}\sqrt{1 - \frac{r^2}{R^2}} \exp\left(e_\alpha \frac{t}{R}\right), 0, 0, e_\alpha \frac{\exp(e_\alpha \frac{t}{R})}{\sqrt{1 - \frac{r^2}{R^2}}}\right\}, \quad (\alpha = 1, 2).$$

Since

$$\| \underset{\alpha}{\psi}{,}^i \| \equiv \sqrt{g_{ij} \underset{\alpha}{\psi}{,}^i \underset{\alpha}{\psi}{,}^j} = \frac{1}{R} \underset{\alpha}{\psi},$$

the components of a 4-velocity vector $\underset{\alpha}{u}^{i}$ of a free particle moving along a geodesic γ_α with tangent vector $\underset{\alpha}{\psi}^{i}$ are

$$\{\underset{\alpha}{u}^{i}\} = \left\{\frac{r}{R}, 0, 0, \frac{e_\alpha}{1 - \frac{r^2}{R^2}}\right\},^{7}$$

and the geodesic γ_α is defined by the equations

$$\frac{dr}{ds} = \frac{r}{R}, \quad \frac{d\vartheta}{ds} = \frac{d\varphi}{ds} = 0, \quad \frac{dt}{ds} = \frac{e_\alpha}{1 - \frac{r^2}{R^2}}.$$

Transferring to a proper time of an observer located at the origin, which coincides with the coordinate time t in these equations, we find the velocity and the acceleration of the αth particle:

$$\frac{dr}{dt} \equiv \dot{r} = e_\alpha \frac{r}{R}\left(1 - \frac{r^2}{R^2}\right) \quad \text{and} \quad \ddot{r} = \frac{r}{R^2}\left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{3r^2}{R^2}\right). \tag{4.17}$$

It is seen from this that a particle that was at the center $r = 0$ at some moment or at the horizon $r = R$ of the universe will always be there. Equations (4.17) are easily integrated:

$$r_\alpha = \frac{1}{\sqrt{\frac{1}{R^2} + q\exp\left(-2e_\alpha \frac{t}{R}\right)}} \quad (\alpha = 1, 2).$$

The condition $r < R$ implies that the constant of integration q is positive: $q > 0$; therefore, it is possible to set $q = 1/a^2$; then

$$r_\alpha = a\frac{1}{\sqrt{\frac{a^2}{R^2} + \exp\left(-2e_\alpha \frac{t}{R}\right)}}. \tag{4.18}$$

Hence the first particle starts from the center in the infinite past ($t = -\infty$) and approaches the horizon $r = R$ of the universe as $t \to \infty$. The particle is accelerated until it reaches the hypersphere $r = \frac{R}{\sqrt{3}}$, and after that, it is damped. The velocity of the particle monotonically increases from zero to $\dot{r}_{\max} = \frac{2}{3\sqrt{3}}$ or (in CGC units) $v_{\max} = \frac{2c}{3\sqrt{3}} \approx 0.38c$, when $r = \frac{R}{\sqrt{3}}$, and then it monotonically decreases until the particle stops near the horizon of the universe.

The second particle moves in a backward direction. It starts in the infinite future from the horizon of the universe, then it is accelerated and, when the particle reaches the hypersphere $r = \frac{R}{\sqrt{3}}$, it begins to damp until its velocity falls to zero near the center.

If we introduce the Lemaitre coordinates in S:

$$r' = r\frac{\exp\left(-e_\alpha \frac{t}{R}\right)}{\sqrt{1 - \frac{r^2}{R^2}}}, \quad \vartheta' = \vartheta, \quad \varphi' = \varphi, \quad t' = t + e_\alpha R \ln\sqrt{1 - \frac{r^2}{R^2}},$$

then the fundamental form ds^2 becomes

$$ds^2 = -\exp\left(2e_\alpha \frac{t}{R}\right)(dr'^2 + r'^2 d\vartheta'^2 + r'^2 \sin^2\vartheta' d\varphi'^2) + dt'^2.$$

The interval ds^2 defines an expanding universe if $e_\alpha = 1$ and a contracting universe if $e_\alpha = -1$. Equation (4.18) in the new coordinates takes the form $r' = a$; hence the particle rests.

Therefore, the trajectories of timelike concircular vector fields coincide with the trajectories of expanding or approaching galaxies that are subordinated to the Weyl hypothesis ([162], §12, p. 7).

In [15], the generators of the Lie algebras of isometric, conformal, and projective motions for S in the Lemaitre coordinates were determined. A part of these generators are used in quantizing spinor fields on the background of the de Sitter universe (Chernikov and Shavokhina).

[7] They are also concircular vector fields.

4.5. Projective and affine concircular vector fields. An infinitesimal transformation X of a pseudo-Riemannian manifold M^n is called a *projective* (in particular, *affine*) *motion of concircular type* or a *projective* (*affine*) *concircular vector field* if it is a concircular vector field and maps geodesics in M^n into geodesics in the transformed $\tilde{M}^n$. This definition generalizes the notion of an affine motion of concircular type introduced by Takano and Okumura ([345, 346, 377, 380, 384]).

Theorem 4.5 (Aminova and Toguleva [70]).

(1) *If a concircular vector field* $X: \nabla X = \rho \cdot \mathrm{id} + X\, d\phi$, $d\phi \neq 0$, *is an affine motion in a pseudo-Riemannian manifold* M^n *with metric* g *and Levi-Civita connection* ∇, *then this field is recurrent, i.e.,* $\rho = 0$. *If* $d\phi = 0$, *then* X *is a concurrent vector field, i.e., an infinitesimal homothety.*

(2) *Every projective motion of recurrent type in* M^n *is an affine motion* $X = \varphi D\psi$, *where* $D\varphi$ *and* $D\psi$ *are parallel vector fields in* M^n $(\nabla^2\varphi = \nabla^2\psi = 0)$.

(3) *If* M^n *admits a nonaffine projective motion* X *of concircular type, then either* (a) *in* M^n, *there exist a parallel vector field* $(D\rho : \nabla^2\rho = 0)$ *and a concurrent vector field* $(Df : \nabla^2 f = g)$ *such that* $X = \rho Df$ *or* (b) *in* M^n, *there exist special concircular vector fields* u *and* v $(\nabla^2 u + Kug = \nabla^2 v + Kvg = 0$, $K = const \neq 0)$ *such that* $X = uDv$.

Theorem 4.5 contains a complete solution of the problem of affine and projective motions of concircular type in pseudo-Riemannian manifolds M^n. This theorem returns us to the equations $\nabla^2\psi + K\psi g = 0$ and $\nabla^2 f = g$, whose solutions play an exceptional role in the origin of symmetry groups of pseudo-Riemannian manifolds (Secs. 4.2 and 4.3). We note that the first equation defines the space $V(K)$ and the second equation defines the space $V(0)$ ([105, 156]).

5. Projective Mappings

5.1. The classic geometrical problem of determining Riemannian metrics g and g' that have corresponding geodesics arose in connection with the dynamic problem on transformations of the equations of motion of mechanical systems in such a way that the trajectories are preserved, and it has been on the agenda for more than a hundred years–since the time of Beltrami, Dini, and Levi-Civita. Among the Russian scientists, the greatest contribution to researching this problem was made by the Kazan, Odessa, and Moscow schools of geometry.

Various ways of identifying the geodesics of a pseudo-Riemannian manifold M with the trajectories of conservative and nonconservative dynamical systems give the possibility of widely applying the results of the theory of projective transformations to physics and mechanics.[8]

In a brief summary, it is not possible to consider all the papers devoted to projective mappings of Riemannian and non-Riemannian spaces and their generalizations, the total number of which has passed a thousand. In view of this, we focus our attention on the basic lines of research in pseudo-Riemannian spaces and leave aside the numerous papers concerning Finsler manifolds and also complex, Kählerian, contact, and other structures. More information can be found in the surveys ([92, 150, 182]) and the monographs ([91, 110, 133, 149, 303, 357, 416]).

The origins of the problem on projective mappings are the classical theorems of Beltrami and Dini, which were published in the 1860s.

[8]Some of these methods are indicated in the work of J. L. Synge [140]. We recall that the trajectories with energy E of a conservative dynamical system with Hamiltonian $H = (1/2)g^{ij}p_ip_j + V(x)$ are in one-to-one correspondence with the geodesic lines of a pseudo-Riemannian space with the fundamental form $ds^2 = 2(E - V)g_{ij}\,dx^i\,dx^j$ and the time is $\tau = (1/2)\int ds/(E - V)$ [111]. [190], Russian p. 335). Geodesics that are extremals of the variational problem $\delta\int(E - V)^k ds = 0$, where ds is the linear element of the 3D Euclidean space, are brachistochrones if $k = -1/2$, they are equilibrium states of a homogeneous inelastic thread in a field of a conservative force with potential V if $k = 1$, and they are paths of light rays in an isotropic medium ([161], Vol. 2), etc. Trajectories of a material point moving on a surface with linear element $d\sigma^2 = a_{ij}\,dy^i\,dy^j$ under the action of a force with potential V are geodesics of the 2D space with linear element $ds^2 = (E - V)a_{ij}\,dy^i\,dy^j$ [111].

Beltrami's theorem (1868, [209]). *Only the spaces which have geodesics that correspond to the geodesics of a space of positive curvature are also spaces of constant curvature.*

Dini's theorem (1869, [240]). *There is a geodesic mapping between two nonoverlapping Riemannian spaces V^2 and $\widetilde{V^2}$ if and only if their linear elements are reducible to the Liouville form*:

$$ds^2 = (u+v)(dx^2+dy^2) \quad and \quad \widetilde{ds^2} = -\left(\frac{1}{u}+\frac{1}{v}\right)\left(\frac{dx^2}{u}-\frac{dy^2}{v}\right), \tag{5.1}$$

where u is a function of x and v is a function of y.

Twenty-seven years after the appearance of Dini's work, Levi-Civita gave a general solution to the problem of geodesic mappings of Riemannian spaces in the following fundamental theorem.

Levi-Civita's theorem (1896, [310]). *There is a geodesic mapping between two Riemannian spaces V and $\tilde{V}$ if and only if their linear elements can be reduced to the form*

$$\begin{aligned} ds^2 &= \sum_{\alpha=1}^{p} {\prod_{\beta}}' |f_\beta - f_\alpha| \, ds_\alpha^2 = \sum_{\alpha=1}^{p} \Phi_\alpha, \\ \widetilde{ds^2} &= a[(f_1+b)\cdots(f_p+b)]^{-1} \sum_{\alpha=1}^{p} (f_\alpha+b)^{-1}\Phi_\alpha, \quad (a, b = \text{const}), \end{aligned} \tag{5.2}$$

where ds_α^2 is an independent quadratic form depending on the variables x^{i_α}, f_α is a function of x^{i_α}, which is just a constant if ds_α^2 is not one-dimensional, $f_\beta \neq f_\alpha$ if $\beta \neq \alpha$, and $\prod'_\beta |f_\beta - f_\alpha|$ denotes the product of the factors $|f_\beta - f_\alpha|$ for all $\beta = 1, \ldots, p$, except $\beta = \alpha$.

The solution of the geodesic mapping problem for pseudo-Riemannian manifolds reduces to the integration of the equations

$$Dg'(Y,Z,W) = 2g'(Y,Z)W\psi + g'(Z,W)Y\psi + g'(Y,W)Z\psi \tag{5.3}$$

on pseudo-Riemannian manifolds of arbitrary signature and dimension.

Seventy years after the appearance of Dini's theorem, P. Shirokov determined all two-dimensional pseudo-Riemannian manifolds with common geodesics (see [188], pp. 383–388). Shirokov's work laid the foundation for systematic research in geodesically corresponding spaces with in definite metrics. Using and developing Shirokov's ideas, Petrov [128] gave a classification of geodesically corresponding pseudo-Riemannian spaces V_3, and his student Golikov [82] determined all four-dimensional Lorentz spaces with the corresponding geodesics. The classification of n-dimensional geodesically corresponding Lorentz spaces was completed by Kruchkovich [106]. The cases listed are all restricted to the three basic types of Segre characteristic $\chi_\nu = \{\nu I \ldots I\}$, $\nu = 1, 2, 3$, for the bilinear form g'. In n-dimensional pseudo-Riemannian spaces of arbitrary signature, the number of basic types unboundedly increases as n increases; each of them is given by a set $\{m_1 \ldots m_\tau\}$ of natural numbers satisfying a single condition $m_1 + \cdots + m_\tau = n$. Clearly, the traditional approach, which is based on the consideration of each type individually, turns out to be not applicable in the general case. Probably that is why the problem of integrating Eqs. (5.3) with a nonzero right-hand side in pseudo-Riemannian spaces M^n of arbitrary signature and dimension was not posed earlier.

In the author's papers ([31], for further references, see [52]), the technique of skew-normal frames was developed, which allows us to find all projectively and affinely equivalent Riemannian connections and to obtain the general solution of the classical geometric problem of determining all pseudo-Riemannian metrics g and g' of arbitrary signature and dimension that have the corresponding geodesics. The solution is given by the following theorem.

Theorem 5.1 (Aminova, 1987 [38]). *Let (M,g) and (M',g') be two n-dimensional pseudo-Riemannian manifolds with common geodesics. Let G and G' be matrix-valued functions on M whose values at each point $p \in M$ are the matrices of the bilinear forms g and g' in an arbitrary basis of T_pM.*

I. *At every point of an open connected set $V \subseteq M$, let the λ-matrix $G^T G'^{-1} G - \lambda G$ have the Segre characteristic $\chi = \{r_1, \dots, r_k\}$, $\quad r_1 + \dots + r_k = n$. Then the contravariant metric tensors g_c and g'_c in V are defined by*

$$g_c\,|_V = \sum_{\alpha=1}^{k} e_\alpha \prod_\beta{}' (f_\beta - f_\alpha)^{-r_\beta} \Phi_\alpha, \tag{5.4}$$

$$g'_c\,|_V = \prod_{\gamma=1}^{k} (f_\gamma)^{r_\gamma} \sum_{\alpha=1}^{k} e_\alpha \prod_\beta{}' (f_\beta - f_\alpha)^{-r_\beta} \Phi_\alpha (f_\alpha \Phi_\alpha + \Lambda_\alpha), \tag{5.5}$$

where $f_1, \dots, f_k$ are pairwise distinct nonzero functions and $\prod'_\beta (f_\beta - f_\alpha)^{-r_\beta}$ denotes the product of $(f_\beta - f_\alpha)^{-r_\beta}$ for all $\beta = 1, \dots, k$, except for $\beta = \alpha$. Around each point $p \in V$, there is a canonical chart (x,U) in which the nonzero components of the contravariant tensor fields $\Phi_\alpha\,|_U$ and $\Lambda_\alpha\,|_U$ are

for $r_\alpha = 1$:

$$\Phi_\alpha^{n_\alpha+1\ n_\alpha+1} = 1, \quad \Lambda_\alpha = 0, \tag{5.6}$$

for $r_\alpha = 2$:

$$\Phi_\alpha^{n_\alpha+1\ n_\alpha+1} = \sigma_1^\alpha, \quad \Phi_\alpha^{n_\alpha+1\ n_\alpha+2} = A^{\alpha-1}, \quad \Lambda_\alpha^{n_\alpha+1\ n_\alpha+1} = 1, \tag{5.7}$$

and

$$\Phi_\alpha^{n_\alpha+1\ n_\alpha+1} = 2A_{r_\alpha-1}^\alpha + \sum_{l=1}^{r_\alpha-2} A_l^\alpha A_{r_\alpha-l-1}^\alpha,$$

$$\Phi_\alpha^{n_\alpha+1\ n_\alpha+p} = A_{r_\alpha-p}^\alpha + (1-p)\varepsilon^\alpha x^{n_\alpha+p-1} A^{\alpha-1} + \sum_{l=1}^{r_\alpha-p} A_l^\alpha A_{r_\alpha-l-p}^\alpha,$$

$$(p = 2, \dots, r_\alpha - 1) \tag{5.8}$$

$$\Phi_\alpha^{n_\alpha+1\ n_\alpha+r_\alpha} = A^{\alpha-1},\ \Phi_\alpha^{n_\alpha+p\ n_\alpha+q} = \sum_{l=0}^{r_\alpha+1-p-q} A_l^\alpha A_{r_\alpha+1-p-q-l}^\alpha,$$

$$(p,q = 2, \dots, r_\alpha - 1, \quad p+q \le r_\alpha + 1),$$

$$\Lambda_\alpha^{n_\alpha+p\ n_\alpha+q} = \sum_{l=0}^{r_\alpha-p-q} A_l^\alpha A_{r_\alpha-p-q-l}^\alpha,$$

$$(p,q = 1, \dots, r_\alpha - 1, \quad p+q \le r_\alpha) \ \text{ for } r_\alpha > 2:$$

where A^α, σ_l^α, and A_l^α are defined by

$$A^\alpha = \varepsilon^\alpha (r_\alpha - 1)(x^{n_\alpha+r_\alpha-1} + \omega^\alpha(x^{n_\alpha+r_\alpha})) + 1 - \varepsilon, \quad \sigma_l^\alpha = \sum_\beta{}' r_\beta (f_\beta - f_\alpha)^{-l},$$

$$A_0^\alpha = 1, \quad A_l^\alpha = \frac{1}{2l} \sum_{s=1}^{l} A_{l-s}^\alpha \sigma_s^\alpha \ \text{ for } l = 1, \dots, r_\alpha - 1,\ \alpha, \beta = 1, \dots, k,$$

$$f_\alpha = f_\alpha(x^{n_\alpha+1}) \ \text{ for } r_\alpha = 1$$

$$\text{and } \ f_\alpha = \varepsilon^\alpha x^{n_\alpha+r_\alpha} + (1-\varepsilon)c_\alpha \ \text{ for } r_\alpha > 1,$$

ε is equal to 0 or 1, c_α is a constant, ω^α is a function of $x^{n_\alpha+r_\alpha}$, $e_\alpha = \pm 1$, $n_1 = 0$, and $n_\gamma = r_1 + \ldots + r_{\gamma-1}$ for $\gamma = 2, \ldots, k$.

In the case $k = 1$, i.e., if $\chi = \{n\}$, we have

$$g'_c\,|_V = (f_1)^n (f_1 g_c\,|_V + \Lambda), \tag{5.9}$$

where the nonzero components of the tensor fields g_c and Λ in the canonical chart (x, U) have the form

$$g^{h\ n+1-h} = e, \quad g^{1h} = e(1-h)\varepsilon x^{h-1} A^{-1}, (h = 2, \ldots, n-1), \tag{5.10}$$

$$g^{1n} = eA^{-1}, \quad \Lambda^{h\ n-h} = e, (h = 1, \ldots, n-1)$$

(here, $e = \pm 1$, $f_1 = \varepsilon x^n + (1-\varepsilon)c_1 \neq 0$, $A = \varepsilon(n-1)(x^{n-1} + \omega(x^n)) + 1 - \varepsilon$, and ε is equal to 0 or 1).

The converse is also true, and, moreover, condition (5.3) is satisfied with the function

$$2\psi = -\sum_{\alpha=1}^{k} r_\alpha \ln |f_\alpha|$$

.

If $(f_\alpha, f_{\alpha+1})$ is a pair of complex-conjugate functions, then $r_\alpha = r_{\alpha+1}$ and the variables $x^{n_\alpha+p} = \overset{*}{x}{}^{n_\alpha+r_\alpha+p}$ are complex when $p = 1, \ldots, r_\alpha$ (denotes the complex conjugation). In order to transfer to the real variables $\bar{x}^{n_\alpha+p}, \bar{x}^{n_\alpha+r_\alpha+p}$, one needs to set $x^{n_\alpha+p} = \bar{x}^{n_\alpha+p} + i\bar{x}^{n_\alpha+r_\alpha+p}$ when $p = 1, \ldots, r_\alpha$ and $f_\alpha = u_\alpha + iv_\alpha$, where u and v are harmonically conjugate functions of the variables $\bar{x}^{n_\alpha+r_\alpha}$ and $\bar{x}^{n_\alpha+2r_\alpha}$, i.e.,*

$$\frac{\partial u_\alpha}{\partial \bar{x}^{n_\alpha+r_\alpha}} = \frac{\partial v_\alpha}{\partial \bar{x}^{n_\alpha+2r_\alpha}}, \frac{\partial u_\alpha}{\partial \bar{x}^{n_\alpha+2r_\alpha}} = -\frac{\partial v_\alpha}{\partial \bar{x}^{n_\alpha+r_\alpha}}.$$

To obtain the contravariant components of the geodesically corresponding metrics g and g' it is necessary to invert the matrices (g^{ij}) and (g'^{ij}). For $k = 1$, we obtain

$$\begin{aligned} eg\,|_U = 2A\,dx^1\,dx^n + \kappa(p-3)\sum_{p=2}^{n-1}((dx^{n+1-p} + 2(n-p)\varepsilon x^{n-p} dx^n) dx^p \\ + \varepsilon(p-1)(n-p)x^{p-1}x^{n-p}dx^{n^2}), \end{aligned} \tag{5.11}$$

$$\begin{aligned} ef_1^{n+1} g'\big|_U = \kappa(n-3)\sum_{i=2}^{n-1}\sum_{s=0}^{i-2}(-f_1)^{-s} dx^{\{} dx^{n+1-i+s\}} \\ + \sum_{i=1}^{n-1} dx^n (2\,dx^i + (\varepsilon(i-1)x^{i-1} - \frac{A}{f_1}\delta^i_{n-1})\,dx^n) \\ \times \big(A(-f_1)^{1-i} + \kappa(i-2)\kappa(n-3)\sum_{s=2}^{i}\varepsilon(n-s)(-f_1)^{s-i}x^{n-s}\big), \end{aligned} \tag{5.12}$$

where $a^{\{i}b^{i\}} \equiv a^i b^i$, $a^{\{i}b^{j\}} \equiv a^i b^j + a^j b^i$, and $\kappa(x)$ is a step function, which is equal to 1 for $x \geq 0$ and 0 for $x < 0$.

In the case $k > 1$, the metrics with common geodesics have the form

$$g\,|_V = \sum_{\alpha=1}^{k} e_\alpha \prod_\beta{}' (f_\beta - f_\alpha)^{r_\beta} ds^2_\alpha, \tag{5.13}$$

$$g'\big|_V = \prod_{\gamma=1}^{k} (f_\gamma)^{-r_\gamma} \sum_{\alpha=1}^{k} e_\alpha \prod_\beta{}' (f_\beta - f_\alpha)^{r_\beta} d\sigma^2_\alpha,$$

where for $r_\alpha = 1$,

$$ds_\alpha^2 = dx^{n_\alpha+1^2}, \quad d\sigma_\alpha^2 = \frac{1}{f_\alpha}\, ds_\alpha^2, \tag{5.14}$$

for $r_\alpha = 2$,

$$ds_\alpha^2 = A^\alpha (2dx^{n_\alpha+1} dx^{n_\alpha+2} - \sigma_1^\alpha A^\alpha dx^{n_\alpha+2^2}, \tag{5.15}$$

$$d\sigma_\alpha^2 = \frac{1}{f_\alpha} ds_\alpha^2 - \frac{1}{f_\alpha^2} A^{\alpha 2} dx^{n_\alpha+2^2},$$

for $r_\alpha > 2$

$$ds_\alpha{}^2 = h_{pq}^\alpha dx^{n_\alpha+p} dx^{n_\alpha+q}, \tag{5.16}$$

$$d\sigma_\alpha{}^2 = \frac{1}{f_\alpha} h'^\alpha_{pq} dx^{n_\alpha+p} dx^{n_\alpha+q}, \quad (p, q = 1, \dots, r_\alpha),$$

and h_{pq}^α *and* h'^α_{pq} *are given by the recurrent relations*

$$h_{1p}^\alpha = h'^\alpha_{1p} = A^\alpha \delta_p^{r_\alpha},$$

$$h_{sp}^\alpha = \delta_p^{r_\alpha+1-s} - \delta_p^{r_\alpha} \Big[A^\alpha A_{s-1}^\alpha + (s - r_\alpha) \varepsilon^\alpha x^{n_\alpha + r_\alpha - s} + A^\alpha \sum_{l=1}^{s-1} A_l^\alpha A_{s-l-1}^\alpha \Big]$$
$$- \kappa(s-3) \sum_{q=2}^{s-1} \sum_{l=0}^{s-q} \big[A_l^\alpha A_{s-q-l}^\alpha \big] h_{pq}^\alpha,$$

$$h_{r_\alpha p}^\alpha = \delta_p^1 A^\alpha - \delta_p^{r_\alpha} A^{\alpha 2} \Big(2A_{r_\alpha-1}^\alpha + \sum_{l=1}^{r_\alpha-2} A_l^\alpha A_{r_\alpha-l-1}^\alpha \Big) - \sum_{q=2}^{r_\alpha-1} \Big[A^\alpha A_{r_\alpha-q}^\alpha$$
$$+ (1-q) \varepsilon^\alpha x^{n_\alpha+q-1} + A^\alpha \sum_{l=1}^{r_\alpha-q} A_l^\alpha A_{r_\alpha-q-l}^\alpha \Big] h_{pq}^\alpha, \tag{5.17}$$

$$h'^\alpha_{sp} = \delta_p^{r_\alpha+1-s} - \delta_p^{r_\alpha} \Big[A^\alpha A_{s-1}^\alpha + (s - r_\alpha) \varepsilon^\alpha x^{n_\alpha+r_\alpha-s} + A^\alpha \sum_{l=1}^{s-1} A_l^\alpha A_{s-l-1}^\alpha$$
$$+ \frac{A^\alpha}{f_\alpha} \sum_{l=0}^{s-2} A_l^\alpha A_{s-l-2}^\alpha \Big] - \kappa(s-3) \sum_{q=2}^{s-1} \Big(\sum_{l=0}^{s-q} A_l^\alpha A_{s-q-l}^\alpha + \frac{1}{f_\alpha} \sum_{l=0}^{s-q-1} A_l^\alpha A_{s-q-l-1}^\alpha \Big) h'^\alpha_{pq},$$

$$h'^\alpha_{r_\alpha p} = \delta_p^1 A^\alpha - \delta_p^{r_\alpha} A^{\alpha 2} \Big(2A_{r_\alpha-1}^\alpha + \sum_{l=1}^{r_\alpha-2} A_l^\alpha A_{r_\alpha-l-1}^\alpha + \frac{1}{f_\alpha} \sum_{l=0}^{r_\alpha-2} A_l^\alpha A_{r_\alpha-l-2}^\alpha \Big)$$
$$- \sum_{q=2}^{r_\alpha-1} \Big[A^\alpha A_{r_\alpha-q}^\alpha + (1-q) \varepsilon^\alpha x^{n_\alpha+q-1} + A^\alpha \sum_{l=1}^{r_\alpha-q} A_l^\alpha A_{r_\alpha-q-l}^\alpha + \frac{A^\alpha}{f_\alpha} \sum_{l=0}^{r_\alpha-q-1} A_l^\alpha A_{r_\alpha-q-l-1}^\alpha \Big] h'^\alpha_{pq},$$
$$(s = 2, \dots, r_\alpha - 1, \quad p = 1, \dots, r_\alpha).$$

II. *If at every point of an open connected set* $V \subseteq M$, *the* λ*-matrix* $G^T G'^{-1} G - \lambda G$ *has the Segre characteristic of the general type,*

$$\chi = \{ r_1 \dots r_{k_0} (m_1^{k_0+1} \dots m_{s_{k_0+1}}^{k_0+1}) \dots (m_1^k \dots m_{s_k}^k) \},$$

where $k > k_0 > 0$, $s_{k_0+1}, \dots, s_k > 1$, *and* $m_1^\gamma \geq \dots \geq m_{s_\gamma}^\gamma$ *if* $\gamma = k_0 + 1, \dots, k$, *then the contravariant metric tensors* g_c *and* g'_c *in* V *are determined by* (5.4), *where* $f_{k_0+1}, \dots, f_k$ *are constants,* $\prod'_\beta (f_\beta - f_\alpha)^{-r_\beta}$

denotes the product of the factors $(f_\beta - f_\alpha)^{-r_\beta}$ *for all* $\beta = 1, \dots, k_0$, *except for* $\beta = \alpha$, *the nonzero contravariant components of the tensor fields* Φ_γ *and* Λ_γ *in terms of the canonical chart* (x, U) *around each point* $p \in V$ *are defined for* $\gamma = 1, \dots, k_0$ *by the formulas* (5.6)–(5.10), *where* k *should be replaced by* k_0, *and are defined for* $\gamma = k_0 + 1, \dots, k$ *by*

$$\Phi_\gamma^{n_t^\gamma + h\ n_p^\gamma + l} = \sum_{s=1}^{s_\gamma} e_s \sum_{q=1}^{m_s^\gamma} \eta_{n_s^\gamma + q}^{n_t^\gamma + h} \eta_{n_s^\gamma + m_s^\gamma + 1 - q}^{n_p^\gamma + l},$$

$$h + l \leq \min(m_t^\gamma + 1, m_p^\gamma + 1), \quad (h = 1, \dots, m_t^\gamma,\ l = 1, \dots, m_p^\gamma),$$

$$\Lambda_\gamma^{n_t^\gamma + h\ n_p^\gamma + l} = \sum_{s=1}^{s_\gamma} e_s \sum_{q=1}^{m_s^\gamma + 1} \eta_{n_s^\gamma + q}^{n_t^\gamma + h} \eta_{n_s^\gamma + m_s^\gamma - q}^{n_p^\gamma + l},$$

$$h + l \leq \min(m_t^\gamma, m_p^\gamma), \quad (h = 1, \dots, m_t^\gamma, \quad l = 1, \dots, m_p^\gamma),$$

$$\eta_{\tau + m_t^\gamma - m_s^\gamma + l}^{\sigma + h} = \sum_{p=h}^{l} \eta_{\tau + m_t^\gamma - m_s^\gamma + p}^{\sigma + h}(x^{i_\gamma}) A_{l-p}^\gamma \quad \text{if} \ \ t < s$$

$$(l = 1, \dots, m_s^\gamma),$$

$$\eta_{\tau + l}^{\sigma + h} = \sum_{p=h}^{l} \zeta_{\tau + l}^{\sigma + h}(x^{i_\gamma}) A_{l-p}^\gamma \quad \text{if} \ \ t \geq s$$

$$(l = 1, \dots, m_t^\gamma)$$

($h \leq l$, $s, t = 1, \dots, s_\gamma$, $\tau \equiv n_t^\gamma$, $\sigma \equiv n_s^\gamma$), *and the condition that the tensor field* Λ_γ *is covariantly constant with respect to the form* $\Phi_{\gamma\ i_\gamma j_\gamma} dx^{i_\gamma} dx^{j_\gamma}$ ($i_\gamma = r_{\gamma-1} + 1, \dots, r_\gamma$, *and when* $s = 1, \dots, s_\gamma$, n_s^γ *is the sum of all numbers preceding* m_s^γ *in the characteristic* χ). *The corresponding covariant components* $\Phi_{\gamma i_\gamma j_\gamma}, \Lambda_{\gamma i_\gamma j_\gamma}$, *where* $\gamma = 1, \dots, k_0$, *are defined by* (5.11)–(5.17). *The converse is also true, and, moreover, condition* (5.3) *is satisfied with the function* $2\psi = -\sum_{\alpha=1}^{k} r_\alpha ln \mid f_\alpha \mid$.

The above theorem reduces the problem of determining the geodesically corresponding pseudo-Riemannian metrics to the problem of determining the metrics that admit parallel symmetric bilinear forms for which the general solution was given by P. Shirokov [183], A. Shirokov [181], Kruchkovich and Solodovnikov [109] (see also [1]). Some aspects of the problem are discussed in the author's papers [31,36] and [52]), where we return, without relying on A. Shirokov's theorem [181], to P. Shirokov's [183] original idea but use a moving skew-normal frame, which is constructed just as simply as an orthonormal frame, but unlike the latter allows us to include the case where the unknown bilinear form has an arbitrary algebraic structure. Such an approach has certain advantages and allows us to obtain a number of new results along with the known results. In particular, it allows us to establish a direct connection between the characteristic of a parallel bilinear form and the existence of parallel vector fields, which greatly simplifies the solution of the problem in many cases (see examples in [31,36], and [52]). It also allows us to give a complete solution of the problem for Lorentz manifolds, and, as a consequence, to indicate some properties of those Ricci-symmetric Lorentz manifolds and Lorentz manifolds that admit nonhomothetic affine motions. In [31], the concept of index of the characteristic of a parallel bilinear form was introduced, and its geometric interpretation was given. A special diagram technique for quirkly writing a parallel bilinear form with given characteristic was also proposed.

An important advantage of the method used is that it does not depend on the choice of a chart, which allows us to apply it to the study of the properties of parallel tensor fields on pseudo-Riemannian manifolds. This fact also stimulated us to turn again to the old problem.

We note that the solutions obtained for the equations of a geodesic mapping include all possible sets of elementary divisors, i.e., all possible Segre characteristics of the bilinear form a; among them is an uncountable set of basic types of Segre characteristic, unlike previous results (Dini, Levi-Civita,

P. Shirokov, Petrov, Golikov, and Kruchkovich), which are all restricted to three basic types of Segre characteristic (the jump $3 \to \infty$ is itself nontrivial). The progress made toward solving the question was impeded by the necessity of considering each type individually. The number of different basic types $\{m_1 \dots m_k\}$, $m_1 + \dots + m_k = n$, grows unboundedly as n increases, and the problem are insoluble.

The technique of skew-normal frames (see Sec. 1), which was developed by author in the framework of the adapted frame method, removes this obstacle. The skew-normal frame is a natural generalization of the orthogonal (orthonormal) frame; however, unlike the latter, it is suited to the consideration of the bilinear form of an arbitrary algebraic structure. Riemannian geometry in a skew-normal frame is developed in a similar way as Riemannian geometry in an orthogonal frame. The technique of integration in a skew-frame differs little from the technique of integration in an orthoframe, which makes it possible to give the general integral for the tensor differential equations for unknown bilinear forms on pseudo-Riemannian manifolds.

The use of a skew-frame allows one to obtain not only a solution of Eqs. (5.3) with nonzero right-hand side and an unknown bilinear form of any characteristic but also allows us to write the general form of solution as a single formula directed by the parameters of the characteristic, unlike the previous papers, where the results were obtained in very particular cases for each characteristic, separately, which led to complicated computations and cumbersome formulas that do not lend themselves to analysis, so that it was difficult to catch anything general in these results and to realize that they may be derived from a single formula. The formulas obtained by using the technique of the skew-normal frame contain recurrence relations, which raises the possibility of applying computer methods.

The metrics and connections we have found can be used for solution of various geometrical problems, for example, the problem of determining symmetric and Ricci-symmetric spaces, investigating geodesics, and so on.

There are various methods for identifying the geodesics of pseudo-Riemannian manifolds with the trajectories of conservative and nonconservative dynamical systems, which gives the possibility of widely applying these results from the study of pseudo-Riemannian manifolds with common geodesics to physics and mechanics.

Also, we recall that the geodesics of a projective connection are the graphs of solutions of a system of second-order ordinary differential equations with right-hand sides that are polynomials of at most third order in derivatives. It was exactly this problem of constructing a satisfactory geometric theory for second-order equations that led E. Cartan to the theory of manifolds with projective connection [221] (see Sec. 7).

A successful application of skew-normal moving frames to the solution of the classical problem with more than a hundred years' history suggests that the use of the skew-frame can be connected in future with progress toward solving old and new geometrical problems, including the study of bilinear forms, and also various problems in general relativity theory, unified field theories, supergravitation, and other branches of modern theoretical physics, where the method of moving frames is widely used.

5.2. Using Levi-Civita's theorem, Solodovnikov [157]–[159] solved the problem of separating Riemannian spaces V^n ($n > 2$) into disjoint classes of geodesically corresponding spaces and determined the geodesic class of the Riemannian spaces $V(K)$. Later Kruchkovich considered in [107] the problem of geodesic mappings of the pseudo-Riemannian spaces $V(K)$.

Some more general problems in the theory of geodesic mappings of pseudo-Riemannian spaces were dealt with in a number of papers of Sinyukov ([143–148] and others) and in his monograph [149], which contains numerous results obtained by him and students of his school and also a bibliography of papers in this area. The following theorem of Sinyukov is a generalization of Beltrami's theorem.

Sinyukov's theorem ([143]). *If a Riemannian space V^n ($n > 2$) and a symmetric Riemannian space $R^i_{jkl,m} = 0$ can be geodesically mapped into one another without preserving the connection, then both spaces are spaces of constant curvature.*

In turn, Sinyukov's theorem was a basis for subsequent generalizations. Soos [374], Roter [355], and Nicolescu [341] proved that this theorem can be extended to projective-symmetric spaces, $W^i_{jkl,m} = 0$, and recurrent spaces, $R^i_{jkl,m} = \lambda_m R^i_{jkl}$. Adati and Miyazawa [193] considered the geodesically corresponding projective-recurrent Riemannian spaces, $W^i_{jkl,m} = k_m W^i_{jkl}$, with the same recurrence vectors k_m and reached a similar conclusion.

If a pseudo-Riemannian space (M, g) admits a geodesic mapping onto an "S-manifold" (M', g') : $R'^p_{ijk,lm} = R'^p_{ijk,ml}$ (the semicolon denotes the covariant differentiation with respect to g'), then either both manifolds are spaces of constant curvature or $\lambda_{ij} \equiv \lambda_{,ij} - \lambda_{,i}\lambda_{,j} = \Delta g_{ij}$, where $\Delta = \text{const}$ and

$$\lambda_{,i} = \frac{1}{2(n+1)} \ln \frac{|\det(g'_{ij})|}{|\det(g_{ij})|},_i$$

(Venzi [399]; for spaces with a birecurrent projective curvature tensor, $W^i_{jkl,mp} = a_{mp} W^i_{jkl}$, see [400]).

If the condition $\lambda_{ij} = \Delta g_{ij}$ is satisfied, one says that the spaces (M, g) and (M', g') are in a special geodesic correspondence [342]. Some properties of such spaces are considered in [400], where, in particular, an example of manifolds with nonconstant curvature which are in a nonaffine special geodesic correspondence is given (see also [402]). For the relationship between a geodesic mapping $(M, g) \to (M', g')$ and the induced mappings $(M, g) \to (M^*, g^*)$ and $(M^*, g^*) \to (M'^*, g'^*)$, where $g^* = e^{2\lambda} g$ and $g'^*_{ij} = e^{2\lambda} g'^{st} g_{si} g_{tj}$, see [403].

Let two Riemannian spaces V^n and V'^n be in a nontrivial, i.e., a nonaffine, geodesic correspondence. If V^n is not reducible and T-birecurrent, $T'^h_{ijk;mr} = b_{mr} T'^h_{ijk}$, $T'^h_{ijk} = R'^h_{ijk} - w(\delta^h_j g'_{ik} - \delta^h_k g'_{ij})$ (w is an arbitrary function), then V and V' are spaces of constant curvature (Florea [258]). A similar result was obtained in the papers of Nicolescu [342], [343] for birecurrent spaces ($w \equiv 0$) and Z'-birecurrent spaces ($w = R'/n(n-1)$, where R' is the scalar curvature).

If V^n is Einstein and V'^n is Z'-birecurrent, then the spaces V and V'^n are of constant curvature and in a special geodesic correspondence [342]. Non-Einstein Ricci 2-symmetric Riemannian spaces, $R_{ij,kl} = 0$ and $R_{ij} \neq \kappa g_{ij}$ of dimension $n > 2$ do not admit nontrivial geodesic mappings (Mikesh [112]). An example of a non-Einstein Ricci generalized symmetric Riemannian space: $R_{ij,k} = R_{ik,j}$ and $R_{ij} \neq \kappa g_{ij}$, that admits a nontrivial geodesic mapping was given by Sobchuk [155]. See the paper of Petrov [131] for projective mappings of Einstein spaces.

Rozenfel'd and Gorbatyi ([85,138]) investigated geodesic mappings of conformally flat and equidistant spaces [145]; see also [113]. Geodesic mappings of special Riemannian spaces "as a whole" were considered by Sinyukova ([151,152]), Kiosak [103], and Mikesh [114]. See [116] for the "degree of mobility" of special Riemannian spaces under nontrivial geodesic mappings.

Sanini [356] investigated projective variations of the Riemannian metric for which the Ricci tensor is a linear combination of the metric tensor and its variation. A simply connected compact Riemannian manifold that admits such a nonhomothetic variation is isometric to a sphere.

Let ∇ and ∇' be projectively equivalent symmetric affine connections ($\nabla'_X Y = \nabla_X Y + w(Y)X + w(X)Y,\ X, Y \in TM$) on the manifold M, and let W be the projective curvature tensor of the connections ∇ and ∇'. If $\nabla'_X W = \mu'(X)W$ and $\nabla_X W = \mu(X)W$, then either $W =$ or $\mu' = \mu - 4w$ (Smaranda [373]).

The projective transformation $\nabla \to \nabla'$ preserves the tensor curvature if and only if $(\nabla_X w)Y - w(X)w(Y) = 0$ (Constantinescu [231]).

Two semisymmetric affine connections are projectively equivalent if and only if their associated symmetric affine connections have this property (Un Hong Yun [395]). See Mikesh and Berezovskii [115] for geodesic mappings from affine connection spaces onto Riemannian spaces.

A projective transformation $T : \nabla \to \nabla'$ that transforms every trajectory of any motion on $M^n(\nabla)$ into the trajectory of a motion on $M'^n(\nabla')$ is called an *index transformation* if the "time" t in M^n is connected with the "time" t' in M'^n by $dt/dt' = \rho \neq 0; \rho$ is called the *index*, and $w = \ln \sqrt{|\rho|}$ is called the *logarithmic index* of T. The properties of such transformations were investigated by De Cicco and Anderson [234]. The concept of Appell index transformations was introduced and a connection was established between such transformations and dynamic systems in M^n and M'^n.

There is a paper of Yano and Ishihara (see [418]) devoted to projective and affine mappings of spaces with symmetric affine connections, where, in particular, conditions are obtained under which a geodesic mapping of a compact connected Riemannian manifold (M, g) into a manifold (N, g') is a homothety or an affine mapping of constant rank. Bachmann and Elster [205] investigated affine mappings of linear subspaces of the Euclidean space $\mathbb{R}^n$. For a classification of the projective mappings on the plane and in space, see the paper [168] of Tsivitsinskii.

If an infinite-dimensional Riemannian space admits a nontrivial geodesic mapping onto an infinite-dimensional locally symmetric space with a torsion-free affine connection, then it is a space of constant curvature (V. E. Fomin [166]).

The distortion theorems proved by Har'El Zvi [274] establish the dependence of the coefficients of diminishing volume and distance for a projective mapping $M \to M'$ with a bound on the Ricci curvatures in the n-dimensional complete Riemannian manifolds M and M'. See [344] for the properties of projective immersions.

If M and M' are pseudo-Riemannian manifolds and $f : M \to M'$ is a geodesic mapping, then the induced mapping of the tangent bundles TM and TM' endowed with the Sasaki metrics can be a geodesic mapping only if the affine connection is preserved (Piliposyan [134]).

See Shandra [173] for geodesic mappings of singular Riemannian spaces with a Levi-Civita pseudo-connection $(\Gamma^i_{jk}, H^i_{jk})$ that preserves the structure of the space.

5.3. For the first time, the problem of determining the Riemannian spaces admitting continuous transformation groups that preserve the geodesics was considered by Sophus Lie for the case of two-dimensional surfaces.

After the appearance of the paper of Levi-Civita ([310], see Sec. 5.1), Fubini published the paper [262] in which the problem of geodesic maps was considered from the point of view of groups.

Having criticized Lie's method which, in Fubini's opinion, could not be used for a general solution of the problem, Fubini developed his own approach based on the Lie differentiation tool. Although the term "Lie derivative" arose later, in the paper [262] of Fubini the Lie derivative of a connection object was, apparently computed for the first time and used effectively. In this paper, a number of basic results, such as the theorem on the dimension of the isometry subgroup of the homothety group (see below), which is traditionally connected with the name of Knebelman [302], is used.

In [262], Fubini began to study the Riemannian spaces V^n, $n \geq 3$, admitting groups of projective transformations that are larger than the homothety groups, and this was continued by Solodovnikov half a century later. The relevant results, including Fubini's deductions, are set out in detail in the paper [156] of Solodovnikov.

If V^n admits a projective motion X that is not a similarity transformation ($L_X g \neq cg$), then its metric reduces to (5.2). The linear element

$$\overset{*}{ds}{}^2 = \sum_{\alpha=1}^{p} {\prod_{\beta}}' |f_\beta - f_\alpha| dy^{\alpha^2}$$

is called the linear element associated with (5.2). The metric (5.2) is called the Levi-Civita metric of basic type if $\overset{*}{ds}{}^2$ does not have a constant curvature; otherwise, it is called an exceptional Levi-Civita metric.

The projective motions of Levi-Civita metrics of basic type were investigated mainly by Fubini, except for the case $p = 2$, which was completely investigated by Solodovnikov. Every exceptional Levi-Civita space is $V(K)$. The theory of $V(K)$ spaces was developed by Solodovnikov ([156,158]) in connection with the study of projective transformations of Riemannian spaces. A generalization of the theory of $V(K)$ spaces to pseudo-Riemannian spaces was given by Kruchkovich [104,107].

Recall that a semireducible Riemannian space with the metric

$$ds^2 = ds_0^2 + \sum_{\gamma=1}^{t} \overset{\gamma}{\sigma}\, ds_\gamma^2, \tag{5.18}$$

where ds_p^2 $(p = 0, 1, \dots, t)$ are independent metrics depending only on the variables x^{i_p} and $\overset{\gamma}{\sigma}$ are positive functions of x^{i_0}, is called a $V(K)$ *space*, and representation (5.18) is called a K-decomposition of the metric ds^2 if the associated metric

$$\overset{*}{ds}{}^2 = ds_0^2 + \sum_{\gamma=1}^{t} \overset{\gamma}{\sigma}\, dy^{q+\gamma^2}$$

is of constant curvature K.

Using the necessary and sufficient test obtained by him for the spaces $V(K)$, Solodovnikov determined the metrics of the Riemannian spaces $V(K)$ and solved the problem of the maximal projective groups in these spaces ([21]; see also [23]).

The deductions of Fubini and Solodovnikov are substantially based on the assumption that the metrics considered are positive definite. If we take a prescribed signature as a condition, this complicates the problem considerably and requires a substantially new approach to its solution. One of these approaches, which involves the consideration of the algebraic structure of the Lie derivative $L_X g$ in the direction of an infinitesimal projective transformation X, was proposed by author as a basis for classifying four-dimensional Lorentz spaces (gravitational fields) in terms of the Lie algebras of projective and affine motions [1–3].

The infinitesimal projective transformations of three-dimensional Lorentz spaces were studied by Zhukova [12]. The approach she used is based on expanding the three-dimensional metrics of Petrov with corresponding geodesics with respect to a small parameter [16] and cannot be applied in the general case.

The remaining research in the field under consideration is devoted either to general problems of the theory of projective transformations or to special projective vector fields or to projective transformations of special Riemannian spaces (see [7]).

5.4. In the author's papers ([19–30], etc.), all pseudo-Riemannian manifolds L^n of Lorentz signature and dimension $n \geq 3$ that admit Lie algebras of infinitesimal projective (in particular, affine) transformations greater than the Lie algebras of infinitesimal homotheties were determined. For each of them, the maximal projective and affine Lie algebras, together with its homothetic and isometric subalgebras, were defined. The metric is constructed for these manifolds, and also the basis vector fields and the structure equations for the maximal projective and affine Lie algebras are given.

All Lorentz manifolds L^n that admit nonhomothetic projective Lie algebras can be divided into five disjoint classes, as is shown in Table 1.

Table 1. Classification of pseudo-Riemannian manifolds L^n, $n \geq 3$, of Lorentz signature with respect to maximal non-homothetic projective Lie algebras (Aminova [30]).

I	II				III	IV
Class	Admissible transformations				Maximal projective Lie algebra (when $n = 4$)	Dimensions of subalgebras (when $n = 4$)
	i.m.	h.m.	a.m.	p.m.		
Ordinary h-spaces	////	////		////	P_7 p.m.	$\underset{p}{r} = \underset{h}{r} + 1$
K-spaces of non-constant curvature ($K \neq 0$)	////			////	P_8 p.m.	$\underset{p}{r} \leq \underset{i}{r} + 3$
K-spaces of non-constant curvature ($K = 0$)	////	////	////	////	A_{10} a.m.	$\underset{p}{r} \leq \underset{a}{r} + 1$ $\underset{a}{r} \leq \underset{h}{r} + 3$
Spaces of constant curvature $K \neq 0$	////			////	P_{24} p.m.	$\underset{p}{r} = \underset{i}{r} + 14$
Spaces of constant curvature $K = 0$	////	////	////	////	P_{24} p.m.	$\underset{p}{r} = \underset{a}{r} + 4 =$ $\underset{h}{r} + 13 =$ $\underset{i}{r} + 14.$

The shaded boxes in column II show that the corresponding class of spaces admits the indicated type of transformations. The symbols $\underset{i}{r}$, $\underset{h}{r}$, $\underset{a}{r}$, and $\underset{p}{r}$ in column IV denote the dimensions of the maximal isometric, homothetic, affine, and projective Lie algebras. For clarity, some information about the case $n = 4$ is given in columns III and IV.

It can be seen from the table that K-spaces of non-constant curvature are essentially different from ordinary h-spaces both in the character and the structure of the maximal projective Lie algebras. They have the greatest projective mobility among all other Lorentz spaces of non-constant curvature, and their properties make them very close to the spaces of constant curvature.

The distinguishing feature of K-spaces is the existence of families of totally geodesic surfaces of constant curvature K in them. On the other hand, every K-space with a non-affine projective Lie algebra admits a concircular vector field $D\psi$ of special form:

$$\nabla^2\psi + K\psi g = 0,$$

which is connected with a solution ψ of the Klein–Gordon wave equation

$$\psi + m^2\psi = 0, \quad m^2 = 4||,$$

which describes the scalar, vector, spinor, and other physical fields.

The projectively mobile Lorentz metrics can serve as an ansatz in constructing field-theoretic models, and the infinitesimal projective transformations they admit can serve as generators of mechanical and field conservation laws in these theories.

The structure constants of projective Lie algebras and the Killing forms defined from them can be used for defining Lagrangians of gauge fields, and left-invariant metrics and left-invariant vector fields in Lie groups that are constructed from the structure constants by using the standard procedure could become the basis of generalized Kaluza–Klein theories [389].

The first integrals of equations for geodesics that are connected with projective motions can be used for studying geodesics that define a large-scale space-time structure [167].

5.5. Petrov's theorem [130] on the three types of gravitation fields [19] and the development by him and his students of a classification of gravitation fields with respect to symmetry groups in the form of isometric (Petrov, Kaygorodov, 1963), conformal (Bilyalov, 1963), projective, and affine (Aminova, 1971) transformations became the basis of the program of searching for exact solutions of Einstein's equations in the general relativity theory. All this formed the beginning of a number of papers in which physical properties of material systems and also those of gravitation, electromagnetic, and other physical fields and transferring interactions are determined by automorphism groups of various objects of geometric or physical nature.

Most of these papers are connected with isometries and conformal transformations of space-time manifolds. At present, homotheties and affine transformations are more and more often included in this scheme.

5.5.1. Isometries. Finley and Plebanski [254] obtained exact solutions for a spinor field in a class of complex space-times with self-dual conformal curvature and Killing vector fields (see also Finley and Plebanski ([255, 256] and Boyer and Finley [213]).

A static spherically symmetric metric with six independent Killing vectors corresponds to a homogeneous mass distribution over the whole space (Qadir and Ziad [352], Bokhari and Qadir [211]).

A homogeneous cosmological model is a four-dimensional space-time that admits a 3-parameter group of motions with spacelike three-dimensional orbits and satisfies the Einstein equations with the stress-energy of a perfect fluid. By using a qualitative theory of homogeneous cosmological models that they developed, Novikov and Bogoyavlenski ([76, 120]) drew, in particular, the conclusion that the properties of a system at arbitrary early stages of the universe's evolution are irreversible; the theory posed the problem on typical initial states of the expanding universe in a new fashion. (See also Mac Callum [318]).

The paper by Osinovski and Teslenko [127] is devoted to the global analysis of Einstein spaces with two-dimensional Abelian group of motions. Siklos [364] investigated global properties of algebraically special Einstein spaces with three-parameter groups of motions. Briginshaw [216] proved that the associated Lorentz group coincides with a causal group, that is, a transformation group that preserves the causality relations between points of the space-time. Ashtecar and Schmidt [202] used the results of the global research in isometries to describe global properties of solutions to Einstein equations that satisfy asymptotically conditions on a null infinity. A survey of the known results (before 1975) in the geometry of four-dimensional manifolds as a whole is contained in the paper of Flaherty [257].

See also Ashtecar and Xanthopoulas [203] for groups of isometries in asymptotically flat space-times. Ashtecar and Magnon Ashtecar [201] proposed a new method for researching the structures of Lie algebras of vector fields. In their opinion, this method can be successfully used in general relativity.

Harris and Zund [280] investigated Osinovski's metrics with motion groups of large dimension and the metrics of homogeneous spaces, which were found by Kruchkovich in 1963.

Daishev ([86, 87]) considered isometric motions of a perfect fluid. Ivano's paper [98] is devoted to Killing vector fields in space-times with non-linear scalar fields (see also Ivanov [99], Ivanov and Chervon [100]).

A number of papers are devoted to Lie groups (Lie algebras) of motions of special Lorentz spaces, including vacuum Einstein spaces with special geodesic congruences (Kerr and Debney [298]), Peres space-times (Navez [337]) and Kasner space-times (Harris and Zund [278]), stationary axial symmetric space-times with perfect and non-perfect fluid (Kitamura [301], Harrison and Stevens [281]), n-dimensional quasi-orthogonal space-times (Volkov [79]), space-times with a special structure of the curvature tensor (Ferzaliev [164]), and so on. See Mitzkevich and Senin [118] for the topology and isometries of the de Sitter universe.

"Internal isometries" acting on 3-dimensional subspaces of a space-time were studied by Szafron [371] and Martinez and Sanz [324].

Spinor fields ψ having the same symmetry group ($\delta\psi \equiv 0$) as a given gravitational field with isometry group were considered by Henneaux [285]. Kolassis [305] studied the action of Killing vector fields on a neutrino field that satisfies the system of the Einstein–Weyl equations. See Sibgatullin [139] for a neutrino electrovacuum with Abelian group of motions.

The papers of Debney [238] and Duggal [242] (a joint treatment of the Einstein–Maxwell equations and Killing vector fields), of Shushpanov [189] (the derivation of some laws of electrodynamics and optics from a Lie algebra of space-time isometries), of Ernst and Plebanski [250] (Killing vector fields and a formulation of the Einstein–Maxwell theory in terms of the complex Ernst potentials), of Pascua [348] (finding solutions to Einstein–Maxwell equations in a space with 7-dimensional group of isometries), Wooley [408], and others are devoted to isometries and electromagnetic fields.

Exact solutions in the unified non-symmetric Einstein and Bonnor field theories for spaces with isometries were found by Aminova and Monakhov [65].

See Collinson and Smith for isometries of a black hole [230]. Volkov, Sorokin, and Tkach [78] discussed the Kremmer–Sherk mechanism of spontaneous compactification in which gauge fields are compared with Killing vector fields of a symmetric space.

See Maia [319] for the unification of internal and external (space-time) symmetries in the framework of the isometry group of the flat space in which a space-time is locally immersed.

Torres [391] used null strings and their complexification to determine Killing, homothetic, and conformal fields in algebraically special space-times. The statement of a dimensional reduction in $d+n$-dimensional space-times that admit n commutating Killing vector fields was proposed by Mansouri and Witten [320]. Krisch [308] used Killing vector fields to generate new exact solutions of the Einstein equations.

Kersten and Martini [299] have shown that a condition of self-duality of SU(3) invariant Yang–Mills theory can be written as the equation of harmonic mapping of a space-time into an 8-dimensional Riemannian space and found 16 Killing vector fields in this space. See Halpern [273], Harnad and Pettitt [275], and Smrz [369] for Killing vector fields and gauge theories of gravity.

Moncrief [333] showed that a space-time is stable with respect to the linearization if and only if it does not admit global Killing vector fields.

Arcidiacono [200] used the isometry group of the de Sitter space as the fundamental group to construct a "projective relativity" in the Klein spirit.

Pommaret [350] established relations between the basic equations of elasticity theory, thermodynamics, and electrodynamics on one hand and the non-linear Spenser series $G_1 \subset G_2 \subset G_3$ on the other hand, where G_1 is the Poincare group, G_2 is the Weyl group, and G_3 is the conformal group in the Minkowski space.

Notions of "Killing" vector fields in the gauge supersymmetry theory and of the "maximal supersymmetry" that is related with these fields were introduced by Kannenberg [294]. He showed that some properties of the ordinary Killing vector fields can be extended to the fields newly defined.

The pioneering papers of Aminova and Mochalov ([66, 67]) are devoted to the development of group invariant methods in supergravity; the supersymmetry is considered in these papers as an automorphism of a supergeometric structure, in particular, as an infinitesimal supertransformation (vector superfield) that leaves the metric of a superspace invariant. The metric itself is defined as an invariant of an appropriate supergroup of supertransformations, in the spirit of Klein's approach, where a symmetry or a transformation group is a fundamental notion that defines the geometry of the space.

See Karger [295] for the use of isometries of three-dimensional space in studying the motions of a robot-manipulator.

See also Ehrlich [247], Held ([283], [284]), Ihrig and Sen [290], Israelit [292], Kauffmann [297], Martin [323], Mensky [332], Nedita [338], Sobczyk [370] and Szabados [372].

5.5.2. Conformal transformations. A method for obtaining exact solutions of the Einstein equations with stress-energy of a viscous fluid by using a conformal transformation of the known vacuum solution has been proposed by Carot and Mass [220]. Depending on the sign of cosmological constant, a vacuum solution of the Einstein equations with a cosmological term and properly conformal vector field is either de

Sitter or anti de Sitter (Garfincle and Tian [252]). See Herrera and Ponce [287] for spherically symmetric solutions of the Einstein equations with stress-energy of a perfect fluid and conformal vector field (see also Herrera, Jimenez, Leal and Ponce [286], as well as Cao [223].

Maartens and Mason ([317, 325]) used a conformal vector field for studying kinetic and dynamic properties of an anisotropic fluid. In particular, they showed that a special conformal vector motion X; $L_X g = 2\psi g$, $\nabla^2\psi = 0$, i.e., a concircular motion (See Sec. 4.1.2), preserves the fluid trajectories (see also Mason and Tsamparlis [326]). See Wainwright and Yaremovicz [404] for solutions of the Einstein–Maxwell equations with perfect fluid as a source in the presence of a conformal vector field.

If a conformally symmetric space-time admits a conformal motion, then it is of Petrov type O or N and describes gravitational waves with parallel rays under some conditions (Sharma [361]). Sinzinkayo and Demaret [367] investigated subgroups of the conformal transformation group of a conformally flat space and found solutions of the Einstein equations and the Einstein–Yang–Mills equations with SU(2) Yang–Mills field as a source of the gravitational field. (See also Branson [214]).

Gauge theories with conformal transformation group of the Minkowski space were considered by Lord and Goswami ([314,315]). Katzin and Levine [296] showed that a charge conservation law in a conformally flat space can be obtained by using a conformal motion and gauge transformations. Moreshi and Sparling [334] used conformal symmetries in studying generalized Kaluza–Klein theories. Pessa [349] constructed a unified theory of the electromagnetic and gravitational fields that is based on the conformal group.

Hussin and Sinzinkayo [289] used space-time conformal vector fields to determine constants of motion of a system of charged particles of spin 1/2 interacting with an external electromagnetic field. They gave an example of the magnetic monopole. Burdet, Patera, Perrin, and Winternitz [218] found all the subalgebras of the 10-dimensional Lie algebra $\mathrm{opt}(3,1)$ and all the closed subgroups of the "optic" group $\mathrm{Opt}(3,1)$, which is a subgroup of the group of conformal motions in the Minkowski space and is used in conformally invariant physical theories.

The paper of Legare [309] is devoted to the determination of subgroups of the conformal group $O(3,1)$ that result in a reduced supersymmetry in the framework of $N = 1$ supersymmetric Yang–Mills theory. Barut [206] treated two aspects of the conformal group as a group of kinematic symmetries of space-time and as a dynamic group acting on internal coordinates of the system being in the rest.

See also Beckers, Harnad, Perroud, and Winternitz, [207]; Cahen [219], Havas and Plebanski [282], and Margulescu [322].

We single out applications of conformal transformation groups to the research in differential equations (Ibragimov [96], Chupakhin [171], Margulescu [321]) and also to the star dynamics and the seismics (Golubyatnikov and Pestov [84]).

5.5.3. Homotheties. A Ricci flat space-time that admits a time-like proper homothetic motion with an orthogonal family of hypersurfaces is a Minkowski space (Sigal [363]).

Infinitesimal homotheties in vacuum Einstein spaces were studied in detail by Halford and Kerr ([268, 269]); see also MacIntosh ([328, 329]).

Eardley and others ([245, 246]) studied the geometry and dynamics of space-homothetic cosmological models, i.e., space-time manifolds with homothety group H^3 transitively acting on space-like hypersurfaces. Exact solutions of the Einstein–Maxwell equations for a charged perfect fluid were found by Wainwright and Yaremovicz [405] provided that infinitesimal homotheties exist in a space-time. Collinson [228] proved that a Petrov-type N metric with rotating geodesic rays does not admit a Lie algebra of infinitesimal homotheties of dimension $r > 2$.

Moschetti [335] considered a space-time with perfect fluid as a source of the gravitational field in which a shock wave is propagated. He found necessary and sufficient conditions under which a linear combination of tangent and normal vectors to a shock-wave front is a vector of a homothetic motion.

Hall [271] proved that fixed points of a 1-parameter Lie group of homotheties are connected with space-time singularities. See Berger [210] for vacuum solutions of the Einstein equations and infinitesimal homotheties.

5.5.4. Affine and projective transformations. Affine transformations of the Robertson–Walker space-times with the linear element

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

were studied by Bedran and Lesche [208] and also by Maartens [316]. Collinson [229] proved that a nonempty Robertson–Walker space-time ($R_{ij} \neq 0$) admitting a proper affine motion X ($L_X g \neq cg$) is static. Hall and Costa [272] considered fixed points of affine transformations in space-times and their holonomy groups.

In the framework of the geometry of superspaces and supersymmetric theories, affine transformations were researched by Tucker [393] and also by Ne'eman and Sherry [339].

See Smrz [368] for an affine group deformed by its subgroup and the Einstein equations.

Iwai [293] gave applications of lifts of infinitesimal projective transformations in a tangent bundle to the Newton dynamics and relativity.

The conditions under which there exists a 1-parameter almost projective group in a reducible gravitational field result in exact solutions of the Einstein–Maxwell equations in the vacuum (electrovacuum spaces) (Aminova [18]).

5.5.5. Curvature collineations, Ricci collineations, and Maxwell collineations. A vector field X on a space-time manifold is called a curvature collineation, Ricci collineation, or Maxwell collineation if a local group of local transformations generated by this field preserves the curvature tensor ($L_X R^i_{jkl} = 0$), the Ricci tensor ($L_X R_{ij} = 0$), or the tensor of electromagnetic field ($L_X F_{ij} = 0$), respectively.

In the vacuum space-time of Petrov type N, the curvature collineation $X = v^i \frac{\partial}{\partial x^i}$ generates the conservation law $[\Phi(v_j p^j)\mu p^i]_{ij} = 0$, where μ is a scalar, Π is an arbitrary function, and p^i is a vector whose direction coincides with a principal null direction of the Ricci tensor (Collinson [226]). By using the tetrad methods, Chinea[9] found curvature collineations in a vacuum space-time of Petrov type N.

Let $X_\alpha = D\varphi_\alpha$, $\alpha = 1, \dots, r$, be parallel vector fields in a Riemannian space, and let Φ_α be functions of $\varphi_1, \dots, \varphi_r$. Levine [311] studied the conditions under which the $D\varphi_\alpha$ span the Lie algebra of curvature collineations. Tariq and Tupper [387] proved that in Einstein spaces ($R_{ij} = \frac{1}{4} g_{ij}$) and in electrovacuum spaces of Petrov types N and O with nonnull electromagnetic field, any curvature collineation is a conformal motion.

Conformal curvature collineations ($L_X g_{ij} = 1\psi g_{ij}$, $L_X r^i_{jkl} = 0$) in gravitational fields created by a perfect fluid lay the conditions on the fluid parameters. Duggal and Sharma [243] found these conditions. Singh and Shri Ram [366] studied curvature collineations of cosmological models. Shri Ram and Pandey [353] determined the curvature collineations for Kazner and Narlikar metrics. (See also Hall [270] and MacIntosh and Halford [331].)

Prasad [351] used Ricci collineations for determining the properties of a heat-conducting, viscous, compressible charged fluid with constant magnetic permittivity and electric conductivity.

Requiring that a space-time with the linear element

$$ds^2 = e^{2P(x^4)} dx^{1^2} + e^{2Q(x^4)} dx^{2^2} + 2dx^3 dx^4$$

admit a Ricci collineation or a Maxwell collineation, Singh and Sharma [365] found solutions of the Einstein–Maxwell equations in this space. The Einstein–Maxwell equations provided that there exists the conformal Ricci collineation, which was treated Faridi [251]. See also Aulestia and others [204].

Davis and Oliver [237] considered collineations defined by the conditions $g^{ij} L_X R_{ij} = 0$ and $\nabla_k (L_X \Gamma^i_{jl}) = 0$ in gravitational fields created by a perfect fluid, dust matter, and so on together with Ricci collineations and curvature collineations.

A survey of group-invariant methods that are used for finding solutions of the Einstein equations is given in a paper of MacIntosh [330].

[9] See F. J. Chinea, *Class. Quantum Gravity* (1986).

In conclusion, we give a brief survey of papers that are carried out by using computer methods.

By using computer programs, Harris and Zund ([276, 277]) verified 18 classes of exact solutions of the Einstein equations with isometries that were found by Kaygorodov (see [133], Chap. IV) and corrected the results.

Reboucas and Aman [354] used a system of algebraic calculations to treat Gödel's metric

$$ds^2 = (dt + H(x)\,dy)^2 - D^2(x)\,dy^2 - dx^2 - dz^2.$$

When $m^2 = 4\Omega^2$, this metric is conformally flat and admits a 7-dimensional Lie algebra of infinitesimal isometries, while for $m^2 - 4\Omega^2 \neq 0$, it is of Petrov type D and admits a 5-dimensional Lie algebra of isometries ($m^2 \equiv D^n/D$ and $2\Omega \equiv H'/D$).

The paper of Cohen, Leringe, and Sundblad [225] contains a survey and a comparative analysis of programs composed for symbolic calculations in general relativity. See also Bona [212] and MacCrea [327].

See Aminova and Kalinin [61–64] and Aminova and Zuev ([59, 60]) for holomorphic-projective and quaternion-projective transformations and the geometry of quantum systems with Kähler and quaternion structures.

6. Groups of Projective Transformations of Two-Dimensional Pseudo-Riemannian Manifolds (Lie Problem)

6.1. Surface of revolution as a dynamic model of a Lagrangian system with one degree of freedom.

6.1.1. As was already mentioned in Sec. 5.3, the problem of determining two-dimensional Riemannian spaces admitting continuous groups of projective transformations was considered by Sophus Lie. However, Fubini wrote in the preface to [262]: "the famous mathematician did not succeed in solving this problem" (which Fubini called the "Lie problem"). Having mentioned the paper of Königs [306], in which surfaces V^2 admitting the first quadratic integrals of geodesic equations were studied in detail, Fubini slightly touched upon the Lie problem, having written Eq. (3.19) for the Liouville metric

$$(U(u) - V(v))\,(du^2 + dv^2),$$

and outlined a program for treating the integrability conditions of these equations.

This section contains a survey of results obtained by the author ([34, 35, 43] and others) and is devoted to the solution of the Lie problem, i.e., to finding all two-dimensional pseudo-Riemannian spaces M^2 admitting nonhomothetic projective motions and the maximal projective and affine Lie algebras for each of them, together with the homothetic and isometric subalgebras. The metric is constructed for these manifolds, and also the basis vector fields and the structure equations for the maximal projective and affine Lie algebras are given in appropriate coordinates.

The solution of this problem reduces to the integration of Eqs. (3.23) and (3.24) on M^2. In Theorem 6.1 (Sec. 6.2), two-dimensional spaces that admit non-trivial solutions $h \neq cg$ of Eq. (3.24) are determined. They are h-spaces of types $\{11\}$ (Liouville surfaces), $\{\overset{*}{1}1\}$, and $\{2\}$.

Integrating Eq. (3.23) for every h-space, we find the nonhomothetic projective motions of all possible types. To determine the maximal projective Lie algebras, one should find all projective motions for each of the h-spaces, i.e., one should find general solutions of Eqs. (3.23) and (3.24) for these spaces.

In Secs. 6.3.1 and 6.3.2, the problem posed is solved for h-spaces of types $\{11\}$ and $\{\overset{*}{\mathrm{II}}\}$. When considering Liouville surfaces, we use the following property.

Königs' theorem ([306]). *Every two-dimensional (pseudo-)Riemannian surface admitting three independent quadratic first integrals of geodesic equations is surface a of revolution.*

Since in the proof of the theorem given by Königs for Riemannian spaces, the property of a fixed signature of the metric form written as (6.37) is not used, the assertion of this theorem is extended to pseudo-Riemannian spaces.

The final part of Sec. 6.3 is devoted to h-spaces of type $\{2\}$, and in Sec. 6.4 the final result on the classification of two-dimensional pseudo-Riemannian manifolds by Lie algebras of projective motions (Theorem 6.7) is stated.

In Sec. 6.5, we present some properties of connected projective Lie groups of two-dimensional pseudo-Riemannian manifolds of nonconstant curvature.

6.1.2. We consider the linear element

$$ds^2 = \Phi^2(x^1)\Big(e_1 dx^{1^2} + e_2 dx^{2^2}\Big), \tag{6.1}$$

where e_1 and e_2 are equal to ± 1, while Φ is a function of x^1. After the change of variables $r = \int \Phi dx^1$, $\varphi = x^2$, form (6.1) becomes

$$ds^2 = e_1 dr^2 + e_2 \eta^2(r) d\varphi^2$$

and if $e_1 = e_2 = +1$, it defines a linear element of the *surface of revolution* given by the equations

$$x = \eta(r)\cos\varphi, \quad y = \eta(r)\sin\varphi, \quad r = \int \sqrt{1 - \eta'^2}\, dr$$

in the three-dimensional Euclidean space with Cartesian coordinates x, y, z ([122], p. 132). The term "surface of revolution" is extended to spaces with the linear element (6.1) of arbitrary signature.

When $e_1 = -e_2 = 1$, the form ds^2 defines the metric of a two-dimensional space-time, which can be considered as linear element of the surface embedded by the equations

$$x = \eta(r)\cosh\varphi, \quad y = \int \sqrt{1 - \eta'^2} dr, \quad z = \text{const}, \quad \tau = \eta(r)\sinh\varphi \tag{6.2}$$

in Minkowski space with the interval

$$\overline{ds}^2 = dx^2 + dy^2 + dz^2 - d\tau^2.$$

When $\eta = r = \text{const}$, (6.2) defines a hyperbolic motion in the special relativistic dynamics [141].

Metric (6.1) admits an isometry (Killing vector field) $X = \frac{\partial}{\partial x^2}$; with this field, one associates the linear integral

$$e_2 \Phi^2 \dot{x}^2 = l \equiv \text{const} \tag{6.3}$$

of the geodesic equation

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 \quad \left(\dot{x}^i \equiv \frac{dx^i}{dt}\right),$$

where t is an *affine parameter*, i.e., the arc length or the canonical parameter for null-geodesics when $e_1 e_2 = -1$ and Γ^i_{jk} are the Christoffel symbols of metric (6.1). Among them, only

$$\Gamma^1_{11} = \Gamma^2_{12} = -e_1 e_2 \Gamma^1_{22} = \frac{\Phi'}{\Phi}$$

are nonzero (the dash denotes the derivative). The curvature tensor of metric (6.1) is

$$R^i_{jkl} = \rho(\delta^i_k g_{jl} - \delta^i_l g_{jk}), \quad \text{where} \quad \rho = -\frac{e_1}{\Phi^2}\Big(\Phi'\Phi\Big)'.$$

If we set $\Phi = e^{x^1}$ and $e_1 = e_2 = +1$, then ρ vanishes and form (6.1) and the linear integral (6.3) in coordinates r, φ become

$$ds^2 = dr^2 + r^2 d\varphi^2, \quad r^2\dot{\varphi} = l \equiv \text{const}.$$

The first relation defines the linear element of the plane in the polar coordinates, and the second relation defines the conservation law of the kinetic moment for the motion in the field of central force ([71], p. 30).

Equations for geodesics on the surface of revolution (6.1) are

$$\ddot{x}^1 + \frac{\Phi'}{\Phi}\left(\dot{x}^{1^2} - e_1 e_2 \dot{x}^{2^2}\right) = 0, \quad \ddot{x}^2 + 2\frac{\Phi'}{\Phi}\dot{x}^1\dot{x}^2 = 0.$$

The second relation is a consequence of the linear integral, while by using the integral

$$\Phi^2\left(e_1\dot{x}^{1^2} + e_2\dot{x}^{2^2}\right) = \epsilon,$$

where ϵ is equal to 0 for null geodesics and ± 1 for nonnull geodesics, it is possible to write the first relation as

$$\ddot{x}^1 = -\frac{\partial U}{\partial x^1},$$

where

$$U = \frac{e_1}{2\Phi^2}\left(e_2 l^2 \Phi^2 - \epsilon\right) + E \quad (E = \text{const}).$$

This equation describes the motion of a Lagrange system with one degree of freedom, the Lagrange function is

$$L = \frac{1}{2}\dot{x}^{1^2} - U(x^1),$$

and total energy is

$$E = \frac{1}{2}\dot{x}^{1^2} + U(x^1);$$

they do not change when a particle moves. Under certain conditions, this motion is oscillating (*one-dimensional oscillator*) with period

$$T = \sqrt{2}\int_{x_1^1}^{x_2^1}\left(E - U(x^1)\right)^{-\frac{1}{2}} dx^1,$$

where x_1^1 and x_2^1 are the roots of the equation $U(x^1) = E$ ([108], p. 38).

Therefore, with every Lagrangian system with one degree of freedom, potential energy $U(x^1)$, and total energy E, one can associate the surface of revolution (6.1) with

$$\Phi^2 = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 8e_1e_2l^2(U - E)}}{4e_1(U - E)} \quad (\epsilon = 0, \pm 1),$$

whose geodesic lines are graphs of the motion of the system, and the first integrals of the geodesic equation define the conservation laws of this system. In view of this, one may consider the surface of revolution as a dynamical model of a mechanical system with one degree of freedom. Since a projective motion gives a first quadratic integral of the system; it defines a conserving quantity, which remains constant along every geodesic, i.e., it gives a mechanical conservation law.

We note that if $e_1 e_2 = -1$, then denoting $m = l^2$ and setting $x^1 = x$, $x^2 = c\tau$, we find that

$$\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + U(x) = E,$$

where

$$U(x) = -\frac{e_1\epsilon c^2\Phi^2}{2} \quad \text{and} \quad E = \frac{mc^2}{2}.$$

Consequently, geodesic lines in the two-dimensional world (M, g), where M is a manifold with the topology of $\mathbb{R}^2$ defined by the coordinates x, t $(-\infty < t < \infty,\ -\infty < x < \infty)$ and the metric

$$g = e_1\Phi^2(x)(dx^2 - c^2 d\tau^2) \quad (e_1 = \pm 1),$$

are the graphs of the motion of a Lagrangian system with one freedom degree and the Lagrangian

$$L = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + \frac{1}{2}e_1\epsilon c^2\Phi^2.$$

If the motion is finite, then the period $T(m)$ of the particle oscillations is

$$T(m) = \frac{2\sqrt{m}}{c} \int_{x_1(m)}^{x_2(m)} \frac{dx}{\sqrt{m + e_1 \epsilon \Phi^2(x)}},$$

where $x_1(m)$ and $x_2(m)$ are the roots of the equation $m + e_1 \epsilon \Phi^2(x) = 0$.

It is known that a mechanical particle-analogue is used for description of two-dimensional static solitons (Christ, Lie, Friedberg, and Coleman; see [136]). If we consider the variable x as a "time" and σ as a coordinate of a point particle of unit mass, then the static soliton equation

$$\sigma''(x) \equiv \frac{\partial^2 \sigma}{\partial x^2} = \frac{\partial u}{\partial \sigma}(x)$$

describes the motion of the particle-analogue. By virtue of the boundary conditions ($U(\sigma) \to 0$, $\sigma' \to 0$ as $x \to \pm\infty$), the total "energy" of the motion E is preserved when x changes, and it is equal to zero [136]. Replacing x^1 by σ and t (arc length or canonical parameter of a geodesic) by x and also $U(x^1)$ by $-U(\sigma)$ in the above formula, we obtain $E = \sigma'^2/2 - U(\sigma) = 0$. Consequently, every geodesic $x = x^1(t) \equiv \sigma(x)$ on the surface of revolution defines a static soliton provided $2U\left(x^1(t)\right) = e_1\left(e_2 l^2 - \epsilon\Phi^2\right)\Phi^4 \to 0$ as $t \to \pm\infty$ (the geodesic is complete).

We indicate one more possible application of the Lie problem. In the theory of two-dimensional self-gravitating nonlinear σ-models, the group symmetries (infinitesimal isometries and homotheties) of a two-dimensional (Riemannian) chiral space S^2 are used for obtaining solutions of instanton and meron types ([169, 170, 224]). The inclusion of a complete projective Lie algebra into consideration along with an extension of the geometry of the chiral space will reduce to generation of new exact solutions of the field equations.

6.2. Integration of Eisenhart's equations. Two-dimensional h-spaces.

6.2.1. Eisenhart's equations in a skew-normal frame. 1. Let M be an n-dimensional Lorentz manifold with metric g and Levi-Civita connection ∇, and let h be a differentiable symmetric bilinear form in M. We say that h satisfies Eisenhart's equation if the condition

$$\nabla h(Y, Z, W) = 2g(Y, Z)W\varphi + g(Y, W)Z\varphi + g(Z, W)Y\varphi \tag{6.4}$$

is satisfied for some 0-form φ and arbitrary vector fields Y, Z, and W on M.

When $h = cg$, Eq. (6.4) implies $c = \text{const}$. If, in M, there exists a nontrivial solution $h \neq cg$ of Eisenhart's equation, then M is called an *h-space*.

Let the differentiable symmetric bilinear form a given on M have a constant *Segre characteristic*

$$\chi = \left\{(\overset{1}{m}_1 \cdots \overset{1}{m}_{s_1}) \dots (\overset{k}{m}_1 \cdots \overset{k}{m}\ s_k)\right\}$$

in the domain $V \subseteq M$, and let a have k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of multiplicity $r_1, \dots, r_k$, which, by definition, coincide with the eigenvalues of the endomorphism $\mathbb{A}\big|_p$ in T_pM, corresponding to a (that is, $g(\mathbb{A}Y, Z) = a(Y, Z)$ for all vector fields Y, Z on V) at every point $p \in V$. If $\{X_i\}$ is a natural frame on $U \subseteq V$, $g_{ij} = g(X_i, X_j)$, and $a_{ij} = a(X_i, X_j)$, then $\chi(p)$ is the Segre characteristic of the λ-matrix $(a_{ij}(p) - \lambda g_{ij}(p))$, defined by the set of its *elementary divisors*

$$(\lambda - \lambda_\alpha(p))^{\overset{\alpha}{m}_1}, \dots, (\lambda - \lambda_\alpha(p))^{\overset{\alpha}{m}_{s_\alpha}}, \quad \alpha = 1, \dots, k$$

([187], §17). The *basis* $\lambda_\alpha(p)$ of elementary divisors is a root of the characteristic equation $\det\left(a_{ij}(p) - \lambda g_{ij}(p)\right) = 0$ of multiplicity $r_\alpha = \overset{\alpha}{m}_1 + \dots + \overset{\alpha}{m}_{s_\alpha}$. It is known that the characteristic equation and eigenvalues do not depend on the choice of the chart and the frame. These invariant elements determine an *algebraic structure* of the bilinear form a in the domain V.

The matrices of the bilinear forms a and g can be simultaneously reduced to the following canonical forms [129] at each point $p \in V$ by using a nonsingular linear transformation:

$$G \equiv (\bar{g}_{hl}) = \mathrm{diag}\{G_1, \dots, G_k\}, \quad A \equiv (\bar{a}_{hl}) = \mathrm{diag}\{A_1, \dots, A_k\}, \tag{6.5}$$

where G_α and A_α are r_α-dimensional block-diagonal matrices consisting of s_α $\overset{\alpha}{m}_s$-dimensional square blocks

$$\overset{\alpha}{G}_s = \begin{pmatrix} & & & \overset{\alpha}{e}_s \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ \overset{\alpha}{e}_s & & & \end{pmatrix} \quad \text{and} \quad \overset{\alpha}{A}_s = \begin{pmatrix} & & & \overset{\alpha}{e}_s\,\lambda_\alpha \\ & & \cdot & \overset{\alpha}{e}_s \\ & \cdot & \cdot & \\ \cdot & \cdot & & \\ \overset{\alpha}{e}_s\,\lambda_\alpha & \overset{\alpha}{e}_s & & \end{pmatrix} \tag{6.6}$$

($\alpha = 1, \dots, k,\ s = 1, \dots, s_\alpha,\ \overset{\alpha}{e}_s = \pm 1$, respectively). This implies that *the canonical moving frame* $\{Y_l\}$ *on* V *determined by the conditions* $g(Y_h, Y_l) = \bar{g}_{hl}$ *and* $a(Y_h, Y_l) = \bar{a}_{hl}$ *at each point* $p \in V$ *is a skew-frame* (see Sec. 1.9). The partitioning

$$I = \bigcup \overset{\alpha}{I}_s$$

is defined by the formulas

$$\begin{gathered} \overset{\alpha}{I}_s = \{h \mid \overset{\alpha}{n}_s + 1 \le h \le \overset{\alpha}{n}_s + \overset{\alpha}{m}_s\}, \\ \sim: h \to \tilde{h} = 2\,\overset{\alpha}{n}_s + \overset{\alpha}{m}_s + 1 - h, \end{gathered} \tag{6.7}$$

$$\overset{\alpha}{n}_1 = 0, \quad \overset{\alpha}{n}_s = \overset{1}{m}_1 + \ldots + \overset{\alpha}{m}_{s-1}, \quad e_h = \overset{\alpha}{e}_s \quad \text{when} \quad h \in \overset{\alpha}{I}_s \quad (\overset{\alpha}{m}_0 \equiv 0),$$

where with each pair $(\lambda_\alpha, \lambda_{\alpha+1})$ of complex-conjugate eigenvalues λ_α, $\lambda_{\alpha+1} = \overset{*}{\lambda}_\alpha$ of multiplicity $r_\alpha = r_{\alpha+1}$ one associates r_α pairs $(Y_h, Y_{r_\alpha+h})$ of complex-conjugate vector fields Y_h, $Y_{r_\alpha+h} = \overset{*}{Y}_h$, where $h \in \bigcup_{s=1}^{s_\alpha} \overset{\alpha}{I}_s \equiv I_\alpha$ and $\overset{\alpha}{e}_s = \overset{\alpha+1}{e}_s = 1$, when $s = 1, \dots, s_\alpha = s_{\alpha+1}$.

Let (M, g) be a two-dimensional h-space, and let h be a differentiable symmetric bilinear form that satisfies Eisenhart's equation (6.4) and has a constant Segre characteristic χ in the domain $V \subseteq M$, where there is one of the following possibilities for the characteristic χ, provided $h \neq cg$:

$$\chi_1 = \{11\}, \ \chi_2 = \{1\overset{*}{1}\}, \ \text{ or } \ \chi_3 = \{2\}.$$

We call $g\,|_V$ an *h-metric of type* χ and (M, g) an *h-space of type* χ.

After making the change of variables

$$h = a + 2\varphi g, \tag{6.8}$$

where a is a symmetric bilinear form with the same Segre characteristic as h, Eq. (6.4) becomes

$$\nabla a(Y, Z.W) = g(Y, W) Z\varphi + g(Z, W) Y\varphi, \tag{6.9}$$

where Y, Z, and W are arbitrary vector fields defined on the domain V.

Let $\{Y_l\}$ be a canonical skew-frame on V defined by the formulas (6.5)–(6.7), where $r_\alpha = \sum_{s=1}^{s_\alpha} \overset{\alpha}{m}_s$ and $\lambda_1, \dots, \lambda_k$ are the eigenvalues of the bilinear form a of multiplicity $r_1, \dots, r_k$, respectively ($r_1 + \cdots + r_k = 2$).

Setting $Y = Y_p$, $Z = Y_q$, and $W = Y_r$ in (6.9), we obtain

$$Y_r \bar{a}_{pq} + \sum_{h=1}^{n} e_h (\bar{a}_{hq} \gamma_{\tilde{h}pr} + \bar{a}_{ph} \gamma_{\tilde{h}qr}) = \bar{g}_{pr} Y_q \varphi + \bar{g}_{qr} Y_p \varphi;$$

this is equivalent to

$$d\bar{a}_{pq} - \bar{a}_{hq} \omega^h{}_p - \bar{a}_{ph} \omega^h{}_q = (Y_q \varphi)\theta_p + (Y_p \varphi)\theta_q,$$

or

$$d\bar{a}_{pq} + \sum_{h=1}^{n} e_h(\bar{a}_{hq}\omega_{p\tilde{h}} + \bar{a}_{ph}\omega_{q\tilde{h}}) = (Y_q\varphi)\theta_p + (Y_p\varphi)\theta_q, \tag{6.10}$$

where $\omega = \omega^i{}_j E_i^j$ is the connection form ∇ and $\theta = \theta^i E_i$ is the canonical form:

$$\omega_{ij} = e_i\omega^{\tilde{i}}{}_j, \quad \theta_i = e_i\theta^{\tilde{i}}.$$

Our problem reduces to the integration of Eq. (6.10). In Sec. 6.2.2, we consider the cases $\chi = \chi_1$ and $\chi = \chi_2$. In Sec. 6.2.3, we find h-metrics of the type χ_3 and formulate the final result.

Multiplying (6.10) by $\bar{g}^{pq} = e_p\delta_p^{\tilde{q}}$ and summing over p and q from 1 to n and using (1.31), we find that

$$\sum_{\alpha=1}^{k} r_\alpha\lambda_\alpha = 2\varphi \tag{6.11}$$

within an inessential additive constant.

We deduce from (1.26), (1.28), (6.5), and (6.6) that

$$g = \sum_{\alpha=1}^{k} g_\alpha \quad a = \sum_{\alpha=1}^{k}(\lambda_\alpha g_\alpha + a_\alpha), \tag{6.12}$$

where

$$g_\alpha = \sum_{p\in I_\alpha} e_p\theta_p\theta_{\tilde{p}} \text{ and } a_\alpha = 0 \text{ if } r_\alpha = 1$$

and

$$a_\alpha = \sum_{p,\widetilde{p+1}\in I_\alpha} e_p\theta_p\theta_{\widetilde{p+1}} = \sum_{s=1}^{s_\alpha}\sum_{h_s=\overset{\alpha}{n}_s+1}^{\overset{\alpha}{n}_s+\overset{\alpha}{m}_s-1} \overset{\alpha}{e}_s\,\theta_{h_s}\theta_{\tilde{h}_s-1} \text{ if } r_\alpha > 1.$$

6.2.2. H-metrics of types $\{11\}$ and $\{1\overset{*}{1}\}$. Let a bilinear form a satisfy (6.10) and have the constant characteristic $\chi_1 = \{11\}$ in the domain V of a two-dimensional pseudo-Riemannian manifold. In this case, $k = 2$ and $r_1 = r_2 = 1$, and formulas (6.11) and (6.12) become

$$g\,|_V = g_1 + g_2 = e_1\theta_1\theta_1 + e_2\theta_2\theta_2 \qquad (e_1, e_2 = \pm 1), \tag{6.13}$$

$$a\big|_V = \lambda_1 g_1 + \lambda_2 g_2 = e_1\lambda_1\theta_1\theta_1 + e_2\lambda_2\theta_2\theta_2, \quad 2\varphi = \lambda_1 + \lambda_2, \tag{6.14}$$

where λ_1 and λ_2 are two distinct eigenvalues of the bilinear form a.

In (6.10), we substitute the corresponding canonical values for $\bar{a}_{pq}$ and consider this equation for all possible values of the subscripts p and q.

If $p = q = 1$ or $p = q = 2$, then $\bar{a}_{pr} = e_p\lambda_p\delta_{rp}$ and (6.10) with $p = q = \tilde{p}$ implies

$$d\lambda_p = 2e_p(Y_p\varphi)\theta_p \quad (p = 1, 2). \tag{6.15}$$

If $p = 1$ and $q = 2$, then $\bar{a}_{pq} = 0$, and from (6.10), we find that for $(pq) = (12)$,

$$\omega_{12} = \frac{1}{\lambda_2 - \lambda_1}\left(\theta_1 Y_2\varphi + \theta_2 Y_1\varphi\right). \tag{6.16}$$

Considering the values of forms (6.15) for vectors of the canonical frame and taking into account that $d\lambda_p(Y_q) = Y_q(\lambda_p)$ and $\theta_1(Y_2) = \theta_2(Y_1) \equiv g(Y_1, Y_2) = 0$ (see (1.25)), we obtain

$$Y_1(\lambda_2) = Y_2(\lambda_1) = 0. \tag{6.17}$$

Let $U \subseteq V$ be a coordinate neighborhood, and let $\{X_i\}$ be a natural frame, $Y_p = \underset{p}{\xi}{}^i X_i \equiv \underset{p}{\xi}{}^i \partial_i$. We denote by u^2 and u^1 solutions of the differential equations $Y_1 u = 0$ and $Y_2 u = 0$, respectively. In the new chart, $\bar{x}^i = u^i(x)$, omitting the bar over x, we obtain $\underset{1}{\xi}{}^2 = \underset{2}{\xi}{}^1 = 0$, i.e.,

$$Y_1 = \frac{1}{\mu}\partial_1, \quad Y_2 = \frac{1}{\nu}\partial_2, \tag{6.18}$$

where μ and ν are functions of the coordinates. In view of this, we find from (6.17) that $\partial_1\lambda_2 = \partial_2\lambda_1 = 0$; hence

$$\lambda_1 = f_1(x^1), \text{ and } \lambda_2 = f_2(x^2) \tag{6.19}$$

and from (6.14), we have

$$2\varphi = f_1(x^1) + f_2(x^2), \tag{6.20}$$

where f_1 and f_2 are functions of the indicated arguments.

Since the 1-form θ_h is dual to the vector field Y_h with respect to $g\,|_V$:

$$< \theta_h, Y_l > \equiv \theta_h(Y_l) = g(Y_h, Y_l) \quad (l = 1, \dots, n),$$

we see that (1.1) and (1.2) imply

$$\theta_h\,|_U \equiv \underset{h}{\xi}{}_i dx^i = e_h \theta^{\tilde{h}}\,|_U, \quad \theta^h_i = e_h \underset{\tilde{h}}{\xi}{}_i \quad (h = 1, \dots, n),$$

where (θ^i_j) is the matrix inverse to $(\underset{j}{\xi}{}^k)$. Computing this matrix via (6.18), we obtain

$$\theta_1 = e_1 \mu\, dx^1 \text{ and } \theta_2 = e_2 \nu\, dx^2. \tag{6.21}$$

Substituting (6.16) and (6.21) in the Cartan structure equations (1.30) written in the form

$$d\theta_1 = e_2 \omega_{21} \wedge \theta_2, \quad d\theta_2 = e_1 \omega_{12} \wedge \theta_1$$

and using (6.18)–(6.20), we find that

$$\frac{d\mu}{\mu} \wedge dx^1 = \frac{1}{2}\frac{df_2}{f_2 - f_1} \wedge dx^1 \text{ and } \frac{d\nu}{\nu} \wedge dx^2 = \frac{1}{2}\frac{df_1}{f_1 - f_2} \wedge dx^2;$$

integrating this, we have

$$\mu = \Phi_1(x^1)|f_1 - f_2|^{1/2} \text{ and } \nu = \Phi_2(x^2)|f_1 - f_2|^{1/2},$$

where Φ_1 and Φ_2 are functions of the indicated arguments. Substituting $\bar{x}^p = e_p \int \Phi_p dx^p$, $p = 1, 2$, we reduce the forms (6.21) to

$$\theta_p = |f_1 - f_2|^{1/2} dx^p \quad (p = 1, 2). \tag{6.22}$$

(Here and in what follows, we assume that after the substitution $x \to \bar{x}$, the bar over the x is omitted. It is not difficult to ascertain in every case that the transformation $x \to \bar{x}$ does not change the results obtained earlier, except for those indicated in the text; we do not comment on this fact in what follows.)

If we substitute (6.22) in (6.8), (6.13), and (6.14), then we obtain coordinate representations of the h-metric of type $\{11\}$ and of the tensor fields h and φ:

$$g\,|_U = |f_1(x^1) - f_2(x^2)| \left(e_1 dx^{1^2} + e_2 dx^{2^2}\right) \quad (e_1, e_2 = \pm 1), \tag{6.23}$$

$$h\,|_U = |f_1(x^1) - f_2(x^2)| \left(e_1 \left(2f_1(x^1) + f_2(x^2)\right) dx^{1^2} + e_2 \left(f_1(x^1) + 2f_2(x^2)\right) dx^{2^2}\right), \tag{6.24}$$

$$2\varphi\,|_U = f_1(x^1) + f_2(x^2). \tag{6.25}$$

If we set $\bar{e}_p = e_p \operatorname{sgn}(f_1 - f_2)$ for $p = 1, 2$, then, omitting the bar, we can write (6.23) and (6.24) in the form

$$g\,|_U = \left(f_1(x^1) - f_2(x^2)\right)\left(e_1 dx^{1^2} + e_2 dx^{2^2}\right) \quad (e_1, e_2 = \pm 1),$$

$$h\,|_U = \left(f_1(x^1) - f_2(x^2)\right)\left(e_1\left(2f_1(x^1) + f_2(x^2)\right)dx^{1^2} + e_2\left(f_1(x^1) + 2f_2(x^2)\right)dx^{2^2}\right).$$

Computing the Christoffel symbols of the h-metric we have found, we verify immediately that the tensor fields g, h, and φ defined above satisfy Eisenhart's equation (3.22).

The arguments given above are extend to the case of the characteristic $\chi_2 = \{1\overset{*}{1}\}$ without change. The corresponding tensor fields g, h, and φ are determined by formulas (6.23)–(6.25), where it is necessary to set $e_1 = e_2 = 1$ and consider f_1 and $f_2 = \overset{*}{f}_1$ as complex-conjugate functions of the complex-conjugate variables x^1 and $x^2 = \overset{*}{x}{}^1$. In order to pass to the real variables $\bar{x}^1$ and $\bar{x}^2$, we need to set $x^1 = \bar{x}^1 + i\bar{x}^2$ and $f_1 = u + iv$, where u and v are harmonic conjugate functions of the variables $\bar{x}^1$ and $\bar{x}^2$, i.e.,

$$\frac{\partial u}{\partial \bar{x}^1} = \frac{\partial v}{\partial \bar{x}^2}, \quad \frac{\partial u}{\partial \bar{x}^2} = -\frac{\partial v}{\partial \bar{x}^1}.$$

Later on, we will make no distinction between the characteristics $\{11\}$ and $\{1\,\overset{*}{1}\}$; in the latter case f_1 and f_2 will be considered as complex-conjugate functions of the complex-conjugate variables z^1 and z^2. It is easy to ascertain that this condition does not have an effect on the calculations and can be taken into account at the last moment.

6.2.3. *H*-metrics of type {2}. If the bilinear form a has the Segre characteristic $\chi_3 = \{2\}$, then $k = 1$, $r_1 = 2$, and (6.11) and (6.12) imply

$$g\,|_V = e_1(\theta_1\theta_2 + \theta_2\theta_1) \quad (e_1 = \pm 1), \tag{6.26}$$

$$a\,|_V = e_1\lambda(\theta_1\theta_2 + \theta_2\theta_1) + e_1\theta_1\theta_1, \quad \varphi = \lambda, \tag{6.27}$$

where λ is a double eigenvalue of the bilinear form a.

We substitute the corresponding canonical forms into Eq. (6.10),

$$(\bar{g}_{pq}) = \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix} \quad \text{and} \quad (\bar{a}_{pq}) = \begin{pmatrix} 0 & e_1\lambda \\ e_1\lambda & e_1 \end{pmatrix},$$

and consider this equation for all possible values (11), (22), and (12) of the subscripts (pq). As a result, we obtain the equations

$$Y_1\lambda = Y_1\varphi = 0, \tag{6.28}$$

$$d\lambda = e_1\theta_1 Y_2\varphi,$$

$$\omega_{21} = \theta_2 Y_2\varphi. \tag{6.29}$$

Let $Y_p = \underset{p}{\xi}{}^i\partial_i$. Denoting by u^2 and u^1 solutions of the differential equations $Y_1u = 0$ and $Y_2u = 0$, respectively, in the new chart $\bar{x}^i = u^i(x)$, omitting the bar, we obtain

$$\underset{1}{\xi}{}^2 = \underset{2}{\xi}{}^1 = 0,$$

and by (6.28), we have $\partial_1\lambda = 0$, i.e., $\lambda = f(x^2)$.

If the function f has a nonzero derivative in the domain V, then taking f as a new variable $\bar{x}^2$ and omitting the bar, we obtain $\lambda = x^2$. Therefore, it is possible to set

$$\lambda = \varphi = \epsilon x^2 + (1 - \epsilon)c,$$

where ϵ is equal to 0 or 1 and c is constant.

In the notation of (6.18), we have

$$\theta_1 = e_1\nu\, dx^2 \quad \text{and} \quad \theta_2 = e_1\mu\, dx^1. \tag{6.30}$$

From the Cartan structure equations

$$d\theta_1 = e_1\omega_{21}\wedge\theta_1, \quad d\theta_2 = e_1\omega_{12}\wedge\theta_2, \tag{6.31}$$

taking into account (6.29) and (6.30), we find that $d\theta_2 = 0$; hence $\partial_2\mu = 0$ and after making the change of the coordinates $\bar{x}^1 = \int \mu\, dx^1$ in order to obtain a new chart, omitting the bar, we obtain

$$\theta_2 = e_1 dx^1, \quad Y_1 = \partial_1. \tag{6.32}$$

Relations (6.18), (6.30), and (6.31) imply

$$e_1 d\theta_1 \equiv (\partial_1\nu)\, dx^1\wedge dx^2 = \epsilon dx^1\wedge dx^2,$$

hence we have $\theta_1 = e_1\left(\epsilon x^1 + w(x^2)\right) dx^2$ here, where w is a function of x^2.

If $\epsilon = 0$, then by using the transformation $\bar{x}^2 = \int w\, dx^2$, the form θ_1 is reduced to the form $e_1\, dx^2$ and by virtue of (6.26) and (6.32), the metric g is flat. Therefore, it will be assumed from now on that $\epsilon = 1$.

If we substitute

$$\lambda = x^2, \quad \theta_1 = e_1\left(x^1 + w(x^2)\right) dx^2, \quad \theta_2 = e_1\, dx^1$$

in (6.26) and (6.27) and make the change $x^2 = e_1\bar{x}^2$, $w(x^2) = \bar{w}(\bar{x}^2)$, then omitting the bar and taking into account that a and φ are determined by Eq. (6.9) to within a constant common factor, we obtain

$$\begin{gathered} g\mid_U = 2\left(x^1 + w(x^2)\right) dx^1 dx^2, \\ h\mid_U = 3x^2 g\mid_U + \left(x^1 + w(x^2)\right)^2 dx^{2^2}, \\ \varphi\mid_U = x^2. \end{gathered}$$

The verification shows that the tensors found above satisfy the Eisenhart equation (3.22). This implies the following theorem.

Theorem 6.1. *Let M^2 be a two-dimensional pseudo-Riemannian manifold with metric g and Levi-Civita connection ∇. Let φ be a 0-form, and let h be a symmetric bilinear form of Segre characteristic χ, defined over M^2 or over some domain $V \subseteq M^2$. For h, g, and φ to satisfy Eisenhart's equation (6.4), i.e., for M^2 to be an h-space of type χ, it is necessary and sufficient that around every point $p \in M^2$ (or $p \in V$), there is a canonical chart (x, U) in which:*

if $\chi = \{11\}$, then

$$g\mid_U = (f_1 - f_2)\left(e_1 dx^{1^2} + e_2 dx^{2^2}\right); \tag{6.33}$$

$$\begin{gathered} h\mid_U = (f_1 - f_2)\left(e_1\left(2f_1 + f_2\right) dx^{1^2} + e_2\left(f_1 + 2f_2\right) dx^{2^2}\right), \\ 2\varphi\mid_U = f_1 + f_2 \\ \left(f_1 = f_1(x^1),\ f_2 = f_2(x^2)\right), \end{gathered}$$

if $\chi = \{1\overset{*}{1}\}$, then

$$\begin{gathered} \sqrt{-1}g\mid_U = (f_1 - f_2)\left(dz^{1^2} + dz^{2^2}\right), \\ \sqrt{-1}h\mid_U = (f_1 - f_2)\left(\left(2f_1 + f_2\right) dz^{1^2} + \left(f_1 + 2f_2\right) dz^{2^2}\right), \\ 2\varphi\mid_U = f_1 + f_2 \\ \left(f_1 = f_1(z^1),\ f_2 = f_2(z^2) = \overset{*}{f}_1,\ z^1 = x^1 + \sqrt{-1}x^2,\ z^2 = x^1 - \sqrt{-1}x^2 = \overset{*}{z}{}^1\right), \end{gathered}$$

if $\chi = \{2\}$, *then*

$$g\,|_U = 2\left(x^1 + w(x^2)\right) dx^1 dx^2,$$
$$h\,|_U = 3x^2 g\,|_U + \left(x^1 + w(x^2)\right)^2 dx^{2^2},$$
$$\varphi\,|_U = x^2,$$

where $*$ *denotes the complex conjugation,* f_1*,* f_2 *and* w *are functions of the indicated arguments, and* $e_1, e_2 = \pm 1$.

The geodesic equation of the metric g *admits the quadratic first integral defined by the formula* (3.25):

$$\sigma(\dot{x}, \dot{x}) \equiv (4\varphi g - h)(\dot{x}, \dot{x}) = \text{const},$$

where

$$\sigma(\dot{x}, \dot{x})\,|_U = (f_1 - f_2)\left(e_1 f_2 \dot{x}^{1^2} + e_2 f_1 \dot{x}^{2^2}\right) \equiv \text{const}$$

for a h-metric of type $\chi = \{11\}$*;*

$$\sqrt{-1}\sigma(\dot{z}, \dot{z})\,|_U = (f_1 - f_2)\left(f_2 \dot{z}^{1^2} + f_1 \dot{z}^{2^2}\right) \equiv \text{const}$$

for an h-metric of type $\chi = \{1\overset{*}{1}\}$ *and, for an h-metric of type* $\chi = \{2\}$*,*

$$\sigma(\dot{x}, \dot{x})\,|_U = 2x^2\left(x^1 + w(x^2)\right)\dot{x}^1\dot{x}^2 - \left(x^1 + w(x^2)\right)^2 \dot{x}^{2^2} \equiv \text{const}$$

($\dot{x}$ ($\dot{z}$) *is a tangent vector to the geodesic).*

6.3. Projective and affine motions of two-dimensional pseudo-Riemannian manifolds.

6.3.1. Projective motions on Liouville surfaces. A *Liouville surface* is a surface whose linear element in suitable coordinates has the form

$$(U(u) - V(v))\,(du^2 + dv^2)$$

([313]; see also [101], Part I, p. 175). We extend this name to surfaces with metrics of the form (6.33) of arbitrary signature. According to Theorem 6.1, *every h-space of type* $\{11\}$ *is a Liouville surface.*

Theorem 6.2. *Every affine motion in a two-dimensional pseudo-Riemannian manifold of nonconstant curvature is an infinitesimal homothety.*

Proof. The curvature tensor of a two-dimensional pseudo-Riemannian manifold M^2 is defined by

$$R^i_{jkl} = \rho(\delta^i_k g_{jl} - \delta^i_l g_{jk}), \tag{6.34}$$

where $\rho \neq$ const for spaces of nonconstant curvature (later on, it will be assumed that this condition is always satisfied).

If X is an affine motion in M^2, then in every chart on M^2, Eqs. (3.18) are satisfied:

$$h_{ij,k} \equiv (L_X g_{ij})_{,k} = 0.$$

The integrability conditions of these equations are obtained by using the Ricci identity (1.3) and have the form

$$h_{im} R^m_{jkl} + h_{jm} R^m_{ikl} = 0,$$

or, in view of (6.34) and inequality $\rho \neq 0$,

$$h_{ik} g_{jl} - h_{il} g_{jk} + h_{jk} g_{il} - h_{jl} g_{ik} = 0;$$

hence, contracting with g^{jl}, we find that $h_{ik} = c g_{ik}$, where $c =$ const. Consequently, X is an infinitesimal homothety. □

We note that it follows from the proof given above that *every symmetric parallel bilinear form on a two-dimensional pseudo-Riemannian manifold of nonconstant curvature is proportional to the metric form.*

We consider the Liouville metric

$$g = (f_1 - f_2)(e_1 dx^{1^2} + e_2 dx^{2^2}) \quad \big(f_1 = f_1(x^1),\ f_2 = f_2(x^2),\ e_1, e_2 = \pm 1\big). \tag{6.35}$$

By Theorem 6.1, it admits two quadratic first integrals of the geodesics equation:

$$g_{ij}\dot{x}^i \dot{x}^j = \text{const} \ \text{ and } \ (f_1 - f_2)(e_1 f_2 \dot{x}^{1^2} + e_2 f_1 \dot{x}^{2^2}) = \text{const}.$$

This implies the following proposition.

Theorem 6.3. *If a Liouville surface is not a surface of revolution, then every projective motion X on it satisfies the equation*

$$L_X g = c_1 h + 2c_2 g, \tag{6.36}$$

where c_1 and c_2 are constants and

$$h = (f_1 - f_2)\left(e_1(2f_1 + f_2)dx^{1^2} + e_2(f_1 + 2f_2)dx^{2^2}\right).$$

Proof. The right-hand side of Eq. (6.36) determines the general solution of (3.24) in the space in question. In fact, if (3.24) has a solution $\bar{h}$ that is not a linear combination of h and g, then the third (independent) integral $(4\bar{\varphi}g - \bar{h})(\dot{x}, \dot{x}) = \text{const}$ is added to the quadratic first integrals of this equation, and, by Königs theorem (see above), the surface turns into a surface of revolution. ☐

Theorem 6.3 implies the following proposition.

Theorem 6.4. *If a Liouville surface L^2 is not a surface of revolution and admits a nonhomothetic projective Lie algebra P_r, then this algebra contains a subalgebra H_{r-1} of infinitesimal homotheties of dimension $r - 1$.*

Proof. Let $E_1, \dots, E_r$ be basis vector fields for a nonhomothetic projective Lie algebra P_r of the Liouville surface L^2. By Theorem 6.3, we have

$$L_{E_\alpha} g = \underset{\alpha}{c}_1 h + \underset{\alpha}{c}_2 g \quad \left(\underset{\alpha}{c}_1, \underset{\alpha}{c}_2 = \text{const},\ \alpha = 1, \dots, r\right).$$

Since, by definition, P_r does not reduce to just homotheties, at least one of the constants $\underset{\alpha}{c}_1$, for example, $\underset{1}{c}_1$, is nonzero. We introduce a new basis

$$\overline{E}_1 = E_1, \quad \overline{E}_\sigma = \underset{1}{c}_1 E_\sigma - \underset{\sigma}{c}_1 E_1 \quad (\sigma = 2, \dots, r)$$

and we show that $r - 1$ vector fields $\overline{E}_\sigma$ are homothetic motions. By the property of the Lie derivative, we have

$$L_{\overline{E}_\sigma} g = \underset{1}{c}_1 L_{E_\sigma} g - \underset{\sigma}{c}_1 L_{E_1} g;$$

hence

$$L_{\overline{E}_\sigma} g = (\underset{1}{c}_1 \underset{\sigma}{c}_2 - \underset{1}{c}_2 \underset{\sigma}{c}_1) g \quad (\sigma = 2, \dots, r).$$

This means that $\overline{E}_2, \dots, \overline{E}_r$ are infinitesimal homotheties, and the Lie algebra P_r has a subalgebra H_{r-1} of homothetic motions. ☐

Let a Liouville surface admit a nonhomothetic projective motion $X = \xi^i \partial_i$. Then in (6.36), one may set $c_1 = 1$. Then by the change

$$f_\alpha \to f_\alpha + \frac{2}{3} c_2 \quad (\alpha = 1, 2),$$

which does not change the Liouville metric, (6.36) reduces to the form

(a) $$\frac{\xi^1 f_1' - \xi^2 f_2'}{f_1 - f_2} + 2\partial_1 \xi^1 = 2f_1 + f_2,$$

(b) $$\frac{\xi^1 f_1' - \xi^2 f_2'}{f_1 - f_2} + 2\partial_2 \xi^2 = f_1 + 2f_2,$$

(c) $$e_1 \partial_2 \xi^1 + e_2 \partial_1 \xi^2 = 0$$

(here and in what follows, by the prime one denotes the differentiation of a function of a single variable with respect to its argument). In the new coordinates,

$$x = \frac{x^1 + \epsilon x^2}{\sqrt{2}}, \quad y = \frac{x^1 - \epsilon x^2}{\sqrt{2}} \quad (\epsilon^2 = -e_1 e_2),$$

the Liouville metric, the projective motion X, and Eqs. (a)–(c) become

$$g = 2e_1(f_1 - f_2)\, dx\, dy \equiv 2\lambda\, dx\, dy, \tag{6.37}$$

$$X = \xi^x \partial_x + \xi^y \partial_y \quad \xi^1 = \frac{\xi^x + \xi^y}{\sqrt{2}}, \quad \xi^2 = \frac{\xi^x - \xi^y}{\epsilon\sqrt{2}},$$

$$\begin{aligned} &4\partial_x \xi^y = e_1 \lambda, \\ &4\partial_y \xi^x = e_1 \lambda, \\ &\xi^x \lambda_{,x} + \xi^y \lambda_{,y} + \lambda(\partial_x \xi^x + \partial_y \xi^y) = \lambda \varphi. \end{aligned}$$

By (6.34), where for the Liouville metric (6.37)

$$\rho = -\frac{1}{\lambda} \partial_{xy}(\ln \lambda),$$

the integrability conditions (3.26) of Eqs. (3.19), which define a projective motion X, are written in the form

$$\rho L_X g_{jl} + (L_X \rho) g_{jl} + \varphi_{,jl} = 0. \tag{6.38}$$

Considering these conditions together with their nearest differential implications, we obtain the following set of linear algebraic equations with respect to the unknowns ξ^x and ξ^y:

$$\xi^x \rho_{,x} + \xi^y \rho_{,y} + \Omega = 0, \quad \xi^x \sigma_{,x} + \xi^y \sigma_{,y} + \Sigma = 0, \quad \xi^x \tau_{,x} + \xi^y \tau_{,y} + \Lambda = 0, \tag{6.39}$$

where we have introduced the following notation:

$$\sigma \equiv \frac{1}{\lambda} \rho_{,x} \rho_{,y}, \quad \tau \equiv \frac{1}{\lambda} \partial_{xy} \rho, \quad \Omega \equiv 3\rho\varphi + \frac{1}{2\lambda} e_1 \partial_{xx} \lambda,$$

$$\Sigma \equiv 9\sigma\varphi - \frac{1}{4} e_1 (\rho_{,x}^2 + \rho_{,y}^2), \quad \Lambda \equiv 6\tau\varphi + \frac{3}{4} e_1 (\lambda_{,x} \rho_{,x} + \lambda_{,y} \rho_{,y}).$$

We note that by differentiating the first equation of (6.39) with respect to x and y, we obtain the relations

$$\partial_x \xi^x \rho_{,x} + \xi^x \partial_{xx} \rho + \xi^y \partial_{xy} \rho + 3\varphi \rho_{,x} - \frac{1}{4} e_1 \lambda \rho_{,y} = 0,$$

$$\partial_y \xi^y \rho_{,y} + \xi^x \partial_{xy} \rho + \xi^y \partial_{yy} \rho + 3\varphi \rho_{,y} - \frac{1}{4} e_1 \lambda \rho_{,x} = 0,$$

from which it follows that $\rho_{,x} = 0$ implies $\rho_{,y} = 0$, and vice versa. Therefore, $\rho_{,x} \rho_{,y} \neq 0$ if the curvature ρ is nonconstant.

By differentiating the equations of system (6.39) and the conditions of its consistency (as a set of algebraic equations with respect to ξ^x, ξ^y), after trivial but lengthy and tedious calculations, which we omit, one can show that the expressions $(\partial_x \xi^x - \partial_y \xi^y)\triangle_1$ and $(\partial_x \xi^x - \partial_y \xi^y)\triangle_2$, where

$$\triangle_1 \equiv \rho_{,x}\sigma_{,y} - \rho_{,y}\sigma_{,x}, \quad \triangle_2 \equiv \rho_{,x}\tau_{,y} - \rho_{,y}\tau_{,x},$$

vanish. Consequently, $\partial_x \xi^x - \partial_y \xi^y$ can be nonzero only under the condition $\triangle_1 = \triangle_2 = 0$. In all remaining cases, we have $\partial_x \xi^x - \partial_y \xi^y = 0$, i.e., $\partial_1 \xi^2 = \partial_2 \xi^1 = 0$.

By $\rho_{,x}\rho_{,y} \neq 0$, the condition $\triangle_1 = \triangle_2 = 0$ implies that σ and τ are functions of ρ. Now let θ be some function of ρ. Since $\rho_{,x}\rho_{,y} = \sigma(\rho)\lambda$ and $\partial_{xy}\rho = \tau(\rho)\lambda$, the following relation holds:

$$\partial_{xy}\theta = \lambda\left(\theta''(\rho)\sigma(\rho) + \theta'(\rho)\tau(\rho)\right).$$

Therefore, taking as θ a nonconstant solution of the equation $\theta''\sigma + \theta'\tau = 0$, we have $\theta(\rho) = \alpha + \beta$, where α is a function of x and β is a function of y, such that $\alpha'\beta' \neq 0$; otherwise $\rho_{,x}\rho_{,y} = 0$. This implies

$$\rho = \kappa(\alpha + \beta), \quad \lambda = \frac{\rho_{,x}\rho_{,y}}{\sigma(\rho)} = \frac{\alpha'\beta'}{\sigma(\rho)\theta'^2(\rho)},$$

and the quadratic form

$$2\lambda\, dx\, dy = \frac{2}{\sigma\theta'^2} d\alpha d\beta \equiv 4\psi(\alpha + \beta) d\alpha d\beta \equiv \psi(\alpha + \beta)\left([d(\alpha + \beta)]^2 - [d(\alpha - \beta)]^2\right)$$

defines the surface of revolution (6.1). We have proved the following lemma.

Lemma 6.1. *If a Liouville surface is not a surface of revolution, then for every nonhomothetic motion* $X = \xi^1\partial_1 + \xi^2\partial_2$ *on it in the chart* (6.35), *the conditions* $\partial_1\xi^2 = \partial_2\xi^1 = 0$ *hold.*

Under these conditions, (c) becomes the identity and (a) and (b) admit separation of variables. As a result, we have

$$\xi^i f_i' = f_i^2 + a f_i + b, \quad 2\xi^{i'} = f_i - a \quad (a, b = \text{const},\ i = 1, 2); \tag{6.40}$$

this implies the equation

$$f_i' = k_i \exp\left(\frac{3}{2}\int \frac{f_i + a}{f_i^2 + a f_i + b} df_i\right) \quad (a, b, k_i = \text{const}), \tag{6.41}$$

which is not integrable in finite form in general.

Now let

$$Y = \eta^1\partial_1 + \eta^2\partial_2 \equiv \eta^x\partial_x + \eta^y\partial_y$$

be a homothetic motion on surface (6.35). From the equations $L_Y g = 2cg$ and $c = \text{const}$ and their integrability conditions, we obtain

$$\partial_x\eta^y = \partial_y\eta^x = 0, \quad \eta^x(\ln\lambda)_{,x} + \eta^y(\ln\lambda)_{,y} + \partial_x\eta^x + \partial_y\eta^y - 2c = 0,$$

$$\eta^x\rho_{,x} + \eta^y\rho_{,y} + 2c\rho = 0, \quad \eta^x\tau_{,x} + \eta^y\tau_{,y} + 4c\tau = 0, \quad \eta^x\sigma_{,x} + \eta^y\sigma_{,y} + 6c\sigma = 0, \tag{6.42}$$

where ρ, σ, and τ were defined above. Applying arguments that are similar to those above to these equations, one can prove that $\partial_x\eta^x - \partial_y\eta^y = 0$, i.e., $\partial_1\eta^2 = \partial_2\eta^1 = 0$, if a Liouville surface is not a surface of revolution. In this case, integrating the equations $L_\eta g = 2cg$ results in the relations

$$\eta^i f_i' = 2(c - k)f_i + l, \quad \eta^{i'} = k, \quad (2(c - k)f_i + l)\, f_i'' = (2c - 3k)f_i'^2 \quad (c, k, l = \text{const}).$$

Comparing them with (6.41), we obtain $c = 0$, i.e., a homothety reduces to an isometry.

For an infinitesimal isometry $Y = \eta^x\partial_x + \eta^y\partial_y \neq 0$, from (6.42) when $c = 0$, we obtain the relation $\triangle_1 = \triangle_2 = 0$, which implies that a Liouville surface is a surface of revolution, as was shown above. Therefore, we have proved the following lemma.

Lemma 6.2. *A Liouville surface admits an infinitesimal isometry if and only if it is a surface of revolution.*

Therefore, if a Liouville surface is not a surface of revolution, it admits at most a one-dimensional projective Lie algebra spanned by nonaffine projective motion $\underset{p}{E}$ with components (6.40):

$$\underset{p}{E} = \frac{f_1^2 + af_1 + b}{f_1'}\partial_1 + \frac{f_2^2 + af_2 + b}{f_2'}\partial_2 \quad (a, b = \text{const}),$$

where $f_1 = f_1(x^1)$ and $f_2 = f_2(x^2)$ are nonconstant functions defined by Eqs. (6.41).

By the remark in Sec. 6.2.2, the result obtained extends to the case of complex-conjugate functions $f_1 = \overset{*}{f}_2$ of complex-conjugate variables $x^1 = \overset{*}{x}{}^2$.

6.3.2. Projective motions on surface of revolution. Let $X = \xi^i\partial_i$ be a projective motion on a surface of revolution M^2 with the linear element

$$ds^2 = \Phi^2(x^1)(e_1\, dx^{1^2} + e_2 d\, x^{2^2}). \tag{6.43}$$

Determination of all projective motions on a surface of revolution reduces to finding a general solution to Eqs. (3.21) and (3.22) for this surface. Equation (3.22) can be replaced by the equivalent equation

$$a_{ij,k} = g_{ik}\varphi_{,j} + g_{jk}\varphi_{,i} \tag{6.44}$$

for a bilinear form $a = (L_X - 2\varphi)g$ with each solution of which it is related a quadratic first integral

$$(a_{ij} - 2\varphi g_{ij})\dot{x}^i\dot{x}^j = \text{const} \tag{6.45}$$

of the geodesic equation.

Equation (6.44) implies the relation $(g^{ij}a_{ij})_{,k} = 2\varphi_{,k}$. Since the function φ is determined by (6.44) to within an additive constant, we may set $a_{ij}g^{ij} = 2\varphi$. In the new variables $c_{ij} = \Phi^{-2}a_{ij}$, the last condition is written in the form $e_1c_{11} = -e_2c_{22} + 2\varphi$, and (6.44) becomes

$$\begin{aligned}
&(\alpha) && \partial_1 c_{22} = 0,\\
&(\beta) && \partial_2 c_{22} = -2e_1e_2\frac{\Phi'}{\Phi}c_{12} + 2e_2\varphi_{,2},\\
&(\gamma) && \partial_1 c_{12} = e_1\varphi_{,2},\\
&(\delta) && \partial_2 c_{12} = 2\frac{\Phi'}{\Phi}(c_{22} - e_2\varphi) + e_2\varphi_{,1},
\end{aligned}$$

From $(\alpha) - (\gamma)$, we have

$$c_{22} = e_2\nu, \quad c_{12} = e_1\Phi\left(\frac{1}{2}\nu'\eta + \sigma\right), \tag{6.46}$$

where ν and σ are functions of x^2 and

$$\eta \equiv \int \frac{dx^1}{\Phi}.$$

(γ) and (δ) imply

$$\varphi = \frac{1}{2}\nu(\eta\Phi)' + \Phi'\int \sigma\, dx^2 + \mu(x^1),$$

$$\frac{1}{2}\nu''\eta + \sigma' + e_1e_2A\int \sigma\, dx^2 + e_1e_2B\nu = \frac{e_1e_2}{\Phi}(\mu' - 2\frac{\Phi'}{\Phi}\mu), \tag{6.47}$$

where μ is a function of x^1,

$$A \equiv 2\frac{\Phi'^2}{\Phi^2} - \frac{\Phi''}{\Phi} = \frac{\eta'''}{\eta'}, \quad B \equiv \frac{1}{2}(\eta A + 3\eta'').$$

It is easy to verify that $A' = e_1\Phi^2\rho'$, where

$$\rho = -\frac{e_1}{\Phi^2}\left(\frac{\Phi'}{\Phi}\right)'$$

is the scalar curvature of metric (6.43). Therefore, the condition $\rho' \neq 0$ is equivalent to the condition $A' \neq 0$. Since we consider spaces of nonconstant curvature, we need to consider A' to be nonzero.

Examining equations obtained by the differentiation of (6.47):

$$\frac{1}{2}\nu'''\eta + \sigma'' + e_1e_2A\sigma + e_1e_2B\nu' = 0, \tag{6.48}$$

$$\frac{1}{2}\nu'''\eta' + e_1e_2A'\sigma + e_1e_2B'\nu' = 0, \tag{6.49}$$

it is not difficult to verify that

$$\sigma + p\nu' = 0, \quad \nu'' + e_1e_2q\nu = 2c_1 \tag{6.50}$$

in the case

$$C \equiv B - pA + \frac{1}{2}q(\eta - 2p) = 0, \tag{6.51}$$

where p, q, and c_1 are constants and $\nu' = \sigma = 0$ in the remaining cases.

Indeed, let $A' = \alpha\eta'$, where α is a constant which is nonzero by virtue of the condition $A' \neq 0$. Then $A = \alpha\eta + \beta$ and $B'/\eta' = (5/2)\alpha\eta + 2\beta$, where $\beta = \text{const}$, and from (6.49), dividing by η' and differentiating with respect to x^1, we obtain $\nu' = 0$; consequently, $\sigma = 0$. If $A' \neq \text{const}\cdot\eta'$, then from (6.49), we find that

$$\nu'\left(\frac{(B'/\eta')'}{(A'/\eta')'}\right)' = 0;$$

hence $\nu' = \sigma = 0$ or $B = pA + (1/2)q\eta + r$, where p, q, and r are constants. After that, (6.48) and (6.49) imply (6.50) and $r = -pq$, which is what we needed to prove.

If $\nu' = \sigma = 0$, then $c_{12} = 0$, and Eqs. (γ) and (δ) become

$$\varphi_{,2} = 0, \quad \varphi_{,1} - 2(\varphi - \nu)\frac{\Phi'}{\Phi} = 0;$$

hence $\varphi = c\Phi^2 + \nu$, where c and ν are constants.

Therefore, when $C \neq 0$, the general solution of Eq. (6.44) in the space (6.43) of nonconstant curvature is given by the formulas

$$a_{ij}\,dx^i\,dx^j = 2e_1c\Phi^4\,dx^{1^2} + \nu ds^2, \quad \varphi = c\Phi^2 + \nu \qquad (c, \nu = \text{const}), \tag{6.52}$$

and the corresponding quadratic first integrals (6.45) of the geodesic equations have the form

$$\Phi^4\dot{x}^{2^2} = \text{const}, \quad \Phi^2(e_1\dot{x}^{1^2} + e_2\dot{x}^{2^2}) = \text{const}.$$

The first integral is a consequence of the linear integral (6.3), and the second one coincides with the integral $g_{ij}\dot{x}^i\dot{x}^j = \text{const}$. Since these integrals exist in every space (6.43), later on they will be not cited.

When $C = 0$, from (6.47) and (6.50), taking into account the equation $\Phi = 1/\eta'$, we find that

$$\mu'\eta' + 2\eta''\mu = e_1e_2c_1(\eta - 2p);$$

from here

$$\mu = \frac{1}{2}e_1e_2c_1\frac{(\eta - 2p)^2}{\eta'^2} + \frac{c_0}{\eta'^2} \quad (c_0 = \text{const}).$$

The condition $C = 0$ is equivalent to the equation

$$\frac{\eta'''}{\eta'} + \frac{3\eta''}{\eta - 2p} - q = 0, \tag{6.53}$$

which after the change $\eta - 2p = \sqrt{\zeta}$ takes the form $\zeta''' - q\zeta' = 0$ and has the following solutions:

$$\begin{aligned}
&\zeta = \alpha_1 x^{1^2} + \alpha_2 x^1 + \alpha_3 && \text{if } \; q = 0,\\
&\zeta = \alpha_1 \left(\alpha + \sin(\lambda x^1 + \alpha_2)\right) && \text{if } \; q = -\lambda^2 < 0,\\
&\zeta = \alpha_1 + \alpha_2 \cosh \lambda x^1 + \alpha_3 \sinh \lambda x^1 && \text{if } \; q = \lambda^2 > 0,
\end{aligned}$$

where α, α_1, α_2 and α_3 are constants.

We consider separately the solutions found above.

When $q = 0$, one should distinguish between two cases: $\alpha_1 = 0$ and $\alpha_1 \neq 0$.

If $\alpha_1 = 0$, then $\alpha_2 \neq 0$ in view of the condition $\eta' \neq 0$, and by a transport of the origin, we can reduce ζ and Φ^2 to the form

$$\zeta = \alpha_2 x^1, \quad \Phi^2 \equiv \frac{4\zeta}{\zeta'^2} = \frac{4x^1}{\alpha_2}.$$

Since similar metrics have the same projective Lie algebra, we can set $\alpha_2 = 4$, and the metric (6.43) becomes

$$ds_{\mathrm{I}}^2 = x^1(e_1 dx^{1^2} + e_2 dx^{2^2}). \tag{6.54}$$

The general solution of Eq. (6.44) in this space is defined by the formulas

$$a_{ij} = \Phi^2 c_{ij}, \quad \Phi = \frac{2\sqrt{\zeta}}{\zeta'},$$

$$c_{11} = e_1(2\varphi - \nu), \quad c_{12} = e_1 \nu' \frac{\zeta}{\zeta'}, \quad c_{22} = e_2 \nu, \tag{6.55}$$

$$\varphi = \nu \left(\frac{\zeta}{\zeta'}\right)' + \Phi^2 \left(c_0 + \frac{1}{2} e_1 e_2 c_1 \zeta\right), \quad \nu'' + e_1 e_2 q \nu = 2c_1;$$

from here, setting $\zeta = 4x^1$ and $q = 0$, we find that

$$\begin{aligned}
a_{ij} dx^i dx^j = x^1 \Big(e_1 \left(2c_0 x^1 + c_1 (4e_1 e_2 x^{1^2} + x^{2^2}) + c_2 x^2\right) dx^{1^2} + 2e_1 x^1 (2c_1 x^2 + c_2) dx^1 dx^2 +\\
+ e_2 x^2 (c_1 x^2 + c_2) dx^{2^2} \Big) + c_3 ds_{\mathrm{I}}^2,
\end{aligned}$$

$$\varphi = c_0 x^1 + c_1 (x^{2^2} + 2e_1 e_2 x^{1^2}) + c_2 x^2 + c_3$$

($\nu = c_1 x^{2^2} + c_2 x^2 + c_3$, here and later on c_0, c_1, c_2, and c_3 are constants). The corresponding quadratic first integrals are

$$x^1 \left(e_1 x^{2^2} \dot{x}^{1^2} - 4e_1 x^1 x^2 \dot{x}^1 \dot{x}^2 + (4e_1 x^{1^2} + e_2 x^{2^2})\, \dot{x}^{2^2} \right) = \text{const},$$

$$x^1 \left(e_1 x^2 \dot{x}^{1^2} - 2e_1 x^1 \dot{x}^1 \dot{x}^2 + e_2 x^2 \dot{x}^{2^2} \right) = \text{const}.$$

If $\alpha_1 \neq 0$, then by the transformation $x^i \to x^i + \text{const} \cdot \delta_1^i$, one can reduce ζ and Φ^2 to the form

$$\zeta = \alpha_1 (x^{1^2} + \alpha), \quad \Phi^2 = \frac{1}{\alpha_1} \left(1 + \frac{\alpha}{x^{1^2}}\right),$$

where α and α_1 are constants. Equating α_1 to 1, we obtain one of the set of similar metrics

$$ds_{\mathrm{II}}^2 = \left(1 + \frac{\alpha}{x^{1^2}}\right)(e_1 dx^{1^2} + e_2 dx^{2^2}) \quad (\alpha = \text{const}). \tag{6.56}$$

The general solution of Eq. (6.44) for the metric (6.56) of nonconstant curvature and the corresponding quadratic first integrals are determined by formulas (6.45) and (6.55) and have the form

$$a_{ij}\,dx^i\,dx^j = \Phi^2\Bigg(e_1\bigg(2c\Phi^2 + c_1\bigg(e_1e_2x^{1^2}\Phi^2 - \frac{\alpha}{x^{1^2}}x^{2^2}\bigg) - c_2\alpha\frac{x^2}{x^{1^2}}\bigg)dx^{1^2} + e_1x^1\Phi^2(2c_1x^2 + c_2)dx^1dx^2 + e_2x^2(c_1x^2+c_2)dx^{2^2}\Bigg) + c_3ds^2_{\mathrm{II}},$$

$$\varphi = c\Phi^2 + \frac{1}{2}c_1\bigg(e_1e_2\big(x^{1^2}+\alpha\big) + \bigg(1-\frac{\alpha}{x^{1^2}}\bigg)x^{2^2}\bigg) + \frac{1}{2}c_2\bigg(1-\frac{\alpha}{x^{1^2}}\bigg)x^2 + c_3,$$

$$\Phi^2\bigg(e_1x^{2^2}\dot{x}^{1^2} - 2e_1x^1x^2\Phi^2\dot{x}^1\dot{x}^2 + \bigg(e_1x^{1^2}\Phi^2 - e_2\alpha\frac{x^{2^2}}{x^{1^2}}\bigg)\dot{x}^{2^2}\bigg) = \mathrm{const},$$

$$\Phi^2\bigg(e_1x^2\dot{x}^{1^2} - e_1x^1\Phi^2\dot{x}^1\dot{x}^2 - e_2\alpha\frac{x^2}{x_{1^2}}\dot{x}^{2^2}\bigg) = \mathrm{const}$$

$\Big(\Phi^2 = 1+\alpha/x^{1^2},\ \nu = c_1x^{2^2} + c_2x^2 + c_3,\ c = c_0 + \frac{1}{2}(e_1e_2c_1 - c_3)\Big).$

When $q = -\lambda^2 < 0$, without changing the metric (6.43), we can transform ζ and Φ^2 to the form

$$\zeta = \alpha_1(\alpha + \sin\lambda x^1), \quad \Phi^2 = \frac{4}{\alpha_1\lambda^2}\frac{\alpha + \sin\lambda x^1}{\cos^2\lambda x^1}.$$

If we set $\alpha_1 = 4/\lambda^2$ and use (6.45) and (6.51), we find that

$$ds^2_{\mathrm{III}} = \frac{\alpha+\sin\lambda x^1}{\cos^2\lambda x^1}\big(e_1dx^{1^2} + e_2dx^{2^2}\big) \quad (\alpha,\lambda = \mathrm{const},\ \lambda\neq 0),$$

$$a_{ij}\,dx^i\,dx^j = \Phi^2e_1\Big(\big(2\tilde{c}_0\Phi^2 + (c_2\psi_1 + c_3\psi_2)(1+2\Phi^2\sin\lambda x^1)\big)\,dx^{1^2} + \frac{2}{\lambda}(c_2\psi_1' + c_3\psi_2')\Phi^2(\cos\lambda x^1)\,dx^1\,dx^2 + e_1e_2(c_2\psi_1 + c_3\psi_2)\,dx^{2^2}\Big) - \tilde{c}_1ds^2_{\mathrm{III}},$$

$$\varphi = \tilde{c}_0\Phi^2 - \tilde{c}_1 + (c_2\psi_1 + c_3\psi_2)(I + \Phi^2\sin\lambda x^1),$$

$$\Phi^2\bigg(e_1\psi_a\dot{x}^{1^2} - 2e_1\frac{\psi_a'}{\lambda}(\cos\lambda x^1)\Phi^2\dot{x}^1\dot{x}^2 + c_2\psi_a(1+2(\sin\lambda x^1)\Phi^2)\dot{x}^{2^2}\bigg) = \mathrm{const},$$

where $a = 1, 2$,

$$\psi_1 = \psi_2^{-1} = e^{\lambda x^2} \qquad \text{if } e_1e_2 = 1,$$

$$\psi_1 = \sin\lambda x^2, \quad \psi_2 = \cos\lambda x^2 \quad \text{if } e_1e_2 = -1.$$

$\big(\Phi^2 = (\alpha+\sin\lambda x^1)/\cos^2\lambda x^1,\ \nu = -\tilde{c}_1 + c_2\psi_1 + c_3,\ \tilde{c}_0 = c_0 + \alpha\tilde{c}_1,\ \tilde{c}_1 = 2e_1e_2c_1/\lambda^2\big).$

When $q = \lambda^2 > 0$, by the change $x^i \to x^i + \mathrm{const}\cdot\delta^i_1$, the expression $\zeta = \alpha_1 + \alpha_2\cosh\lambda x^1 + \alpha_3\sinh\lambda x^1$ and the function Φ^2 are reduced to the form

$$\zeta = \alpha_0(\alpha + e^{\lambda x^1}), \qquad \Phi^2 = \frac{4}{\alpha_0\lambda^2}e^{-\lambda x^1}(1+\alpha e^{-\lambda x^1}) \qquad \text{if } \alpha_2 = \alpha_3,$$

$$\zeta = \alpha_0(\alpha + e^{-\lambda x^1}), \qquad \Phi^2 = \frac{4}{\alpha_0\lambda^2}e^{\lambda x^1}(1+\alpha e^{\lambda x^1}) \qquad \text{if } \alpha_2 = -\alpha_3,$$

$$\zeta = \alpha_0(\alpha + \sinh\lambda x^1), \quad \Phi^2 = \frac{4}{\alpha_0\lambda^2}(\alpha+\sinh\lambda x^1)/\cosh^2\lambda x^1 \quad \text{if } |\alpha_2| < |\alpha_3|,$$

$$\zeta = \alpha_0(\alpha + ch\lambda x^1), \qquad \Phi^2 = \frac{4}{\alpha_0\lambda^2}(\alpha+\cosh\lambda x^1)/\sinh^2\lambda x^1 \quad \text{if } |\alpha_2| > |\alpha_3|,$$

where α_0 and α are constants. Choosing in each case from the set of similar linear elements those for which $\alpha_0 = 4/\lambda^2$, we obtain the following metrics together with the corresponding quadratic first integrals (6.45) and solutions of Eq. (6.44), which will be the general solutions if the metrics have nonconstant curvature:

$$ds^2_{\text{IV}} = e^{-\lambda x^1}(1+\alpha e^{-\lambda x^1})(e_1 dx^{1^2} + e_2 dx^{2^2}) \quad (\alpha, \lambda = \text{const},\ \lambda \neq 0), \tag{6.57}$$

$$\begin{aligned} a_{ij}\,dx^i\,dx^j &= \Phi^2\Big(e_1\big(2\tilde{c}_0\Phi^2 - (c_2\phi_1 + c_3\phi_2)(1+2\alpha e^{-\lambda x^1})\big)dx^{1^2} \\ &\quad + 2\frac{e_1}{\lambda}(c_2\phi_1' + c_3\phi_2')e^{\lambda x^1}\Phi^2 dx^1 dx^2 + e_2(c_2\phi_1 + c_3\phi_2)dx^{2^2}\Big) + \tilde{c}_1 ds^2_{\text{IV}}, \end{aligned}$$

$$\varphi = \tilde{c}_0\Phi^2 + \tilde{c}_1 - (c_2\phi_1 + c_3\phi_2)\alpha e^{-\lambda x^1},$$

$$\Phi^2\left(e_1\phi_a \dot{x}^{1^2} - 2\frac{e_1}{\lambda}\phi_a' e^{\lambda x^1}\Phi^2\dot{x}^1\dot{x}^2 - e_2\phi_a(1+2\alpha e^{-\lambda x^1})\dot{x}^{2^2}\right) = \text{const}, \quad \left(\Phi^2 = e^{-\lambda x^1}(1+\alpha e^{-\lambda x^1})\right),$$

$$ds^2 = e^{\lambda x^1}(1+\alpha e^{\lambda x^1})(e_1 dx^{1^2} + e_2 dx^{2^2}) \quad (\alpha, \lambda = \text{const},\ \lambda \neq 0)$$

(by the transformation $x^1 \to -x^1$, $x^2 \to x^2$, this metric is reduced to the previous one and is not considered in what follows),

$$ds^2_{\text{V}} = \frac{\alpha + \sinh\lambda x^1}{\cosh^2\lambda x^1}(e_1 dx^{1^2} + e_2 dx^{2^2}) \quad (\alpha, \lambda = \text{const},\ \lambda \neq 0),$$

$$\begin{aligned} a_{ij}\,dx^i\,dx^j &= \Phi^2\Big(e_1\big(2\tilde{c}_0\Phi^2 + (c_2\phi_1 + c_3\phi_2)(1 - 2\Phi^2\sinh\lambda x^1)\big)dx^{1^2} \\ &\quad + 2\frac{e_1}{\lambda}(c_2\phi_1' + c_3\phi_2')(\cosh\lambda x^1)\Phi^2 dx^1 dx^2 + e_2(c_2\phi_1 + c_3\phi_2)dx^{2^2}\Big) + \tilde{c}_1 ds^2_{\text{V}}, \end{aligned}$$

$$\varphi = \tilde{c}_0\Phi^2 + \tilde{c}_1 + (c_2\phi_1 + c_3\phi_2)(1 - \Phi^2\sinh\lambda x^1),$$

$$\Phi^2\left(e_1\phi_a\dot{\phi}^{1^2} - 2\frac{e_1}{\lambda}\phi_a'(\cosh\lambda x^1)\Phi^2\dot{x}^1\dot{x}^2 + e_2\phi_a(1 - 2\Phi^2\sinh\lambda x^1)\dot{x}^{2^2}\right) = \text{const},$$

$$\left(\Phi^2 = (\alpha + \sinh\lambda x^1)/\cosh^2\lambda x^1\right),$$

$$ds^2_{\text{VI}} = \frac{\alpha + \cosh\lambda x^1}{\sinh^2\lambda x^1}(e_1 dx^{1^2} + e_2 dx^{2^2}) \quad (\alpha, \lambda - \text{const},\ \lambda \neq 0),$$

$$\begin{aligned} a_{ij}\,dx^i\,dx^j &= \Phi^2\Big(e_1\big(2\tilde{c}_0\Phi^2 + (c_2\phi_1 + c_3\phi_2)(1 - 2\Phi^2\cosh\lambda x^1)\big)dx^{1^2} \\ &\quad + 2\frac{e_1}{\lambda}(c_2\phi_1' + c_3\phi_2')\Phi^2(\sinh\lambda x^1)dx^1 dx^2 + e_2(c_2\phi_1 + c_3\phi_2)dx^{2^2}\Big) + c_1 ds^2_{\text{VI}}, \end{aligned}$$

$$\varphi = \tilde{c}_0\Phi^2 + \tilde{c}_1 + (c_2\phi_1 + c_3\phi_2)(1 - \Phi^2\cosh\lambda x^1),$$

$$\Phi^2\left(e_1\phi_a\dot{x}^{1^2} - 2\frac{e_1}{\lambda}\phi_a'\Phi^2(\sinh\lambda x^1)\dot{x}^1\dot{x}^2 + e_2\phi_a(1 - 2\Phi^2\cosh\lambda x^1)\dot{x}^{2^2}\right) = \text{const},$$

where $a = 1, 2$,

$$\begin{aligned} &\phi_1 = \sin\lambda x^2, \quad \phi_2 = \cos\lambda x^2 && \text{if } e_1 e_2 = 1, \\ &\phi_1 = \phi_2^{-1} = e^{\lambda x^2} && \text{if } e_1 e_2 = -1 \end{aligned}$$

$\big(\Phi^2 = (\alpha + \cosh\lambda x^1)/\sinh^2\lambda x^1,\ \nu = \tilde{c}_1 + c_2\phi_1 + c_3\phi_2\big)$.

The problem of finding the general solution of Eq. (6.44) in space (6.43) of nonconstant curvature ρ has been solved. We note that the condition $\rho' \neq 0$, which is equivalent, as shown above, to the condition $(\eta'''/\eta')' \neq 0$, is equivalent, by virtue of (6.53), to the inequality $(\eta''/(\eta - 2p))' \neq 0$, or $\Big((2\zeta\zeta'' - \zeta'^2)/\zeta^2\Big)' \neq 0$, where $\zeta = (\eta - 2p)^2$. From here, it is easy to deduce that ds^2_{I} and ds^2_{V} are the metrics of nonconstant curvature, and the conditions under which the remaining metrics have nonconstant curvature are $\alpha \neq 0$ for ds^2_{II} and ds^2_{IV} with $\lambda \neq 0$, $\alpha^2 \neq 1$ for $ds^2_{\text{III}}\ and ds^2_{\text{VI}}$ with $\lambda \neq 0$, $(\Phi'/\Phi)'/\Phi^2 \neq \text{const}$ for the metric (6.43).

We turn to the generalized Killing equations (3.21) for a projective motion $X = \xi^1\partial_1 + \xi^2\partial_2$. Taking into account that $L_X\rho = \xi^1\rho'$ and using (3.7) and (6.55), for the metric (6.43), we write Eqs. (3.21) with $h_{ij} = 2\varphi g_{ij} + a_{ij}$ together with conditions (6.38) of their integrability:

$$\text{(a)} \qquad \partial_1\xi^1 + \frac{\Phi'}{\Phi}\xi^1 = 2\varphi - \frac{\nu}{2},$$

$$\text{(b)} \qquad \partial_2\xi^2 + \frac{\Phi'}{\Phi}\xi^1 = \varphi + \frac{\nu}{2},$$

$$\text{(c)} \qquad e_1\partial_2\xi^1 + e_2\partial_1\xi^2 = c_{12},$$

$$\text{(d)} \qquad \partial_{12}\varphi - \frac{\Phi'}{\Phi}\partial_2\varphi + \Phi^2 c_{12}\rho = 0,$$

$$\text{(e)} \qquad \partial_{11}\varphi - \frac{\Phi'}{\Phi}\partial_1\varphi + e_1\Phi^2\left(\xi^1\rho' + (4\varphi - e_2c_{22})\rho\right) = 0.$$

Differentiating (e) with respect to x^2 and (d) with respect to x^1 and subtracting, with the help of (c) and (β) and (γ) (see above), we obtain the condition $\rho'\Phi^2\partial_1\xi^2 = 0$, which, by virtue of the inequality $\rho'\Phi^2 \neq 0$, implies $\partial_1\xi^2 = 0$, i.e., ξ^2 depends only on x^2 and, from (a) and (b),

$$\partial_2\xi^1 = e_1c_{12}, \quad \partial_1\xi^1 = \varphi - \nu + \xi^{2'}. \tag{6.58}$$

Now, equating the mixed second derivatives of ξ^1 and using (γ), we find that

$$\xi^2 = \int \nu dx^2 + a_1x^2 + a_2, \quad \varphi = \partial_1\xi^1 - a_1 \qquad (a_1, a_2 = \text{const}). \tag{6.59}$$

Integrating (a), we obtain

$$\xi^1 = \left((2a_1 + \frac{\nu}{2})(\eta - 2p) + \tau\right)\Phi,$$

where p is the constant from (6.50) and τ is a function of x^2.

Substituting the expression found above in the first equality of (6.58), we have

$$\left(\nu'(\eta - 2p) + 2\tau'\right)\Phi = 2e_1c_{12}.$$

If $C \neq 0$, i.e., if condition (6.51) does not hold, then, as shown, $\nu' = c_{12} = 0$ and from the above equation, it follows that $\tau = a_3 = \text{const}$. In the case $C = 0$, we come to the same conclusion with the help of (6.46) and (6.50). This implies

$$\xi^1 = \left(\left(2a_1 + \frac{\nu}{2}\right)(\eta - 2p) + a_3\right)\Phi = (4a_1 + \nu)\frac{\zeta}{\zeta'} + a_3\Phi, \tag{6.60}$$

$$\varphi = \left(\left(2a_1 + \frac{\nu}{2}\right)(\eta - 2p) + a_3\right)\Phi' + \frac{1}{2}\nu + a_1 = \nu\left(\frac{\zeta}{\zeta'}\right)' + (2a_1(\eta - 2p) + a_3)\Phi' + a_1. \tag{6.61}$$

If $C \neq 0$ and $\rho' \neq 0$, then, according to (6.52), $\varphi = c\Phi^2 + \nu$, where c and ν are constants. Since every symmetric bilinear form on a two-dimensional pseudo-Riemannian manifold a of non-constant curvature is proportional to its metric (see the remark after the proof of Theorem 6.2), then under the assumptions made, every projective motion X on a surface of revolution satisfies the condition $L_Xg = c(a + 2\Phi^2 g) + \text{const} \cdot g$, where a is a solution of Eq. (6.44) with $\varphi = \Phi^2$. From here, repeating the deduction of Theorem 6.3, it is easy to prove that in this case, the homothetic subalgebra either coincides with the maximal projective Lie algebra or has codimension one in it.

Since ξ^i and φ are determined from Eqs. (3.21) and (3.22) to within a constant factor and $c \neq 0$, one can set $\varphi = \Phi^2 + \nu$ (otherwise, $\varphi = \text{const}$, i.e., a projective motion is affine and, by Theorem 6.2, reduces to an infinitesimal homothety).

Now from (6.61), we obtain the equation

$$((2a-b)(\eta-2p)+a_3)\,\Phi' = \Phi^2 - a + b \qquad \left(a = 3a_1+\nu,\ b = 4a_1 + \frac{3}{2}\nu\right),$$

which implies

$$\Phi'' = \frac{2\Phi^2-2a+b}{\Phi(\Phi^2-a+b)}\Phi'^2,$$

hence

$$\Phi' = k\exp\int\frac{2\Phi^2-2a+b}{\Phi(\Phi^2-a+b)}d\Phi \qquad (a,b,k=\text{const},\ k\neq 0). \tag{6.62}$$

In the general case this equation is not integrable in finite form.

Equations (6.59) and (6.60) imply that every space (6.43) under the condition (6.62) admits a two-dimensional non-affine projective Lie algebra with structure equation $[E_1, E_2] = (4b-5a)E_1$ and basis including an infinitesimal isometry $\underset{i}{E}_1$ and a non-affine projective motion $\underset{p}{E}_2$:

$$\underset{i}{E}_1 = \partial_2, \quad \underset{p}{E}_2 = \frac{\Phi}{\Phi'}(\Phi^2-a+b)\partial_1 + (4b-5a)x^2\partial_2 \qquad (\Phi'\neq 0).$$

From the expressions for the scalar curvature ρ and its derivative ρ' in the space (6.43), which is defined by (6.62):

$$\rho = \frac{e_1(a-\Phi^2)\Phi'^2}{\Phi^4(\Phi^2-a+b)}, \quad \rho' = \frac{-2e_1ab\Phi'^3}{\Phi^5(\Phi^2-a+b)^2},$$

one can see that among the spaces under consideration, there occur spaces of constant curvature (only when $ab\Phi'=0$) and spaces for which condition (6.51) holds (only when $a=-2$, $b=-3$, when the metric reduces to ds^2_{II}).

We show that when $ab\Phi'\neq 0$ and $(a,b)\neq(-2,-3)$, the Lie algebra we have found above is maximal. Indeed, in this case, according to the above remark, the basis for the maximal projective Lie algebra can be obtained by taking a non-homothetic projective motion, for example, $\underset{p}{E}_2$, together with a basis for the homothetic subalgebra, which, in turn, is obtained by taking a non-isometric infinitesimal homothety, if it exists, together with a basis for the isometric subalgebra, because in a pseudo-Riemannian space, an isometric subgroup either coincides with a group of homotheties or has codimension one in it.[10]

It is easy to prove that a fundamental form of every two-dimensional surface that admits an infinitesimal isometry X reduces to the form

$$ds^2 = e_1\,dy^{1^2} + g_{22}\,dy^{2^2} \qquad (X=\partial_2),$$

where g_{22} does not depend on the coordinate y^2 (see [191], p. 291). Hence, if we make the transformation of coordinates

$$x^1 = \int\frac{dy^1}{\sqrt{|g_{22}|}}, \quad x^2 = y^2$$

and set $|g_{22}| = \Phi^2(x^1)$ and $e_2 = \mathrm{sgn}(g_{22})$, then we obtain the linear element (6.43).

If X is an infinitesimal isometry in (6.43), when $L_X g_{jl} = 0$ and $\varphi = \text{const}$, then (6.38) implies $L_X\rho = \xi^1\rho' = 0$, i.e., $\xi^1 = 0$ when $\rho'\neq 0$ and from Eqs. (a)–(c) with zero right-hand sides, we find that $\xi^2 = \text{const}$. Therefore, the *maximal isometric Lie algebra in the space* (6.43) *of non-constant curvature is the one-dimensional Lie algebra spanned by* ∂_2. In view of this property, the *group of isometries of a two-dimensional pseudo-Riemannian space of non-constant curvature is at most one-dimensional.*

All that remains is to solve a problem about infinitesimal homotheties. Since, as remarked above, in a pseudo-Riemannian space, the isometric subgroup either coincides with the group of homotheties or has codimension one in it, what has gone before implies that the *affine group of a two-dimensional*

[10] This fact, which is traditionally connected with the name of Knebelman [302], was first established by Fubini [262].

pseudo-Riemannian space M^2 with non-constant curvature, which coincides, by Theorem 6.2, *with its homothetic subgroup, is at most two-dimensional.*

If X is an infinitesimal homothety in the metric (6.43), then the following equations are satisfied: $L_X g_{ij} = 2\kappa g_{ij}$, $\kappa = \text{const}$, and (6.38) when $\varphi = \text{const}$ implies $\xi^1\rho' + 2\kappa\rho = 0$; hence $\partial_2\xi^1 = 0$ when $\rho' \neq 0$. Therefore, integrating the equations $L_X g = 2\kappa g$, we find that

$$X = (ax^1 + b)\partial_1 + (ax^2 + d)\partial_2, \quad \Phi = s\left|x^1 + \frac{b}{a}\right|^{\frac{\kappa}{a}-1} \qquad (a, b, d, s = \text{const},\ a, s \neq 0).$$

Without loss of generality, we may set $a = s = 1$. Then denoting $\alpha \equiv \kappa - 1$ and making the change $x^1 + b \to x^1$, $x^2 + d \to x^2$, we have

$$ds^2 = \left(x^1\right)^{2\alpha}(e_1 dx^{1^2} + e_2 dx^{2^2}).$$

This metric admits the two-dimensional Lie algebra H_2 of homothetic motions which is spanned by the vector fields

$$\underset{i}{E}_1 = \partial_2, \quad \underset{h}{E}_2 = x^1\partial_1 + x^2\partial_2 \qquad ([E_1, E_2] = E_1),$$

where $\underset{h}{E}_2$ is a nonisometric infinitesimal homothety. When $\alpha \neq 0, -1$ (under this condition, the metric has nonconstant curvature) the above Lie algebra is maximal, and the metric we have found defines a *two-dimensional pseudo-Riemannian space of nonconstant curvature of the maximal affine mobility.*

If we substitute the solution we have found in (6.62), then we obtain $\alpha = 0$, $\Phi = \text{const}$ or $\alpha + 1 = b = 0$, i.e., $\rho = \text{const}$. Consequently, *spaces* (6.43) *and* (6.62) *of nonconstant curvature do not admit nonisometric infinitesimal homotheties, and the maximal projective Lie algebra in this space when* $(a, b) \neq (-2, -3)$ *is spanned by the vector fields* $\underset{i}{E}_1$ *and* $\underset{p}{E}_2$.

If X is an infinitesimal homothety in M^2, then, reducing X to the form $X = \partial_1$, which is always possible ([191], p. 14), and integrating the equations

$$L_X g_{ij} = 2\kappa g_{ij} \qquad (\kappa = \text{const}),$$

we obtain

$$ds^2 = e^{2\kappa x^1}\left(E\, dx^{1^2} + 2F\, dx^1\, dx^2 + G\, dx^{2^2}\right),$$

where E, F, and G are functions of x^2. By changing the variables

$$\bar{x}^1 = x^1 + \int \frac{F}{E}\, dx^2, \quad \bar{x}^2 = \int e^{-\kappa \int \frac{F}{E}\, dx^2} \sqrt{\left|G - \frac{F^2}{E}\right|}\, dx^2$$

ds^2 reduces to the form (the bar has been omitted)

$$ds^2 = e^{2\kappa x^1}\left(f(x^2) dx^{1^2} + e_2 dx^{2^2}\right) \qquad (e_2 = \pm 1),$$

where f is a function of x^2, and, moreover, X does not change. The metric of every two-dimensional pseudo-Riemannian space admitting a nonisometric homothetic motion is transformed to such a form. If the scalar curvature of this metric,

$$\rho = \frac{1}{4f^2} e_2 e^{-2\kappa x^1}\left(f'^2 - 2ff''\right),$$

is constant, then it is zero (one can ascertain this by differentiating ρ with respect to x^1) and $f = (c_1 x^2 + c_2)^2$.

We consider the metrics $ds^2_{\mathrm{I}} - ds^2_{\mathrm{VI}}$ under condition (6.51). In this case, (6.55) with $\zeta(\eta - 2p)^2$ and $\Phi = 1/\eta'$ and (6.61) with $\Phi' = -\eta''/\eta'^2$ imply the relation

$$c_0 + \frac{1}{2} e_1 e_2 c_1 \bar{\eta}^2 = a_1 \eta'^2 - (2a_1\bar{\eta} + a_3)\eta'' \qquad (\bar{\eta} \equiv \eta - 2p),$$

from which, by differentiating and using (6.53), we find that

$$(e_1e_2c_1 + 2a_1q)\bar{\eta}^2 + qa_3\bar{\eta} = 3(2a_1\bar{\eta} + a_3)\eta''.$$

When $2a_1\bar{\eta} + a_3 \neq 0$, we have

$$3\eta'' = \frac{\bar{\eta}(s\bar{\eta} + qa^3)}{2a_1\bar{\eta} + a_3} \qquad (s \equiv e_1e_2c_1 + 2a_1q).$$

If we find the expression η'''/η' and substitute it together with the above expression into (6.53), we obtain the identity

$$4a_1(2s - 3a_1q)\bar{\eta}^2 + a_3(5s - 6a_1q)\bar{\eta} + a_3^2q = 0,$$

which, by $\eta' \neq 0$, implies $qa_3^2 = 0$. If $a_3 \neq 0$, then $q = 0$ and the identity becomes $8a_1s\bar{\eta}^2 + 5a_3s\bar{\eta} = 0$; hence $s = 0$, and, consequently, $\eta'' = 0$. If $a_3 = 0$, then from (6.62), we have $6\eta'' = s\bar{\eta}/a_1$, i.e., $\eta'''/\eta' = \text{const}$. In both cases, the space has constant curvature, which contradicts our assumption. Therefore, $2a_1\eta + a_3 = 0$, i.e., $a_1 = a_3 = c_0 = c_1 = 0$.

The above results imply that *all projective motions in a space with metrics* $ds_I^2 - ds_{VI}^2$ *of non-constant curvature are defined by formulas* (6.59) and (6.60), *in which it is necessary to set* $a_1 = a_3 = 0$:

$$X = \xi^1\partial_1 + \xi^2\partial_2 = \nu\frac{\zeta}{\zeta'}\partial_1 + \left(\int \nu dx^2 + a_2\right)\partial_2,$$

where $\nu = c_2x^2 + c_3$ for ds_{I}^2 and ds_{II}^2, $\quad \nu = c_2\psi_1 + c_3\psi_2$ for ds_{III}^2, and $\nu = c_2\phi_1 + c_3\phi_2$ for $ds_{\text{IV}}^2 - ds_{\text{VI}}^2$. Since the constants a_2, c_2 and c_3 are arbitrary, the *dimension of the maximal projective Lie algebras in the spaces with metrics* $ds_{\text{I}}^2 - ds_{\text{VI}}^2$ *of nonconstant curvature is equal to* 3. The corresponding elements of the bases and the structure equations are determined in Theorem 6.7.

6.3.3. Projective motions in h-spaces of type {2}. By Theorem 4.1, in appropriate local coordinates, the metric of an h-space of type {2} reduces to

$$ds^2 = 2\left(x^1 + w(x^2)\right)dx^1dx^2 \equiv 2Adx^1dx^2. \tag{6.63}$$

The components of the curvature tensor of this metric are determined by Eq. (6.34), where $\rho = w'A^{-3}$. If $\rho = \text{const}$, then $\partial_1\rho = 0$; hence $w' = \rho = 0$. Therefore, *every space* (6.63) *of constant curvature is a flat space defined by the condition* $w' = 0$. Therefore, from now on it will be assumed that $w' \neq 0$, and Eqs. (6.44) take the form

$$\begin{aligned}
&1^\circ \quad \partial_1a_{11} = \frac{2}{A}a_{11}, && 4^\circ \quad \partial_2a_{12} - \frac{w'}{A}a_{12} = A\varphi_{,2},\\
&2^\circ \quad \partial_2a_{11} = 2A\varphi_{,1}, && 5^\circ \quad \partial_1a_{22} = 2A\varphi_{,2},\\
&3^\circ \quad \partial_1a_{12} - \frac{1}{A}a_{12} = A\varphi_{,1}, && 6^\circ \quad \partial_2a_{22} - 2w'A^{-1}a_{22} = 0.
\end{aligned}$$

From 1° and 2°, we obtain $a_{11} = \alpha_2(x^2)A^2$ and $2\varphi_{,1} = 2\alpha_2w' + \alpha_2'$. After that, from 5° and 6° we find that $a_{22} = \alpha_1(x^1)A^2$ and $2\varphi_{,2} = 2\alpha_1 + \alpha_1'A$, and by 3° and 4° we have $a_{12} = (\varphi + 3\mu)A$, where μ are constants. By equating the mixed second derivatives of the function φ, we have

$$2\alpha_2w'' + 3\alpha_2'w' + \alpha_2''A = 3\alpha_1' + \alpha_1''A;$$

from here, separating the variables, we obtain $\alpha_1 = c_4x^{1^2} + c_5x^1 + c_1$, $\alpha_2 = 4c_4x^{2^2} + c_2x^2 + c_3$ and $2\alpha_2w'' + 3w'\alpha_2' + 6c_4w = 3c_5$, where $c_1, \ldots, c_5$ are constants.

Under these conditions, integrating the generalized Killing equations (3.21) with $h_{ij} = a_{ij} + 2\varphi g_{ij}$, we obtain the relations $c_4 = c_5 = 0$ and

$$\xi^1 = (\varphi + \mu)A + \xi^2\left(\frac{1}{3}\frac{w''}{w'}A - w'\right), \quad \xi^2 = \frac{1}{4}\alpha_2A^2 + \sigma(x^2),$$

$$\varphi = \alpha_2 w' A + \frac{1}{4}\alpha_2' A^2 + \psi(x^2),$$

$$2\alpha_2 w'' + 3\alpha_2' w' = 0, \quad \alpha_1 = c_1, \quad \alpha_2 = c_2 x^2 + c_3, \tag{6.64}$$

$$\sigma' + \frac{2}{3}\frac{w''}{w'}\sigma = \psi + \mu, \tag{6.65}$$

where σ and ψ are functions of x^2. We consider separately the following three cases:

(1) $\alpha_2' \neq 0$,
(2) $\alpha_2' = 0$, $\alpha_2 \neq 0$, and
(3) $\alpha_2 = 0$.

(1) In the case $\alpha_2' \neq 0$, from (6.64) we find that

$$w = \gamma + \beta \left| x^2 + \frac{c_3}{c_2} \right|^{-\frac{1}{2}},$$

where β and γ are constants. After making the transformation of coordinates $\bar{x}^1 = x^1 + \gamma$ and $\bar{x}^2 = x^2 + c_3/c_2$, which does not change the form of the metric (6.63), omitting the bar, we have

$$A \equiv x^1 + w = x^1 + \beta |x^2|^{-1/2}, \quad \alpha_2 = c_2 x^2,$$
$$\varphi = c_1 x^2 + (1/4) c_2 x^{1^2} + \nu,$$
$$\psi = c_1 x^2 + (1/4) c_2 \beta^2 |x^2|^{-1} + \nu.$$

where ν is a constant. If we substitute the last expression into (6.65), we obtain

$$\sigma = \left(c_1 x^2 - \frac{1}{4} c_2 \beta^2 |x^2|^{-1} + (\mu + \nu) \ln |x^2| + a_1 \right) x^2 \qquad (a_1 = \text{ const}).$$

As a result, we have

$$\xi^1 = (1/2) c_1 x^2 \left(x^1 + 2\beta |x^2|^{-1/2} \right) + (1/8) c_2 x^{1^3} - (1/2) a_1 x^1,$$
$$\xi^2 = c_1 x^{2^2} + (1/4) c_2 x^1 x^2 \left(x^1 + 2\beta |x^2|^{1/2} \right) + a_1 x^2.$$

These formulas define the general solution of Eqs. (3.21) in the case considered. Since the constants a_1, c_1, and c_2 are arbitrary, the dimension of the projective Lie algebra is equal to 3, and its basis can be chosen in the following way:

$$\underset{i}{E}_1 = -x^1 \partial_1 + 2x^2 \partial_2,$$
$$\underset{p}{E}_2 = x^2 \left(x^1 + 2\beta |x^2|^{-1/2} \right) \partial_1 + 2x^{2^2} \partial_2,$$
$$\underset{p}{E}_3 = x^{1^3} \partial_1 + 2x^1 x^2 \left(x^1 + 2\beta |x^2|^{-1/2} \right) \partial_2,$$
$$\left([E_1, E_2] = 2E_2, \quad [E_3, E_1] = 2E_3, \quad [E_2, E_3] = 8\beta^2 \operatorname{sgn}(x^2) E_1 \right),$$

where $\underset{i}{E}_1$ is the isometric motion, whereas $\underset{p}{E}_2$ and $\underset{p}{E}_3$ are projective motions, no one of which can be obtained from the other by combining with isometric motions.

The first integrals of the equations for geodesics generated by the infinitesimal transformations E_1, E_2, and E_3, are

$$A(2x^2 \dot{x}^1 - x^1 \dot{x}^2) = \text{const},$$
$$A\dot{x}^2 (A\dot{x}^2 - 2x^2 \dot{x}^1) = \text{const},$$
$$4A\dot{x}^1 (2Ax^2 \dot{x}^1 - x^{1^2} \dot{x}^2) = \text{const}.$$

To these integrals it is necessary to add the integral $g_{ij}\dot{x}^i\dot{x}^j = \text{const}$, i.e.,

$$2A\dot{x}^1\dot{x}^2 = \epsilon,$$

where ϵ is equal to 0, +1, or −1 for null, timelike, and spacelike geodesics, respectively.

(2) Let $\alpha_2' = 0$ and $\alpha_2 \neq 0$. In this case, from (6.64) we obtain $w = \beta x^2 + \delta$, where δ is a constant. After the transformation of the coordinates $\bar{x}^1 = x^1 + \delta$, $\bar{x}^2 = x^2$, omitting the bar, we have

$$\begin{aligned} &A \equiv x^1 + w = x^1 + \beta x^2, \\ &\varphi = c_1 x^2 + c_3 \beta x^1 + \nu, \\ &X = \xi^1\partial_1 + \xi^2\partial_2 = a_1 E_1 + a_2 E_2 + (1/4)c_3 E_3, \end{aligned}$$

where the infinitesimal transformations

$$\begin{aligned} &\underset{i}{E}{}_1 = -\beta\partial_1 + \partial_2, \\ &\underset{h}{E}{}_2 = x^1\partial_1 + x^2\partial_2, \\ &\underset{p}{E}{}_3 = \beta(3x^{1^2} - 2\beta x^1 x^2 - \beta^2 x^{2^2})\partial_1 + (x^{1^2} + 2\beta x^1 x^2 - 3\beta^2 x^{2^2})\partial_2 \end{aligned}$$

form the basis for the three-dimensional Lie algebra with structure equations

$$[E_1, E_2] = E_1, \quad [E_1, E_3] = -8\beta^2 E_2, \quad [E_2, E_3] = E_3$$

($\beta \neq 0$ and ν are constants). The corresponding first integrals of the equation for geodesics are

$$\begin{aligned} A\left(A\dot{x}^{1^2} - 2\beta(x^1 - \beta x^2)\dot{x}^1\dot{x}^2 - \beta^2 A\dot{x}^{2^2}\right) &= \text{const}, \\ A(\dot{x}^1 - \beta\dot{x}^2) &= \text{const}, \\ 2A\dot{x}^1\dot{x}^2 &= \epsilon \qquad (\epsilon = 0, \pm 1). \end{aligned}$$

(3) In the case $\alpha_2 = 0$, we have

$$w = k\int \exp\left(\frac{3}{2}\int \frac{a_1 - a_2 - x^2}{x^{2^2} + a_2 x^2 + a^3}dx^2\right)dx^2, \tag{6.66}$$

where a_1, a_2, a_3, and k are constants.

Every space defined by formulas (6.63) and (6.66), admits a one-dimensional Lie algebra of projective motions spanned by the vector field

$$\underset{p}{E} = \left((x^2 + 3a_1 - a_2)(x^1 + w) - 2(x^{2^2} + a_2 x^2 + a_3)w'\right)\partial_1 + 2(x^{2^2} + a_2 x^2 + a_3)\partial_2.$$

The corresponding first integral of the equation for geodesics is

$$A\dot{x}^2\left(A\dot{x}^2 - 2(x^2 + a_1)\dot{x}^1\right) = \text{const}.$$

Using the first integrals that we have found to study the properties of timelike and null geodesics ofthe two-dimensional space-times under consideration, one can determine the physical structure of these spaces.

We note that when $A = x^1 + \beta x^2$ and $A = x^1 + \beta|x^2|^{-1/2}$, the metric (6.63) admits infinitesimal isometries, and, consequently, reduces to the form (6.43) (see Sec. 6.3.2). By making the change of variables

$$\bar{x}^1 = \ln|x^{1^2}x^2|, \quad \bar{x}^2 = \ln\frac{x^{1^2}}{|x^2|}$$

the metric

$$ds^2 = 2\left(x^1 + \frac{\beta}{\sqrt{|x^2|}}\right)dx^1 dx^2 \qquad (\beta = \text{const} \neq 0)$$

can be reduced to the form (the bar is omitted)

$$ds^2 = (\beta/4)e^{x^1/2}\left(1+\alpha e^{x^1/2}\right)\left(dx^{1^2} - dx^{2^2}\right),$$

which is similar to (6.57) with $\alpha = 1/\beta$, $\lambda = -1/2$, and $e_1 = -e_2 = 1$, while the metric

$$ds^2 = 2(x^1 + \beta x^2)dx^1 dx^2 \qquad (\beta = \text{const} \neq 0),$$

by making the change of variables $x^1 + \beta x^2 \to x^1$, $x^1 - \beta x^2 \to x^2$, is reduced to the form

$$ds^2 = \frac{1}{2\beta}x^1\left(dx^{1^2} - dx^{2^2}\right),$$

which is similar to (6.54) with $e_1 = -e_2 = 1$.

6.4. Classification of two-dimensional pseudo-Riemannian manifolds with respect to the Lie algebras of projective and affine motions. The results we have obtained in Secs. 6.2 and 6.3 imply the following theorem.

Theorem 6.5 (Aminova [43]). *The dimension of the maximal projective Lie algebra in a two-dimensional pseudo-Riemannian manifold M^2 of nonconstant curvature is less than or equal to three. An affine Lie algebra in M^2 consists of at most infinitesimal homotheties, and its dimension does not exceed the number 2. If M^2 admits an infinitesimal projective transformation, then the linear element $ds^2 = g_{ij}\,dx^i\,dx^j$ of this space and the maximal projective (affine) Lie algebra P acting in it are determined in coordinate neighborhoods by the following conditions, in which $\underset{i}{E}_i$ are infinitesimal isometries, $\underset{h}{E}_j$ are non-isometric infinitesimal homotheties, and $\underset{p}{E}_k$ are non-affine infinitesimal projective transformation, spanning P.* [11]

dim $P = 3$*:*

$$ds^2_{\mathrm{I}} = x^1\left(e_1 dx^{1^2} + e_2 dx^{2^2}\right),$$

$$\underset{i}{E}_1 = \partial_2, \quad \underset{h}{E}_2 = x^1\partial_1 + x^2\partial_2, \quad \underset{p}{E}_3 = 2x^1x^2\partial_1 + x^{2^2}\partial_2 ([E_1, E_2] = E_1, \quad [E_1, E_3] = 2E_2, \quad [E_2, E_3] = E_3);$$

$$ds^2_{\mathrm{II}} = \left(1 + \frac{\alpha}{x^{1^2}}\right)\left(e_1\,dx^{1^2} + e_2 dx^{2^2}\right) \qquad (\alpha = \mathit{const} \neq 0),$$

$$\underset{i}{E}_1 = \partial_2, \quad \underset{p}{E}_2 = \left(x^1 + \alpha/x^1\right)\partial_1 + 2x^2\partial_2, \quad \underset{p}{E}_3 = x^2\left(x^1 + \alpha/x^1\right)\partial_1 + x^{2^2}\partial_2$$

$$([E_1, E_2] = 2E_1, \quad [E_1, E_3] = E_2, \quad [E_2, E_3] = 2E_3);$$

$$ds^2_{\mathrm{III}} = \left[(\alpha + \sin\lambda x^1)/\cos^2\lambda x^1\right]\left(e_1\,dx^{1^2} + e_2\,dx^{2^2}\right),$$

$$ds^2_{\mathrm{IV}} = e^{-\lambda x^1}\left(1 + \alpha e^{-\lambda x^1}\right)\left(e_1\,dx^{1^2} + e_2\,dx^{2^2}\right),$$

$$ds^2_{\mathrm{V}} = \left[(\alpha + \sinh\lambda x^1)/\cosh^2\lambda x^1\right]\left(e_1\,dx^{1^2} + e_2\,dx^{2^2}\right),$$

$$ds^2_{\mathrm{VI}} = \left[(\alpha + \cosh\lambda x^1)/\sinh^2\lambda x^1\right]\left(e_1 dx^{1^2} + e_2 dx^{2^2}\right)$$

$(\alpha, \lambda = \text{const},\ \lambda \neq 0)$,

$$\underset{i}{E}_1 = \partial_2, \quad \underset{p}{E}_2 = \Lambda\sin\lambda x^2\partial_1 - \cos\lambda x^2\partial_2, \quad \underset{p}{E}_3 = \Lambda\cos\lambda x^2\partial_1 + \sin\lambda x^2\partial_2$$

$$([E_1, E_2] = \lambda E_3, \quad [E_3, E_1] = \lambda E_2, \quad [E_3, E_2] = \lambda E_1)$$

in the case ds^2_{III} with $e_1e_2 = -1$ and $ds^2_{\mathrm{IV}} - ds^2_{\mathrm{VI}}$ with $e_1e_2 = +1$,

$$\underset{i}{E}_1 = \partial_2, \quad \underset{p}{E}_2 = e^{\lambda x^2}(\Lambda\partial_1 + \partial_2), \quad \underset{p}{E}_3 = e^{-\lambda x^2}(\Lambda\partial_1 - \partial_2)$$

[11] In all cases where the basis for a Lie algebra P contains more than one projective motion $\underset{p}{E}$, no one of them can be obtained from the others by combining with infinitesimal homotheties and isometries.

$$([E_1, E_2] = \lambda E_2, \quad [E_3, E_1] = \lambda E_3, \quad [E_2, E_3] = 2\lambda E_1)$$

in the case ds^2_{III} *with* $e_1 e_2 = +1$ *and* $ds_{\mathrm{IV}} - ds^2_{\mathrm{VI}}$ *with* $e_1 e_2 = -1$,

$$\Lambda = (\alpha + \sin\lambda x^1)/\cos\lambda x^1 \quad for \quad ds^2_{\mathrm{III}}, \quad 1 + \alpha e^{-\lambda x^1} \quad for \quad ds^2_{\mathrm{IV}},$$

$$(\alpha + \sinh\lambda x^1)/\cosh\lambda x^1 \;\; for \;\; ds^2_{\mathrm{V}}, \quad (\alpha + \cosh\lambda x^1)/\sinh\lambda x^1 \quad for \quad ds^2_{\mathrm{VI}}.$$

dim $P = 2$*:*

$$ds^2_{\mathrm{VII}} = \Phi^2(x^1)\left(e_1\, dx^{1^2} + e_2\, dx^{2^2}\right),$$

$$\frac{d\Phi}{dx^1} \equiv \Phi' = k \exp \int \frac{2\Phi^2 - 2a + b}{\Phi\left(\Phi^2 - a + b\right)} d\Phi,$$

$$\underset{i}{E}\,1 = \partial_2, \quad \underset{p}{E}\,2 = \left(\Phi/\Phi'\right)\left(\Phi^2 - a + b\right)\partial_1 + (4b - 5a)x^2\partial_2$$

$$([E_1, E_2] = (4b - 5a)E_1 \quad (a, b, k = const, \quad ab(a+2)(b+3)k \neq 0);$$

$$ds^2_{\mathrm{VIII}} = \left(x^1\right)^{2\alpha}\left(e_1 dx^{1^2} + e_2\, dx^{2^2}\right) \quad (\alpha = \text{const}),$$

$$\underset{i}{E}\,1 = \partial_2, \quad \underset{h}{E}\,2 = x^1\partial_1 + x^2\partial_2 \quad ([E_1, E_2] = E_1)$$

(*the maximal affine* (*homothetic*) *Lie algebra in a space* M^2 *of nonconstant curvature when* $\alpha \neq 0, -1$).

Each of the spaces defined by the linear elements ds^2_{IX}–ds^2_{XIII} *admits a one-dimensional projective Lie algebra, which is spanned by* E_1*:*

$$ds^2_{\mathrm{IX}} = 2\left(x^1 + w(x^2)\right)\, dx^1\, dx^2,$$

$$w(x^2) = k \int \exp\left(\frac{3}{2}\int \frac{c - a - x^2}{x^{2^2} + ax^2 + b}\, dx^2\right) dx^2 \qquad (a, b, c, k = \text{const}, \;\; k \neq 0),$$

$$\underset{p}{E}\,1 = \left(\frac{1}{2}\left(x^1 + w(x^2)\right)\left(x^2 + 3c - a\right) - \left(x^{2^2} + ax^2 + b\right) w'(x^2)\right)\partial_1 + \left(x^{2^2} + ax^2 + b\right)\partial_2;$$

$$ds^2_{\mathrm{X}} = \left(f_1(x^1) - f_2(x^2)\right)\left(e_1 dx^{1^2} + e_2 dx^{2^2}\right),$$

$$\frac{df_i}{dx^i} \equiv f_i' = k_i \exp\left(\frac{3}{2}\int \frac{f_i + a}{f_i^2 + af_i + b} df_i\right) \qquad (a, b, k_i = \text{const}, \;\; k_i \neq 0, \;\; i = 1, 2),$$

$$\underset{p}{E}\,1 = \frac{f_1^2 + af_1 + b}{f_1'}\partial_1 + \frac{f_2^2 + af_2 + b}{f_2'}\partial_2$$

;

$$\sqrt{-1} ds^2_{\mathrm{XI}} = \left(f_1(z^1) - f_2(z^2)\right)\left(dz^{1^2} + dz^{2^2}\right),$$

$$\frac{df_1}{dz^1} \equiv f_1' = k \exp\left(\frac{3}{2}\int \frac{f_1 + a}{f_1^2 + af_1 + b}\, df_1\right) \qquad (k, a, b = \text{const}, \;\; k \neq 0),$$

$$\underset{p}{E}\,1 = \Re\frac{f_1^2 + af_1 + b}{f_1'}\partial_1 + \Im\frac{f_1^2 + af_1 + b}{f_1'}\partial_2$$

$(z^1 = x^1 + ix^2 = \overset{*}{z}{}^2, \; f_1 = f_1(z^1) \equiv u(x^1, x^2) + iv(x^1, x^2) = \overset{*}{f}_2, \; \partial_1 u = \partial_2 v, \; \partial_2 u = -\partial_1 v)$*;*

$$ds^2_{\mathrm{XII}} = e^{2cx^1}\left(f(x^2)dx^{1^2} + e_2 dx^{2^2}\right) \qquad (c = \text{const}, \;\; e_2 = \pm 1).$$

$$\underset{h}{E}\,1 = \partial_1,$$

$$ds^2_{\mathrm{XIII}} = \Phi^2(x^1)\left(e_1 dx^{1^2} + e_2 dx^{2^2}\right),$$

$$\underset{i}{E}\,1 = \partial_2$$

(*the maximal isometric Lie algebra in a space* M^2 *of non-constant curvature when* $\left(\Phi'/\Phi\right)'/\Phi^2 \neq \text{const}$).

We mention the solution of the problem for the two-dimensional space of constant curvature $K \neq 0$ with the linear element in the Riemannian form

$$ds^2_{\mathrm{XIV}} = \frac{e_1 dx^{1^2} + e_2 dx^{2^2}}{\sigma^2}, \quad \text{where} \;\; \sigma = 1 + \frac{K}{4}\left(e_1 x^{1^2} + e_2 x^{2^2}\right) \qquad (e_1, e_2 = \pm 1).$$

The maximal projective Lie algebra in this space has dimension 8. Its basis includes three infinitesimal isometries and five nonaffine projective motions:

$$\underset{i}{E}_{\,i} = x^i(x^1\partial_1 + x^2\partial_2) + (2e_i/K)(2-\sigma)\partial_i, \quad \underset{i}{E}_{\,3} = e_1 x^2\partial_1 - e_2 x^1\partial_2, \quad \underset{p}{E}_{\,3+i} = \left(x^i/\sigma\right)(x^1\partial_1 + x^2\partial_2),$$

$$\underset{p}{E}_{\,4+i+j} = \left(x^i x^j/\sigma\right)(x^1\partial_1 + x^2\partial_2) - (2e_j/K)x^i\partial_j \qquad (i,j = 1,2,\; i \leq j),$$

whose commutators are given in the tables below, where the commutator $[E_r, E_s]$ stands at the intersection of the rth row and the sth column.

Table 2. The structure of the 8-dimensional projective Lie algebra of the two-dimensional space of constant curvature. The commutator $[E_r, E_s]$ of the elements E_r, E_s of the basis for the algebra stands at the intersection of the rth row and the sth column.

$\hookrightarrow$	1	2	3	4
1	0	$\frac{4}{K}E_3$	$-e_1E_2$	$-2E_6 - e_1e_2E_8$
2		0	e_2E_1	$-E_7$
3			0	e_1E_5

$\hookrightarrow$	5	6	7	8
1	$\frac{2}{K}E_3 - E_7$	$\frac{2}{K}e_1(4E_4 - E_1)$	$\frac{2}{K}e_1(2E_5 - E_2)$	0
2	$-2E_8 - e_1e_2E_6$	0	$\frac{4}{K}e_2E_4$	$\frac{2}{K}e_2(4E_5 - E_2)$
3	$-e_2E_4$	$2e_1(E_7 - \frac{1}{K}E_3)$	$e_1E_8 - e_2E_6$	$2e_2(\frac{1}{K}E_3 - E_7)$
4	0	$\frac{2}{K}e_1E_4$	0	0
5	0	0	$\frac{2}{K}e_2E_4$	$\frac{2}{K}e_2E_5$
6		0	$-\frac{2}{K}e_1E_7$	0
7			0	$-\frac{2}{K}e_2E_7$

The 8-dimensional projective Lie algebra of a (pseudo)-Euclidean plane with the linear element

$$ds^2_{\mathrm{XV}} = e_1 dx^{1^2} + e_2 dx^{2^2}$$

is spanned by the vector fields

$$\underset{i}{E}_{\,i} = \partial_i, \quad \underset{i}{E}_{\,3} = e_1x^2\partial_1 - e_2x^1\partial_2, \quad \underset{h}{E}_{\,4} = x^1\partial_1 + x^2\partial_2, \quad \underset{a}{E}_{\,4+i} = x^i\partial_1, \quad \underset{p}{E}_{\,6+i} = x^i\left(x^1\partial_1 + x^2\partial_2\right)$$
$$(i = 1,2),$$

whose commutators are given in the following table.

Table 3. The structure of the 8-dimensional projective Lie algebra of a (pseudo-)Euclidean plane. The commutator $[E_r, E_s]$ of the elements E_r, E_s of the basis for the algebra stands at the intersection of the rth row and the sth column.

$\hookrightarrow$	1	2	3	4	5	6	7	8
1	0	0	$-e_2E_2$	E_1	E_1	0	E_4+E_5	E_6
2		0	e_1E_1	E_2	0	E_1	$e_1e_2E_6-e_2E_3$	$2E_4-E_5$
3			0	0	$2e_1E_6-E_3$	$e_2E_4-2e_2E_5$	e_1E_8	$-e_2E_7$
4				0	0	0	E_7	E_8
5					0	$-E_6$	E_7	0
6						0	E_8	0
7							0	0

6.5. Groups of projective transformations of two-dimensional pseudo-Riemannian manifolds of non-constant curvature. We list some properties of connected projective Lie groups of two-dimensional pseudo-Riemannian manifolds with non-constant curvature (projective groups of pseudo-Riemannian manifolds of constant curvature have been investigated by Solodovnikov [156]).

Let P_N be a projective Lie algebra of a two-dimensional surface with linear element ds_N^2 defined by Theorem 6.1. Let $\widehat{P}_N$ be a connected projective Lie group with the Lie algebra P_N, and let $\widetilde{P}_N$ be its universal covering. We denote by $\mathcal{O}(P_N)$ the set of representers of the classes of Lie groups that are isomorphic to each other and have the Lie algebra P_N. Then the following propositions are true.

$P_{\rm I}$, $P_{\rm II}$, and also $P_{\rm III}$ when $e_1e_2=+1$ and $P_{\rm IV}-P_{\rm VI}$ when $e_1e_2=-1$ are the 3-dimensional simple non-solvable Lie algebras that are isomorphic to the Lie algebra $\{\{Y_1,Y_2,Y_3\}\}$ of Bianchi type VIII with the structure equations (cf., for example, [133], p. 72)

$$[Y_1,Y_2]=Y_1,\quad [Y_1,Y_3]=2Y_2,\quad [Y_2,Y_3]=Y_3.$$

The group of rotations of the 3-dimensional pseudo-Euclidean space of the (Lorentz) signature 1, the group of isometric motions of a non-Euclidean plane, the group of bilinear (projective) transformations of the straight line, the group of projective transformations of a projective plane which remain a conic-section invariant, the group of unimodular anti-quaternions, and the group $\mathrm{SL}(2,\mathbb{R})$ of second-order real unimodular matrices have such a Lie algebra (cf. [135], p. 429, [180], and also [137], Chap. XII, §4).

$\mathcal{O}(P_{\rm I})$, $\mathcal{O}(P_{\rm II})$, and also $\mathcal{O}(P_{\rm III})$ when $e_1e_2=+1$ and $\mathcal{O}(P_{\rm IV})-\mathcal{O}(P_{\rm VI})$ when $e_1e_2=-1$ contain the simply connected (covering) group $\widetilde{G}$ that is homeomorphic to $\mathbb{R}^3$, whose center is isomorphic to $\mathbb{Z}$, and infinite series of non-connected groups $\widehat{G}_n$, which are homeomorphic to $\mathbb{R}^2T^1$ and have the fundamental group $\mathbb{Z}$. The center of the group $\widehat{G}_n$ consists of n elements ($n=1,2,3,\dots$).

$P_{\rm III}$ when $e_1e_2=-1$ and $P_{\rm IV}-P_{\rm VI}$ when $e_1e_2=+1$ are the 3-dimensional simple non-solvable Lie algebras that are isomorphic to a Lie algebra $\{\{Y_1,Y_2,Y_3\}\}$ of Bianchi type IX with the structure equations

$$[Y_1,Y_2]=Y_3,\quad [Y_2,Y_3]=Y_1,\quad [Y_3,Y_1]=Y_2.$$

The simply connected group $\mathrm{Sp}(1)$ of unimodular quaternions which is homeomorphic to the sphere $\mathbb{S}^3$ ([180]. has such a Lie algebra.

$\mathcal{O}(P_{\rm III})$ when $e_1e_2=-1$ and $\mathcal{O}(P_{\rm IV})-\mathcal{O}(P_{\rm VI})$ when $e_1e_2=+1$ contain two compact groups: the group $\mathrm{Sp}(1)$ which is homeomorphic to $\mathbb{S}^3$ and the group $\mathrm{SU}(2)$ which is homeomorphic to the projective space $\mathbb{P}^3$.

$P_{\rm VII}$ when $4b=5a$ is a two-dimensional commutative Lie algebra. $\mathcal{O}(P_{\rm VII})$ consists of the simply connected group $\mathbb{R}\oplus R$, which is homeomorphic to the plane $\mathbb{R}^2$, the group $\mathbb{R}\oplus T$ which is homeomorphic to the cylinder $\mathbb{R}^1T^1$, and the group $\mathbb{T}\oplus T$ which is homeomorphic to a two-dimensional torus ([135]).

$P_{\rm VII}$ when $4b\neq 5a$ and $P_{\rm VIII}$ are 2-dimensional non-Abelian Lie algebras. $\mathcal{O}(P_{\rm VII})$ when $4b\neq 5a$ and $\mathcal{O}(P_{\rm VIII})$ consist of the simply connected group $\widetilde{P}_{\rm IX}$ which is homeomorphic to the plane $\mathbb{R}^2$. The center and the fundamental groups are trivial.

$P_{\rm IX}$–$P_{\rm XIII}$ are one-dimensional Abelian Lie algebras. $\mathcal{O}(P_{\rm IX})$–$\mathcal{O}(P_{\rm XIII})$ consist of the simply connected group $\mathbb{R}^1$ which is homeomorphic to the straight line and the compact group $\mathbb{T}^1$ which is homeomorphic to the circle [126] ($\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{T}$ are the additive groups of integers, real numbers, and real numbers modulo 1, respectively).

In Table 4, some information is given about projective Lie algebras and projective Lie groups in M^2.

Table 4. The classification of maximal non-homothetic projective Lie algebras of the two-dimensional pseudo-Riemannian manifolds M^2 of nonconstant curvature.

I	II				III	IV
Maximal projective Lie algebras	Admissible transformations				Lie subalgebras	Connected Max. project. Lie groups
	i.m.	h.m.	a.m.	p.m.		
P_3 simple non-solvable Bianchi types VIII, IX	////	////		////	$P_3 \supset A_1 \equiv H_1 \equiv I_1$ $P_3 \supset A_2 \equiv H_2 \supset I_1$	$\mathrm{SL}(2,\mathbb{R})$ $\mathrm{Sp}(1)$ $\mathrm{SU}(2)$ $\mathrm{SO}(3)$ $\mathrm{SO}(1,2)$
P_2 Abelian non-Abelian	////			////	$P_2 \supset A_1 \equiv H_1 \equiv I_1$	$\mathbb{R} \oplus \mathbb{R}$ $\mathbb{R} \oplus \mathbb{T}$ $\mathbb{T} \oplus \mathbb{T}$
P_1				////		$\mathbb{R}$, $\mathbb{T}$

The shaded boxes in column II show that the corresponding class of spaces admits the indicated type of motions. The symbols I_i, H_i, A_i in column III denote the maximal isometric, homothetic, and affine subalgebras of the maximal projective Lie algebra P_r, respectively. In column IV, connected Lie groups with the Lie algebra P_r are given.

Every two-dimensional pseudo-Riemannian manifold of constant curvature K admits the projective Lie algebra $P_8 \supset A_3 \equiv H_3 \equiv I_3$ when $K \neq 0$ and $P_8 \supset A_6 \supset H_4 \supset I_3$ when $K = 0$.

We note that among the projectively mobile two-dimensional spaces, there are 1- and 2-parameter sets of surfaces, which makes them particularly attractive from the point of view of geometric research and its applications.

The first quadratic integrals of equations for geodesics generated by projective motions (see Secs. 6.2 and 6.3) can be used to study properties of geodesics and to construct computer models similar to the Lobachevskii plane or Poincaré model of Lobachevskii geometry ([121], p. 46), as well as solid representations of two-dimensional surfaces that admit groups of projective transformations. These models can be used in engineering when calculating elements of modern constructions (geodesic winding of glass filament when manufacturing details of rocket bodies and rocket engines, ect.).

Figure 1 contains the volume figures of some sets of surfaces of revolution admitting projective transformations (the visualization was carried out by A. Korneev).

In the following section, applications of the Lie problem to differential equations are considered. A classification of the second-order ordinary differential equations with cubic right-hand sides and systems of second-order equations with two unknowns and quadratic right-hand sides (the equations for geodesics) with respect to Lie algebras of infinitesimal symmetries is given and the conditions under which integral curves of the above equations and systems are straight lines are found.

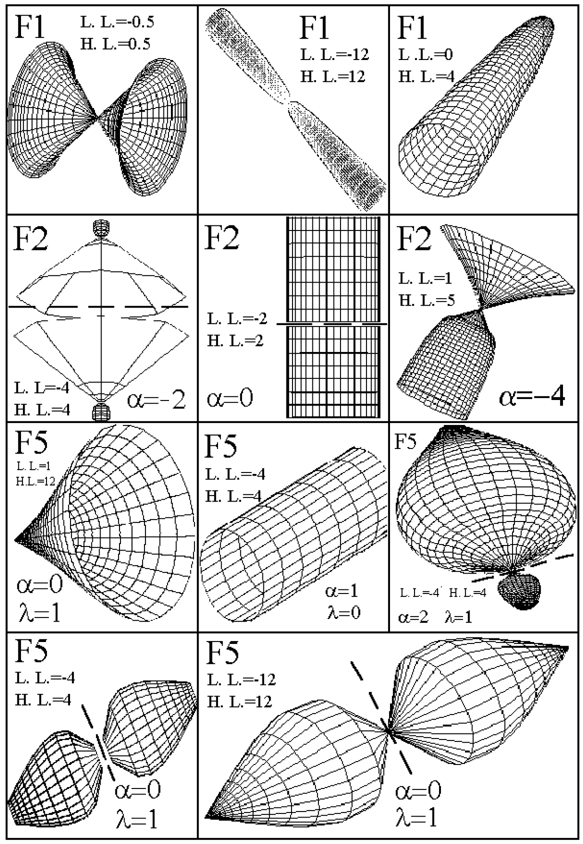

Fig. 1. L.L. - Low Limit, H.L. - High Limit

7. Projective Geometry of a Differential System

7.1. Projective symmetries of Hamiltonian systems. Geodesic lines (with affine parametrization) in (M, g) define the geodesic flow on the tangent bundle TM, i.e., the Hamiltonian system

$$\dot{x}^i \equiv \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} = \{x^i, H\}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i} = \{p_i, H\} \tag{7.1}$$

with the *Hamiltonian*

$$H(x,p) = \frac{1}{2} g^{ij} p_i p_j, \tag{7.2}$$

where x^i and $p_i \equiv g_{ij}\dot{x}^j$ are canonical local coordinates and $\{,\}$ is the canonical *Poisson bracket*

$$\{u, v\} = \frac{\partial u}{\partial x^i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial x^i}. \tag{7.3}$$

If u is a function of t, x, and p, then

$$\frac{du}{dt} = \{u, H\} + \frac{\partial u}{\partial t}. \tag{7.4}$$

The canonical Poisson bracket defines a *Poisson structure* on TM and makes TM a *Poisson* (*symplectic*) *manifold.*

Let $H : TM \longrightarrow \mathbb{R}$ be a smooth function.

The unique smooth vector field on TM satisfying the condition

$$\xi_H(F) = \{F, H\} = -\{H, F\} = \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}\right) F \tag{7.5}$$

for each smooth function $F : TM \longrightarrow \mathbb{R}$ is called the *Hamiltonian vector field* ξ_H.

The Hamilton equations (7.1) are the equations of the flow of the vector field ξ_H ([71, 72, 89, 124].

The Hamiltonian vector field corresponding to the Poisson bracket of the functions F and H coincides up to a sign with the Lie bracket of Hamiltonian vector fields ξ_F and ξ_H:

$$\xi_{\{F,H\}} = -[\xi_F, \xi_H] = [\xi_H, \xi_F]. \tag{7.6}$$

For each a, the flow $\exp(a\xi_H) : TM \longrightarrow TM$ defines (locally) a mapping of TM into itself that preserves the Poisson bracket (7.3) (a *Poisson mapping*).

We call a function $F(t, x, p)$ a (*first*) *integral of the Hamiltonian system* (7.1) or a *first integral* (a *constant*) *of the motion* if

$$\frac{\partial F}{\partial t} + \{F, H\} = 0 \tag{7.7}$$

does not explicitly depend on t; then the last condition becomes

$$\{F, H\} = 0. \tag{7.8}$$

In view of (7.4), the value of F is constant along each trajectory of the Hamiltonian system (7.1). Since the Hamiltonian (7.1) does not depend explicitly on t, it is itself invariant. In view of the Jacobi identity, the Poisson bracket of the first two integrals is also a first integral (Poisson's theorem) ([71, 80, 89, 90]).

If $F(t, x, p)$ is a first integral of the Hamiltonian system (7.1), then the Hamiltonian vector field ξ_F generates a one-parameter symmetry group of this system, which we call a *Hamiltonian symmetry group* [125]. It follows from (3.25) that the function

$$F(x, p) = \left(4\varphi g^{ij} - h^{ij}\right)(x) p_i p_j$$

is a first integral of the Hamiltonian system (7.1) with Hamiltonian (7.2) if h, g, and φ satisfy (3.24). Hence *each infinitesimal projective transformation X in M generates a Hamiltonian symmetry group* $\exp(a\xi_F)$ *of the Hamiltonian system* (7.1), (7.2) *with*

$$\xi_F = 2\left(4\varphi g^{ij} - g^{ki}g^{lj}L_X g_{kl}\right) p_j \frac{\partial}{\partial x^i} - \partial_s \left(4\varphi g^{ij} - g^{ki}g^{lj}L_X g_{kl}\right) p_i p_j \frac{\partial}{\partial p_s}. \tag{7.9}$$

Associated with the Hamiltonian system (7.1) (with Hamiltonian $H(t, x, p)$) is the Hamilton–Jacobi partial differential equation [80]

$$\frac{\partial S}{\partial t} + H\left(t, x, \frac{\partial S}{\partial x}\right) = 0. \tag{7.10}$$

A function $F(t, x, p)$ is a first integral of Hamiltonian equations (7.1) if and only if the Hamilton–Jacobi equation (7.10) admits a *canonical Lie–Bäcklund operator* ([73, 95], [96])

$$Y = F\left(t, x, \frac{\partial S}{\partial x}\right)\frac{\partial}{\partial S}$$

(see [259, 260]). Consequently, *with each projective motion X in the pseudo-Riemannian manifold* (M, g), *one can associate the canonical Lie–Bäcklund operator*

$$Y = \left(4\varphi g^{ij} - g^{ki}g^{lj}L_X g_{kl}\right)\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j}\frac{\partial}{\partial S} \tag{7.11}$$

of the Hamilton–Jacobi equation (7.10) *with Hamiltonian* (7.2).

The operator (7.11) generates a group of local nonpoint symmetries of the Hamilton–Jacobi equation. Corresponding to it, there is a 1-parameter group of *point*, or *geometric symmetries* [125] of Hamilton equations (7.1) generated by the vector field

$$Z = \frac{\partial}{\partial t} + \frac{\partial F(x,p)}{\partial x^i}\frac{\partial}{\partial p_i} - \frac{\partial F(x,p)}{\partial p_i}\frac{\partial}{\partial x^i}$$

and acting in the space of the dependent (x, p) and independent (t) variables of the Hamiltonian system [260].

7.2. Lie problem and differential equations. Sophus Lie laid the basis for the theory of continuous groups of transformations with the purpose of propagating the methods of the Galois theory of solution of algebraic equations in radicals to the problem of integration of differential equations.

In Olver's opinion [125], for half a century the "ascendancy" in mathematics of the "global abstract reformulation of differential geometry and Lie group theory championed by E. Cartan" resulted in the relegation to the background of the mastering of the deep ideas of Lie and Noether relating to the applications of Lie groups to differential equations (Birkhoff [75], Ovsyannikov [124], Ibragimov [96], Anderson and Ibragimov [199], Vinogradov, Krasilshchik, and Lychagin [77], Mishchenko and Fomenko [119], Dubrovin, Krichever, and Novikov [89]). Notwithstanding, we feel that Cartan's ideas concerning the construction of a geometric theory of differential equations [221] have unfairly been kept in the shade.

Cartan was looking for a generalization of a metric space which enables one to consider the integral curves of a system of second-order ordinary differential equations as the geodesic lines of the generalized space.

Lie worked toward giving a clear geometrical character to symmetries of differential equations. While developing the theory of spaces with a projective connection, E. Cartan stressed persistently its importance for the study of differential equations and considered it to be "le point le plus interessant" of his theory.

"La notion de connexion projective confere... a la theorie des invariants differentiels d'une equation differentielle du second ordre vis-a-vis du groupe ponctuel, ... (which was the subject "d'un important Memoire de M. A. Tresse," [392]), ... est un aspect assez inattendu" – wrote Cartan. Since the normal projective connection and the geodesic lines determine each other unambiguously, "c'est donne la notion de connexion projective normale qui seule permet de faire une theorie geometrique satisfaisante des equations de la forme

$$\frac{d^2u^\alpha}{du^{n^2}} = -P^\alpha\left(\frac{du^\beta}{du^n}\right) + \frac{du^\alpha}{du^n}P^n\left(\frac{du^\beta}{du^n}\right) \qquad (\alpha, \beta = 1, \dots, n-1)\text{"}$$

[221].[12]

Methods of differential geometry, in particular, methods of Cartan's theory, provide tools for developing a systematic geometric approach to defining and investigating point and nonpoint symmetries of large classes of differential equations and partial differential equations and obtaining their solutions ([48, 56]).

7.3. Symmetry groups of a geodesic equation in the configuration space (x)**.** Rewritten for the parameter x^n, Eqs. (2.3) become

$$\nabla^\alpha \equiv \ddot{x}^\alpha + a_{\beta\gamma}(x)\dot{x}^\alpha\dot{x}^\beta\dot{x}^\gamma + b^\alpha_{\beta\gamma}(x)\dot{x}^\beta\dot{x}^\gamma + c^\alpha_\beta(x)\dot{x}^\beta + d^\alpha(x) = 0, \tag{7.12}$$

where α, β, $\gamma = 1, \dots, n-1$,

$$a_{\beta\gamma} = -\Pi^n_{\beta\gamma}, \quad b^\alpha_{\beta\gamma} = \Pi^\alpha_{\beta\gamma} - \Pi^n_{n\gamma}\delta^\alpha_\beta - \Pi^n_{n\beta}\delta^\alpha_\gamma, \quad c^\alpha_\beta = 2\Pi^\alpha_{n\beta} - \Pi^n_{nn}\delta^\alpha_\beta, \quad d^\alpha = \Pi^\alpha_{nn}, \tag{7.13}$$

the dot over the x denotes the differentiation with respect to x^n (so that $\ddot{x}^n = 0$), Π^i_{jk} are the Thomas projective parameters (2.5), and the Greek indices range from 1 to $n-1$ and the Latin ones from 1 to n.

[12] Dryuma [88] used the Cartan normal projective connection to study the invariants of various equations appearing in theoretical physics, mechanics, and chemistry.

Since we can unambiguously solve (7.13) with respect to the parameters Π^i_{jk} as

$$\Pi^\alpha_{\beta\gamma} = b^\alpha_{\beta\gamma} - \frac{1}{n+1}(b^\delta_{\delta\beta}\delta^\alpha_\gamma + b^\delta_{\delta\gamma}\delta^\alpha_\beta), \quad 2\Pi^\alpha_{n\beta} = c^\alpha_\beta - \frac{1}{n+1}c^\delta_\delta\delta^\beta_\alpha, \tag{7.14}$$
$$\Pi^n_{\beta\gamma} = -a_{\beta\gamma}, \quad \Pi^\alpha_{nn} = d^\alpha, \quad \Pi^n_{n\gamma} = -\frac{1}{n+1}b^\delta_{\delta\gamma}, \quad \Pi^n_{nn} = -\frac{1}{n+1}c^\delta_\delta,$$

there is a one-to-one correspondence between the coefficients of (7.12) and the components of the projective connections defined by the Thomas parameters. We note that the coefficients of (7.12) define an affine connection Γ^i_{jk} up to a summand of the form $\delta^i_j p_k + \delta^i_k p_j$ determining the geodesic class of the space M^n, i.e., the set of spaces M'^n corresponding geodesically to the space M^n.

We recall that the group of (point) symmetries of a system of differential equations of order k

$$\Delta^\nu(t, x^{(k)}) = 0, \quad \nu = 1, \dots, s, \tag{7.15}$$

is the local Lie transformation group G ([135], § 60) acting on an open subset N of the space $T \times \Xi$ of the independent (t) and the dependent (x) variables of the system and having the following property. If $x = f(t)$ is a solution of (7.15) defined in a domain $\Omega \subset T$, and if for $h \in G$ the composite $h \circ f$ is well defined, then $x = h \circ f(t)$ is also a solution of the system. The Lie algebra of G consists of "infinitesimal symmetries," which are vector fields on N generating 1-parameter groups of the system's symmetries.

System (7.15) defines a submanifold F in the space of k-jets $T \times \Xi$ on which the coordinate functions are the independent variables, the dependent variables, and all the derivatives of the dependent variables up to order k. A connected local Lie transformation group G is the symmetry group of a nondegenerate ([125], Definition 2.70) system (7.15) if and only if the kth prolongation $pr^{(k)}G$ leaves the submanifold F invariant. To this end, it is necessary and sufficient that the kth prolongation $pr^{(k)}X$ of each vector field X in the Lie algebra of the group G satisfy the conditions ([96, 124, 125])

$$\mathrm{pr}^{(k)}X[\nabla^\nu(t, x^{(k)})] = 0 \ \text{ when } \ \nabla^\nu(t, x^{(k)}) = 0 \qquad (\nu = 1, \dots, s,).$$

For the system (7.12), these conditions become

$$\mathrm{pr}^{(2)}X[\nabla^\alpha]|_{\nabla^\alpha=0} = 0, \quad (\alpha = 1, \dots, n-1), \tag{7.16}$$

where (see, e.g., [124])

$$X = \xi^i\partial_i, \quad \mathrm{pr}^{(1)}X = X + (\partial_n\xi^\alpha + \dot{x}^\beta\partial_\beta\xi^\alpha - \dot{x}^\alpha\partial_n\xi^n - \dot{x}^\alpha\dot{x}^\beta\partial_\beta\xi^n)\frac{\partial}{\partial\dot{x}\alpha}, \tag{7.17}$$
$$\begin{aligned}\mathrm{pr}^{(2)}X = \mathrm{pr}^{(1)}X + \big[&\partial_{nn}\xi^\alpha + \dot{x}^\beta(2\partial_{\beta n}\xi^\alpha - \partial_{nn}\xi^n\delta^\alpha_\beta)\\ &+ \dot{x}^\beta\dot{x}^\gamma(\partial_{\beta\gamma}\xi^\alpha - \partial_{\beta n}\xi^n\delta^\alpha_\gamma - \partial_{\gamma n}\xi^n\delta^\alpha_\beta)\\ &+ \ddot{x}^\beta(\partial_\beta\xi^\alpha - 2\partial_n\xi^n\delta^\alpha_\beta) - \dot{x}^\alpha\dot{x}^\beta\dot{x}^\gamma\partial_{\beta\gamma}\xi^n - 2\ddot{x}^\alpha\dot{x}^\beta\partial_\beta\xi^n - \dot{x}^\alpha\ddot{x}^\beta\partial_\beta\xi^n\big]\frac{\partial}{\partial\ddot{x}^\alpha}.\end{aligned}$$

Substituting (7.17) in (7.16) and using (7.12), we obtain a set of equations the left-hand sides of which are polynomials of third order in $\dot{x}^\alpha$. Equating the coefficients of these polynomials to zero, we obtain equations that, using the formula (3.9), can be reduced to the form (3.19), where $\phi_\alpha = L_X\Gamma^n_{\alpha n}$ and $2\phi_n = L_X\Gamma^n_{nn}$. Hence X is a projective motion in M^n. Thus, we have proved the following result.

Theorem 7.1. *The symmetry group for the equations of geodesics with Cartan parametrization* (7.12), (7.13)*, and* (2.5) *in the space M^n with affine connection $\nabla(\Gamma^i_{jk})$ is the group of projective transformations in this space.*

Remark 7.1. If the Weyl tensor (2.7) of projective connection (7.14) vanishes, then for $n > 2$, this connection is projectively flat and in projective coordinates all its components vanish (Sec. 2.4); therefore, Eqs. (7.12) are reduced by change of the variables to the equations $y^{\alpha''} = 0$, of which integral curves are straight lines.

Remark 7.2. If the projective connection (7.14) is induced by a Riemannian one, then this connection will be projectively flat if and only if the (pseudo-)Riemannian metric has constant curvature. In this and only in this case does (7.12) have in appropriate coordinates linear solutions only.

7.4. Classification of symmetry groups of 2D geodesic equations in the configuration space (x)**.** For $n = 2$, the formulas (7.12)–(7.14) and (7.17) become

$$\Delta \equiv \ddot{x}^1 + a(x^1, x^2)\dot{x}^{1^3} + b(x^1, x^2)\dot{x}^{1^2} + c(x^1, x^2)\dot{x}^1 + d(x^1, x^2) = 0, \tag{7.18}$$

$$a = -\Pi^2_{11}, \quad b = \Pi^1_{11} - 2\Pi^2_{21}, \quad c = 2\Pi^1_{21} - \Pi^2_{22}, \quad d = \Pi^1_{22}$$

(cf. [101], Part 2, p. 421),

$$\Pi^1_{11} = \frac{1}{3}b, \quad 2\Pi^1_{21} = \frac{2}{3}c,$$

$$\Pi^2_{11} = -a, \quad \Pi^1_{22} = d, \quad \Pi^2_{12} = -\frac{1}{3}b, \quad \Pi^2_{22} = -\frac{1}{3}c,$$

$$X = \xi^i \partial_i, \quad \mathrm{pr}^{(1)} X = X + (\partial_2 \xi^1 + \dot{x}^1 \partial_1 \xi^1 - \dot{x}^1 \partial_2 \xi^2 - \dot{x}^1 \dot{x}^1 \partial_1 \xi^2) \frac{\partial}{\partial \dot{x}^1},$$

$$\begin{aligned}\mathrm{pr}^{(2)} X = \mathrm{pr}^{(1)} X + \big[&\partial_{22}\xi^1 + \dot{x}^1(2\partial_{12}\xi^1 - \partial_{22}\xi^2\delta^1_1) \\ &+ \dot{x}^1\dot{x}^1(\partial_{11}\xi^1 - \partial_{12}\xi^2\delta^1_1 - \partial_{12}\xi^2\delta^1_1) + \\ &+ \ddot{x}^1(\partial_1\xi^1 - 2\partial_2\xi^2\delta^1_1) - \dot{x}^1\dot{x}^1\dot{x}^1\partial_{11}\xi^2 - 2\ddot{x}^1\dot{x}^1\partial_1\xi^2 - \dot{x}^1\ddot{x}^1\partial_1\xi^2\big]\frac{\partial}{\partial \ddot{x}^1}\end{aligned}$$

(the dot over the x^1 denotes differentiation with respect to x^2).

According to Theorem 7.1, the determination of a symmetry group of (7.18) can be reduced to the definition of a projective group of an associated pseudo-Riemannian manifold M^2. The last task is precisely the Lie problem that was considered by S. Lie and solved by author (see Sec. 6). Theorems 6.7 and 7.1 imply the following theorem.

Theorem 7.2. *If Eq.* (7.18) *of geodesics with Cartan parametrization in a two-dimensional pseudo-Riemannian space* M^2 *admits an* r*-dimensional symmetry group, then this equation and the basis vector fields* $E_1, \dots, E_r$ *of its symmetry Lie algebra* P *belong to one of the types below* (y *is an unknown function of an independent variable* x *and a prime denotes differentiation*).

Type I. $\dim P = 3$:

$$y'' - \frac{1}{2y}y'^2 \pm \frac{1}{2y} = 0,$$

$$E_1 = \partial_x, \quad E_2 = y\partial_y + x\partial_x, \quad E_3 = yx\partial_y + \frac{1}{2}x^2\partial_x.$$

A type-I *equation cannot be reduced to* $\tilde{y}''_{\tilde{x}} = 0$ *by the change of variables* $(x, y) \to (\tilde{x}, \tilde{y})$*.*

Type II. $\dim P = 3$:

$$y'' + \frac{\alpha}{y^3 + \alpha y}\left(y'^2 \pm 1\right) = 0 \qquad (\alpha = \mathrm{const} \neq 0),$$

$$E_1 = \partial_x, \quad E_2 = \left(y + \frac{\alpha}{y}\right)\partial_y + 2x\partial_x, \quad E_3 = x\left(y + \frac{\alpha}{y}\right)\partial_y + x^2\partial_x.$$

When $\alpha \neq 0$ *an equation of type* II *cannot be reduced to* $\tilde{y}''_{\tilde{x}} = 0$ *by the change of variables* $(x, y) \to (\tilde{x}, \tilde{y})$*.*

Types III–VI. $\dim P = 3$:

$$y'' - \frac{f'(y)}{2f(y)}\left(y'^2 + \epsilon\right) = 0 \qquad (\epsilon = \pm 1),$$

where

$$f(y)=(\alpha+\sin\lambda y)\,/\cos^2\lambda y \;\; \textit{for type} \;\; \mathrm{III}, \quad e^{-\lambda y}\left(1+\alpha e^{-\lambda y}\right) \;\; \textit{for type} \;\; \mathrm{IV},$$

$$(\alpha+\sinh\lambda y)\,/\cosh^2\lambda y \;\; \textit{for type} \;\; \mathrm{V}, (\alpha+\cosh\lambda y)\,/\sinh^2\lambda y \;\; fortype \;\; \mathrm{VI},$$

$$E_1=\partial_x, \quad E_2=\Lambda\sin\lambda x\partial_y-\cos\lambda x\partial_x, \quad E_3=\Lambda\cos\lambda x\partial_y+\sin\lambda x\partial_x$$

for type III *with* $\epsilon=-1$ *and types* IV–VI *with* $\epsilon=+1$,

$$E_1=\partial_x, \quad E_2=e^{\lambda x}(\Lambda\partial_y+\partial_x), \quad E_3=e^{-\lambda x}(\Lambda\partial_y-\partial_x)$$

for type III *with* $\epsilon=+1$ *and types* IV–VI *with* $\epsilon=-1$,

$$\Lambda=(\alpha+\sin\lambda y)\,/\cos\lambda y \;\; \textit{for the type} \;\; \mathrm{III}, \quad 1+\alpha e^{-\lambda y} \;\; \textit{for the type} \;\; \mathrm{IV},$$

$$(\alpha+\sinh\lambda y)\,/\cosh\lambda y \;\; \textit{for the type} \;\; \mathrm{V}, \quad (\alpha+\cosh\lambda y)\,/\sinh\lambda y \;\; \textit{for the type} \;\; \mathrm{VI}.$$

Equations of types III, VI *with* $\alpha=\pm1$ *and an equation of type* IV *with* $\alpha=0$ *are reduced to* $\tilde{y}''_{\tilde{x}}=0$ *by a change of variables. For any* α, *an equation of type* V *cannot be reduced to such a form.*

Type VII. $\dim P=2$*:*

$$y''-\frac{\Phi'(y)}{\Phi(y)}\left(y'^2\pm1\right)=0,$$

where

$$\Phi'=k\exp\int\left(2\Phi^2-2a+b\right)\Phi^{-1}\left(\Phi^2-a+b\right)^{-1}d\Phi$$

$$(a,b,k=\mathrm{const},\; ab(a+2)(b+3)k\neq0),$$

$$E_1=\partial_x, \quad E_2=\frac{\Phi}{\Phi'}\left(\Phi^2-a+b\right)\partial_y+(4b-5a)x\partial_x.$$

Type VIII. $\dim P=2$:

$$y''-\frac{\alpha}{y}\left(y'^2\pm1\right)=0 \qquad (\alpha=\mathrm{const}),$$

$$E_1=\partial_x, \quad E_2=y\partial_y+x\partial_x.$$

By a change of variables an equation of type VIII *is reduced to* $\tilde{y}''_{\tilde{x}}=0$ *if and only if* $\alpha=0,\,-1$.

Type IX. $\dim P=1$:

$$y''+\frac{1}{y+w(x)}y'^2-\frac{w'(x)}{y+w(x)}y'=0,$$

$$w(x)=k\int\exp\left(\frac{3}{2}\int(c-a-x)\left(x^2+ax+b\right)^{-1}dx\right)dx,$$

$$E_1=\left(\frac{1}{2}(y+w(x))(x+3c-a)-\left(x^2+ax+b\right)w'(x)\right)\partial_y+\left(x^2+ax+b\right)\partial_x.$$

When $k\neq0$ *an equation of type* IX *cannot be reduced to* $\tilde{y}''_{\tilde{x}}=0$ *by a change of variables.*

Type X. $\dim P=1$:

$$y''+\frac{1}{2}\frac{1}{f_2(x)-f_1(y)}\left(\epsilon f_2'(x)y'^3+f_1'(y)y'^2+f_2'(x)y'+\epsilon f_1'(y)\right)=0 \qquad (\epsilon=\pm1),$$

$$\frac{df_i}{dx^i}\equiv f_i'=k_i\exp\left(\frac{3}{2}\int(f_i+a)\left(f_i^2+af_i+b\right)^{-1}df_i\right)$$

$$(k_i,a,b=\mathrm{const}, \quad k_i\neq0, \quad i=1,2, \quad x_1\equiv y,\; x_2\equiv x),$$

$$\underset{p}{E}_1=\frac{f_1^2+af_1+b}{f_1'}\partial_y+\frac{f_2^2+af_2+b}{f_2'}\partial_x,$$

Type XI. dim $P = 1$:

$$y'' - \frac{1}{2v}\left(\partial_x v\ y'^3 + \partial_y v\ y'^2 - \partial_x v\ y' - \partial_y v\right) = 0,$$

$$\frac{df_1}{dz^1} \equiv f_1' = k \exp\left(\frac{3}{2}\int (f_1 + a)\left(f_1^2 + af_1 + b\right)^{-1} df_1\right), \quad (a, b, k = \text{const}, \quad k \neq 0),$$

$$\underset{p}{E}_1 = \Re\frac{f_1^2 + af_1 + b}{f_1'}\partial_y + \Im\frac{f_1^2 + af_1 + b}{f_1'}\partial_x.$$

$$(z^1 = y + ix = \overset{*}{z}{}^2,\ f_1 = f_1(z^1) \equiv u(y,x) + iv(y,x) = \overset{*}{f}_2,\ \partial_y u = \partial_x v, \quad \partial_x u = -\partial_y v).$$

Type XII. dim $P = 1$:

$$y'' + \frac{1}{2}e_2 f'(x) y'^3 - ay'^2 + \frac{f'(x)}{f(x)}y' - \frac{e_2 a}{f(x)} = 0 \qquad (a = \text{const}),$$

$$E_1 = \partial_y.$$

An equation of type XII *is reduced to* $\tilde{y}''_{\tilde{x}} = 0$ *by a change of variables if and only if* $f = (c_1 x + c_2)^2$, *where* c_1 *and* c_2 *are constants.*

Type XIII. dim $P = 1$:

$$y'' - \frac{\Phi'(y)}{\Phi(y)}\left(y'^2 \pm 1\right) = 0,$$

$$E_1 = \partial_x.$$

An equation of type XIII *is reduced to* $\tilde{y}''_{\tilde{x}} = 0$ *by a change of variables if and only if* $(\Phi'/\Phi)'/\Phi^2 =$ const.

Type XIV. dim $P = 8$:

$$y'' - \frac{2K}{4 + K(e_1 y^2 + e_2 x^2)}\left(e_1 x y'^3 - e_1 y y' x^2 + e_2 x y' - e_2 y\right) = 0,$$

$$E_i = x^i(y\partial_y + x\partial_x) + (2e_i/K)(2 - \sigma)\partial_i, \quad E_3 = e_1 x\partial_y - e_2 y\partial_x, \quad E_{3+i} = \left(x^i/\sigma\right)(y\partial_y + x\partial_x),$$

$$E_{4+i+j} = \left(x^i x^j/\sigma\right)(y\partial_y + x\partial_x) - (2e_j/K)\, x^i\partial_j, \quad (i, j = 1, 2,\ i \leq j, x^1 \equiv y, x^2 \equiv x).$$

By a change of variables, an equation of type XIV *is reduced to* $\tilde{y}''_{\tilde{x}} = 0$.

Type XV. dim $P = 8$:

$$y'' = 0,$$

$$E_i = \partial_i, \quad E_3 = e_1 x\partial_y - e_2 y\partial_x, \quad E_4 = x^1\partial_1 + x^2\partial_2, \quad E_{4+i} = x^i\partial_y, \quad E_{6+i} = x^i\left(x^1\partial_1 + x^2\partial_2\right)$$

$$\left(i, j = 1, 2, \quad i \leq j, \quad x^1 \equiv y, \quad x^2 \equiv x\right).$$

Remark 7.3. The change of variables $x \leftrightarrow y$ in the above equations does not change the form of generators of symmetry groups of these equations.

One can find in Table 4 some information about the maximal connected Lie groups of symmetries of Eq. (7.18).

7.5. Symmetry groups of a geodesic equation in the extended configuration space (t, x). Let M^n be a pseudo-Riemannian manifold with Riemannian connection $\nabla(\Gamma^i_{jk})$. Each one-parameter symmetry group for Eqs. (2.3) of geodesics in M^n with affine parametrization t is generated by a vector field

$$X = \Lambda\partial_t + \xi^i\partial_i,$$

where Λ and $\xi^1, \ldots, \xi^n$ are functions of $t, x^1, \ldots, x^n$. By conditions (7.16) written for Eqs. (2.3), we obtain

$$\partial_{tt}\xi^i = 0, \quad (\partial_t\xi^i)_{,j} = \frac{1}{2}\delta^i_j\partial_{tt}\Lambda,$$
$$\Lambda_{,jk} = 0, \quad L_{X-\Lambda\partial_t}\Gamma^i_{jk} = \delta^i_j(\partial_t\Lambda)_{,k} + \delta^i_k(\partial_t\Lambda)_{,j}.$$

Hence

$$\Lambda = a + \tau(x) + \sigma(x)t + \frac{1}{2}\rho(x)t^2, \quad \tau_{,jk} = \sigma_{,jk} = \rho_{,jk} = 0, \quad \xi^i\partial_i = \eta^i(x)\partial_i + t\zeta^i(x)\partial_i \equiv Y + tZ,$$
$$\zeta^i_{,j} = \frac{1}{2}\delta^i_j\rho(x), \quad L_Y\Gamma^i_{jk} \equiv \eta^i_{,jk} + R^i_{jlk}\eta^l = \delta^i_j\sigma_{,k} + \delta^i_k\sigma_{,j}, \tag{7.19}$$
$$L_Z\Gamma^i_{jk} \equiv \zeta^i_{,jk} + R^i_{jlk}\zeta^l = \delta^i_j\rho_{,k} + \delta^i_k\rho_{,j} \quad (a = \text{const}).$$

Contracting with respect to i and j, from the last three equations we obtain $\sigma = \frac{1}{n+1}\operatorname{div} Y + b$ and $\rho = 2c$, where b and c are constants. The relation $\zeta^i_{,j} = \rho\delta^i_j$ defines a special concircular (concurrent when $\rho =$ constant) vector field Z (see Sec. 3.4) and absolutely parallel vector field $Z_0 : \nabla Z_0 = 0$, when $\rho = 0$. A concurrent vector field is an infinitesimal homothety, and a parallel vector field is an infinitesimal isometry. Thus, we have proved the following theorem.

Theorem 7.3. *Each* 1*-parameter symmetry group of Eqs.* (2.3) *of geodesics with affine parametrization t in the space M^n with Riemannian connection $\nabla(\Gamma^i_{jk})$ is generated by an infinitesimal symmetry*

$$X = \left(a + \tau(x) + \left(\frac{1}{n+1}\operatorname{div} Y(x) + b\right)t + ct^2\right)\partial_t + Y(x) + t\left(Z(x) + Z_0(x)\right),$$

where Y is a projective motion in M^n that is simultaneously a curvature collineation ($L_Y R^i_{jkl} = 0$):

$$\nabla_W(L_Y - \nabla_Y) = R(Y, W) - \left(d\left(\frac{1}{n+1}\operatorname{div} Y\right)\right)(W)\cdot \operatorname{id} - W d\left(\frac{1}{n+1}\operatorname{div} Y\right),$$
$$\nabla^2(\operatorname{div} Y) = 0,$$

Z is a special concircular (*concurrent*) *vector field in M^n : $\nabla Z = \operatorname{id}$, Z_0 is a parallel vector field: $\nabla Z_0 = 0$, τ is a scalar field whose gradient is a parallel* 1*-form: $\nabla^2\tau = 0$, and a, b, and c are constants. If the connection ∇ admits no parallel* 1*-forms* (*respectively, no concurrent vector fields*)*, then in place of* div Y *and τ we consider constants and assume that Y is an affine motion in M^n* (*respectively, we set $c = 0$*).

The statements below are corollaries of Theorem 7.3.

Theorem 7.4. *Each symmetry group of Eqs.* (2.3) *of geodesics in the space M^n with Riemannian connection $\nabla(\Gamma^i_{jk})$ contains a two-dimensional subgroup*

$$\exp[(a + bt)\partial_t] : (t, x) \longrightarrow (a + te^b, x),$$

isomorphic to the group of affine transformations $t \longrightarrow a + te^b$ of a straight line, i.e., to the group of affine reparametrizations of geodesics.

Theorem 7.5. *Each projective motion Y in the space M^n with Riemannian connection $\nabla(\Gamma^i_{jk})$ that is simultaneously a curvature collineation,* ($\nabla^2(\operatorname{div} Y) = 0$, *defines an infinitesimal symmetry*

$$X = \frac{1}{n+1}(\operatorname{div} Y)td_t + Y$$

of Eqs. (2.3) *of geodesics with affine parametrization* t *in* M^n.

Theorem 7.6. *Each affine motion* Y *in the space* M^n *with Riemannian connection is an infinitesimal symmetry of Eqs.* (2.3) *of geodesics with affine parametrization* t *in* M^n.

Theorem 7.7. *Each concurrent vector field* Z $(\nabla Z = \mathrm{id})$ *in the space* M^n *with Riemannian connection* $\nabla(\Gamma^i_{jk})$ *defines an infinitesimal symmetry*

$$X = t^2\partial_t + tZ$$

of Eqs. (2.3) *of geodesics with affine parametrization* t *in* M^n.

Theorem 7.8. *Each parallel vector field* Z_0 $(\nabla Z_0 = 0)$ *in the space* M^n *with Riemannian connection* $\nabla(\Gamma^i_{jk})$ *defines an infinitesimal symmetry* $X = tZ_0$ *of Eqs.* (2.3) *of geodesics with affine parametrization* t *in* M^n.

Theorem 7.9. *Each parallel exact* 1*-form* $d\tau(\nabla(d\tau) = 0)$ *in the space* M^n *with Riemannian connection* $\nabla(\Gamma^i_{jk})$ *defines an infinitesimal symmetry*

$$X = \tau\partial_t$$

of Eqs. (2.3) *of geodesics with affine parametrization* t *in* M^n.

Remark 7.4. The results of this section are directly extended to non-Riemannian spaces with an affine connection.

7.6. Classification of symmetries of a 2D geodesic equation in the extended configuration space (t, x)**.** From the integrability conditions for the equation $\zeta^i_{,j} = 2c\delta^i_j$ in 2D space granting the special structure of 2D curvature tensor $R^i_{jkl} = \rho(\delta^i_k g_{jl} - \delta^i_l g_{jk})$, we obtain $\rho\zeta_k = 0$; hence a 2D space of nonzero curvature admits no parallel 1-forms and no concurrent vector fields. The same is true for symmetric bilinear forms. It follows that each affine motion in 2D space is an infinitesimal isometry. Hence Theorems 6.2, 6.7, and 7.2 imply the following proposition.

Theorem 7.10. *Each* 1*-parameter subgroup of the symmetry group* G *of Eqs.* (2.3) *of geodesics with affine parametrization* t *in a two-dimensional space* M^2 *with Riemannian connection* $\nabla(\Gamma^i_{jk})$ *of non-zero curvature is generated by an infinitesimal symmetry*

$$X = (a + bt)\partial_t + Y,$$

where Y *is an infinitesimal homothety in* M^2. G *has the dimension* $\dim G = 2 + \dim H \le 5$, *where* H *is the homothety group in* M^2. *The corresponding geodesic equations and generators of a symmetry group are listed below (the dot over* x *denotes differentiation with respect to* t*).*

$\dim G = 5$:

$$\ddot{x}^1 - \frac{K}{2\sigma}\left(e_1x^1\dot{x}^{1^2} + 2e_2x^2\dot{x}^1\dot{x}^2 - e_2x^1\dot{x}^{2^2}\right) = 0,$$
$$\ddot{x}^2 + \frac{K}{2\sigma}\left(e_1x^2\dot{x}^{1^2} - 2e_1x^1\dot{x}^1\dot{x}^2 - e_2x^2\dot{x}^{2^2}\right) = 0$$
$$\left(\sigma = 1 + \frac{K}{4}\left(e_1x^{1^2} + e_2x^{2^2}\right), \quad e_1, e_2 = \pm 1\right),$$

$$E_1 = \partial_t, \quad E_2 = t\partial_t, \quad E_{2+i} = x^i(x^1\partial_{x^1} + x^2\partial_{x^2}) + \frac{2}{K}e_i(2-\sigma)\partial_{x^i}, \quad (i = 1, 2) \quad E_5 = e_1x^2\partial_{x^1} - e_2x^1\partial_{x^2},$$

$\dim G = 4$:

$$\ddot{x}^1 + \left(\alpha/x^1\right)\left(\dot{x}^{1^2} \pm \dot{x}^{2^2}\right) = 0, \quad \ddot{x}^2 + 2\left(\alpha/x^1\right)\dot{x}^1\dot{x}^2 = 0,$$

$$E_1 = \partial_t, \quad E_2 = t\partial_t, \quad E_3 = \partial_{x^2}, \quad E_4 = x^1\partial_{x^1} + x^2\partial_{x^2}.$$

$\dim G = 3$:

$$\ddot{x}^1 + a\dot{x}^{1^2} + (1/f)\left(f'\dot{x}^1\dot{x}^2 - \epsilon a\dot{x}^{2^2}\right) = 0, \quad \ddot{x}^2 - (\epsilon/2)f'\dot{x}^{1^2} + 2a\dot{x}^1\dot{x}^2 = 0$$
$$(\epsilon = \pm 1, \quad a = \text{const}, \quad f = f(x^2)),$$
$$E_1 = \partial_t, \quad E_2 = t\partial_t, \quad E_3 = \partial_{x^1},$$

$\dim G = 3$:

$$\ddot{x}^1 + f(x^1)\left(\dot{x}^{1^2} \pm \dot{x}^{2^2}\right) = 0, \quad \ddot{x}^2 + 2f(x^1)\dot{x}^1\dot{x}^2 = 0,$$
$$E_1 = \partial_t, \quad E_2 = t\partial_t, \quad E_3 = \partial_{x^2},$$

$\dim G = 2$:

$$\ddot{x}^1 + f_{,1}\left(\dot{x}^{1^2} - \epsilon\dot{x}^{2^2}\right) + 2f_{,2}\dot{x}^1\dot{x}^2 = 0, \quad \ddot{x}^2 - f_{,2}\left(\epsilon\dot{x}^{1^2} - \dot{x}^{2^2}\right) + 2f_{,1}\dot{x}^1\dot{x}^2 = 0 \quad \left(f = f(x^1, x^2),\ \epsilon = \pm 1\right),$$
$$E_1 = \partial_t, \quad E_2 = t\partial_t.$$

7.7. Prolongation of a projective connection to the extended configuration space (t, x)**.** If we put $t \equiv x^{n+1}$, $\Lambda \equiv \xi^{n+1}$, and $X = \xi^1\partial_1 + \cdots + \xi^{n+1}\partial_{n+1}$ and define in a space, referred to coordinates $x^1, \ldots, x^{n+1}$, a projective connection with coefficients

$$\Pi^i_{jk} = \Gamma^i_{jk} + \delta^i_j p_k + \delta^i_k p_j, \quad \Pi^{n+1}_{n+1\ k} = p_k, \quad \Pi^{n+1}_{jk} = \Pi^i_{n+1\ n+1} = \Pi^{n+1}_{n+1\ n+1} = \Pi^i_{j\ n+1} = 0, \tag{7.20}$$

where

$$p_k \equiv -\frac{1}{n+2}\Gamma^l_{lk} \tag{7.21}$$

(the small Latin indices range from 1 to n), then, owing to formula (3.12), Eqs. (7.19) become

$$L_X \Pi^A_{BC} = 0 \quad (A, B, C = 1, \ldots, n+1).$$

Hence X is an automorphism of a projective structure defined by the coefficients (7.20), (7.21), that is, a projective transformation. This implies the following theorem.

Theorem 7.11. *The symmetry group of geodesic equations* (2.3) *in the space M^n with affine connection $\nabla(\Gamma^i_{jk})$ is a group of projective transformations in $n+1$-dimensional space with the projective connection Π^A_{BC}, defined by coefficients* (7.20), (7.21).

The maximal dimension $(n+1)^2 + 2(n+1) \equiv n^2 + 4n + 3$ of a projective group in $(n+1)$-dimensional space is attained in a projectively flat space. The necessary and sufficient condition in order that a space M^{n+1}, $n > 1$, be projectively flat is the vanishing of its Weyl tensor of projective curvature (2.7).

It is easy to verify that nonzero components of the Weyl tensor of the connection (7.20), (7.21) are

$$W^i_{jkl} = R^i_{jkl} - (1/n)\delta^i_k(R_{jl} + p_{jl}) + (1/n)\delta^i_l(R_{jk} + p_{jk}) + \delta^i_j p_{kl}, \quad W^{n+1}_{jl\ n+1} = (1/n)(R_{jl} + p_{jl}),$$
$$W^{n+1}_{n+1\ kl} = p_{kl}, \quad (p_{jk} \equiv \partial_j p_k - \partial_k p_j,\ i, j, k, l = 1, \ldots, n), \tag{7.22}$$

where R^i_{jkl} and R_{jk} are the curvature tensor and the Ricci tensor of the affine connection $\nabla(\Gamma^i_{jk})$, respectively. By equating the Weyl tensor to zero, we obtain $R^i_{jkl} = 0$. This implies the following theorem.

Theorem 7.12. *The maximal group $G_{\max}$ of symmetries of the geodesic equation* (2.3) *in the space M^n with affine connection $\nabla(\Gamma^i_{jk})$ has dimension $(n+1)^2 + 2(n+1) \equiv n^2 + 4n + 3$ and is attained if and only if M^n is flat. In appropriate coordinates, Eq.* (2.3) *and the generators $T_A, E_A, and E_A B$ of the group $G_{\max}$ are as follows*:

$$\frac{d^2x^i}{dt^2} = 0 \quad (i = 1, \ldots, n), \tag{7.23}$$

$$T_A = \partial_A, \quad E_{AB} = x^A\partial_B, \quad E_A = x^A x^C\partial_C, \tag{7.24}$$

$$
\begin{gathered}
[T_A, E_{AB}] = T_B, \\
[T_A, E_B] = E_{BA} \quad (A \neq B), \\
[T_A, E_A] = E_{AA} + \sum_{C=1}^{n+1} E_{CC},
\end{gathered}
\tag{7.25}
$$

$$
[E_{AB}, E_{CD}] = \delta_C^B E_{AD} - \delta_D^A E_{CB}, \quad [E_{AB}, E_B] = E_A \quad (A, B, C, D = 1, \ldots, n+1, \quad x^{n+1} \equiv t)
$$

(*the remaining commutators are zero*).

A direct consequence of Theorem 7.12 is the following proposition.

Theorem 7.13. *The dimension r of the symmetry group $G_{\max}$ of the geodesic equation* (2.3) *in a two-dimensional* (*pseudo-*)*Riemannian space M^2 is at most* 15. *The maximal dimension $r = 15$ is attained only for a flat space. In appropriate coordinates, Eq.* (2.3), *generators of the group $G_{\max}$, and its structure equations are defined by formulas* (7.23)*–*(7.25) *where we should set $n = 2$.*

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