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# AN INTRODUCTION TO LIE GROUPS AND LIE ALGEBRAS, WITH APPLICATIONS*

J. G. BELINFANTE†, B. KOLMAN‡, AND H. A. SMITH§

CONTENTS



**1. Introduction.** The importance of Lie algebras and Lie groups for applied mathematics and for physics has become increasingly evident in recent years. The Lie theory has become a powerful tool for studying differential equations [7], [21], [29], [35], special functions [17], [43], [48], classical and quantum mechanics, perturbation theory [33], atomic spectroscopy [38], nuclear physics [13], [32], elementary particle physics [18], [27], [28] and solid state physics [47]. Moreover, recent work in pattern recognition has used the Lie theory to advantage. The contemporary literature on the applications of the Lie theory uses material which hitherto has been available in research papers and treatises

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† Department of Physics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania.

‡ Department of Mathematics, Drexel Institute of Technology, Philadelphia, Pennsylvania.

§ Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania. Presently at the Institute for Defense Analyses, Arlington, Virginia. The work of this author was carried out while he held a National Science Foundation Faculty Fellowship at the University of Pennsylvania during the year 1964–65.

written for the specialist [25], [30], [37], [41], [44], [49]. The purpose of this paper is to make the basic material of this theory available to a wider audience, and to enable the reader to use the current literature on applications of the Lie theory. No advanced mathematical knowledge is presumed beyond the usual undergraduate mathematical training including linear algebra, advanced calculus, and the basic elements of abstract algebra and topology.

The first part of this paper is an expository presentation of basic facts about the structure of Lie groups and Lie algebras. This is followed by a survey of some illustrative applications to mathematics and physics. Although the theory of representations is crucial for many applications, the theory is too extensive to be included in the present paper beyond cursory references. We hope to deal with these topics in a more leisurely fashion in a later paper.

**2. Lie groups.** A Lie group is the structure which naturally results when analytic flesh is put on the bones of abstract group theory. Curvilinear coordinates, derivatives, and power series become available as tools for the study of the resulting structure. Essentially what is required is that the elements of the group be specified by some curvilinear coordinate system, and that the group operations should be expressible analytically. More formally, a *Lie group* can be regarded as a set of points (for example, a surface) in a Euclidean space. About each point of the Lie group there is required to be some ball (spherical neighborhood) such that all of the elements of the group lying within this ball are located by a nonsingular system of differentiable curvilinear coordinates. We will refer to the part of the group within such a ball as a *coordinate neighborhood.* It is further required that where two of these coordinate neighborhoods have a point in common, so that this point may be located in two possibly distinct coordinate systems, the coordinates in each system be expressible as analytic functions of those in the other. (A function is said to be analytic at a point when it can be expressed as a Taylor's series in some ball about the point.) The group operation is required to be analytic in terms of these coordinates. Let $x$ and $y$ be points of the group and let $\circ$ denote the group operation. Then if $z = x \circ y$, we require that the coordinates of $z$ be analytic functions of those of $x$ and $y$. Further, the coordinates of $x^{-1}$ must be analytic functions of those of $x$. For a more extensive discussion of the definition of a Lie group, based on the theory of analytic manifolds, see [5], [8], [10].

**2.1. Matrix groups.** Lie groups occur naturally in the study of groups of matrices, the group operation being multiplication. The matrices of a group must be nonsingular square matrices of a fixed size in order that the group operation be defined and an inverse for each element exist. The largest group of $n \times n$ matrices, consisting of all nonsingular complex $n \times n$ matrices, is called the complex *general linear group*, denoted by $GL(n, C)$. All complex or real matrix groups are subgroups of the general linear groups. Important subgroups of $GL(n, C)$ include the group of real nonsingular matrices $GL(n, R)$, the group of complex orthogonal matrices $O(n, C)$, the unitary matrices $U(n)$, and the special linear group $SL(n, C)$, consisting of matrices with determinant one. Further subgroups, obtained by taking intersections of these, are: (a) the real orthogonal group, $O(n) = O(n, C) \cap GL(n, R) = U(n) \cap GL(n, R) = U(n) \cap O(n, C)$; (b) the proper orthogonal or rotation group, $SO(n) = O(n) \cap SL(n, C)$; (c) the special (or unimodular) real linear group, $SL(n, R) = SL(n, C) \cap GL(n, R)$; (d) the special unitary group, $SU(n) = SL(n, C) \cap U(n)$; (e) the special complex orthogonal group, $SO(n, C) = O(n, C) \cap SL(n, C)$. There are many other matrix groups. The reader who wishes to see a more extensive catalog of them is referred to [19].

Matrices of the types listed above will be familiar from elementary linear algebra and the reader can easily verify that in each case they form a group. (Some related, but less familiar, groups of matrices have important applications and will be discussed later in this paper.) To see that these groups of matrices are Lie groups, we imbed them in a Euclidean space and simultaneously erect coordinate systems simply by noting that the $n^2$ entries of a matrix can be used as the coordinates of a point in Euclidean space of dimension $n^2$. (It is sometimes convenient to treat real and imaginary parts of complex entries as separate coordinates and so to use a real Euclidean space of dimension $2n^2$ rather than a complex space of dimension $n^2$.) Each point of this Euclidean space is thus associated with a unique matrix, and we may identify the matrix with the point. These coordinates serve as curvilinear coordinates for the general linear group, for all points sufficiently near a point representing a nonsingular matrix also represent nonsingular matrices because the determinant is a continuous function of the coordinates of the point. The general linear group fills the space of dimension $n^2$ except for a hypersurface, dividing the space, which consists of the points corresponding to singular matrices. The subgroups discussed previously, however, are hypersurfaces or intersections of hypersurfaces in the space, so not all of the coordinates are free to vary independently when we describe these subgroups. In the general linear group, with the chosen coordinates, the group operation is ordinary matrix multiplication and the coordinates of the product of two matrices are polynomials in those of the two factors. The coordinates of the inverse of a matrix are rational functions of the coordinates of the matrix. Since polynomials and rational functions are analytic, we see that the general linear group satisfies the conditions required of a Lie group. For all of the matrix groups listed previously it can be shown that appropriate curvilinear coordinate systems exist and that matrix multiplication and inversion will be analytic in terms of these coordinates. These groups are therefore Lie groups.

While we have discussed the matrix groups as examples of Lie groups, they are by no means the simplest ones. Indeed every Euclidean space can be regarded as a Lie group where the group operation is vector addition and any rectilinear coordinate system can be used.

**2.2. Local structure of Lie groups.** In a certain sense the global structure of a Lie group is almost completely determined by the local structure, by what happens in an arbitrarily small coordinate neighborhood. Suppose $U$ to be an arbitrary coordinate neighborhood and $g$ to be an element in the interior of $U$; then the set $g^{-1} \circ U$, obtained by operating (on the left) on all the elements of $U$ by the inverse of $g$, will contain a coordinate neighborhood of the identity element

$e$. By this procedure, every coordinate neighborhood can be transported back to the identity, and it suffices to study what happens close to the identity. For convenience, coordinates in a neighborhood of the identity will be chosen so that the coordinates of the identity are zero. If $x$ is a member of the group, we denote its coordinates by $x^i$. If $x \circ y = z$, we have $z^i(x^1, x^2, \cdots, x^n; y^1, y^2, \cdots, y^n)$, by hypothesis, an analytic function. Expanding in a Taylor's series about the origin (the identity) we have

$$\begin{aligned} z^i(x, y) &= z^i(0,0) + \sum_{j=1}^{n} \frac{\partial z^i(0,0)}{\partial x^j} x^j + \sum_{j=1}^{n} \frac{\partial z^i(0,0)}{\partial y^j} y^j \\ &\quad + \frac{1}{2} \sum_{j,k=1}^{n} \left\{ \frac{\partial^2 z^i(0,0)}{\partial x^j \partial x^k} x^j x^k + 2 \frac{\partial^2 z^i(0,0)}{\partial x^j \partial y^k} x^j y^k + \frac{\partial^2 z^i(0,0)}{\partial y^j \partial y^k} y^j y^k \right\} + \cdots, \end{aligned} \tag{1}$$

where the remaining terms are of higher order. Now

$$z^i(0,0) = (e \circ e)^i = 0,$$

$$\frac{\partial z^i(0,0)}{\partial y^j} = \frac{\partial (e \circ y)^i}{\partial y^j} = \delta_j^{\ i},$$

$$\frac{\partial z^i(0,0)}{\partial x^j} = \frac{\partial (x \circ e)^i}{\partial x^j} = \delta_j^{\ i},$$

$$\frac{\partial^2 z^i(0,0)}{\partial x^j \partial x^k} = \frac{\partial^2 z^i(0,0)}{\partial y^j \partial y^k} = 0,$$

so (1) becomes

$$z^i = x^i + y^i + \sum_{j,k=1}^{n} a^i_{jk} x^j y^k + \cdots . \tag{2}$$

If the coordinate neighborhood is taken sufficiently small, the terms indicated by dots become arbitrarily small with respect to the terms written out. Since such a neighborhood can be transported along any curve in arbitrarily small steps (much as one does classical analytic continuation), it might seem reasonable to ask whether the structure demanded of a Lie group is sufficiently restrictive to be completely determined by the tensor

$$a^i_{jk} = \frac{\partial^2 z^i(0,0)}{\partial x^j \partial y^k} .$$

The answer to this question is yes, provided the Lie group is simply connected.* Further, only the antisymmetric part of the tensor is of importance. Define the tensor $c^i_{jk} = a^i_{jk} - a^i_{kj}$ ; then there is only one simply connected Lie group yielding this tensor [8], [37].

It is of interest to compute the inverse of $x$ as a power series. Since $x \circ x^{-1} = e$, we have $z^i(x, x^{-1}) = 0$, or using (2),

$$0 = x^i + (x^{-1})^i + \sum_{j,k=1}^{n} a^i_{jk} x^j (x^{-1})^k + \cdots ;$$

* The terms *connected* and *simply connected* are defined in §2.5.

solving for $(x^{-1})^i$, we obtain by repeated substitution

$$(x^{-1})^i = -x^i + \sum_{j,k=1}^{n} a^i_{jk} x^j x^k + \cdots . \tag{3}$$

The commutator of two group elements $x$ and $y$ is defined by $x \circ y \circ x^{-1} \circ y^{-1} = (x \circ y) \circ (y \circ x)^{-1}$ and essentially determines whether $x$ and $y$ commute. Expanding the commutator in series we have

$$\begin{aligned} z^i(z(x, y), z^{-1}(y, x)) &= z^i(x, y) + (z^{-1}(y, x))^i \\ &\quad + \sum_{j,k=1}^{n} a^i_{jk} z^j(x, y)[z^{-1}(y, x)]^k + \cdots \\ &= \sum_{j,k=1}^{n} (a^i_{jk} - a^i_{kj}) x^j y^k + \cdots \qquad (4) \\ &= \sum_{j,k=1}^{n} c^i_{jk} x^j y^k + \cdots . \end{aligned}$$

Thus the tensor $c^i_{jk}$ determines the predominant part of the commutator.

These results can be summarized conveniently in terms of the tangent space to the Lie group at $e$. Let $q^1, \cdots, q^n$ be the coordinates in a neighborhood of a point $x$ of the Lie group. If $\mathbf{v} = \mathbf{v}(q^1, \cdots, q^n)$ is the position vector (with respect to the origin of the surrounding Euclidean space) of the points in the neighborhood of $x$ in the Lie group, then the vectors $(\partial \mathbf{v}/\partial q^i)_x$ form a basis for the tangent space at $x$ and a vector in the space is of the form

$$\sum_{i=1}^{n} u^i \left. \frac{\partial \mathbf{v}}{\partial q^i} \right|_x .$$

The coordinates $x^i$, $y^i$, $z^i$ of the points of the Lie group close to the origin, used in the previous expressions, can be regarded as components of vectors in the tangent space at $e$. That is, we construct the vectors

$$\mathbf{x} = \sum_{i=1}^{n} x^i \left. \frac{\partial \mathbf{v}}{\partial q^i} \right|_e , \qquad \mathbf{y} = \sum_{i=1}^{n} y^i \left. \frac{\partial \mathbf{v}}{\partial q^i} \right|_e ,$$

and

$$\mathbf{z} = \sum_{i=1}^{n} z^i \left. \frac{\partial \mathbf{v}}{\partial q^i} \right|_e .$$

The computations given previously can now be interpreted as operations on vectors in this tangent space. To the lowest order of approximation, (2) shows that the group operation corresponds to vector addition in the tangent space, and (3) shows that taking the inverse of a group element corresponds to taking the negative of the corresponding vector. The operation of taking the commutator of two group elements is more complicated; it corresponds in the tangent space approximately to

$$z^i = \sum_{j,k=1}^{n} c^i_{jk} x^j y^k . \tag{5}$$

We can regard this as a form of vector multiplication and introduce the notation

$$\mathbf{z} = [\mathbf{x}, \mathbf{y}] \tag{6}$$

for (5). Square brackets will only be used for this notation hereafter. The tangent vector space at the origin, with this multiplication, is called the *Lie algebra* of the Lie group. (In the older literature the Lie algebra was called the *infinitesimal group.*) The components of the tensor $c^i_{jk}$ are called the *structure constants* of the Lie algebra.

It is clear that not any tensor $c^i_{jk}$ can serve as structure constants of a Lie algebra. Antisymmetry in the lower indices,

$$c^i_{jk} = -c^i_{kj}, \tag{7}$$

is necessary from the definition of the tensor. We might ask what further properties $c^i_{jk}$ must have. By extending the power series (1) to higher terms and equating the corresponding terms of each side of the equation $(w \circ x) \circ y = w \circ (x \circ y)$, one obtains expressions relating the $a^i_{jk}$ to coefficients of higher order terms. By permuting the indices cyclically and summing, one can eliminate the coefficients of the higher order terms, obtaining the relation

$$\sum_{k=1}^{n} c^i_{jk}c^k_{lm} + c^i_{lk}c^k_{mj} + c^i_{mk}c^k_{jl} = 0. \tag{8}$$

Equations (7) and (8) are the only conditions on the structure constants. Given any set of constants satisfying (7) and (8) there are a simply connected Lie group and a local coordinate system in a neighborhood of the identity such that the given constants are the structure constants in that coordinate system. The proof of this theorem is essentially based on the existence of solutions to certain systems of partial differential equations. Returning to the Lie algebra, it is of interest to note the implications of (7) and (8). Straightforward computations show that (7) is equivalent to *anticommutativity*,

$$[x, y] = -[y, x], \tag{9}$$

while (8) is equivalent to

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{10}$$

which is called the *Jacobi identity.* The idea of a Lie algebra has been generalized to arbitrary vector spaces but we will only be concerned here with real or complex vector spaces. Any vector space endowed with a bilinear vector multiplication is called an *algebra.* If the vector multiplication is associative we speak of an *associative algebra.*

A complex or real algebra with the vector multiplication satisfying identities (9) and (10) is called a *Lie algebra.* The results recounted in this section can be restated by saying that there is a one-to-one correspondence between (real or complex) Lie algebras and simply connected Lie groups.

The Lie algebra of a Lie group is written in the small letters corresponding to the ones used to express the group. For example, the Lie algebra of $SU(n)$ is denoted $su(n)$.

The local structure of a Lie group (the structure in a sufficiently small neighborhood) is completely determined by the Lie algebra. This is of great importance in applications, for when properties of a local nature are being studied one need only consider the Lie algebra.

**2.3. Examples.** A Lie group is abelian if $a \circ b = b \circ a$ for all elements $a$, $b$. The corresponding Lie algebra has all its structure constants equal to zero, so that it is abelian, i.e., $[a, b] = 0$ for all elements $a$, $b$. Any vector space may be made into an abelian Lie algebra by introducing this trivial multiplication. A real or complex vector space is an abelian Lie group under the operation of vector addition. Obviously such a space is its own tangent space, so with the trivial multiplication, it is its own Lie algebra. Since vector spaces are simply connected, we can conclude from the results given in the previous section that such spaces are the only simply connected abelian Lie groups.

Consider the general linear group of real or complex $N \times N$ matrices. As was mentioned previously, the matrix elements are satisfactory coordinates. In order to have coordinates which vanish at the identity it is only necessary to express the matrix in terms of its difference from the identity. We then have for the matrix $x^{ij}$ the new coordinates $\bar{x}^{ij} = x^{ij} - \delta^{ij}$. The product of two matrices in these coordinates is given by

$$\bar{z}^{ij} = \bar{x}^{ij} + \bar{y}^{ij} + \sum_{k,l,m,n=1}^{N} \delta_k{}^i \delta_{lm} \delta_n{}^j \bar{x}^{kl} \bar{y}^{mn},$$

so we have $a^{ij}_{kl,mn} = \delta_k{}^i \delta_{lm} \delta_n{}^j$. Therefore the structure constants for the general linear group are given by

$$c^{ij}_{kl,mn} = \delta_k{}^i \delta_{lm} \delta_n{}^j - \delta_m{}^i \delta_{nk} \delta_l{}^j.$$

The tangent space has for a basis the matrices

$$\frac{\partial \bar{x}^{ij}}{\partial \bar{x}^{kl}} = \delta_k{}^i \delta_l{}^j$$

and can be identified with the space of all $N \times N$ matrices (not just the nonsingular ones.) If $x$ and $y$ are tangent vectors (matrices) with components $\bar{x}^{kl}$ and $\bar{y}^{mn}$, then we have $[x, y] = xy - yx$, where the usual matrix multiplication is meant on the right side. We have determined the structure of the Lie algebra of the general linear group of any order. It should be noted that the real general linear group is not connected, since a surface separates the elements which have a positive determinant from those which have a negative determinant.

Undoubtedly the most familiar Lie algebra consists of the vectors in real three-dimensional space with the usual vector cross-product. Anticommutativity of the vector cross product is familiar, and the Jacobi identity can be readily verified by calculating the vector triple products involved. We will show later that this Lie algebra is $so(3, R)$, the Lie algebra of the group of rotations in space.

It is convenient to consider any group consisting of isolated points in Euclidean space to be a Lie group in a trivial sense. We simply assign constants as local coordinates at each point. The resulting Lie algebra will, of course, consist only of the zero vector. Such a Lie group will be referred to as a *discrete group*.

**2.4. Subgroups and homomorphisms.** If the terms *subgroup* and *homomorphism* are to be applied to Lie groups it is necessary to demand more than the usual algebraic definitions. A subgroup, for instance, must not only be a subgroup in the algebraic sense, it must also be a Lie group, possessing its own appropriate local coordinate systems. Further, in order to be able to relate the subgroup to the group in a useful manner it is necessary that the local coordinates in the subgroup be analytic functions of those in the original group. Similarly, a homomorphism of one Lie group to another will only be manageable if the coordinates of the image of a point are analytic functions of those of the point. When speaking of a subgroup or a homomorphism of a Lie group one occasionally uses the terms *analytic subgroup* or *analytic homomorphism* for emphasis.

As one might expect, a form of the fundamental theorem of homomorphisms holds for Lie groups. Every homomorphism of a Lie group has for its kernel a closed normal subgroup. Further, every closed normal subgroup of a Lie group generates a natural homomorphism onto a quotient group which is a Lie group. The Lie algebra behaves correspondingly. The Lie algebra of a subgroup is a subalgebra of that of the original group, a homomorphism of one Lie group to another induces a homomorphism of the corresponding Lie algebras. The kernel of this Lie algebra homomorphism is an ideal, i.e., a subalgebra which has the property that a multiple of a member by any member of the original algebra is in the subalgebra. Explicitly, a subalgebra $I$ is an ideal of a Lie algebra $A$ if $[i, a]$ is in $I$ for every $i$ in $I$ and every $a$ in $A$. The kernel of the induced Lie algebra homomorphism will be the Lie algebra of the closed normal subgroup which is the kernel of the group homomorphism.

**2.5. Global structure of Lie groups.** We noted previously that a simply connected Lie group is completely determined by its Lie algebra. By this, of course, we mean that it is determined to within isomorphism. We now discuss the possibilities for Lie groups which are not simply connected.

We say a Lie group is *connected* if every two points of it can be joined by an arc lying in the group. If a Lie group is connected, we call it *simply connected* when every simple closed curve in the group can be continuously shrunk to a point without any part of it passing outside the group in the process. By a *component* of a Lie group we mean a maximal connected subset, i.e., all the elements which can be connected to some given element by arcs in the group.

Let us first consider the situation where the Lie group is not connected. It can be shown that the component containing the identity is always a closed normal subgroup of the Lie group and that the components are precisely the cosets of this normal subgroup. We can regard this collection of cosets, forming the quotient group, as an abstract group. (Indeed, if we take it to be a discrete group, the natural mapping is analytic.) The study of the algebraic structure of a Lie group which is not connected can almost be broken into two parts:

the structure of the connected subgroup forming the component of the identity and the structure of the discrete quotient group. (One also needs to know the action of certain inner automorphisms restricted to the identity component in order to analyze the complete structure of the Lie group.) Frequently, for Lie groups of practical importance, the structure of the quotient group is extremely simple, and knowledge of it together with knowledge of the structure of the identity component is all that is needed to determine the group structure completely. (For more explicit results on the structure of groups which are not connected see [23].) Our analysis can therefore be directed toward the structure of the identity component, which is a connected Lie group.

If a connected Lie group is simply connected, then it is completely determined to within isomorphism by the structure of its Lie algebra. The correspondence between connected Lie groups and Lie algebras, however, is many-to-one, so one must ask how the various groups corresponding to the same Lie algebra are related. The answer to this question is rather simple. Every connected Lie group corresponding to a given Lie algebra is a homomorphic image of the simply connected Lie group determined by the algebra. The kernel of this homomorphism is a discrete normal subgroup of the simply connected Lie group. Conversely, given any Lie group and any discrete normal subgroup, there is a homomorphism, having the discrete subgroup as its kernel, onto a new Lie group having the same Lie algebra. Further, to every point of the kernel in the simply connected Lie group there corresponds, in the homomorphic image, a closed path which cannot be shrunk continuously to a point without passing outside the image group, and to each such path there is a point of the kernel. Two paths correspond to the same point of the kernel if and only if one can be continuously distorted into the other without passing outside the group.

We have seen that in a sense the Lie algebra almost determines its Lie group by determining a corresponding simply connected group from which the given group differs only by discrete factors.

**2.6. Examples.** We can now determine all of the connected real abelian Lie groups. In §2.4 we found that the simply connected abelian Lie groups are merely the vector spaces. According to the previous section, every connected real abelian Lie group will be a homomorphic image of a real vector space, the kernel of the homomorphism being a discrete subgroup of the vector space. But the discrete subgroups are simply the lattices of vectors of the form

$$\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_k\mathbf{e}_k ,$$

where the $a_i$ run over all integers and the $\mathbf{e}_i$ are fixed linearly independent vectors. It is thus easy to see that every real abelian Lie group is the direct product of lines and circles. The simply connected one-dimensional real Lie group is a line. Any discrete subgroup of the line consists of equally spaced points, all the integer multiples of a fixed vector. Identifying points modulo this subgroup is equivalent to winding the line about a circle of circumference equal to the spacing of the points. If $k$ is the least positive member of the discrete subgroup, then $f(x) = \exp(2\pi ix/k)$ maps the line homomorphically onto the unit circle in the complex plane (which is a group under multiplication) and has for kernel the multiples of $k$. Thus the line and the circle are the only one-dimensional real abelian Lie groups. Similarly one can show that the only two-dimensional real abelian Lie groups are the plane, the cylinder, and the torus—and so on for higher dimensions.

Previously we determined the Lie algebra of the general linear group. The correspondence between subgroups of a Lie group and subalgebras of its Lie algebra means that we can find the Lie algebra of any matrix group simply by finding the tangent space at the identity. The tangent space will be some subspace of the tangent space of the general linear group, which consisted of all matrices. The Lie product will always be the commutator, just as for the general linear group.

To illustrate this, consider the group consisting of real matrices of the form

$$\begin{pmatrix} e^a & b \\ 0 & 1 \end{pmatrix}.$$

A basis for the tangent space consists of the matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The Lie product is simply

$$[e_1 , e_2] = e_1e_2 - e_2e_1 = e_2 .$$

In fact, it can easily be shown that any nonabelian two-dimensional Lie algebra has this structure if the basis is chosen appropriately. This group can be regarded as a plane with $a$ and $b$ as coordinates. Since the Lie algebra given above is the only nonabelian two-dimensional one, we have the result that any connected two-dimensional nonabelian real Lie group is equivalent to this matrix group modulo a discrete normal subgroup. By writing out the form of a conjugate class it is easily seen that no normal subgroup can be discrete.

Thus the plane with this multiplication is the only nonabelian two-dimensional real Lie group. We have now determined all possible connected real two-dimensional Lie groups. They are the plane, the cylinder and the torus. The plane admits two group structures making it a Lie group, one abelian and one nonabelian. The torus and the cylinder only admit an abelian group structure.

As the dimension increases, the problem of characterizing all connected Lie groups of that dimension becomes more difficult because more Lie algebras are possible and the simply connected Lie groups have more complicated lattices of discrete normal subgroups. Later we will discuss in detail the progress which has been made in classifying the possible Lie algebras.

As a final elementary example, let us consider the Lie group $SO(3)$ of rotations in ordinary three-dimensional Euclidean space. Small rotations can be composed of rotations about three mutually perpendicular axes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so $\phi$, $\theta$, and $\psi$ are suitable local coordinates. If $R$ denotes the product of these matrices we have the basis of tangent vectors at the identity given by

$$\mathbf{e}_1 = \frac{\partial R}{\partial \phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{e}_2 = \frac{\partial R}{\partial \theta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{e}_3 = \frac{\partial R}{\partial \psi} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and an element of the Lie algebra has the form

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3 = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

These vectors $\boldsymbol{\omega}$ of the Lie algebra are commonly referred to as "angular velocity" vectors.

The Lie product is given by

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3,$$

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1,$$

$$[\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2.$$

If we identify $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ of vector analysis, we recognize this to be the usual vector product. Since $SO(3)$ can be shown to be connected but not simply connected, it is the homomorphic image of a simply connected group having the same Lie algebra. The simply connected group with the same Lie algebra is $SU(2)$. The homomorphism of $SU(2)$ onto $SO(3)$ is of fundamental importance in the Pauli theory of electron spin. It is easily verified that the group $SU(2)$ consists precisely of the $2 \times 2$ matrices of determinant one which can be written in the form $aI + ib\sigma_1 + ic\sigma_2 + id\sigma_3$, where $a, b, c, d$ are real, $I$ is the identity matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are frequently called the *Pauli matrices*. Since the determinant of such a matrix is $a^2 + b^2 + c^2 + d^2$, we see that $SU(2)$ can be identified with the unit sphere in real four-dimensional Euclidean space with coordinates $a$, $b$, $c$, $d$. Thus $SU(2)$ is a simply connected real three-dimensional Lie group. For any $Q$ in $SU(2)$, we can define elements $a_{ij}$ of a $3 \times 3$ matrix $A$ by $a_{ij} = \frac{1}{2}$ Tr $(Q\sigma_i Q^* \sigma_j)$, where $Q^*$ is the adjoint (complex conjugate transpose) of $Q$ and Tr denotes the trace, the sum of the elements on the principal diagonal. It is easily verified that the matrix $A$ is in $SO(3)$, that every member $A$ of $SO(3)$ can be generated by a member $Q$ of $SU(2)$ using this formula, and that two distinct members $Q$ and $Q'$ of $SU(2)$ yield the same matrix $A$ if and only if $Q' = -Q$. To verify that the mapping of $SU(2)$ onto $SO(3)$ is a homomorphism it is convenient to use the identity

$$\tfrac{1}{2}\mathrm{Tr}\ (xy) = \sum_{i=1}^{3} \tfrac{1}{2}\mathrm{Tr}\ (\sigma_i x) \cdot \tfrac{1}{2}\mathrm{Tr}\ (\sigma_i y),$$

satisfied by any pair of $2 \times 2$ matrices, $x$ and $y$, having zero trace. By the fundamental theorem of homomorphisms, we then have an isomorphism of $SO(3)$ with $SU(2)/Z_2$, where $Z_2$ is the discrete subgroup consisting of the two elements $\pm I$.

**2.7. Exponentiation and matrix groups.** If $A$ is a square matrix and $I$ is the identity matrix, then by exp $(A)$ we mean the matrix power series $I + A + A^2/2! + \cdots + A^n/n! + \cdots$, each element of which converges uniformly and absolutely. Thus exp $(0) = I$. It is easily shown that the inverse of exp $(A)$ is exp $(-A)$, and that if $\lambda$ is an eigenvalue of $A$, then exp $(\lambda)$ is an eigenvalue of exp $(A)$ with the same eigenspace. Thus the determinant of exp $(A)$, the product of its eigenvalues, is exp (Tr $(A)$), where Tr $(A)$ is the trace, the sum of the eigenvalues, of $A$.

Suppose each matrix in some neighborhood of the identity of a group can be expressed as exp $(A)$, for $A$ in some linear subspace of the space of square matrices. It is then apparent from the power series expansion that this subspace of square matrices constitutes the tangent space to the group. Thus, when endowed with the commutator multiplication, $[A, B] = AB - BA$, this linear subspace of square matrices must be the Lie algebra. This device frequently allows one to determine the Lie algebra of a matrix group.

From the inverse function theorem, it is easily seen that every matrix in some neighborhood of the identity in $GL(n, C)$ can be expressed as exp $(A)$ with $A$ some complex square matrix of order $n$. For $GL(n, R)$, we can restrict the matrix $A$ to be real. The matrix exp $(A)$ will be in $O(n, C)$ if and only if the transpose of exp $(A)$ is its inverse. But the transpose of the exponential is the exponential of the transpose, while we remarked previously that the inverse of the exponential is the exponential of the negative, so the condition for exp $(A)$ to be in $O(n, C)$ is that $A$ be antisymmetric. The Lie algebra $o(n, C)$ thus consists of all complex antisymmetric matrices of order $n$, while the Lie algebra $o(n)$ consists of the real ones. Similarly the complex conjugate of exp $(A)$ is the exponential of the complex conjugate so that exp $(A)$ is in $U(n)$ if and only if its transposed conjugate is its negative. The Lie algebra $u(n)$ therefore consists of the anti-Hermitian matrices. As remarked before, the determinant of exp $(A)$ will be one if and only if the trace of $A$ is zero. Thus the Lie algebra $sl(n, C)$ consists of the matrices of trace zero. (The reader can readily verify that if $A$ and $B$ are matrices both of which are antisymmetric, anti-Hermitian or of trace zero, then $AB - BA$ also has this property, so the spaces of matrices referred to actually do form Lie algebras with this multiplication.)

Since the tangent space to the intersection of two manifolds at a point is the intersection of the tangent spaces, the Lie algebras of the matrix groups obtained by intersecting $GL(n, C)$, $GL(n, R)$, $O(n, C)$, $U(n)$ and $SL(n, C)$ are the intersections of their Lie algebras, which were just described. Incidentally, since the dimension of the linear subspaces forming the Lie algebras is easily computed, we can obtain the dimensions of these matrix groups.

## 3. Lie algebras and their structure.

**3.1. Lie algebras and associative algebras.** In the course of computing the Lie algebras of matrix groups, we found that the Lie multiplication can be defined in terms of the associative matrix multiplication. This procedure can be extended to any associative algebra and a Lie product can be defined by the *commutator*, $[a, b] = ab - ba$, making it a Lie algebra. (The reader can easily check that (9) and (10) are satisfied.)

By a *representation* of a Lie algebra, we mean a homomorphism of the given Lie algebra onto a Lie algebra of linear transformations on a vector space with the commutator multiplication. One such representation occurs in an especially natural manner. Any fixed member, $a$, of a Lie algebra defines a linear transformation $\mathrm{ad}(a): x \to [a, x]$ of the Lie algebra into itself. The mapping $\mathrm{ad}: a \to \mathrm{ad}(a)$ is a linear transformation. To show that it is a representation we need only verify that

$$[\mathrm{ad}(a), \mathrm{ad}(b)] = \mathrm{ad}([a, b]).$$

But for any $x$ in the Lie algebra,

$$\begin{aligned}[\mathrm{ad}(a), \mathrm{ad}(b)]x &= \mathrm{ad}(a)\,\mathrm{ad}(b)x - \mathrm{ad}(b)\,\mathrm{ad}(a)x \\ &= [a, [b, x]] - [b, [a, x]].\end{aligned}$$

Using Jacobi's identity (10) and antisymmetry, (9), we have

$$[\mathrm{ad}(a), \mathrm{ad}(b)]x = [[a, b], x] = \mathrm{ad}([a, b])x,$$

which is the desired result, so the mapping ad is a representation. It is called the *adjoint representation.* A representation is called *faithful* if it is an isomorphism. Ado has proved that every finite-dimensional Lie algebra has a faithful finite-dimensional representation, so every finite-dimensional Lie algebra is isomorphic with a Lie algebra of linear transformations and hence, upon choosing a basis, with a Lie algebra of matrices. Similarly, a finite-dimensional representation of a Lie group is a homomorphism of the group onto a group of linear transformations on a finite-dimensional vector space. Again, if a fixed basis is chosen, this is a homomorphism onto a matrix group. Since a homomorphism of one Lie group onto another induces a homomorphism of its Lie algebra onto that of the image and since the Lie algebra of a matrix group is a Lie algebra of matrices, every representation of a Lie group induces a representation of its Lie algebra. The correspondence between representations of Lie groups and those of Lie algebras is many-to-one, however, for nonisomorphic groups may have the same Lie algebra. Nevertheless, the major portion of work involved in studying the group representations is frequently done when one has found those of the Lie algebra.

The operator $\mathrm{ad}(a)$ behaves just like a first order differential operator in the sense that it obeys the Leibnitz rule,

$$\mathrm{ad}(a)[b, c] = [\mathrm{ad}(a)b, c] + [b, \mathrm{ad}(a)c],$$

which is just another way of viewing the Jacobi identity. It is however frequently very convenient in computations to think of the analogy with differential operators.

**3.2. Solvable and semisimple Lie algebras.** If $M_1$ and $M_2$ are subspaces of a Lie algebra, we denote by $[M_1, M_2]$ the space spanned by all vectors $[m_1, m_2]$ where $m_1 \in M_1$ and $m_2 \in M_2$. Thus, an *ideal* $I$ of a Lie algebra $L$ is a subspace which satisfies $[I, L] \subset I$, and a *subalgebra* $A$ is a subspace which satisfies $[A, A] \subset A$. With any Lie algebra $L$, there is associated a *derived series* of ideals defined by $L' = [L, L]$; $L'' = [L', L']$; $\cdots$ ; and in general $L^{(k+1)} = [L^{(k)}, L^{(k)}]$. That $L^{(k)}$ is indeed an ideal of $L$ can be readily proved by induction on $k$ using the Jacobi identity. We have $L^{(k+1)} \subset L^{(k)}$. If the derived series $L \supset L' \supset L'' \supset \cdots$ is eventually zero, we say that $L$ is a *solvable* Lie algebra. For example, consider the Lie algebra $L$ of a nonabelian two-dimensional Lie group. In this algebra we can choose a basis $e_1, e_2$ such that $[e_1, e_2] = e_1$. Then $L'$ consists of the multiples of $e_1$, and we have $L'' = 0$, so that $L$ is solvable. Solvability can be characterized in the following remarkable way. A Lie algebra $L$ is solvable if and only if the trace of the square of the matrix $\mathrm{ad}(x)$ vanishes for all $x$ in $L'$. A Lie algebra is called *semisimple* if it has no solvable nonzero ideals, and a Lie algebra is called *simple* if it is nonabelian and has no proper ideals whatever. The Lie algebra $so(3)$, three-dimensional vector space with the usual vector cross-product, is an example of a simple Lie algebra. Every simple Lie algebra $L$ is, as one would expect, semisimple, for the only nonzero ideal is the whole algebra; if $L' = L$, this is not solvable, while if $L' = 0$, then $L$ is abelian and consequently not simple.

Every Lie algebra has a unique maximal solvable ideal which is called its *radical*. A Lie algebra is, of course, semisimple if and only if its radical is zero. Levi has shown that every Lie algebra is the direct sum of its radical and a semisimple subalgebra (i.e., every element can be written uniquely as a sum of an element in the radical and an element of the semisimple subalgebra). While this decomposition of the algebra is not unique, in that there may be several suitable semisimple subalgebras, Mal'cev, Goto and Harish-Chandra have shown that if there are two, then there is an automorphism of the whole Lie algebra carrying one onto the other. The structure of a Lie algebra can be even more precisely studied, for every semisimple Lie algebra is the direct sum of its simple ideals. Thus a Lie algebra can always be written as the direct sum

$$L = S \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_k,$$

where $S$ is the maximal solvable ideal (the radical) and the $L_i$ are simple ideals of their direct sum (but not necessarily ideals of $L$). The problem of studying the structure of a Lie algebra is thus almost reduced to studying the simple Lie algebras and the solvable Lie algebras. The elements of distinct $L_i$ must commute since a product of elements from distinct $L_i$ must be in their intersection which is zero. The only question of structure remaining is to characterize the action of the various $L_i$ upon $S$, for multiplication by $x$ in $L_i$ maps $S$ into itself by the transformation $\operatorname{ad}(x)$ restricted to $S$. In particular the structure of a semisimple algebra is completely characterized if one can determine the structure of the simple ideals, since $S$ is zero in that case.

**3.3. Nilpotent subalgebras and Cartan subalgebras.** We shall now discuss some ideas needed for the analysis of the structure of simple Lie algebras. Many of these ideas date back to Cartan [6]. A Lie algebra $L$ is said to be *nilpotent* if the lower central series of ideals

$$L \supset L^2 = [L, L] \supset \cdots \supset L^{k+1} = [L^k, L] \supset \cdots$$

is eventually zero. It is necessary to distinguish this sequence from the derived series used in defining solvability. It can readily be shown that $L^{(k)} \subset L^{2^k}$, so every nilpotent Lie algebra is solvable. On the other hand, we saw that the nonabelian two-dimensional Lie algebra defined by $[e_1, e_2] = e_1$ is solvable but it is clearly not nilpotent, for $L^k$ is the space spanned by $e_1$ for all $k \geqq 1$. Engel has shown that a Lie algebra is nilpotent if and only if $\operatorname{ad}(x)$ is a nilpotent linear transformation (i.e., some power of the matrix of $\operatorname{ad}(x)$ vanishes) for every $x$ in the Lie algebra.

Suppose $H$ is a nilpotent subalgebra of a Lie algebra $L$. If we restrict the adjoint representation to the subalgebra we have as its image a collection of linear mappings $\operatorname{ad}_L H$ of $L$ into itself. Denote by $L_0$ the subspace of $L$ on which every member of $\operatorname{ad}_L H$ is nilpotent. Since $H$ is nilpotent, by Engel's theorem, $H \subset L_0$. A nilpotent subalgebra $H$ for which $H = L_0$ is called a *Cartan subalgebra* of $L$. Every Lie algebra contains a Cartan subalgebra, and all Cartan subalgebras of a given Lie algebra $L$ have the same dimension, which is called the *rank* of $L$. If $L$ is semisimple, then a Cartan subalgebra will be a maximal commutative subalgebra, but every maximal commutative subalgebra need not be a Cartan subalgebra. We will denote the rank of $L$ by $l$.

For the remainder of this chapter we will let $L$ be a semisimple Lie algebra over the complex numbers, with Cartan subalgebra $H$. Let $H^*$ denote the dual space of $H$, that is, the set of all linear forms which map $H$ into the set of all complex numbers. A linear form $\alpha \in H^*$ is called a *root* if there exists a nonzero element $x \in L$ such that $[h, x] = \alpha(h)x$ for all $h \in H$.

Let $H_R^*$ denote the subspace of $H^*$ spanned by all the real linear combinations of roots. The subspace $H_R{}^*$ is of dimension $l$. Let $\gamma_1, \cdots, \gamma_l$ be an arbitrary (ordered) basis for $H_R{}^*$. If $\xi \in H_R{}^*$, then

$$\xi = r_1\gamma_1 + \cdots + r_l\gamma_l,$$

where $r_1, \cdots, r_l$ are real numbers, the components of $\xi$. We define a *lexicographic ordering* in $H_R{}^*$ by saying that $\xi > 0$ if the first nonzero component of $\xi$ is positive. We say that $\xi > \eta$ if $\xi - \eta > 0$. A root $\alpha$ is called a *positive root* if $\alpha > 0$. A positive root is called a *simple root* if it is not the sum of two positive roots. There are exactly $l$ simple roots. The simple roots $\alpha_1, \cdots, \alpha_l$ form a basis for $H^*$. Moreover, if $\alpha$ is any root, then we can write

$$\alpha = k_1\alpha_1 + \cdots + k_l\alpha_l,$$

where $k_1, \cdots, k_l$ are integers of the same sign. If $\alpha > 0$, then $k_1 \geqq 0, \cdots, k_l \geqq 0$; and if $\alpha < 0$, then $k_1 \leqq 0, \cdots, k_l \leqq 0$. The sum $k_1 + \cdots + k_l$ is called the *level* of the root $\alpha$. The difference $\alpha - \beta$ between two simple roots cannot be a root. If $\alpha$ is a nonzero root, the only integral multiples of $\alpha$ which are roots are $\alpha$, 0, and $-\alpha$. Associated with any nonzero root $\alpha$, there is a unique (up to a factor) eigenvector $e_\alpha$ common to all members of $\mathrm{ad}_L H$ such that

$$\mathrm{ad}(h)e_\alpha = [h, e_\alpha] = \alpha(h)e_\alpha.$$

By Jacobi's identity, if $e_\alpha$ and $e_\beta$ are such eigenvectors then

$$[h, [e_\alpha, e_\beta]] = [\alpha(h) + \beta(h)][e_\alpha, e_\beta],$$

so $[e_\alpha, e_\beta]$ is a multiple of $e_{\alpha+\beta}$ if $\alpha + \beta$ is a root. Otherwise it is zero. It can be shown that $[e_\alpha, e_{-\alpha}]$ is always a member of $H$. Finally, $L$ is the direct sum of $H$ and the one-dimensional subalgebras $L_H{}^\alpha$ spanned by the $e_\alpha$:

$$L = H \oplus L_H{}^{\alpha_1} \oplus L_H{}^{\alpha_2} \oplus \cdots \oplus L_H{}^{\alpha_{n-l}},$$

where $\alpha_1, \cdots, \alpha_l$ are all the nonzero roots. Thus, a basis for $H$ together with the $e_\alpha$ determines a basis for $L$. The $e_\alpha$ are called *root vectors* [11].

**3.4. The Killing form.** We can introduce a symmetric bilinear form in any Lie algebra by defining $(x, y) = \mathrm{Tr}\ (\mathrm{ad}(x)\ \mathrm{ad}(y))$. This form is due to Killing. Its importance arises partly from the fact that a Lie algebra is semisimple if and only if its Killing form is *nonsingular*. This means that if $(x, y) = 0$ for all $y \in L$, then $x = 0$. The Killing form then yields an indefinite inner product on the Lie algebra and permits geometric ideas such as orthogonality to be used. We will say that the vectors $a$ and $b$ are *orthogonal* if $(a, b) = 0$. It must be borne in mind, however, that $(a, a) = 0$ does not imply $a = 0$, since we are discussing Lie algebras over the complex number field. The Killing form is invariant under every automorphism of the Lie algebra: if $\alpha$ is any automorphism then $(\alpha x, \alpha y) = (x, y)$.

It can be shown that if $\alpha$ and $\beta$ are any two roots of a Cartan subalgebra $H$ in $L$, and if $\alpha + \beta \neq 0$, then the corresponding eigenvectors $e_\alpha$ and $e_\beta$ are orthogonal. The eigenvectors $e_\alpha$ and $e_{-\alpha}$ are never orthogonal. The Cartan subalgebra $H$ is a *nonsingular subspace* with respect to the Killing form. That is, if

$h' \in H$ satisfies $(h, h') = 0$ for all $h \in H$, then $h' = 0$. For every linear form $\alpha \in H^*$, there exists a unique vector $h_\alpha \in H$ such that $(h_\alpha, h) = \alpha(h)$. In this sense $H^*$ and $H$ may be identified. In particular, we may define $(\alpha, \beta) = (h_\alpha, h_\beta)$. If $\alpha$, $\beta$ are nonzero roots, then $(\alpha, \beta)$ is real, and $(\alpha, \alpha)$ is positive nonzero. The Killing form is thus positive definite on $H_R{}^*$, so that $H_R{}^*$ is a Euclidean space with the Killing form as inner product. We have also

$$[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha .$$

**3.5. Simple Lie algebras and Dynkin diagrams.** The structure of a semisimple Lie algebra is determined up to an isomorphism by its system of roots. Let the rank of the Lie algebra be denoted by $l$, and let $\alpha_1, \alpha_2, \cdots, \alpha_l$ be a system of simple roots. The corresponding root vectors will be denoted by $e_i = e_{\alpha_i}$. We also denote

$$f_i = \frac{2e_{-\alpha_i}}{(\alpha_i, \alpha_i)(e_{\alpha_i}, e_{-\alpha_i})},$$

$$h_i = \frac{2h_{\alpha_i}}{(\alpha_i, \alpha_i)}.$$

(The choice of normalization simplifies later computation.) The vectors $e_1, \cdots, e_l$ may be called *simple raising operators*, and $f_1, \cdots, f_l$ *simple lowering operators*. Together, these simple raising and lowering operators generate the whole Lie algebra. The elements $h_1, \cdots, h_l$ form a basis for the chosen Cartan subalgebra of the semisimple Lie algebra. The Lie products are given by

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = A_{ij}e_j, \qquad [h_i, f_j] = -A_{ij}f_j, \qquad [e_i, f_j] = \delta_{ij}h_i .$$

The matrix $A_{ij}$, defined by $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ is called the *Cartan matrix* of the Lie algebra. It plays an important role in the structure of a simple Lie algebra.

The relations written above do not yet completely define all products, for $[e_i, e_j]$ and $[f_i, f_j]$ have not been specified. The complete set of relations can nevertheless be derived from the above set by an algorithm which we shall illustrate for a special case in §3.7.

Let $\beta$ and $\alpha$ be roots. The sequence of linear forms $\beta - p\alpha, \cdots, \beta - \alpha, \beta, \beta + \alpha, \cdots, \beta + q\alpha$ is called an *$\alpha$-ladder* through $\beta$ if all these linear forms are roots and if $\beta - (p + 1)\alpha$ and $\beta + (q + 1)\alpha$ are not roots. Then $p$, $q$ are related by $p - q = 2(\alpha, \beta)/(\alpha, \alpha)$. If $\alpha$, $\beta$ are simple roots, then $p = 0$, and the Cartan matrix determines the $\alpha$-ladder through $\beta$. From a knowledge of the Cartan matrix we can determine by an algorithm (using induction on the level of the root $\beta$) whether or not a given linear combination of simple roots,

$$\beta = \sum_{i=1}^{l} k_i\alpha_i ,$$

is a positive root. Thus the full root system is determined by the Cartan matrix.

Let us denote

$$e_{i_1 \cdots i_s} = [e_{i_1}, \cdots [e_{i_{s-1}}, e_{i_s}] \cdots ],$$

$$f_{i_1 \cdots i_s} = [f_{i_1}, \cdots [f_{i_{s-1}}, f_{i_s}] \cdots ].$$

The elements $h_i$, $e_{i_1 \cdots i_s}$ and $f_{i_1 \cdots i_s}$ span the Lie algebra. All linear relations and commutation relations for these elements are computable from the Cartan matrix. Thus the Cartan matrix determines the structure of a semisimple Lie algebra.

The systems of vectors which can represent root systems of simple Lie algebras are restricted by the following conditions:

1. If $\alpha$ is a root, then $-\alpha$ is a root, but no other nonzero multiples of $\alpha$ are roots.
2. $| 2(\alpha, \beta)/(\alpha, \alpha) | = 0, 1, 2,$ or $3$.
3. Reflection of a root $\beta$ in the hyperplane through the origin perpendicular to any root $\alpha$ yields another root, $\beta - (2(\beta, \alpha)/(\alpha, \alpha))\alpha$. (This is called a *Weyl reflection.*)

These conditions determine the possible root systems, and hence the possible simple Lie algebras. While the problem of determining all simple Lie algebras has thus been reduced to a geometrical problem, Dynkin [11] has found an even simpler method of doing this using diagrams in the plane. He represents the simple roots $\alpha_1, \cdots, \alpha_l$ of a simple system by points (vertices of the diagram) in the plane. The points representing $\alpha_i$ and $\alpha_j$ are connected by

$$A_{ij} A_{ji} = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} = 4 \cos^2 \measuredangle\, \alpha_i,\alpha_j$$

lines. The number of lines determines the angle between the roots, since $(\alpha_i, \alpha_j) \leqq 0$ for simple roots. Thus we have

$$\measuredangle\, \alpha_i,\alpha_j = 90° \text{ if no line connects the } i\text{th, } j\text{th vertices,}$$

$$\measuredangle\, \alpha_i,\alpha_j = 120° \text{ if one line connects the } i\text{th, } j\text{th vertices,}$$

$$\measuredangle\, \alpha_i,\alpha_j = 135° \text{ if two lines connect the } i\text{th, } j\text{th vertices,}$$

$$\measuredangle\, \alpha_i,\alpha_j = 150° \text{ if three lines connect the } i\text{th, } j\text{th vertices.}$$

The relative lengths of the roots are indicated by using light circles and dark circles to represent the points. If all roots are of the same length, all the circles are light. The only other possibility is that there are roots of two different lengths. The long roots are then assigned light circles, the short roots dark circles. The relative lengths of two roots is determined by the angle between them. If the angle is 120°, the roots are of equal length. If the angle is 135°, the relative lengths are in the ratio $\sqrt{2}:1$. If the angle is 150° the ratio is $\sqrt{3}:1$. The ratio of the squares of the lengths of the $i$th and $j$th roots is

$$\frac{(\alpha_i, \alpha_i)}{(\alpha_j, \alpha_j)} = \frac{A_{ji}}{A_{ij}}.$$

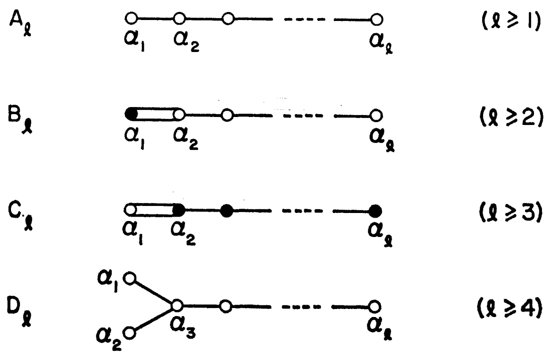


FIG. 1. *The four main classes of simple Lie algebras*

Since the product $A_{ij}A_{ji}$ and ratio $A_{ij}/A_{ji}$ can be determined from the Dynkin diagram, we can obviously calculate all the $A_{ij}$. The sign is always minus for off-diagonal elements, and the diagonal elements are clearly $A_{ii} = +2$. Thus the Dynkin diagram determines the Cartan matrix uniquely.

The three conditions on the root system listed above can be used to restrict the possible Dynkin diagrams of simple Lie algebras. We find four general classes of Dynkin diagrams and five exceptions. The four main classes are shown in Fig. 1. Each diagram has $l$ vertices, where $l$ is the rank of the algebra. The five exceptional Dynkin diagrams are shown in Fig. 2. These diagrams represent the only possible simple Lie algebras of finite dimension over the complex number field. Each of these possibilities is actually realized by a Lie algebra which can be explicitly constructed from the diagram.

The problem of classifying real simple Lie algebras is studied by generating them from the complex ones. A given simple Lie algebra over the complex numbers in general yields several real ones, so that the Dynkin diagrams must be supplemented with further considerations. This problem has been studied, for example, by Gantmacher [14].

**3.6. Identification of simple Lie algebras.** It is of interest to identify the Lie algebras corresponding to the diagrams $A_l$, $B_l$, $C_l$, and $D_l$. It can be shown that $A_l = sl(l + 1, C)$, $B_l = o(2l + 1, C)$, $D_l = o(2l, C)$.

The Lie algebras corresponding to the diagrams $C_l$ are the Lie algebras of a set of matrix groups known as the complex symplectic group, denoted $Sp(l, C)$. Thus $C_l = sp(l, C)$. These matrix groups are very similar to the orthogonal groups. Just as the orthogonal group may be regarded as the set of transformations which leave invariant a symmetric nondegenerate bilinear form, the symplectic group may be regarded as the transformations which leave invariant an antisymmetric nondegenerate bilinear form.

Alternately, we may proceed as follows. Let $I_n$ denote the $n \times n$ identity matrix, and let $J$ be the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

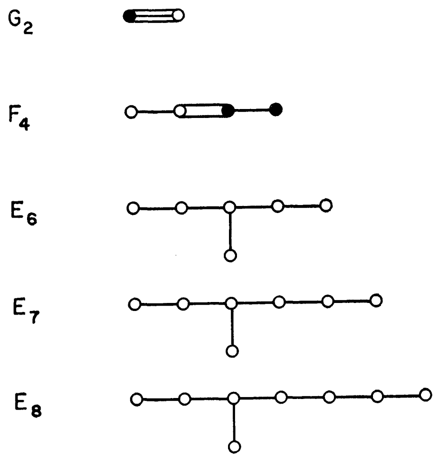


FIG. 2. *The five exceptional simple Lie algebras*

Then $Sp(n, C)$ is the set of all complex matrices $A$ which satisfy $A^T J A = J$. The group nature of $Sp(n, C)$ is easily verified.

Closely related to the complex symplectic group $Sp(n, C)$ are some other matrix groups obtained by taking intersections. In particular we mention the real symplectic group $Sp(n, R) = Sp(n, C) \cap GL(2n, R)$, and the unitary symplectic group $Sp(n) = Sp(n, C) \cap U(2n)$. The exceptional Lie algebras $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ can be associated in various ways with the Cayley numbers. The Lie algebras of the symplectic groups play an important role in nuclear physics [20] and the Lie algebra $G_2$ has been applied by Racah to certain problems in atomic spectroscopy.

For the low-rank Lie algebras there are a number of simple isomorphisms which we note:

$$sl(2, C) \cong o(3, C) \cong sp(1, C),$$

$$o(4, C) \cong o(3, C) \oplus o(3, C),$$

$$o(5, C) \cong sp(2, C),$$

$$o(6, C) \cong sl(4, C).$$

A more detailed list is given in [19].

**3.7. Construction of the Lie algebra $A_2$.** As an illustration of the procedure for determining the structure constants of a Lie algebra from its Dynkin diagram, we shall consider here the case of the simple Lie algebra $A_2$ (or $sl(3, C)$) in some detail. The Dynkin diagram is shown in Fig. 3. The two simple roots $\alpha_1$ and $\alpha_2$ are of equal length, and form an angle of 120°. Hence the Cartan matrix, with components $A_{ij}$, of the Lie algebra $A_2$ is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

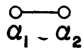


FIG. 3. *Dynkin diagram for* $A_2$

Consider now the $\alpha_1$-ladder through $\alpha_2$. Since the difference of two simple roots obviously cannot be a root, we have $p = 0$. Hence $q = -(p - q) = -A_{12} = 1$. The ladder in question is thus composed of the roots $\alpha_2$ and $\alpha_2 + \alpha_1$. Bv considering further ladders, we obtain nothing new. Hence the only positive roots are $\alpha_1$, $\alpha_2$ and $\alpha_1 + \alpha_2$. The negative roots are $-\alpha_1$, $-\alpha_2$ and $-(\alpha_1 + \alpha_2)$. The Euclidean space $H_R^*$ spanned by $\alpha_1$ and $\alpha_2$ is a plane, so it is therefore possible to draw the root diagram of the Lie algebra $A_2$ as shown in Fig. 4. The symmetry under Weyl reflections is immediately evident from this picture.

We introduce the simple raising and lowering operators $e_1$, $e_2$, $f_1$, $f_2$ and the elements $h_1$, $h_2$. Their commutation relations are

$$[h_1, e_1] = 2e_1, \quad [h_1, f_1] = -2f_1, \quad [h_2, e_1] = -e_1, \quad [h_2, f_1] = +f_1,$$
$$[h_1, e_2] = -e_2, \quad [h_1, f_2] = f_2, \quad [h_2, e_2] = 2e_2, \quad [h_2, f_2] = -2f_2,$$
$$[e_1, f_1] = h_1, \quad [e_2, f_1] = 0, \quad [e_1, f_2] = 0, \quad [e_2, f_2] = h_2,$$

and of course $[h_i, h_j] = 0$. We supplement these six elements with the two further elements $e_{12} = [e_1, e_2]$, $f_{12} = [f_1, f_2]$, which are root vectors corresponding to the roots $\pm(\alpha_1 + \alpha_2)$. These eight vectors form a basis for $A_2$.

The commutation relations involving the elements $e_{12}$ and $f_{12}$ are obtained by use of the Jacobi identity. For example,

$$[h_1, e_{12}] = [h_1, [e_1, e_2]] = [[h_1, e_1], e_2] + [e_1, [h_1, e_2]] = 2e_{12} - e_{12} = e_{12}.$$

In this fashion we obtain the following further commutation relations:

$$[h_1, e_{12}] = e_{12}, \quad [h_1, f_{12}] = -f_{12}, \quad [h_2, e_{12}] = e_{12}, \quad [h_2, f_{12}] = -f_{12},$$
$$[e_1, e_{12}] = 0, \quad [e_1, f_{12}] = f_2, \quad [e_2, e_{12}] = 0, \quad [e_2, f_{12}] = -f_1,$$
$$[f_1, e_{12}] = e_2, \quad [f_1, f_{12}] = 0, \quad [f_2, e_{12}] = -e_1, \quad [f_2, f_{12}] = 0,$$
$$[e_{12}, f_{12}] = -(h_1 + h_2).$$

Thus starting from the Dynkin diagram for $A_2$ we have completely determined the products of the members of a basis. This determines the structure constants and hence the structure of the algebra. A similar procedure can be carried through to construct the Lie algebra corresponding to any Dynkin diagram.

**4. Applications.**

**4.1. Differential equations.** Since Sophus Lie originated his theory of "continuous groups" in order to study the problem of solving differential equations, it may seem odd to speak of the application of Lie algebras and Lie groups to differential equations as a contemporary field of endeavor, but recent work has dealt with quite different problems and techniques from those studied by Lie. Since Lie's work is not widely known we present a brief indication of his

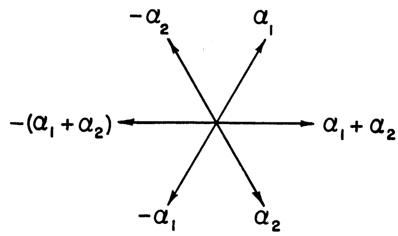


FIG. 4. *Root diagram for* $A_2$

methods before turning to the contemporary applications. A more detailed exposition is given in [9] and, of course, in Lie's works.

Suppose we are presented with a first order differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

and suppose further that we know a one-dimensional Lie group of transformations of the $(x, y)$-plane which acts analytically and leaves the solutions of the differential equation invariant. Will knowledge of the action of the group on the plane aid us in finding the solutions of the differential equation? (Consider the path, or orbit, of a given point as the group of transformations acts. In saying that the group *acts analytically* we mean that the $x$ and $y$ coordinates along any orbit are analytic functions of the local coordinates in the group. In saying that the *solutions are invariant* we mean that under the action of a member of the group each point of a given solution curve is carried into a point of another fixed solution curve.) Lie was able to show that knowing the orbits one can set up canonical coordinates $(u, v)$ in the plane such that the action of the group leaves $u$ invariant and such that $v$ is additive in the sense that if the point $(u, v)$ is carried by the group element $g_1$ to $(u, v + \delta_1)$ and by $g_2$ to $(u, v + \delta_2)$ then $g_2 \circ g_1$ carries it to $(u, v + \delta_1 + \delta_2)$. In these coordinates Lie showed that the variables separate, so the differential equation can be solved by a quadrature. Later he was able to give an integrating factor in terms of the tangent vectors to the orbits. If $P(x, y)$ and $Q(x, y)$ are the components of the tangent vector to an orbit through $(x, y)$ then $1/(PM + QN)$ is an integrating factor for the differential equation, and the solution is again obtained by quadrature. If one knows two distinct groups of transformations (or a two-dimensional group) leaving the differential equation invariant, the quadrature can be eliminated, for the solutions are obtained immediately by setting the ratio of the two integrating factors equal to an arbitrary constant.

If we associate with the group the operator

$$U = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

and define an operator associated with the differential equation by

$$A = N\frac{\partial}{\partial x} - M\frac{\partial}{\partial y},$$

then a necessary and sufficient condition for the differential equation to be invariant under the group is

$$[U, A] = UA - AU = \lambda(x, y)A.$$

While every first order differential equation can be shown to be invariant under some one-dimensional Lie group, this is not the case for higher order differential equations. When a higher order differential equation is invariant under a one-dimensional group of transformations, however, Lie was able to give a process of introducing new coordinates which reduces the order of the differential equation by one. Essentially, one needs to find nontrivial functions $u(x, y)$, $v(x, y, dy/dx)$ which are invariant under the group. When the differential equation is reformulated in terms of $u$ and $v$ it is of order one lower than that of the original equation. Repetition of this procedure reduces the problem to that of solving a first order differential equation. If the dimension of the Lie group under which the differential equation remains invariant is greater than one, the integration procedure can be simplified by introducing canonical coordinates. These canonical coordinates and the integration procedure depend in general on the structure of the Lie algebra of the group of transformations and on the relative dimensions of the group and the orbits. (For instance the group $O(3)$ is a three-dimensional group of transformations on three-dimensional Euclidean space while its nontrivial orbits are the spheres and are two-dimensional.)

In addition to giving methods of reducing differential equations invariant under a Lie group to quadratures, Lie was also able to determine the differential equations which would be invariant under a given group. This procedure has been used to advantage in the current work on pattern recognition [22] which is discussed in §4.7.

More recently the theory of Lie algebras has been applied to the study of equations of the type

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

where $\mathbf{x}$ and $\mathbf{A}$ are operators and $\mathbf{x}(0) = I$, the identity. Magnus [29] has shown that this problem has a formal solution

$$\mathbf{x} = \exp \mathbf{\Omega}\,(t),$$

which actually converges for some interval about $t = 0$ if the operators are matrices and $\mathbf{A}$ is a continuous function of $t$.

Wichmann [50] and Wei and Norman [35], [45] have considered the case

$$\mathbf{A}(t) = \sum_{i=0}^{n} a_i(t)\mathbf{X}_i\,,$$

where the constant operators $\mathbf{X}_i$ generate a Lie algebra of finite dimension $l$ under commutation. In that case the solutions can be given locally in the form treated by Magnus as

$$\mathbf{x} = \exp\left(\sum_{i=1}^{l} f_i(t)\mathbf{X}_i\right).$$

Wei and Norman show that a solution can also be written

$$\mathbf{x} = \exp(g_1(t)\mathbf{X}_1)\,\exp(g_2(t)\mathbf{X}_2)\,\cdots\,\exp(g_l(t)\mathbf{X}_l).$$

These authors have given various conditions under which the solutions converge globally, i.e., for all $t$.

The point of departure for all of this work is a pair of formulas of Baker and Hausdorff which give $z$ and $w$ in $\exp(x)\,\exp(y) = \exp(z)$ and $y\,\exp(x) = \exp(x)w$ in terms of commutators of $x$ and $y$. In particular,

$$w = \exp(\mathrm{ad}(x))y = y + [x, y] + \frac{[x,[x, y]]}{2!} + \cdots,$$

$$z = x + y + \frac{[x, y]}{2} + \frac{[[x, y], y]}{12} + \frac{[[y, x], x]}{12} + \cdots,$$

where the omitted terms involve successively higher commutators of $x$ and $y$. Magnus gives $\mathbf{\Omega}(t)$ by developing a "continuous analogue" of the Baker-Hausdorff formula for $z$. The work of Wei and Norman utilizes the formula for $w$ to find the coefficients $g_i(t)$. Conditions for the solutions to be global can be stated in terms of the structure of the Lie algebra generated by $\mathbf{A}(t)$.

Wichmann has utilized the Levi decomposition of finite-dimensional Lie algebras to show that the problem of solving the differential equation can actually be reduced to the case where the Lie algebra generated by $\mathbf{A}(t)$ is simple. Recall that a finite-dimensional Lie algebra can be decomposed as the direct sum of its radical and a semisimple subalgebra. This subalgebra in turn is a direct sum of its simple ideals. Wichmann decomposes $\mathbf{A}$ as

$$\mathbf{A}(t) = \mathbf{A}_1(t) + \mathbf{A}_2(t),$$

where $\mathbf{A}_1$ is in the radical and $\mathbf{A}_2$ in the semisimple subalgebra. He then decomposes $\mathbf{A}_2$ as

$$\mathbf{A}_2(t) = \mathbf{A}_{21}(t) + \mathbf{A}_{22}(t) + \cdots + \mathbf{A}_{2k}(t),$$

where the $\mathbf{A}_{2i}$ are in the simple ideals. He shows that the problem,

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}, \qquad \mathbf{x}(0) = I,$$

can be reduced to solving

$$\frac{d\mathbf{x}_2}{dt} = \mathbf{A}_2(t)\mathbf{x}_2, \qquad \mathbf{x}_2(0) = I,$$

and

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{x}_2^{-1}(t)\mathbf{A}_1(t)\mathbf{x}_2(t)\mathbf{x}_1\,, \qquad \mathbf{x}_1(0) = I,$$

the second of which is shown to be solvable by quadratures. The solution of the first of these equations is then shown to reduce to solving the problems

$$\frac{d\mathbf{x}_{2i}}{dt} = \mathbf{A}_{2i}(t)\mathbf{x}_{2i}, \qquad \mathbf{x}_{2i}(0) = I.$$

In [7] Chen has considered the problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{U}(t, \mathbf{x}).$$

This can be regarded as the problem of studying a flow with variable velocity field. An infinitely differentiable vector field can be regarded as an operator on the infinitely differentiable scalar functions by identifying its direction with the directional derivative. Thus the vector field $\mathbf{U}(t, \mathbf{x})$ would be identified with its dot product $\mathbf{U}\cdot\nabla$ with the gradient operator. The set of such operators forms a Lie algebra under commutation, and therefore by this identification we can also regard the vector fields themselves as forming a Lie algebra. Chen shows that if the Lie algebra generated in this manner by the velocity field is finite dimensional, then the Levi structure theorem can be used to obtain the type of decomposition which Wichmann exhibited for the more special problem. This depends basically on the development of a more general transformation on $\mathbf{A}_1$ playing the role of the similarity transformation appearing in Wichmann's differential equation for $\mathbf{A}_1$.

**4.2. Poisson brackets in classical mechanics.** Lie algebras play a fundamental role in the study of conservation laws in both classical and quantum mechanics. In this section we recall how these Lie algebras arose historically, and how they can be used.

In classical mechanics, a system can be described in terms of generalized coordinates $q_1, q_2, \cdots, q_n$, and the time $t$ by specifying an infinitely differentiable function, the *Lagrangian*, $L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$. The *generalized momenta* associated with the corresponding coordinates are defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad j = 1, \cdots, n,$$

and the function

$$H = \sum_{j=1}^{n} p_j\dot{q}_j - L$$

is called the *Hamiltonian* of the system. By a *dynamical variable* we shall mean any infinitely differentiable function of the coordinates, momenta and time. The dynamical variables form an infinite dimensional Lie algebra with the vector multiplication

$$[A, B] = \sum_{j=1}^{n} \left\{\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial q_j}\frac{\partial A}{\partial p_j}\right\}.$$

The Lie product $[A, B]$ of two dynamical variables $A$ and $B$ is a dynamical variable called the *Poisson bracket* of $A$ and $B$. By direct computation one may show that the Poisson bracket satisfies the three axioms for a Lie product: linearity, antisymmetry and the Jacobi identity. The Lie algebra of all dynamical variables contains no physical information about the system beyond the number of degrees of freedom. All systems with the same number of coordinates and momenta have isomorphic Lie algebras.

By a *Hamiltonian system* we shall mean a classical system for which the Hamiltonian $H$ can be expressed in terms of the variables $q_i$, $p_i$, $t$ by eliminating all the velocities $\dot{q}_i$. We also assume that none of the momenta are constrained. The space with coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$ is called the *phase space*. A particular physical system describes a trajectory in phase space determined by Hamilton's equations of motion,

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \qquad j = 1, \cdots, n,$$

together with suitable initial conditions. The rate of change of a dynamical variable $A$ during the motion of a system along this trajectory is found by use of the equations of motion and the chain rule for differentiation to be given by the expression

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + [A, H].$$

The dynamical variable $A$ is said to be *conserved* if $dA/dt = 0$ identically along every trajectory. In particular, if the Hamiltonian is not explicitly a function of time, then it is conserved. (If there are no constraints, this is the law of conservation of energy.) For any two dynamical variables $A$, $B$ the preceding equation can be used together with the Jacobi identity to show that

$$\frac{d[A, B]}{dt} = \left[\frac{dA}{dt}, B\right] + \left[A, \frac{dB}{dt}\right].$$

It follows that if $A$ and $B$ are conserved, then the Poisson bracket of $A$ and $B$ is conserved. Also any real linear combination of conserved quantities is conserved, and hence the set of all conserved dynamical variables forms a subalgebra of the Lie algebra of all dynamical variables. This subalgebra describes the physical symmetries of the system, and is usually of primary interest. The Lie algebra of all conserved dynamical variables is usually still infinite-dimensional, however, since the algebra is closed not only under the Poisson bracket, but also under ordinary multiplication. One can frequently obtain useful results by working with a finite-dimensional subalgebra. We shall give several examples in the following sections. One fairly general method to obtain finite-dimensional Lie algebras, for example, is to restrict one's attention to conserved currents. The study of conserved currents as generalizations of the conservation of the electric current has recently been of great interest in the strong interactions of elementary particles. Because the underlying dynamics in this case is completely unknown, the use of symmetry considerations has been particularly valuable in the classification of the strongly interacting elementary particles, and in the understanding of their mass spectrum and other simple properties [3], [15].

Although many of the contemporary applications of Lie algebras are relevant to quantum dynamics, there is a close relation between the classical and quantum descriptions. Due to an ingenious discovery by Dirac, there is a simple and convenient correspondence principle which relates classical mechanical Poisson brackets to quantum-mechanical commutators. By using this correspondence principle, we can often discuss symmetries of analogous systems in a parallel fashion. For example, the Kepler problem of planetary motion and the hydrogen atom are analogous systems and so have the same symmetries (the four-dimensional orthogonal group). We shall discuss this further below.

It is of interest to consider quantum mechanics from the point of view of the Lie algebra of dynamical variables. In order to formulate this idea more precisely, we shall be obliged to introduce some of the basic ideas and language of elementary quantum mechanics, and in particular the concept of a Hilbert space of states. In quantum mechanics, physical states are not described by points in phase space; instead the states are described by vectors in a Hilbert space. We may recall, to be absolutely precise, that a *Hilbert space* is simply a vector space (possibly infinite-dimensional) over the field of complex numbers, in which there is defined an inner product, and having the property that every Cauchy sequence of vectors converges to some vector in the space. The inner product of two vectors $\alpha$ and $\beta$ is often denoted by $(\alpha, \beta)$. By an *operator* in Hilbert space we mean a linear transformation whose domain and range are subspaces of the Hilbert space. An operator $H$ is called *Hermitian* if $(H\alpha, \beta) = (\alpha, H\beta)$ for all $\alpha$ and $\beta$ in the domain of $H$. The Hermitian operators in a Hilbert space form a real Lie algebra if the Lie product $[A, B]$ is defined as a purely imaginary multiple of the commutator $AB - BA$. The correspondence principle requires that we define the Lie product as

$$[A, B] = \frac{(BA - AB)2\pi i}{h},$$

where $h$ is Planck's constant. The relation between classical mechanics and quantum mechanics found by Dirac may then be formally described as a homomorphism from the classical Lie algebra of dynamical variables into the Lie algebra of Hermitian operators in a Hilbert space. To each classical dynamical variable, $a$, there corresponds a Hermitian operator, $A$, in Hilbert space. For the constant dynamical variables, the corresponding operators are the constant multiples of the identity operator, $I$. A measurement of the dynamical variable $a$ will yield a definite value, say $\alpha$, only in states which are described by eigenvectors of $A$ with the eigenvalue $\alpha$.

To amplify the above discussion, let us consider the description of a nonrelativistic point particle. If $x_i$ and $p_i$ are the ordinary cartesian coordinates and momenta of the particle, then the classical Poisson brackets are $[x_i, p_j] = \delta_{ij}$. The corresponding operator equation in quantum mechanics must be $[X_i, P_j] = \delta_{ij}I$, where $X_i$ is the operator corresponding to $x_i$, etc. If we take the operator $X_i$ to be the operation of multiplying an appropriate function $\psi(x)$ by $x_i$, and if we take $P_i$ to be $h/(2\pi i)$ times the directional derivative operator $\partial/\partial x_i$, then purely formal computation indicates that these commutation relations are fulfilled. In the Schrödinger formulation of nonrelativistic quantum mechanics, these operators are interpreted as Hermitian operators on an appropriate Hilbert space of equivalence classes of functions. Such an explicit interpretation does not seem to be desirable for relativistic quantum mechanics. One works with the formal commutation relations without asking for an explicit construction of a Hilbert space.

**4.3. Symmetry of the Kepler problem.** The eigenvalues of the Hamiltonian operator are energy levels corresponding to the states of the system which are represented by the corresponding eigenvectors. An eigenvalue is called degenerate when the dimension of the space spanned by the corresponding eigenvectors is greater than one, i.e., more than one state corresponds to the given energy level. Perturbations of the Hamiltonian, such as are produced by the presence of an additional electric or magnetic field, may cause the eigenvalues to separate, so that the degeneracy disappears. The different states then correspond to distinct energy levels, causing what had previously been single lines of the atomic spectrum to split into several lines. The symmetry of the Hamiltonian under a four-dimensional rotation group can be used to analyze the degeneracy of the hydrogen spectrum.

When the Hamiltonian is not explicitly dependent on time there is a general relationship between the existence of conserved quantities and degeneracy of spectra, for in that case conserved dynamical variables commute with the Hamiltonian and the corresponding operators must also commute. This means that any operator corresponding to a conserved dynamical variable maps an eigenvector of the Hamiltonian operator corresponding to a given eigenvalue (energy level) into another eigenvector corresponding to the same eigenvalue. Thus the space spanned by the eigenvectors corresponding to a particular eigenvalue of the Hamiltonian is invariant under the algebra of operators corresponding to conserved dynamical variables. Restricting the operators to this subspace produces a representation of the algebra.

The first derivation of the spectrum of atomic hydrogen within the framework of quantum mechanics was given by Pauli [36], who succeeded in finding the spectrum by diagonalizing the Hamiltonian operator, making use of symmetry of this operator under a certain four-dimensional rotation group. He was also able to obtain by this method the splitting of the levels in a uniform magnetic field (Zeeman effect) and the splitting in a uniform electric field (Stark effect). A more recent discussion of the effect of the presence of an electric field has been given by Redmond [39]. Biedenharn [4] has advocated using the same symmetry group, which he calls the geometrization of the Coulomb field, to solve certain problems involving the Coulomb potential, for instance, the no-energy-loss Coulomb excitation of nuclear levels.

The problem of a nonrelativistic electron moving around the proton in a hydrogen atom is the quantum mechanical analog of the familiar classical problem, dating back to Kepler, of a planet moving about the sun in a central gravitational force field which is inversely proportional to the square of the distance from the sun. The solution of this problem leads to two types of motion: bound orbits, in which the planet traces out an elliptical path with the sun at one focal point, and scattering trajectories, corresponding to parabolic or hyperbolic motion. For the bound orbits, the system is invariant under a certain four-dimensional rotation group, while the appropriate symmetry group for the scattering trajectories turns out to be the Lorentz group.

For definiteness, let us discuss the classical problem, realizing that a completely analogous discussion can be given for the quantum-mechanical problem. The Hamiltonian for the system is

$$H = \frac{p^2}{2m} - \frac{k}{r},$$

where $p$ is relative momentum, $r$ is the radius from the common center of mass, $m$ is the reduced mass, and $k$ is a constant. Due to the manifest rotational invariance of the Hamiltonian, it follows immediately that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved, as one may also verify directly. In addition, there is a less obvious conserved quantity, the *Runge-Lenz vector*, defined by

$$\mathbf{A} = \frac{1}{\sqrt{2m \mid H \mid}} \left( \mathbf{L} \times \mathbf{p} + \frac{km\mathbf{r}}{r} \right).$$

It is a vector with the dimensions of angular momentum pointing along the major axis of the elliptic orbit of the planet. The magnitude of $A$ is equal to $k(m/2 \mid H \mid)^{1/2}$ times the eccentricity of the ellipse. By straightforward calculation one can obtain the Poisson bracket relations

$$[J_\alpha^+, J_\beta^+] = \epsilon_{\alpha\beta\gamma} J_\gamma^+, \qquad [J_\alpha^-, J_\beta^-] = \epsilon_{\alpha\beta\gamma} J_\gamma^-, \qquad [J_\alpha^+, J_\beta^-] = 0,$$

where $J_\alpha^\pm = \frac{1}{2}(L_\alpha \pm A_\alpha)$ and $\epsilon_{\alpha\beta\gamma}$ is $+1$ $(-1)$ if $(\alpha, \beta, \gamma)$ is an even (odd) permutation of $(1, 2, 3)$ and 0 otherwise. Hence $\mathbf{L}$ and $\mathbf{A}$ together generate a Lie algebra which is the direct sum of two ideals, each isomorphic to $so(3, R)$. This Lie algebra may be identified with

$$so(4, R) \cong so(3, R) \oplus so(3, R).$$

Biedenharn's paper deals with the use of the Clebsch-Gordan or Wigner coefficients of the group $SO(4, R)$ for solving problems involving a $1/r$ potential. He discusses how one may use the Wigner-Eckart theorem to express matrix elements of operators in terms of these coefficients.

**4.4. The rotation group and the Lorentz group.** The theory of the rotation group has become increasingly important in nuclear and atomic physics. At present it is the most widely used application of Lie algebra in physics. Textbooks have been devoted to this topic alone.

The quantum-mechanical operator corresponding to the classical angular momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ is obtained by replacing $\mathbf{p}$ by $(h/2\pi i)\boldsymbol{\nabla}$:

$$\mathbf{L} = \frac{h}{2\pi i}\mathbf{r} \times \boldsymbol{\nabla}.$$

This gives us only the orbital angular momentum of a single particle. To obtain the total angular momentum, we must add also the spin $\mathbf{S}$ for which there is no classical analog. Thus for a single particle we have $\mathbf{J} = \mathbf{L} + \mathbf{S}$ for the total angular momentum. One then postulates that the operators $\mathbf{J}$ are Hermitian and satisfy the same commutation relations as $\mathbf{L}$, to wit,

$$J_\alpha J_\beta - J_\beta J_\alpha = i\epsilon_{\alpha\beta\gamma}J_\gamma ,$$

where we have set $h = 2\pi$ for convenience and $\epsilon_{\alpha\beta\gamma}$ is as in §4.3.

The formal theory of angular momentum can be built up from the basis of these commutation relations and the Hermitian requirements alone [12], [40]. One can, however, give a more general treatment by dropping the assumption that $\mathbf{J}$ is Hermitian, which is not essential for most of the theory. When this assumption is dropped, the theory can also be applied to the Lorentz group. In that case one need only study the Lie algebra $A_1$, for the algebra of the homogeneous Lorentz group is just a real form of the semisimple Lie algebra $A_1 \oplus A_1$.

To discuss the representation theory of $A_1$ one customarily defines operators $J_\pm = J_1 \pm iJ_2$ so that the commutation relations become

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_-, \qquad [J_+, J_-] = 2J_3.$$

This is just a special case of the general prescriptions given earlier in §3.5 and illustrated in §3.7. The operator $J_3$ corresponds to the Cartan subalgebra, and the operators $J_\pm$ are just the root vectors $e_{\pm\alpha}$.

If $\psi$ is an eigenvector in Hilbert space such that $J_3\psi = m\psi$, then $J_\pm\psi$ are eigenvectors of $J_3$ with eigenvalues $m \pm 1$. This shows that $J_+$ raises the eigenvalue by one unit and $J_-$ lowers the eigenvalue by one unit; hence we have the terms *raising* and *lowering operators*.

We now discuss the relativistic generalization. In order to obtain the commutation relations for the generators $M^{jk}$ of the homogeneous Lorentz group, we proceed heuristically. We generalize the equation $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ to the equation

$$m^{jk} = x^j p^k - x^k p^j + (\text{spin terms}), \qquad j, k = 0, 1, 2, 3,$$

where $x^0 = ct$, $p^0 = E/c$ ($E$ being total energy and $c$ the speed of light), while the other $x^i$ and $p^i$ are as usual and the spin terms will not be written explicitly. For the corresponding operators, one is led to postulate commutation relations

$$X^jP^k - P^jX^k = ig^{jk},$$

which leads to

$$M^{jk}M^{st} - M^{st}M^{jk} = -i\{g^{ks}M^{jt} - g^{js}M^{kt} + g^{jt}M^{ks} - g^{kt}M^{js}\},$$

where the metric tensor $g^{ij} = \delta^{ij}$ if either $i$ or $j$ is nonzero and $g^{00} = -1$. These commutation relations are the correct relations on which to base the discussion of the homogeneous Lorentz group.

The operator $M^{jk}$ is antisymmetric, so it has six independent nonzero components which include the three components of ordinary angular momentum,

$$J_k = \tfrac{1}{2} \sum_{l,m=1}^{4} \epsilon_{klm} M^{lm},$$

as well as three further components which have been called boosting operators,

$$K_l = M^{l0}.$$

The operators $\mathbf{J}$ are the generators of pure rotations, while the operators $\mathbf{K}$ are the generators of pure Lorentz transformations.

The commutation relations acquire a particularly simple form if we set

$$\mathbf{J}^{\pm} = \tfrac{1}{2}(\mathbf{J} \pm i\mathbf{K}).$$

We then find that the complex Lie algebra of the Lorentz group is just $A_1 \oplus A_1$ :

$$[J_m^{\pm}, J_n^{\pm}] = i\epsilon_{mnp} J_p^{\pm}, \qquad [J_m^{+}, J_n^{-}] = 0.$$

Note that although $J$ and $K$ are Hermitian operators, the operators $J^{\pm}$ are not. (As a consequence of this, the Lorentz group can have infinite-dimensional irreducible unitary representations [34]. The rotation group, being compact, can have only finite-dimensional irreducible unitary representations [37]. The physical significance of the infinite-dimensional representations is at present debatable.)

The theory of the Lorentz group which finds more practical application in elementary particle physics is that of the inhomogeneous Lorentz group. The Lie algebra of this group is spanned by the ten operators of angular momentum $M^{\mu\nu}$ and momentum $P^{\nu}$. This theory leads to the so-called *helicity representation*, which is much used for partial-wave analyses in high energy scattering experiments.

**4.5. Harmonic oscillators and currents.** As we discussed in §4.2, Lie algebras arise in the study of conservation laws in classical mechanics. The modern theory of conservation laws must be formulated in terms of quantum field theory. Harmonic oscillators play a fundamental role in quantum mechanics mainly because the theory of the harmonic oscillator underlies the formal apparatus of second quantization. Lie algebras arise naturally as currents in quantum field theory.

We begin by discussing harmonic oscillators. The unitary groups arise naturally in the discussion of harmonic oscillators. We adopt the quantum-mechanical point of view because it is simpler, but the same type of discussion can be given in classical mechanics. The Hamiltonian for a system of $n$ identical noninteracting harmonic oscillators is

$$H = \sum_{\alpha=1}^{n} \left\{ \frac{p_\alpha^2}{2m} + \frac{k}{2} q_\alpha^2 \right\}.$$

For convenience we choose units in which $m = k = 1$ as well as $h = 2\pi$.

We define creation and annihilation operators by

$$a_\alpha = \frac{q_\alpha + ip_\alpha}{\sqrt{2}}, \qquad a_\alpha^* = \frac{q_\alpha - ip_\alpha}{\sqrt{2}},$$

in terms of which the Hamiltonian may be written as

$$H = (\text{const.}) + \sum_{\alpha=1}^{n} a_\alpha^* a_\alpha .$$

The creation and annihilation operators satisfy the commutation relations

$$[a_\alpha, a_\beta] = 0, \qquad [a_\alpha, a_\beta^*] = \delta_{\alpha\beta}, \qquad [a_\alpha^*, a_\beta^*] = 0,$$

with each other, and

$$[H, a_\alpha^*] = a_\alpha^*, \qquad [H, a_\alpha] = -a_\alpha,$$

with the Hamiltonian. These latter commutation relations imply that if $\psi$ is an eigenvector of $H$ with eigenvalue $E$, then $a_\alpha\psi$ is an eigenvector with eigenvalue $E - 1$, and $a_\alpha^*\psi$ is an eigenvector with eigenvalue $E + 1$.

Both the Hamiltonian and the commutation relations are invariant under a unitary transformation,

$$a_\alpha' = \sum_\beta U_{\alpha\beta} a_\beta ,$$

so $U(n)$ is a symmetry group of the $n$-dimensional harmonic oscillator. The corresponding Lie algebra $u(n)$ is generated by the elements $a_\alpha^* a_\beta$, which commute with the Hamiltonian $H$ [1], [2].

We now discuss the conservation of currents. In quantum field theory one introduces for each type of particle a set of *creation and annihilation operators* on a Hilbert space (the Fok space). If $\psi$ is a certain state (vector in Hilbert space), then the state $a_j^*\psi$ is another state which differs from the original state by the addition of a single particle of type $j$. Similarly, $a_j\psi$ is the state obtained by removing a single particle of type $j$.

Let $a_1, \cdots, a_n$ denote a set of annihilation operators; their Hermitian conjugates $a_1^*, \cdots, a_n^*$ are creation operators. If we limit discussion to half-integer spin particles (fermions), then these operators satisfy the anticommutation relations,

$$\{a_i, a_j\} = 0, \qquad \{a_i, a_j^*\} = \delta_{ij}, \qquad \{a_i^*, a_j^*\} = 0,$$

where $\{x, y\} = xy + yx$. The physical significance of these relations is the *Pauli exclusion principle*: no two fermions can occupy the same quantum state. Similarly, if the creation and annihilation operators refer to integer spin particles (bosons) then these operators satisfy commutation relations which are identical to the commutation relations written down in the case of a harmonic oscillator.

One can treat the case of fermions and the case of bosons simultaneously. In either case one can verify the commutation relations,

$$[a_i{}^* a_j\,,\; a_k{}^* a_l] = \delta_{jk} a_i{}^* a_l - \delta_{il} a_k{}^* a_j\,,$$

which form the basis for our discussion. These relations show that the elements $a_i{}^* a_j$ always span a Lie algebra, called the *Lie algebra of currents* [16].

Any operator of the form

$$J = \sum_{i,j} a_i{}^* c_{ij} a_j\,,$$

where the $c_{ij}$ are complex numbers, will be called a (generalized) *current.* One reason for this name is that ordinary electric current corresponds to such an operator in quantum field theory. There are, however, many other currents, for example, the total spin of a many-electron system and the total kinetic energy of a system of particles. A current $J$ which does not depend explicitly on the time is said to be a *conserved current* if it commutes with the Hamiltonian, $[H, J] = 0$. The set of conserved currents forms a Lie algebra. It is just this Lie algebra which one studies, for instance, in elementary particle physics.

The formula for the current defines a homomorphism from any Lie algebra of $n \times n$ matrices to the Lie algebra of currents. Let us write the definition above in matrix notation as $J = a^*Ca$. If $K$ is another current, say $K = a^*Da$, then $[J, K] = a^*[C, D]a$ follows directly from the commutation relations written down for $a_i{}^* a_j$ with $a_k{}^* a_l$, so the mapping $h: C \to a^*Ca$ is a homomorphism.

**4.6. Lie algebras and special functions.** A considerable unification of the theory of the special functions can be achieved by the use of Lie algebraic ideas. The types of functions that can be considered include Bessel functions, Hermite polynomials, parabolic cylinder functions, and Legendre polynomials. A systematic catalog of these functions has been given in a paper by Infield and Hull [24]. The method used by these authors involved factorization of the appropriate second order differential equation. For the Hermite polynomials (related to the harmonic oscillator) this leads to the creation and annihilation operators considered in §4.5.

Infield and Hull were able to reduce the entire list of possibilities to six overlapping classes. More recently, Miller [31] has shown that all of these six classes arise naturally in the representation theory of four particular Lie algebras, namely, the Lie algebras of the three-dimensional rotation group, the group of Euclidean motions in a plane, the Euclidean group in three-dimensional space, and a certain four-dimensional solvable Lie algebra.

We illustrate the general technique by a simple example showing how the theory of Bessel functions is related to the Lie algebra of the group of Euclidean motions in a plane [42], [51]. The Lie algebra of this group is three-dimensional. Any motion can be composed of a rotation and a translation, the angle $\phi$ of the rotation and the orthogonal components $(x, y)$ of the translation give local coordinates for a neighborhood of the identity. The tangent vectors corresponding to the subgroup of translations in the $x$ direction will be denoted $\mathbf{T}_x$, that corresponding to translations in the $y$ direction $\mathbf{T}_y$, while the tangent vector to the subgroup of rotations will be $\mathbf{L}$. The vectors $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{L}$ are a basis for the Lie algebra. As in previous sections, we can identify the vectors with operators.

Setting

$$\mathbf{T}_x = \frac{\partial}{\partial x}, \qquad \mathbf{T}_y = \frac{\partial}{\partial y}, \qquad \mathbf{L} = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial \phi}$$

yields an isomorphism of the Lie algebra of the group with a Lie algebra of operators on the infinitely differentiable functions on the plane, the Lie multiplication being, as usual, commutation. If we define

$$\mathbf{T}_\pm = \mathbf{T}_x \pm i\mathbf{T}_y = \frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y} = \exp(\pm i\phi)\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial \phi}\right),$$

then we find that the commutation relations take the particularly simple form,

$$[\mathbf{T}_+, \mathbf{T}_-] = 0, \qquad [\mathbf{L}, \mathbf{T}_\pm] = \pm\mathbf{T}_\pm .$$

Note that $\mathbf{T}_+\mathbf{T}_- = \mathbf{T}_-\mathbf{T}_+ = \nabla^2$ corresponds to the Laplacian operator. If we set

$$\psi_m = \psi_m(r, \phi) = \exp(im\phi)\, J_m(r),$$

then the differential equations which define the Bessel functions may be written

$$(\mathbf{T}_+\mathbf{T}_- + 1)\psi_m = 0, \qquad \mathbf{L}\psi_m = m\psi_m .$$

Thus $\psi_m$ is an eigenvector of $\mathbf{L}$. The operators $\mathbf{T}_\pm$ are raising and lowering operators,

$$\mathbf{T}_\pm\psi_m = \mp\, \psi_{m\pm1} .$$

These equations yield the recursion relations for the Bessel functions,

$$rJ_m'(r) = \pm(mJ_m(r) - rJ_{m\pm1}(r)).$$

Many other properties of Bessel functions can be obtained by use of the Lie algebra; for example, addition formulas can be obtained by using the Baker-Hausdorff formulas. Weisner [46] has shown how to obtain generating functions for special functions by using Lie algebraic ideas.

**4.7. Pattern recognition.*** Recently, Hoffman [22] has made an attempt to apply the theory of Lie groups to the pattern recognition problem. On the basis of physiological studies of electrical activity in the brain, he postulates that the visual integrative process is based on a first order differential equation, with electrical patterns in the brain embodying information about the isoclines of the differential equation. In order for recognition of the pattern as an entity (as required by the Gestalt theory) to occur, it is postulated that the differential equation involved must be invariant under the group of transformations which leave the pattern recognizable. Hoffman is able to determine the class of differential equations which remain invariant under the Lie group which includes translations, rotations, and magnifications. The results of this analysis are used to explain and interpret various visual phenomena, including developmental

* The authors are indebted to Dr. L. Kanal and Mr. L. Buchsbaum for bringing this application to their attention.

dyslexia, the whirling spirals evoked under flicker, the alpha rhythm and its desynchronization, and Mackay's complementary after-images.

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