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AN INTRODUCTION TO LIE GROUPS AND LIE ALGEBRAS, WITH APPLICATIONS. III: COMPUTATIONAL METHODS AND APPLICATIONS OF REPRESENTATION THEORY*

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1. Introduction. This is the third of a series of papers giving a survey of the theory of Lie groups and Lie algebras, and their current applications. In the first paper [13] we introduced the basic notions of Lie groups and Lie algebras, and we discussed how one can obtain the Lie algebra of a given Lie group. We then concentrated our attention on the problem of studying the structure of Lie algebras. Levi's theorem says that a Lie algebra is the direct sum, as vector spaces, of its radical (the maximal solvable ideal of the Lie algebra) and a semisimple subalgebra, the latter in turn being a direct sum of its simple ideals. To each semisimple Lie algebra there is associated a unique Dynkin diagram, in general disconnected, summarizing information about the lengths and angles between the simple roots. The connected components of the Dynkin diagram correspond to the simple ideals. From a knowledge of the Dynkin diagram one can completely reconstruct the Lie algebra, so that the Dynkin diagram completely characterizes a given semisimple Lie algebra. There are four main classes of simple Lie algebras over the complex numbers, called A_l, B_l, C_l and D_l. There are also five exceptional simple Lie algebras: G_2, F_4, E_6, E_7 and E_8.

The second paper [14] was devoted to studying the concepts of representations and modules over a Lie algebra. We discussed the basic module operations, including the concepts of direct sum and tensor product. The Clebsch–Gordan series $M \otimes M' = \oplus M''$ which reduce the tensor products of irreducible modules as a direct sum of irreducible modules play an important role in certain applications of Lie algebras. We introduced the notions of weights and characters, discussing how the Clebsch–Gordan series can be obtained from a knowledge of the characters of the modules. We also classified the modules over semisimple Lie algebras, obtaining as one of our main results the fact that each irreducible representation is characterized by the highest weight.

In this third paper, we go one step further and discuss effective methods for carrying out computations involving representations. One of us (J.G.B.) has implemented the methods to be discussed below in the form of explicit FORTRAN V routines which have been run on the UNIVAC 1108 computer at Carnegie-Mellon University. These programs are available upon request. Some of the programs have also been translated into UNIVAC 1107/8 ALGOL procedures by Professor Vishnu K. Agrawala and are also available. These programs are all written in integer-mode arithmetic available on this machine.

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Let M be a module over a semisimple Lie algebra L with Cartan subalgebra H. The character χ of M, which is a complex-valued function defined on H by

$$\chi(h) = \mathrm{Tr}_M \exp h \qquad \text{for all } h \in H,$$

satisfies the basic properties:

$$\chi(0) = \dim M,$$
$$\chi_{M \oplus M'}(h) = \chi_M(h) + \chi_{M'}(h),$$
$$\chi_{M \otimes M'}(h) = \chi_M(h) \cdot \chi_{M'}(h).$$

Given some method to compute the characters, these formulas can be used to obtain the Clebsch–Gordan series and dimensions of modules. We may also recall that the character χ of a module M may be written as

$$\chi(h) = \sum_{\mu} n_\mu e^{\mu(h)},$$

where the summation goes over all the weights μ of H in M, and $n_\mu = \dim M_H^\mu$ is the multiplicity of the weight μ. Thus a knowledge of the character of a representation is equivalent to a knowledge of its weight diagram and the multiplicities of all the weights. We are naturally interested then in practical methods for computing characters and multiplicities of weights.

In this paper we present first a simple algorithm for computing weight diagrams and then Freudenthal's algorithm for computing the multiplicities of the weights. The use of Freudenthal's algorithm may be simplified in practice by making use of the Weyl group, since weights related by Weyl reflections have the same multiplicity. We also present Weyl's formula for computing characters of irreducible modules over semisimple Lie algebras. Weyl's formula gives the character as a ratio of two girdles. From Weyl's formula we also obtain a formula for computing the dimensions of the irreducible modules over semisimple Lie algebras. We also deal with Kostant's formula, which provides a closed form expression for the multiplicity of a weight for an irreducible module with a given highest weight λ, and Steinberg's formula which enables one to compute Clebsch–Gordan series. We finally discuss briefly more complicated computational problems such as the calculations of Clebsch–Gordan coefficients. For the calculation of Clebsch–Gordan and Racah coefficients, graphical techniques prove useful [1].

In this paper we have included a brief discussion of some applications requiring computational methods to illustrate some of the material covered in the paper. The authors are grateful to Prof. H. A. Smith for a number of helpful suggestions and for his continued interest in this effort.

2. Computational methods.

2.1. Computational preliminaries. Some basic tools needed to perform calculations on the finite-dimensional representations of a complex simple Lie algebra by high-speed electronic computers are discussed in this section. The problems considered here have to do with some technicalities involved in computing the Cartan–Killing metric automatically on a computing machine. We do not bore the reader with all the tedious details of these programs, but are content to outline the ideas involved.

We describe a method which uses only integer-mode arithmetic. The avoidance of floating point arithmetic is desirable because much of the later computation requires tests for equality of numbers which could be seriously affected if roundoff errors were permitted. Weights can be represented as integer arrays, since any weight μ may be written as a linear combination of the basic weights $\lambda_1, \cdots, \lambda_1$,

$$\mu = m_1\lambda_1 + \cdots + m_l\lambda_l,$$

with integer coefficients $m_1, \cdots, m_l$. The simple roots $\alpha_1, \cdots, \alpha_l$ may likewise be represented in this fashion, the kth component of the ith simple root being the entry A_{ki} of the Cartan matrix,

$$\alpha_i = \sum_k \lambda_k A_{ki}.$$

Our next problem is to compute the Cartan–Killing metric $g_{ij} = (\lambda_i, \lambda_j)$, which turns out to be a rational matrix, and hence must be represented as a ratio of an integer matrix G_{ij} over some common denominator D,

$$g_{ij} = \frac{G_{ij}}{D}.$$

Once this has been computed, we can calculate any inner product; for example, if $\mu = \sum m_i\lambda_i$ and $\nu = \sum n_j\lambda_j$, then $(\mu, \nu) = \sum g_{ij}m_in_j$. Perhaps the simplest way to compute the metric g_{ij} is to start from the matrix (α_i, α_j) which can be read off directly except for an unknown overall normalization factor N from the Dynkin diagram. Moreover, there exists an integer N such that we obtain an integer array M_{ij}:

$$M_{ij} = N \cdot (\alpha_i, \alpha_j).$$

It is most convenient to start with M_{ij} and compute N later on. The integer Cartan matrix A_{ij} can be computed as $A_{ij} = 2M_{ij}/M_{ii}$, and its inverse can be computed by a modification of the Gauss–Jordan reduction procedure. Overflow problems can be avoided here by using the smallest possible nonzero integers as pivot elements. The inverse $(A^{-1})_{ij}$ is a rational matrix and must be represented in the machine as an integer matrix B_{ij} over a common denominator d

$$(A^{-1})_{ij} = \frac{B_{ij}}{d}.$$

Given M, B, d, we can compute G by

$$G_{ij} = B_{ji}M_{jj} \qquad \text{(no sum)}.$$

This gives the metric up to the overall normalization factor D. The overall normalization factor D can be computed in two ways. One way is to make use of the fact that the sum of the squares of all the roots of a semisimple Lie algebra is equal to its rank. This method requires a prior computation of the root-diagram. For simple Lie algebras, a better procedure is to make use of the fact that the second-order Casimir operator has the value 1 in the adjoint representation. Thus, if the Dynkin indices of the adjoint representation are $r_1, \cdots, r_l$, then

$$D = \sum_{i,j=1}^{l} G_{ij} r_i (r_j + 2).$$

This latter method is computationally much simpler. Note that once D is known, we can compute $N = D/2d$.

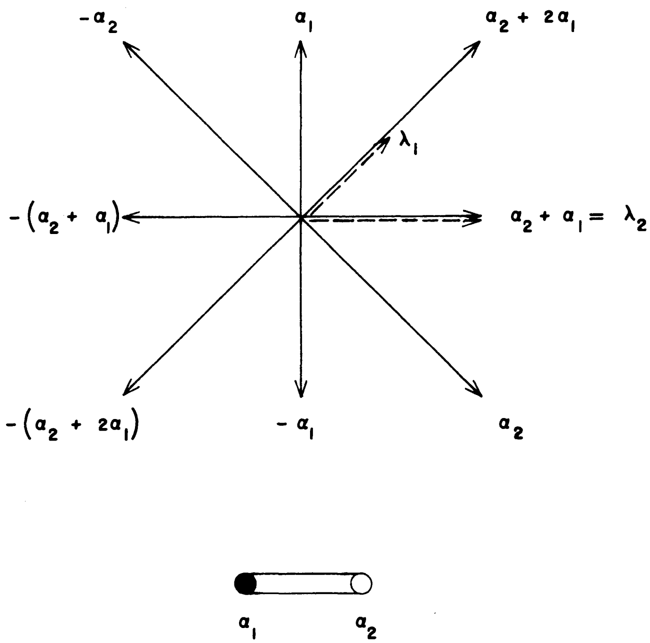


FIG. 1. *Dynkin and Root diagrams of B_2, showing also the basic weights*

We may illustrate the method on the simple Lie algebra B_2 corresponding to the Lie groups $SO(5, R)$ and $Sp(2)$. The Dynkin diagram is shown in Fig. 1. From this we can write down the matrix $M_{ij} = N(\alpha_i, \alpha_j)$ by inspection

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

and the Cartan matrix is thus

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

The inverse can be readily computed, and it is

$$A^{-1} = \frac{B}{d} = \frac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

Then we compute the metric up to normalization as

$$G = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.$$

The normalization constant is computed as

$$D = \sum G_{ij} r_i (r_j + 2) = 24,$$

where we used the fact that the Dynkin indices of the adjoint representation of B_2 are (2, 0). Hence, finally, the metric $g_{ij} = (\lambda_i, \lambda_j)$ is given by

$$(g^{ij}) = \frac{1}{12}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note also that $N = 24/2^2 = 6$ can now be computed as well.

A general program to compute the metric of any simple Lie algebra can be written, and we shall assume that this is given in the following sections.

2.2. Algorithm for computing weight systems. Given the highest weight of a module, it is a simple task to find all the other weights. For low rank semisimple Lie algebras this can be done most easily geometrically. We need only apply Weyl reflections to the highest weight to obtain the general shape of the weight diagram and then fill in the rest of the weight diagram by using ladders of weights [23].

To illustrate this procedure, consider again the simple Lie algebra B_2 corresponding to the Lie groups $SO(5, R)$ and $Sp(2)$. The root and Dynkin diagrams of B_2 are shown in Fig. 1. The Weyl group of this algebra is generated by the Weyl reflections w_1 and w_2 associated with the simple roots α_1 and α_2. If we introduce Cartesian coordinates (x, y) in the plane of the root diagram, then w_1 and w_2 are the following maps:

$$s = w_1 : (x, y) \to (x, -y) \text{ reflection in the } x\text{-axis},$$

$$w_2 : (x, y) \to (y, x) \text{ reflection about the line } x = y.$$

The product $r = w_1 w_2$ is a 90° rotation. The Weyl group then consists of the following elements:

$$e, r, r^2, r^3, s, sr, sr^2, sr^3,$$

where e is the identity map. Note that we have $s^2 = r^4 = e$, and $rs = sr^{-1}$. The Weyl group W is thus a non-Abelian group of order 8, which we can identify [22] as the dihedral group of a square, usually denoted D_4. The weight diagrams for the 4- and 5-dimensional modules, which are the two basic modules of this Lie algebra, are easily obtained by the process of applying Weyl reflections and filling in ladders. The results are shown in Fig. 2.

For higher rank semisimple Lie algebras, the geometrical method is not practical because we cannot easily draw figures in many dimensions. What we want then is a simple algorithm which could be used, say, in a program for an electronic computer.

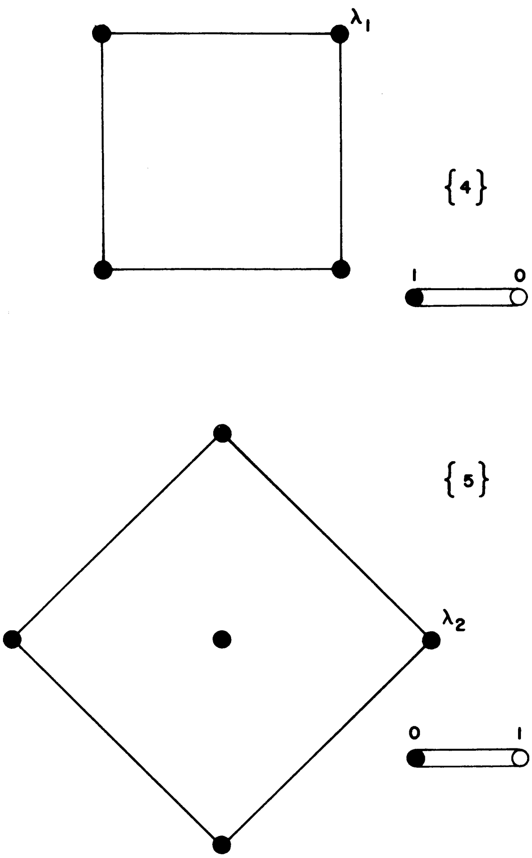


FIG. 2. *Weight diagrams for the basic modules of* B_2

Let μ be a weight and α_i a simple root, and let $\mu - p\alpha_i, \cdots, \mu + q\alpha_i$ be the α_i-ladder through μ. It is then clear that the top and bottom weights of this ladder are related by a Weyl reflection; if $w_i = w_{\alpha_i}$ is the Weyl reflection corresponding to the simple root in question, then we have $w_i(\mu + q\alpha_i) = \mu - p\alpha_i$. Since $w_i\mu = \mu - m_i\alpha_i$, and since also $w_i\alpha_i = -\alpha_i$, we obtain the result that the reflection of the top weight is $\mu - (q + m_i)\alpha_i$. Therefore $p = q + m_i$.

We can now set up the following algorithm [26] for computing the weight diagram, given the highest weight. We split up the weight diagram into layers, the zeroth layer consisting of the highest weight alone. The weights on any given layer are those which can be obtained from some weight of the previous layer by subtracting some simple root. The general procedure for obtaining the sth layer, given all the preceding layers is as follows. Consider the α_i-ladders corresponding to simple roots α_i passing through all the weights $\mu = m_1\lambda_1 + \cdots + m_l\lambda_l$ in the $(s - 1)$st layer. All the weights lying above μ in this ladder are known because they all lie in previous layers. Hence, the value of q for this ladder is known. The value of p for the ladder is given by $q + m_i$. If this is positive, then $\mu - \alpha_i$ belongs to the sth layer, otherwise it does not. This algorithm is easily programmed and seems to be fairly efficient in practice.

We shall illustrate this algorithm for another module over the simple Lie algebra B_2. Again, we could obtain the weight diagram simply by use of Weyl reflections and filling in the ladders, but we shall instead use the algorithm. Consider the (16-dimensional) irreducible representation of B_2 whose highest weight is (1, 1). The Dynkin diagram for this irreducible module is shown in Fig. 3. Starting from the highest weight, we now compute all the other weights for M. The computation is shown in Fig. 3. Note that from the transpose of the Cartan matrix we can read off the simple roots, which we discover are represented by (2, −1) and

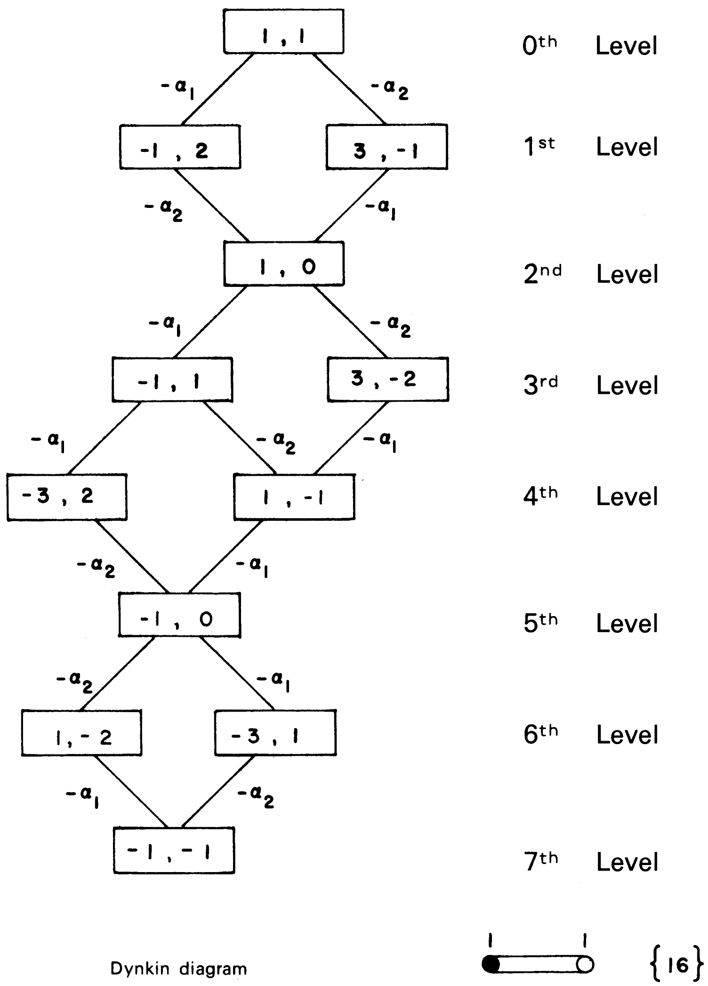


FIG 3. *Computation of weights for the* 16-*dimensional irreducible module for* B_2 *with highest weight* $\lambda = \lambda_1 + \lambda_2$

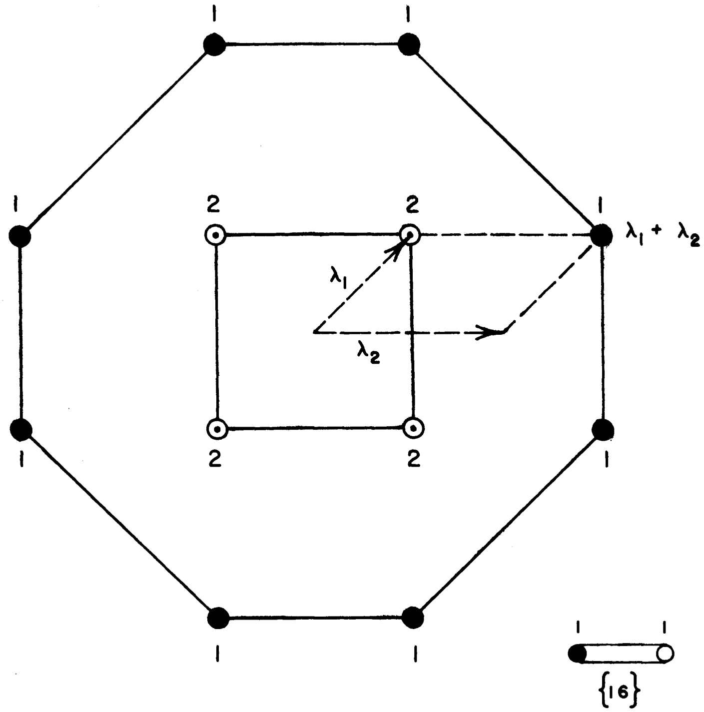


FIG. 4. *Weight diagram (showing multiplicities) for the irreducible module for* B_2 *with highest weight* $\lambda = \lambda_1 + \lambda_2$

$(-2, 2)$. There are a total of 12 weights, and the weight diagram for these is shown in Fig. 4.

For several subsequent computations it is necessary to know all of the positive roots. The easiest procedure to obtain these is to use the preceding algorithm for computing the weight diagram for the adjoint representation and then perform a lexicographic ordering of these.

The process of lexicographic ordering can be done automatically by using the matrix B related to the inverse of the Cartan matrix. Suppose we are given the Dynkin components of a weight $\mu = (m_1, \cdots, m_l)$ and we wish to know whether μ is positive or not. To determine this, we compute a new array $(p_1, \cdots, p_l)$ defined by

$$p_i = \sum B_{ki} m_i.$$

Then $\mu > 0$ if the first nonzero entry in the array $(p_1, \cdots, p_l)$ is positive.

2.3. Freudenthal's algorithm for multiplicity of weights. If L is a semisimple Lie algebra and M is an L-module, then M is a direct sum of irreducible submodules of M:

$$M = \bigoplus_{\mu \in \Lambda} M_H^{\mu},$$

where Λ is the system of weights of M. We can thus concentrate our attention on the irreducible submodules of M. For the remainder of this chapter, let L be a semisimple Lie algebra. If M is an irreducible L-module with highest weight λ, then we need a method for computing χ_λ, the character of M. For a semisimple Lie algebra L we have also seen that starting with the highest weight we can compute all other weights. The obvious question which comes up at this point is that of finding the multiplicity of the weights. If μ is an integral combination of basic weights, we recall that the multiplicity n_μ of μ is 0 if μ is not a weight, and otherwise $n_\mu = \dim M_H^\mu$, where M_H^μ is the weight space in M of H corresponding to the weight μ. Recall also that for the highest weight λ we have $n_\lambda = 1$. Freudenthal's formula gives a recursive method for finding n_μ in terms of the $n_{\mu'}$ for $\mu' > \mu$. The weights are simply ordered with respect to the lexicographic ordering of H_R^* determined by the choice of the simple system of roots.

We now turn to Freudenthal's algorithm [29]. Let M be an irreducible L-module with highest weight λ; and let $\delta = \frac{1}{2}\sum_{\alpha>0}\alpha$ denote half the sum of the positive roots. Then we have the following expression

$$\{(\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta)\}n_\mu$$
$$= 2\sum_{\alpha>0}\sum_{j=1}^{\infty} n_{\mu+j\alpha}(\mu + j\alpha, \alpha),$$

relating the multiplicity of the weight μ to the multiplicities of the higher weights. To use this expression recursively to compute n_μ, we start out with n_λ which we know is 1. Suppose now that μ is a weight other than the highest weight, which we may write as $\mu = \lambda - \sum_i k_i\alpha_i$ where the k_i are nonnegative integers. Suppose further that we already know the multiplicities for weights $\mu' > \mu$, that is, for weights which can be written as $\mu' = \lambda - \sum_i k_i'\alpha_i$, where the k_i' are integers satisfying $0 \leqq k_i' \leqq k_i$. Then each term on the right-hand side of the preceding expression is known. One can show that for any weight $\mu \neq \lambda$, we have $(\mu + \delta, \mu + \delta) < (\lambda + \delta, \lambda + \delta)$. If the quantity $(\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta)$ is nonzero, we can solve for n_μ, and if it is zero, then μ is not a weight and thus $n_\mu = 0$. In actual practice, the use of Freudenthal's formula can be a lengthy process, but this process can be simplified by making use of Weyl reflections.

As an example of the use of Freudenthal's formula, we consider the simple Lie algebra B_2. We have already computed the metric:

$$(\lambda_1, \lambda_1) = \tfrac{1}{12},$$
$$(\lambda_1, \lambda_2) = \tfrac{1}{12},$$
$$(\lambda_2, \lambda_2) = \tfrac{1}{6}.$$

Now consider again the module M over B_2 with highest weight $\lambda = \lambda_1 + \lambda_2 = (1, 1)$ which we previously considered in § 2.1. The weight diagram consists of an octagon with a square inside it. The eight weights of the octagon are related to each other by the Weyl group and therefore will have the same multiplicity. This multiplicity is one since the highest weight has multiplicity one and the highest weight is one of those eight weights. The four weights of the square inside the octagon are also related to each other by Weyl reflections, and it is therefore really only necessary to compute the multiplicity of one of these. Thus we only need to use Freudenthal's formula once, applying it to the case $\mu = \lambda_1 = (1, 0)$. Note that the positive roots are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2$, and half their sum is given by $\delta = \lambda_1 + \lambda_2$. Then $(\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta) = \frac{5}{6}$. In the summation over j in Freudenthal's formula only the term with $j = 1$ survives and we obtain $n_\mu = 2$. Thus the four weights forming the square inside the octagon all have multiplicity 2. The dimension of the module is therefore $\dim M = 8 \cdot 1 + 4 \cdot 2 = 16$. When calculating by computer, as opposed to calculating by hand, it is easier to use the Freudenthal algorithm repeatedly, ignoring the simplifications obtainable by use of the Weyl group.

2.4. Weyl's formula and dimensions of modules. One way to calculate characters is to make use of Freudenthal's algorithm for computing the multiplicity n_μ of a weight μ in a given irreducible module with given highest weight, since we can write the character as

$$\chi(h) = \sum_{\mu} n_\mu e^{\mu(h)}.$$

Another formula for computing characters has been given by Weyl, which has the advantage of being conceptually simpler, although Freudenthal's algorithm actually turns out to be easier to implement on an electronic computer. Weyl's formula also gives, as a corollary, a simple and useful formula for calculating the dimensions of the irreducible modules, using the fact that $\chi(0) = \dim M$.

Weyl's formula expresses the character $\chi_\lambda(h)$ of the irreducible module (over a semisimple Lie algebra) having the highest weight λ as a ratio of two girdles, each girdle also being a linear combination of $e^{\mu(h)}$'s. The coefficients of the $e^{\mu(h)}$'s in a girdle are ± 1, so that the girdles are simpler than the characters themselves. The main computational problem in using Weyl's formula is not in the computation of the girdles but just in the division of the one girdle by the other. Examples of such computations for rank 2 simple Lie algebras have been given by Behrends et al. [11]. For rank 3, some of these computations are also available in the literature [38]. The reproduction of these results by computer requires only a few seconds to execute and has been used as a check for this program.

The coefficients ± 1 occurring in the formula for a girdle are the parities of elements of the Weyl group. A reflection may be regarded as an orthogonal linear transformation with determinant equal to -1. The Weyl group W is composed of reflections and products of reflections. The parity of an element w in W, defined by

$$\delta_w = \det w,$$

is equal to $+1$ if w is a product of an even number of reflections and is equal to -1 if w is a product of an odd number of reflections.

Now let δ be half the sum of the positive roots of a semisimple Lie algebra:

$$\delta = \tfrac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{i=1}^{l} \lambda_i.$$

Then the λ-girdle is defined by

$$\xi_\lambda(h) = \sum_{w \in W} (\det w) e^{[w(\lambda + \delta)](h)},$$

and Weyl's formula for the characters of irreducible representations may be written as

$$\chi_\lambda(h) = \frac{\xi_\lambda(h)}{\xi_0(h)}.$$

The 0-girdle appearing in the denominator plays a key role also in the derivation of Freudenthal's algorithm.

To clarify the Weyl formula, we introduce some further terminology. The idea here is to work with the algebra of polynomials in the quantities $e^{\mu(h)}$ and allow the Weyl group to operate directly on this algebra [37].

If G is an infinite group, then the group algebra $A(G)$ over the complex number field is defined as the set of all formal sums

$$\sum_{g\in G} c_g g,$$

where c_g are complex numbers and $c_g = 0$ for all but a finite number of elements g in the group G. We then define the following operations in $A(G)$:

$$\sum_{g\in G} c_g g + \sum_{g\in G} d_g g = \sum_{g\in G} (c_g + d_g) g,$$

$$\alpha\left(\sum_{g\in G} c_g g\right) = \sum_{g\in G} (\alpha c_g) g, \qquad \alpha \in C,$$

$$\left(\sum_{g\in G} c_g g\right)\left(\sum_{h\in G} d_h h\right) = \sum_{g\in G}\left(\sum_{h\in G} c_{gh} d_{h^{-1}}\right) g.$$

Since the group is infinite, the algebra $A(G)$ will have an infinite basis, but no convergence questions arise in the preceding sums since each element of the algebra is a finite linear combination of basis elements.

The quantities $e^{\mu(h)}$ form a group under multiplication, and the character $\chi(h)$ may be regarded as an element of the group algebra of this group. Let $\lambda_1, \cdots, \lambda_l$ be the basic weights for a semisimple Lie algebra L of rank l, and let J be the set of all integral linear combinations

$$\sum_{i=1}^{l} k_i \lambda_i$$

of these basic weights, where the k_i are integers which may be positive, zero or negative. The weights μ of any module belong to J. It is clear that J is an additive group which may be regarded as a direct sum of l infinite cyclic groups generated by the basic weights. The quantities $e^{\mu(h)}$ form a multiplicative group which is isomorphic to J, so the algebra of polynomials in these quantities may be identified with the group algebra $A(J)$. Let e^μ denote the function which assigns the complex value $e^{\mu(h)}$ to the element $h \in H$, that is, $e^\mu(h) = e^{\mu(h)}$. Then we may write the character as

$$\chi = \sum_{\mu\in J} n_\mu e^\mu,$$

where we have formally extended the sum so that μ varies over J. This is possible since $n_\mu = 0$ when μ is not a weight. We define a product $e^\mu e^\nu$ by

$$(e^\mu e^\nu)(h) = e^\mu(h)e^\nu(h),$$

so that $e^\mu e^\nu = e^{\mu+\nu}$, establishing the isomorphism of our polynomial algebra with the group algebra $A(J)$. The identity element of our algebra is e^0, and it is clear that the algebra is both commutative and associative.

If M is a finite-dimensional irreducible L-module with highest weight λ, we call the character χ_λ a primitive character. We can write

$$\chi_\lambda = e^\lambda + \sum_{\mu<\lambda} n_\mu e^\mu.$$

Since the elements e^λ form a basis for the group algebra A of J we note that if the characters χ_{γ_1}, χ_{γ_2} of two irreducible L-modules M_1, M_2 are equal then $\gamma_1 = \gamma_2$, which means that M_1 and M_2 are isomorphic. Conversely, if M_1 and M_2 are isomorphic then the characters are equal. Moreover, if M is a finite-dimensional L-module, then M is completely reducible, which means that M is a direct sum of m_1 irreducible modules with character χ_{γ_1}, m_2 with character χ_{γ_2} and so forth. We then have for the character χ of M: $\chi = m_1\chi_{\gamma_1} + m_2\chi_{\gamma_2} + \cdots + m_r\chi_{\gamma_r}$. We also observe that if χ and the primitive characters are known, then the m_i can be determined, since the expression for χ is unique; the primitive characters χ_λ are linearly independent.

We may now regard the algebra $A(J)$ as a module over the Weyl group in the following way. For any element w of the Weyl group W, we define the action of w on an element of the group algebra by

$$w\left(\sum_{\mu\in J} c_\mu e^\mu\right) = \sum_{\mu\in J} c_\mu e^{w\mu}.$$

Now let us define an antisymmetizer,

$$\mathscr{A} = \sum_{w\in W} \delta_w w,$$

which may be regarded as a linear transformation on the group algebra $A(J)$. Then the λ-girdles may be defined by

$$\xi_\lambda = \mathscr{A} e^{\lambda+\delta},$$

where δ is half the sum of the positive roots of the Lie algebra.

As a corollary of Weyl's formula, we obtain an expression giving the dimension of M. We have already noted that $\dim M = \chi_\lambda(0)$. Now the Weyl formula gives the character as a quotient of two girdles

$$\chi_\lambda(h) = \frac{\xi_\lambda(h)}{\xi_0(h)}.$$

We cannot let $h = 0$ in this expression, for $\xi_0(0) = 0$. We must then evaluate the limit as $h \to 0$; $\dim M = \lim_{h\to 0}(\xi_\lambda(h)/\xi_0(h))$. Since H_R^* is a Euclidean space, we can regard H as a complexified Euclidean space, and then questions of analysis—

like taking limits—are well-defined. We can thus use L'Hospitals' rule to compute dim M. First, one can show that the 0-girdle can be written in product form as

$$\xi_0(h) = e^{-\delta(h)} \prod_{\alpha>0} (e^{\alpha(h)} - 1).$$

We thus see that for small h

$$\xi_0(h) \approx \prod_{\alpha>0} \alpha(h).$$

To obtain $\lim_{h\to 0} \xi_\lambda(h)$, we first let $h = th_\delta$, where $(h, h_\delta) = \delta(h)$. After replacing w by w^{-1}, for $w \in W$, one can then write the λ-girdle in terms of the 0-girdle as

$$\xi_\lambda(th_\delta) = \xi_0(th_{\lambda+\delta}).$$

We then obtain

$$\lim_{h\to 0} \xi_\lambda(h) = \lim_{t\to 0} \xi_\lambda(th_\delta) = \lim_{t\to 0} \xi_0(th_{\lambda+\delta}) = \prod_{\alpha>0} \{t(\alpha, \lambda + \delta)\}.$$

Hence, we obtain as the formula for dim M

$$\dim M = \prod_{\alpha>0} \left[\frac{(\alpha, \lambda + \delta)}{(\alpha, \delta)}\right].$$

As an example of Weyl's formulas, we consider any irreducible module over A_1. The root system for A_1 is $-\alpha, 0, \alpha$. There is only one positive root and thus

$$\delta = \tfrac{1}{2} \sum_{\alpha>0} \alpha = \tfrac{1}{2}\alpha.$$

Now the space H_R^* is one-dimensional and contains also the basic weight $\lambda_1 = \alpha/2$. The highest weight of an irreducible module over A_1 must then be of the form $\lambda = n\lambda_1$, where n is an integer. Recall that (cf. [27]) $j = n/2$ is the spin of the module; j can be either an integer or half an integer. Hence $\lambda = n\alpha/2 = j\alpha$. Let M_j be any irreducible module with spin j. We compute the dimension of M_j by Weyl's formula. Since there is only one positive root, we have only one factor:

$$\dim M_j = \frac{(\alpha, \lambda + \alpha/2)}{(\alpha, \alpha/2)} = 2j + 1.$$

The character χ_j of the module M_j is given by the Weyl formula as a ratio of the girdle ξ_j to ξ_0. Now the Weyl group for A_1 is $W = \{1, w_\alpha\}$. Hence, the formula for the girdle reduces to two terms. Also, $w_\alpha \mu = -\mu$, and we then have

$$\begin{aligned}\xi_\lambda &= \sum_{w\in W} \delta_w e^{w(\lambda+\delta)} = e^{(\lambda+\alpha/2)} - e^{-(\lambda+\alpha/2)} \\ &= e^{(j+1/2)\alpha} - e^{-(j+1/2)\alpha}.\end{aligned}$$

Then

$$\chi_j = \frac{\xi_\lambda}{\xi_0} = \frac{e^{(j+1/2)\alpha} - e^{-(j+1/2)\alpha}}{e^{\alpha/2} - e^{-\alpha/2}} = \sum_{m=-j}^{j} e^{m\alpha}.$$

Thus, the weights are $-j\alpha, -(j-1)\alpha, \cdots, +j\alpha$ each with multiplicity 1.

As a less trivial example of Weyl's formula, we consider the simple Lie algebra A_2 related to the Lie group $SU(3)$. Let Δ be the weight diagram for the irreducible module M over A_2 having highest weight λ, and let

$$\psi_\lambda = \sum_{\mu \in \Delta} e^\mu.$$

Thus ψ_λ is defined exactly like the character except that n_μ is replaced by unity for μ belonging to the weight diagram. Thus ψ_λ can be calculated without knowledge of multiplicities of weights.

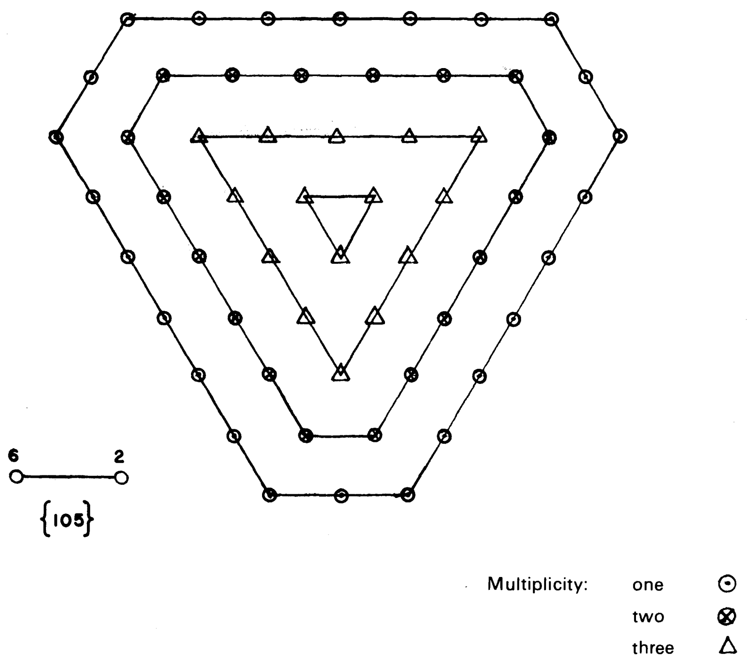


FIG. 5. *Weight diagram of the* 105-*dimensional representation* $D(6, 2)$, *showing the multiplicity of each weight*

The weight diagram in general has a hexagonal shape, as shown in Fig. 5, obtained from the highest weight $\lambda = p\lambda_1 + q\lambda_2$ by the process of Weyl reflection and filling in ladders.

The girdles satisfy the recursion relation [2]

$$\xi_\lambda - \xi_{\lambda-\delta} = \xi_0 \psi_\lambda.$$

From this we obtain by iteration a formula for the character. For triangular weight diagrams (p or q zero) we have $\chi_\lambda = \psi_\lambda$, and for nontriangular weight diagrams, we can use the recursion relation to obtain the formula

$$\chi_{\lambda+n\delta} = \psi_\lambda + \psi_{\lambda+\delta} + \cdots + \psi_{\lambda+n\delta}$$

which is valid whenever λ is a multiple of either λ_1 or λ_2 and n is any positive integer. Here $\delta = \lambda_1 + \lambda_2$ is half the sum of the positive roots. If $p > q$, we may let $\lambda = (p - q)\lambda_1$ and $n = q$, and we obtain $\lambda + n\delta = p\lambda_1 + q\lambda_2$. For $p < q$ we simply interchange the roles of p and q throughout, setting $\lambda = (q - p)\lambda_2$ and $n = p$.

These considerations lead to the following prescriptions [15] for obtaining the character of any irreducible module over A_2. We first note that it suffices to consider only the case where $p \geqq q$, because $D(q, p) = D(p, q)^*$. Thus, the weight diagram of $D(q, p)$ can be obtained by inversion with respect to the origin of the corresponding weight diagram of $D(p, q)$. Of course, this just amounts to turning the diagram upside down. For $p \geqq q$ the construction is as follows. First draw a hexagon, with sides of alternating lengths of p and q units. (Each unit corresponds to the length of the nonzero roots $= \sqrt{1/3}$.) The horizontal side at the top has length p. The interior angle between two adjacent sides is 120°. On every side whose length is p(respectively q) we then place $p + 1$ (respectively $q + 1$) dots spaced at unit intervals. (See Fig. 5.) If $q = 0$ the figure degenerates into a triangle. The interior of this figure is now filled with dots in a regular triangular array. The distance between any two adjacent dots is one unit. This gives us the weight diagram; we must now specify the multiplicities. The weight diagram may be thought of as a series of hexagons of decreasing size one contained in the other, eventually degenerating into triangles. All dots on the outer perimeter of the figure are assigned multiplicity one. One then proceeds inward, increasing the multiplicity each time by one unit until the multiplicity reaches the value $q + 1$, at which point we stop increasing the multiplicity. Thus we stop increasing the multiplicity at the stage just after the hexagons have degenerated into triangles.

2.5. Kostant and Steinberg's formulas. Kostant's formula provides a closed form expression for the multiplicity n_μ of the weight μ in the irreducible module with highest weight λ. We first define a partition function $P(\mu)$ for $\mu \in J$, the set of all integral linear forms on H_R^*, as the number of ways of writing μ as a sum of positive roots with nonnegative integer coefficients [49]. That is, if $\mu \in J$, $P(\mu)$ is the number of ways of writing $\mu = \sum_{\alpha > 0} k_\alpha \alpha$, where k_α is a nonnegative integer and α is a root. Note that $P(0) = 1$, and $P(\mu) = 0$ unless $\mu = \sum_i m_i \lambda_i$, where the m_i are nonnegative integers and the λ_i are the basic weights. Kostant's formula is then [37a]

$$n_\mu = \sum_{w \in W} \delta_w P(w(\lambda + \delta) - (\mu + \delta)).$$

Applications of Kostant's formula to calculating characters are given by Antoine and Speiser [3], [4].

The characters can also be used to obtain the multiplicities of the irreducible components in the tensor product of two irreducible modules, using a formula developed by Steinberg for calculating Clebsch–Gordan series [48]. If M_1 and M_2 are L-modules with highest weights λ_1 and λ_2 and $M = M_1 \otimes M_2$ is the tensor product of M_1 and M_2, then M is completely reducible and we can obtain its structure in terms of its irreducible constituents by considering the characters of the modules. Thus,

$$\chi_{\lambda_1}\chi_{\lambda_2} = \sum m_\lambda \chi_\lambda$$

summed over all $\lambda \in J$, where m_λ is the multiplicity in M of the irreducible submodule with highest weight λ. Steinberg's expression for m_λ is then

$$m_\lambda = \sum_{v,w \in W} \delta_{vw} P[v(\lambda_1 + \delta) + w(\lambda_2 + \delta) - (\lambda + 2\delta)],$$

where $P(\mu)$ is the partition function defined earlier [37].

The Steinberg formula is a generalization of the original Clebsch–Gordan formula for computing the tensor product of modules over the simple Lie algebra A_1. Let M_j be an irreducible module with highest weight (spin) j, dimension $2j + 1$. The tensor product $M_j \otimes M_{j'}$ of two such modules is completely reducible. To find the reduction, we use the characters, recalling that

$$\chi_j = x^j + x^{j-1} + \cdots + x^{-j} = \sum_{m=-j}^{+j} x^m,$$

where $x = e^\alpha$. Therefore the character of the product module is

$$\begin{aligned} \chi_j \chi_{j'} &= \left(\sum_{m=-j}^{+j} x^m \right) \left(\sum_{m'=-j'}^{+j'} x^{m'} \right) \\ &= \sum_{j''=|j-j'|}^{j+j'} \chi_{j''}. \end{aligned}$$

Hence, the Clebsch–Gordon series for A_1 is [27]

$$M_j \otimes M_{j'} = \bigoplus_{j''=|j-j'|}^{j+j'} M_{j''}.$$

To illustrate the use of Steinberg's formula we now consider the Lie algebra A_2, the Lie algebra of the Lie group $SU(3)$, which has been widely used in physical applications (cf. [13, pp. 30–31]).

The Cartan matrix of A_2 is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and since its inverse is

$$\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix},$$

the basic weights are $\lambda_1 = (2\alpha_1 + \alpha_2)/3$ and $\lambda_2 = (\alpha_1 + 2\alpha_2)/3$. The positive roots are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$; the root diagram is shown in Fig. 6. Then $\delta = (1/2)[\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)] = \alpha_1 + \alpha_2 = \lambda_1 + \lambda_2$. Now every weight is a linear combination of the basic weights, i.e., $\lambda = m_1\lambda_1 + m_2\lambda_2$. We shall denote the irreducible module M with highest weight $\lambda = m_1\lambda_1 + m_2\lambda_2$ by $D^n(m_1, m_2)$, where n is the dimension of M. The dimension of M can be computed by using Weyl's formula

$$\dim M = \prod_{\alpha > 0} \left[\frac{(\alpha, \lambda + \delta)}{(\alpha, \delta)} \right].$$

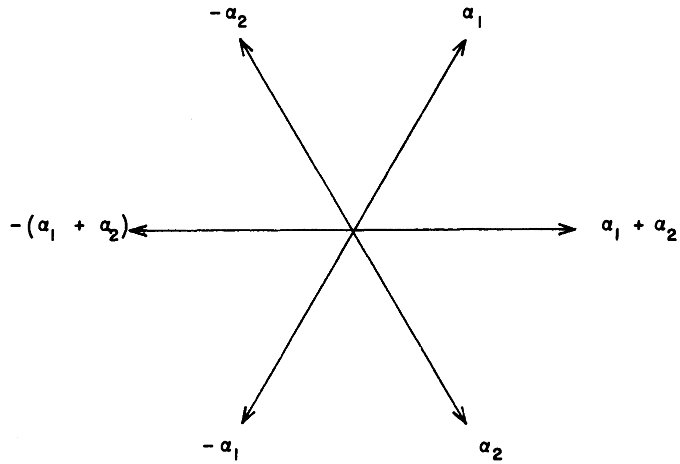


FIG. 6. *Root diagram for* A_2

In this case we find $\prod_{\alpha>0}(\alpha, \lambda + \delta) = (1/8)[(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)]$ and $\prod_{\alpha>0}(\alpha, \delta) = 1/4$. Thus, $\dim M = (1/2)[(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)]$. If $m_1 = 1$ and $m_2 = 0$, we then find that $\dim M = 3$.

Now we show by means of Steinberg's formula that $D^3(1, 0) \otimes D^3(0, 1) = D^1(0, 0) \oplus D^8(1, 1)$. The Weyl group of A_2 is $W = \{1, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1\}$, where $w_1w_2w_1 = w_2w_1w_2$. W is isomorphic to the symmetric group S_3 on 3 elements, the correspondence being given by

$$
\begin{aligned}
1 &\leftrightarrow (1)(2)(3),\\
w_1 &\leftrightarrow (1)(23),\\
w_2 &\leftrightarrow (1\ 3)(2),\\
w_1w_2 &\leftrightarrow (1\ 2\ 3),\\
w_2w_1 &\leftrightarrow (1\ 3\ 2),\\
w_1w_2w_1 &\leftrightarrow (1\ 2)(3).
\end{aligned}
$$

We also need the action of the Weyl group on the positive roots and this is given in Fig. 7. By computing dimensions of the corresponding modules, we can show that the only possible choices for m_1, m_2 are $m_1 = 0$ and $m_2 = 0$, which gives $D^1(0, 0)$, and $m_1 = 1$ and $m_2 = 1$, which gives $D^8(1, 1)$. To find the multiplicity of $D^1(0, 0)$ we use Steinberg's formula. We find $P(\alpha_2) = 1$, $P(\alpha_1 + \alpha_2) = 2$, and $P(\alpha_2 + 4\alpha_1) = 2$. Thus, the multiplicities of $D^1(0, 0)$ and $D^8(1, 1)$ are 1.

	α_1	α_2	$\alpha_1+\alpha_2$
1	α_1	α_2	$\alpha_1+\alpha_2$
w_1	$-\alpha_1$	$\alpha_1+\alpha_2$	α_2
w_2	$\alpha_1+\alpha_2$	$-\alpha_2$	α_1
$w_1 w_2$	α_2	$-(\alpha_1+\alpha_2)$	$-\alpha_1$
$w_2 w_1$	$-(\alpha_1+\alpha_2)$	α_1	$-\alpha_2$
$w_1 w_2 w_1$	$-\alpha_2$	$-\alpha_1$	$-(\alpha_1+\alpha_2)$

FIG. 7. *Action of Weyl group on positive roots* $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ *of* A_2

2.6. Tensor analysis. We have two distinct reasons for discussing tensor analysis. The first has to do with the construction of the modules for a given Lie algebra. The construction of all modules has been pushed back from general modules to irreducible modules and from irreducible modules to basic modules. By the use of tensor analysis, we can continue this process back one more step, arriving finally at the concept of an elementary module [25], [26] and [46].

The elementary modules are the natural building blocks out of which to construct all possible representations. They correspond to the classic notion of a vector, or a tensor with a single index. By taking tensor powers of the elementary modules we obtain something corresponding to tensors with any number of indices. Finally, by applying processes of symmetrization and contraction, we arrive at the irreducible modules.

Our second purpose for discussing tensor analysis is just to describe the classical notions from a modern point of view, giving a general classification of tensors according to their symmetry properties and degree of tracelessness. For this reason we shall sometimes give several different ways of constructing the same irreducible module; the knowledge of several different ways to construct a particular module is not a useless luxury since it may well prove more useful in some particular calculation to use one construction rather than another. Some flexibility is often welcome.

We shall begin our discussion of tensor analysis by considering Lie algebras of type A_l. The basic modules for these algebras are specified by their highest weights as shown in the Dynkin diagrams in Fig. 8. The Lie algebra A_l consists of all traceless linear operators in an $(l+1)$-dimensional space M. The space M is itself an irreducible module with highest weight λ_1, and we may identify $M = M_1$. The other basic modules $M_2, M_3, \cdots, M_l$ can all be constructed by antisymmetrization of the tensor powers of M, so that we may call M an elementary module of A_l. One could also start with the dual module $M^* \simeq M_l$, so that the dual module M^* may also be regarded as an elementary module.

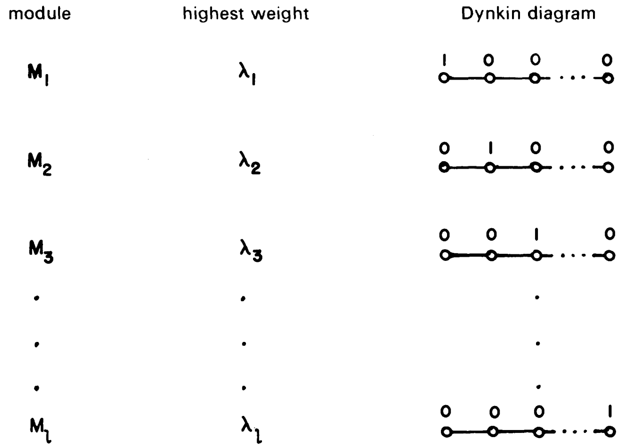


FIG. 8. *Basic modules for the simple Lie algebra A_l*

Now consider the subspace of $M \otimes M$ consisting of all elements of the form $\xi \otimes \eta - \eta \otimes \xi$, where $\xi, \eta \in M$, plus all linear combinations of elements of this form. This subspace, which we may denote by $M \Lambda M$, is isomorphic to the basic module M_2. Similarly, the basic module M_3 is isomorphic to the subspace $\mathrm{M\Lambda M\Lambda M}$ of $M \otimes M \otimes M$ consisting of all linear combinations of the form $\xi_1 \Lambda \xi_2 \Lambda \xi_3$, with ξ_1, ξ_2 and ξ_3 in M, and where

$$\xi_1 \Lambda \xi_2 \Lambda \xi_3 = \sum_{\pi} \delta_\pi \xi_{i_1} \otimes \xi_{i_2} \otimes \xi_{i_3},$$

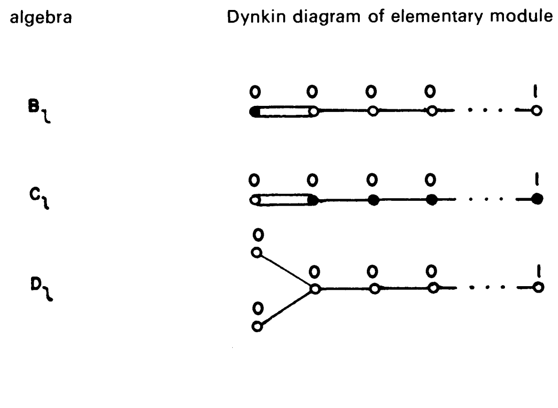


FIG. 9. *Elementary modules for tensor analysis in the simple Lie algebras B_l, C_l, D_l*

the sum going over all permutations $\pi = (i_1, i_2, i_3)$ of the indices $(1, 2, 3)$. Here δ_π is a sign factor which depends on the parity of the permutation. It is $+1$ if the permutation is even and -1 if the permutation is odd. Proceeding in this fashion, we obtain all the basic modules $M_1, M_2, M_3, \cdots, M_l$ with $M_n \simeq \Lambda^n M \simeq M\Lambda \cdots \Lambda M(n$ copies).

Essentially the same procedure can also be applied to the simple Lie algebras of types $B_l(l > 1)$, $C_l(l > 2)$, and $D_l(l > 3)$. For these algebras one can choose the elementary module M to be the module which corresponds to the Dynkin diagram listed in Fig. 9. One then constructs the antisymmetrized tensor powers $\Lambda^n M$, and one obtains in this way all except one or two of the basic modules. The omitted modules are the basic spinor module of B_l and the two basic semispinor modules of D_l. These are discussed in § 2.7. Another further exception also arises for the algebra C_3 which we shall spell out.

We study the modules over the simple Lie algebra C_3 more for the purpose of providing an example of the use of tensor analysis in the construction of basic modules than to discuss the rather minor exceptional features which arise. This Lie algebra is related to the symplectic Lie group $SP(3)$, which consists of 6 by 6 matrices, and therefore the 6-dimensional basic irreducible module is the natural starting point for building up the others. Although this module is not an elementary module in the technical sense discussed previously, it still plays a fundamental role

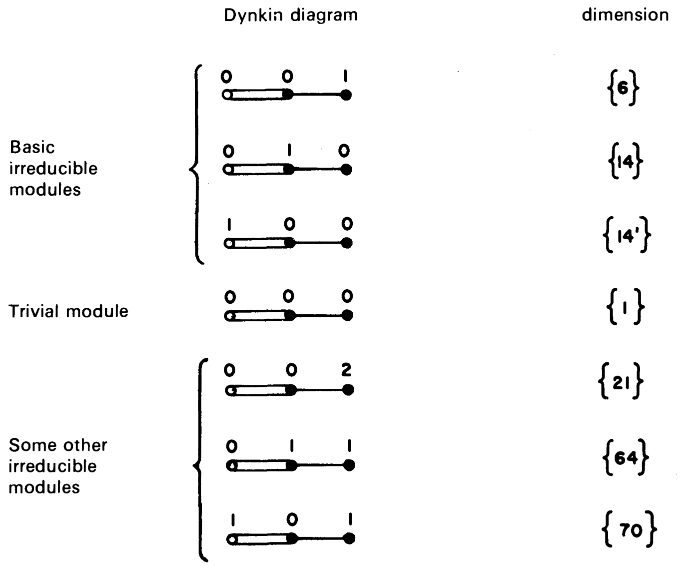


FIG. 10. *Dynkin diagrams for some of the irreducible modules for the Lie algebra* C_3

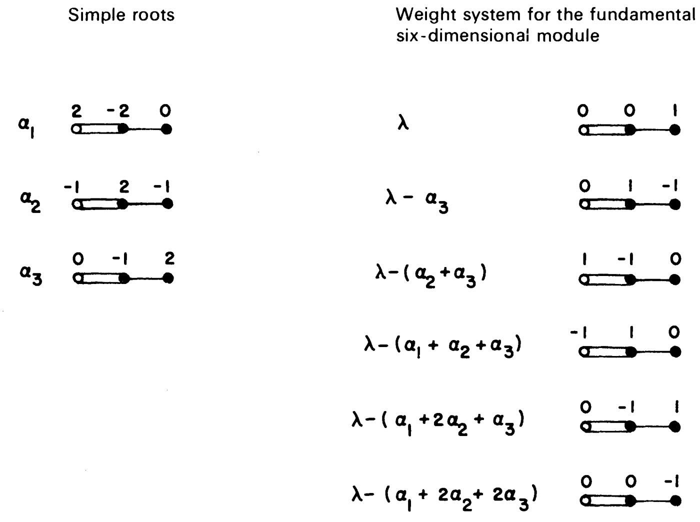


FIG. 11. *Roots and fundamental weights for the simple Lie algebra* C_3

in the representation theory of this algebra. The Dynkin diagrams of this module and the other two basic modules, as well as some others, are shown in Fig. 10. In Fig. 11, we have given the complete weight system of the fundamental 6-dimensional module, as well as the Dynkin diagrams of the simple roots. The weight system here was calculated by using the algorithm discussed in § 2.2.

The following Clebsch–Gordan series can be obtained, for instance, from a study of characters.

$$\{6\} \otimes \{6\} \simeq \{1\} \oplus \{14\} \oplus \{21\},$$

$$\{6\} \otimes \{14\} \simeq \{6\} \oplus \{14'\} \oplus \{64\},$$

$$\{6\} \otimes \{14'\} \simeq \{14\} \oplus \{70\}.$$

The Dynkin diagrams for the various modules which appear here are among those included in Fig. 10.

The decomposition of the antisymmetrized tensor powers can be written down by inspection. The module $\{6\} \otimes \{6\}$ is a 36-dimensional space, which breaks up into a symmetric part with $(6 \times 7)/(1 \times 2) = 21$ dimensions and an antisymmetric part which has $(5 \times 6)/(1 \times 2) = 15$ dimensions. From these dimensional considerations and the Clebsch–Gordan series decomposition given previously, it is clear that the decomposition of the antisymmetric part is given by

$$\Lambda^2\{6\} \simeq \{6\}\Lambda\{6\} \simeq \{1\} \oplus \{14\}.$$

The antisymmetrized cube of the fundamental module $\Lambda^3\{6\}$ is contained in $\{6\} \oplus \Lambda^2\{6\} \simeq \{6\} \oplus \{6\} \oplus \{14'\} \oplus \{64\}$. Since it has $(4 \times 5 \times 6)/(1 \times 2 \times 3) = 20$ dimensions, its decomposition must be given by

$$\Lambda^3\{6\} \simeq \{6\}\Lambda\{6\}\Lambda\{6\} \simeq \{6\} \oplus \{14'\}.$$

By similar dimension counting arguments and use of Clebsch–Gordan series, we see that the remaining antisymmetrized tensor powers of the fundamental module are given by $\Lambda^4\{6\} \simeq \{1\} \oplus \{14\}$; $\Lambda^5\{6\} \simeq \{6\}$; $\Lambda^6\{6\} = \{1\}$. All higher order antisymmetrized tensor powers are zero.

From the preceding discussion we note that the exceptional feature of the representation theory of the algebra C_3 is just that the antisymmetrized tensor powers of the fundamental module fail to be irreducible. For algebras of type C_l with $l \geqq 4$ the corresponding antisymmetrized tensor powers of the elementary module are all irreducible modules. In the case of C_3 we see that the basic irreducible modules can be constructed from the fundamental module, but in addition to antisymmetrizing the tensor powers, some additional process must be used to separate off the accompanying 1 or 6-dimensional modules.

Having constructed the basic modules from the elementary ones, it is possible to construct all irreducible modules by using the Cartan composition procedure discussed earlier, which made use of the enveloping algebra. In this way, in principle, the entire representation theory is reduced to a single elementary module.

In practice one would like to bypass the Cartan composition procedure if possible and construct all of the irreducible modules from the elementary module directly. This can often be accomplished in a way similar to the construction of the basic modules. What is involved is the theory of symmetrization of the tensor powers of the elementary module, making use of the theory of Young tableaux. For algebras of type A_l the procedure is fairly straight-forward, and we discuss this case first [34], [36].

The theory of Young tableaux is essentially a way of studying the group algebra of the permutation group. A systematic discussion of this is given in [42] to which we refer for details, as well as proofs of facts we shall use in our discussion. Let us consider the tensor power module

$$M^n = \otimes^n M = M \otimes M \otimes M \otimes \cdots \otimes M, \qquad n \text{ copies}.$$

If $\xi_1, \xi_2, \cdots, \xi_l$ is a basis for the elementary module M over the simple Lie algebra A_l, then a basis for M^n is given by the vectors $\xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_n}$, where $i_k = 1, 2, \cdots, l + 1$. We may consider M^n not only as a module over the Lie algebra A_l, but also as a module over the permutation group S_n. For each permutation π in S_n, we define

$$\pi\{\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}\} = \xi_{i_{\pi(1)}} \otimes \cdots \otimes \xi_{i_{\pi(n)}}.$$

That is, the permutation permuted the various factors in the tensor product.

The idea now is to construct projection operators by using the permutation group. For example, if we let δ_π denote, as before, the parity sign factor of a given permutation, then we can define an antisymmetrizer

$$\mathscr{A} = \sum_{\pi \in S_n} \delta_\pi \pi$$

and we have

$$\Lambda^n M = \mathscr{A} \otimes^n M.$$

In addition to the antisymmetrizer, we can construct other operators which project irreducible components out of the tensor powers of the elementary module. We can in fact find a basis e^{α}_{ij} for the group algebra of the permutation group S_n such that

$$e^{\alpha}_{ij}e^{\beta}_{kl} = \delta^{\alpha\beta}\delta_{jk}e^{\alpha}_{il}.$$

For a given α, the elements e^{α}_{ij} generate a simple ideal of the (associative) group algebra of the permutation group. The elements $e^{\alpha}_{r} = e^{\alpha}_{rr}$ are seen to be projection operators, $(e^{\alpha}_{r})^2 = e^{\alpha}_{r}$, and give a resolution of the unity element of the group algebra:

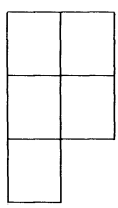
$$1 = \sum_{\alpha,r} e^{\alpha}_{r}.$$

The construction of the basis elements e^{α}_{ij} proceeds as follows. Let $[\alpha]$ denote a partition of n. That is, $[\alpha]$ is a set of numbers $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ which are non-negative integers such that

$$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n \geqq 0,$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = n.$$

The partition $[\alpha]$ can be denoted by a Young shape consisting of n boxes arranged so that α_k boxes occur in the kth row. If there are several α's equal, then we may use an abbreviated notation, where α^k means that α is repeated k times. For example the partition $[2^2 1]$ is represented by the Young shape



A Young tableau is a Young shape in which the boxes have been numbered from $1, \cdots, n$ is some order. A standard Young tableau is a tableau with the property that if all but the boxes labeled $1, \cdots, h$ with $h < n$ are erased, then these remaining h boxes by themselves form a Young tableau. An example of a standard Young tableau is

1	3
2	5
4	

The standard Young tableaux corresponding to a given shape $[\alpha]$ can be labeled $[\alpha]_r$, where $r = 1, \cdots, f_\alpha$. The particular method of numbering the standard tableaux is irrelevant and just a matter of convention; the number of standard tableaux for a given shape $[\alpha]$ is f_α.

For each tableau we can define a positive and a negative group of permutations. The positive group $P[\alpha]_r$ contains all permutations which permute only the numbers within each row; the negative group $N[\alpha]_r$ contains all permutations which permute only numbers within each column of the tableau. For example, the permutation (13)(25) belongs to the positive group of the tableau given in our example, while (124) belongs to the negative group. We define a symmetrizer P_r^α, which is the sum of all the elements of the positive group, and an antisymmetrizer N_r^α equal to the sum of all the elements of the negative group, each element being multiplied by its parity sign factor. Explicitly, the symmetrizer on all the elements in rows of the tableau is given by

$$P_r^\alpha = \sum_{\pi \in P[\alpha]_r} \pi,$$

and the antisymmetrizer on the columns of the tableau is given by

$$N_r^\alpha = \sum_{\pi \in N[\alpha]_r} \delta_\pi \pi.$$

Let σ_{rs}^α denote the permutation which transforms the tableau $[\alpha]_s$ into the tableau $[\alpha]_r$:

$$[\alpha]_r = \sigma_{rs}^\alpha [\alpha]_s,$$

and define $E_{rs}^\alpha = P_r^\alpha \sigma_{rs}^\alpha N_s^\alpha$. The elements E_{rs}^α form a basis for the group algebra of the permutation group but do not have simple multiplication laws. To obtain simpler basis elements, we expand E_{rs}^α in terms of all the permutations π in S_n:

$$E_{rs}^\alpha = \sum_{\pi \in S_n} \xi_{rs}^\alpha(\pi) \cdot \pi,$$

and we denote by ξ_{rs}^α the coefficient of the identity permutation e in this expansion:

$$\xi_{rs}^\alpha = \xi_{rs}^\alpha(e).$$

Let $(\xi^\alpha)^{-1} = \eta^\alpha$ denote the matrix inverse to the matrix whose elements are the coefficients ξ_{rs}^α. Then the elements

$$e_{rs}^\alpha = \frac{f_\alpha}{n!} \sum_t E_{rt}^\alpha \eta_{ts}^\alpha$$

form a basis of the group algebra of the permutation group having the simple multiplication properties announced earlier. It turns out that the matrix η^α is in fact very close to being a unit matrix, and for the elements with $r = s$, with which we are primarily concerned here, these coefficients effectively disappear, and we have simply

$$e_r^\alpha = e_{rr}^\alpha = \frac{f_\alpha}{n!} E_{rr}^\alpha = \frac{f_\alpha}{n!} P_r^\alpha N_r^\alpha.$$

Thus the projection operators can be computed directly from the positive and negative groups of the tableau, without requiring a calculation of the η_{rs}^α coefficients [42].

The permutations π in S_n commute with the elements x in the Lie algebra and hence so do the elements e_r^α. From this it follows immediately that $e_r^\alpha M^n$ is a submodule of the module M^n. We can show that the modules $e_r^\alpha M^n$ for the same α but different r are isomorphic:

$$e_r^\alpha M^n \simeq e_s^\alpha M^n.$$

One can further show that for algebras of type A_l, these modules are all irreducible, and moreover each irreducible module can be obtained in this way. We thus have here a method of constructing all the irreducible modules for these algebras by projecting out appropriately symmetrized portions of the tensor powers of the elementary module. The irreducible modules can now be characterized either by the Young shape $[\alpha]$ or by the highest weight λ.

For Lie algebras of type A_l, we can limit ourselves to Young shapes with only l rows at most. For a given shape $[\alpha] = [\alpha_1, \cdots, \alpha_l]$ with a total of n boxes, the modules $e_r^\alpha M^n$ have highest weight equal to $\lambda = \sum (\alpha_i - \alpha_{i+1})\lambda_i$, where $\lambda_1, \cdots, \lambda_l$ are the basic weights, and we have set $\alpha_{l+1} = 0$ by convention. Since the projection operators e_r^α form a resolution of the identity, we also have $M^n = \oplus e_r^\alpha M^n$, giving a complete reduction of the tensor powers of the elementary module into its irreducible components. This is equivalent to the classical reduction of a tensor with an arbitrary number of indices to a sum of tensors with definite symmetry properties.

For Lie algebras of type B_l, C_l or D_l one can similarly reduce tensor powers of the elementary module M into symmetrized tensor powers $e_r^\alpha M^n$, but in general these submodules will fail to be irreducible, and hence the reduction process is not yet complete [1a]. To remedy this, we need to discuss the process of contraction of tensors and the removal of traces [41].

The elementary module M is the space which is traditionally used to define the algebras B_l, C_l, D_l. It has $2l + 1$ dimensions for the "odd" orthogonal Lie algebras B_l, and it has $2l$ dimensions for the "even" orthogonal algebras D_l and for the symplectic Lie algebras C_l. In the elementary module M there is defined a nondegenerate bilinear form (ξ, η) which is symmetric in the case of the orthogonal Lie algebras and antisymmetric in the case of the symplectic Lie algebras. In any case the elements x of the Lie algebra are antisymmetric with respect to the bilinear form: $(x\xi, \eta) = -(\xi, x\eta)$. For $i, j = 1, \cdots, n$, we define a contraction operator $c_{ij}: M^n \to M^{n-2}$ which satisfies

$$c_{ij}(\xi_1 \otimes \cdots \otimes \xi_n) = (\xi_i, \xi_j)\xi_1 \otimes \cdots \hat{\xi}_i \cdots \hat{\xi}_j \cdots \otimes \xi_n.$$

A hat accent is used here to indicate that the corresponding factor is to be omitted. The contraction operators are easily seen to commute with all elements of the Lie algebra.

One can now classify tensors according to their degree of tracelessness, arriving at the decomposition

$$M^n = \bigoplus_{k=0}^{[n/2]} M_k^n.$$

Here M_1^n consists of completely traceless tensors, that is, elements of M^n which are annihilated by any contraction. The space M_2^n consists of tensors which are orthogonal to all traceless tensors but which are annihilated by a product of any two contractions and so forth. Finally the space M_0^n is the "remainder," if any, which is orthogonal to all of the spaces M^n_k with $k = 1, 2, \cdots, [n/2]$. (Note: Orthogonality of tensors is defined by

$$(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n) = (\xi_1, \eta_1)(\xi_2, \eta_2) \cdots (\xi_n, \eta_n).)$$

The spaces M^n_k are all mutually orthogonal and linearly independent. Each of them is a submodule of the tensor power M^n but, in general, is not irreducible. The submodules M^n_k are modules over the permutation group S_n as well as over the Lie algebra. We can thus carry out a further reduction by symmetrization. The spaces $e^\alpha_r M^n_k$ are irreducible modules over the Lie algebra [34]. They are not all distinct, and there are various interesting module isomorphisms, which we shall not discuss, however. In addition, there are modules (spinor–tensor modules) which are not obtained at all.

2.7. Spinor analysis. In this section we discuss the spinor and semispinor modules for the Lie algebras corresponding to the orthogonal Lie groups $SO(n, R)$. For the simple Lie algebras of type B_l, there is a single basic spinor module, while for D_l there are two basic semispinor modules, all of which cannot be obtained by the methods of tensor analysis. Once we are given these basic spinors, an arbitrary irreducible module can be constructed by combining tensor and spinor analysis, using the Cartan composition method. The Dynkin diagrams of the basic spinor and semispinor modules are given in Fig. 12.

The fundamental representation of $SO(n, R)$ is clearly the n-dimensional representation. Let M be an n-dimensional vector space over the complex numbers, and let (x, y) be a nondegenerate symmetric bilinear form on M. Let L be the complex Lie algebra of linear transformations on M which are antisymmetric with respect to the bilinear form (x, y). That is, for all a in L we have $(ax, y) = -(x, ay)$.

If n is an odd integer, $n = 2l + 1, l \geqq 1$, then L is B_l and if n is an even integer, $n = 2l, l \geqq 1$, then L is D_l.

We now form the tensor algebra $T(M)$ over M defined by $T(M) = C \oplus M \oplus M^2 + \cdots$, where $M^r = \otimes^r M = M \otimes M \otimes \cdots \otimes M$ (r times). Let K be the ideal in $T(M)$ generated by all elements of the form $z \otimes z - (z, z)1$. The quotient algebra $C = T/K$ is defined to be the (first) Clifford algebra of M with respect to

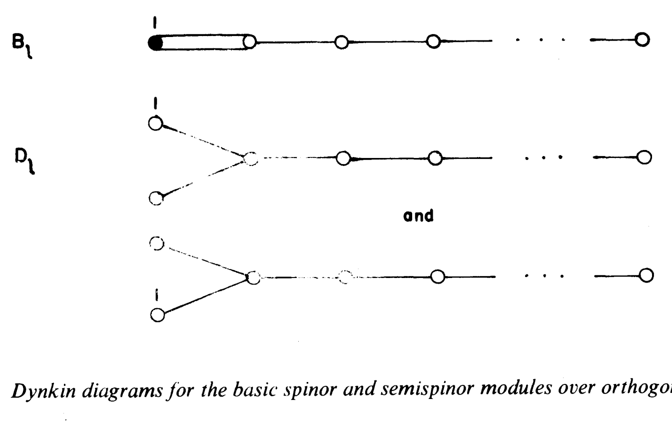


FIG. 12. *Dynkin diagrams for the basic spinor and semispinor modules over orthogonal Lie algebras B_l and D_l*

the bilinear form (x, y) [5]. One can show that the canonical mapping of M into the associative algebra $C = T/K$ is an isomorphism, and we can thus identify M with its image in C. The product in C will be denoted by $x \cdot y$ for x and y in M. One can also show that $\dim C = 2^n$. If $x_1, \cdots, x_n$ is a basis for M, then $x_1^{r_1} \cdots x_n^{r_n}$, $r_i = 0, 1$, is a basis for C. Moreover, in C we have the following "Jordan relations," $\{x, y\}_+ = xy + yx = 2(x, y)1$, for x, y in M.

Now let $z_1, z_2, \cdots, z_r$ be elements of M. We define an r-fold product $[z_1, z_2, \cdots, z_r]$ in the Clifford algebra C as follows:

$$[z_1] = z_1,$$

$$[z_1, z_2]_- = z_1 z_2 - z_2 z_1,$$

$$\vdots$$

$$[z_1, z_2, \cdots, z_{2k-1}, z_{2k}] = [[z_1, z_2, \cdots, z_{2k-1}], z_{2k}]_-,$$

$$[z_1, z_2, \cdots, z_{2k}, z_{2k+1}] = \{[z_1, z_2, \cdots, z_{2k}], z_{2k+1}\}_+.$$

In other words we alternately use the commutator and the anticommutator. We can show that

$$[z_1, z_2, z_3] = -[z_1, z_3, z_2]$$

by using the Jordan relation. In general $[z_1, z_2, \cdots, z_r]$ is antisymmetric in any pair z_i, z_j. Now let

$$M_0 = \mathbf{C}1, \quad M_1 = M, \quad M_2 = [M_1, M_1]_-, \cdots, \quad M_r = [M_1, M_{r-1}]_\pm .$$

We then have $C = \bigoplus_{r=0}^n M_r$, and $\dim M_r = \binom{n}{r}$. Thus, C is a graded vector space.

Next, we can establish the relation $[[x, y], z] = 4(y, z)x - 4(x, z)y$ for x, y, z in M. This can be shown by using the Jordan relation on the right-hand side and noting that x commutes with (y, z) so that, for instance, $4(y, z)x = 2(y, z)x + 2x(y, z) = (yz + zy)x + x(yz + zy)$. An immediate corollary of this result is that $[M_1, M_2] \subset M_1$. It then follows that M_2 is a subalgebra of the Lie algebra C_L, (C_L is the Lie algebra associated with the associative algebra C) for

$$[M_2, M_2] = [M_2, [M_1, M_1]] \subset [[M_2, M_1], M_1] \subset [M_1, M_1] \subset M_2,$$

by Jacobi's identity and the corollary just mentioned. Using this, one can then easily show that $M + M_2$ is also a Lie subalgebra of C_L. If $a \in M_2$ we define $f(a)$ as the following linear transformation on $M: f(a)x = [a, x]$ for all $x \in M$. It then follows that $(f(a)x, y) = -(x, f(a)y)$, which implies that $f(M_2) \subset L$, the orthogonal Lie algebra determined by the bilinear form (x, y) (L is B_l or D_l). Moreover, one can show that $f: M_2 \to f(M_2)$ is a one-one homomorphism. Now $\dim M_2 = \binom{n}{2} = \dim f(M_2)$, and since $\dim L = \binom{n}{2}$, we have $L = f(M_2)$. Hence, $L = f(M_2) \simeq (M_2)_L$.

Now we consider the subspace C^+ of C spanned by the even elements of C. C^+ is a subalgebra and we have $C^+ = C \oplus M_2 \oplus M_4 \oplus \cdots$; C^+ is called [5] the second Clifford algebra of (x, y); its dimensionality is 2^{n-1}. We now distinguish two cases: $n = 2l + 1$ which gives $L = B_l$ and $n = 2l$ which gives $L = D_l$. In the case of $B_l (l \geqq 2)$, we consider a basis $x_0, x_{\pm 1}, x_{\pm 2}, \cdots, x_{\pm 2l}$ for M, such that the following relations hold:

$$(x_\alpha, x_\beta) = \delta_{\alpha, -\beta}.$$

We then have

$$x_i x_{-j} + x_{-j} x_i = 2\delta_{ij}, \qquad i, j = 1, \cdots, l,$$

$$\{x_i, x_j\}_+ = 0, \quad \{x_{-i}, x_{-j}\}_+ = 0, \quad \{x_0, x_{\pm i}\}_+ = 0, \quad x_0^2 = 1.$$

Let $u_i = x_0 x_i$, $v_i = x_0 x_{-i}$, $i = 1, 2, \cdots, l$. Then u_i and v_i generate C^+, and we have $\{u_i, v_j\}_+ = -2\delta_{ij}$; $\{u_i, u_j\}_+ = 0$; $\{v_i, v_j\}_+ = 0$. A basis for C^+ consists of the set of all elements

$$u_{i_1} u_{i_2} \cdots u_{i_r} v_{j_1} v_{j_2} \cdots v_{j_s}, \quad i_1 < i_2 < \cdots < i_r, \quad j_1 < j_2 < \cdots < j_s.$$

That is, $C^+ = UV$, where U, V are generated by the u's and v's respectively. Let N be the subspace of C^+ spanned by the elements $(v_1 v_2 \cdots v_l)$ and

$$u_{i_1} u_{i_2} \cdots u_{i_r} (v_1 v_2 \cdots v_l), \quad 1 \leqq i_1 < \cdots < i_r \leqq l.$$

That is, $N = U \cdot (v_1 v_2 \cdots v_l)$ is a 2^l-dimensional left ideal of C^+. Thus $M_2 N \subset N$, which implies that N is a module over M_2. Now $(M_2)_L \simeq L$, and hence N can be regarded as a module over L. N is called the spin module for B_l, and it is the irreducible module with highest weight λ_l. The elements of N are called spinors of M_2, and the representation corresponding to N is called the spin representation.

In the special case of $B_2 \simeq C_2$ corresponding to the Lie groups $SO(5, R) \approx Sp(2)$ the fundamental module is the 5-dimensional module corresponding to $SO(5, R)$, whereas the spinor module is the 4-dimensional module corresponding to $Sp(2)$. The CG series $4 \otimes 4 = 1 \oplus 5 \oplus 10$ shows that the "fundamental" module can be built up from the spinor module, whereas the converse is not possible. Thus the spinor module is in some respects more elementary than the fundamental module.

In the case of D_l, $l \geqq 4$, the spin module in the Clifford algebra decomposes as a direct sum of two irreducible modules, and these two irreducible modules (of dimension 2^{l-1}) are the basic irreducible modules whose highest weights are shown in Fig. 12. These are called the semispinor modules for D_l.

The spinor modules are related to the compact, simply connected covering groups of the orthogonal groups $SO(n, R)$. The Lie groups obtained from B_l and C_l by exponentiation of their compact real form are not the orthogonal groups but their covering groups, which are sometimes called the "spin" groups Spin (n) [20].

3. Applications of computational methods.

3.1. Theory of tensor operators. Lie algebraic ideas have been extensively used in quantum mechanics, the main application being the theory of the rotation group, or the quantum theory of angular momentum. The Hilbert space of all state vectors with a given total angular momentum j is a module over the complex Lie algebra A_1 of the three-dimensional rotation group. The theory of irreducible tensor operators has played an important role in the development of a practical calculus for atomic and nuclear physics. [6]–[9], [17]–[19] and [24]. We shall generalize this concept below to an arbitrary Lie algebra. But first we shall illustrate the use of the ideas of tensor operators to an even simpler problem considered in our first paper [13, pp. 35–39], namely, the classical Kepler problem.

The concept of a tensor operator has an analogue in classical mechanics. The three components L_x, L_y, L_z of the angular momentum form a basis for a real Lie algebra isomorphic to the Lie algebra $so(3, \mathbf{R})$ of the rotation group. We recall the Poisson bracket relations

$$[L_\alpha, L_\beta] = \varepsilon_{\alpha\beta\gamma} L_\gamma,$$

where $\varepsilon_{\alpha\beta\gamma}$ is the usual Levi–Civita tensor. A dynamical variable S will be called a scalar dynamical variable if the Poisson bracket of S with the angular momentum vanishes: $[L_\alpha, S] = 0$. A set of three dynamical variables $\mathbf{V} = (V_x, V_y, V_z)$ is called a vector dynamical variable if the Poisson bracket relations

$$[L_\alpha, V_\beta] = \varepsilon_{\alpha\beta\gamma} V_\gamma$$

hold. One may easily prove that the dot product of two vector dynamical variables is a scalar dynamical variable while the cross product is a vector dynamical variable. In particular one may show that $\mathbf{r}$ and $\mathbf{p}$ are vector dynamical variables, and hence it immediately follows, for example, that $\mathbf{L} \times \mathbf{p}$ is also a vector dynamical variable. This shows that the Runge–Lenz vector is a vector dynamical variable, a fact that we used in our discussion of the Kepler problem. Recently a number of papers have appeared which discuss the "tensor operator" nature of the various quantities appearing in the Kepler problem, leading to even larger Lie algebras than the algebra $so(4, \mathbf{R})$ which we considered in our first paper. One of these algebras is a noncompact real form $so(4, 2; \mathbf{R})$ of the six-dimensional rotation group [10]. These algebras do not describe symmetries of the Kepler problem but only the transformation properties of the quantities which appear.

We have recently shown [1] that it is possible to develop diagrammatic methods for dealing with tensor operators for any compact group. An analogous discussion can be given for any Lie algebra L. Let $\operatorname{lin}(M_1, M_2)$ denote the space of all linear transformations from the vector space M_1 to the vector space M_2. If M_1 and M_2 are modules over L, then we may regard $\operatorname{lin}(M_1, M_2)$ as a module over L by defining the module product lt of any two elements $l \in L$, $t \in \operatorname{lin}(M_1, M_2)$ to be the linear transformation $lt \in \operatorname{lin}(M_1, M_2)$ such that

$$(lt)x = l(tx) - t(lx)$$

for all $x \in M_1$. Any submodule $T \subset \operatorname{lin}(M_1, M_2)$ is said to be a tensor operator module from M_1 to M_2. Note that in this case we have a homomorphism from $T \otimes M_1$ to M_2, and hence M_2 is isomorphic to a submodule of the tensor product $T \otimes M_1$. The theory of tensor operators [28] is closely related to the study of Clebsch–Gordan series and Clebsch–Gordan coefficients, this relationship being called the Wigner–Eckart Theorem [22a].

The simplest example of a tensor operator module is the trivial case for which $lt = 0$ for all $l \in L$ and $t \in T$. In this case the preceding equation says that $0 = l(tx) - t(lx)$, so that each element $t \in T$ in this case is a module homomorphism. We call t in this case a scalar operator, and Schur's lemma can be applied to give the structure of such scalar operators [1].

3.2. Symmetries of elementary particles. Elementary particles engage in four main classes of interactions—strong, electromagnetic, weak and gravitational [21], [51]. The strong interactions possess greater symmetry than the others, and it is here that Lie group theory enters particle physics most directly. However, even for the weak and electromagnetic interactions, Lie group theory is important because the high symmetry of the strong interactions will be broken in a definite way for these weaker interactions. Let us, however, consider the strong interactions only. Not all particles engage in strong interactions; those that do are called hadrons, while those that do not include the leptons (neutrino, electron, muon), their antiparticles, the photon and the (hypothetical) graviton. The hadrons are divided into two groups: the baryons (half integer spin) and the mesons (integer spin).

The higher symmetry of the strong interactions is manifested experimentally most strikingly in the classification of the hadrons and in their mass spectrum [31], [32] and [52]. The earliest evidence of this sort is the near equality of the proton and neutron masses, leading to the theory of isotopic spin [35]. The idea basically is that the state vectors for the proton and neutron form the basis of a two-dimensional representation of the group $SU(2)$. The isotropic spin operators are a basis for the corresponding Lie algebra A_1. More recently [30], [47] it has been suggested that the group $SU(2)$ could be extended to $SU(3)$. The proton and neutron, together with the lambda Λ^0, sigma $\Sigma^+\Sigma^0\Sigma^-$ and xi hyperons $\Xi^0\Xi^-$, would then form an eight-dimensional representation of $SU(3)$. Here the masses are not all equal, but there is a simple formula [39] which relates them:

$$2(m_N + m_\Xi) = 3m_\Lambda + m_\Sigma.$$

In addition to this baryon octet, there are several other meson and baryon multiplets and corresponding mass formulas, all of which seem to be consistent with the idea that these multiplets span modules over the $SU(3)$ Lie algebra A_2.

The simplest nontrivial representations of $SU(3)$ are clearly the three-dimensional representations, of which there are two, the one being isomorphic to the dual of the other. There is widespread speculation at present that these representations correspond to hypothetical fractionally charged elementary particles called "quarks." At present there is, however, no experimental observation of these particles. The quarks would have spin one half, so that if one counted the spin up and spin down states separately, one would have six quarks [50]. This idea led to the hypothesis of $SU(6)$ symmetry [33], [40] and [43]. It is possible that the $SU(6)$ symmetry could hold as an approximate self-consistent static limit of the baryon-meson interaction even if it turns out that quarks do not exist [12], [16].

To illustrate how we can use the methods of representation theory for working with the group $SU(3)$, we shall work out some of the Clebsch–Gordan coefficients for this group. Since our purpose is only to illustrate the general technique, we shall not give here a complete discussion of this problem, but remain content to just give one nontrivial example which shows the general method which can be used.

The eight-dimensional representation whose Dynkin diagram is given by

$$\overset{1}{\circ}\!\!-\!\!-\!\!-\!\!\overset{1}{\circ}\ \{\mathbf{8}\}$$

plays an important role in applications. The CG series

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{10^*} \oplus \mathbf{27}$$

can be obtained by the use of characters, or by using the Young tableau technique. The two **8**'s in the decomposition can be distinguished readily because one is symmetric and the other antisymmetric:

$$(\mathbf{8} \otimes \mathbf{8})_{\text{sym}} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{27},$$
$$(\mathbf{8} \otimes \mathbf{8})_{\text{antisym}} = \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{10^*}.$$

We shall work out the CG coefficients for the symmetric coupling [23]. For the antisymmetric coupling one can perform a similar analysis, or one can note that the antisymmetric coupling coefficients are related directly to the structure constants of the Lie algebra. As a basis for **8**, we can choose vectors u_1, u_2, u_{12}, v_1, v_2, v_{12}, w_1, w_2, which correspond respectively to the basis e_1, e_2, e_{12}, f_1, f_2, f_{12}, h_1, h_2 of the Lie algebra. The action of the Lie algebra on the module can then be written down immediately by inspecting the commutation relations for A_2 written down in [13, p. 31]. For example, since $[e_1, e_2] = e_{12}$, we can immediately write down $e_1 u_2 = u_{12}$ and $e_2 = -u_{12}$. We list now all module products that we need:

$$\begin{aligned} e_2 u_{12} &= 0, & f_2 w_1 &= -v_2, \\ e_2 u_1 &= -u_{12}, & e_1 u_{12} &= 0, \\ e_2 w_1 &= u_2, & e_1 w_1 &= -2u_1, \\ e_2 w_2 &= -2u_2, & e_1 w_2 &= u_1, \\ f_2 u_2 &= -w_2, & e_1 u_2 &= u_{12}, \\ f_2 w_2 &= -2v_2, & f_2 u_{12} &= -u_1. \end{aligned}$$

Let us use primes to distinguish the three **8**'s involved in our computation: $\mathbf{8} \otimes \mathbf{8}' \to \mathbf{8}''$. By inspection of the weight diagrams we can immediately write down the general form for the highest weight vector of $\mathbf{8}''$:

$$\begin{aligned} u''_{12} = {} & A(u_1 \otimes u'_2 + u_2 \otimes u'_1) \\ & + B(u_{12} \otimes w'_1 + w_1 \otimes u'_{12}) \\ & + C(u_{12} \otimes w'_2 + w_2 \otimes u'_{12}). \end{aligned}$$

All we have used here is the rule $M_H^{\mu} \otimes N_H^{\nu} \subset (M \otimes N)_H^{\mu+\nu}$. The coefficients A, B, C can be determined by making use of the the fact that u''_{12} is an extreme vector. Thus

$$e_1 u''_{12} = e_2 u''_{12} = 0.$$

We obtain, by a simple computation,

$$0 = e_1 u''_{12} = (A - 2B + C)(u_1 \otimes u'_{12} + u_{12} \otimes u'_1).$$

$$0 = e_2 u''_{12} = (-A + B - 2C)(u_{12} \otimes u'_2 + u_2 \otimes u'_{12}).$$

Thus

$$A - 2B + C = 0,$$

$$-A + B - 2C = 0.$$

The overall normalization is left undetermined, since we have not specified any particular normalization for our basis. Thus we might as well choose $B = 1$, say. Then we obtain $A = 3$, $B = 1$, $C = -1$.

Thus,

$$u''_{12} = 3(u_1 \otimes u'_2 + u_2 \otimes u'_1) + u_{12} \otimes (w'_1 - w'_2) + (w_1 - w_2) \otimes u'_{12}.$$

The remaining vectors of **8**$''$ can now be obtained by applying the lowering algebra. For example, the equation $[f_2, e_{12}] = -e_1$ allows us to write down $u''_1 = -f_2 u''_{12}$. Using our formula for u''_{12}, we then can immediately calculate u''_1, obtaining

$$u''_1 = 3(u_{12} \otimes v'_2 + v_2 \otimes u'_{12}) + u_1 \otimes (w'_1 + 2w'_2) + (w_1 + 2w_2) \otimes u'_1.$$

An interesting feature of our computation of *CG* coefficients in this example is that only integers appear. The reason that it is possible for us to obtain integer *CG* coefficients is essentially that we have not used the normalization conventions which are customarily imposed [23]. Being able to work just with integers is extremely useful for programming automatic electronic digital computers to compute *CG* coefficients. A general proof that it is always possible to choose bases in which the *CG* coefficients are integers for any simple Lie group has been given recently by Chevalley [20a], [41a], [43a], [44], [45].

It should be noted that in the course of solving the equations which determine the extreme subspace of a given reduction, it sometimes happens that we obtain more than one extreme vector with the same weight. This is sometimes referred to as the "multiplicity problem" for *CG* coefficients. In the preceding, for example, there are two 8-submodules in $8 \otimes 8$, and we distinguish them on the basis of symmetry. But even if there is no symmetry to distinguish them, we can find a basis for the extreme subspace in integer mode by use of the Smith algorithm [45] so that all multiplicity problems can be resolved.

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