Acta Applicandae Mathematicae **64:** 75–109, 2000.

Some Geometric Aspects of Integrability of Differential Equations in Two Independent Variables

In memory of Julia Hermosilla Pardo (1911–2000)

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(Received: 8 June 1999; in final form: 27 June 2000)

Abstract. The relation between scalar evolution equations which are the integrability condition of sl(2, **R**)-valued linear problems with parameter ('kinematic' integrability) and those which possess recursion operators ('formal' integrability) is studied: using that kinematically integrable equations describe one-parameter families of pseudo-spherical surfaces and vice versa, it is shown that every second order formally integrable evolution equation is kinematically integrable, and that this result cannot be extended as proven to the third-order case.

Conservation laws of kinematically integrable equations obtained from their underlying pseudo-spherical structure are compared with the ones one finds from the 'Riccati equation' version of their associated linear problems. Symmetries (generalized/nonlocal) for these equations are also studied, by considering infinitesimal deformations of the associated pseudo-spherical surfaces.

Finally, conservation laws for equations describing pseudo-spherical surfaces *immersed* in a flat three-space are found, and the class of 'equations describing Calapso–Guichard surfaces' is introduced.

Mathematics Subject Classifications (2000): 35Q53, 37K05, 37K10, 37K25.

Key words: pseudo-spherical surfaces, Calapso–Guichard surfaces, kinematic integrability, geometric integrability, formal integrability, symmetries, conservation laws.

1. Introduction

The issue considered here is classical (Zakharov, 1991): What is integrability in the context of *partial* differential equations? Many alternative answers have been suggested, each of them trying to capture an essential aspect of what it should mean. The following notions will be studied here: formal integrability (Mikhailov, Shabat, and Yamilov (1987), Mikhailov, Shabat, and Sokolov (1991)), kinematic integrability (Faddeev and Takhtajan, 1987), and geometric integrability, the approach favoured by Chern and Tenenblat (1986).

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The main goal of this work is to compare kinematic and formal integrability, to be introduced in Sections 2 and 3, respectively. Interestingly, Chern and Tenenblat's geometric point of view – based on the notion of an 'equation describing pseudo-spherical surfaces' to be rigorously defined in Section 2 – can be thought of as a 'bridge' between them. This fact allows one to prove the following theorem (Section 3). If a second order autonomous *evolution* equation is formally integrable, it is kinematically integrable. This result cannot be extended, as proven in this paper, to third order autonomous evolution equations in complete generality: only a special case, and the reason why this extension fails, will be reported here.

Sections 2 and 3 form the first part of this article. They are partially based on Reyes (1998a, b).

What about the converse? A natural path to take in order to investigate this question is to study integrability from the Chern and Tenenblat (1986) point of view, gain a deeper understanding of its properties, and then translate these properties into statements about kinematically integrable equations. One would expect this approach to yield a proof or counterexample of the implication 'kinematic integrability $\Rightarrow$ formal integrability', because formal integrability manifests itself at very fundamental levels: if an evolution equation of the form $u_t = F(u, u_x, \ldots, u_{x^m})$ possesses either an infinite number of generalized symmetries or an infinite number of local conservation laws, it is formally integrable (Ibragimov and Shabat, 1980, Svinolupov and Sokolov, 1982). One is therefore led to study conservation laws and symmetries of equations integrable in the Chern–Tenenblat sense. The following results have been obtained:

First, Chern–Tenenblat integrable equations do possess a host of conservation laws, but they may have a nonlocal character. These conservation laws are obtained in several ways: by means of an algorithm developed by Chern and Tenenblat (1986) and Cavalcante and Tenenblat (1988), by means of a generalization of these works carried out by the present author, and by means of infinitesimal deformations along (generalized, nonlocal) symmetries of conservation laws already known (see Reyes, 1998b). The first two methods are reviewed here, in Sections 4 and 6 respectively. Second (Section 4) the Cavalcante–Chern–Tenenblat collection of conservation laws corresponds exactly to the one obtained by means of the 'Riccati equation form of the associated linear problem' familiar from the early literature on inverse scattering (Wadati, Sanuki and Konno, 1975). Third (Section 5) equations integrable by the Chern–Tenenblat method (and satisfying a technical assumption) possess *many* symmetries, but, again, they may be of a nonlocal nature.

Thus, there is strong evidence for the validity of the implication 'kinematic integrability $\Rightarrow$ formal integrability', *if* one is willing to admit nonlocal symmetries and conservation laws in the theory, as Kaptsov (1982) does, and as it has been advocated by Krasil'shchik and Vinogradov (1989) in the context of the theory of coverings developed by them.

Sections 4 and 5 form the second part of this paper. Section 4 is partially based on Reyes (1998b, 2000). The results presented in Section 5 have only appeared in the author's thesis (Reyes, 1998b) and so complete proofs are provided. It is important to note that, as it will be explained in Section 2, solutions (satisfying a generic condition) of equations in the Chern–Tenenblat class determine pseudo-spherical surfaces described *intrinsically* by one-forms $\overline{\omega}^\alpha$, $\alpha = 1, 2, 3$. Sections 4 and 5 depend *only* on the intrinsic geometry of these surfaces.

Equations integrable in the Chern–Tenenblat sense can be also studied from an *extrinsic* point of view: one can consider the pseudo-spherical surfaces they determine as (locally and isometrically) immersed submanifolds of a three-dimensional flat space. This approach is used in Section 6 to introduce the second method to construct conservation laws for geometrically integrable equations which was mentioned above, and in Section 7 to prove that every scalar equation which describes pseudo-spherical surfaces can be deformed into a system of equations which is – in principle – kinematically integrable. Sections 6 and 7 complement the analysis made in (Reyes, 2000). They form the third and last part of this work.

Of course, an important aspect of the theory of differential equations which is often discussed in relation with integrability is the existence of Hamiltonian structures for them. A classical result (Magri, 1978) states that if an equation possesses two such Hamiltonian structures, and they satisfy a compatibility condition, the equation is formally integrable. Thus, yet another avenue to study the implication 'kinematic integrability $\Rightarrow$ formal integrability', is to investigate the existence of Hamiltonian structures for equations integrable in the Chern and Tenenblat sense. One can, in principle, construct Hamiltonian operators for this class of equations by using some interesting work due to Hojman (1996) (Reyes, 1998b). However, these operators are defined, in general, on *coverings* of the equation manifold determined by the equation at hand, so that the remark made before on the implication above still stands. This Hamiltonian approach will not be pursued here.

The main tool used in this article – besides classical differential geometry of surfaces (Eisenhart, 1909) – is the formal geometry of differential equations (Olver, 1993). Thus, for instance, (a) a scalar differential equation $\Xi(x, t, u, \ldots) = 0$ in two independent variables will be often identified (Krasil'shchik, Lychagin, and Vinogradov, 1986, Krasil'shchik and Verbovetsky, 1998) with a sub-bundle S^∞ of the infinite jet bundle $J^\infty E$, in which $E \to M$ is a trivial bundle with base the space of independent variables and typical fiber the space of the dependent variable, and (b) local solutions of $\Xi = 0$ will be considered to be infinite prolongations $j^\infty(s)$ of local sections $s\colon (x, t) \mapsto (x, t, u(x, t))$ of E such that $j^\infty(s)$ is a section of S^∞.

The function u and its derivatives will be henceforth denoted by

$$z_0 := u, \qquad z_{0,x^m t^n} = \frac{\partial^{m+n} u}{\partial x^m \partial t^n}, \quad \text{and} \quad z_j := \frac{\partial^j u}{\partial x^j} \quad \text{for } j, m, n \geqslant 1 \tag{1}$$

if considered as jet coordinates.

2. Kinematic and Chern–Tenenblat Integrability

The notions of geometric (*à la* Chern–Tenenblat) and kinematic integrability are introduced in this section, and it is shown that one can study the latter by means of the former. Chern and Tenenblat's (1986) approach is based on the following structure:

DEFINITION 1. Let Ξ be a smooth function on the infinite jet bundle $J^\infty E$ equipped with local coordinates determined by (1). Consider the differential equation

$$\Xi(x, t, z_0, \ldots, z_{0,x^m t^n}) = 0, \tag{2}$$

and let S^∞ be its equation manifold. Equation (2) is said to describe pseudo-spherical surfaces if there exist smooth functions $f_{\alpha\beta}$ $(\alpha = 1, 2, 3;\ \beta = 1, 2)$ on $J^\infty E$ for which (a) $f_{11} f_{22} - f_{12} f_{21} \not\equiv 0$, and (b) the pull-back of the one-forms $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$ by local holonomic sections $j^\infty(s)$ of S^∞, $\overline{\omega}^\alpha$ say, satisfy the structure equations of a surface of constant Gaussian curvature equal to -1 with metric $(\overline{\omega}^1)^2 + (\overline{\omega}^2)^2$ and connection one-form $\overline{\omega}_{12} = \overline{\omega}^3$, namely,

$$\mathrm{d}\overline{\omega}^1 = \overline{\omega}_{12} \wedge \overline{\omega}^2, \qquad \mathrm{d}\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}_{12}, \qquad \text{and} \qquad \mathrm{d}\overline{\omega}_{12} = \overline{\omega}^1 \wedge \overline{\omega}^2. \tag{3}$$

Note that if $s\colon (x, t) \mapsto (x, t, z_0(x, t))$ is a local section of E such that $j^\infty(s)$ is a *generic* solution of an equation describing pseudo-spherical surfaces with associated one-forms ω^α, that is (Kamran and Tenenblat, 1995) $(j^\infty(s))^*(\omega^1 \wedge \omega^2) \neq 0$, its graph Q is a submanifold of S^∞ possessing the structure of a pseudo-spherical surface, and this structure is determined in the (x, t) coordinates by the one-forms $\overline{\omega}^\alpha = (j^\infty(s))^*\omega^\alpha$.

Remarks. (a) Definition 1 can be extended *mutatis mutandis* to the case in which $\Xi = 0$ represents a system of partial differential equations in two independent variables. This fact will be assumed and used hereafter without further notice.

(b) The notations ω_{12} and ω^3 will be used for the one-form $f_{31}\,\mathrm{d}x + f_{32}\,\mathrm{d}t$. The former is classical, and useful when an extrinsic approach is taken (Sections 6 and 7). The latter is typographically economical.

(c) The expression 'PSS equation' will be sometimes utilized as an abbreviation of the phrase 'equation describing pseudo-spherical surfaces'.

DEFINITION 2. A differential equation (or system of equations) is geometrically integrable (*à la* Chern–Tenenblat) if it describes nontrivial one-parameter families of pseudo-spherical surfaces.

Chern and Tenenblat (1986) introduced this integrability notion motivated by the fact that if an equation is integrable by the AKNS inverse scattering scheme, it describes one-parameter families of pseudo-spherical surfaces whenever the spectral parameter is real (Sasaki, 1979). An interesting example of a geometrically integrable equation which is not of AKNS type is the Cavalcante and Tenenblat (1988) equation

$$u_t = (u_x^{-1/2})_{xx} + u_x^{3/2}.$$

Associated one-forms are $\omega^1 = \eta \sinh u \, \mathrm{d}x + \eta((u_x^{-1/2})_x \cosh u + (u_x^{1/2} - \eta u_x^{-1/2}) \sinh u) \, \mathrm{d}t$, $\omega^2 = \eta \, \mathrm{d}x - \eta^2 u_x^{-1/2} \, \mathrm{d}t$ and $\omega^3 = \eta \cosh u \, \mathrm{d}x + \eta((u_x^{-1/2})_x \sinh u + (u_x^{1/2} - \eta u_x^{-1/2}) \cosh u) \, \mathrm{d}t$, in which $\eta \neq 0$ is a parameter. Two other examples (Burgers' and cylindrical KdV equations) appear at the end of this section.

Kinematic integrability is defined after Faddeev and Takhtajan (1987):

DEFINITION 3. A differential equation $\Xi = 0$ is kinematically integrable if it is the integrability condition of a nontrivial one-parameter family of linear problems

$$\frac{\partial v}{\partial x} = U(\eta)v, \qquad \frac{\partial v}{\partial t} = V(\eta)v, \tag{4}$$

in which $U(\eta)$ and $V(\eta)$ are smooth $\mathrm{sl}(2, \mathbf{R})$-valued functions on $J^\infty E$ for all values of the parameter η.

Thus, a differential equation $\Xi(x, t, z_0, \ldots) = 0$ is kinematically integrable if the matrix equation

$$\frac{\partial U(\eta)}{\partial t} - \frac{\partial V(\eta)}{\partial x} + [U(\eta), V(\eta)] = 0 \tag{5}$$

is identically satisfied whenever $z_0(x, t)$ is a solution of $\Xi = 0$, and the 2×2 matrices $U(\eta)$ and $V(\eta)$ satisfy trace $U(\eta) =$ trace $V(\eta) = 0$.

One also says that a differential equation is *strictly* kinematically integrable if it is kinematically integrable and the diagonal entries of the matrix $U(\eta)$ introduced above are η and $-\eta$. For example, AKNS integrable equations are kinematically integrable in the strict sense, but it will be shown that (in principle) the cylindrical KdV equation is not. If an equation is integrable in the strict sense, it is customary to call η the 'spectral' parameter, in agreement with the fact that a strictly kinematically integrable equation is the integrability condition of a linear problem of the form

$$\left(\frac{\partial}{\partial x} - U_0\right)v = \eta \, \mathrm{diag}(1, -1)v, \qquad \frac{\partial v}{\partial t} = V(\eta)v,$$

in which $U_0 = U(\eta) - \eta \, \mathrm{diag}(1, -1)$, and that therefore one may hope to solve it by inverse scattering techniques.

Remark. One can interpret kinematic integrability in terms of an $\mathrm{SL}(2, \mathbf{R})$ connection. Consider a trivial vector bundle $M \times V \to M$, in which V is some vector space of dimension two, associated to a principal fiber bundle $M \times \mathrm{SL}(2, \mathbf{R}) \to M$. The parameter-dependent linear problem (4) can be re-written as

$$\mathrm{d}v = (U(\eta)\,\mathrm{d}x + V(\eta)\,\mathrm{d}t)v, \tag{6}$$

and the $\mathrm{sl}(2, \mathbf{R})$-valued one-form appearing in (6),

$$\Omega = U(\eta)\,\mathrm{d}x + V(\eta)\,\mathrm{d}t, \tag{7}$$

can be thought of, whenever $z_0(x, t)$ is a solution of the equation $\Xi = 0$, as a connection on the principal bundle $M \times \mathrm{SL}(2, \mathbf{R}) \to M$. In this context, Equation (5) is saying that – whenever $z_0(x, t)$ is a solution of $\Xi = 0$ – the linear system (6) is compatible if and only if the connection Ω is flat. In the classical literature on kinematic integrability (Faddeev and Takhtajan, 1987) Equation (5) is indeed called the *zero curvature condition*. The reader is referred to a recent paper by Rybnikov (1999) for a careful analysis of kinematic integrability in terms of connections on higher-order jet bundles.

The relation between kinematic and geometric integrability is now this: If a differential equation $\Xi = 0$ describes nontrivial one-parameter families of pseudo-spherical surfaces with associated one-forms $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$, it is kinematically integrable, while strict kinematic integrability amounts to the condition $f_{21} = \eta$. Indeed, one easily sees that Equations (3) hold if and only if the linear problem

$$\begin{pmatrix} \mathrm{d}v_1 \\ \mathrm{d}v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{8}$$

is integrable whenever $z_0(x, t)$ is a solution of $\Xi = 0$. Conversely, if $\Xi(x, t, z_0, \ldots) = 0$ is kinematically integrable, and its parameter-depending associated linear problem (4) is real, one can find a one-parameter family of one-forms ω^α, $\alpha = 1, 2, 3$, satisfying the requirements of Definition 1 by applying (again, whenever $z_0(x, t)$ is a solution of $\Xi = 0$) a gauge transformation to the connection $U\,\mathrm{d}x + V\,\mathrm{d}t$ so that the condition $\omega^1 \wedge \omega^2 \not\equiv 0$ holds, and setting, using an obvious notation, $\omega^1 = (U_{21} + U_{12})\,\mathrm{d}x + (V_{21} + V_{12})\,\mathrm{d}t$, $\omega^2 = 2U_{11}\,\mathrm{d}x + 2V_{11}\,\mathrm{d}t$, and $\omega^3 = (U_{21} - U_{12})\,\mathrm{d}x + (V_{21} - V_{12})\,\mathrm{d}t$.

This section ends with two further examples of kinematically integrable equations.

EXAMPLE. The generalized Burgers' equation

$$z_{0,t} = z_2 + z_0 z_1 + h(x) \tag{9}$$

describes pseudo-spherical surfaces (Reyes, 1998). Indeed, it is *strictly* kinematically integrable, as one can check that the η-dependent one-forms

$$\omega^1 = \left(\frac{1}{2}z_0 - \frac{\beta}{\eta}\right)\mathrm{d}x + \left(\frac{1}{2}z_1 + \frac{1}{4}z_0^2 + \frac{1}{2}\int h(x)\,\mathrm{d}x\right)\mathrm{d}t, \tag{10}$$

$$\omega^2 = \eta\,\mathrm{d}x + \left(\frac{\eta}{2}z_0 + \beta\right)\mathrm{d}t, \tag{11}$$

and

$$\omega^3 = -\eta\, \mathrm{d}x + \left(-\frac{\eta}{2} z_0 - \beta\right) \mathrm{d}t, \tag{12}$$

in which β is a solution of the Riccati equation

$$\beta^2 - \eta\beta_x + \frac{\eta^2}{2} \int h(x)\, \mathrm{d}x = 0, \tag{13}$$

satisfy the structure equations (3) whenever $z_0(x, t)$ is a solution of (9).

EXAMPLE. The cylindrical KdV equation

$$\frac{\partial v}{\partial \sigma} = -\frac{\partial^3 v}{\partial \xi^3} - v\frac{\partial v}{\partial \xi} - \frac{1}{2\sigma} v \tag{14}$$

describes pseudo-spherical surfaces. Indeed, it is kinematically integrable, although (in principle) not in a strict sense. This follows (Reyes, 1998) from an interesting transformation between the KdV and cylindrical KdV equations written down by Fuchssteiner (1993). The reader is also referred to the paper by Burtsev, Zakharov, and Mikhailov (1987) from which some earlier history of this transformation result may be inferred. First of all one sees that the KdV equation

$$z_{0,t} = z_3 + z_0 z_1 \tag{15}$$

describes pseudo-spherical surfaces with associated one-forms

$$\omega^1 = (1 - z_0)\, \mathrm{d}x + \left(-z_2 + \eta z_1 - \eta^2 z_0 - \tfrac{1}{3} z_0^2 + \eta^2 - \tfrac{2}{9} z_0 + \tfrac{5}{9}\right) \mathrm{d}t, \tag{16}$$

$$\omega^2 = \eta\, \mathrm{d}x + \left(\eta^3 + \tfrac{1}{3}\eta z_0 - \tfrac{1}{3} z_1 + \tfrac{5}{9}\eta\right) \mathrm{d}t, \tag{17}$$

and

$$\omega^3 = \left(\tfrac{2}{3} - z_0\right) \mathrm{d}x - \left(z_2 - \eta z_1 + \eta^2 z_0 + \frac{z_0^2}{3} - \frac{2}{3}\eta^2 + \frac{z_0}{3} - \frac{10}{27}\right) \mathrm{d}t. \tag{18}$$

Now, the invertible point transformation

$$x = \frac{\xi}{2^{1/3}\sqrt{\sigma}}, \qquad t = \frac{1}{\sqrt{\sigma}}, \quad \text{and} \quad z_0 = 2^{2/3}\sigma v - 2^{-1/3}\xi, \tag{19}$$

takes the cylindrical KdV equation (14) into Equation (15). This means that the pull-back of the one-forms (16), (17), and (18) by the invertible map (19) determines one-forms $\widehat{\omega}^\alpha = \widehat{f_{\alpha 1}}\, \mathrm{d}\xi + \widehat{f_{\alpha 2}}\, \mathrm{d}\sigma$, $\alpha = 1, 2, 3$, satisfying the structure equations of a pseudo-spherical surface whenever $v(\xi, \sigma)$ is a solution of (14).

Note that

$$\widehat{f}_{21} = \frac{\eta}{2^{1/3}\sqrt{\sigma}},$$

so that indeed Equation (14) is not strictly kinematically integrable.

3. Formal Integrability and PSS Equations

This section is devoted to the relation between formal and kinematic integrability. Formal integrability is at the core of the 'formal symmetry approach' to integrability developed by A. B. Shabat and his collaborators (see Mikhailov and Shabat, 1985, Mikhailov, Shabat, and Yamilov, 1987, and Mikhailov, Shabat, and Sokolov, 1991). Before stating its rigorous definition, it is useful to recall the following (Olver, 1993):

(1) The total derivatives with respect to x and t on $J^\infty E$ are the operators

$$D_x = \frac{\partial}{\partial x} + \sum_{m,n} z_{0,x^{m+1}t^n} \frac{\partial}{\partial z_{0,x^m t^n}}$$

and (20)

$$D_t = \frac{\partial}{\partial t} + \sum_{m,n} z_{0,x^m t^{n+1}} \frac{\partial}{\partial z_{0,x^m t^n}}.$$

(2) If $z_{0,t} = F$ is an evolution equation with equation manifold S^∞, the total derivatives with respect to x and t restricted to S^∞ are the operators

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} z_{k+1} \frac{\partial}{\partial z_k} \quad \text{and} \quad D_t = \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} D_x^k(F) \frac{\partial}{\partial z_k}. \tag{21}$$

(3) If f is a smooth function on $J^\infty E$, the *formal linearization of* f is the differential operator f_* on $J^\infty E$ whose action on smooth functions g on $J^\infty E$ is given by

$$f_*(g)\,(j^\infty(s)(x,t)) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \big(f \circ j^\infty(s_g)\big)(x,t), \tag{22}$$

in which s_g is the section $s_g\colon (x,t) \mapsto (x, t, (j^\infty(s))^*(z_0 + \tau g))$. In particular, if $f(x, t, z_0, \ldots, z_k)$ is a smooth function on the reduced jet space with coordinates $x, t, z_i, i \geqslant 0$, the formal linearization of f is the operator

$$f_* = \sum_{i=0}^{k} \frac{\partial f}{\partial z_i} D_x^i. \tag{23}$$

Formal integrability is introduced as follows (Mikhailov and Shabat, 1985, Olver, 1993):

DEFINITION 4. Let $z_{0,t} = F(x, z_0, \ldots, z_m)$ be an autonomous evolution equation with equation manifold S^∞. It is formally integrable if there exists a formal pseudo-differential operator

$$\Psi = \sum_{-\infty<k\leqslant N} f_k D_x^k$$

in which f_k, $-\infty < k \leqslant N$, are smooth functions on $J^\infty E$ such that (a) Ψ is not trivial, that is, it is not merely multiplication by a constant, and (b) the Lax-type equation

$$\Psi_t = [F_*, \Psi], \tag{24}$$

holds identically on S^∞.

In (24), F_* is the formal linearization of F, and Ψ_t is the formal pseudo-differential operator

$$\Psi_t = \sum_{-\infty<k\leqslant N} D_t(f_k) D_x^k.$$

Equation (24) is the equation characterizing the important ‘recursion operators’ first defined by P. Olver (see Olver, 1993 and references therein). It guarantees that if G is a (generalized) symmetry of the equation $z_{0,t} = F$, then so is $\Psi(G)$. Of course, one is assuming here that $\Psi(G)$ is a well defined smooth function on $J^\infty E$, a nontrivial hypothesis which does hold in many interesting cases, as proven recently by Wang (1998). In principle therefore, a pseudo-differential operator Ψ satisfying (24), generates a sequence of generalized symmetries of the equation $z_{0,t} = F$ which depend on arbitrarily large numbers of x-derivatives of the dependent variable z_0.

One can also prove (Mikhailov *et al.*, 1991, Olver, 1993) the following two important facts. First, if Ψ satisfies (24), then the coefficient f_{-1} is a conserved density of the autonomous equation $z_{0,t} = F$. Second, if Ψ satisfies (24), so does $\Psi^{(m/n)}$ for every m and $n > 0$. Thus, if an autonomous equation is formally integrable, it possesses an infinite number of local conservation laws. One concludes then that the existence of a nontrivial formal pseudo-differential operator Ψ satisfying (24) may be certainly advocated as an indicator of integrability.

Remark. (a) The conservation laws referred to in the last paragraph are not necessarily nontrivial: Burgers’ equation is formally integrable, but, it does not possess nontrivial local conservation laws other than itself.

(b) Formal integrability has been studied in the context of nonautonomous evolution equations by Abellanas and Galindo (1985), Svinolupov and Sokolov (1990), and Hernández Heredero, Sokolov, and Svinolupov (1995). It has also been generalized to systems of evolution equations (Mikhailov and Shabat, 1985, Mikhailov, Shabat, and Yamilov, 1987, and Mikhailov, Shabat, and Sokolov, 1991).

(c) The author is grateful to a referee for pointing out that the term 'formal integrability' was used by D. Spencer, H. Goldschmidt, and their co-workers already around 1960, in a context independent of symmetry considerations. See (Krasil'shchik and Verbovetsky, 1998) for details and original references.

Now one is ready to formulate and prove the results on the relationship between kinematic and formal integrability discussed in the Introduction. The first theorem of this section is this (Reyes, 1998):

THEOREM 1. *Every autonomous second-order evolution equation* $z_{0,t} = F(x, z_0, z_1, z_2)$ *which is formally integrable, is kinematically integrable. Moreover, up to invertible contact transformations of the form*

$$x' = \phi(x, z_0, z_1), \qquad t' = t, \qquad z_0' = \psi(x, z_0, z_1), \tag{25}$$

in which $(D_x\phi)\psi_{z_1} = (D_x\psi)\phi_{z_1}$, *and* $\psi_{z_0}\phi_{z_1} - \phi_{z_0}\psi_{z_1} \neq 0$, *there exists an exhaustive list of representatives of autonomous formally integrable second-order equations which are kinematically integrable in the strict sense.*

Proof. There exists an exhaustive list of formally integrable autonomous second-order equations. Up to contact transformations (25), they are (Mikhailov, Shabat and Yamilov, 1987, Mikhailov, Shabat and Sokolov, 1991, Olver, 1993):

$$z_{0,t} = z_2 + a(x)z_0, \tag{26}$$
$$z_{0,t} = z_2 + z_0z_1 + h(x), \tag{27}$$
$$z_{0,t} = D_x(z_1z_0^{-2} + \alpha x z_0 + \beta z_0), \quad \alpha, \beta \in \mathbf{R}, \tag{28}$$

and

$$z_{0,t} = D_x(z_1z_0^{-2} + x). \tag{29}$$

Now, these four equations describe pseudo-spherical surfaces:

(1) The equation $z_{0,t} = z_2 + a(x)z_0$ describes pseudo-spherical surfaces with associated functions $f_{11} = f_{31} = \gamma(x)z_0$, $f_{21} = \eta$, $f_{22} = 0$, and $f_{12} = f_{32} = \gamma z_1 - (\eta\gamma + \gamma_x)z_0 + E$, in which γ is given by the ordinary differential equation $\eta(-\eta\gamma - \gamma_x) + (-\eta\gamma - \gamma_x)_x = \gamma a(x)$, and E is a solution of $\eta E + E_x = 0$.

(2) The equation $z_{0,t} = z_2 + z_0z_1 + h(x)$ was shown to describe pseudo-spherical surfaces in the last section.

(3) The equation $z_{0,t} = D_x(z_1z_0^{-2} + \alpha x z_0 + \beta z_0)$ describes pseudo-spherical surfaces with associated functions $f_{11} = \exp(-\eta\lambda x)z_0$, $f_{21} = \eta$, $f_{22} = 0$, $f_{31} = \lambda\exp(-\eta\lambda x)z_0$, $f_{12} = \exp(-\eta\lambda x)z_0^{-2}z_1 + \exp(-\eta\lambda x)(\alpha x + \beta)z_0$, and $f_{32} = \lambda\exp(-\eta\lambda x)z_0^{-2}z_1 + \lambda\exp(-\eta\lambda x)(\alpha x + \beta)z_0$, in which $\lambda^2 = 1$.

(4) The equation $z_{0,t} = D_x(z_1 z_0^{-2} + x) = z_0^{-2} z_2 - 2z_0^{-3} z_1^2 + 1$ describes pseudo-spherical surfaces with associated functions $f_{11} = \exp(-\lambda\eta x)z_0$, $f_{21} = \eta$, $f_{22} = 0$, $f_{31} = \lambda \exp(-\lambda\eta x)z_0$, $f_{12} = \exp(-\lambda\eta x)z_0^{-2} z_1 + \delta(x)$, and finally $f_{32} = \lambda \exp(-\lambda\eta x)z_0^{-2} z_1 + \lambda\delta(x)$, in which the function $\delta(x)$ is determined by the equation $\lambda\eta\delta + \delta_x = \exp(-\lambda\eta x)$, and $\lambda^2 = 1$.

It follows then that these four equations are kinematically integrable in a strict sense. This finishes the proof. □

EXAMPLE. The equation

$$w_T = -\frac{1}{w_{XX}} - X(-Xw_X + w) \tag{30}$$

is formally integrable, as it reduces to Burgers' equation $z_{0,t} = z_2 + z_0 z_1$ under the Legendre transformation

$$X = -z_1, \quad T = t, \quad w = -xz_1 + z_0, \quad w_X = x, \quad w_T = z_{0,t}, \tag{31}$$

a classical example of a contact transformation of the form (25). Theorem 1 implies that Equation (30) is kinematically integrable. One-forms associated with it can be obtained from the one-forms (10), (11), and (12) associated with Burgers' equation by means of the invertible transformation (31). Use of the MAPLE package VESSIOT developed at Utah State University by Charles Miller and Ian Anderson (See the web-page http://www.math.usu.edu/~fg_mp/Pages/Symbolics.html for this interesting package) yields the following formulae:

$$\begin{aligned}\sigma^1 \;=\; & \left(-\tfrac{1}{2}\, w_{XX} X w_X + \tfrac{1}{2}\, w_{XX} w\right) \mathrm{d}X + \\ & + \left(-\tfrac{1}{4}\, X^2 {w_X}^2 + \tfrac{1}{2}\, X w_X w - \tfrac{1}{2}\, w_{XX} X^3 w_X - \tfrac{1}{2}\, \frac{X w_X w_{XXX}}{{w_{XX}}^2} - \right. \\ & \left. - \tfrac{1}{4}\, w^2 + \tfrac{1}{2}\, w_{XX} w X^2 + \tfrac{1}{2}\, \frac{w w_{XXX}}{{w_{XX}}^2} - \tfrac{1}{2}\, X \right) \mathrm{d}T,\end{aligned} \tag{32}$$

$$\sigma^2 = w_{XX}\eta\, \mathrm{d}X + \left(\tfrac{1}{2}\, \eta\, X w_X - \tfrac{1}{2}\, \eta\, w + \eta\, w_{XX} X^2 + \frac{\eta\, w_{XXX}}{{w_{XX}}^2} \right) \mathrm{d}T, \tag{33}$$

$$\begin{aligned}\sigma_{12} \;=\; & -w_{XX}\eta\, \mathrm{d}X + \\ & + \left(-\tfrac{1}{2}\, \eta\, X w_X + \tfrac{1}{2}\, \eta\, w - \eta\, w_{XX} X^2 - \frac{\eta\, w_{XXX}}{{w_{XX}}^2} \right) \mathrm{d}T.\end{aligned} \tag{34}$$

The method used to discover the one-forms appearing in the proof of Theorem 1 was this. First, *classify* (Reyes, 1998) the equations of the form

$z_{0,t} = F(x, t, z_0, z_1, z_2, \ldots, z_m)$ which describe pseudo-spherical surfaces under the assumption that $f_{21} = \eta$, a nonzero parameter. Second, check that the formally integrable Equations (26)–(29) appear in this classification. This check is not straightforward. Details appear in (Reyes, 1998).

Remark. The classification results for PSS equations used to construct the pseudo-spherical structures appearing in the proof of Theorem 1 – as well as those found in earlier papers, see (Reyes, 1998) and references therein – were obtained under a technical assumption ((Kamran and Tenenblat, 1995) and Section 5 below) which implies that if an evolution equation describes pseudo-spherical surfaces with associated one-forms $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$, one can constrain *a priori* the dependence of the functions $f_{\alpha\beta}$ on the variables z_k, $k \geqslant 1$. These constraints are important, not only in terms of classification results: they have been used by Chern and Tenenblat (1986) to study Bäcklund transformations for PSS equations, and it will be seen in Section 5 that they allow one to prove the existence of (generalized, nonlocal) symmetries of equations describing pseudo-spherical surfaces. It is an interesting open problem to check whether this technicality is unavoidable.

Now, one wonders if a theorem analogous to Theorem 1 is true for third order autonomous evolution equations. In fact, the proof of the implication 'formal integrability $\Rightarrow$ kinematic integrability' sketched above cannot be extended from second- to third-order equations in complete generality. By using the classification results mentioned above, one can check (Reyes, 1998) that the equation

$$u_t = (u_x + \delta)^3 u_{xxx}, \quad \delta \in \mathbf{R},$$

one of the two equations of the form $u_t = f(t, u, u_x)u_{xxx}$ which are formally integrable (Abellanas and Galindo, 1985), is not kinematically integrable in the strict sense. However, the following result holds:

THEOREM 2. *Every autonomous formally integrable third order evolution equation of the form*

$$z_{0,t} = z_0^{-3} z_3 + a_2(x, z_0, z_1) z_2^2 + a_1(x, z_0, z_1) z_2 + a_0(x, z_0, z_1) \tag{35}$$

is kinematically integrable. Moreover, there exists an exhaustive list of representatives (*up to contact transformations of the form* (25)) *of formally integrable third order equations of the form* (35) *which are kinematically integrable in the strict sense.*

In contrast with the last theorem, the proof of Theorem 2 is very short. See (Reyes, 1998) for details.

The results reviewed in this section clarify one aspect of the relationship between kinematic and formal integrability. One would like to proceed in the opposite direction, and study whether kinematic integrability implies, at least in some sense, formal integrability. For this, as pointed out in the Introduction, it is natural to consider conservation laws and symmetries of geometrically integrable equations. This is done in the following sections.

4. Conservation Laws for PSS Equations: Intrinsic Version

Let $\Omega^1(S^\infty)$ be the space of differential one-forms on the equation manifold S^∞ of a differential equation $\Xi = 0$. Recall that (Krasil'shchik, Lychagin, and Vinogradov, 1986, Olver, 1993, Krasil'shchik and Verbovetsky, 1998) S^∞ is equipped with a flat connection, and that this connection determines a splitting of $\Omega^1(S^\infty)$ into *horizontal* and *vertical* one-forms, and also a splitting of the exterior derivative operator into an *horizontal* exterior derivative d_H and a *vertical* exterior derivative d_V. Local conservation laws of $\Xi = 0$ are then defined as those horizontal one-forms on S^∞ which are d_H-closed. 'Nonlocal' conservation laws, for instance conservation laws which depend on integrations of the dependent variable z_0, can be also considered (Krasil'shchik and Vinogradov, 1989, Krasil'shchik and Verbovetsky, 1998). In the context of PSS equations, it is natural to study both cases simultaneously (Cavalcante and Tenenblat, 1988, Tenenblat, 1998), and this is what will be done here.

One obtains conservation laws of equations describing pseudo-spherical surfaces by using the following fact (Chern and Tenenblat, 1986):

Given an arbitrary coframe $\{\overline{\omega}^1, \overline{\omega}^2\}$ and corresponding connection one-form $\overline{\omega}_{12}$ on a smooth surface M equipped with the Riemannian metric $\mathrm{d}s^2 = (\overline{\omega}^1)^2 + (\overline{\omega}^2)^2$, there exists a new coframe $\{\overline{\theta}^1, \overline{\theta}^2\}$ and a correspondingly new connection one-form $\overline{\theta}_{12}$ satisfying the structure equations

$$\mathrm{d}\overline{\theta}^1 = 0, \qquad \mathrm{d}\overline{\theta}^2 = \overline{\theta}^2 \wedge \overline{\theta}^1, \quad \text{and} \quad \overline{\theta}_{12} + \overline{\theta}^2 = 0, \tag{36}$$

if and only if the surface M is pseudo-spherical. This is seen thus: one assumes that the orthonormal frames dual to the coframes $\{\overline{\omega}^1, \overline{\omega}^2\}$ and $\{\overline{\theta}^1, \overline{\theta}^2\}$ possess the same orientation. Since M is in principle an arbitrary Riemannian two-manifold, the structure group of the principal bundle of orthonormal oriented frames of M is SO(2). This means that the 'old' and 'new' one-forms, $\overline{\omega}^1, \overline{\omega}^2, \overline{\omega}_{12}$ and $\overline{\theta}^1, \overline{\theta}^2, \overline{\theta}_{12}$ respectively, are connected by means of a rotation in an angle $\rho(x,t)$,

$$\overline{\theta}^1 = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho, \tag{37}$$

$$\overline{\theta}^2 = -\overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho, \tag{38}$$

and by a gauge transformation

$$\overline{\theta}_{12} = \overline{\omega}_{12} + \mathrm{d}\rho. \tag{39}$$

It follows that a coframe $\{\overline{\theta}^1, \overline{\theta}^2\}$ and corresponding connection one-form $\overline{\theta}_{12}$ satisfying (36) exists if and only if the Pfaffian system

$$\overline{\omega}_{12} + \mathrm{d}\rho - \overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho = 0 \tag{40}$$

on the space of coordinates (x, t, ρ) is completely integrable, and one can check that this happens if and only if M is pseudo-spherical.

The analytic importance of Chern and Tenenblat's (1986) observation is this. If the differential equation

$$\Xi(x, t, z_0, \ldots, z_{0,x^m t^n}) = 0 \qquad (41)$$

describes pseudo-spherical surfaces with associated one-forms $\omega^1, \omega^2, \omega_{12}$, Equation (40) implies that the Pfaffian system

$$\overline{\omega}_{12} + \mathrm{d}\rho - \overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho = 0, \qquad (42)$$

in which $\overline{\omega}^b = (j^\infty(s))^*\omega^b$ for $b = 1, 2$, and $\overline{\omega}_{12} = (j^\infty(s))^*\omega_{12}$, is completely integrable for $\rho(x, t)$ whenever $s\colon (x, t) \mapsto (x, t, z_0(x, t))$ is a local section of E such that $j^\infty(s)$ is a local solution of Equation (41). Equations (36)–(38) then imply that for each solution $z_0(x, t)$ of (41) and a corresponding solution $\rho(x, t)$ of (42), the one-form

$$\theta = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho \qquad (43)$$

is closed. If Equation (41) is assumed to be geometrically integrable, and the associated functions $f_{\alpha\beta}$ can be formally expanded as a power series in a parameter η, the function $\rho(x, t)$ given by (42) can be also expanded in powers of η. The one-form θ then determines a sequence of one-forms which are closed whenever $z_0(x, t)$ is a solution of Equation (41), and which are d_H-closed horizontal one-forms on the equation manifold S^∞ of Equation (41) if they belong to $\Omega^1(S^\infty)$ (see (Cavalcante and Tenenblat, 1988) for an explicit construction). Thus, geometrically integrable equations possess an infinite number of conservation laws, but they may well be of a nonlocal nature.

What is the relationship between these conservation laws and the ones obtained from the associated linear system (8) by using the 'Riccati equation method' as in (Wadati, Sanuki, and Konno, 1975)? (See (Reyes, 1998) for a recent review in the context of PSS equations.) One can prove that they are actually the same in two steps:

First, one compares Chern and Tenenblat's (1986) one-form (43) with the ones appearing in (Sasaki, 1979). The following theorem holds (Reyes, 2000):

THEOREM 3. *Let* $\Xi = 0$ *be a geometrically integrable equation with associated one-forms* $\{\omega^1, \omega^2, \omega_{12}\}$. *The conservation laws of this equation which are obtained from the Chern–Tenenblat one-form* (43), *coincide with those obtained by Sasaki* (1979) *up to a rotation, nonessential constants, invertible changes of variables, and addition of exact differential forms.*

This theorem is proven as follows: under the changes of variables $\Gamma = \tan(\rho/2)$ and $\hat{\Gamma} = \cot(\rho/2)$, the completely integrable Pfaffian system (42) and the one-form

(43) become, respectively,

$$-2\,\mathrm{d}\Gamma = (\overline{\omega}_{12} + \overline{\omega}^2) - 2\Gamma\overline{\omega}^1 + \Gamma^2(\overline{\omega}_{12} - \overline{\omega}^2), \tag{44}$$

$$\Theta = \overline{\omega}^1 - \Gamma(\overline{\omega}_{12} - \overline{\omega}^2) \quad \text{(up to an exact differential form)} \tag{45}$$

and

$$2\,\mathrm{d}\widehat{\Gamma} = (\overline{\omega}_{12} - \overline{\omega}^2) - 2\widehat{\Gamma}\overline{\omega}^1 + \hat{\Gamma}^2(\overline{\omega}_{12} + \overline{\omega}^2), \tag{46}$$

$$\hat{\Theta} = -\overline{\omega}^1 + \hat{\Gamma}(\overline{\omega}_{12} + \overline{\omega}^2) \quad \text{(up to an exact differential form).} \tag{47}$$

One then rotates the coframe $\{\overline{\omega}^1, \overline{\omega}^2\}$ in $\rho_0 = \pi/2$, and performs some elementary changes of variables in the resulting Pfaffian systems and closed one-forms.

Second, one checks (Reyes, 2000) that Sasaki's (1979) conservation laws coincide with the ones obtained by the classical method of Wadati, Sanuki, and Konno (1975). Explicitly, some easy manipulations using the rotated versions of (45) and (47) yield the following two expressions for Sasaki's conservation laws,

$$D_t\left(\frac{f_{21}}{2} + \phi_1\right) = D_x\left(\frac{f_{22}}{2} + \phi_1\frac{f_{12} - f_{32}}{f_{11} - f_{31}}\right) \quad \text{if } f_{11} \neq f_{31}, \tag{48}$$

and

$$D_t\left(\frac{f_{21}}{2} + \phi_2\right) = D_x\left(\frac{f_{22}}{2} + \phi_2\frac{f_{12} + f_{32}}{f_{11} + f_{31}}\right) \quad \text{if } f_{11} \neq -f_{31}. \tag{49}$$

These are Wadati, Sanuki and Konno's (1975) formulae if $f_{21} = \eta$, see Reyes (1998). Here, the functions ϕ_1 and ϕ_2 are determined, whenever $z_0(x,t)$ is a solution of the equation $\Xi = 0$, by the Riccati equations

$$D_x\phi_1 = \frac{1}{4}(f_{11}^2 - f_{31}^2) + \left(\frac{D_x(f_{11} - f_{31})}{f_{11} - f_{31}} - f_{21}\right)\phi_1 - \phi_1^2, \tag{50}$$

and

$$D_x\phi_2 = -\frac{1}{4}(f_{11}^2 - f_{31}^2) + \left(\frac{D_x(f_{11} + f_{31})}{f_{11} + f_{31}} + f_{21}\right)\phi_2 + \phi_2^2, \tag{51}$$

respectively, both of which follow from the rotated versions of the Pfaffian systems (44) and (46).

Examples of conservation laws obtained by using formulae (48)–(51) have appeared in (Reyes, 1998, 2000).

As pointed out in the Introduction one can extend Chern and Tenenblat's result, and obtain (at least in principle) more conservation laws for PSS equations from purely geometric considerations. In order to do this, one considers the surfaces determined by solutions of PSS equations as immersed in some higher dimensional space. This analysis is postponed until Section 6, after a second fundamental property of PSS equations, the existence of (generalized, nonlocal) symmetries, is investigated.

5. Symmetries for PSS Equations

Suppose that the evolutionary PSS equation

$$z_{0,t} = F(x, t, z_0, z_1, \dots, z_m) \tag{52}$$

determines a submanifold S^m of $J^m E$, and that its equation manifold $S^\infty \hookrightarrow J^\infty E$ is equipped with local coordinates $(x, t, z_0, \dots, z_m, \dots)$. It will be assumed in this section that one can choose one-forms $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$ associated with Equation (52) so that they belong to $\Omega(J^m E)$, the space of differential forms on $J^m E$, that no variables of the form $z_{0,x^i t^j}$, $j \geqslant 1$, appear in the expressions for the functions $f_{\alpha\beta}$, and that (Kamran and Tenenblat, 1995) the differential ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\Omega(J^m E)$ generated by the two-forms

$$\Omega_1 = \mathrm{d}\omega^1 - \omega^3 \wedge \omega^2, \qquad \Omega_2 = \mathrm{d}\omega^2 - \omega^1 \wedge \omega^3, \qquad \Omega_3 = \mathrm{d}\omega^3 - \omega^1 \wedge \omega^2,$$

and

$$\mathrm{d}z_0 \wedge \mathrm{d}x + F\,\mathrm{d}x \wedge \mathrm{d}t, \qquad \mathrm{d}z_i \wedge \mathrm{d}t - z_{i+1}\,\mathrm{d}x \wedge \mathrm{d}t,\ 1 \leqslant i \leqslant m-1$$

respectively, satisfy

$$\mathcal{I}_1 = \mathcal{I}_2. \tag{53}$$

Note that the local solutions of Equation (52) are in a one-to-one correspondence with the local integral manifolds of the exterior differential system $\{\mathcal{I}_2, \mathrm{d}x \wedge \mathrm{d}t\}$. It follows from (53) that the *equation ideal* $\mathcal{I}_2$ is algebraically equivalent to a system of differential forms satisfying the pseudo-spherical structure equations whenever $z_0(x, t)$ is a solution of Equation (52). In particular, this means that Equation (52) is necessary and sufficient for the structure equations $\Omega_i = 0$, $i = 1, 2, 3$, to hold. The following lemma is crucial:

LEMMA 1. *Necessary and sufficient conditions for the evolution equation* (52) *to describe pseudo-spherical surfaces under the assumption* (53) *are the following*:

(1) *The functions* $f_{\alpha\beta}$ *satisfy the constraints*

$$f_{i1,z_a} = 0, a \geqslant 1,\ \ f_{i2,z_k} = 0;\ \ i = 1, 2, 3, \tag{54}$$

and

$$f_{11,z_0}^2 + f_{21,z_0}^2 + f_{31,z_0}^2 \neq 0. \tag{55}$$

(2) *The right-hand side of Equation* (52) *is related to the functions* $f_{\alpha\beta}$ *and their derivatives by the identities*

$$-f_{11,z_0}F + \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} + f_{21}f_{32} - f_{31}f_{22} + f_{12,x} - f_{11,t} = 0, \tag{56}$$

$$-f_{21,z_0}F + \sum_{i=0}^{k-1} z_{i+1} f_{22,z_i} + f_{12}f_{31} - f_{11}f_{32} + f_{22,x} - f_{21,t} = 0, \qquad (57)$$

and

$$-f_{31,z_0}F + \sum_{i=0}^{k-1} z_{i+1} f_{32,z_i} + f_{21}f_{12} - f_{11}f_{22} + f_{32,x} - f_{31,t} = 0. \qquad (58)$$

Proof. This lemma is proven along the lines of Lemma 1 of Reyes (1998) and Lemma 4 of Kamran and Tenenblat (1995). □

DEFINITION 5. A generalized symmetry of an evolution equation $z_{0,t} = F$ with equation manifold $S^\infty \hookrightarrow J^\infty E$, is a smooth function G on $J^\infty E$ such that for any section $s\colon (x,t) \mapsto (x,t,z_0(x,t))$ of E for which $j^\infty(s)$ is a local solution of $z_{0,t} = F$, the infinite prolongation of the section

$$s_G\colon (x,t) \mapsto (x,t,z_0(x,t) + \tau(j^\infty(s))^* G) \qquad (59)$$

of E, satisfies the equation $z_{0,t} = F$ to first order in τ.

In other words, a smooth function G on $J^\infty E$ is a generalized symmetry of $z_{0,t} = F$ if and only if the equation

$$D_t G = F_* G \qquad (60)$$

(see (21) and (23)) holds identically on S^∞ once all the derivatives with respect to t appearing in G have been replaced by means of $z_{0,t} = F$.

Generalized symmetries of PSS equations for which the assumption (53) holds may be characterized in a very appealing way, as it will be seen presently. One needs the following straightforward lemma:

LEMMA 2. *Suppose that the one-forms $\overline{\omega}^\alpha$, $\alpha = 1, 2, 3$, satisfy* (3), *the structure equations of a pseudo-spherical surface with metric $(\overline{\omega}^1)^2 + (\overline{\omega}^2)^2$ and connection form $\overline{\omega}^3$. The deformed one-forms $\overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha$ satisfy the structure equations of a pseudo-spherical surface up to terms of order τ^2 if and only if*

$$\mathrm{d}\overline{\Lambda}_1 = \overline{\omega}^3 \wedge \overline{\Lambda}_2 + \overline{\Lambda}_3 \wedge \overline{\omega}^2, \qquad (61)$$

$$\mathrm{d}\overline{\Lambda}_2 = \overline{\omega}^1 \wedge \overline{\Lambda}_3 + \overline{\Lambda}_1 \wedge \overline{\omega}^3, \qquad (62)$$

and

$$\mathrm{d}\overline{\Lambda}_3 = \overline{\omega}^1 \wedge \overline{\Lambda}_2 + \overline{\Lambda}_1 \wedge \overline{\omega}^2. \qquad (63)$$

The analytical content of Lemma 2 is seen as follows. Let

$$z_{0,t} = F(x,t,z_0,\ldots,z_m) \qquad (64)$$

be an equation describing pseudo-spherical surfaces with equation manifold $S^\infty \hookrightarrow J^\infty E$ and associated one-forms ω^α, $\alpha = 1, 2, 3$. Assume that (53) holds. Consider an arbitrary local section $s: (x, t) \mapsto (x, t, z_0(x, t))$ of E such that $j^\infty(s)$ is a local solution of Equation (64), set $\overline{G} = (j^\infty(s))^* G$, in which G is any smooth function on $J^\infty E$, and let s_G be the section $s_G: (x, t) \mapsto (x, t, z_0(x, t) + \tau\overline{G})$ defined in (59). What one does is to study the one-forms $(j^\infty(s_G))^*\omega^\alpha$.

Expand these one-forms in powers of τ about $\tau = 0$. One obtains, to first order in τ, an infinitesimal deformation of the one-forms

$$\overline{\omega}^\alpha = (j^\infty(s))^*\, \omega^\alpha, \tag{65}$$

namely,

$$\overline{\omega}^\alpha \;\mapsto\; \overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha, \tag{66}$$

in which

$$\overline{\Lambda}_\alpha = \overline{g}_{\alpha 1}\, \mathrm{d}x + \overline{g}_{\alpha 2}\, \mathrm{d}t, \tag{67}$$

and the functions $\bar{g}_{\alpha\beta}$ are given by the formulae

$$\bar{g}_{\alpha 1} = (j^\infty(s))^* f_{\alpha 1, z_0}\, \overline{G} \quad \text{and} \quad \bar{g}_{\alpha 2} = \sum_{i=0}^{m-1} (j^\infty(s))^* f_{\alpha 2, z_i} \frac{\partial^i \overline{G}}{\partial x^i}, \qquad \alpha = 1, 2, 3, \tag{68}$$

as a straightforward computation using Lemma 1 shows.

THEOREM 4. *Suppose that Equation* (64) *describes pseudo-spherical surfaces with associated one-forms* $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$, $\alpha = 1, 2, 3$, *satisfying* (53). *Let* G *be a smooth function on* $J^\infty E$, *and consider the deformed one-forms* $\overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha$ *given by Equations* (66)–(68). *They satisfy the structure equations of a pseudo-spherical surface up to terms of order* τ^2 *if and only if* G *is a generalized symmetry of* (64).

Proof. Generalizing the foregoing notation (Equations (65)–(68)) it will be assumed that a line (‘$-$’) over an expression E indicates that E has been pulled-back by means of $j^\infty(s)$.

Lemma 2 implies that the one-forms $\overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha$ satisfy the structure equations of a pseudo-spherical surface to first order in τ if and only if Equations (61), (62) and (63) hold, that is, if and only if

$$-\frac{\partial \bar{g}_{11}}{\partial t} + \frac{\partial \bar{g}_{12}}{\partial x} = \bar{f}_{31}\bar{g}_{22} - \bar{f}_{32}\bar{g}_{21} + \bar{f}_{22}\bar{g}_{31} - \bar{f}_{21}\bar{g}_{32}, \tag{69}$$

$$-\frac{\partial \bar{g}_{21}}{\partial t} + \frac{\partial \bar{g}_{22}}{\partial x} = \bar{f}_{11}\bar{g}_{32} - \bar{f}_{12}\bar{g}_{31} + \bar{f}_{32}\bar{g}_{11} - \bar{f}_{31}\bar{g}_{12}, \tag{70}$$

and

$$-\frac{\partial \bar{g}_{31}}{\partial t} + \frac{\partial \bar{g}_{32}}{\partial x} = \bar{f}_{11}\bar{g}_{22} - \bar{f}_{12}\bar{g}_{21} + \bar{f}_{22}\bar{g}_{11} - \bar{f}_{21}\bar{g}_{12}. \tag{71}$$

Now, since Equation (64) describes pseudo-spherical surfaces, Equations (56), (57) and (58) of Lemma 1 are *identities* on $J^{\infty}E$. One can then pull them back by the infinite prolongation of the section s_G introduced above, Equation (59). Taking derivatives with respect to τ at $\tau = 0$, one obtains the equations

$$\begin{aligned} &-\bar{f}_{11,z_0z_0}\overline{G}\,\overline{F} - \bar{f}_{11,z_0}\overline{F}_*(\overline{G}) + \frac{\partial \bar{g}_{12}}{\partial x} + \\ &\quad + \bar{g}_{21}\bar{f}_{32} + \bar{f}_{21}\bar{g}_{32} - \bar{g}_{31}\bar{f}_{22} - \bar{f}_{31}\bar{g}_{22} - \bar{f}_{11,tz_0}\overline{G} = 0, \end{aligned} \tag{72}$$

$$\begin{aligned} &-\bar{f}_{21,z_0z_0}\overline{G}\,\overline{F} - \bar{f}_{21,z_0}\overline{F}_*(\overline{G}) + \frac{\partial \bar{g}_{22}}{\partial x} + \bar{g}_{12}\bar{f}_{31} + \\ &\quad + \bar{f}_{12}\bar{g}_{31} - \bar{g}_{11}\bar{f}_{22} - \bar{f}_{11}\bar{g}_{32} - \bar{f}_{21,tz_0}\overline{G} = 0, \end{aligned} \tag{73}$$

$$\begin{aligned} &-\bar{f}_{31,z_0z_0}\overline{G}\,\overline{F} - \bar{f}_{31,z_0}\overline{F}_*(\overline{G}) + \frac{\partial \bar{g}_{32}}{\partial x} + \bar{g}_{21}\bar{f}_{12} + \\ &\quad + \bar{f}_{21}\bar{g}_{12} - \bar{g}_{11}\bar{f}_{22} - \bar{f}_{11}\bar{g}_{22} - \bar{f}_{31,tz_0}\overline{G} = 0, \end{aligned} \tag{74}$$

in which

$$\overline{F}_*(\overline{G}) = \sum_{i=0}^{m} \frac{\partial \overline{F}}{\partial z_i}\,\frac{\partial^i \overline{G}}{\partial x^i}. \tag{75}$$

Substituting (72), (73) and (74) into Equations (69), (70), and (71), one finds that (61), (62) and (63) hold if and only if

$$-\bar{f}_{11,z_0}\frac{\partial \overline{G}}{\partial t} + \bar{f}_{11,z_0}\overline{F}_*(\overline{G}) \;=\; 0, \tag{76}$$

$$-\bar{f}_{21,z_0}\frac{\partial \overline{G}}{\partial t} + \bar{f}_{21,z_0}\overline{F}_*(\overline{G}) \;=\; 0, \tag{77}$$

and

$$-\bar{f}_{31,z_0}\frac{\partial \overline{G}}{\partial t} + \bar{f}_{31,z_0}\overline{F}_*(\overline{G}) = 0. \tag{78}$$

Since the constraint (55) holds, one concludes that Equations (61), (62) and (63) of Lemma 2 are equivalent to

$$\frac{\partial \overline{G}}{\partial t} = \overline{F}_*(\overline{G}),$$

and therefore they are satisfied if and only if G is a generalized symmetry of the PSS equation (64). □

This theorem relates (generalized) symmetries of PSS equations to the geometry of the pseudo-spherical surfaces themselves. It will be seen presently that one can use it to show the existence of (generalized, nonlocal) symmetries of geometrically integrable equations.

Assume that the equation

$$z_{0,t} = F(x, t, z_0, z_1, \ldots, z_m) \tag{79}$$

is geometrically integrable. As pointed out in Section 4, the Pfaffian system defined in Equation (42), namely

$$\overline{\omega}^3 - \mathrm{d}\rho + \overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho = 0, \tag{80}$$

in which, for convenience, the variable ρ appearing in (42) has been replaced by $-\rho$ and again, $\overline{\omega}^\alpha = (j^\infty(s))^*\omega^\alpha$, $\alpha = 1, 2, 3$, is completely integrable for $\rho(x, t, \eta)$ whenever $j^\infty(s)$ is a local solution of Equation (79). Let $\overline{G}(x, t, \eta)$ be a (for now arbitrary) function of the independent variables and the parameter η, and set

$$\frac{\partial}{\partial\tau} z_0(x, t) = \overline{G}(x, t, \eta). \tag{81}$$

The Pfaffian system (80) can be 'linearized at z_0 in the direction $\overline{G}$' that is, one can compute its τ-derivative at $\tau = 0$. One obtains a new Pfaffian system,

$$\overline{\Lambda}_3 - \mathrm{d}\sigma + \sigma(\cos\rho)\overline{\omega}^1 + (\sin\rho)\overline{\Lambda}_1 - \sigma(\sin\rho)\overline{\omega}^2 + (\cos\rho)\overline{\Lambda}_2 = 0, \tag{82}$$

in which

$$\sigma(x, t, \eta) = \frac{\partial}{\partial\tau}\rho(x, t, \eta),$$

$\overline{\Lambda}_\alpha = \bar{g}_{\alpha 1}\,\mathrm{d}x + \bar{g}_{\alpha 2}\,\mathrm{d}t$, and the functions $\bar{g}_{\alpha\beta}$ are given by

$$\bar{g}_{\alpha\beta} = \sum_{i=0}^{k-1} (j^\infty(s))^*(f_{\alpha\beta, z_i}) \frac{\partial^i \overline{G}}{\partial x^i},$$

compare with Equations (68). One would like to study the complete integrability of this system whenever $z_0(x, t)$ is a solution of Equation (79) and $\rho(x, t, \eta)$ is a corresponding solution of the completely integrable Pfaffian system (80).

LEMMA 3. *The exterior derivative of* (82) *is given by*

$$\begin{aligned}&(\mathrm{d}\overline{\Lambda}_3 - \overline{\omega}^1 \wedge \overline{\Lambda}_2 - \overline{\Lambda}_1 \wedge \overline{\omega}^2) + \cos\rho\,(\mathrm{d}\overline{\Lambda}_2 - \overline{\omega}^1 \wedge \overline{\Lambda}_3 - \overline{\Lambda}_1 \wedge \overline{\omega}^3) + \\ &\quad + \sin\rho\,(\mathrm{d}\overline{\Lambda}_1 - \overline{\omega}^3 \wedge \overline{\Lambda}_2 - \overline{\Lambda}_3 \wedge \overline{\omega}^2) = 0\end{aligned} \tag{83}$$

whenever $z_0(x, t)$ *is a solution of Equation* (79) *and* $\rho(x, t, \eta)$ *is a corresponding solution of system* (80).

Proof. A direct computation using the structure equations of a pseudo-spherical surface and Equations (80) and (82). □

The analytical interpretation of this lemma is as follows:

PROPOSITION 1. *Assume that Equation* (79) *is geometrically integrable, and that the hypothesis* (53) *holds. Set* $z_{0,\tau} = \overline{G}$ *and* $\rho_\tau = \sigma$*, as above. The Pfaffian system* (82) *is completely integrable for* $\sigma(x,t,\eta)$ *whenever* $z_0(x,t)$ *is a solution of Equation* (79)*, and* $\rho(x,t,\eta)$ *is a corresponding solution of the completely integrable Pfaffian system* (80)*, if and only if* $\overline{G}$ *satisfies the equation*

$$\frac{\partial \overline{G}}{\partial t} = \sum_{i=0}^{k} (j^\infty(s))^* \left(\frac{\partial F}{\partial z_i}\right) \frac{\partial^i \overline{G}}{\partial x^i}, \tag{84}$$

in which s *is the section* $s\colon (x,t) \mapsto (x,t,z_0(x,t))$.

Proof. Consider the identities (56), (57), and (58) of Lemma 2 and pull them back by means of the infinite prolongation $j^\infty(s)$ (compare with the proof of Theorem 4). Take derivatives with respect to τ of the resulting identities by means of (81). One obtains equations analogous to (72), (73), and (74), in which $\bar{f}_{\alpha 1} = (j^\infty(s))^* f_{\alpha 1}$, and $\overline{F}_*(\overline{G})$ is defined as in (75). It follows that Equation (83) is equivalent to

$$\bar{f}_{31,z_0}\left(-\frac{\partial \overline{G}}{\partial t} + \overline{F}_*(\overline{G})\right) + \cos\rho\, \bar{f}_{21,z_0}\left(-\frac{\partial \overline{G}}{\partial t} + \overline{F}_*(\overline{G})\right) + \\ + \sin\rho\, \bar{f}_{11,z_0}\left(-\frac{\partial \overline{G}}{\partial t} + \overline{F}_*(\overline{G})\right) = 0.$$

The *if* part is then clear. On the other hand, if Equation (84) were not satisfied, it would follow that

$$\bar{f}_{31,z_0} + \cos\rho\, \bar{f}_{21,z_0} + \sin\rho\, \bar{f}_{11,z_0} = 0$$

whenever $z_0(x,t)$ is a solution of Equation (79) and $\rho(x,t,\eta)$ is a corresponding solution of the completely integrable Pfaffian system (80). But this equation certainly does not hold for arbitrary geometrically integrable evolution equations, as the examples of Cavalcante and Tenenblat (1988) show. □

Thus, if $j^\infty(s)$ is a local solution of Equation (79), and the Pfaffian systems (80) and (82) are completely integrable, the function $\overline{G}$ must satisfy precisely the pull-back by $j^\infty(s)$ of the equation $D_t G = F_* G$ which characterizes (generalized) symmetries of Equation (79).

This function $\overline{G}$ can be found as follows. The Pfaffian system (82) is, explicitly,

$$\frac{\partial \sigma}{\partial x} + \sigma(\bar{f}_{21}\sin\rho - \bar{f}_{11}\cos\rho) = \bar{g}_{31} + \bar{g}_{11}\sin\rho + \bar{g}_{21}\cos\rho, \tag{85}$$

$$\frac{\partial \sigma}{\partial t} + \sigma(\bar{f}_{22}\sin\rho - \bar{f}_{12}\cos\rho) = \bar{g}_{32} + \bar{g}_{12}\sin\rho + \bar{g}_{22}\cos\rho. \tag{86}$$

Introduce functions b and A^β, $\beta = 1, 2$, as follows:

$$b = \bar{f}_{31,z_0} + \bar{f}_{11,z_0}\sin\rho + \bar{f}_{21,z_0}\cos\rho, \quad \text{and}$$
$$A^\beta = \bar{f}_{2\beta}\sin\rho - \bar{f}_{1\beta}\cos\rho. \tag{87}$$

A straightforward computation shows that Equations (85) and (86) are equivalent to

$$\overline{G} = \frac{1}{b}\left(\frac{\partial\sigma}{\partial x} + A^1\sigma\right), \tag{88}$$

$$\frac{\partial\sigma}{\partial t} + A^2\sigma = \sum_{i=0}^{k-1}\left(\bar{f}_{32,z_i} + \bar{f}_{12,z_i}\sin\rho + \bar{f}_{22,z_i}\cos\rho\right)\frac{\partial^i}{\partial x^i}\left(\frac{1}{b}\left(\frac{\partial\sigma}{\partial x} + A^1\sigma\right)\right). \tag{89}$$

The last proposition allows one to conclude the following:

PROPOSITION 2. *Assume that Equation* (79) *is geometrically integrable, and that* (53) *holds. Set* $z_{0,\tau} = \overline{G}$ *and* $\rho_\tau = \sigma$, *as above. The Pfaffian system*

$$\overline{\Lambda}_3 - \mathrm{d}\sigma + \sigma(\cos\rho)\overline{\omega}^1 + (\sin\rho)\overline{\Lambda}_1 - \sigma(\sin\rho)\overline{\omega}^2 + (\cos\rho)\overline{\Lambda}_2 = 0 \tag{90}$$

is completely integrable for $\sigma(x, t, \eta)$ *whenever* $z_0(x, t)$ *is a solution of Equation* (79) *and* $\rho(x, t, \eta)$ *is a corresponding solution of the completely integrable Pfaffian system*

$$\overline{\omega}^3 - \mathrm{d}\rho + \overline{\omega}^1\sin\rho + \overline{\omega}^2\cos\rho = 0, \tag{91}$$

if the function $\overline{G}(x, t, \eta)$ *satisfies Equation* (84), *namely, if*

$$\frac{\partial\overline{G}}{\partial t} = \sum_{i=0}^{k}(j^\infty(s))^*\left(\frac{\partial F}{\partial z_i}\right)\frac{\partial^i\overline{G}}{\partial x^i}. \tag{92}$$

Moreover, every solution $\sigma(x, t, \eta)$ *of the Pfaffian system* (90) *satisfies Equations* (88) *and* (89).

If $\overline{G}(x, t, \eta)$ *is defined by* (88), *in which* $\sigma(x, t, \eta)$ *is a solution of* (89) *whenever* $z_0(x, t)$ *is a solution of* (79) *and* $\rho(x, t, \eta)$ *is a corresponding solution of* (91), *then* $\sigma(x, t, \eta)$ *is a solution of the Pfaffian system* (90). *Moreover,* $\overline{G}$ *satisfies Equation* (92).

Now, Equation (89) is simply a linear equation for $\sigma(x, t, \eta)$ whenever $z_0(x, t)$ is a solution of Equation (79). Thus, one can, in principle, find $\sigma(x, t, \eta)$ and hence $\overline{G}$ as formal power series in η. It would then follow that (79) possesses an infinite number of solutions of the linearized equation $D_t G = F_* G$ whenever $z_0(x, t)$ is a solution of (79). Note, however, that one cannot conclude that $\overline{G}$ will give rise to an infinite sequence of *generalized* symmetries, since one is working with the Pfaffian system (80). They may well be of a nonlocal nature.

Standard facts about existence of solutions of linear differential equations yield the following result on the existence of (generalized/nonlocal) symmetries for geometrically integrable equations:

THEOREM 5. *Let $z_{0,t} = F$ be a geometrically integrable equation, and assume that the hypothesis* (53) *hold. If the associated functions $f_{\alpha\beta}(x,t,\eta)$ are analytic functions whenever $z_0(x,t)$ is a solution of $z_{0,t} = F$, then so are $\rho(x,t,\eta)$, the solution $\sigma(x,t,\eta)$ of Equation* (89) *and the solution $\overline{G}(x,t,\eta)$ of* (88). *Thus, in particular, if $\rho(x,t,\eta) \neq 0$, a solution $\sigma(x,t,\eta)$ of* (89) *gives rise to an infinite number of* (*generalized, nonlocal*) *symmetries of the equation $z_{0,t} = F$ by means of* (88).

This result is analogous to Cavalcante and Tenenblat's (1988) result on conservation laws for geometrically integrable equations. As in the conservation laws case, one can re-write the linear system (88) and (89) in a way more amiable for computations:

Consider again the change of coordinates $\Gamma = \tan(\rho/2)$ used in Section 4. One easily obtains from Equation (80) that Γ satisfies the Pfaffian system

$$2\,\mathrm{d}\Gamma = (\overline{\omega}^3 + \overline{\omega}^2) + 2\Gamma\overline{\omega}^1 + \Gamma^2(\overline{\omega}^3 - \overline{\omega}^2), \tag{93}$$

and that $\Sigma := \Gamma_\tau$ and $\mathrm{d}\Sigma$ are given by the formulae

$$\Sigma = \frac{1+\Gamma^2}{2}\sigma, \tag{94}$$

and

$$\mathrm{d}\Sigma = \frac{1+\Gamma^2}{2}\left(\mathrm{d}\sigma + \frac{4\Sigma\Gamma}{(1+\Gamma^2)^2}\,\mathrm{d}\Gamma\right). \tag{95}$$

Instead of the functions b and A^β defined in (87) one now uses

$$b_0 = (1+\Gamma^2)\bar{f}_{31,z_0} + (1-\Gamma^2)\bar{f}_{21,z_0} + 2\Gamma\bar{f}_{11,z_0}, \tag{96}$$

and

$$A_0^\beta = \bar{f}_{1\beta} + \Gamma(\bar{f}_{3\beta} - \bar{f}_{2\beta}), \quad \beta = 1, 2, \tag{97}$$

and it follows easily that the linear system (88) and (89) now reads

$$\overline{G} = \frac{2}{b_0}\left(\frac{\partial\Sigma}{\partial x} - A_0^1\Sigma\right), \tag{98}$$

and

$$\frac{\partial \Sigma}{\partial t} - A_0^2 \Sigma = \sum_{i=0}^{k-1} \big((1+\Gamma^2)\bar{f}_{32,z_i} + (1-\Gamma^2)\bar{f}_{22,z_i} + 2\Gamma \bar{f}_{12,z_i} \big) \times \\ \times \frac{\partial^i}{\partial x^i}\left(\frac{1}{b_0}\left(\frac{\partial \Sigma}{\partial x} - A_0^1 \Sigma \right)\right). \tag{99}$$

Remark. Results more general than the ones presented in this section are obtained if one uses the theory of coverings of Krasil'shchik and Vinogradov (1989). See (Reyes, 1998b) for details.

EXAMPLE (A nonlocal symmetry of Burgers' equation). Burgers' equation

$$z_{0,t} = z_2 + z_0 z_1 \tag{100}$$

is geometrically integrable, of course. Associated one-forms for it are obtained from Equations (10)–(12) by setting $\beta(x) = h(x) = 0$. Equations (93) for Γ, (98) for $\overline{G}$, and (99) for Σ become, whenever $z_0(x, t)$ is a solution of (100),

$$2\Gamma_x = z_0 \Gamma - 2\eta \Gamma^2, \tag{101}$$

$$2\Gamma_t = \left(\frac{z_0^2}{2} + z_1 \right)\Gamma - \eta z_0 \Gamma^2, \tag{102}$$

$$\overline{G} = \frac{1}{\Gamma}(2\Sigma_x + 4\eta\Gamma\Sigma - z_0\Sigma), \tag{103}$$

and

$$2\Sigma_t + 2\eta\Gamma\Sigma z_0 - \Sigma\left(\frac{z_0^2}{2} + z_1 \right) \\ = (-\eta\Gamma + z_0)(2\Sigma_x + 4\eta\Gamma\Sigma - z_0\Sigma) + \Gamma \frac{\partial}{\partial x}\left(\frac{1}{\Gamma}(2\Sigma_x + 4\eta\Gamma\Sigma - z_0\Sigma) \right) \tag{104}$$

respectively. The function Γ can be easily computed from (101). Indeed, one finds that

$$\Gamma = \sum_{k=0}^{\infty} \Gamma_k \eta^k$$

is determined by the recursion relation

$$z_0 \Gamma_0 = 2\Gamma_{0,x}, \tag{105}$$

and

$$z_0 \Gamma_k - 2 \sum_{j=0}^{k-1} \Gamma_j \Gamma_{k-1-j} = 2\Gamma_{k,x}, \quad k \geqslant 1 \tag{106}$$

whenever $z_0(x,t)$ is a solution of Burgers' equation. Now write $\Sigma = \sum_{k=0}^{\infty} S_k \eta^k$. One obtains from (104) the following equation for S_0:

$$2S_{0,t} - S_0\left(\frac{z_0^2}{2} + z_1\right) = 2z_0 S_{0,x} - z_0^2 S_0 + \frac{\Gamma_0}{\Gamma_{0,x}}(2S_{0,x} - z_0 S_0) + 2S_{0,xx} - (z_0 S_0)_x$$

whenever $z_0(x,t)$ is a solution of (100). It is straightforward to check that this last equation can be re-written in the form

$$S_{0,t} = S_{0,xx} + \frac{(z_0^2+4)}{2}\left(\frac{1}{z_0}S_{0,x} - \frac{1}{2}S_0\right). \tag{107}$$

Now, expand $\overline{G}$, as given by Equation (103), in powers of η, $\overline{G} = \sum_{n=0}^{\infty} G_n \eta^n$, say. At zero order in η, Equation (105) implies that

$$G_0 = \frac{1}{\Gamma_0}(2S_{0,x} - z_0 S_0), \tag{108}$$

that is,

$$G_0 = (2S_{0,x} - z_0 S_0)\mathrm{e}^{-\frac{1}{2}\int z_0\,\mathrm{d}x}. \tag{109}$$

If, following Krasil'shchik and Vinogradov (1989), one now asks for solutions $z_0(x,t)$ of Burgers' equation which are invariant under the (still not completely determined) nonlocal symmetry G_0, these solutions must also satisfy (Olver, 1993) $G_0 = 0$, or in other words, the equation

$$2S_{0,x} - z_0 S_0 = 0 \tag{110}$$

must hold. But if this is so, Equation (107) implies that S_0 must be a solution of the heat equation! Thus, Equation (110) is a transformation between solutions of Burgers' equation and the heat equation: it is, of course, the classical Cole–Hopf transformation. It follows that if S_0 satisfies $S_{0,t} = S_{0,xx}$, G_0 given by (109) is a nonlocal symmetry of (100).

6. Conservation Laws for PSS Equations: Extrinsic Version

In this and the next section, the pseudo-spherical surfaces determined by PSS equations are considered to be immersed in a flat three-dimensional space. This extrinsic approach allows one to generalize the construction of conservation laws considered in Section 4. As an application, conservation laws of a PSS equation discovered by Kamran and Tenenblat (1995) will be computed. To begin with, one needs the following fact (Reyes, 2000):

PROPOSITION 3. *Suppose that M is a smooth Riemannian surface locally and isometrically immersed in a flat three-space E^3 equipped with a metric of signature*

$(1, 1, \epsilon)$, $\epsilon = \pm 1$, *and let* $S(\xi)$ *and* $C(\xi)$ *be two real-valued functions such that* $C(\xi)^2 + \epsilon S(\xi)^2 = 1$. *The surface* M *is pseudo-spherical if and only if for any number* ξ *with* $C(\xi) \neq 0$, *and any unit vector* v_0 *tangent to* M *at* $p_0 \in M$, *there exists an orthonormal moving frame* $\{e'_1, e'_2, e^{\perp}\}$ *locally defined on* $M \hookrightarrow E^3$ *with* $e'_1(p_0) = v_0$, *and such that the corresponding moving coframe* $\overline{\theta}^b$ *and connection one-forms* $\overline{\theta}_{ij}$ $(b = 1, 2, i, j = 1, 2, 3)$ *satisfy the equation*

$$\overline{\theta}_{12} C(\xi) + \overline{\theta}^2 + \overline{\theta}_{31} S(\xi) = 0. \tag{111}$$

In this case,

$$\overline{\theta}^1 - \overline{\theta}_{32} S(\xi) \tag{112}$$

is a one-parameter family of closed one-forms.

Proof. (Sketch) As in Section 4, one finds that the one-forms $\{\overline{\theta}^1, \overline{\theta}^2, \overline{\theta}_{12}, \overline{\theta}_{13}, \overline{\theta}_{23}\}$ appearing in the proposition are related to an arbitrarily chosen coframe $\{\overline{\omega}^1, \overline{\omega}^2\}$ on M and corresponding connection one-forms $\{\overline{\omega}_{12}, \overline{\omega}_{13}, \overline{\omega}_{23}\}$ by means of (assuming that the frames associated to $\{\overline{\theta}^1, \overline{\theta}^2\}$ and $\{\overline{\omega}^1, \overline{\omega}^2\}$ have the same orientation) formulas (37), (38), and (39), and by the equations

$$\overline{\theta}_{31} = \overline{\omega}_{31} \cos\rho + \overline{\omega}_{32} \sin\rho, \tag{113}$$

and

$$\overline{\theta}_{32} = -\overline{\omega}_{31} \sin\rho + \overline{\omega}_{32} \cos\rho. \tag{114}$$

The structure equations of a surface immersed in E^3,

$$\begin{aligned} &\mathrm{d}\overline{\omega}^1 = \overline{\omega}_{12} \wedge \overline{\omega}^2, \qquad \mathrm{d}\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}_{12}, \\ &0 = \overline{\omega}^1 \wedge \overline{\omega}_{13} + \overline{\omega}^2 \wedge \overline{\omega}_{23}, \end{aligned} \tag{115}$$

and

$$\begin{aligned} &\mathrm{d}\overline{\omega}_{12} = -\epsilon \overline{\omega}_{13} \wedge \overline{\omega}_{23}, \qquad \mathrm{d}\overline{\omega}_{13} = \overline{\omega}_{12} \wedge \overline{\omega}_{23}, \\ &\mathrm{d}\overline{\omega}_{23} = \overline{\omega}_{13} \wedge \overline{\omega}_{12}, \end{aligned} \tag{116}$$

imply that the Pfaffian system for ρ obtained from (111) by means of (37)–(39) and (113), (114), is completely integrable if and only if M is pseudo-spherical. □

Proposition 3 then implies the following result (Reyes, 2000):

THEOREM 6. *Let* $\Xi = 0$ *be an equation with equation manifold* $S^\infty \hookrightarrow J^\infty E$ *which describes pseudo-spherical surfaces locally and isometrically immersed in a flat three-space equipped with a metric of signature* $(1, 1, \epsilon)$, $\epsilon = \pm 1$, *by means of one-forms* $\omega^\alpha = f_{\alpha 1}\, \mathrm{d}x + f_{\alpha 2}\, \mathrm{d}t$, $\alpha = 1, 2, 3$, *and 'extrinsic connection one-forms'* $\omega_{3i} = h_{i1}\, \mathrm{d}x + h_{i2}\, \mathrm{d}t$, $i = 1, 2$. *Let* $s\colon (x, t) \mapsto (x, t, z_0(x, t))$ *be a section of the trivial bundle* E, *denote* ω^3 *by* ω_{12}, *and set*

$$\overline{\omega}^a = (j^\infty(s))^* \omega^a, \quad \text{and} \quad \overline{\omega}_{ij} = (j^\infty(s))^* \omega_{ij}, \quad a = 1, 2,\ i, j = 1, 2, 3.$$

Consider two functions $S(\xi)$ and $C(\xi)$ satisfying $C(\xi)^2 + \epsilon S(\xi)^2 = 1$. Then, for each ξ for which $C(\xi) \neq 0$, the Pfaffian system

$$C(\xi)(\overline{\omega}_{12} + \mathrm{d}\rho) = \overline{\omega}^1 \sin\rho - \overline{\omega}^2 \cos\rho - S(\xi)(\overline{\omega}_{31}\cos\rho + \overline{\omega}_{32}\sin\rho), \quad (117)$$

on the space of coordinates (x, t, ρ), is completely integrable for $\rho(x,t)$ whenever $j^\infty(s)$ is a local solution of $\Xi = 0$. Moreover, for each solution of this equation and a corresponding solution $\rho(x,t)$ of (117)*, the one-form*

$$\theta' = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho - S(\xi)(-\overline{\omega}_{31}\sin\rho + \overline{\omega}_{32}\cos\rho) \quad (118)$$

is closed.

Thus, θ' determines, in principle (no claim on their nontriviality has been made) conservation laws of the equation $\Xi = 0$. A simpler representative of the conservation law $\mathrm{d}\theta' = 0$ can be found as in the original Chern–Tenenblat case studied in Section 4, by means of the substitution $\Phi = \tan(\rho/2)$. After addition of an exact one-form, (118) becomes

$$\Theta' = \overline{\omega}^1 + \Phi(\overline{\omega}^2 - C(\xi)\overline{\omega}_{12}) + S(\xi)(\Phi\overline{\omega}_{31} - \overline{\omega}_{32}), \quad (119)$$

in which Φ is now determined by the Pfaffian system

$$\begin{aligned}(-C(\xi)\overline{\omega}_{12} - \overline{\omega}^2 - S(\xi)\overline{\omega}_{31}) + 2\Phi(\overline{\omega}^1 - S(\xi)\overline{\omega}_{32}) +\\ + \Phi^2(-C(\xi)\overline{\omega}_{12} + \overline{\omega}^2 + S(\xi)\overline{\omega}_{31}) = 2C(\xi)\,\mathrm{d}\Phi.\end{aligned} \quad (120)$$

Note that Theorem 6 implies that one can find a one-parameter family of conservation laws for the PSS equation $\Xi = 0$ even if the associated one-forms ω^b, ω_{ij} do not depend on a parameter! This is illustrated in the example below.

EXAMPLE (Conservation laws of the Kamran–Tenenblat equation). Kamran and Tenenblat (1995) proved that the equation

$$z_{0,t} = D_x h + \lambda z_0 + \delta, \quad (121)$$

in which λ, δ are real constants with $\lambda \neq 0$, and h is *any* smooth function on $J^\infty E$ depending on the variables z_i, $i \geqslant 0$, describes pseudo-spherical surfaces. If $\delta^2 \geqslant \lambda^2$, and $\delta \neq 0$, one can check (for instance, using the package VESSIOT mentioned in Section 3) that the one-forms $\omega^\alpha = f_{\alpha 1}\,\mathrm{d}x + f_{\alpha 2}\,\mathrm{d}t$, in which

$$\omega^1 = \frac{\sqrt{\delta^2 - \lambda^2}(\lambda\, z_0 + \delta)}{\delta}\,\mathrm{d}x + \frac{(\lambda\sqrt{\delta^2 - \lambda^2}\,h + \delta^2)}{\delta}\,\mathrm{d}t, \quad (122)$$

$$\omega^2 = \frac{\lambda(\lambda z_0 + \delta)}{\delta}\,\mathrm{d}x + \frac{\lambda^2 h}{\delta}\,\mathrm{d}t, \quad (123)$$

and

$$\omega^3 = (\lambda z_0 + \delta)\,\mathrm{d}x + \left(\lambda h + \sqrt{\delta^2 - \lambda^2}\right)\mathrm{d}t, \quad (124)$$

satisfy the structure equations of a pseudo-spherical surface whenever $z_0(x,t)$ is a solution of (121). One would like to apply Theorem 6. The following elementary result is useful:

PROPOSITION 4. *A pseudo-spherical surface M described locally by one-forms $\overline{\omega}^b$ $(b = 1,2)$ and $\overline{\omega}_{12}$, can be locally and isometrically immersed into a three-space E^3 equipped with a flat metric of signature $(1,1,-1)$, by setting*

$$\overline{\omega}_{13} = \overline{\omega}^1, \quad \textit{and} \quad \overline{\omega}_{23} = \overline{\omega}^2. \tag{125}$$

It follows that, whenever $z_0(x,t)$ is a solution of Equation (121), the function Φ determined by the completely integrable Pfaffian system (120) can be found from the Riccati equation

$$\begin{aligned}(-C(\xi)f_{31} - f_{21} + S(\xi)f_{11}) + 2\Phi(f_{11} + S(\xi)f_{21}) - \\ -\Phi^2(C(\xi)f_{31} - f_{21} + S(\xi)f_{11}) = 2C(\xi)\Phi_x,\end{aligned} \tag{126}$$

in which $C(\xi) = \cosh\xi$, and $S(\xi) = \sinh\xi$. In order to compute Φ, one can use (Cavalcante and Tenenblat, 1988). Expand C and S as power series in ξ, assume that

$$\Phi = \sum_{n=0}^{\infty} \Phi_n \xi^n,$$

and replace into (126). One obtains the recurrence relation

$$A_0 + \Phi_0 D_0 + \Phi_0^2 B_0 = 2\Phi_{0,x} C_0, \tag{127}$$

$$A_n + \sum_{i=0}^{n} \Phi_i D_{n-i} + \sum_{j=0}^{n} \left(\sum_{i=0}^{j} \Phi_i \Phi_{j-i} \right) B_{n-j} = 2 \sum_{i=0}^{n} \Phi_{i,x} C_{n-i}, \tag{128}$$

in which

$$C(\xi) = \sum_{n=0}^{\infty} C_n \xi^n, \qquad S(\xi) = \sum_{n=0}^{\infty} S_n \xi^n,$$

and the coefficients A_n, B_n, and D_n are determined by the equations

$$-Cf_{31} - f_{21} + Sf_{11} = \sum_{n=0}^{\infty} A_n \xi^n,$$

$$-Cf_{31} + f_{21} - Sf_{11} = \sum_{n=0}^{\infty} B_n \xi^n,$$

$$2(f_{11} + Sf_{21}) = \sum_{n=0}^{\infty} D_n \xi^n.$$

Substituting the formulae for $f_{\alpha\beta}$ into (127), one finds that, whenever z_0 is a solution of Equation (121), Φ_0 is a solution of the Riccati equation

$$-(\lambda z_0+\delta)\left[\left(1+\frac{\lambda}{\delta}\right)-2\Phi_0\sqrt{1-\frac{\lambda^2}{\delta^2}}+\Phi_0^2\left(1-\frac{\lambda}{\delta}\right)\right]=2\Phi_{0,x},$$

that is, Φ_0 is given implicitly by the equation

$$-\frac{1}{2}\int(\lambda z_0+\delta)\,\mathrm{d}x=\int\frac{\Phi_0}{\left(\Phi_0\sqrt{1-\frac{\lambda}{\delta}}-\sqrt{1+\frac{\lambda}{\delta}}\right)^2}\,\mathrm{d}\Phi_0,$$

while the coefficients Φ_n are determined in terms of Φ_0 by linear ordinary differential equations. Substitution into the conservation law $\mathrm{d}\Theta'=0$ yields a sequence of conservation laws for Equation (121), as claimed.

7. Equations Describing Calapso–Guichard Surfaces

This final section, which complements the last part of (Reyes, 2000), also illustrates the extrinsic approach used in Section 6. A class of 'large' deformations of scalar equations of pseudo-spherical type will be presented. Its construction is suggested by the fact that the Gauss–Codazzi equations of Calapso–Guichard surfaces in isothermal spherical coordinates generalize the sine-Gordon equation, and can be interpreted as the integrability condition of a two-dimensional linear problem with spectral parameter (Gürses and Nutku, 1981 and references therein).

Calapso–Guichard surfaces are obtained classically thus: one considers the metric $\mathrm{d}s^2_{CG}$ and second fundamental form φ_{CG} given respectively by

$$\mathrm{d}s^2_{CG}=(\mathrm{e}^{\xi}\cos\theta)^2\,\mathrm{d}x^2+(\mathrm{e}^{\xi}\sin\theta)^2\,\mathrm{d}t^2 \tag{129}$$

and

$$\varphi_{CG}=\mathrm{e}^{\xi}\cos\theta(\sin\theta+h\cos\theta)\,\mathrm{d}x^2+\mathrm{e}^{\xi}\sin\theta(-\cos\theta+h\sin\theta)\,\mathrm{d}t^2, \tag{130}$$

and imposes the Gauss–Codazzi equations (116). One finds three nonlinear equations for the functions ξ, h, and θ. These are the *Calapso–Guichard equations*, and the surfaces determined by their solutions, the *Calapso–Guichard surfaces*. Explicitly, the Calapso–Guichard equations are

$$(\theta_t-\xi_t\cot\theta)_t-(\theta_x+\xi_x\tan\theta)_x=(-\cos\theta+h\sin\theta)(\sin\theta+h\cos\theta), \tag{131}$$

$$h_x=\xi_x(h+\tan\theta), \tag{132}$$

and

$$h_t=\xi_t(h-\cot\theta), \tag{133}$$

which clearly generalize the sine-Gordon equation $\theta_{xx}-\theta_{tt}=\sin\theta\cos\theta$. In order to extend this construction to arbitrary PSS equations, one proceeds as follows:

Let $\Xi(x, t, z_0, \dots, z_{0,x^m t^n}) = 0$ be a differential equation describing pseudo-spherical surfaces with equation manifold $S^\infty \hookrightarrow J^\infty E$, and associated one-forms

$$\omega^1 = f_{11}\,\mathrm{d}x + f_{12}\,\mathrm{d}t, \qquad \omega^2 = f_{21}\,\mathrm{d}x + f_{22}\,\mathrm{d}t,$$
$$\omega_{12} = f_{31}\,\mathrm{d}x + f_{32}\,\mathrm{d}t, \tag{134}$$

in which $\omega^1 \wedge \omega^2 \not\equiv 0$. Let $\overline{E}$ be a trivial fiber bundle with base coordinatized by the independent variables x, t, and three-dimensional fiber with coordinates z_0, ξ, and h. Set

$$\sigma^1 = \mathrm{e}^{\xi}\omega^1 \quad \text{and} \quad \sigma^2 = \mathrm{e}^{\xi}\omega^2. \tag{135}$$

One can uniquely determine a one-form $\sigma_{12} = h_{31}\,\mathrm{d}x + h_{32}\,\mathrm{d}t$ as the solution to the equations

$$\mathrm{d}_H\sigma^1 = \sigma_{12} \wedge \sigma^2 \quad \text{and} \quad \mathrm{d}_H\sigma^2 = \sigma^1 \wedge \sigma_{12} \tag{136}$$

on $J^\infty\overline{E}$. Here, d_H denotes the horizontal derivative on $J^\infty\overline{E}$, see (Olver, 1993). Suppose now that there exist connection one-forms $\{\omega_{13}, \omega_{23}\}$ such that the one-forms ω^1, ω^2, ω_{12}, ω_{13}, and ω_{23} satisfy the structure equations (115), (116) of a pseudo-spherical surface immersed in a flat space E^3 whenever $z_0(x, t)$ is a solution of $\Xi = 0$, and set

$$\sigma_{13} = \omega_{13} + h\omega^1 \quad \text{and} \quad \sigma_{23} = \omega_{23} + h\omega^2. \tag{137}$$

The one-forms $\overline{\sigma}^i$ and $\overline{\sigma}_{ij}$ (i.e., σ^i and σ_{ij} pulled back by the infinite prolongation of a local section $s\colon (x, t) \mapsto (x, t, z_0(x, t), \xi(x, t), h(x, t))$ of $\overline{E}$) will determine a surface immersed in E^3 *if and only if* the compatibility condition

$$\overline{\sigma}_{ij} + \overline{\sigma}_{ji} = 0, \quad i, j = 1, 2, 3 \tag{138}$$

and the structure equations (115), (116) (with the letter 'ω' replaced by 'σ' everywhere) are satisfied. The first three structure equations are identities, but the last three determine a system of nonlinear equations for $z_0(x, t)$, $\xi(x, t)$, and $h(x, t)$. This motivates the following definition (Reyes, 2000).

DEFINITION 6. Let $\Xi = 0$ be a scalar equation which describes pseudo-spherical surfaces locally and isometrically immersed in a flat three-space E^3 equipped with a metric of signature $(1, 1, \epsilon)$, $\epsilon = \pm 1$, by means of one-forms $\omega^1, \omega^2, \omega_{12}$ given by (134), and extrinsic connection one-forms

$$\omega_{13} = h_1^{13}\,\mathrm{d}x + h_2^{13}\,\mathrm{d}t \quad \text{and} \quad \omega_{23} = h_1^{23}\,\mathrm{d}x + h_2^{23}\,\mathrm{d}t. \tag{139}$$

Let σ^a, and σ_{ij} ($a = 1, 2,\ i, j = 1, 2, 3$) be the one-forms defined by Equations (135), (136), and (137), and consider the two-forms

$$\Sigma_1 = \mathrm{d}_H\sigma_{12} + \epsilon\sigma_{13} \wedge \sigma_{23}, \qquad \Sigma_2 = \mathrm{d}_H\sigma_{13} - \sigma_{12} \wedge \sigma_{23},$$
$$\text{and} \quad \Sigma_3 = \mathrm{d}_H\sigma_{23} - \sigma_{13} \wedge \sigma_{12}. \tag{140}$$

Assume that Σ_i are two-forms on $J^m\overline{E}$ for some $m \geqslant 1$. The system of equations $\Xi_i = 0$ describing Calapso–Guichard surfaces of type Ξ is the locus

$$\mathcal{L} = \{j^m(\bar{s})(x,t) \mid \Sigma_i(j^\infty(\bar{s})(x,t)) = 0,\ i = 1,2,3\}. \tag{141}$$

The locus (141) determines a submanifold $\overline{S}^m$ of $J^m\overline{E}$, and a sub-bundle $\overline{S}^\infty$ of $J^\infty\overline{E}$, the equation manifold of the system of equations $\Xi_i = 0$. Explicitly, in local coordinates, the nonlinear equations $\Xi_i = 0$ are

$$\begin{aligned} -h_{31,t} + h_{32,x} &= -\epsilon\big[h_1^{13}h_2^{23} - h_1^{23}h_2^{13} + h^2(f_{11}f_{22} - f_{12}f_{21}) + \\ &\qquad + h(h_1^{13}f_{22} - h_2^{13}f_{21} + f_{11}h_2^{23} - f_{12}h_1^{23})\big], \end{aligned} \tag{142}$$

$$-[h_1^{13} + hf_{11}]_t + [h_2^{13} + hf_{12}]_x = h_{31}(h_2^{23} + hf_{22}) - h_{32}(h_1^{23} + hf_{21}), \tag{143}$$

$$-[h_1^{23} + hf_{21}]_t + [h_2^{23} + hf_{22}]_x = h_{32}(h_1^{13} + hf_{11}) - h_{31}(h_2^{13} + hf_{12}). \tag{144}$$

In particular, one easily sees that Equations (131), (132), and (133) are obtained from (142), (143), and (144) if one takes $\Xi = 0$ to be the sine-Gordon equation.

Now, the Gauss–Codazzi equations (116) (with 'ω' replaced by 'σ' everywhere) are the integrability condition of the Gauss–Weingarten linear problem

$$\mathrm{d}e_i = \overline{\sigma}_i^j e_j, \quad i, j = 1, 2, 3,$$

in which indices are reaised and lowered by means of the metric tensor $\eta_{ib} = \delta_{ib}$, $\eta_{i3} = \epsilon\delta_{i3}$, $i = 1, 2, 3$, $b = 1, 2$, see (Eisenhart, 1909, Tenenblat, 1998). One obtains the following theorem (Reyes, 2000):

THEOREM 7. *Let $\Xi = 0$ be a scalar equation describing pseudo-spherical surfaces immersed in a flat three-space E^3 equipped with a metric of signature* $(1, 1, \epsilon)$, $\epsilon = \pm 1$, *with associated one-forms ω^1, ω^2, ω_{12}, ω_{13}, and ω_{23} given by* (134) *and* (139), *and let S^∞ be its equation manifold. Assume that*

$$\Xi_i(x, t, z_0, \ldots, z_{0,x^n t^m}, \xi, \ldots, \xi_{tt}, h, \ldots, h_{tt}) = 0, \quad i = 1, 2, 3, \tag{145}$$

is the system of equations which describes Calapso–Guichard surfaces of type Ξ with equation manifold $\overline{S}^\infty$. Then, Equations (145) *are the integrability condition of a* sl(2, **R**)*-valued linear problem. Furthermore,* (145) *reduces to $\Xi = 0$ in the following sense: if ξ is constant and $h = 0$, holonomic sections of S^∞* (*i.e. local solutions of* $\Xi = 0$) *are also holonomic sections of $\overline{S}^\infty$* (*i.e. local solutions of* $\Xi_i = 0$).

EXAMPLE (a generalized KdV equation). Consider the KdV equation $z_{0,t} = -6z_0z_1 - z_3$ with associated one-forms

$$\omega^1 = (z_0 - 2)\,\mathrm{d}x + (-z_2 - 2z_0^2 + 8)\,\mathrm{d}t, \tag{146}$$

$$\omega^2 = 2z_1\,\mathrm{d}t, \tag{147}$$

$$\omega_{12} = z_0\,\mathrm{d}x - (z_2 + 2z_0^2 + 4z_0)\,\mathrm{d}t. \tag{148}$$

Set $\Delta = z_{0,t} + 6z_0z_1 + z_3$, and

$$E = f_{11}^2 + f_{21}^2 = (z_0 - 2)^2, \tag{149}$$
$$F = f_{11}f_{12} + f_{21}f_{22} = -(z_0 - 2)(z_2 + 2z_0^2 - 8), \tag{150}$$
$$G = f_{12}^2 + f_{22}^2 = 4z_1^2 + (-z_2 - 2z_0^2 + 8)^2. \tag{151}$$

The one-form σ_{12} satisfying Equations (136) for $\sigma^b = \mathrm{e}^{\xi}\omega^b$ is given by

$$\sigma_{12} = \omega_{12} - \left(\frac{\Delta}{2z_1} - \frac{\xi_x F - \xi_t E}{2z_1(z_0 - 2)}\right)\mathrm{d}x + \\ + \left(\frac{\Delta(z_2 + 2z_0^2 - 8) + (\xi_x G - \xi_t F)}{2z_1(z_0 - 2)}\right)\mathrm{d}t,$$

while the one-forms σ_{13} and σ_{23} determined by (137) read

$$\sigma_{13} = (1 + h)\omega^1 \quad \text{and} \quad \sigma_{23} = (1 + h)\omega^2,$$

in which Proposition 4 has been used to find appropriate one-forms ω_{13} and ω_{23}. It is straightforward to check that Equations (142)–(144) become:

$$2z_1(2h + h^2)(z_0 - 2) \\ = -\Delta + \left(\frac{\Delta}{2z_1} - \frac{\xi_x F - \xi_t E}{2z_1(z_0 - 2)}\right)_t + \\ + \left(\frac{\Delta(z_2 + 2z_0^2 - 8) + (\xi_x G - \xi_t F)}{2z_1(z_0 - 2)}\right)_x, \tag{152}$$

$$h_x(-z_2 - 2z_0^2 + 8) - h_t(z_0 - 2) = (1 + h)\left(\frac{1}{(z_0 - 2)}(\xi_x F - \xi_t E)\right), \tag{153}$$

and

$$2h_x z_1 = (1 + h)\left[\frac{\xi_x G - \xi_t F}{2z_1} + (z_2 + 2z_0^2 - 8)\left(\frac{\xi_x F - \xi_t E}{2z_1(z_0 - 2)}\right)\right]. \tag{154}$$

Note that, as asserted in the last theorem, a solution $z_0(x, t)$ of the KdV equation is also a solution of the system (152), (153), and (154) if ξ is constant and $h = 0$. It is very satisfactory indeed that equations as intricate as (152), (153), and (154) are the integrability condition of two-dimensional linear problems and can be studied geometrically, for example, by using the one-forms (155) below.

Are equations which describe Calapso–Guichard surfaces kinematically integrable? Gürses and Nutku (1981) proved that the classical Calapso–Guichard equations are *strictly* kinematically integrable by using the gauge freedom of the connection introduced in Section 2. One can use the same argument here. If $\epsilon = -1$, one can proceed as follows:

Equations (145) describe pseudo-spherical surfaces with associated one-forms

$$\alpha^1 = \sigma_{13}, \qquad \alpha^2 = \sigma_{23}, \quad \text{and} \quad \alpha_{12} = \sigma_{12}, \tag{155}$$

in which the one-forms σ_{ij} are defined by (136) and (137). Now (the author gratefully acknowledges an enlightening discussion with Ian Anderson on this point), the basic structure equations (3) (with 'ω' replaced by 'α') are invariant on solutions of (145) not only under the transformation (37), (38), and (39), but also under the 'Lorentz boosts'

$$\hat{\alpha}^1 = \cosh\rho\,\alpha^1 - \sinh\rho\,\alpha_{12}, \qquad \hat{\alpha}^2 = \alpha^2 + \mathrm{d}\rho,$$
$$\hat{\alpha}_{12} = -\sinh\rho\alpha^1 + \cosh\rho\alpha_{12},$$

and

$$\hat{\alpha}^1 = \alpha^2 + \mathrm{d}\rho, \qquad \hat{\alpha}^2 = \cosh\rho\alpha^2 + \sinh\rho\alpha_{12},$$
$$\hat{\alpha}_{12} = \sinh\rho\alpha^2 + \cosh\rho\alpha_{12},$$

in which ρ is a smooth function on $J^\infty\overline{E}$. Thus, in order to obtain $\hat{\alpha}^2 = \eta\,\mathrm{d}x + \beta\,\mathrm{d}t$, for some function β, it is enough to use the second boost and choose ρ so that the equation

$$\cosh\rho\,\alpha^2 + \sinh\rho\,\alpha_{12} = \eta\,\mathrm{d}x + \beta\,\mathrm{d}t$$

holds identically.

This report ends with two remarks:

First, no explicit connection between the theory exposed here and the symplectic and Lie theoretical approaches to integrability of partial differential equations (Adams, Harnad and Hurtubise, 1993 and references therein; Terng, 1997, Terng and Uhlenbeck, 1997) appears to be known. In particular, while it is a classical fact that some kinematically integrable equations can be interpreted as Hamiltonian systems on coadjoint orbits of a loop group, it is not known whether one can understand the class of geometrically integrable equations in the same terms. To investigate this issue is an important open problem.

Second, R. Beals and K. Tenenblat have studied systems of equations in n independent variables which are related to metrics on open subsets of $\mathbf{R}^n$ with constant sectional curvature. They obtained intrinsic generalizations of the sine-Gordon and wave equations (see (Tenenblat, 1998) for original references) which are integrable by inverse scattering methods. It appears, however, that little else is known about generalizations of the geometrical point of view considered in this work to equations in more than two independent variables. Further work along these lines would be extremely interesting.

Acknowledgements

The author is very grateful to Ian Anderson and Mark Fels for very many interesting discussions and comments, and to Niky Kamran for his patience and much encouragement. He thanks Robert Milson for bringing the papers by Terng (1997) and Terng and Uhlenbeck (1997) to his attention, and also thanks the referees for their thoughtful comments.

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