

AN EXISTENCE THEOREM FOR G -STRUCTURE PRESERVING AFFINE IMMERSIONS

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ABSTRACT. We prove an existence result for local and global G -structure preserving affine immersions between affine manifolds. Several examples are discussed in the context of Riemannian and semi-Riemannian geometry, including the case of isometric immersions into Lie groups endowed with a left-invariant metric, and the case of isometric immersions into products of space forms.

1. INTRODUCTION

In this paper we prove an existence result for G -structure preserving affine immersions between affine manifolds, with special attention to the class of isometric immersions in the context of Riemannian and semi-Riemannian geometry. The original idea was to find a unifying language for several isometric immersion theorems that appear in the classical literature [2] (immersions into Riemannian manifolds with constant sectional curvature, immersions into Kähler manifolds of constant holomorphic curvature), and also some recent results (see for instance [3, 4]) concerning the existence of isometric immersions in more general Riemannian manifolds. Given an isometric immersion, the celebrated equations of Gauss, Codazzi and Ricci relate the curvature of the environment with the curvature of the submanifold, the curvature of the normal bundle and the second fundamental form (and its covariant derivative). A folk theorem says that such equations are necessary conditions for the existence of isometric immersions, however the reader should observe that, unless the isometric immersion has already been given, the equations cannot in general even be written down. Nevertheless, when the target manifold is “sufficiently homogeneous” (or, more precisely, *infinitesimally homogeneous* in the sense of Definition 6.1), the Gauss, Codazzi and Ricci equations do make sense *a priori* and then they are indeed necessary conditions for the existence of the isometric immersion. Sufficient conditions for the existence of an isometric immersion involve additional assumptions depending on the context; the starting point of our theory was precisely the interpretation of such additional assumptions in terms of G -structures and *inner torsion*, which is a kind of covariant derivative of a G -structure.

The central result of the paper is an affine immersion theorem into infinitesimally homogeneous affine manifolds endowed with a G -structure. *Infinitesimally*

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homogeneous means that the curvature and the torsion of the connection, as well as the inner torsion of the G -structure, can be written uniquely in terms of the G -structure, i.e., are constant in frames that belong to the G -structure. For instance, consider the case that M is a Riemannian manifold endowed with the Levi-Civita connection of its metric tensor, G is the orthogonal group and the G -structure is given by the set of orthonormal frames. Since the Levi-Civita connection is compatible with the Riemannian metric, the inner torsion of this G -structure is zero. The condition that the curvature tensor should be constant in orthonormal frames is equivalent to the condition that M has constant sectional curvature, and we recover in this case the classical “fundamental theorem of isometric immersions in spaces of constant curvature”. Similarly, if M is a Riemannian manifold endowed with an orthogonal almost complex structure, then one has a G -structure on M , where G is the unitary group, by considering the set of orthonormal complex frames of TM . In this case, the inner torsion of the G -structure relatively to the Levi-Civita connection of the Riemannian metric is the covariant derivative of the almost complex structure, which vanishes if and only if M is Kähler. Requiring that the curvature tensor be constant in orthonormal complex frames means that M has constant holomorphic curvature; in this context, our immersion theorem reproduces the classical result of isometric immersions into Kähler manifolds of constant holomorphic curvature. Another interesting example of G -structure that will be considered in some detail is the case of Riemannian manifolds endowed with a distinguished unit vector field ξ ; in this case, we obtain an immersion theorem into Riemannian manifolds with the property that both the curvature tensor and the covariant derivative of the vector field at a general point p can be written in terms only of the Riemannian metric at p and of the vector $\xi(p)$. This is the case in a number of important examples, like for instance all manifolds that are Riemannian products of a space form with a copy of the real line, as well as all homogeneous, simply-connected 3-dimensional manifolds whose isometry group has dimension 4. These examples were first considered in [3]. Two more examples will be studied in some detail. First, we will consider isometric immersions into Lie groups endowed with a left invariant semi-Riemannian metric tensor. These manifolds have an obvious 1-structure, given by the choice of a distinguished orthonormal left invariant frame; clearly, the curvature tensor is constant in this frame. Moreover, the inner torsion of the structure is simply the Christoffel tensor associated to this frame, which is also constant. It should be observed that a different immersion theorem into a class of nilpotent and solvable Lie groups has been recently proved in [8]. In spite of many analogies both in the statement and in the proof of the result, the setup considered by the author in [8] does not fit into the infinitesimally homogeneous case considered in the present paper. Another example discussed is the case of isometric immersions into products of manifolds with constant sectional curvature; in this situation, the G -structure considered is the one consisting of orthonormal frames adapted to such product. More generally, products of infinitesimally homogeneous affine manifolds with G -structures are infinitesimally homogeneous.

The proof of (the local version of) the main theorem relies on an application of Frobenius in the language of differential forms. More precisely, assume that we

are given affine manifolds (M, ∇) , $(\overline{M}, \overline{\nabla})$ and a vector bundle E^0 over M . Given a suitable set of data (a connection ∇^0 on E^0 , and “second fundamental forms” α^0 and A^0) we assemble a connection $\widehat{\nabla}$ on the Whitney sum $\widehat{E} = TM \oplus E$ and we look for an immersion $f : M \rightarrow \overline{M}$ and a connection preserving vector bundle isomorphism $L : \widehat{E} \rightarrow f^*T\overline{M}$ such that $L|_{TM} = df$. We assume that G -structures \widehat{P} and \overline{P} are given on \widehat{E} and on $T\overline{M}$, respectively, and we require that L be G -structure preserving. Given a smooth local frame of \widehat{E} (in \widehat{P}), the problem of determining L is reduced to the problem of determining a smooth map $F : U \rightarrow \overline{P}$, with U open in M , such that F pulls back the canonical form and the connection form of \overline{P} to, respectively, the canonical form and the connection form of \widehat{P} . We then employ a version of the Frobenius theorem that allows one to guarantee the existence of a smooth map F satisfying a PDE of the form $F^*\lambda^{\overline{M}} = \lambda^M$, where $\lambda^M, \lambda^{\overline{M}}$ are vector-valued 1-forms taking values in the same vector space. The integrability condition for such PDE corresponds to the Gauss, Codazzi and Ricci equations, as well as to some *torsion equations*; moreover, the condition that $\lambda^M, \lambda^{\overline{M}}$ take value in the same vector space corresponds to an equation relating the inner torsions of \widehat{P} and of \overline{P} . Finally, the proof of the global version of the affine immersion theorem employs a general globalization principle stated in the language of pre-sheafs.

2. NOTATIONS AND TERMINOLOGY

Vector spaces. Let V be a real finite-dimensional vector space. We denote by $GL(V)$ the general linear group of V and by $\mathfrak{gl}(V)$ its Lie algebra. If W is another real finite-dimensional vector space and $p : V \rightarrow W$ is a linear isomorphism then $\mathcal{I}_p : GL(V) \rightarrow GL(W)$ denotes the Lie group isomorphism given by conjugation with p and $\text{Ad}_p = d\mathcal{I}_p(\text{Id}) : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(W)$ denotes the Lie algebra isomorphism given by conjugation with p . By $\text{Lin}(V, W)$ we denote the space of linear maps from V to W .

Vector bundles, frame bundles and connections. Let E be a vector bundle over a differentiable manifold M . We denote by $\Gamma(E)$ the set of all smooth sections of E . Given a connection ∇ on E then the *curvature* of ∇ is the smooth tensor $R \in \Gamma(TM^* \otimes TM^* \otimes E^* \otimes E)$ defined by:

$$R(X, Y)\epsilon = \nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon - \nabla_{[X, Y]}\epsilon,$$

for all $X, Y \in \Gamma(TM)$, $\epsilon \in \Gamma(E)$; if $\iota : TM \rightarrow E$ is a vector bundle morphism then the ι -*torsion* of ∇ is the smooth tensor $T^\iota \in \Gamma(TM^* \otimes TM^* \otimes E)$ defined by:

$$T^\iota(X, Y) = \nabla_X(\iota(Y)) - \nabla_Y(\iota(X)) - \iota([X, Y]),$$

for all $X, Y \in \Gamma(TM)$. When $E = TM$ and ι is the identity $T = T^\iota$ coincides with the usual *torsion* of ∇ .

Let k be the *rank* of E , i.e., the dimension of the fibers of E . We denote by $\text{FR}(E) = \bigcup_{x \in M} \text{FR}(E_x)$ the frame bundle of E , which is the set of all linear

isomorphisms $p : \mathbb{R}^k \rightarrow E_x$, with $x \in M$. The frame bundle $\text{FR}(E)$ is a $\text{GL}(\mathbb{R}^k)$ -principal bundle over M . A local section $s : U \rightarrow \text{FR}(E)$ (where U is an open subset of M) is called a *local frame* for E . A smooth local frame $s : U \rightarrow \text{FR}(E)$ defines a connection \mathbb{d}^s on $E|_U$ which corresponds via the trivialization of $E|_U$ defined by s to the standard derivative. More explicitly, we set:

$$\mathbb{d}_v^s \epsilon = s(x)(d\tilde{\epsilon}_x(v)),$$

for all $x \in U$, $v \in T_x M$, $\epsilon \in \Gamma(E|_U)$, where $\tilde{\epsilon} : U \rightarrow \mathbb{R}^k$ is defined by $\tilde{\epsilon}(x) = s(x)^{-1}(\epsilon(x))$, for all $x \in U$. If ∇ is a connection in E then the *Christoffel tensor* of ∇ with respect to the smooth local frame s is the tensor $\Gamma = \nabla - \mathbb{d}^s$; more explicitly, $\Gamma : U \rightarrow TM^* \otimes E^* \otimes E$ is the smooth local section such that:

$$\nabla_v \epsilon = \mathbb{d}_v^s \epsilon + \Gamma_x(v, \epsilon(x)),$$

for all $x \in U$, $v \in T_x M$ and all $\epsilon \in \Gamma(E|_U)$. For $x \in U$, $v \in T_x M$, we also set $\Gamma_x(v) = \Gamma_x(v, \cdot) \in \mathfrak{gl}(E_x)$, so that $\Gamma_x : T_x M \rightarrow \mathfrak{gl}(E_x)$ is a linear map. If Hor is the horizontal distribution in $\text{FR}(E)$ corresponding to ∇ and ω is the $\mathfrak{gl}(\mathbb{R}^k)$ -valued connection form on $\text{FR}(E)$ whose kernel is Hor then, setting $\bar{\omega} = s^* \omega$, we have:

$$(2.1) \quad \Gamma_x(v) = \text{Ad}_{s(x)}(\bar{\omega}_x(v)),$$

for all $x \in U$ and all $v \in T_x M$. If $\iota : TM \rightarrow E$ is a vector bundle morphism then the ι -*canonical form* of $\text{FR}(E)$ is the \mathbb{R}^k -valued 1-form θ on $\text{FR}(E)$ defined by:

$$\theta_p(\zeta) = p^{-1}[\iota(d\Pi_p(\zeta))] \in \mathbb{R}^k,$$

for all $p \in \text{FR}(E)$, $\zeta \in T_p \text{FR}(E)$, where $\Pi : \text{FR}(E) \rightarrow M$ denotes the projection. When $E = TM$ and ι is the identity then θ is simply the usual *canonical form* of $\text{FR}(TM)$. The ι -*torsion form* Θ and the *curvature form* Ω are defined respectively by:

$$(2.2) \quad \Theta = d\theta + \omega \wedge \theta, \quad \Omega = d\omega + \omega \wedge \omega,$$

where the wedge product in $\omega \wedge \omega$ is taken with respect to the associative product of $\mathfrak{gl}(\mathbb{R}^k)$. The following equalities hold:

$$(2.3) \quad p(\Theta_p(\zeta_1, \zeta_2)) = T_x^\iota(d\Pi_p(\zeta_1), d\Pi_p(\zeta_2)) \in E_x,$$

$$(2.4) \quad \text{Ad}_p(\Omega_p(\zeta_1, \zeta_2)) = R_x(d\Pi_p(\zeta_1), d\Pi_p(\zeta_2)) \in \mathfrak{gl}(E_x),$$

for all $p \in \text{FR}(E)$, $\zeta_1, \zeta_2 \in T_p \text{FR}(E)$, where $x = \Pi(p)$.

Covariant derivative along curves. Let $p : I \rightarrow \text{FR}(E)$ be a smooth curve and set $\gamma = \Pi \circ p$, where $\Pi : \text{FR}(E) \rightarrow M$ denotes the projection. For all $t \in I$, we denote by $(\nabla_1 p)(t) \in \text{Lin}(\mathbb{R}^k, E_{\gamma(t)})$ the *covariant derivative* of p at the instant t , which is just the vertical component of $p'(t) \in T_{p(t)} \text{FR}(E)$ (observe that the vertical space of $\text{FR}(E)$ at $p(t)$ is identified with $\text{Lin}(\mathbb{R}^k, E_{\gamma(t)})$). We have:

$$(2.5) \quad \omega_{p(t)}(p'(t)) = p(t)^{-1} \circ (\nabla_1 p)(t),$$

for all $t \in I$. If $\varepsilon : I \rightarrow E$ is a smooth section of E along γ (i.e., $\varepsilon(t) \in E_{\gamma(t)}$, for all $t \in I$), we denote by $(\nabla_1 \varepsilon)(t) \in E_{\gamma(t)}$ the covariant derivative of ε along γ at

the instant t . Given a smooth curve $u : I \rightarrow \mathbb{R}^k$ then $\varepsilon : I \ni t \mapsto p(t) \cdot u(t) \in E_{\gamma(t)}$ is a smooth section of E along γ and the following ‘‘Leibniz rule’’ holds:

$$(\nabla_1 \varepsilon)(t) = (\nabla_1 p)(t) \cdot u(t) + p(t) \cdot u'(t),$$

for all $t \in I$. If $s : U \rightarrow \text{FR}(E)$ is a smooth local section with $\gamma(I) \subset U$ and if $\bar{\omega} = s^* \omega$ then:

$$(2.6) \quad (\nabla_1 p)(t) = s(\gamma(t)) \circ [\tilde{p}'(t) + \bar{\omega}_{\gamma(t)}(\gamma'(t)) \circ \tilde{p}(t)],$$

for all $t \in I$, where $\tilde{p} : I \rightarrow \text{GL}(\mathbb{R}^k)$ is defined by $\tilde{p}(t) = s(\gamma(t))^{-1} \circ p(t)$.

Vector subbundles. If F is a vector subbundle of E then the *absolute second fundamental form* of F in E with respect to the connection ∇ is the tensor $\alpha^F \in \Gamma(TM^* \otimes F^* \otimes (E/F))$ defined by $\alpha^F(X, \epsilon) = \mathfrak{q}(\nabla_X \epsilon)$, for all $X \in \Gamma(TM)$, $\epsilon \in \Gamma(F)$, where $\mathfrak{q} : E \rightarrow E/F$ denotes the quotient map.

3. AFFINE IMMERSIONS AND THEIR INVARIANTS

Let (M, ∇) , $(\bar{M}, \bar{\nabla})$ be affine manifolds and $f : M \rightarrow \bar{M}$ be a smooth immersion. We identify the differential $df : TM \rightarrow T\bar{M}$ with a (injective) vector bundle morphism $df : TM \rightarrow f^*T\bar{M}$. A vector subbundle E of $f^*T\bar{M}$ with $f^*T\bar{M} = df(TM) \oplus E$ will be called a *normal bundle* for f . Let a normal bundle E for f be fixed; we denote by $\pi^E : f^*T\bar{M} \rightarrow E$ the projection onto E corresponding to the decomposition $f^*T\bar{M} = df(TM) \oplus E$ and by $\pi^{TM} : f^*T\bar{M} \rightarrow TM$ the composition of the projection onto $df(TM)$ with the isomorphism $df^{-1} : df(TM) \rightarrow TM$. Given smooth vector fields $X, Y \in \Gamma(TM)$ in M , we set $\alpha(X, Y) = \pi^E(\bar{\nabla}_X df(Y)) \in \Gamma(E)$, so that α is identified with a smooth section of $TM^* \otimes TM^* \otimes E$. We call α the *second fundamental form* of the immersion f with respect to the normal bundle E . Notice that in the case of Riemannian (or semi-Riemannian) geometry, f has a canonical normal bundle (the orthogonal complement of $df(TM)$ with respect to the metric), so there is also a canonical notion of second fundamental form. Also, if (M, g) and (\bar{M}, \bar{g}) are semi-Riemannian, ∇ and $\bar{\nabla}$ are the corresponding Levi-Civita connections, f is an isometric immersion and E is the orthogonal complement of $df(TM)$ in $f^*T\bar{M}$ with respect to \bar{g} then:

$$(3.1) \quad \bar{\nabla}_X df(Y) = df(\nabla_X Y) + \alpha(X, Y),$$

for all $X, Y \in \Gamma(TM)$. In the general affine case, we say that f is an *affine immersion with respect to E* if (3.1) holds, for all $X, Y \in \Gamma(TM)$. Following [11] we say simply that f is an *affine immersion* if there exists a normal bundle E for f such that f is an affine immersion with respect to E . We define the *normal connection* ∇^\perp of the immersion f corresponding to the normal bundle E by setting $\nabla_X^\perp \epsilon = \pi^E(\bar{\nabla}_X \epsilon)$, for all $X \in \Gamma(TM)$ and all $\epsilon \in \Gamma(E)$. In the semi-Riemannian case, α and ∇^\perp are the only invariants associated to an isometric immersion. In the general affine case, we have an additional invariant associated to the immersion. We set $A(X, \epsilon) = \pi^{TM}(\bar{\nabla}_X \epsilon)$, for all $X \in \Gamma(TM)$ and all $\epsilon \in \Gamma(E)$; clearly A is identified with a smooth section of $TM^* \otimes E^* \otimes TM$. We call A the *Weingarten*

form of the immersion f relatively to E and for all $x \in M$ and all $e \in E_x$, the linear endomorphism $A(e) = A_x(\cdot, e) : T_xM \rightarrow T_xM$ is called the *Weingarten operator* in the direction of e . In the semi-Riemannian case α and A are related by the equality:

$$\bar{g}_x(\alpha_x(v, w), e) = -\bar{g}_x(A(e) \cdot v, w), \quad x \in M, v, w \in T_xM, e \in E_x,$$

so that A is determined by α . In the affine case, there is no relation between α and A ; moreover, α is not in general symmetric unless the connection $\bar{\nabla}$ is symmetric.

We are interested in studying the existence of affine immersions with prescribed invariants ∇^\perp , α and A . More precisely, let (M, ∇) , $(\bar{M}, \bar{\nabla})$ be affine manifolds, E^0 be a vector bundle over M , ∇^0 be a connection in E^0 and α^0, A^0 be smooth sections of $TM^* \otimes TM^* \otimes E^0$ and $TM^* \otimes (E^0)^* \otimes TM$ respectively. We look for an affine immersion $f : M \rightarrow \bar{M}$, a normal bundle E for f and a connection preserving vector bundle isomorphism $S : (E^0, \nabla^0) \rightarrow (E, \nabla^\perp)$ such that $S(\alpha^0(\cdot, \cdot)) = \alpha$ and $A(\cdot, S \cdot) = A^0$. The pair (f, S) will be called a *solution for the affine immersion problem with data ∇^0, α^0 and A^0* . More generally, if $f : U \rightarrow \bar{M}$ is an affine immersion defined in an open subset U of M and $S : E^0|_U \rightarrow E|_U$ is a connection preserving vector bundle isomorphism such that $S(\alpha^0(\cdot, \cdot)) = \alpha$ and $A(\cdot, S \cdot) = A^0$ then the pair (f, S) will be called a *local solution for the affine immersion problem with data ∇^0, α^0, A^0 and with domain U* .

An important special situation is the one of isometric immersions. Assume that (M, g) , (\bar{M}, \bar{g}) are semi-Riemannian manifolds, E^0 is a vector bundle over M endowed with a semi-Riemannian structure g^0 (i.e., g^0 is a smooth section of $(E^0)^* \otimes (E^0)^*$ and g_x^0 is a nondegenerate symmetric bilinear form on E_x^0 , for all $x \in M$), g^0 is ∇^0 -parallel and α^0 is a smooth symmetric section of the vector bundle $TM^* \otimes TM^* \otimes E^0$; by a *solution for the isometric immersion problem with data ∇^0, α^0, g^0* we mean a pair (f, S) where $f : M \rightarrow \bar{M}$ is an isometric immersion, $S : (E^0, \nabla^0, g^0) \rightarrow (E, \nabla^\perp, g^\perp)$ is a connection preserving vector bundle isometry and $S(\alpha^0(\cdot, \cdot)) = \alpha$, where E denotes the orthogonal complement of $df(TM)$ in $f^*T\bar{M}$ with respect to \bar{g} and g^\perp denotes the restriction of \bar{g} to E . As in the affine case, one defines the concept of *local solution for the isometric immersion problem* by replacing M with an open subset U of M .

Notice that if (f, S) is a (local) solution for the isometric immersion problem, (M, g) and (\bar{M}, \bar{g}) are endowed with their respective Levi-Civita connections ∇ , $\bar{\nabla}$ and a smooth section A^0 of $TM^* \otimes (E^0)^* \otimes TM$ is defined by the equality:

$$(3.2) \quad g_x^0(\alpha_x^0(v, w), e) = -g_x(A_x^0(e) \cdot v, w), \quad x \in M, v, w \in T_xM, e \in E_x^0,$$

then (f, S) is also a (local) solution for the affine immersion problem with data ∇^0, α^0 and A^0 .

4. THE COMPONENTS OF A CONNECTION

The following general construction gives a convenient language for discussing the theory of affine immersions. Let \mathcal{E} be a vector bundle over a differentiable manifold M endowed with a connection ∇ and let $\mathcal{E} = \mathcal{E}^1 \oplus \mathcal{E}^2$ be a direct sum

decomposition of \mathcal{E} ; denote by $\pi^i : \mathcal{E} \rightarrow \mathcal{E}^i, i = 1, 2$, the projections. We set:

$$\begin{aligned} \nabla_X^i \epsilon_i &= \pi^i(\nabla_X \epsilon_i), \quad i = 1, 2 \\ \alpha^2(X, \epsilon_1) &= \pi^2(\nabla_X \epsilon_1), \quad \alpha^1(X, \epsilon_2) = \pi^1(\nabla_X \epsilon_2), \end{aligned}$$

for all $X \in \Gamma(TM), \epsilon_i \in \Gamma(\mathcal{E}^i), i = 1, 2$, so that ∇^i is a connection in $\mathcal{E}^i, i = 1, 2$, and α^2 (resp., α^1) is identified with a smooth section of $TM^* \otimes (\mathcal{E}^1)^* \otimes \mathcal{E}^2$ (resp., of $TM^* \otimes (\mathcal{E}^2)^* \otimes \mathcal{E}^1$). We call $\nabla^1, \nabla^2, \alpha^2$ and α^1 the *components* of the connection ∇ with respect to the decomposition $\mathcal{E} = \mathcal{E}^1 \oplus \mathcal{E}^2$. Clearly, one recovers ∇ from its components using the formula:

$$\nabla_X \epsilon = \nabla_X^1(\pi^1(\epsilon)) + \alpha^1(X, \pi^2(\epsilon)) + \nabla_X^2(\pi^2(\epsilon)) + \alpha^2(X, \pi^1(\epsilon)),$$

where $X \in \Gamma(TM)$ and $\epsilon \in \Gamma(\mathcal{E})$.

Notice that if $f : (M, \nabla) \rightarrow (\overline{M}, \overline{\nabla})$ is an affine immersion with respect to a normal bundle E then, identifying for a moment TM with $df(TM)$, the components of the connection $f^*\overline{\nabla}$ on $f^*T\overline{M}$ with respect to the decomposition $f^*T\overline{M} = df(TM) \oplus E$ are the connection ∇ of M , the normal connection ∇^\perp , the second fundamental form α and the Weingarten form A of f with respect to E .

Remark 4.1. If g^1, g^2 are semi-Riemannian structures on E^1 and E^2 , respectively and if g is the semi-Riemannian structure on E given by the orthogonal direct sum of g^1 and g^2 then a connection ∇ with components $\nabla^1, \nabla^2, \alpha^2, \alpha^1$ is compatible with g (i.e., $\nabla g = 0$) if and only if ∇^i is compatible with $g^i, i = 1, 2$, and the following relation between α^2 and α^1 holds:

$$(4.1) \quad g_x(\alpha_x^2(v, e_1), e_2) + g_x(e_1, \alpha_x^1(v, e_2)) = 0,$$

for all $x \in M, v \in T_x M, e_1 \in \mathcal{E}_x^1$ and $e_2 \in \mathcal{E}_x^2$. Notice that relation (4.1) implies that α^1 is uniquely determined from α^2 , so that in a context where we are dealing with connections compatible with a semi-Riemannian structure, we will talk only about the components ∇^1, ∇^2 and α^2 of ∇ , where one should understand implicitly that α^1 is determined by condition (4.1).

Denote by R, R^1, R^2 the curvature tensors of ∇, ∇^1 and ∇^2 , respectively. A straightforward computation gives the following:

$$(4.2) \quad \pi^1(R_x(v, w)e_1) = R_x^1(v, w)e_1 + \alpha_x^1(v, \alpha_x^2(w, e_1)) - \alpha_x^1(w, \alpha_x^2(v, e_1)),$$

$$(4.3) \quad \pi^2(R_x(v, w)e_2) = R_x^2(v, w)e_2 + \alpha_x^2(v, \alpha_x^1(w, e_2)) - \alpha_x^2(w, \alpha_x^1(v, e_2)),$$

for all $x \in M, e_1 \in \mathcal{E}_x^1, e_2 \in \mathcal{E}_x^2$ and all $v, w \in T_x M$. Moreover, given a connection ∇^M on TM with torsion T and denoting by ∇^\otimes the induced connections on $TM^* \otimes (\mathcal{E}^2)^* \otimes \mathcal{E}^1$ and on $TM^* \otimes (\mathcal{E}^1)^* \otimes \mathcal{E}^2$ then:

$$(4.4) \quad \begin{aligned} \pi^2(R_x(v, w)e_1) &= (\nabla^\otimes \alpha^2)_x(v, w, e_1) - (\nabla^\otimes \alpha^2)_x(w, v, e_1) \\ &\quad + \alpha_x^2(T_x(v, w), e_1), \end{aligned}$$

$$(4.5) \quad \begin{aligned} \pi^1(R_x(v, w)e_2) &= (\nabla^\otimes \alpha^1)_x(v, w, e_2) - (\nabla^\otimes \alpha^1)_x(w, v, e_2) \\ &\quad + \alpha_x^1(T_x(v, w), e_2), \end{aligned}$$

for all $x \in M$, $e_1 \in \mathcal{E}_x^1$, $e_2 \in \mathcal{E}_x^2$ and all $v, w \in T_x M$. If $\iota = (\iota^1, \iota^2) : TM \rightarrow \mathcal{E} = \mathcal{E}^1 \oplus \mathcal{E}^2$ is a vector bundle morphism then the ι -torsion T^ι of ∇ satisfies the following identities:

$$(4.6) \quad \pi^1(T_x^\iota(v, w)) = T_x^{\iota^1}(v, w) + \alpha_x^1(v, \iota^2(w)) - \alpha_x^1(w, \iota^2(v)),$$

$$(4.7) \quad \pi^2(T_x^\iota(v, w)) = T_x^{\iota^2}(v, w) + \alpha_x^2(v, \iota^1(w)) - \alpha_x^2(w, \iota^1(v)),$$

for all $x \in M$, $v, w \in T_x M$, where T^{ι^1} , T^{ι^2} denote respectively the ι^1 -torsion of ∇^1 and the ι^2 -torsion of ∇^2 .

Let us look at equations (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7) in the context of affine immersions. More precisely, let $f : (M, \nabla) \rightarrow (\overline{M}, \overline{\nabla})$ be an affine immersion, E be a normal bundle for f , and ∇^\perp , α and A denote respectively the normal connection, the second fundamental form and the Weingarten form. Since ∇ , ∇^\perp , α and A are (up to the identification of TM with $df(TM)$) the components of $f^*\overline{\nabla}$ with respect to the decomposition $f^*T\overline{M} = df(TM) \oplus E$, equation (4.2) gives:

$$(4.8) \quad \pi^{TM}[\overline{R}_{f(x)}(df_x(v), df_x(w))df_x(u)] = R_x(v, w)u + A_x(v, \alpha_x(w, u)) - A_x(w, \alpha_x(v, u)),$$

for all $x \in M$ and all $v, w, u \in T_x M$, where R and \overline{R} denote the curvature tensors of ∇ and $\overline{\nabla}$ respectively. We call (4.8) the *Gauss equation* of the affine immersion f with respect to E . Similarly, equation (4.3) gives:

$$(4.9) \quad \pi^E[\overline{R}_{f(x)}(df_x(v), df_x(w))e] = R_x^\perp(v, w)e + \alpha_x(v, A_x(w, e)) - \alpha_x(w, A_x(v, e)),$$

for all $x \in M$, $v, w \in T_x M$ and all $e \in E_x$, where R^\perp denotes the curvature tensor of the normal connection ∇^\perp . We call (4.9) the *Ricci equation* of the affine immersion f with respect to E . Equations (4.4) and (4.5) (with $\nabla^M = \nabla$) give:

$$(4.10) \quad \pi^E[\overline{R}_{f(x)}(df_x(v), df_x(w))u] = (\nabla^\otimes \alpha)_x(v, w, u) - (\nabla^\otimes \alpha)_x(w, v, u) + \alpha_x(T_x(v, w), u),$$

$$(4.11) \quad \pi^{TM}[\overline{R}_{f(x)}(df_x(v), df_x(w))e] = (\nabla^\otimes A)_x(v, w, e) - (\nabla^\otimes A)_x(w, v, e) + A_x(T_x(v, w), e),$$

for all $x \in M$, $v, w, u \in T_x M$ and all $e \in E_x$, where T denotes the torsion tensor of ∇ . We call (4.10) and (4.11) the *Codazzi equations* of the affine immersion f with respect to E . Finally, if $\iota : TM \rightarrow f^*T\overline{M} = df(TM) \oplus E$ is the map identified with $df : TM \rightarrow T\overline{M}$ then $\iota^1 = df : TM \rightarrow df(TM)$, $\iota^2 = 0$ and equations (4.6) and (4.7) read¹:

$$(4.12) \quad \pi^{TM}[\overline{T}_{f(x)}(df_x(v), df_x(w))] = T_x(v, w),$$

$$(4.13) \quad \pi^E[\overline{T}_{f(x)}(df_x(v), df_x(w))] = \alpha_x(v, w) - \alpha_x(w, v),$$

¹By taking such map ι , the ι -torsion of $f^*\overline{\nabla}$ is just the pull-back by f of the torsion of $\overline{\nabla}$.

for all $x \in M$, $v, w \in T_x M$, where \overline{T} denotes the torsion of $\overline{\nabla}$. We call (4.12) and (4.13) the *torsion equations* of the affine immersion f with respect to E .

Now assume that we are given affine manifolds (M, ∇) , $(\overline{M}, \overline{\nabla})$, a vector bundle E^0 over M endowed with a connection ∇^0 and smooth sections α^0, A^0 of the vector bundles $TM^* \otimes TM^* \otimes E^0$ and $TM^* \otimes (E^0)^* \otimes TM$ respectively. Assume that there exists a local solution (f, S) for the affine immersion problem with data ∇^0, α^0, A^0 defined on an open subset U of M , where $S : E^0|_U \rightarrow E|_U$ and E is a normal bundle for f . Clearly, (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) imply:

$$(4.14) \quad \pi^{TM} [\overline{R}_{f(x)}(df_x(v), df_x(w))df_x(u)] = R_x(v, w)u + A_x^0(v, \alpha_x^0(w, u)) - A_x^0(w, \alpha_x^0(v, u)),$$

$$(4.15) \quad \pi^E [\overline{R}_{f(x)}(df_x(v), df_x(w))S_x(e)] = S_x[R_x^0(v, w) + \alpha_x^0(v, A_x^0(w, e)) - \alpha_x^0(w, A_x^0(v, e))],$$

$$(4.16) \quad \pi^E [\overline{R}_{f(x)}(df_x(v), df_x(w))df_x(u)] = S_x[(\nabla^{\otimes} \alpha^0)_x(v, w, u) - (\nabla^{\otimes} \alpha^0)_x(w, v, u) + \alpha_x^0(T_x(v, w), u)],$$

$$(4.17) \quad \pi^{TM} [\overline{R}_{f(x)}(df_x(v), df_x(w))S_x(e)] = (\nabla^{\otimes} A^0)_x(v, w, e) - (\nabla^{\otimes} A^0)_x(w, v, e) + A_x^0(T_x(v, w), e),$$

$$(4.18) \quad \pi^{TM} [\overline{T}_{f(x)}(df_x(v), df_x(w))] = T_x(v, w),$$

$$(4.19) \quad \pi^E [\overline{T}_{f(x)}(df_x(v), df_x(w))] = S_x(\alpha_x^0(v, w) - \alpha_x^0(w, v)),$$

for all $x \in U$, $v, w, u \in T_x M$ and all $e \in E_x^0$, where R^0 denotes the curvature tensor of ∇^0 .

Notice that in the case of isometric immersions, the torsion equation (4.18) is trivial and (4.19) says that α^0 is symmetric; moreover, using (3.2), it can be seen that the Codazzi equations (4.16) and (4.17) are equivalent to each other.

One may think that Gauss, Ricci, Codazzi and the torsion equations are “necessary conditions” for the existence of a solution (f, S) of the affine immersion problem, although such statement is obviously meaningless because one cannot write down equations (4.14), (4.15), (4.16), (4.17), (4.18) and (4.19) unless f and S are already given. Notice that in the special case that $(\overline{M}, \overline{g})$ is a semi-Riemannian manifold with constant sectional curvature $c \in \mathbb{R}$ and (f, S) is a solution for the isometric immersion problem then the lefthand side of equations (4.14), (4.15), (4.16) and (4.17) can be written only in terms of c and g , i.e., without using f and S ; more explicitly, the lefthand side of (4.14) is $c(g_x(w, u)v - g_x(v, u)w)$, while the lefthand sides of (4.15), (4.16) and (4.17) are zero (the possibility of writing down Gauss, Ricci and Codazzi equations without using f and S depends on the fact that the curvature tensor \overline{R} of the target manifold is *constant in orthonormal frames*). Thus, in this case, the Gauss, Ricci and Codazzi equations are indeed necessary conditions for the existence of a solution of the isometric immersion problem (f, S) . By the celebrated fundamental theorem of isometric immersions

into space forms (see, for instance, [2, 5, 12, 14]), the Gauss, Ricci and Codazzi equations are also sufficient conditions for the existence of local solutions for the isometric immersion problem (provided that one assumes that α^0 is symmetric and that g^0 is ∇^0 -parallel).

Using the notion of *infinitesimally homogeneous affine manifold with G-structure* introduced in Section 6 we will describe a very general situation in which the lefthand side of (4.14), (4.15), (4.16), (4.17), (4.18) and (4.19) can be described without explicit use of f and S .

5. G-STRUCTURES AND INNER TORSION

Let E be a vector bundle of rank k over a differentiable manifold M . If G is a Lie subgroup of $GL(\mathbb{R}^k)$ then by a *G-structure* on E we mean a G -principal subbundle P of $FR(E)$. By a *G-structure* on M we mean a G -structure on the tangent bundle of M . Let ∇ be a connection in E . We denote by Hor the corresponding horizontal distribution on $FR(E)$ and by ω the $\mathfrak{gl}(\mathbb{R}^k)$ -valued connection form on $FR(E)$ whose kernel is Hor . We say that ∇ is *compatible* with a G -structure P if $\text{Hor}_p \subset T_p P$, for all $p \in P$, i.e., if parallel transport carries frames in P to frames in P . In the general case, there is a tensor that measures the lack of compatibility of ∇ with P called the *inner torsion* of P with respect to ∇ , which is defined as follows.

For each $x \in M$, denote by G_x the subgroup of $GL(E_x)$ consisting of *G-structure preserving maps*, i.e., maps $\sigma : E_x \rightarrow E_x$ such that $\sigma \circ p \in P_x$ for some (and hence for all) $p \in P_x$. Clearly $G_x = \mathcal{I}_p(G)$, for all $p \in P_x$, so that G_x is a Lie subgroup of $GL(E_x)$. We denote by $\mathfrak{g}_x \subset \mathfrak{gl}(E_x)$ the Lie algebra of G_x , so that $\text{Ad}_p(\mathfrak{g}) = \mathfrak{g}_x$, for all $p \in P_x$, where $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^k)$ denotes the Lie algebra of G . For each $x \in M$ and each $p \in FR(E_x)$, we can identify the tangent space $T_p FR(E)$ with the direct sum $T_x M \oplus \mathfrak{gl}(\mathbb{R}^k)$ via the isomorphism $(d\Pi_p, \omega_p)$, where $\Pi : FR(E) \rightarrow M$ denotes the projection. For $p \in P$, the subspace $\mathcal{V}_p = (d\Pi_p, \omega_p)(T_p P)$ of $T_x M \oplus \mathfrak{gl}(\mathbb{R}^k)$ corresponding to $T_p P$ satisfies the conditions $\text{pr}_1(\mathcal{V}_p) = T_x M$ and $\mathcal{V}_p \cap (\{0\} \oplus \mathfrak{gl}(\mathbb{R}^k)) = \{0\} \oplus \mathfrak{g}$, where $\text{pr}_1 : T_x M \oplus \mathfrak{gl}(\mathbb{R}^k) \rightarrow T_x M$ denotes the first projection. Thus, there exists a unique linear map $\mathcal{L}_p : T_x M \rightarrow \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g}$ such that:

$$(5.1) \quad \mathcal{V}_p = \{(v, X) \in T_x M \oplus \mathfrak{gl}(\mathbb{R}^k) : \mathcal{L}_p(v) = X + \mathfrak{g}\}.$$

If $s : U \rightarrow P$ is a smooth local section with $x \in U$ and $\bar{\omega} = s^* \omega$ then \mathcal{L}_p is the composition of $\bar{\omega}_x : T_x M \rightarrow \mathfrak{gl}(\mathbb{R}^k)$ with the quotient map $\mathfrak{gl}(\mathbb{R}^k) \rightarrow \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g}$. It follows from the usual properties of connection forms that, given $p, q \in P_x$, the maps \mathcal{L}_p and \mathcal{L}_q are related by $\mathcal{L}_q = \overline{\text{Ad}}_g \circ \mathcal{L}_p$, where $g \in G$ is such that $p = q \circ g$, and $\overline{\text{Ad}}_g : \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g} \rightarrow \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g}$ is obtained from $\text{Ad}_g : \mathfrak{gl}(\mathbb{R}^k) \rightarrow \mathfrak{gl}(\mathbb{R}^k)$ by passing to the quotient. It follows that the linear map $\mathcal{J}_x^P : T_x M \rightarrow \mathfrak{gl}(E_x)/\mathfrak{g}_x$ defined by:

$$(5.2) \quad \mathcal{J}_x^P = \overline{\text{Ad}}_p \circ \mathcal{L}_p, \quad p \in P_x,$$

does not depend on the choice of $p \in P_x$; here $\overline{\text{Ad}}_p : \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g} \rightarrow \mathfrak{gl}(E_x)/\mathfrak{g}_x$ is obtained from $\text{Ad}_p : \mathfrak{gl}(\mathbb{R}^k) \rightarrow \mathfrak{gl}(E_x)$ by passing to the quotient. We call \mathcal{J}_x^P the

inner torsion of the G -structure P at the point x with respect to the connection ∇ . Obviously, $\mathfrak{J}^P = 0$ if and only if ∇ is compatible with P . It follows from (2.1) that if $s : U \rightarrow P$ is a smooth local section with $x \in U$ and Γ denotes the Christoffel tensor of ∇ with respect to s then the inner torsion \mathfrak{J}_x^P is precisely the composition of the map $\Gamma_x : T_x M \rightarrow \mathfrak{gl}(E_x)$ with the quotient map $\mathfrak{gl}(E_x) \rightarrow \mathfrak{gl}(E_x)/\mathfrak{g}_x$. This observation gives a simple method for computing inner torsions.

Let us compute inner torsions in some specific examples.

5.1. Example. If E is trivial and $s : M \rightarrow \text{FR}(E)$ is a smooth global frame then $P = s(M)$ is a G -structure on E with $G = \{\text{Id}_{\mathbb{R}^k}\}$. For each $x \in M$, we have $G_x = \{\text{Id}_{E_x}\}$ and $\mathfrak{g}_x = \{0\}$; the inner torsion $\mathfrak{J}_x^P : T_x M \rightarrow \mathfrak{gl}(E_x)$ is equal to the Christoffel tensor $\Gamma_x : T_x M \rightarrow \mathfrak{gl}(E_x)$ of ∇ with respect to s .

5.2. Example. Let g be a semi-Riemannian structure on E of index r , i.e., g is a smooth section of $E^* \otimes E^*$ such that g_x is a nondegenerate symmetric bilinear form on E_x of index r , for all $x \in M$; denote by $\langle \cdot, \cdot \rangle_r$ the *standard Minkowski inner product of index r in \mathbb{R}^k* defined by:

$$(5.3) \quad \langle v, w \rangle_r = \sum_{i=1}^{k-r} v_i w_i - \sum_{i=k-r+1}^k v_i w_i, \quad v, w \in \mathbb{R}^k.$$

We denote by $\text{FR}^\circ(E)$ the set of all $p \in \text{FR}(E)$ that are linear isometries, so that $P = \text{FR}^\circ(E)$ is a G -structure on E , where $G = \text{O}(k-r, r)$ denotes the Lie group of linear isometries of $(\mathbb{R}^k, \langle \cdot, \cdot \rangle_r)$. Let $x \in M$ be fixed. Clearly, G_x is the group of linear isometries of (E_x, g_x) and \mathfrak{g}_x is the Lie algebra of linear endomorphisms of E_x that are g_x -anti-symmetric. We identify $\mathfrak{gl}(E_x)/\mathfrak{g}_x$ with the space $\text{sym}(E_x)$ of all linear endomorphisms of E_x that are g_x -symmetric via the map:

$$(5.4) \quad \mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto \frac{1}{2}(T + T^*) \in \text{sym}(E_x),$$

where $T^* : E_x \rightarrow E_x$ denotes the transpose of T with respect to g_x . Thus, the inner torsion \mathfrak{J}_x^P is identified with a linear map from $T_x M$ to $\text{sym}(E_x)$. Let $s : U \rightarrow P$ be a smooth local section with $x \in U$ and let $e, e' \in E_x$ be fixed; consider the local sections $\epsilon, \epsilon' : U \rightarrow E$ such that $\epsilon(x) = e$, $\epsilon'(x) = e'$ and such that the \mathbb{R}^k -valued maps $y \mapsto s(y)^{-1}(\epsilon(y))$, $y \mapsto s(y)^{-1}(\epsilon'(y))$ are constant. Then $d\mathbb{I}^s \epsilon = 0$, $d\mathbb{I}^s \epsilon' = 0$ and:

$$\nabla_v \epsilon = \Gamma_x(v) \cdot e, \quad \nabla_v \epsilon' = \Gamma_x(v) \cdot e',$$

for all $v \in T_x M$. Since $s(y) : (\mathbb{R}^k, \langle \cdot, \cdot \rangle_r) \rightarrow (E_y, g_y)$ is a linear isometry for all $y \in U$, the real-valued map $g(\epsilon, \epsilon')$ is constant. Thus:

$$\begin{aligned} 0 &= v(g(\epsilon, \epsilon')) = (\nabla_v g)(e, e') + g_x(\nabla_v \epsilon, e') + g_x(e, \nabla_v \epsilon') \\ &= (\nabla_v g)(e, e') + g_x(\Gamma_x(v) \cdot e, e') + g_x(e, \Gamma_x(v) \cdot e'), \end{aligned}$$

for all $v \in T_x M$. Then:

$$g_x[(\Gamma_x(v) + \Gamma_x(v)^*) \cdot e, e'] = -(\nabla_v g)(e, e')$$

and (using (5.4)):

$$g_x(\mathfrak{J}_x^P(v), \cdot) = \frac{1}{2} g_x[(\Gamma_x(v) + \Gamma_x(v)^*), \cdot] = -\frac{1}{2} \nabla_v g,$$

for all $x \in M$, $v \in T_x M$. Using g_x to identify $\nabla_v g : E_x \times E_x \rightarrow \mathbb{R}$ with a linear endomorphism of E_x we obtain:

$$\mathfrak{J}_x^P(v) = -\frac{1}{2}\nabla_v g.$$

Thus, the inner torsion of P is essentially the covariant derivative of the semi-Riemannian structure g . In particular, $\mathfrak{J}^P = 0$ if and only if $\nabla g = 0$.

5.3. Example. Let F be a vector subbundle of E of rank l . For $x \in M$, set:

$$\text{FR}(E_x; F_x) = \{p \in \text{FR}(E_x) : p(\mathbb{R}^l \oplus \{0\}) = F_x\}$$

and $\text{FR}(E; F) = \bigcup_{x \in M} \text{FR}(E_x; F_x)$. Then $P = \text{FR}(E; F)$ is a G -structure on E with G the Lie subgroup of $\text{GL}(\mathbb{R}^k)$ consisting of linear isomorphisms that preserve $\mathbb{R}^l \oplus \{0\}$. Let $x \in M$ be fixed. Clearly G_x is the Lie group of linear isomorphisms of E_x that preserve F_x and \mathfrak{g}_x is the Lie algebra of linear endomorphisms of E_x that preserve F_x . We identify the quotient $\mathfrak{gl}(E_x)/\mathfrak{g}_x$ with the space $\text{Lin}(F_x, E_x/F_x)$ via the map:

$$(5.5) \quad \mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto \mathfrak{q} \circ T|_{F_x} \in \text{Lin}(F_x, E_x/F_x),$$

where $\mathfrak{q} : E_x \rightarrow E_x/F_x$ denotes the quotient map. Thus, the inner torsion \mathfrak{J}_x^P is identified with a linear map from $T_x M$ to $\text{Lin}(F_x, E_x/F_x)$. Let $s : U \rightarrow P$ be a smooth local section with $x \in U$ and $e \in F_x$ be fixed. Define ϵ as in Example 5.2, so that $\nabla_v \epsilon = \Gamma_x(v) \cdot e$, for all $v \in T_x M$. We have $\epsilon(U) \subset F$ and therefore:

$$\Gamma_x(v) \cdot e + F_x = \nabla_v \epsilon + F_x = \alpha_x^F(v, e) \in E_x/F_x,$$

where α^F denotes the absolute second fundamental form of the vector subbundle F . Hence, using (5.5):

$$\mathfrak{J}_x^P(v) = \alpha_x^F(v, \cdot) \in \text{Lin}(F_x, E_x/F_x),$$

for all $x \in M$ and all $v \in T_x M$. In particular, $\mathfrak{J}^P = 0$ if and only if $\alpha^F = 0$, i.e., if and only if the covariant derivative of any smooth section of F is a smooth section of F .

5.4. Example. Let g be a semi-Riemannian structure on E of index r and F be a vector subbundle of E such that the restriction of g to F is a semi-Riemannian structure on F of index $s \leq r$; denote by F^\perp the g -orthogonal complement of F in E , so that $E = F \oplus F^\perp$. Define $\langle \cdot, \cdot \rangle_r$ as in (5.3) and fix any subspace F_0 of \mathbb{R}^k such that the restriction of $\langle \cdot, \cdot \rangle_r$ to F_0 is a nondegenerate symmetric bilinear form of index s . Then $P = \text{FR}^\circ(E; F) = \bigcup_{x \in M} \text{FR}^\circ(E_x; F_x)$, where

$$\text{FR}^\circ(E_x; F_x) = \{p \in \text{FR}^\circ(E_x) : p(F_0) = F_x\},$$

is a G -structure on E where G is the Lie group of linear isometries of $(\mathbb{R}^k, \langle \cdot, \cdot \rangle_r)$ that preserve F_0 . Denote by $\mathfrak{q} : E \rightarrow F^\perp$ the projection with respect to the decomposition $E = F \oplus F^\perp$. Let $x \in M$ be fixed. Clearly G_x is the Lie group

of linear isometries of (E_x, g_x) that preserve F_x and \mathfrak{g}_x is the Lie algebra of g_x -anti-symmetric linear endomorphisms of E_x that preserve F_x . We have an isomorphism:

$$\begin{aligned} \mathfrak{gl}(E_x)/\mathfrak{g}_x &\longrightarrow \text{sym}(E_x) \oplus \text{Lin}(F_x, F_x^\perp) \\ T + \mathfrak{g}_x &\longmapsto \left(\frac{1}{2}(T + T^*), \frac{1}{2}\mathfrak{q}_x \circ (T - T^*)|_{F_x}\right), \end{aligned}$$

so that we identify \mathfrak{J}_x^P with a linear map from $T_x M$ to $\text{sym}(E_x) \oplus \text{Lin}(F_x, F_x^\perp)$. Consider the component $\alpha \in \Gamma(TM^* \otimes F^* \otimes F^\perp)$ of the connection ∇ with respect to the decomposition $E = F \oplus F^\perp$. Arguing as in Examples 5.2 and 5.3, one easily computes:

$$\mathfrak{J}_x^P(v) = \left(-\frac{1}{2}\nabla_v g, \alpha_x(v, \cdot) + \frac{1}{2}\mathfrak{q} \circ \nabla_v g|_{F_x}\right),$$

for all $x \in M$, $v \in T_x M$, where $\nabla_v g$ is identified with a linear endomorphism of E_x using g_x . In particular, $\mathfrak{J}^P = 0$ if and only if $\nabla g = 0$ and $\alpha = 0$, i.e., if and only if $\nabla g = 0$ and the covariant derivative of any smooth section of F is a smooth section of F .

5.5. Example. Let $\epsilon \in \Gamma(E)$ be a smooth section of E with $\epsilon(x) \neq 0$, for all $x \in M$. Fix a nonzero vector $e_0 \in \mathbb{R}^k$; then:

$$P = \bigcup_{x \in M} \{p \in \text{FR}(E_x) : p(e_0) = \epsilon(x)\}$$

is a G -structure on E where G is the subgroup of $\text{GL}(\mathbb{R}^k)$ consisting of isomorphisms that fix e_0 . Let $x \in M$ be fixed. Then G_x is the subgroup of $\text{GL}(E_x)$ consisting of isomorphisms that fix $\epsilon(x)$ and \mathfrak{g}_x is the Lie algebra of linear endomorphisms $T : E_x \rightarrow E_x$ such that $T(\epsilon(x)) = 0$. We identify the quotient $\mathfrak{gl}(E_x)/\mathfrak{g}_x$ with E_x via the map:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto T(\epsilon(x)) \in E_x,$$

so that \mathfrak{J}_x^P is identified with a linear map from $T_x M$ to E_x . Let $s : U \rightarrow P$ be a smooth local section with $x \in U$. We have $s(y)^{-1}(\epsilon(y)) = e_0$, for all $y \in U$, so that $\text{d}\mathfrak{f}_y^s \epsilon = 0$ and $\nabla_v \epsilon = \Gamma_x(v) \cdot \epsilon(x)$, for all $v \in T_x M$. Then:

$$\mathfrak{J}_x^P(v) = \nabla_v \epsilon,$$

for all $v \in T_x M$, i.e., the inner torsion \mathfrak{J}^P is identified with the covariant derivative of ϵ . In particular, $\mathfrak{J}^P = 0$ if and only if the section ϵ is parallel.

Assume now that g is a semi-Riemannian structure on E of index r , $\langle \cdot, \cdot \rangle_r$ is defined as in (5.3) and that $g_x(\epsilon(x), \epsilon(x)) = \langle e_0, e_0 \rangle_r$, for all $x \in M$. Then:

$$P' = \bigcup_{x \in M} \{p \in \text{FR}^\circ(E_x) : p(e_0) = \epsilon(x)\}$$

is a G -structure on E where G is the Lie subgroup of $\text{O}(k - r)$ consisting of linear isometries that fix e_0 . Let $x \in M$ be fixed. Then G_x is the Lie group

of linear isometries of (E_x, g_x) that fix $\epsilon(x)$ and \mathfrak{g}_x is the Lie algebra of g_x -anti-symmetric linear endomorphisms T of E_x such that $T(\epsilon(x)) = 0$. We have the following linear isomorphism:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto \left(\frac{1}{2}(T + T^*), \frac{1}{2}(T - T^*) \cdot \epsilon(x)\right) \in \text{sym}(E_x) \oplus \epsilon(x)^\perp$$

where $\epsilon(x)^\perp$ denotes the g_x -orthogonal complement of $\epsilon(x)$ in E_x . Arguing as before, we obtain:

$$(5.6) \quad \mathfrak{J}_x^{P'}(v) = \left(-\frac{1}{2}\nabla_v g, \nabla_v \epsilon + \frac{1}{2}(\nabla_v g)(\epsilon(x))\right),$$

for all $x \in M$ and all $v \in T_x M$. In particular, $\mathfrak{J}^{P'} = 0$ if and only if $\nabla g = 0$ and $\nabla \epsilon = 0$.

5.6. Example. Assume $k = 2l$ and let J be an almost complex structure on E , i.e., J is a smooth section of $E^* \otimes E$ and J_x is a complex structure on E_x for all $x \in M$. Consider the complex structure $J_0 : \mathbb{R}^k \cong \mathbb{R}^l \oplus \mathbb{R}^l \ni (v, w) \mapsto (-w, v)$ on \mathbb{R}^k and set $\text{FR}^c(E) = \bigcup_{x \in M} \text{FR}^c(E_x)$, where for each $x \in M$, $\text{FR}^c(E_x)$ denotes the set of all complex linear isomorphisms $p : (\mathbb{R}^k, J_0) \rightarrow (E_x, J_x)$. Then $P = \text{FR}^c(E)$ is a G -structure on E where $G = \text{GL}(\mathbb{R}^k, J_0)$ is the Lie group of complex linear isomorphisms of (\mathbb{R}^k, J_0) . Let $x \in M$ be fixed. Then $G_x = \text{GL}(E_x, J_x)$ and \mathfrak{g}_x is the Lie algebra of complex linear endomorphisms of (E_x, J_x) . We have an isomorphism:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto [T, J_x] \in \overline{\text{Lin}}(E_x, J_x),$$

where $[T, J_x] = T \circ J_x - J_x \circ T$ and $\overline{\text{Lin}}(E_x, J_x)$ denotes the space of linear maps $T : E_x \rightarrow E_x$ such that $T \circ J_x + J_x \circ T = 0$. Let $s : U \rightarrow P$ be a smooth local section with $x \in U$ and let $e \in E_x$ be fixed. We define a local section $\epsilon : U \rightarrow E$ as in Example 5.2, so that $\nabla_v \epsilon = \Gamma_x(v) \cdot e$, for all $v \in T_x M$. Since $U \ni y \mapsto s(y)^{-1}(J_y \cdot \epsilon(y))$ is constant, it follows that $\text{d}_v^s(J(\epsilon)) = 0$ and $\nabla_v(J(\epsilon)) = \Gamma_x(v) \cdot (J_x(e))$; then:

$$\Gamma_x(v) \cdot (J_x(e)) = \nabla_v(J(\epsilon)) = (\nabla_v J)(e) + J_x(\Gamma_x(v) \cdot e),$$

for all $v \in T_x M$. We therefore obtain:

$$\Gamma_x(v) \circ J_x = \nabla_v J + J_x \circ \Gamma_x(v)$$

and hence:

$$\mathfrak{J}_x^P(v) = \nabla_v J,$$

for all $x \in M$ and all $v \in T_x M$. In particular, $\mathfrak{J}^P = 0$ if and only if J is parallel.

5.7. Example. Assume $k = 2l$. Let J be an almost complex structure on E , g be a semi-Riemannian structure on E of index $r = 2s$, J_0 be the complex structure on \mathbb{R}^k considered in Example 5.6 and $\langle \cdot, \cdot \rangle$ be the nondegenerate symmetric bilinear form of index r on \mathbb{R}^k defined by:

$$\langle v, w \rangle = \sum_{i=1}^{l-s} v_i w_i - \sum_{i=l-s+1}^l v_i w_i + \sum_{i=l+1}^{k-s} v_i w_i - \sum_{i=k-s+1}^k v_i w_i, \quad v, w \in \mathbb{R}^k;$$

observe that J_0 is anti-symmetric with respect to $\langle \cdot, \cdot \rangle$. Assume that J_x is g_x -anti-symmetric for all $x \in M$. Set $\text{FR}^u(E) = \bigcup_{x \in M} \text{FR}^u(E_x)$, where for each $x \in M$, $\text{FR}^u(E_x)$ is the set of complex linear isometries from $(\mathbb{R}^k, J_0, \langle \cdot, \cdot \rangle)$ to (E_x, J_x, g_x) . Then $P = \text{FR}^u(E)$ is a G -structure on E where $G = \text{U}(s, l - s)$ is the Lie group of complex linear isometries of $(\mathbb{R}^k, J_0, \langle \cdot, \cdot \rangle)$. Let $x \in M$ be fixed. Clearly G_x is the Lie group of complex linear isometries of (E_x, J_x, g_x) and \mathfrak{g}_x is the Lie algebra of complex linear g_x -anti-symmetric endomorphisms of (E_x, J_x) . We have a linear isomorphism:

$$\begin{aligned} \mathfrak{gl}(E_x)/\mathfrak{g}_x &\longrightarrow \text{sym}(E_x) \oplus \overline{\text{Lin}}_a(E_x, J_x) \\ T + \mathfrak{g}_x &\longmapsto \left(\frac{1}{2}(T + T^*), \frac{1}{2}[T - T^*, J_x]\right), \end{aligned}$$

where $\overline{\text{Lin}}_a(E_x, J_x)$ denotes the space of g_x -anti-symmetric linear endomorphisms T of E_x such that $T \circ J_x + J_x \circ T = 0$. Arguing as in Examples 5.2 and 5.6 we obtain:

$$\mathfrak{J}_x^P(v) = \left(-\frac{1}{2}\nabla_v g, \nabla_v J - [\nabla_v g, J_x]\right),$$

for all $x \in M$ and all $v \in T_x M$. In particular, $\mathfrak{J}^P = 0$ if and only if both g and J are parallel.

We conclude the section with a technical lemma that will be used later on.

Lemma 5.1. *Let (M_1, ∇^1) , (M_2, ∇^2) be n -dimensional affine manifolds, G be a Lie subgroup of $\text{GL}(\mathbb{R}^n)$ and $P^1 \subset \text{FR}(TM_1)$, $P^2 \subset \text{FR}(TM_2)$ be G -structures on M_1 and M_2 , respectively. Assume that for all $x \in M_1$, $y \in M_2$ and for every G -structure preserving map $\sigma : T_x M_1 \rightarrow T_y M_2$ we have $\mathfrak{J}_y^{P^2} \circ \sigma = \overline{\text{Ad}}_\sigma \circ \mathfrak{J}_x^{P^1}$. Let $\gamma : I \rightarrow M_1$, $\mu : I \rightarrow M_2$ be smooth curves and $p : I \rightarrow \text{FR}(TM_1)$, $q : I \rightarrow \text{FR}(TM_2)$ be horizontal liftings of γ and μ , respectively. For each $t \in I$, set $\sigma(t) = q(t) \circ p(t)^{-1} : T_{\gamma(t)} M_1 \rightarrow T_{\mu(t)} M_2$. If $\sigma(t)(\gamma'(t)) = \mu'(t)$, for all $t \in I$ and if $\sigma(t_0)$ is G -structure preserving for some $t_0 \in I$ then $\sigma(t)$ is G -structure preserving for all $t \in I$.*

PROOF. By partitioning I , we may assume without loss of generality that there are smooth local sections $s_1 : U \rightarrow P^1$, $s_2 : V \rightarrow P^2$ with $\gamma(I) \subset U$, $\mu(I) \subset V$ and such that U is the domain of a local chart of M_1 . Let ω^i denote the connection form of $\text{FR}(TM_i)$ and set $\bar{\omega}^i = s_i^* \omega$, $i = 1, 2$. Since p, q are horizontal, (2.6) gives us:

$$(5.7) \quad \tilde{p}'(t) + \bar{\omega}_{\gamma(t)}^1(\gamma'(t)) \circ \tilde{p}(t) = 0, \quad \tilde{q}'(t) + \bar{\omega}_{\mu(t)}^2(\mu'(t)) \circ \tilde{q}(t) = 0,$$

for all $t \in I$, where $\tilde{p}(t) = s_1(\gamma(t))^{-1} \circ p(t)$ and $\tilde{q}(t) = s_2(\mu(t))^{-1} \circ q(t)$. Now set $L(t) = s_2(\mu(t))^{-1} \circ \sigma(t) \circ s_1(\gamma(t)) = \tilde{q}(t) \circ \tilde{p}(t)^{-1}$, so that $\sigma(t)$ is G -structure preserving if and only if $L(t) \in G$. Now (5.7) implies:

$$(5.8) \quad L'(t) = L(t) \circ \bar{\omega}_{\gamma(t)}^1(\gamma'(t)) - \bar{\omega}_{\mu(t)}^2(\mu'(t)) \circ L(t),$$

for all $t \in I$. Since U is the domain of a local chart of M_1 , there exists a smooth time-dependent vector field $X : I \times U \rightarrow TM_1$ in U such that γ is an integral curve of X (for instance, let $X(t, x) \in T_x M_1$ be the vector that has the same coordinates

as $\gamma'(t)$ in a local chart). We define a smooth time-dependent vector field \mathcal{X} in $U \times V \times \text{GL}(\mathbb{R}^n)$ by setting:

$$\mathcal{X}(t, x, y, A) = \left(X(t, x), \widehat{A}(X(t, x)), A \circ \bar{\omega}_x^1(X(t, x)) - \bar{\omega}_y^2[\widehat{A}(X(t, x))] \circ A \right),$$

for all $t \in I$, $x \in U$, $y \in V$, $A \in \text{GL}(\mathbb{R}^n)$, where $\widehat{A} : T_x M_1 \rightarrow T_y M_2$ is defined by $\widehat{A} = s_2(y) \circ A \circ s_1(x)^{-1}$. Notice that if $A = L(t)$, $x = \gamma(t)$ and $y = \mu(t)$ then $\widehat{A} = \sigma(t)$, $X(t, x) = \gamma'(t)$ and $\widehat{A}(X(t, x)) = \mu'(t)$; thus, by (5.8), $(\gamma, \mu, L) : I \rightarrow U \times V \times \text{GL}(\mathbb{R}^n)$ is an integral curve of \mathcal{X} . We claim that $\mathcal{X}(t, x, y, A) \in T(M_1 \times M_2 \times G)$ for all $t \in I$, $x \in M_1$, $y \in M_2$ and all $A \in G$. Namely, for all $x \in U$, $y \in V$, $v \in T_x M_1$, $w \in T_y M_2$, we have:

$$(5.9) \quad \bar{\omega}_x^1(v) + \mathfrak{g} = \overline{\text{Ad}}_{s_1(x)}^{-1}(\mathfrak{J}_x^{P_1}(v)), \quad \bar{\omega}_y^2(w) + \mathfrak{g} = \overline{\text{Ad}}_{s_2(y)}^{-1}(\mathfrak{J}_y^{P_2}(w)).$$

If $A \in G$ then $\widehat{A} : T_x M_1 \rightarrow T_y M_2$ is G -structure preserving and therefore:

$$(5.10) \quad \mathfrak{J}_y^{P_2}(\widehat{A}(v)) = \overline{\text{Ad}}_{\widehat{A}}(\mathfrak{J}_x^{P_1}(v)).$$

From (5.9) and (5.10) it follows that:

$$\text{Ad}_A(\bar{\omega}_x^1(v)) - \bar{\omega}_y^2(\widehat{A}(v)) = A \circ \bar{\omega}_x^1(v) \circ A^{-1} - \bar{\omega}_y^2(\widehat{A}(v)) \in \mathfrak{g}.$$

This proves that $A \circ \bar{\omega}_x^1(v) - \bar{\omega}_y^2(\widehat{A}(v)) \circ A \in T_A G$ and concludes the proof of the claim. We have established that the set of instants $t \in [a, b]$ such that $L(t) \in G$ is open. The conclusion follows from the elementary lemma below. \square

Lemma 5.2. *Let \mathcal{G} be a Lie group, G be a (not necessarily closed) Lie subgroup of \mathcal{G} and $\lambda : I \rightarrow \mathcal{G}$ be a continuous curve. If $\lambda^{-1}(G)$ is open in I and nonempty then $\lambda(I) \subset G$.*

PROOF. It suffices to consider the case where $I = [a, b]$ and $\lambda(a) \in G$. Let $t_0 \in]a, b]$ denote the supremum of the set of all $t \in [a, b]$ such that $\lambda([a, t]) \subset G$. Clearly $\lambda([a, t_0[) \subset G$ and the conclusion will follow once we show that $\lambda(t_0) \in G$. Let $\varphi : U \subset \mathcal{G} \rightarrow \tilde{U} \subset \mathbb{R}^n$ be a local chart with $\lambda(t_0) \in U$ and such that the intersection of any left coset of G with U is a countable union of slices $U_c = \varphi^{-1}(\tilde{U} \cap \pi^{-1}(c))$, $c \in \mathbb{R}^{n-k}$, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ denotes the projection onto the first $n - k$ coordinates, $n = \dim(\mathcal{G})$, $k = \dim(G)$ (this is a standard construction for Lie groups; see, for instance, [15, Chapter 3]). For $\varepsilon > 0$ sufficiently small, $\lambda([t_0 - \varepsilon, t_0]) \subset U$ and thus $\lambda([t_0 - \varepsilon, t_0[) \subset U \cap G$ is contained in a countable union of slices; since $(\pi \circ \varphi \circ \lambda)([t_0 - \varepsilon, t_0[)$ is a connected countable subset of \mathbb{R}^{n-k} , it follows that $\lambda([t_0 - \varepsilon, t_0[)$ is contained in a single slice U_c . Since U_c is closed in U , it follows that $\lambda(t_0) \in U_c \subset G$. \square

6. INFINITESIMALLY HOMOGENEOUS AFFINE MANIFOLDS WITH G-STRUCTURE

Let (M, ∇) be an n -dimensional affine manifold, let G be a Lie subgroup of $\text{GL}(\mathbb{R}^n)$ and let $P \subset \text{FR}(TM)$ be a G -structure on M . Denote by T and R respectively the torsion and the curvature tensors of ∇ . Recall that, given $x, y \in$

M , a map $\sigma : T_x M \rightarrow T_y M$ is *G-structure preserving* if $\sigma \circ p \in P_y$ for some (and hence for all) $p \in P_x$. A smooth map $f : M \rightarrow M$ is said to be *G-structure preserving* if $df_x : T_x M \rightarrow T_{f(x)} M$ is G-structure preserving for all $x \in M$.

Definition 6.1. We say that the triple (M, ∇, P) is an *infinitesimally homogeneous affine manifold with G-structure* if for all $x, y \in M$ it is the case that every G-structure preserving map $\sigma : T_x M \rightarrow T_y M$ relates T_x with T_y , R_x with R_y and \mathfrak{J}_x^P with \mathfrak{J}_y^P , i.e., $T_y(\sigma \cdot, \sigma \cdot) = \sigma \circ T_x$, $R_y(\sigma \cdot, \sigma \cdot) = \sigma \circ R_x(\cdot, \cdot) \circ \sigma^{-1}$ and $\mathfrak{J}_y^P \circ \sigma = \overline{\text{Ad}}_\sigma \circ \mathfrak{J}_x^P$.

It is easy to see that (M, ∇, P) is infinitesimally homogeneous if and only if there exist bilinear maps $T_o : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $R_o : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathfrak{gl}(\mathbb{R}^n)$ and a linear map $\mathfrak{J}_o : \mathbb{R}^n \rightarrow \mathfrak{gl}(\mathbb{R}^n)/\mathfrak{g}$ such that for every $x \in M$, every $p \in P_x$ relates T_o with T_x , R_o with R_x and \mathfrak{J}_o with \mathfrak{J}_x^P . We will refer to T_o , R_o and \mathfrak{J}_o collectively as the *characteristic tensors* of the infinitesimally homogeneous affine manifold with G-structure (M, ∇, P) .

Remark 6.2. Clearly, the characteristic tensors T_o , R_o and \mathfrak{J}_o are invariant by the action of G . This implies that one can induce “versions” of the tensors T_o , R_o , \mathfrak{J}_o on every vector space endowed with a G-structure. More precisely, let Z be a real n -dimensional vector space endowed with a G-structure P_Z , i.e., an orbit of the right action of G on $\text{FR}(Z)$. Denote by $G_Z \subset \text{GL}(Z)$ the Lie group of all G-structure preserving automorphisms of Z and by $\mathfrak{g}_Z \subset \mathfrak{gl}(Z)$ its Lie algebra. Given any $p \in P_Z$, there exists a unique triple of tensors $T_Z : Z \times Z \rightarrow Z$, $R_Z : Z \times Z \rightarrow \mathfrak{gl}(Z)$, $\mathfrak{J}_Z : Z \rightarrow \mathfrak{gl}(Z)/\mathfrak{g}_Z$ that are related to T_o , R_o , \mathfrak{J}_o by p . The G-invariance of T_o , R_o , \mathfrak{J}_o implies that T_Z , R_Z , \mathfrak{J}_Z do not depend on the choice of $p \in P_Z$.

Definition 6.3. We say that the triple (M, ∇, P) is *locally homogeneous* if for every $x, y \in M$ and every G-structure preserving map $\sigma : T_x M \rightarrow T_y M$ there exists an open neighborhood $U \subset M$ of x , an open neighborhood $V \subset M$ of y and a smooth G-structure preserving affine diffeomorphism $f : U \rightarrow V$ such that $f(x) = y$ and $df_x = \sigma$. If for every $x, y \in M$ and every G-structure preserving map $\sigma : T_x M \rightarrow T_y M$ there exists a smooth G-structure preserving affine diffeomorphism $f : M \rightarrow M$ with $f(x) = y$ and $df_x = \sigma$ we say that the triple (M, ∇, P) is *(globally) homogeneous*.

Clearly every (locally) homogeneous affine manifold with G-structure is also infinitesimally homogeneous. The converse also holds, as is proven below.

Proposition 6.4. *Let (M, ∇, P) be an infinitesimally homogeneous affine manifold with G-structure. Then (M, ∇, P) is locally homogeneous. If, in addition, (M, ∇) is geodesically complete and M is (connected and) simply-connected then (M, ∇, P) is globally homogeneous.*

PROOF. Let $x_0, y_0 \in M$ be fixed and let $\sigma_0 : T_{x_0} M \rightarrow T_{y_0} M$ be a G-structure preserving map. We will use the Cartan–Ambrose–Hicks (CAH) theorem (see, for instance, [16]) to produce a smooth G-structure preserving affine diffeomorphism

$f : U \rightarrow V$ with U an open neighborhood of x_0 , V an open neighborhood of y_0 , $f(x_0) = y_0$ and $df_{x_0} = \sigma_0$. In order to verify the hypotheses of the CAH theorem, we consider a geodesic $\gamma : [a, b] \rightarrow M$ with $\gamma(a) = x_0$ and a geodesic $\mu : [a, b] \rightarrow M$ with $\mu(a) = y_0$ and $\mu'(a) = \sigma_0(\gamma'(a))$. Let $p : [a, b] \rightarrow \text{FR}(TM)$, $q : [a, b] \rightarrow \text{FR}(TM)$ be horizontal liftings of γ and μ respectively such that $q(a) = \sigma_0 \circ p(a)$; set $\sigma(t) = q(t) \circ p(t)^{-1} : T_{\gamma(t)}M \rightarrow T_{\mu(t)}M$. Lemma 5.1 implies that $\sigma(t)$ is G -structure preserving for all $t \in [a, b]$; namely, $\sigma(a) = \sigma_0$ is G -structure preserving and, since γ' and μ' are parallel, $\sigma(t)(\gamma'(t)) = \mu'(t)$, for all $t \in [a, b]$. Now, by the infinitesimal homogeneity hypothesis, we have that $\sigma(t)$ relates $T_{\gamma(t)}$ with $T_{\mu(t)}$ and $R_{\gamma(t)}$ with $R_{\mu(t)}$, for all $t \in [a, b]$. The CAH theorem now gives us a smooth affine diffeomorphism $f : U \rightarrow V$ with $f(x_0) = y_0$ and $df_{x_0} = \sigma_0$, where U, V are open neighborhoods of x_0 and y_0 , respectively. The differentials of such map f are given by the maps $\sigma(t)$ considered above and thus f is G -structure preserving. Finally, under the assumption of simply-connectedness and geodesical completeness of (M, ∇) , one can easily apply the global version of the CAH theorem to conclude that M is homogeneous. \square

6.1. Example. Let (M, g) be an n -dimensional semi-Riemannian manifold with constant sectional curvature equal to $c \in \mathbb{R}$, i.e.:

$$g_x(R_x(v, w)v, w) = -c(g_x(v, v)g_x(w, w) - g_x(v, w)^2),$$

for all $x \in M$ and all $v, w \in T_xM$; denote by r the index of g . Setting $G = O(n - r, r)$ and $P = \text{FR}^o(TM)$ then (M, ∇, P) is infinitesimally homogeneous, where ∇ denotes the Levi-Civita connection of g . Given an n -dimensional real vector space Z endowed with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of index r (which amounts to giving a G -structure on Z) then the “versions” T_Z, R_Z, \mathfrak{J}_Z of the characteristic tensors of (M, ∇, P) are given by $T_Z = 0, \mathfrak{J}_Z = 0$ and:

$$R_Z(z_1, z_2)z_3 = c(\langle z_2, z_3 \rangle z_1 - \langle z_1, z_3 \rangle z_2),$$

for all $z_1, z_2, z_3 \in Z$.

6.2. Example. Let (M, g) be an n -dimensional semi-Riemannian manifold with Levi-Civita connection ∇ , J be a ∇ -parallel g -antisymmetric almost complex structure on M ; in this case we call (M, g, J) a *semi-Riemannian Kähler* manifold. Assume that (M, g, J) has constant holomorphic sectional curvature equal to $c \in \mathbb{R}$, i.e., $g_x[R_x(v, J_x(v))v, J_x(v)] = -cg_x(v, v)^2$, for all $x \in M$ and all $v \in T_xM$. Denoting by r the index of g and setting $G = U(\frac{1}{2}r, \frac{1}{2}(n - r))$, $P = \text{FR}^u(TM)$ (recall Example 5.7) then (M, ∇, P) is infinitesimally homogeneous. If Z is a n -dimensional real vector space endowed with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of index r and a $\langle \cdot, \cdot \rangle$ -antisymmetric complex structure J_0 then the “versions” of the characteristic tensors of (M, ∇, P) are given by $T_Z = 0, \mathfrak{J}_Z = 0$ and:

$$\begin{aligned} R_Z(z_1, z_2)z_3 = & -\frac{c}{4}(\langle z_1, z_3 \rangle z_2 - \langle z_2, z_3 \rangle z_1 - \langle z_1, J_0(z_3) \rangle J_0(z_2) \\ & + \langle z_2, J_0(z_3) \rangle J_0(z_1) - 2\langle z_1, J_0(z_2) \rangle J_0(z_3)), \end{aligned}$$

for all $z_1, z_2, z_3 \in Z$.

6.3. Example. Let \mathcal{G} be an n -dimensional Lie group with Lie algebra \mathfrak{g} and let $p_0 : \mathbb{R}^n \rightarrow \mathfrak{g}$ be a linear isomorphism. Let ∇ be a left-invariant connection on \mathcal{G} and set $P = \{dL_g(1) \circ p_0 : g \in \mathcal{G}\}$, where $L_g : \mathcal{G} \rightarrow \mathcal{G}$ denotes left multiplication by g . Then P is a G -structure on \mathcal{G} with $G = \{\text{Id}_{\mathbb{R}^n}\}$ the trivial group. The affine manifold with G -structure (\mathcal{G}, ∇, P) is homogeneous, since the left translations are affine G -structure preserving diffeomorphisms. Let $\Gamma : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ be the linear map defined by $\Gamma(X)Y = \nabla_X Y$, for all $X, Y \in \mathfrak{g}$, where we identify the elements of the Lie algebra \mathfrak{g} with left invariant vector fields in \mathcal{G} . The characteristic tensors of (\mathcal{G}, ∇, P) are given by:

$$\begin{aligned} \mathfrak{I}_o(v) &= p_0^{-1} \circ \Gamma(p_0(v)) \circ p_0 \in \mathfrak{gl}(\mathbb{R}^n), \\ \mathfrak{T}_o(v, w) &= p_0^{-1} \left(\Gamma(p_0(v))p_0(w) - \Gamma(p_0(w))p_0(v) - [p_0(v), p_0(w)] \right) \in \mathbb{R}^n, \\ R_o(v, w) &= p_0^{-1} \circ \left([\Gamma(p_0(v)), \Gamma(p_0(w))] - \Gamma([p_0(v), p_0(w)]) \right) \circ p_0 \in \mathfrak{gl}(\mathbb{R}^n), \end{aligned}$$

for all $v, w \in \mathbb{R}^n$. If Z is a real n -dimensional vector space then a G -structure on Z is just a linear isomorphism from \mathbb{R}^n to Z , which can be used to push-forward $\mathfrak{I}_o, \mathfrak{T}_o, R_o$ respectively to $\mathfrak{I}_Z, \mathfrak{T}_Z$ and R_Z . If ∇ is the Levi-Civita connection of the semi-Riemannian left invariant metric on \mathcal{G} corresponding to the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} then Γ is given by:

$$\langle \Gamma(X)Y, Z \rangle = \frac{1}{2}(-\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle),$$

for all $X, Y, Z \in \mathfrak{g}$.

6.4. Example. Let (M, g) be an oriented three-dimensional Riemannian manifold with Levi-Civita connection ∇ such that:

- there exists a Riemannian submersion of (M, g) onto a two-dimensional Riemannian manifold of constant sectional curvature equal to $\kappa \in \mathbb{R}$;
- there exists a unitary smooth vector field ξ on M , vertical with respect to the submersion, and a real number τ such that $\nabla_v \xi = \tau v \times \xi(x)$, for every $x \in M, v \in T_x M$, where \times denotes the vector product on $T_x M$ (determined by the given inner product g_x and the given orientation).

For instance, all homogeneous three-dimensional Riemannian manifolds having a Lie group of isometries of dimension greater than or equal to four satisfy the conditions above (see [4]). Let $P = \bigcup_{x \in M} P_x$, where P_x denotes the set of all positively oriented linear isometries $p : \mathbb{R}^3 \rightarrow T_x M$ with $p(1, 0, 0) = \xi(x)$; then P is a G -structure on M , where $G \cong \text{SO}(\mathbb{R}^2)$ is the group of positively oriented linear isometries of \mathbb{R}^3 that fix $(1, 0, 0)$. Using (5.6), we see that the inner torsion $\mathfrak{I}_x^P : T_x M \rightarrow \text{sym}(T_x M) \oplus \xi(x)^\perp$ is given by:

$$\mathfrak{I}_x^P(v) = (0, \tau v \times \xi(x)),$$

for all $x \in M, v \in T_x M$. The curvature tensor R of ∇ can be computed easily and the resulting formulas show that R can be written only in terms of κ, τ, ξ and the metric g (see, for instance, [4]). Hence (M, ∇, P) is infinitesimally homogeneous. Given a three dimensional real vector space Z endowed with a G -structure (i.e., an

orientation, an inner product $\langle \cdot, \cdot \rangle$ and a unit vector $\mathfrak{z} \in Z$) then the “versions” of the characteristic tensors of (M, ∇, P) are given by:

$$(6.1) \quad \mathfrak{J}_Z(v) = (0, \tau v \times \mathfrak{z}) \in \text{sym}(Z) \oplus \mathfrak{z}^\perp,$$

$$(6.2) \quad \mathbb{T}_Z = 0,$$

$$(6.3) \quad R_Z(v, w)u = (\kappa - 3\tau^2)(\langle v, u \rangle w - \langle w, u \rangle v) + (\kappa - 4\tau^2)(\langle w, \mathfrak{z} \rangle \langle u, \mathfrak{z} \rangle v - \langle w, \mathfrak{z} \rangle \langle v, \mathfrak{z} \rangle \mathfrak{z} - \langle v, u \rangle \langle w, \mathfrak{z} \rangle \mathfrak{z} - \langle v, \mathfrak{z} \rangle \langle u, \mathfrak{z} \rangle w) \in Z,$$

for all $v, w, u \in Z$.

6.5. Example. Let $(M_1, \nabla_1), (M_2, \nabla_2)$ be affine manifolds, with $\dim(M_i) = n_i$, $i = 1, 2$; let G_i be a Lie subgroup of $\text{GL}(\mathbb{R}^{n_i})$ and let P_i be a G_i -structure on M_i , $i = 1, 2$. Denote by ∇ the connection on $M_1 \times M_2$ naturally induced by ∇_1, ∇_2 ; the product $G = G_1 \times G_2$ can be identified (diagonally) with a Lie subgroup of $\text{GL}(\mathbb{R}^{n_1+n_2})$ and the product $P = P_1 \times P_2$ is a G -structure on M . If (M_1, ∇_1, P_1) and (M_2, ∇_2, P_2) are homogeneous (resp., infinitesimally homogeneous) then also (M, ∇, P) is homogeneous (resp., infinitesimally homogeneous). The characteristic tensors of (M, ∇, P) are given by the obvious direct sums of the corresponding characteristic tensors of (M_1, ∇_1, P_1) and (M_2, ∇_2, P_2) . Notice that if Z is an $(n_1 + n_2)$ -dimensional real vector space then a G -structure on Z is determined by a direct sum decomposition $Z = Z_1 \oplus Z_2$ with $\dim(Z_i) = n_i$, $i = 1, 2$, by a G_1 -structure on Z_1 and by a G_2 -structure on Z_2 .

7. EXISTENCE OF G -STRUCTURE PRESERVING AFFINE IMMERSIONS

Throughout the section we consider fixed the following objects: affine manifolds (M, ∇) and $(\overline{M}, \overline{\nabla})$, a vector bundle E^0 over M , a connection ∇^0 on E^0 and smooth sections α^0, A^0 of $TM^* \otimes TM^* \otimes E^0$ and $TM^* \otimes (E^0)^* \otimes TM$ respectively; we denote by \mathbb{T} and R respectively the torsion and the curvature tensors of ∇ , by $\overline{\mathbb{T}}$ and \overline{R} respectively the torsion and the curvature tensors of $\overline{\nabla}$, by R^0 the curvature tensor of ∇^0 and by ∇^\otimes the connection induced by ∇ and ∇^0 on the vector bundles $TM^* \otimes TM^* \otimes E^0$ and $TM^* \otimes (E^0)^* \otimes TM$. We are interested in studying conditions under which there is a solution (f, S) for the affine immersion problem with data ∇^0, α^0 and A^0 , satisfying a given initial condition (i.e., $f(x_0), df(x_0)$ and S_{x_0} should be prescribed, for some $x_0 \in M$).

Set $n = \dim(M)$, $k = \text{rank}(E^0)$, $\bar{n} = \dim(\overline{M})$,

$$\widehat{E} = TM \oplus E^0,$$

and assume that $\bar{n} = n + k$, i.e., that $\text{rank}(\widehat{E}) = \bar{n}$; we denote by $\pi^{TM} : \widehat{E} \rightarrow TM$, $\pi^{E^0} : \widehat{E} \rightarrow E^0$ the projections. Let $\widehat{\nabla}$ be the connection in \widehat{E} whose components are $\nabla, \nabla^0, \alpha^0$ and A^0 . If (f, S) is a solution for the affine immersion problem with data ∇^0, α^0 and A^0 , we define a vector bundle isomorphism $L : \widehat{E} \rightarrow f^*T\overline{M}$ by setting:

$$(7.1) \quad L|_{TM} = df : TM \rightarrow f^*T\overline{M}, \quad L|_{E^0} = S : E^0 \rightarrow f^*T\overline{M}.$$

The vector bundle isomorphism L is connection preserving. Conversely, given a smooth map $f : M \rightarrow \overline{M}$ and a connection preserving vector bundle isomorphism $L : \widehat{E} \rightarrow f^*T\overline{M}$ such that $L|_{TM} = df$ then, setting $S = L|_{E^0}$, we obtain a solution (f, S) for the affine immersion problem with data ∇^0, α^0 and A^0 .

Denote by $\omega^M, \omega^{\overline{M}}, \theta^M, \theta^{\overline{M}}$ respectively the $\mathfrak{gl}(\mathbb{R}^{\bar{n}})$ -valued connection form of $\widehat{\nabla}$ in $\text{FR}(\widehat{E})$, the $\mathfrak{gl}(\mathbb{R}^{\bar{n}})$ -valued connection form of $\overline{\nabla}$ in $\text{FR}(T\overline{M})$, the $\mathbb{R}^{\bar{n}}$ -valued ι -canonical form of $\text{FR}(\widehat{E})$ and the $\mathbb{R}^{\bar{n}}$ -valued canonical form of $\text{FR}(T\overline{M})$, where $\iota : TM \rightarrow \widehat{E}$ denotes the inclusion. Let $s : V \rightarrow \text{FR}(\widehat{E})$ be a smooth local section, where V is an open subset of M . If $f : M \rightarrow \overline{M}$ is a smooth map and $L : \widehat{E} \rightarrow f^*T\overline{M}$ is a vector bundle isomorphism then we define a smooth map $F : V \rightarrow \text{FR}(T\overline{M})$ by setting:

$$(7.2) \quad F(x) = L_x \circ s(x),$$

for all $x \in V$. It is easy to see that:

$$(7.3) \quad \begin{aligned} F^*\theta^{\overline{M}} = s^*\theta^M &\iff L|_{TM} = df, \\ F^*\omega^{\overline{M}} = s^*\omega^M &\iff L \text{ is connection preserving.} \end{aligned}$$

The conditions $F^*\theta^{\overline{M}} = s^*\theta^M$ and $F^*\omega^{\overline{M}} = s^*\omega^M$ can be summarized by writing:

$$(7.4) \quad F^*\lambda^{\overline{M}} = \lambda^M,$$

where:

$$(7.5) \quad \lambda^{\overline{M}} = (\theta^{\overline{M}}, \omega^{\overline{M}}), \quad \lambda^M = s^*(\theta^M, \omega^M).$$

Hence if (f, S) is a solution for the affine immersion problem with data ∇^0, α^0 and A^0 and if L and F are defined as in (7.1) and (7.2) then, by (7.3), equality (7.4) holds. Notice also that, conversely, if a smooth map $F : V \rightarrow \text{FR}(T\overline{M})$ is given then one can obtain a smooth map $f : V \rightarrow \overline{M}$ and a vector bundle isomorphism $L : \widehat{E}|_V \rightarrow f^*T\overline{M}$ by setting:

$$(7.6) \quad f = \overline{\Pi} \circ F, \quad L_x = F(x) \circ s(x)^{-1}$$

for all $x \in V$, where $\overline{\Pi} : \text{FR}(T\overline{M}) \rightarrow \overline{M}$ denotes the projection; in particular, if F satisfies (7.4) then one obtain a local solution (f, S) for the affine immersion problem with data ∇^0, α^0 and A^0 by setting $S = L|_{E^0}$.

We prove a general result concerning uniqueness of solutions of the affine immersion problem.

Proposition 7.1. *Assume that M is connected. If $(f^1, S^1), (f^2, S^2)$ are both solutions of the affine immersion problem with data ∇^0, α^0 and A^0 and if there exists $x_0 \in M$ with:*

$$f^1(x_0) = f^2(x_0), \quad df^1(x_0) = df^2(x_0), \quad S_{x_0}^1 = S_{x_0}^2,$$

then $(f^1, S^1) = (f^2, S^2)$.

PROOF. We will show that the set A of points $x \in M$ with $f^1(x) = f^2(x)$, $df^1(x) = df^2(x)$ and $S_x^1 = S_x^2$ is open and closed in M (clearly $x_0 \in A$). Let $x \in M$ be fixed and let $s : V \rightarrow \text{FR}(\widehat{E})$ be a smooth local section, where V is a connected open neighborhood of x in M . For $i = 1, 2$, let $L^i : \widehat{E} \rightarrow (f^i)^*T\overline{M}$, $F^i : V \rightarrow \text{FR}(T\overline{M})$ be the maps defined from (f^i, S^i) as in (7.1) and (7.2). As it was observed in (7.4):

$$(F^i)^*\lambda^{\overline{M}} = \lambda^M.$$

Clearly, for $y \in V$, we have $y \in A$ if and only if $F^1(y) = F^2(y)$. Thus, if x is not in A then $F^1(x) \neq F^2(x)$ and therefore $F^1(y) \neq F^2(y)$ for y near x . This proves that A is closed. If $x \in A$ then $F^1(x) = F^2(x)$ and, keeping in mind that $\lambda_{\overline{p}}^{\overline{M}} : T_{\overline{p}}\text{FR}(T\overline{M}) \rightarrow \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}})$ is an isomorphism for all $\overline{p} \in \text{FR}(T\overline{M})$, we apply Lemma 7.2 below and obtain that $F^1 = F^2$. Thus $V \subset A$ and we are done. \square

Lemma 7.2. *Let \mathcal{M}, \mathcal{N} be differentiable manifolds, Z be a real finite-dimensional vector space and $\lambda, \tilde{\lambda}$, be Z -valued smooth 1-forms on \mathcal{M} and on \mathcal{N} respectively; assume that \mathcal{M} is connected and that $\tilde{\lambda}_y : T_y\mathcal{N} \rightarrow Z$ is an isomorphism, for all $y \in \mathcal{N}$. If $F_1 : \mathcal{M} \rightarrow \mathcal{N}$, $F_2 : \mathcal{M} \rightarrow \mathcal{N}$ are smooth maps with $F_i^*\tilde{\lambda} = \lambda$, $i = 1, 2$, and if $F_1(x_0) = F_2(x_0)$ for some $x_0 \in \mathcal{M}$ then $F_1 = F_2$.*

PROOF. If $\gamma : I \rightarrow \mathcal{M}$ is a smooth curve such that $F_1(\gamma(t_0)) = F_2(\gamma(t_0))$ for some $t_0 \in I$ then $F_1 \circ \gamma = F_2 \circ \gamma$; namely, both $F_1 \circ \gamma$ and $F_2 \circ \gamma$ are integral curves of the smooth time-dependent vector field:

$$I \times \mathcal{N} \ni (t, y) \longmapsto \tilde{\lambda}_y^{-1}[\lambda_{\gamma(t)}(\gamma'(t))] \in T_y\mathcal{N}.$$

The conclusion follows from the observation that, since \mathcal{M} is connected, any two points of \mathcal{M} can be joined by a piecewise smooth curve. \square

In the remainder of this section we will also fix a Lie subgroup G of $\text{GL}(\mathbb{R}^{\bar{n}})$ with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^{\bar{n}})$, a G -structure \widehat{P} on \widehat{E} and a G -structure \overline{P} on \overline{M} .

Definition 7.3. A (possibly local) solution (f, S) for the affine immersion problem with data ∇^0 , α^0 and A^0 is said to be *G -structure preserving* if the vector bundle isomorphism $L : \widehat{E} \rightarrow f^*T\overline{M}$ defined in (7.1) is G -structure preserving, i.e., if $L_x \circ p \in \overline{P}_{f(x)}$, for all $x \in M$ and all $p \in \widehat{P}_x$.

If $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous then every \bar{n} -dimensional real vector space Z endowed with a G -structure inherits “versions” $\overline{T}_Z : Z \times Z \rightarrow Z$, $\overline{R}_Z : Z \times Z \rightarrow \mathfrak{gl}(Z)$ and $\overline{\mathcal{T}}_Z : Z \rightarrow \mathfrak{gl}(Z)/\mathfrak{g}_Z$ of the characteristic tensors $\overline{T}_0 : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^{\bar{n}}$, $\overline{R}_0 : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \rightarrow \mathfrak{gl}(\mathbb{R}^{\bar{n}})$ and $\overline{\mathcal{T}}_0 : \mathbb{R}^{\bar{n}} \rightarrow \mathfrak{gl}(\mathbb{R}^{\bar{n}})/\mathfrak{g}$ of $(\overline{M}, \overline{\nabla}, \overline{P})$ (recall Remark 6.2).

Theorem 7.4. *Assume that $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous with characteristic tensors $\overline{\mathbb{T}}_o$, \overline{R}_o and $\overline{\mathbb{J}}_o$ and that the equalities:*

$$(7.7) \quad \begin{aligned} \pi^{TM}(\overline{R}_{\widehat{E}_x}(v, w)u) &= R_x(v, w)u + A_x^0(v, \alpha_x^0(w, u)) \\ &\quad - A_x^0(w, \alpha_x^0(v, u)), \end{aligned}$$

$$(7.8) \quad \begin{aligned} \pi^{E^0}(\overline{R}_{\widehat{E}_x}(v, w)e) &= R_x^0(v, w) + \alpha_x^0(v, A_x^0(w, e)) \\ &\quad - \alpha_x^0(w, A_x^0(v, e)), \end{aligned}$$

$$(7.9) \quad \begin{aligned} \pi^{E^0}(\overline{R}_{\widehat{E}_x}(v, w)u) &= (\nabla^\otimes \alpha^0)_x(v, w, u) - (\nabla^\otimes \alpha^0)_x(w, v, u) \\ &\quad + \alpha_x^0(\mathbb{T}_x(v, w), u), \end{aligned}$$

$$(7.10) \quad \begin{aligned} \pi^{TM}(\overline{R}_{\widehat{E}_x}(v, w)e) &= (\nabla^\otimes A^0)_x(v, w, e) - (\nabla^\otimes A^0)_x(w, v, e) \\ &\quad + A_x^0(\mathbb{T}_x(v, w), e), \end{aligned}$$

$$(7.11) \quad \pi^{TM}(\overline{\mathbb{T}}_{\widehat{E}_x}(v, w)) = \mathbb{T}_x(v, w),$$

$$(7.12) \quad \pi^{E^0}(\overline{\mathbb{T}}_{\widehat{E}_x}(v, w)) = \alpha_x^0(v, w) - \alpha_x^0(w, v),$$

$$(7.13) \quad \overline{\mathbb{J}}_{\widehat{E}_x}(v) = \mathfrak{J}_x^{\widehat{P}}(v),$$

hold, for all $x \in M$, $v, w, u \in T_x M$ and all $e \in E_x^0$. Then, for every $x_0 \in M$, every $y_0 \in \overline{M}$ and every G -structure preserving linear map $\sigma_0 : \widehat{E}_{x_0} \rightarrow T_{y_0} \overline{M}$ there exists a G -structure preserving local solution (f, S) for the affine immersion problem with data ∇^0 , α^0 and A^0 whose domain is an open neighborhood U of x_0 in M , $f(x_0) = y_0$ and $\sigma_0 = L_{x_0}$, where L is as in (7.1). Moreover, if M is (connected and) simply-connected and $(\overline{M}, \overline{\nabla})$ is geodesically complete then there exists a unique global solution (f, S) for the affine immersion problem with data ∇^0 , α^0 and A^0 such that $f(x_0) = y_0$ and $\sigma_0 = L_{x_0}$.

PROOF OF THE EXISTENCE OF LOCAL SOLUTIONS. Let $s : V \rightarrow \widehat{P}$ be a smooth local section defined in an open neighborhood V of x_0 in M and consider the $\mathfrak{gl}(\mathbb{R}^{\bar{n}})$ -valued 1-forms λ^M and $\lambda^{\overline{M}}$ defined in (7.5). We will look for a smooth map $F : U \rightarrow \widehat{P} \subset \text{FR}(T\overline{M})$ such that $F^* \lambda^{\overline{M}} = \lambda^M|_U$ and $F(x_0) = \sigma_0 \circ s(x_0)$, where U is an open neighborhood of x_0 in V . Once such map F is found, we define $f : U \rightarrow \overline{M}$ and $L : \widehat{E}|_U \rightarrow f^* T\overline{M}$ as in (7.6) and we set $S = L|_{E^0}$. It will then follow from (7.3) that (f, S) is a local solution for the affine immersion problem with data ∇^0 , α^0 and A^0 ; moreover, the fact that s takes values in \widehat{P} and F takes values in \overline{P} will imply that L is G -structure preserving.

In order to find the map F we will use the version of the Frobenius theorem stated in Lemma 7.5 below. We claim that for each $y \in \overline{M}$ and each $\bar{p} \in \overline{P}_y$, the linear map $\lambda_{\bar{p}}^{\overline{M}} : T_{\bar{p}} \overline{P} \rightarrow \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}})$ is an isomorphism onto the space:

$$Z = \{(u, X) \in \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}}) : \overline{\mathbb{J}}_o(u) = X + \mathfrak{g}\}.$$

Namely, $(\theta_{\bar{p}}^{\overline{M}}, \omega_{\bar{p}}^{\overline{M}}) = (\bar{p}^{-1} \oplus \text{Id}) \circ (\text{d}\overline{\Pi}_{\bar{p}}, \omega_{\bar{p}}^{\overline{M}})$ and the isomorphism $(\text{d}\overline{\Pi}_{\bar{p}}, \omega_{\bar{p}}^{\overline{M}})$ carries $T_{\bar{p}}\overline{P}$ onto the space (recall (5.1) and (5.2)):

$$\mathcal{V}_{\bar{p}} = \{(v, X) \in T_y\overline{M} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}}) : \overline{\text{Ad}}_{\bar{p}}^{-1}(\mathcal{J}_y^{\overline{P}}(v)) = X + \mathfrak{g}\},$$

and $\overline{\mathcal{J}}_{\circ} = \overline{\text{Ad}}_{\bar{p}}^{-1} \circ \mathcal{J}_y^{\overline{P}} \circ \bar{p}$. We now claim that for all $x \in V$, the linear map $\lambda_x^M : T_x M \rightarrow \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}})$ takes values in Z . Namely, for all $\hat{p} \in \widehat{P}_x$, the linear map $(\theta_{\hat{p}}^M, \omega_{\hat{p}}^M) = ((\hat{p}^{-1} \circ \iota_x) \oplus \text{Id}) \circ (\text{d}\widehat{\Pi}_{\hat{p}}, \omega_{\hat{p}}^M) : T_{\hat{p}}\text{FR}(\widehat{E}) \rightarrow \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}})$ takes $T_{\hat{p}}\widehat{P}$ to the space:

$$(7.14) \quad \{(u, X) \in \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}}) : \hat{p}(u) \in T_x M, \overline{\text{Ad}}_{\hat{p}}^{-1}[\mathcal{J}_x^{\widehat{P}}(\hat{p}(u))] = X + \mathfrak{g}\},$$

where $\widehat{\Pi} : \text{FR}(\widehat{E}) \rightarrow M$ denotes the projection. By assumption (7.13) we may replace $\mathcal{J}_x^{\widehat{P}}$ with $\overline{\mathcal{J}}_{\widehat{E}_x}$ in (7.14) and, since $\overline{\text{Ad}}_{\hat{p}}^{-1} \circ \overline{\mathcal{J}}_{\widehat{E}_x} \circ \hat{p} = \overline{\mathcal{J}}_{\circ}$, we conclude that $(\theta_{\hat{p}}^M, \omega_{\hat{p}}^M)(T_{\hat{p}}\widehat{P}) \subset Z$. The proof of the claim is completed by setting $\hat{p} = s(x)$ and by observing that $\lambda_x^M(T_x M)$ is contained in $(\theta_{\hat{p}}^M, \omega_{\hat{p}}^M)(T_{\hat{p}}\widehat{P})$. We have so far checked the validity of the hypotheses of Lemma 7.5; we will now verify that condition (b) in the statement of the lemma holds and this will give us the desired map $F : U \rightarrow \overline{P}$ with $F^*\lambda^{\overline{M}} = \lambda^M|_U$ and $F(x_0) = \sigma_0 \circ s(x_0)$. Let $x \in V$, $y \in \overline{M}$, $\bar{p} \in \overline{P}_y$ be fixed and set $\tau = (\lambda_{\bar{p}}^{\overline{M}})^{-1} \circ \lambda_x^M : T_x M \rightarrow T_{\bar{p}}\overline{P}$. We have to check that:

$$(7.15) \quad \tau^* \text{d}\theta_{\bar{p}}^{\overline{M}} = \text{d}(s^*\theta^M)_x, \quad \tau^* \text{d}\omega_{\bar{p}}^{\overline{M}} = \text{d}(s^*\omega^M)_x;$$

since $\tau^*\lambda_{\bar{p}}^{\overline{M}} = \lambda_x^M$, i.e., $\tau^*\theta_{\bar{p}}^{\overline{M}} = s^*\theta_x^M$ and $\tau^*\omega_{\bar{p}}^{\overline{M}} = s^*\omega_x^M$, equalities (2.2) imply that (7.15) is equivalent to:

$$(7.16) \quad \tau^*\Theta_{\bar{p}}^{\overline{M}} = (s^*\Theta^M)_x, \quad \tau^*\Omega_{\bar{p}}^{\overline{M}} = (s^*\Omega^M)_x,$$

where $\Omega^M, \Omega^{\overline{M}}, \Theta^M, \Theta^{\overline{M}}$ denote respectively the curvature form of $\widehat{\nabla}$ in $\text{FR}(\widehat{E})$, the curvature form of $\overline{\nabla}$ in $\text{FR}(T\overline{M})$, the ι -torsion form of $\widehat{\nabla}$ in $\text{FR}(\widehat{E})$ and the curvature form of $\overline{\nabla}$ in $\text{FR}(T\overline{M})$. It is easy to see that $\text{d}\overline{\Pi}_{\bar{p}} \circ \tau = \bar{p} \circ s(x)^{-1}|_{T_x M}$; using this equality, and equalities (2.3) and (2.4), we get that (7.16) is equivalent to:

$$(7.17) \quad \bar{p}^{-1} \left(\overline{\mathcal{T}}_y [\bar{p}(s(x)^{-1}(v)), s(x)^{-1}(w)] \right) = s(x)^{-1}(\widehat{\mathcal{T}}_x^\iota(v, w)),$$

$$(7.18) \quad \bar{p}^{-1} \circ \overline{R}_y [\bar{p}(s(x)^{-1}(v)), s(x)^{-1}(w)] \circ \bar{p} = s(x)^{-1} \circ \widehat{R}_x(v, w) \circ s(x),$$

for all $v, w \in T_x M$, where $\widehat{\mathcal{T}}^\iota$ and \widehat{R} denote respectively the ι -torsion tensor and the curvature tensor of $\widehat{\nabla}$. Clearly (7.17) and (7.18) are equivalent to:

$$(7.19) \quad \overline{\mathcal{T}}_{\circ}(s(x)^{-1}(v), s(x)^{-1}(w)) = s(x)^{-1}(\widehat{\mathcal{T}}_x^\iota(v, w)),$$

$$(7.20) \quad \overline{R}_{\circ}(s(x)^{-1}(v), s(x)^{-1}(w)) = s(x)^{-1} \circ \widehat{R}_x(v, w) \circ s(x).$$

Now (7.19) and (7.20) are equivalent to:

$$(7.21) \quad \overline{T}_{\widehat{E}_x}(v, w) = \widehat{T}_x^\iota(v, w), \quad v, w \in T_x M,$$

$$(7.22) \quad \overline{R}_{\widehat{E}_x}(v, w) = \widehat{R}_x(v, w), \quad v, w \in T_x M.$$

Finally, (7.21) is equivalent to assumptions (7.11), (7.12) (recall (4.6) and (4.7)), while (7.22) is equivalent to assumptions (7.7), (7.8), (7.9) and (7.10) (recall (4.2), (4.3), (4.4) and (4.5)). This concludes the proof of the existence of local solutions of the affine immersion problem. \square

Lemma 7.5. *Let \mathcal{M}, \mathcal{N} be differentiable manifolds, Z be a real finite-dimensional vector space and $\lambda, \tilde{\lambda}$, be Z -valued smooth 1-forms on \mathcal{M} and on \mathcal{N} respectively; assume that $\tilde{\lambda}_y : T_y \mathcal{N} \rightarrow Z$ is an isomorphism, for all $y \in \mathcal{N}$. The following conditions are equivalent:*

- (a) *for all $x \in \mathcal{M}, y \in \mathcal{N}$ there exists a smooth map $F : U \rightarrow \mathcal{N}$ defined in an open neighborhood U of x in \mathcal{M} with $F(x) = y$ such that $F^* \tilde{\lambda} = \lambda|_U$;*
- (b) *for all $x \in \mathcal{M}, y \in \mathcal{N}$, $\tau_{xy}^* d\tilde{\lambda}_y = d\lambda_x$, where $\tau_{xy} : T_x \mathcal{M} \rightarrow T_y \mathcal{N}$ is the linear map defined by $\tau_{xy} = \tilde{\lambda}_y^{-1} \circ \lambda_x$.*

PROOF. It is a simple application of the classical Frobenius theorem to the distribution on $\mathcal{M} \times \mathcal{N}$ annihilated by the Z -valued 1-form $\theta = \text{pr}_2^* \tilde{\lambda} - \text{pr}_1^* \lambda$, where $\text{pr}_1 : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M}$, $\text{pr}_2 : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{N}$ denote the projections. \square

In order to prove the existence of global solutions to the affine immersion problem, we will employ a very general globalization technique that is explained below. Let X be a topological space and let \mathfrak{P} be a pre-sheaf of sets over X , i.e., a cofunctor from the category of open subsets of X (partially ordered by inclusion) to the category of sets and maps; for each open subset U of X we denote by $\mathfrak{P}(U)$ the corresponding set and for every open subset V of U we denote by $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \rightarrow \mathfrak{P}(V)$ the corresponding map. We say that the pre-sheaf \mathfrak{P} has the *localization property* if, given a family $(U_i)_{i \in I}$ of open subsets of X and setting $U = \bigcup_{i \in I} U_i$ then the map:

$$(7.23) \quad \mathfrak{P}(U) \ni f \longmapsto (\mathfrak{P}_{U,U_i}(f))_{i \in I} \in \prod_{i \in I} \mathfrak{P}(U_i)$$

is injective and its image consists of the families $(f_i)_{i \in I}$ in $\prod_{i \in I} \mathfrak{P}(U_i)$ such that $\mathfrak{P}_{U_i, U_i \cap U_j}(f_i) = \mathfrak{P}_{U_j, U_i \cap U_j}(f_j)$, for all $i, j \in I$. We say that the pre-sheaf \mathfrak{P} has the *uniqueness property* if for every connected open subset $U \subset X$ and every nonempty open subset $V \subset U$ the map $\mathfrak{P}_{U,V}$ is injective. We say that an open subset $U \subset X$ has the *extension property with respect to the pre-sheaf \mathfrak{P}* if for every connected nonempty open subset V of U the map $\mathfrak{P}_{U,V}$ is surjective. We say that the pre-sheaf \mathfrak{P} has the *extension property* if X can be covered by open sets having the extension property with respect to \mathfrak{P} .

If \tilde{X}, X are topological spaces and $\pi : \tilde{X} \rightarrow X$ is a local homeomorphism then an open subset U of X is said to be *fundamental* for π if $\pi^{-1}(U)$ is a disjoint union $\bigcup_{i \in I} U_i$ of open subsets U_i of \tilde{X} such that π maps U_i homeomorphically

onto U for all $i \in I$. An open subset U of X is said to be *quasi-fundamental* for π if for every $x \in U$ and every $\tilde{x} \in \pi^{-1}(x)$ there exists a continuous local section $s : U \rightarrow \tilde{X}$ of π such that $s(x) = \tilde{x}$. Clearly, every fundamental open subset is also quasi-fundamental; conversely, if \tilde{X} is Hausdorff then it is easy to check that every connected quasi-fundamental open subset of X is fundamental (take the U_i 's to be the images of the continuous local sections of π defined in U).

We have the following general globalization principle:

Proposition 7.6. *Let X be a topological space and \mathfrak{P} be a pre-sheaf of sets over X . Assume that X is Hausdorff, locally arc-connected, connected, simply-connected and that the pre-sheaf \mathfrak{P} has the localization property, the uniqueness property and the extension property. Then for every connected nonempty open subset $V \subset X$ and every $\mathfrak{f} \in \mathfrak{P}(V)$ there exists a unique $\bar{\mathfrak{f}} \in \mathfrak{P}(X)$ with $\mathfrak{P}_{X,V}(\bar{\mathfrak{f}}) = \mathfrak{f}$.*

PROOF. Let $\pi : \mathcal{S} \rightarrow X$ be the sheaf of germs associated to the pre-sheaf \mathfrak{P} and for each open subset U of X , each $\mathfrak{f} \in \mathfrak{P}(U)$ and each $x \in U$, denote by $[\mathfrak{f}]_x \in \mathcal{S}_x = \pi^{-1}(x)$ the germ of \mathfrak{f} at x . The fact that \mathfrak{P} has the localization property means that for every open subset U of X and every continuous local section $s : U \rightarrow \mathcal{S}$ of π there exists a unique $\mathfrak{f} \in \mathfrak{P}(U)$ such that $s(x) = [\mathfrak{f}]_x$, for all $x \in U$. Since X is locally connected, it follows that every open subset of X having the extension property with respect to \mathfrak{P} is quasi-fundamental for π . Moreover, since X is Hausdorff and locally connected, the uniqueness property of \mathfrak{P} implies that \mathcal{S} is Hausdorff. Thus, every quasi-fundamental connected open subset of X is fundamental and we obtain that π is a covering map. Let $x \in V$ be fixed and let \mathcal{S}' denote the connected component of \mathcal{S} containing $[\mathfrak{f}]_x$; since X is locally arc-connected, connected and simply-connected, it follows that the restriction of π to \mathcal{S}' is a homeomorphism onto X . The inverse of $\pi|_{\mathcal{S}'}$ is a continuous global section $s : X \rightarrow \mathcal{S}$ of π with $s(x) = [\mathfrak{f}]_x$. As observed in the beginning of the proof, there exists a unique $\bar{\mathfrak{f}} \in \mathfrak{P}(X)$ such that $[\bar{\mathfrak{f}}]_y = s(y)$, for all $y \in X$; in particular, we have $[\bar{\mathfrak{f}}]_x = [\mathfrak{f}]_x$. Since V is connected and \mathfrak{P} has the uniqueness property, it follows that $\mathfrak{P}_{X,V}(\bar{\mathfrak{f}}) = \mathfrak{f}$. \square

We can now prove the global part of the statement of Theorem 7.4.

PROOF OF THE EXISTENCE OF GLOBAL SOLUTIONS. By considering the universal covering, there is no loss of generality in assuming that \bar{M} is connected and simply-connected; thus, it follows from Proposition 6.4 that $(\bar{M}, \bar{\nabla}, \bar{P})$ is homogeneous.

For each open subset U of M let $\mathfrak{P}(U)$ be the set of all G -structure preserving local solutions for the affine immersion problem with data ∇^0, α^0, A^0 and with domain U ; if V is an open subset of U , we let $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \rightarrow \mathfrak{P}(V)$ be the restriction map. It is obvious that the pre-sheaf of sets \mathfrak{P} over M has the localization property and it follows from Proposition 7.1 that \mathfrak{P} has the uniqueness property. We will prove that \mathfrak{P} has the extension property and then the aimed result will follow from the local part of the statement of Theorem 7.4 and from Proposition 7.6. In order to prove that \mathfrak{P} has the extension property we will show that every open

subset U of M such that $\mathfrak{P}(U)$ is nonempty has the extension property for \mathfrak{P} (observe also that the local part of the statement of Theorem 7.4 implies that every point of M has an open neighborhood U such that $\mathfrak{P}(U)$ is nonempty). Let U be an open subset of M such that $\mathfrak{P}(U)$ is nonempty, V be a connected nonempty open subset of U and let (f, S) be an element of $\mathfrak{P}(V)$. We have to find (\bar{f}, \bar{S}) in $\mathfrak{P}(U)$ such that (f, S) is the restriction of (\bar{f}, \bar{S}) to V . Let (\tilde{f}, \tilde{S}) be an arbitrary element of the nonempty set $\mathfrak{P}(U)$ and let $x \in V$ be an arbitrary point. Denote by $L : \hat{E}|_V \rightarrow f^*T\bar{M}$, $\tilde{L} : \hat{E}|_U \rightarrow \tilde{f}^*T\bar{M}$ the G -structure preserving connection preserving vector bundle isomorphisms corresponding respectively to (f, S) and (\tilde{f}, \tilde{S}) , as in (7.1). Let $\tau : T_{\tilde{f}(x)}\bar{M} \rightarrow T_{f(x)}\bar{M}$ be the linear isomorphism such that $\tau \circ \tilde{L}_x = L_x$; clearly, τ is G -structure preserving and thus there exists a global G -structure preserving affine diffeomorphism $\phi : \bar{M} \rightarrow \bar{M}$ such that $\phi(\tilde{f}(x)) = f(x)$ and $d\phi_{\tilde{f}(x)} = \tau$. We set $\bar{f} = \phi \circ \tilde{f} : U \rightarrow \bar{M}$ and we consider the G -structure preserving connection preserving vector bundle isomorphism $\bar{L} : \hat{E}|_U \rightarrow \bar{f}^*T\bar{M}$ defined by $\bar{L}_y = d\phi_{\tilde{f}(y)} \circ \tilde{L}_y$, for all $y \in U$. Setting $\bar{S} = \bar{L}|_{E^0}$ then (\bar{f}, \bar{S}) is in $\mathfrak{P}(U)$. Moreover, since $\bar{f}(x) = f(x)$ and $\bar{L}_x = L_x$, it follows from Proposition 7.1 and from the connectedness of V that (f, S) is the restriction of (\bar{f}, \bar{S}) to V . This concludes the proof. \square

7.1. Example. Assume that \bar{M} is a Lie group \mathcal{G} with Lie algebra \mathfrak{g} and that $\bar{\nabla}$ is a left invariant connection corresponding to a linear map $\Gamma : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ as in Example 6.3; let $p_0 : \mathbb{R}^{\bar{n}} \rightarrow \mathfrak{g}$ be a linear isomorphism and let \bar{P} be the $\{\text{Id}_{\mathbb{R}^{\bar{n}}}\}$ -structure on \mathcal{G} defined by $\bar{P} = \{dL_g(1) \circ p_0 : g \in \mathcal{G}\}$. Then $(\bar{M}, \bar{\nabla}, \bar{P})$ is homogeneous. Assume that we are given a global smooth frame $s : M \rightarrow \text{FR}(\hat{E})$ of \hat{E} , so that $\hat{P} = s(M)$ is a $\{\text{Id}_{\mathbb{R}^{\bar{n}}}\}$ -structure on \hat{E} . An application of Theorem 7.4 shows that, under assumptions (7.7), (7.8), (7.9), (7.10), (7.11), (7.12), (7.13), we can find local solutions (f, S) (with adequately prescribed initial conditions $f(x_0), df(x_0), S_{x_0}$) for the affine immersion problem with data ∇^0, α^0 and A^0 , where the vector bundle isomorphism $L : \hat{E} \rightarrow f^*T\mathcal{G}$ defined by (7.1) satisfies $L_x \circ s(x) = dL_{f(x)}(1) \circ p_0$, for all $x \in M$. Observe that the tensors $\bar{R}_{\hat{E}_x}, \bar{T}_{\hat{E}_x}, \bar{\mathcal{J}}_{\hat{E}_x}$ that appear in the lefthand side of the assumptions are the push-forward by the linear isomorphism $s(x) : \mathbb{R}^{\bar{n}} \rightarrow \hat{E}_x$ of the respective characteristic tensors R_o, T_o and \mathcal{J}_o that appear in Example 6.3. Equation (7.13) means simply that the Christoffel tensor $\hat{\Gamma}_x : T_x M \rightarrow \mathfrak{gl}(\hat{E}_x)$ of $\hat{\nabla}$ with respect to the frame s is the restriction to $T_x M$ of the push-forward to \hat{E}_x of \mathcal{J}_o by $s(x)$.

8. EXISTENCE OF G -STRUCTURE PRESERVING ISOMETRIC IMMERSIONS

As a special case of the theory developed in Section 7, in this section we will consider semi-Riemannian manifolds endowed with G -structures contained in the orthonormal frame bundle, aiming at an existence theorem for isometric immersions. More precisely, we consider fixed the following objects: an \bar{n} -dimensional semi-Riemannian manifold (\bar{M}, \bar{g}) , where the semi-Riemannian metric \bar{g} has index

\bar{r} , an n -dimensional semi-Riemannian manifold (M, g) , where the semi-Riemannian metric g has index r , a vector bundle E^0 of rank k over M endowed with a semi-Riemannian structure g^0 of index s , where $\bar{n} = n + k$ and $\bar{r} = r + s$, a connection ∇^{E^0} on E^0 compatible with g^0 , a smooth symmetric section α^0 of $TM^* \otimes TM^* \otimes E^0$, a Lie subgroup G of $O(\bar{n} - \bar{r}, \bar{r})$ with Lie algebra $\mathfrak{g} \subset \mathfrak{so}(\bar{n} - \bar{r}, \bar{r})$, a G -structure \bar{P} on \bar{M} contained in the orthonormal frame bundle $FR^o(T\bar{M})$ and a G -structure \hat{P} on the direct sum $\hat{E} = TM \oplus E$ contained in $FR^o(\hat{E})$, where \hat{E} is endowed with the semi-Riemannian structure \hat{g} of index \bar{r} given by the orthogonal direct sum of g and g^0 . We denote by ∇ and $\bar{\nabla}$ respectively the Levi-Civita connections of g and \bar{g} , by $\hat{\nabla}$ the connection on \hat{E} compatible with \hat{g} with components ∇ , ∇^{E^0} and α^0 (see Remark 4.1), by ∇^\otimes the connection induced by ∇ and ∇^0 on the vector bundle $TM^* \otimes TM^* \otimes E^0$ and by R and R^0 the curvature tensors of ∇ and ∇^0 respectively.

If $(\bar{M}, \bar{\nabla}, \bar{P})$ is infinitesimally homogeneous with characteristic tensors \bar{T}_o , \bar{R}_o and $\bar{\mathcal{J}}_o$ then obviously $\bar{T}_o = 0$; moreover, since the connection $\bar{\nabla}$ is compatible with the semi-Riemannian metric \bar{g} , the linear map $\bar{\mathcal{J}}_o$ takes values in $\mathfrak{so}(\bar{n} - \bar{r}, \bar{r})/\mathfrak{g}$. Recall that if Z is any \bar{n} -dimensional real vector space endowed with a G -structure then we have “versions” \bar{R}_Z and $\bar{\mathcal{J}}_Z$ of the characteristic tensors \bar{R}_o and $\bar{\mathcal{J}}_o$ of $(\bar{M}, \bar{\nabla}, \bar{P})$ (recall Remark 6.2).

Theorem 8.1. *Assume that $(\bar{M}, \bar{\nabla}, \bar{P})$ is infinitesimally homogeneous with characteristic tensors \bar{T}_o , \bar{R}_o and $\bar{\mathcal{J}}_o$ and that the equalities:*

$$(8.1) \quad \hat{g}_x(\bar{R}_{\hat{E}_x}(v, w)u, z) = g_x(R_x(v, w)u, z) - g_x^0(\alpha_x^0(w, u), \alpha_x^0(v, z)) \\ + g_x^0(\alpha_x^0(v, u), \alpha_x^0(w, z)),$$

$$(8.2) \quad \hat{g}_x(\bar{R}_{\hat{E}_x}(v, w)u, e) = g_x^0((\nabla^\otimes \alpha^0)_x(v, w, u), e) \\ - g_x^0((\nabla^\otimes \alpha^0)_x(w, v, u), e),$$

$$(8.3) \quad \hat{g}_x(\bar{R}_{\hat{E}_x}(v, w)e, e') = g_x(R_x^0(v, w)e, e') + g_x(A_x^0(e) \cdot v, A_x^0(e')^* \cdot w) \\ - g_x(A_x^0(e) \cdot w, A_x^0(e') \cdot v),$$

$$(8.4) \quad \bar{\mathcal{J}}_{\hat{E}_x}(v) = \mathfrak{J}_x^{\hat{P}}(v),$$

for all $x \in M$, $v, w, u, z \in T_x M$, $e, e' \in E_x^0$, where A^0 is defined by (3.2). Then, for every $x_0 \in M$, every $y_0 \in \bar{M}$ and every G -structure preserving linear map $\sigma_0 : \hat{E}_{x_0} \rightarrow T_{y_0} \bar{M}$ there exists a G -structure preserving local solution (f, S) for the isometric immersion problem with data ∇^0 , α^0 , g^0 whose domain is an open neighborhood U of x_0 in M , $f(x_0) = y_0$ and $\sigma_0 = L_{x_0}$, where L is as in (7.1). Moreover, if M is (connected and) simply-connected and $(\bar{M}, \bar{\nabla})$ is geodesically complete then there exists a unique global solution (f, S) for the isometric immersion problem with data ∇^0 , α^0 , g^0 such that $f(x_0) = y_0$ and $\sigma_0 = L_{x_0}$.

PROOF. Apply Theorem 7.4, observing that (8.1) implies (7.7), (8.2) implies both (7.9) and (7.10), (8.3) implies (7.8), (7.11) holds trivially and (7.12) follows from the symmetry of α^0 . \square

8.1. Example (semi-Riemannian manifolds with constant sectional curvature). Assume that the semi-Riemannian manifold $(\overline{M}, \overline{g})$ has constant sectional curvature equal to $c \in \mathbb{R}$ and set $G = O(\overline{n} - \overline{r}, \overline{r})$, $\overline{P} = \text{FR}^o(T\overline{M})$, so that $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous (recall Example 6.1). In this case, setting $\widehat{P} = \text{FR}^o(\widehat{E})$, Theorem 8.1 reproduces the classical Fundamental Theorem of Isometric Immersions (see, for instance, [2, 14]). More explicitly, equations (8.1), (8.2) and (8.3) become, respectively, the standard Gauss, Codazzi and Ricci equations (observe that the lefthand side of (8.2) and (8.3) vanish); moreover, equation (8.4) (whose lefthand side also vanishes) says that the connection $\widehat{\nabla}$ is compatible with the semi-Riemannian structure \widehat{g} of \widehat{E} , so that the equation is always satisfied.

8.2. Example (semi-Riemannian Kähler manifolds with constant holomorphic sectional curvature). Let \overline{J} be a $\overline{\nabla}$ -parallel \overline{g} -antisymmetric almost complex structure on \overline{M} , so that $(\overline{M}, \overline{g}, \overline{J})$ is a semi-Riemannian Kähler manifold. Assume that $(\overline{M}, \overline{g}, \overline{J})$ has constant holomorphic sectional curvature equal to $c \in \mathbb{R}$. Setting $G = U(\frac{1}{2}\overline{r}, \frac{1}{2}(\overline{n} - \overline{r}))$ and $\overline{P} = \text{FR}^u(T\overline{M})$ then $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous (recall Example 6.2). Consider a ∇ -parallel g -antisymmetric almost complex structure J on M and a ∇^0 -parallel g^0 -antisymmetric almost complex structure J^0 on E^0 ; set $\widehat{J} = J \oplus J^0$ and $\widehat{P} = \text{FR}^u(\widehat{E})$. In this case, Theorem 8.1 becomes a well-known result about existence of isometric immersions into Kähler manifolds of constant holomorphic sectional curvature. Notice that the lefthand sides of (8.2) and (8.4) vanish (but the lefthand side of (8.3) does not!); equation (8.4) says that both \widehat{g} and \widehat{J} are $\widehat{\nabla}$ -parallel, which happens if and only if $\alpha_x^0 : T_x M \times T_x M \rightarrow E_x^0$ is complex bilinear, for all $x \in M$.

Recent advances in the theory of minimal and constant mean curvature submanifolds (see for instance [1, 3, 9]) have triggered an increasing interest in the study of immersions into product manifolds ([6, 13]). We discuss the case of isometric immersions into product of manifolds with constant sectional curvature.

8.3. Example (products of manifolds with constant sectional curvature). Let us consider semi-Riemannian manifolds $(\overline{M}_1, \overline{g}_1)$, $(\overline{M}_2, \overline{g}_2)$ such that $\dim(\overline{M}_i) = \overline{n}_i$, the index of \overline{g}_i is \overline{r}_i and the sectional curvature of $(\overline{M}_i, \overline{g}_i)$ is constant and equal to c_i , $i = 1, 2$. Assume that $(\overline{M}, \overline{g})$ is the orthogonal cartesian product of $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, so that $\overline{n} = \overline{n}_1 + \overline{n}_2$ and $\overline{r} = \overline{r}_1 + \overline{r}_2$. Let $\mathbb{R}^{\overline{n}} = \mathbb{R}^{\overline{n}_1} \oplus \mathbb{R}^{\overline{n}_2}$ be endowed with the orthogonal direct sum of the standard Minkowski inner product of index \overline{r}_1 in $\mathbb{R}^{\overline{n}_1}$ and the standard Minkowski inner product of index \overline{r}_2 in $\mathbb{R}^{\overline{n}_2}$. Let $G \cong O(\overline{r}_1, \overline{n}_1 - \overline{r}_1) \times O(\overline{r}_2, \overline{n}_2 - \overline{r}_2)$ be the Lie group of linear isometries of $\mathbb{R}^{\overline{n}}$ that preserve $F_0 = \mathbb{R}^{\overline{n}_1} \oplus \{0\}$ and set $\overline{P} = \text{FR}^o(T\overline{M}, (T\overline{M}_1) \times \overline{M}_2)$ (recall the notation introduced in Example 5.4). Then $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous (this is obtained by applying the product construction explained in Example 6.5). If Z is a \overline{n} -dimensional real vector space then a G -structure on Z

can be identified with a pair $(\langle \cdot, \cdot \rangle, Z_1)$, where $\langle \cdot, \cdot \rangle$ is a nondegenerate symmetric bilinear form of index \bar{r} on Z and Z_1 is a subspace of Z on which $\langle \cdot, \cdot \rangle$ is a nondegenerate symmetric bilinear form of index \bar{r}_1 . The “versions” of the characteristic tensors on Z become $\bar{\mathcal{J}}_Z = 0$ and:

$$\begin{aligned} \bar{R}_Z(z_1 + z'_1, z_2 + z'_2)(z_3 + z'_3) &= c_1(\langle z_2, z_3 \rangle z_1 - \langle z_1, z_3 \rangle z_2) \\ &\quad + c_2(\langle z'_2, z'_3 \rangle z'_1 - \langle z'_1, z'_3 \rangle z'_2), \end{aligned}$$

for all $z_1, z_2, z_3 \in Z_1$ and all $z'_1, z'_2, z'_3 \in Z_1^\perp$.

In Example 8.3 one can consider for instance the case where \bar{M} is an orthogonal product of a space form with the real line \mathbb{R} , obtaining the isometric immersion theorems of [3].

8.4. Example. An isometric immersion theorem into Lie groups endowed with left-invariant semi-Riemannian metrics is easily obtained from Example 7.1 by considering a linear isometry $p_0 : \mathbb{R}^{\bar{n}} \rightarrow \mathfrak{g}$ and an orthonormal global smooth frame $s : M \rightarrow \text{FR}^o(\hat{E})$.

Remark 8.2. A different set up for isometric immersions into Lie groups has been discussed in [8] where the author considers isometric immersions into some classes of solvable and nilpotent Lie groups. In our terminology, the result in [8] is associated to the G -structure on the target manifold determined by an orthonormal basis of the center of the Lie algebra. Such G -structure is not in general infinitesimally homogeneous, so the result in [8] is not a direct corollary of Theorem 8.1.

8.5. Example. Assume that (\bar{M}, \bar{g}) is a three-dimensional Riemannian manifold as in Example 6.4. Let \bar{P} denote the G -structure of positively oriented linear isometries that send $(1, 0, 0)$ to ξ , where $G \cong \text{SO}(\mathbb{R}^2)$ is as in Example 6.4; then $(\bar{M}, \bar{\nabla}, \bar{P})$ is infinitesimally homogeneous. Assume that (M, g) is an oriented two-dimensional Riemannian manifold and that $E = M \times \mathbb{R}$ is the trivial linear bundle over M . Consider a global smooth unitary section of \hat{E} , which is determined by a smooth vector field X on M and a smooth function $\nu : M \rightarrow \mathbb{R}$ with $g_x(X(x), X(x)) + \nu(x)^2 = 1$, for all $x \in M$. We consider the G -structure \hat{P} on \hat{E} of positively oriented linear isometries that send $(1, 0, 0)$ to (X, ν) . In this context, a local solution (f, S) for the isometric immersion problem with data ∇^0, α^0, g^0 is G -structure preserving if $\xi(f(x)) = df_x(X(x)) + \nu(x)S_x(1)$, for all $x \in M$ (notice that $x \mapsto S_x(1)$ is a unit normal vector field). In this context, we have (omitting the obvious zero corresponding to the covariant derivative of the metric tensor):

$$\bar{\mathcal{J}}_{\hat{E}_x}(v) = \tau v \times (X(x), \nu(x)) = \tau(-\nu(x)J_x(v), \langle J_x(v), X(x) \rangle),$$

for all $x \in M, v \in T_x M$, where $J_x : T_x M \rightarrow T_x M$ denotes the canonical complex structure of the two-dimensional oriented Riemannian manifold (M, g) . Moreover:

$$\mathcal{J}_x^{\hat{P}}(v) = \hat{\nabla}_v(X, \nu) = \left(\nabla_v X + \nu(x)A_x^0(1) \cdot v, \alpha_x^0(v, X(x)) + d\nu_x(v) \right).$$

In this case, Theorem 8.1 reproduces the isometric immersion theorem of [4].

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