Analytic Solutions of Partial Differential Equations

MATH3414

School of Mathematics, University of Leeds

- 15 credits
- Taught Semester 1
- Year running 2006/07
- Pre-requisites MATH2360 or MATH2420 or equivalent
- Co-requisites none
- Objectives: To provide an understanding of, and methods of solution for, the most important types of partial differential equations that arise in Mathematical Physics. On completion of this module, students should be able to: a) use the method of characteristics to solve first-order equations; b) classify a second order PDE as elliptic, parabolic or hyperbolic; c) use Green's functions to solve elliptic equations; d) have a basic understanding of diffusion; e) obtain a priori bounds for reaction-diffusion equations.
- Syllabus: The majority of physical phenomena can be described by partial differential equations (e.g., the Navier-Stokes equation of fluid dynamics, Maxwell's equations of electromagnetism). This module considers the properties of, and analytical methods of solution for some of the most common first and second order PDEs of Mathematical Physics. In particular, we shall look in detail at elliptic equations (Laplace's equation), describing steady-state phenomena and the diffusion/heat conduction equation describing the slow spread of concentration or heat. The topics covered are: First order PDEs. Semilinear and quasilinear PDEs; method of characteristics. Characteristics crossing. Second order PDEs. Classification and standard forms. Elliptic equations: weak and strong minimum and maximum principles; Green's functions. Parabolic equations: exemplified by solutions of the diffusion equation. Bounds on solutions of reaction-diffusion equations.
- Form of teaching
Lectures: 26 hours and 7 examples classes.
- Form of assessment
One 3 hour examination at end of semester (100%).

Details:

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Schedule: three lectures every week, for eleven weeks (from 25/09 to 08/12).

Tuesday	11:00–12:00	EC Stoner 08.62
Wednesday	10:00–11:00	RSLT 07
Friday	12:00–13:00	RSLT 04

Pre-requisite: elementary differential calculus and several variables calculus (e.g., partial differentiation with change of variables, parametric curves, integration), elementary algebra (e.g., partial fractions, linear eigenvalue problems), ordinary differential equations (e.g., change of variable, integrating factor), and vector calculus (e.g., vector identities, Green's theorem).

Outline of course:

Introduction:

- definitions
- examples

First order PDEs:

- linear & semilinear
- characteristics
- quasilinear
- nonlinear
- system of equations

Second order linear PDEs:

- classification
- elliptic
- parabolic

Book list:

P. Prasad & R. Ravindran, "Partial Differential Equations", Wiley Eastern, 1985.

W. E. Williams, "Partial Differential Equations", Oxford University Press, 1980.

P. R. Garabedian, "Partial Differential Equations", Wiley, 1964.

Thanks to Dr. E. Kersalé, Prof. D. W. Hughes, Prof. J. H. Merkin and Dr. R. Sturman for their lecture notes.

Course Summary

- Definitions of different types of PDE (linear, semilinear, quasilinear, nonlinear).
- Existence and uniqueness of solutions.
- Solving PDEs analytically is generally based on finding a change of variables to transform the equation into something soluble or on finding an integral form of the solution.

First order PDEs

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c.$$

Linear equations: change coordinates using $\eta(x, y)$, defined by the characteristic equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a},$$

and $\xi(x, y)$ independent (usually $\xi = x$) to transform the PDE into an ODE.

Quasilinear equations: change coordinates using the solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = b \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s} = c$$

to get an implicit form of the solution $\phi(x, y, u) = F(\psi(x, y, u))$.

Nonlinear waves: region of solution.

System of linear equations: linear algebra to decouple equations.

Second order PDEs

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g.$$

Classification	Type	Canonical form	Characteristics
$b^2 - ac > 0$	Hyperbolic	$\frac{\partial^2 u}{\partial \xi \partial \eta} + \ldots = 0$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}$
$b^2 - ac = 0$	Parabolic	$\frac{\partial^2 u}{\partial \eta^2} + \ldots = 0$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}, \eta = x$ (say)
$b^2 - ac < 0$	Elliptic	$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + \ldots = 0$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}, \begin{cases} \alpha = \xi + \eta \\ \beta = i(\xi - \eta) \end{cases}$

Elliptic equations: (Laplace equation.) Maximum Principle. Solutions using Green's functions (uses new variables and the Dirac δ-function to pick out the solution). Method of images.

Parabolic equations: (heat conduction, diffusion equation.) Derive a fundamental solution in integral form or make use of the similarity properties of the equation to find the solution in terms of the diffusion variable

$$\eta = \frac{x}{2\sqrt{t}}.$$

First and Second Maximum Principles and Comparison Theorem give bounds on the solution, and can then construct invariant sets.

Contents

Chapter 1

Introduction

Contents

1.1 Motivation

Why do we study partial differential equations (PDEs) and in particular analytic solutions?

We are interested in PDEs because most of mathematical physics is described by such equations. For example, fluid dynamics (and more generally continuous media dynamics), electromagnetic theory, quantum mechanics, traffic flow. Typically, a given PDE will only be accessible to numerical solution (with one obvious exception — exam questions!) and analytic solutions in a practical or research scenario are often impossible. However, it is vital to understand the general theory in order to conduct a sensible investigation. For example, we may need to understand what type of PDE we have to ensure the numerical solution is valid. Indeed, certain types of equations need appropriate boundary conditions; without a knowledge of the general theory it is possible that the problem may be ill-posed and that the method of solution is erroneous.

1.2 Reminder

Partial derivatives: The differential (or differential form) of a function f of n independent variables, $(x_1, x_2, \dots, x_n)$, is a linear combination of the basis form $(\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n)$

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \ldots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n,$$

where the <u>partial derivatives</u> are defined by

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_i + h, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{h}.$$

The usual differentiation identities apply to the partial differentiations (sum, product, quotient, chain rules, etc.)

Notations: I shall use interchangeably the notations

$$\frac{\partial f}{\partial x_i} \equiv \partial_{x_i} f \equiv f_{x_i}, \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} \equiv \partial^2_{x_i x_j} f \equiv f_{x_i x_j},$$

for the first order and second order partial derivatives, respectively. We shall also use interchangeably the notations

$$\vec{u} \equiv \underline{u} \equiv \mathbf{u},$$

for vectors.

Vector differential operators: in three dimensional Cartesian coordinates $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ we consider $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ and $[u_x(x, y, z), u_y(x, y, z), u_z(x, y, z)] : \mathbb{R}^3 \to \mathbb{R}^3$.

Gradient: $\nabla f = \partial_x f\, \mathbf{i} + \partial_y f\, \mathbf{j} + \partial_z f\, \mathbf{k}$.

Divergence: $\operatorname{div} \mathbf{u} \equiv \nabla \cdot \mathbf{u} = \partial_x u_x + \partial_y u_y + \partial_z u_z$.

Curl: $\nabla \times \mathbf{u} = (\partial_y u_z - \partial_z u_y)\, \mathbf{i} + (\partial_z u_x - \partial_x u_z)\, \mathbf{j} + (\partial_x u_y - \partial_y u_x)\, \mathbf{k}$.

Laplacian: $\Delta f \equiv \nabla^2 f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$.

Laplacian of a vector: $\Delta \mathbf{u} \equiv \nabla^2 \mathbf{u} = \nabla^2 u_x\, \mathbf{i} + \nabla^2 u_y\, \mathbf{j} + \nabla^2 u_z\, \mathbf{k}$.

Note that these operators are different in other coordinate systems (cylindrical or spherical, say)

1.3 Definitions

A partial differential equation (PDE) is an equation for some quantity u (dependent variable) which depends on the independent variables $x_1, x_2, x_3, \ldots, x_n$, $n \geq 2$, and involves derivatives of u with respect to at least some of the independent variables.

$$F(x_1, \ldots, x_n, \partial_{x_1} u, \ldots, \partial_{x_n} u, \partial^2_{x_1} u, \partial^2_{x_1 x_2} u, \ldots, \partial^n_{x_1 \ldots x_n} u) = 0.$$

Note:

1. In applications x_i are often space variables (e.g., x, y, z) and a solution may be required in some region Ω of space. In this case there will be some conditions to be satisfied on the boundary $\partial\Omega$; these are called boundary conditions (BCs).

2. Also in applications, one of the independent variables can be time (t say), then there will be some initial conditions (ICs) to be satisfied (i.e., u is given at $t = 0$ everywhere in Ω).

3. Again in applications, systems of PDEs can arise involving the dependent variables $u_1, u_2, u_3, \ldots, u_m$, $m \geq 1$ with some (at least) of the equations involving more than one u_i.

The order of the PDE is the order of the highest (partial) differential coefficient in the equation.
As with ordinary differential equations (ODEs) it is important to be able to distinguish between linear and nonlinear equations.
A linear equation is one in which the equation and any boundary or initial conditions do not include any product of the dependent variables or their derivatives; an equation that is not linear is a nonlinear equation.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{first order linear PDE (simplest wave equation),}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Phi(x, y), \quad \text{second order linear PDE (Poisson equation).}$$

A nonlinear equation is semilinear if the coefficients of the highest derivative are functions of the independent variables only.

$$(x + 3)\frac{\partial u}{\partial x} + xy^2\frac{\partial u}{\partial y} = u^3,$$

$$x\frac{\partial^2 u}{\partial x^2} + (xy + y^2)\frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial y} = u^4.$$

A nonlinear PDE of order m is quasilinear if it is linear in the derivatives of order m with coefficients depending only on the independent variables and derivatives of order $< m$.

$$\left[1 + \left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y}\,\frac{\partial^2 u}{\partial x \partial y} + \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]\frac{\partial^2 u}{\partial y^2} = 0.$$

Principle of superposition: A linear equation has the useful property that if u_1 and u_2 both satisfy the equation then so does $\alpha u_1 + \beta u_2$ for any $\alpha, \beta \in \mathbb{R}$. This is often used in constructing solutions to linear equations (for example, so as to satisfy boundary or initial conditions; c.f., Fourier series methods). This is not true for nonlinear equations, which helps to make this sort of equations more interesting, but much more difficult to deal with.

1.4 Examples

1.4.1 Wave Equations

Waves on a string, sound waves, waves on stretch membranes, electromagnetic waves, etc.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

or more generally

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u,$$

where c is a constant (wave speed).

1.4.2 Diffusion or Heat Conduction Equations

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2},$$

or more generally

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u,$$

or even

$$\frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u),$$

where κ is a constant (diffusion coefficient or thermometric conductivity).
Both those equations (wave and diffusion) are linear equations and involve time (t). They require some initial conditions (and possibly some boundary conditions) for their solution.

1.4.3 Laplace's Equation

Another example of a second order linear equation is the following:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

or more generally

$$\nabla^2 u = 0.$$

This equation usually describes steady processes and is solved subject to some boundary conditions.

One aspect that we shall consider is: why do similar looking equations describe essentially different physical processes? What is there about the equations that make this the case?

1.4.4 Other Common Second Order Linear PDEs

Poisson's equation is just Lapace's equation (homogeneous) with a known source term (e.g., electric potential in the presence of a density of charge):

$$\nabla^2 u = \Phi.$$

The Helmholtz equation may be regarded as a stationary wave equation:

$$\nabla^2 u + k^2\, u = 0.$$

The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behaviour; Schrödinger wave equation is a PDE that describes how the wavefunction of a physical system evolves over time:

$$-\nabla^2 u + V\, u = i\frac{\partial u}{\partial t}.$$

1.4.5 Nonlinear PDEs

An example of a nonlinear equation is the equation for the propagation of reaction-diffusion waves:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \quad (2^{\text{nd}} \text{ order}),$$

or for nonlinear wave propagation:

$$\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = 0; \quad (1^{\text{st}} \text{ order}).$$

The equation

$$y\frac{\partial u}{\partial x} + (x^3+y)\frac{\partial u}{\partial y} = u^3$$

is an example of semilinear equation and

$$x^2 u\frac{\partial u}{\partial x} + (y+u)\frac{\partial u}{\partial y} = u^3$$

is an example of a quasilinear equation.

1.4.6 System of PDEs

Maxwell equations constitute a system of linear PDEs:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}, \qquad \nabla \times \mathbf{B} = \mu \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

In empty space (free of charges and currents) this system can be rearranged to give the equations of propagation of the electromagnetic field,

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}, \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B}.$$

Incompressible magnetohydrodynamic (MHD) equations combine Navier-Stokes equation (including the Lorentz force), the induction equation as well as the solenoidal constraints,

$$\begin{aligned}
&\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} = -\nabla\Pi + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{U} + \mathbf{F},\\
&\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}) + \eta\nabla^2\mathbf{B},\\
&\nabla\cdot\mathbf{U} = 0, \qquad \nabla\cdot\mathbf{B} = 0.
\end{aligned}$$

Both systems involve space and time; they require some initial and boundary conditions for their solution.

1.5 Existence and Uniqueness

Before attempting to solve a problem involving a PDE we would like to know if a solution exists, and, if it exists, if the solution is unique. Also, in problems involving time, whether a solution exists $\forall t > 0$ (global existence) or only up to a given value of t — i.e., only for $0 < t < t_0$ (finite time blow-up, shock formation). As well as the equation there could be certain boundary and initial conditions. We would also like to know whether the solution of the problem depends continuously of the prescribed data — i.e., small changes in boundary or initial conditions produce only small changes in the solution.

Illustration from ODEs:

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We say that the PDE with boundary or initial conditions is well-formed (or well-posed) if its solution exists (globally), is unique and depends continuously on the assigned data. If any of these three properties (existence, uniqueness and stability) is not satisfied, the problem (PDE, BCs and ICs) is said to be ill-posed. Usually problems involving linear systems are well-formed but this may not always be the case for nonlinear systems (bifurcation of solutions, etc.)

Example 1: A simple example of showing uniqueness is provided by:

$$\nabla^2 u = F \ \text{ in } \Omega \quad \text{(Poisson's equation)},$$

with $u = 0$ on $\partial\Omega$, the boundary of Ω, and F is some given function of $\mathbf{x}$.

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Chapter 2

First Order Equations

Contents

2.1 Linear and Semilinear Equations

2.1.1 Method of Characteristic

We consider linear first order partial differential equations in two independent variables:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y), \tag{2.1}$$

where a, b, c and f are continuous in some region of the plane and we assume that $a(x,y)$ and $b(x,y)$ are not zero for the same (x,y).
In fact, we could consider semilinear first order equations (where the nonlinearity is present only in the right-hand side) such as

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = \kappa(x,y,u), \tag{2.2}$$

instead of a linear equation as the theory of the former does not require any special treatment as compared to that of the latter.

The key to the solution of equation (2.1) is to find a change of variables (or a change of coordinates)

$$\xi \equiv \xi(x,y), \eta \equiv \eta(x,y)$$

which transforms (2.1) into the simpler equation

$$\frac{\partial w}{\partial \xi} + h(\xi,\eta)w = F(\xi,\eta) \tag{2.3}$$

where $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$.

We shall define this transformation so that it is one-to-one, at least for all (x,y) in some set D of points in the (x-y) plane. Then, on D we can (in theory) solve for x and y as functions of ξ, η. To ensure that we can do this, we require that the Jacobian of the transformation does not vanish in D:

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} \neq \{0, \infty\}$$

for (x, y) in D. We begin looking for a suitable transformation by computing derivatives via the chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

We substitute these into equation (2.1) to obtain

$$a\left(\frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x}\right) + b\left(\frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y}\right) + cw = f.$$

We can rearrange this as

$$\left(a\frac{\partial \xi}{\partial x} + b\frac{\partial \xi}{\partial y}\right)\frac{\partial w}{\partial \xi} + \left(a\frac{\partial \eta}{\partial x} + b\frac{\partial \eta}{\partial y}\right)\frac{\partial w}{\partial \eta} + cw = f. \tag{2.4}$$

This is close to the form of equation (2.3) if we can choose $\eta \equiv \eta(x, y)$ so that

$$a\frac{\partial \eta}{\partial x} + b\frac{\partial \eta}{\partial y} = 0 \quad \text{for} \quad (x, y) \quad \text{in} \quad D.$$

Provided that $\partial\eta/\partial y \neq 0$ we can express this required property of η as

$$\frac{\partial_x \eta}{\partial_y \eta} = -\frac{b}{a}.$$

Suppose we can define a new variable (or coordinate) η which satisfies this constraint. What is the equation describing the curves of constant η? Putting $\eta \equiv \eta(x, y) = k$ (k an arbitrary constant), then

$$\mathrm{d}\eta = \frac{\partial \eta}{\partial x}\mathrm{d}x + \frac{\partial \eta}{\partial y}\mathrm{d}y = 0$$

implies that $\mathrm{d}y/\mathrm{d}x = -\partial_x\eta/\partial_y\eta = b/a$. So, the equation $\eta(x, y) = k$ defines solutions of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x, y)}{a(x, y)}. \tag{2.5}$$

Equation (2.5) is called the characteristic equation of the linear equation (2.1). Its solution can be written in the form $F(x, y, \eta) = 0$ (where η is the constant of integration) and defines a family of curves in the plane called characteristics or characteristic curves of (2.1). (More on characteristics later.) Characteristics represent curves along which the independent variable η of the new coordinate system (ξ, η) is constant.

So, we have made the coefficient of $\partial w/\partial \eta$ vanish in the transformed equation (2.4), by choosing $\eta \equiv \eta(x, y)$, with $\eta(x, y) = k$ an equation defining the solution of the characteristic equation (2.5). We can now choose ξ arbitrarily (or at least to suit our convenience), providing we still have $J \neq 0$. An easy choice is

$$\xi \equiv \xi(x, y) = x.$$

Then

$$J = \begin{vmatrix} 1 & 0 \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \frac{\partial \eta}{\partial y},$$

and we have already assumed this non-zero.

Now we see from equation (2.4) that this change of variables,

$$\xi = x, \eta \equiv \eta(x, y),$$

transforms equation (2.1) to

$$\alpha(x, y)\frac{\partial w}{\partial \xi} + c(x, y)w = f(x, y),$$

where $\alpha = a\partial\xi/\partial x + b\partial\xi/\partial y$. To complete the transformation to the form of equation (2.3), we first write $\alpha(x,y), c(x,y)$ and $f(x,y)$ in terms of ξ and η to obtain

$$A(\xi,\eta)\frac{\partial w}{\partial \xi} + C(\xi,\eta)w = \rho(\xi,\eta).$$

Finally, restricting the variables to a set in which $A(\xi,\eta) \neq 0$ we have

$$\frac{\partial w}{\partial \xi} + \frac{C}{A}w = \frac{\rho}{A},$$

which is in the form of (2.3) with

$$h(\xi,\eta) = \frac{C(\xi,\eta)}{A(\xi,\eta)} \quad \text{and} \quad F(\xi,\eta) = \frac{\rho(\xi,\eta)}{A(\xi,\eta)}.$$

The characteristic method applies to first order semilinear equations (2.2) as well as linear equations (2.1); a similar change of variables and basic algebra transform equation (2.2) to

$$\frac{\partial w}{\partial \xi} = \frac{\mathcal{K}}{A},$$

where the nonlinear term $\mathcal{K}(\xi,\eta,w) = \kappa(x,y,u)$ and restricting again the variables to a set in which $A(\xi,\eta) = \alpha(x,y) \neq 0$.

Notation: It is very convenient to use the function u in places where rigorously the function w should be used. E.g., the equation here above can identically be written as $\partial u/\partial\xi = \mathcal{K}/A$.

Example A: Consider the linear first order equation

$$x^2\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + xyu = 1.$$

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2.1.2 Equivalent set of ODEs

The point of this transformation is that we can solve equation (2.3). Think of

$$\frac{\partial w}{\partial \xi} + h(\xi,\eta)w = F(\xi,\eta)$$

as a linear first order ordinary differential equation in ξ, with η carried along as a parameter. Thus we use an integrating factor method

$$e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\frac{\partial w}{\partial \xi} + h(\xi,\eta)\,e^{\int h(\xi,\eta)\,\mathrm{d}\xi}w = F(\xi,\eta)\,e^{\int h(\xi,\eta)\,\mathrm{d}\xi},$$

$$\frac{\partial}{\partial \xi}\left(e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\,w\right) = F(\xi,\eta)\,e^{\int h(\xi,\eta)\,\mathrm{d}\xi}.$$

Now we integrate with respect to ξ. Since η is being carried as a parameter, the constant of integration may depend on η

$$e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\,w = \int F(\xi,\eta)\,e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\,\mathrm{d}\xi + g(\eta)$$

in which g is an arbitrary differentiable function of one variable. Now the general solution of the transformed equation is

$$w(\xi,\eta) = e^{-\int h(\xi,\eta)\,\mathrm{d}\xi} \int F(\xi,\eta)\, e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\,\mathrm{d}\xi + g(\eta)\, e^{-\int h(\xi,\eta)\,\mathrm{d}\xi}.$$

We obtain the general form of the original equation by substituting back $\xi(x,y)$ and $\eta(x,y)$ to get

$$u(x,y) = e^{\alpha(x,y)}\left[\beta(x,y) + g(\eta(x,y))\right]. \tag{2.6}$$

A certain class of first order PDEs (linear and semilinear PDEs) can then be reduced to a set of ODEs. This makes use of the general philosophy that ODEs are easier to solve than PDEs.

Example B: Consider the constant coefficient equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0$$

where $a, b, c \in \mathbb{R}$.

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Exercise: Verify the solution by substituting back into the PDE.

Note: Consider the difference between general solution for linear ODEs and general solution for linear PDEs. For ODEs, the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = p(x)$$

contains an arbitrary constant of integration. For different constants one gets different curves of solution in the (x-y)-plane. To choose a unique solution one uses some initial condition (say $y(x_0) = y_0$) to specify the constant.
For PDEs, if u is the general solution to equation (2.1), then $z = u(x,y)$ defines a family of integral surfaces in 3D-space, each surface corresponding to a choice of arbitrary function g in (2.6). We need some kind of information to pick out a unique solution; i.e., to chose the arbitrary function g.

2.1.3 Characteristic Curves

We investigate the significance of characteristics which, defined by the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y)}{a(x,y)},$$

represent a one parameter family of curves whose tangent at each point is in the direction of the vector $\mathbf{e} = (a,b)$. (Note that the left-hand side of equation (2.2) is the derivative of u in the direction of the vector $\mathbf{e}$, $\mathbf{e}\cdot\nabla u$.) Their parametric representation is $(x = x(s), y = y(s))$ where $x(s)$ and $y(s)$ satisfy the pair of ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x,y), \quad \frac{\mathrm{d}y}{\mathrm{d}s} = b(x,y). \tag{2.7}$$

The variation of u with respect to $x = \xi$ along these characteristic curves is given by

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\partial u}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + \frac{b}{a}\frac{\partial u}{\partial y}, \\ &= \frac{\kappa(x,y,u)}{a(x,y)} \quad \text{from equation (2.2)},\end{aligned}$$

such that, in terms of the curvilinear coordinate s, the variation of u along the curves becomes

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \kappa(x,y,u).$$

The one parameter family of characteristic curves is parameterised by η (each value of η represents one unique characteristic curve). The solution of equation (2.2) reduces to the solution of the family of ODEs

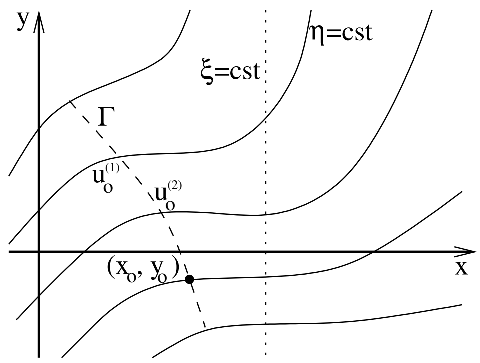
$$\frac{\mathrm{d}u}{\mathrm{d}s} = \kappa(x,y,u) \quad \left(\text{or similarly} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{\kappa(x,y,u)}{a(x,y)}\right) \tag{2.8}$$

along each characteristics (i.e., for each value of η).
Characteristic equations (2.7) have to be solved together with equation (2.8), called the compatibility equation, to find a solution to the semilinear equation (2.2).

Cauchy Problem: Consider a curve Γ in (x,y)-plane whose parametric form is $(x = x_0(\sigma), y = y_0(\sigma))$. The Cauchy problem is to determine a solution of the equation

$$F(x,y,u,\partial_x u, \partial_y u) = 0$$

in a neighbourhood of Γ such that u takes prescribed values $u_0(\sigma)$ called Cauchy data on Γ.



Notes:

1. u can only be found in the region between the characteristics drawn through the end-points of Γ.
2. Characteristics are curves on which the values of u combined with the equation are not sufficient to determine the normal derivative of u.
3. A discontinuity in the initial data propagates onto the solution along the characteristics. These are curves across which the derivatives of u can jump while u itself remains continuous.

Existence & Uniqueness: Why do some choices of Γ in (x,y)-space give a solution and other give no solution or an infinite number of solutions? It is due to the fact that the Cauchy data (initial conditions) may be prescribed on a curve Γ which is a characteristic of the PDE. To understand the definition of characteristics in the context of existence and uniqueness of the solution, return to the general solution (2.6) of the linear PDE:

$$u(x,y) = e^{\alpha(x,y)}\left[\beta(x,y) + g(\eta(x,y))\right].$$

Consider the Cauchy data, u_0, prescribed along the curve Γ whose parametric form is $(x = x_0(\sigma), y = y_0(\sigma))$ and suppose $u_0(x_0(\sigma), y_0(\sigma)) = q(\sigma)$. If Γ is not a characteristic, the problem is well-posed and there is a unique function g which satisfies the condition

$$q(\sigma) = e^{\alpha(x_0(\sigma),y_0(\sigma))}\left[\beta(x_0(\sigma), y_0(\sigma)) + g(x_0(\sigma), y_0(\sigma))\right].$$

If on the other hand $(x = x_0(\sigma), y = y_0(\sigma))$ is the parametrisation of a characteristic ($\eta(x, y) = k$, say), the relation between the initial conditions q and g becomes

$$q(\sigma) = e^{\alpha(x_0(\sigma),y_0(\sigma))}\left[\beta(x_0(\sigma), y_0(\sigma)) + G\right], \tag{2.9}$$

where $G = g(k)$ is a constant; the problem is ill-posed. The functions $\alpha(x, y)$ and $\beta(x, y)$ are determined by the PDE, so equation (2.9) places a constraint on the given data function $q(x)$. If $q(\sigma)$ is not of this form for any constant G, then there is no solution taking on these prescribed values on Γ. On the other hand, if $q(\sigma)$ is of this form for some G, then there are infinitely many such solutions, because we can choose for g any differentiable function so that $g(k) = G$.

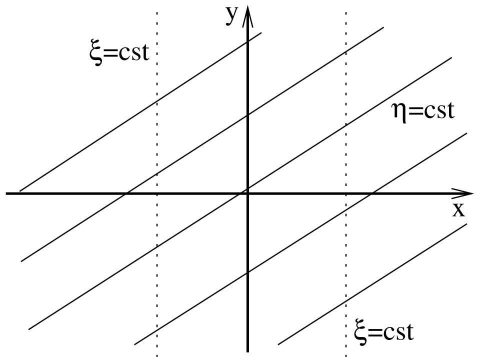
Example 1: Consider

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} + 8u = 0.$$

The characteristic equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}$$

and the characteristics are the straight line graphs $3x - 2y = c$. Hence we take $\eta = 3x - 2y$ and $\xi = x$.



(We can see that the lines of constant η and constant ξ cross only once; they are independent, i.e., $J \neq 0$; η and ξ have been properly chosen.)

This gives the solution

$$u(x, y) = e^{-4x}\, g(3x - 2y)$$

where g is a differentiable function defined over the real line. Simply specifying the solution at a given point (as in ODEs) does not uniquely determine g; we need to take a curve of initial conditions.
Suppose we specify values of $u(x, y)$ along a curve Γ in the plane. For example, let us choose Γ as the x-axis and give values of $u(x, y)$ at points on Γ, say

$$u(x, 0) = \sin(x).$$

Then we need

$$u(x, 0) = e^{-4x}\, g(3x) = \sin(x) \quad \text{i.e.,} \quad g(3x) = \sin(x)e^{4x},$$

and putting $t = 3x$,

$$g(t) = \sin(t/3)\, \mathrm{e}^{4t/3}.$$

This determines g and the solution satisfying the condition $u(x, 0) = \sin(x)$ on Γ is

$$u(x, y) = \sin(x - 2y/3)\, e^{-8y/3}.$$

We have determined the unique solution of the PDE with u specified along the x-axis.
We do not have to choose an axis — say, along $x = y$, $u(x, y) = u(x, x) = x^4$. From the general solution this requires,

$$u(x, x) = e^{-4x}\, g(x) = x^4, \quad \text{so} \quad g(x) = x^4\, e^{4x}$$

to give the unique solution

$$u(x, y) = (3x - 2y)^4 e^{8(x-y)}$$

satisfying $u(x, x) = x^4$.
However, not every curve in the plane can be used to determine g. Suppose we choose Γ to be the line $3x - 2y = 1$ and prescribe values of u along this line, say

$$u(x, y) = u(x, (3x - 1)/2) = x^2.$$

Now we must choose g so that

$$e^{-4x}\, g(3x - (3x - 1)) = x^2.$$

This requires $g(1) = x^2\, e^{4x}$ (for all x). This is impossible and hence there is no solution taking the value x^2 at points (x, y) on this line.
Last, we consider again Γ to be the line $3x - 2y = 1$ but choose values of u along this line to be

$$u(x, y) = u(x, (3x - 1)/2) = e^{-4x}.$$

Now we must choose g so that

$$e^{-4x}\, g(3x - (3x - 1)) = e^{-4x}.$$

This requires $g(1) = 1$, condition satisfied by an infinite number of functions and hence there is an infinite number of solutions taking the values e^{-4x} on the line $3x - 2y = 1$.
Depending on the initial conditions, the PDE has one unique solution, no solution at all or an infinite number or solutions. The difference is that the x-axis and the line $y = x$ are not the characteristics of the PDE while the line $3x - 2y = 1$ is a characteristic.

Example 2:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u \quad \text{with} \quad u = x^2 \quad \text{on} \quad y = x, 1 \le y \le 2$$

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Alternative approach to solving example 2:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u \quad \text{with} \quad u = x^2 \quad \text{on} \quad y = x, 1 \le y \le 2$$

This method is not suitable for finding general solutions but it works for Cauchy problems. The idea is to integrate directly the characteristic and compatibility equations in curvilinear coordinates. (See also "alternative method for solving the characteristic equations" for quasilinear equations hereafter.)
The solution of the characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -y$$

gives the parametric form of the characteristic curves, while the integration of the compatibility equation

$$\frac{\mathrm{d}u}{\mathrm{d}s} = u$$

gives the solution $u(s)$ along these characteristics curves.
The solution of the characteristic equations is

$$x = c_1\,\mathrm{e}^{s} \quad \text{and} \quad y = c_2\,\mathrm{e}^{-s},$$

where the parametric form of the data curve Γ allows us to find the two constants of integration c_1 and c_2 in terms of the curvilinear coordinate along Γ.
The curve Γ is described by

$$x_0(\theta) = \theta \quad \text{and} \quad y_0(\theta) = \theta \quad \text{with} \quad \theta \in [1,2]$$

and we consider the points on Γ to be the origin of the coordinate s along the characteristics (i.e., $s = 0$ on Γ). So,

$$\text{on } \Gamma \ (s=0) \quad \left| \begin{array}{l} x_0 = \theta = c_1 \\ y_0 = \theta = c_2 \end{array} \right\} \Rightarrow \left| \begin{array}{l} x(s,\theta) = \theta\,\mathrm{e}^{s} \\ y(s,\theta) = \theta\,\mathrm{e}^{-s} \end{array} \right\}, \quad \forall\theta \in [1,2].$$

For linear or semilinear problems we can solve the compatibility equation independently of the characteristic equations. (This property is not true for quasilinear equations.) Along the characteristics u is determined by

$$\frac{\mathrm{d}u}{\mathrm{d}s} = u \Rightarrow u = c_3\,\mathrm{e}^{s}.$$

Now we can make use of the Cauchy data to determine the constant of integration c_3,

$$\text{on } \Gamma, \text{ at } s = 0, \quad u_0(x_0(\theta), y_0(\theta)) \equiv u_0(\theta) = \theta^2 = c_3.$$

Then, we have the parametric forms of the characteristic curves and the solution

$$x(s,\theta) = \theta\,\mathrm{e}^{s}, \quad y(s,\theta) = \theta\,\mathrm{e}^{-s} \quad \text{and} \quad u(s,\theta) = \theta^2\,\mathrm{e}^{s},$$

in terms of two parameters, s the curvilinear coordinate along the characteristic curves and θ the curvilinear coordinate along the data curve Γ. From the first two ones we get s and θ in terms of x and y.

$$\frac{x}{y} = \mathrm{e}^{2s} \Rightarrow s = \ln\sqrt{\frac{x}{y}} \quad \text{and} \quad xy = \theta^2 \Rightarrow \theta = \sqrt{xy} \quad (\theta \geq 0).$$

Then, we substitute s and θ in $u(s,\theta)$ to find

$$u(x,y) = xy\,\exp\left(\ln\sqrt{\frac{x}{y}}\right) = xy\sqrt{\frac{x}{y}} = x\sqrt{xy}.$$

2.2 Quasilinear Equations

Consider the first order quasilinear PDE

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \tag{2.10}$$

where the functions a, b and c can involve u but not its derivatives.

2.2.1 Interpretation of Quasilinear Equation

We can represent the solutions $u(x,y)$ by the integral surfaces of the PDE, $z = u(x,y)$, in (x,y,z)-space. Define the Monge direction by the vector (a,b,c) and recall that the normal to the integral surface is $(\partial_x u, \partial_y u, -1)$. Thus, the quasilinear equation (2.10) says that the normal to the integral surface is perpendicular to the Monge direction; i.e., integral surfaces are surfaces that at each point are tangent to the Monge direction,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \partial_x u \\ \partial_y u \\ -1 \end{pmatrix} = a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} - c(x,y,u) = 0.$$

With the field of Monge direction, with direction numbers (a, b, c), we can associate the family of Monge curves which at each point are tangent to that direction field. These are defined by

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c\,\mathrm{d}y - b\,\mathrm{d}z \\ a\,\mathrm{d}z - c\,\mathrm{d}x \\ b\,\mathrm{d}x - a\,\mathrm{d}y \end{pmatrix} = 0 \Leftrightarrow \frac{\mathrm{d}x}{a(x,y,u)} = \frac{\mathrm{d}y}{b(x,y,u)} = \frac{\mathrm{d}z}{c(x,y,u)} (= \mathrm{d}s),$$

where $\mathrm{d}\mathbf{l} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is an arbitrary infinitesimal vector parallel to the Monge direction.
In the linear case, characteristics were curves in the (x, y)-plane (see § 2.1.3). For the quasilinear equation, we consider Monge curves in (x, y, u)-space defined by

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x, y, u),$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = b(x, y, u),$$
$$\frac{\mathrm{d}u}{\mathrm{d}s} = c(x, y, u).$$

Characteristic equations $(\mathrm{d}\{x, y\}/\mathrm{d}s)$ and the compatibility equation $(\mathrm{d}u/\mathrm{d}s)$ are simultaneous first order ODEs in terms of a dummy variable s (curvilinear coordinate along the characteristics); we cannot solve the characteristic equations and compatibility equation independently as we have done for a semilinear equation. Note that in cases where $c \equiv 0$, the solution remains constant on the characteristics.
The rough idea in solving the PDE is thus to build up the integral surface from the Monge curves, obtained by solution of the ODEs.

Note that we make the difference between Monge curve or direction in (x, y, z)-space and characteristic curve or direction, their projections in (x, y)-space.

2.2.2 General solution

Suppose that the characteristic and compatibility equations that we have defined have two independent first integrals (function, $f(x, y, u)$, constant along the Monge curves)

$$\phi(x, y, u) = c_1 \quad \text{and} \quad \psi(x, y, u) = c_2.$$

Then the solution of equation (2.10) satisfies $F(\phi, \psi) = 0$ for some arbitrary function F (equivalently, $\phi = G(\psi)$ for some arbitrary G), where the form of F (or G) depends on the initial conditions.

Proof:

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Example 1:

$$(y + u)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x - y \quad \text{in} \quad y > 0, -\infty < x < \infty,$$
$$\text{with} \quad u = 1 + x \quad \text{on} \quad y = 1.$$

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Example 2: using the same procedure solve

$$x(y-u)\frac{\partial u}{\partial x}+y(x+u)\frac{\partial u}{\partial y}=(x+y)u \quad \text{with} \quad u=x^2+1 \quad \text{on} \quad y=x.$$

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Alternative approach to example 1:

$$(y+u)\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x-y \quad \text{in} \quad y>0, -\infty<x<\infty,$$
$$\text{with} \quad u=1+x \quad \text{on} \quad y=1.$$

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Summary: Solving the characteristic equations — two approaches.

1. Manipulate the equations to get them in a "directly integrable" form, e.g.,

$$\frac{1}{x+u}\frac{\mathrm{d}}{\mathrm{d}s}(x+u)=1$$

and find some combination of the variables which differentiates to zero (first integral), e.g.,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x+u}{y}\right)=0.$$

2. Solve the equations with respect to the dummy variable s, and apply the initial data (parameterised by θ) at $s=0$. Eliminate θ and s; find invariants by solving for constants.

2.3 Wave Equation

We consider the equation

$$\frac{\partial u}{\partial t}+(u+c)\frac{\partial u}{\partial x}=0 \quad \text{with} \quad u(0,x)=f(x),$$

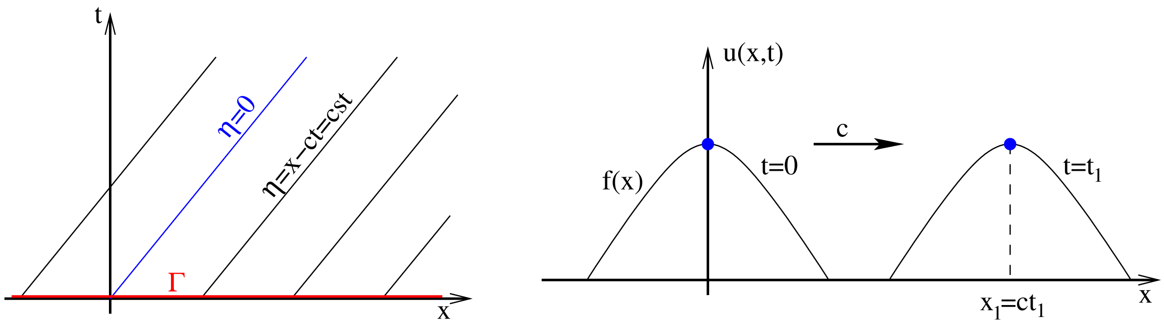
where c is some positive constant.

2.3.1 Linear Waves

If u is small (i.e., $u^2 \ll u$), then the equation can be approximated by the linear wave equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0 \quad \text{with} \quad u(x,0)=f(x).$$

The solution of the equation of characteristics, $\mathrm{d}x/\mathrm{d}t=c$, gives the first integral of the PDE, $\eta(x,t)=x-ct$, and then the general solution $u(x,t)=g(x-ct)$, where the function g is determined by the initial conditions. Applying $u(x, 0) = f(x)$ we find that the linear wave equation has the solution $u(x, t) = f(x - ct)$, which represents a wave (unchanging shape) propagating with constant wave speed c.



Note that u is constant where $x - ct$ =constant, i.e., on the characteristics.

2.3.2 Nonlinear Waves

For the nonlinear equation (quasilinear),

$$\frac{\partial u}{\partial t} + (u + c)\frac{\partial u}{\partial x} = 0,$$

the characteristics are defined by

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = c + u \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s} = 0,$$

which we can solve to give two independent first integrals $\phi = u$ and $\psi = x - (u + c)t$. So,

$$u = f[x - (u + c)t],$$

according to the initial conditions $u(x, 0) = f(x)$. This is similar to the previous result, but now the "wave speed" involves u.

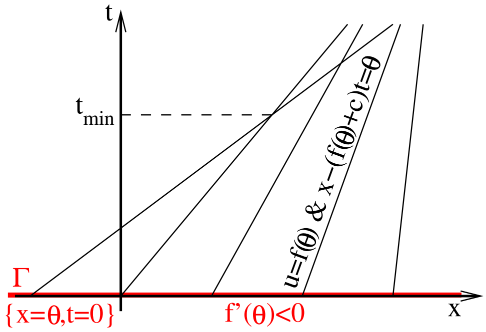
However, this form of the solution is not very helpful; it is more instructive to consider the characteristic curves. (The PDE is homogeneous, so the solution u is constant along the Monge curves — this is not the case in general — which can then be reduced to their projections in the (x, t)-plane.) By definition, $\psi = x - (c + u)t$ is constant on the characteristics (as well as u); differentiate ψ to find that the characteristics are described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + c.$$

These are straight lines,

$$x = (f(\theta) + c)t + \theta,$$

expressed in terms of a parameter θ. (If we make use of the parametric form of the data curve Γ: $\{x = \theta, t = 0, \theta \in \mathbb{R}\}$ and solve directly the Cauchy problem in terms of the coordinate $s = t$, we similarly find, $u = f(\theta)$ and $x = (u + c)t + \theta$.) The slope of the characteristics, $1/(c + u)$, varies from one line to another, and so, two curves can intersect.



Consider two crossing characteristics expressed in terms of θ_1 and θ_2,

$$\text{i.e.,}\quad x = (f(\theta_1) + c)t + \theta_1,$$
$$x = (f(\theta_2) + c)t + \theta_2.$$

(These correspond to initial values given at $x = \theta_1$ and $x = \theta_2$.) These characteristics intersect at the time

$$t = -\frac{\theta_1 - \theta_2}{f(\theta_1) - f(\theta_2)},$$

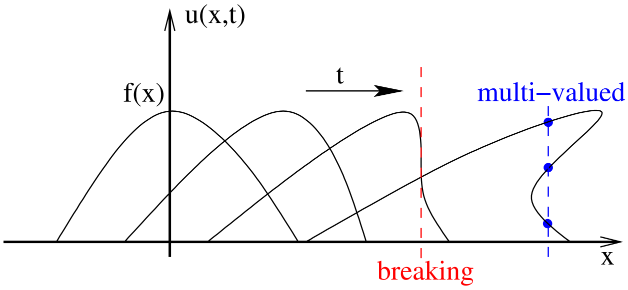
and if this is positive it will be in the region of solution. At this point u will not be single-valued and the solution breaks down. By letting $\theta_2 \to \theta_1$ we can see that the characteristics intersect at

$$t = -\frac{1}{f'(\theta)},$$

and the minimum time for which the solution becomes multi-valued is

$$t_{\min} = \frac{1}{\max[-f'(\theta)]},$$

i.e., the solution is single valued (i.e., is physical) only for $0 \le t < t_{\min}$. Hence, when $f'(\theta) < 0$ we can expect the solution to exist only for a finite time. In physical terms, the equation considered is purely advective; in real waves, such as shock waves in gases, when very large gradients are formed then diffusive terms (e.g., $\partial_{xx} u$) become vitally important.



To illustrate finite time solutions of the nonlinear wave equation, consider

$$f(\theta) = \theta(1-\theta), \quad (0 \le \theta \le 1),$$
$$f'(\theta) = 1 - 2\theta.$$

So, $f'(\theta) < 0$ for $1/2 < \theta < 1$ and we can expect the solution not to remain single-valued for all values of t. ($\max[-f'(\theta)] = 1$ so $t_{\min} = 1$). Now,

$$u = f(x - (u+c)t),$$
$$\text{so}\quad u = [x - (u+c)t] \times [1 - x + (u+c)t], \quad (ct \le x \le 1 + ct),$$

which we can express as

$$t^2u^2 + (1 + t - 2xt + 2ct^2)u + (x^2 - x - 2ctx + ct + c^2t^2) = 0,$$

and solving for u (we take the positive root from initial data)

$$u = \frac{1}{2t^2}\left(2t(x-ct) - (1+t) + \sqrt{(1+t)^2 - 4t(x-ct)}\right).$$

Now, at $t = 1$,

$$u = x - (c+1) + \sqrt{1 + c - x},$$

so the solution becomes singular as $t \to 1$ and $x \to 1 + c$.

2.3.3 Weak Solution

When wave breaking occurs (multi-valued solutions) we must re-think the assumptions in our model. Consider again the nonlinear wave equation,

$$\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = 0,$$

and put $w(x,t) = u(x,t) + c$; hence the PDE becomes the inviscid Burger's equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0,$$

or equivalently in a conservative form

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{w^2}{2}\right) = 0,$$

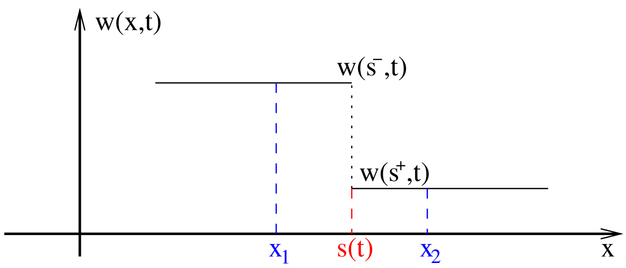
where $w^2/2$ is the flux function. We now consider its integral form,

$$\int_{x_1}^{x_2}\left[\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{w^2}{2}\right)\right]\mathrm{d}x = 0 \quad\Leftrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} w(x,t)\,\mathrm{d}x = -\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{w^2}{2}\right)\mathrm{d}x$$

where $x_2 > x_1$ are real. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} w(x,t)\,\mathrm{d}x = \frac{w^2(x_1,t)}{2} - \frac{w^2(x_2,t)}{2}.$$

Let us now relax the assumption regarding the differentiability of the solution; suppose that w has a discontinuity in $x = s(t)$ with $x_1 < s(t) < x_2$.



Thus, splitting the interval $[x_1, x_2]$ in two parts, we have

$$\begin{aligned}\frac{w^2(x_1,t)}{2} - \frac{w^2(x_2,t)}{2} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{s(t)} w(x,t)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)}^{x_2} w(x,t)\,\mathrm{d}x,\\ &= w(s^-,t)\,\dot{s}(t) + \int_{x_1}^{s(t)}\frac{\partial w}{\partial t}\,\mathrm{d}x - w(s^+,t)\,\dot{s}(t) + \int_{s(t)}^{x_2}\frac{\partial w}{\partial t}\,\mathrm{d}x,\end{aligned}$$

where $w(s^-(t),t)$ and $w(s^+(t),t)$ are the values of w as $x \to s$ from below and above, respectively; $\dot{s} = \mathrm{d}s/\mathrm{d}t$.

Now, take the limit $x_1 \to s^-(t)$ and $x_2 \to s^+(t)$. Since $\partial w/\partial t$ is bounded, the two integrals tend to zero. We then have

$$\frac{w^2(s^-,t)}{2} - \frac{w^2(s^+,t)}{2} = \dot{s}\left(w(s^-,t) - w(s^+,t)\right).$$

The velocity of the discontinuity is the shock velocity $U = \dot{s}$. If $[\,]$ indicates the jump across the shock then this condition may be written in the form

$$U\,[w] = \left[\frac{w^2}{2}\right].$$

The shock velocity for Burger's equation is

$$U = \frac{1}{2}\frac{w^2(s^+) - w^2(s^-)}{w(s^+) - w(s^-)} = \frac{w(s^+) + w(s^-)}{2}.$$

The problem then reduces to fitting shock discontinuities into the solution in such a way that the jump condition is satisfied and multi-valued solutions are avoided. A solution that satisfies the original equation in regions and which satisfies the integral form of the equation is called a weak solution or generalised solution.

Example 1: Consider the inviscid Burger's equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0,$$

with initial conditions

$$w(x=\theta, t=0) = f(\theta) = \begin{cases} 1 & \text{for} \quad \theta \le 0, \\ 1-\theta & \text{for} \quad 0 \le \theta \le 1, \\ 0 & \text{for} \quad \theta \ge 1. \end{cases}$$

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2.4 Systems of Equations

2.4.1 Linear and Semilinear Equations

These are equations of the form

$$\sum_{j=1}^{n}\left(a_{ij}\,u_x^{(j)} + b_{ij}\,u_y^{(j)}\right) = c_i, \quad i = 1,2,\ldots,n \quad \left(\frac{\partial u}{\partial x} = u_x\right),$$

for the unknowns $u^{(1)}, u^{(j)}, \ldots, u^{(n)}$ and when the coefficients a_{ij} and b_{ij} are functions only of x and y. (Though the c_i could also involve $u^{(k)}$.)
In matrix notation

$$A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{c},$$

where

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}, \qquad B = (b_{ij}) = \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(n)} \end{bmatrix}.$$

E.g.,

$$\begin{aligned} u_x^{(1)} - 2u_x^{(2)} + 3u_y^{(1)} - u_y^{(2)} &= x + y, \\ u_x^{(1)} + u_x^{(2)} - 5u_y^{(1)} + 2u_y^{(2)} &= x^2 + y^2, \end{aligned}$$

can be expressed as

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_x^{(1)} \\ u_x^{(2)} \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} u_y^{(1)} \\ u_y^{(2)} \end{bmatrix} = \begin{bmatrix} x+y \\ x^2+y^2 \end{bmatrix},$$

or $A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{c}$ where

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} x+y \\ x^2+y^2 \end{bmatrix}.$$

If we multiply by $A^{-1} \left(= \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} \right)$

$$\begin{aligned} & A^{-1}A\,\mathbf{u}_x + A^{-1}B\,\mathbf{u}_y = A^{-1}\mathbf{c}, \\ \text{we obtain} \quad & \mathbf{u}_x + D\mathbf{u}_y = d, \end{aligned}$$

where $D = A^{-1}B \left(= \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 7/3 & 1 \\ -8/3 & 1 \end{bmatrix} \right)$ and $d = A^{-1}c$.

We now assume that the matrix A is non-singular (i.e., the inverse A^{-1} exists) — at least there is some region of the (x, y)-plane where it is non-singular. Hence, we need only to consider systems of the form

$$\mathbf{u}_x + D\mathbf{u}_y = \mathbf{d}.$$

We also limit our attention to totally hyperbolic systems, i.e., systems where the matrix D has n distinct real eigenvalues (or at least there is some region of the plane where this holds). D has the n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ where $\det(\lambda_i I - D) = 0$ $(i = 1, \ldots, n)$, with $\lambda_i \neq \lambda_j$ $(i \neq j)$ and the n corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ so that

$$D\mathbf{e}_i = \lambda_i \mathbf{e}_i.$$

The matrix $P = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]$ diagonalises D via $P^{-1}DP = \Lambda$,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & \ldots & \ddots & \ldots & 0 \\ 0 & \ldots & 0 & \lambda_{n-1} & 0 \\ 0 & \ldots & \ldots & 0 & \lambda_n \end{bmatrix}.$$

We now put $\mathbf{u} = P\mathbf{v}$,

$$\begin{aligned} \text{then} \quad & P\mathbf{v}_x + P_x\mathbf{v} + DP\mathbf{v}_y + DP_y\mathbf{v} = \mathbf{d}, \\ \text{and} \quad & P^{-1}P\mathbf{v}_x + P^{-1}P_x\mathbf{v} + P^{-1}DP\mathbf{v}_y + P^{-1}DP_y\mathbf{v} = P^{-1}\mathbf{d}, \end{aligned}$$

which is of the form

$$\mathbf{v}_x + \Lambda\mathbf{v}_y = \mathbf{q},$$

where

$$\mathbf{q} = P^{-1}\mathbf{d} - P^{-1}P_x\mathbf{v} - P^{-1}DP_y\mathbf{v}.$$

The system is now of the form

$$v_x^{(i)} + \lambda_i\, v_y^{(i)} = q_i \quad (i = 1, \ldots, n),$$

where q_i can involve $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ and with n characteristics given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_i.$$

This is the canonical form of the equations.

Example 1: Consider the linear system

$$\left.\begin{aligned} u_x^{(1)} + 4\, u_y^{(2)} &= 0, \\ u_x^{(2)} + 9\, u_y^{(1)} &= 0, \end{aligned}\right\} \quad \text{with initial conditions } \mathbf{u} = [2x, 3x]^{\mathrm{T}} \text{ on } y = 0.$$

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Example 2: Reduce the linear system

$$\mathbf{u}_x + \begin{bmatrix} 4y - x & 2x - 2y \\ 2y - 2x & 4x - y \end{bmatrix} \mathbf{u}_x = 0$$

to canonical form in the region of the (x, y)-space where it is totally hyperbolic.

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2.4.2 Quasilinear Equations

We consider systems of n equations, involving n functions $u^{(i)}(x, y)$ $(i = 1, \ldots, n)$, of the form

$$\mathbf{u}_x + D\mathbf{u}_y = \mathbf{d},$$

where D as well as $\mathbf{d}$ may now depend on $\mathbf{u}$. (We have already shown how to reduce a more general system $A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{c}$ to that simpler form.) Again, we limit our attention to totally hyperbolic systems; then

$$\Lambda = P^{-1}DP \Longleftrightarrow D = P\Lambda P^{-1},$$

using the same definition of P, P^{-1} and the diagonal matrix Λ, as for the linear and semilinear cases. So, we can transform the system in its normal form as,

$$P^{-1}\mathbf{u}_x + \Lambda P^{-1}\mathbf{u}_y = P^{-1}\mathbf{d},$$

such that it can be written in component form as

$$\sum_{j=1}^{n} P_{ij}^{-1}\left(\frac{\partial}{\partial x}u^{(j)} + \lambda_i \frac{\partial}{\partial y}u^{(j)}\right) = \sum_{j=1}^{n} P_{ij}^{-1} d_j, \quad (i = 1, \ldots, n)$$

where λ_i is the ith eigenvalue of the matrix D and wherein the ith equation involves differentiation only in a single direction — the direction $\mathrm{d}y/\mathrm{d}x = \lambda_i$. We define the ith characteristic, with curvilinear coordinate s_i, as the curve in the (x, y)-plane along which

$$\frac{\mathrm{d}x}{\mathrm{d}s_i} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}s_i} = \lambda_i \quad \text{or equivalently} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_i.$$

Hence, the directional derivative parallel to the characteristic is

$$\frac{\mathrm{d}}{\mathrm{d}s_i} u^{(j)} = \frac{\partial}{\partial x} u^{(j)} + \lambda_i \frac{\partial}{\partial y} u^{(j)},$$

and the system in normal form reduces to n ODEs involving different components of $\mathbf{u}$

$$\sum_{j=1}^{n} P_{ij}^{-1} \frac{\mathrm{d}}{\mathrm{d}s_i} u^{(j)} = \sum_{j=1}^{n} P_{ij}^{-1} d_j \quad (i = 1, \ldots, n).$$

Example 1: Unsteady, one-dimensional motion of an inviscid compressible adiabatic gas. Consider the equation of motion (Euler equation)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x},$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0.$$

If the entropy is the same everywhere in the motion then $P\rho^{-\gamma} = \text{constant}$, and the motion equation becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial x} = 0,$$

where $c^2 = \mathrm{d}P/\mathrm{d}\rho = \gamma P/\rho$ is the sound speed.

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Chapter 3

Second Order Linear and Semilinear Equations in Two Variables

Contents

3.1 Classification and Standard Form Reduction

Consider a general second order linear equation in two independent variables

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+2b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}+d(x,y)\frac{\partial u}{\partial x}+e(x,y)\frac{\partial u}{\partial y}+f(x,y)u=g(x,y).$$

In the case of a semilinear equation, the coefficients d, e, f and g could be functions of $\partial_x u, \partial_y u$ and u as well.
Recall, for a first order linear and semilinear equation, $a\,\partial u/\partial x + b\,\partial u/\partial y = c$, we could define new independent variables, $\xi(x,y)$ and $\eta(x,y)$ with $J = \partial(\xi,\eta)/\partial(x,y) \neq \{0,\infty\}$, to reduce the equation to the simpler form, $\partial u/\partial\xi = \kappa(\xi,\eta)$. For the second order equation, can we also transform the variables from (x,y) to (ξ,η) to put the equation into a simpler form?

So, consider the coordinate transformation $(x,y)\to(\eta,\xi)$ where ξ and η are such that the Jacobian,

$$J=\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y}\\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y}\end{vmatrix}\neq\{0,\infty\}.$$

Then by the inverse theorem, there is an open neighbourhood of (x,y) and another neighbourhood of (ξ,η) such that the transformation is invertible and one-to-one on these neighbourhoods. As before we compute the following chain rule derivations

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x},\quad \frac{\partial u}{\partial y}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2+2\frac{\partial^2 u}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial^2 u}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}+\frac{\partial u}{\partial\eta}\frac{\partial^2\eta}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}.$$

The equation becomes

$$A\frac{\partial^2 u}{\partial \xi^2} + 2B\frac{\partial^2 u}{\partial \xi \partial \eta} + C\frac{\partial^2 u}{\partial \eta 2} + F(u_\xi, u_\eta, u, \xi, \eta) = 0, \tag{3.1}$$

where

$$A = a\left(\frac{\partial \xi}{\partial x}\right)^2 + 2b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2,$$
$$B = a\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + b\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + c\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},$$
$$C = a\left(\frac{\partial \eta}{\partial x}\right)^2 + 2b\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + c\left(\frac{\partial \eta}{\partial y}\right)^2.$$

We write explicitly only the principal part of the PDE, involving the highest-order derivatives of u (terms of second order). It is easy to verify that

$$(B^2 - AC) = (b^2 - ac)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2,$$

where $(\partial_x \xi \partial_y \eta - \partial_y \xi \partial_x \eta)^2$ is just the Jacobian squared. So, provided $J \neq 0$ we see that the sign of the discriminant $b^2 - ac$ is invariant under coordinate transformations. We can use this invariance properties to classify the equation.

Equation (3.1) can be simplified if we can choose ξ and η so that some of the coefficients A, B or C are zero. Let us define,

$$D_\xi = \frac{\partial \xi/\partial x}{\partial \xi/\partial y} \quad \text{and} \quad D_\eta = \frac{\partial \eta/\partial x}{\partial \eta/\partial y};$$

then we can write

$$A = \left(aD_\xi^2 + 2bD_\xi + c\right)\left(\frac{\partial \xi}{\partial y}\right)^2,$$
$$B = \left(aD_\xi D_\eta + b\left(D_\xi + D_\eta\right) + c\right)\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},$$
$$C = \left(aD_\eta^2 + 2bD_\eta + c\right)\left(\frac{\partial \eta}{\partial y}\right)^2.$$

Now consider the quadratic equation

$$aD^2 + 2bD + c = 0, \tag{3.2}$$

whose solution is given by

$$D = \frac{-b \pm \sqrt{b^2 - ac}}{a}.$$

If the discriminant $b^2 - ac \neq 0$, equation (3.2) has two distinct roots; so, we can make both coefficients A and C zero if we arbitrarily take the root with the negative sign for D_ξ and the one with the positive sign for D_η,

$$D_\xi = \frac{\partial \xi/\partial x}{\partial \xi/\partial y} = \frac{-b - \sqrt{b^2 - ac}}{a} \Rightarrow A = 0, \tag{3.3}$$
$$D_\eta = \frac{\partial \eta/\partial x}{\partial \eta/\partial y} = \frac{-b + \sqrt{b^2 - ac}}{a} \Rightarrow C = 0.$$

Then, using $D_\xi D_\eta = c/a$ and $D_\xi + D_\eta = -2b/a$ we have

$$B = \frac{2}{a}\left(ac - b^2\right)\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \Rightarrow B \neq 0.$$

Furthermore, if the discriminant $b^2 - ac > 0$ then D_ξ and D_η as well as ξ and η are real. So, we can define two families of one-parameter characteristics of the PDE as the curves described by the equation $\xi(x,y) =$ constant and the equation $\eta(x,y) =$ constant. Differentiate ξ along the characteristic curves given by $\xi =$ constant,

$$\mathrm{d}\xi = \frac{\partial \xi}{\partial x}\mathrm{d}x + \frac{\partial \xi}{\partial y}\mathrm{d}y = 0,$$

and make use of (3.3) to find that these characteristics satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b + \sqrt{b^2 - ac}}{a}. \tag{3.4}$$

Similarly we find that the characteristic curves described by $\eta(x,y) =$ constant satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b - \sqrt{b^2 - ac}}{a}. \tag{3.5}$$

If the discriminant $b^2 - ac = 0$, equation (3.2) has one unique root and if we take this root for D_ξ say, we can make the coefficient A zero,

$$D_\xi = \frac{\partial \xi/\partial x}{\partial \xi/\partial y} = -\frac{b}{a} \Rightarrow A = 0.$$

To get η independent of ξ, D_η has to be different from D_ξ, so $C \neq 0$ in this case, but B is now given by

$$B = \left(-a\frac{b}{a}D_\eta + b\left(-\frac{b}{a} + D_\eta\right) + c\right)\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = \left(-\frac{b^2}{a} + c\right)\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},$$

so that $B = 0$. When $b^2 - ac = 0$ the PDE has only one family of characteristic curves, for $\xi(x,y) =$ constant, whose equation is now

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}. \tag{3.6}$$

Thus we have to consider three different cases.

1. If $b^2 > ac$ we can apply the change of variable $(x,y) \to (\eta,\xi)$ to transform the original PDE to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + (\text{lower order terms}) = 0.$$

   In this case the equation is said to be hyperbolic and has two families of characteristics given by equation (3.4) and equation (3.5).

2. If $b^2 = ac$, a suitable choice for ξ still simplifies the PDE, but now we can choose η arbitrarily — provided η and ξ are independent — and the equation reduces to the form

$$\frac{\partial^2 u}{\partial \eta^2} + (\text{lower order terms}) = 0.$$

   The equation is said to be parabolic and has only one family of characteristics given by equation (3.6).

3. If $b^2 < ac$ we can again apply the change of variables $(x, y) \to (\eta, \xi)$ to simplify the equation but now these functions will be complex conjugate. To keep the transformation real, we apply a further change of variables $(\xi, \eta) \to (\alpha, \beta)$ via

$$\alpha = \xi + \eta = 2\,\Re(\eta),$$
$$\beta = i(\xi - \eta) = 2\,\Im(\eta),$$

$$\text{i.e.,}\quad \frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} \qquad \text{(via the chain rule);}$$

so, the equation can be reduced to

$$\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} + \text{(lower order terms)} = 0.$$

In this case the equation is said to be elliptic and has no real characteristics.

The above forms are called the canonical (or standard) forms of the second order linear or semilinear equations (in two variables).

Summary:

$b^2 - ac$	> 0	$= 0$	< 0
Canonical form	$\frac{\partial^2 u}{\partial\xi\partial\eta} + \ldots = 0$	$\frac{\partial^2 u}{\partial\eta^2} + \ldots = 0$	$\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} + \ldots = 0$
Type	Hyperbolic	Parabolic	Elliptic

E.g.,

- The wave equation,
$$\frac{\partial^2 u}{\partial t^2} - c_w^2 \frac{\partial^2 u}{\partial x^2} = 0,$$
is hyperbolic ($b^2 - ac = c_w^2 > 0$) and the two families of characteristics are described by $\mathrm{d}x/\mathrm{d}t = \pm c_w$ i.e., $\xi = x - c_w t$ and $\eta = x + c_w t$. So, the equation transforms into its canonical form $\partial^2 u/\partial\xi\partial\eta = 0$ whose solutions are waves travelling in opposite direction at speed c_w.
- The diffusion (heat conduction) equation,
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0,$$
is parabolic ($b^2 - ac = 0$). The characteristics are given by $\mathrm{d}t/\mathrm{d}x = 0$ i.e., $\xi = t = \text{constant}$.
- Laplace's equation,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
is elliptic ($b^2 - ac = -1 < 0$).

The type of a PDE is a local property. So, an equation can change its form in different regions of the plane or as a parameter is changed. E.g., Tricomi's equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (b^2 - ac = 0 - y = -y)$$

is elliptic in $y > 0$, parabolic for $y = 0$ and hyperbolic in $y < 0$.
For small disturbances in incompressible (inviscid) flow we have the following equation:

$$\frac{1}{1-m^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (b^2 - ac = -\frac{1}{1-m^2})$$

is elliptic if $m < 1$ and hyperbolic if $m > 1$.

Example 1: Reduce to canonical form

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{xy}\left(y^3\frac{\partial u}{\partial x} + x^3\frac{\partial u}{\partial y}\right).$$

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Example 2: Reduce to canonical form and then solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} + 1 = 0 \quad \text{in } 0 \le x \le 1, y > 0, \text{ with } u = \frac{\partial u}{\partial y} = x \text{ on } y = 0.$$

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Example 3: Reduce to canonical form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

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3.2 Extensions of the Theory

3.2.1 Linear second order equations in n variables

There are two obvious ways in which we might wish to extend the theory.

(i) To consider quasilinear second order equations (still in two independent variables.) Such equations can be classified in an invariant way according to rules analogous to those developed above for linear equations. However, since a, b and c are now functions of $\partial_x u, \partial_y u$ and u, its type turns out to depend in general on the particular solution searched and not just on the values of the independent variables.

(ii) To consider linear second order equations in more than two independent variables. In such cases it is not usually possible to reduce the equation to a simple canonical form. However, for the case of an equation with constant coefficients such a reduction is possible. Although this seems a rather restrictive class of equations, we can regard the classification obtained as a local one, at a particular point.
Consider the linear PDE

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i} + cu = d.$$

Without loss of generality we can take the matrix $A = (a_{ij}),\ i, j = 1 \cdots n$, to be symmetric (assuming derivatives commute). For any real symmetric matrix A, there is an associate orthogonal matrix P

such that $P^T AP = \Lambda$ where Λ is a diagonal matrix whose element are the eigenvalues, λ_i, of A and the columns of P the linearly independent eigenvectors of A, $\mathbf{e}_i = (e_{1i}, e_{2i}, \cdots, e_{ni})$. So

$$P = (e_{ij}) \quad \text{and} \quad \Lambda = (\lambda_i \delta_{ij}), \;\; i, j = 1, \cdots, n.$$

Now consider the transformation $\mathbf{x} = P\boldsymbol{\xi}$, i.e., $\boldsymbol{\xi} = P^{-1}\mathbf{x} = P^T\mathbf{x}$ (P orthogonal), where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$; this can be written as

$$x_i = \sum_{j=1}^{n} e_{ij}\,\xi_j \quad \text{and} \quad \xi_j = \sum_{i=1}^{n} e_{ij}\,x_i.$$

So,

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k}\, e_{ik} \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,r=1}^{n} \frac{\partial^2 u}{\partial \xi_k \partial \xi_r}\, e_{ik}\, e_{jr}.$$

The original equation becomes,

$$\sum_{i,j=1}^{n} \sum_{k,r=1}^{n} \frac{\partial^2 u}{\partial \xi_k \partial \xi_r}\, a_{ij}\, e_{ik}\, e_{jr} + (\text{lower order terms}) = 0.$$

But by definition of the eigenvectors of A,

$$\mathbf{e}_k^T\, A\, \mathbf{e}_r = \sum_{i,j=1}^{n} e_{ik}\, a_{ij}\, e_{jr} \equiv \lambda_r\, \delta_{rk}.$$

The equation simplifies to

$$\sum_{k=1}^{n} \lambda_k\, \frac{\partial^2 u}{\partial \xi_k^2} + (\text{lower order terms}) = 0.$$

We are now in a position to classify the equation.

- Equation is elliptic if and only if all λ_k are non-zero and have the same sign. E.g., Laplace's equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$
- When all the λ_k are non-zero and have the same sign except for precisely one of them, the equation is hyperbolic. E.g., the wave equation
$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0.$$
- When all the λ_k are non-zero and there are at least two of each sign, the equation ultra-hyperbolic. E.g.,
$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2};$$
such equations do not often arise in mathematical physics.
- If any of the λ_k vanishes the equation is parabolic. E.g., the heat equation
$$\frac{\partial u}{\partial t} - \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0.$$

3.2.2 The Cauchy Problem

Consider the problem of finding the solution of the equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + F(\partial_x u, \partial_y u, u, x, y) = 0,$$

which takes prescribed values on a given curve Γ which we assume is represented parametrically in the form

$$x = \phi(\sigma), \quad y = \theta(\sigma),$$

for $\sigma \in I$, where I is an interval, $\sigma_0 \leq \sigma \leq \sigma_1$, say. (Usually consider piecewise smooth curves.)
We specify Cauchy data on Γ: u, $\partial u/\partial x$ and $\partial u/\partial y$ are given $\forall \sigma \in I$, but note that we cannot specify all these quantitie
To show this, suppose u is given on Γ by $u = f(\sigma)$; then the derivative tangent to Γ, $\mathrm{d}u/\mathrm{d}\sigma$, can be calculated from $\mathrm{d}u/\mathrm{d}\sigma = f'(\sigma)$ but also

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\sigma} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\sigma} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\sigma}, \\ &= \phi'(\sigma)\frac{\partial u}{\partial x} + \theta'(\sigma)\frac{\partial u}{\partial y} = f'(\sigma),\end{aligned}$$

so, on Γ, the partial derivatives $\partial u/\partial x, \partial u/\partial y$ and u are connected by the above relation. Only derivatives normal to Γ and u can be prescribed independently.

So, the Cauchy problem consists in finding the solution $u(x, y)$ which satisfies the following conditions

$$\left.\begin{array}{rl} & u(\phi(\sigma), \theta(\sigma)) = f(\sigma) \\ \text{and} & \dfrac{\partial u}{\partial n}(\phi(\sigma), \theta(\sigma)) = g(\sigma) \end{array}\right\},$$

where $\sigma \in I$ and $\partial/\partial n = \mathbf{n} \cdot \nabla$ denotes a normal derivative to Γ (e.g., $\mathbf{n} = [\theta', -\phi']^T$); the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are uniquely determined on Γ by these conditions.

Set, $p = \partial u/\partial x$ and $q = \partial u/\partial y$ so that on Γ, p and q are known; then

$$\frac{\mathrm{d}p}{\mathrm{d}\sigma} = \frac{\partial^2 u}{\partial x^2}\frac{\mathrm{d}x}{\mathrm{d}\sigma} + \frac{\partial^2 u}{\partial x \partial y}\frac{\mathrm{d}y}{\mathrm{d}\sigma} \qquad \text{and} \qquad \frac{\mathrm{d}q}{\mathrm{d}\sigma} = \frac{\partial^2 u}{\partial x \partial y}\frac{\mathrm{d}x}{\mathrm{d}\sigma} + \frac{\partial^2 u}{\partial y^2}\frac{\mathrm{d}y}{\mathrm{d}\sigma}.$$

Combining these two equations with the original PDE gives the following system of equations for $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$ and $\partial^2 u/\partial y^2$ on Γ (in matrix form),

$$\mathcal{M}\begin{pmatrix} \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial^2 u}{\partial x \partial y} \\ \dfrac{\partial^2 u}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -F \\ \dfrac{\mathrm{d}p}{\mathrm{d}\sigma} \\ \dfrac{\mathrm{d}q}{\mathrm{d}\sigma} \end{pmatrix} \qquad \text{where} \quad \mathcal{M} = \begin{pmatrix} a & 2b & c \\ \dfrac{\mathrm{d}x}{\mathrm{d}\sigma} & \dfrac{\mathrm{d}y}{\mathrm{d}\sigma} & 0 \\ 0 & \dfrac{\mathrm{d}x}{\mathrm{d}\sigma} & \dfrac{\mathrm{d}y}{\mathrm{d}\sigma} \end{pmatrix}.$$

So, if $\det(\mathcal{M}) \neq 0$ we can solve the equations uniquely and find $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$ and $\partial^2 u/\partial y^2$ on Γ. By successive differentiations of these equations it can be shown that the derivatives of u of all orders are uniquely determined at each point on Γ for which $\det(\mathcal{M}) \neq 0$. The values of u at neighbouring points can be obtained using Taylor's theorem.

So, we conclude that the equation can be solved uniquely in the vicinity of Γ provided $\det(\mathcal{M}) \neq 0$ (Cauchy-Kowaleski theorem provides a majorant series ensuring convergence of Taylor's expansion).

Consider what happens when $\det(\mathcal{M}) = 0$, so that $\mathcal{M}$ is singular and we cannot solve uniquely for the second order derivatives on Γ. In this case the determinant $\det(\mathcal{M}) = 0$ gives,

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}\sigma}\right)^2 - 2b\frac{\mathrm{d}x}{\mathrm{d}\sigma}\frac{\mathrm{d}y}{\mathrm{d}\sigma} + c\left(\frac{\mathrm{d}x}{\mathrm{d}\sigma}\right)^2 = 0.$$

But,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\sigma}{\mathrm{d}x/\mathrm{d}\sigma}$$

and so (dividing through by $(\mathrm{d}x/\mathrm{d}\sigma)^2$), $\mathrm{d}y/\mathrm{d}x$ satisfies the equation,

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2b\,\frac{\mathrm{d}y}{\mathrm{d}x} + c = 0, \qquad \text{i.e.,} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}.$$

The exceptional curves Γ, on which, if u and its normal derivative are prescribed, no unique solution can be found satisfying these conditions, are the characteristics curves.

Chapter 4

Elliptic Equations

Contents

4.1 Definitions

Elliptic equations are typically associated with steady-state behaviour. The archetypal elliptic equation is Laplace's equation

$$\nabla^2 u = 0, \quad \text{e.g.,} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in 2-D},$$

and describes

- steady, irrotational flows,
- electrostatic potential in the absence of charge,
- equilibrium temperature distribution in a medium.

Because of their physical origin, elliptic equations typically arise as boundary value problems (BVPs). Solving a BVP for the general elliptic equation

$$\mathcal{L}[u] = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = F$$

(recall: all the eigenvalues of the matrix $A = (a_{ij}), i, j = 1 \cdots n$, are non-zero and have the same sign) is to find a solution u in some open region Ω of space, with conditions imposed on $\partial\Omega$ (the boundary of Ω) or at infinity. E.g., inviscid flow past a sphere is determined by boundary conditions on the sphere ($\mathbf{u} \cdot \mathbf{n} = 0$) and at infinity ($\mathbf{u} = \text{constant}$).
There are three types of boundary conditions for well-posed BVPs,

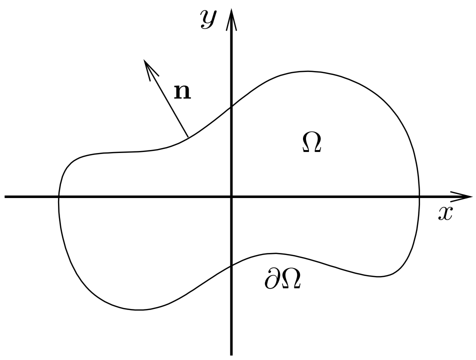
1. Dirichlet condition — u takes prescribed values on the boundary $\partial\Omega$ (first BVP).
2. Neumann conditions — the normal derivative, $\partial u/\partial n = \mathbf{n} \cdot \nabla u$ is prescribed on the boundary $\partial\Omega$ (second BVP).

In this case we have compatibility conditions (i.e., global constraints):
E.g., suppose u satisfies $\nabla^2 u = F$ on Ω and $\mathbf{n}\cdot\nabla u = \partial_n u = f$ on $\partial\Omega$. Then,

$$\int_\Omega \nabla^2 u \,\mathrm{d}V = \int_\Omega \nabla\cdot\nabla u \,\mathrm{d}V = \int_{\partial\Omega} \nabla u\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,\mathrm{d}S \qquad \text{(divergence theorem)},$$

$$\Rightarrow \int_\Omega F\,\mathrm{d}V = \int_{\partial\Omega} f\,\mathrm{d}S \qquad \text{for the problem to be well-defined.}$$

3. Robin conditions — a combination of u and its normal derivative such as $\partial u/\partial n + \alpha u$ is prescribed on the boundary $\partial\Omega$ (third BVP).



Sometimes we may have a mixed problem, in which u is given on part of $\partial\Omega$ and $\partial u/\partial n$ given on the rest of $\partial\Omega$.
If Ω encloses a finite region, we have an interior problem; if, however, Ω is unbounded, we have an exterior problem, and we must impose conditions "at infinity".
Note that initial conditions are irrelevant for these BVPs and the Cauchy problem for elliptic equations is not always well-posed (even if Cauchy-Kowaleski theorem states that the solution exist and is unique).

As a general rule, it is hard to deal with elliptic equations since the solution is global, affected by all parts of the domain. (Hyperbolic equations, posed as initial value or Cauchy problem, are more localised.)

From now, we shall deal mainly with the Helmholtz equation $\nabla^2 u + P\,u = F$, where P and F are functions of $\mathbf{x}$, and particularly with the special one if $P = 0$, Poisson's equation, or Laplace's equation, if $F = 0$ too. This is not too severe a restriction; recall that any linear elliptic equation can be put into the canonical form

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} + \cdots = 0$$

and that the lower order derivatives do not alter the overall properties of the solution.

4.2 Properties of Laplace's and Poisson's Equations

Definition: A continuous function satisfying Laplace's equation in an open region Ω, with continuous first and second order derivatives, is called an harmonic function. Functions u in $\mathcal{C}^2(\Omega)$ with $\nabla^2 u \geq 0$ (respectively $\nabla^2 u \leq 0$) are call subharmonic (respectively superharmonic).

4.2.1 Mean Value Property

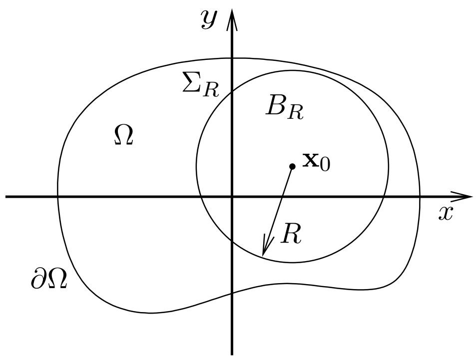
Definition: Let $\mathbf{x}_0$ be a point in Ω and let $B_R(\mathbf{x}_0)$ denote the open ball having centre $\mathbf{x}_0$ and radius R. Let $\Sigma_R(\mathbf{x}_0)$ denote the boundary of $B_R(\mathbf{x}_0)$ and let $A(R)$ be the surface area of $\Sigma_R(\mathbf{x}_0)$. Then a function u has the mean value property at a point $\mathbf{x}_0 \in \Omega$ if

$$u(\mathbf{x}_0) = \frac{1}{A(R)} \int_{\Sigma_R} u(\mathbf{x})\, \mathrm{d}S$$

for every $R > 0$ such that $B_R(\mathbf{x}_0)$ is contained in Ω. If instead $u(\mathbf{x}_0)$ satisfies

$$u(\mathbf{x}_0) = \frac{1}{V(R)} \int_{B_R} u(\mathbf{x})\, \mathrm{d}V,$$

where $V(R)$ is the volume of the open ball $B_R(\mathbf{x}_0)$, we say that $u(\mathbf{x}_0)$ has the second mean value property at a point $\mathbf{x}_0 \in \Omega$. The two mean value properties are equivalent.



Theorem: If u is harmonic in Ω, an open region of $\mathbb{R}^n$, then u has the mean value property on Ω.

Proof: We need to make use of Green's theorem which says,

$$\int_S \left(v\, \frac{\partial u}{\partial n} - u\, \frac{\partial v}{\partial n} \right) \mathrm{d}S = \int_V \left(v\, \nabla^2 u - u\, \nabla^2 v \right) \mathrm{d}V. \tag{4.1}$$

(Recall: Apply divergence theorem to the function $v\nabla u - u\nabla v$ to state Green's theorem.)

Since u is harmonic, it follows from equation (4.1), with $v = 1$, that

$$\int_S \frac{\partial u}{\partial n}\, \mathrm{d}S = 0.$$

Now, take $v = 1/r$, where $r = |\mathbf{x} - \mathbf{x}_0|$, and the domain V to be $B_R(\mathbf{x}_0) - B_\varepsilon(\mathbf{x}_0)$, $0 < \varepsilon < R$. Then, in $\mathbb{R}^n - \mathbf{x}_0$,

$$\nabla^2 v = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \right) = 0$$

so v is harmonic too and equation (4.1) becomes

$$\int_{\Sigma_R} u\, \frac{\partial v}{\partial n}\, \mathrm{d}S + \int_{\Sigma_\varepsilon} u\, \frac{\partial v}{\partial n}\, \mathrm{d}S = \int_{\Sigma_R} u\, \frac{\partial v}{\partial r}\, \mathrm{d}S - \int_{\Sigma_\varepsilon} u\, \frac{\partial v}{\partial r}\, \mathrm{d}S = 0$$

$$\Rightarrow \int_{\Sigma_\varepsilon} u\, \frac{\partial v}{\partial r}\, \mathrm{d}S = \int_{\Sigma_R} u\, \frac{\partial v}{\partial r}\, \mathrm{d}S \quad \text{i.e} \quad \frac{1}{\varepsilon^2} \int_{\Sigma_\varepsilon} u\, \mathrm{d}S = \frac{1}{r^2} \int_{\Sigma_R} u\, \mathrm{d}S.$$

Since u is continuous, then as $\varepsilon \to 0$ the LHS converges to $4\pi\, u(x_0, y_0, z_0)$ (with $n = 3$, say), so

$$u(\mathbf{x}_0) = \frac{1}{A(R)} \int_{\Sigma_R} u\, \mathrm{d}S.$$

Recovering the second mean value property (with $n = 3$, say) is straightforward

$$\int_0^R \rho^2 \, u(\mathbf{x}_0) \, \mathrm{d}\rho = \frac{R^3}{3} \, u(\mathbf{x}_0) = \frac{1}{4\pi} \int_0^R \int_{\Sigma_\rho} u \, \mathrm{d}S \, \mathrm{d}\rho = \frac{1}{4\pi} \int_{B_R} u \, \mathrm{d}V.$$

The inverse of this theorem holds too, but is harder to prove. If u has the mean value property then u is harmonic.

4.2.2 Maximum-Minimum Principle

One of the most important features of elliptic equations is that it is possible to prove theorems concerning the boundedness of the solutions.

Theorem: Suppose that the subharmonic function u satisfies

$$\nabla^2 u = F \quad \text{in } \Omega, \text{ with } F > 0 \text{ in } \Omega.$$

Then $u(x, y)$ attains its maximum on $\partial\Omega$.

Proof: (Theorem stated in 2-D but holds in higher dimensions.) Suppose, for contradiction, that u attains its maximum at an interior point (x_0, y_0) of Ω. Then at (x_0, y_0),

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial x^2} \leq 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \leq 0,$$

since it is a maximum. So,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \leq 0, \quad \text{which contradicts } F > 0 \text{ in } \Omega.$$

Hence u must attain its maximum on $\partial\Omega$, i.e., if $u \leq M$ on $\partial\Omega$, $u < M$ in Ω.

Theorem: The weak Maximum-Minimum Principle for Laplace's equation.
Suppose that u satisfies

$$\nabla^2 u = 0 \quad \text{in a bounded region } \Omega;$$

if $m \leq u \leq M$ on $\partial\Omega$, then $m \leq u \leq M$ in Ω.

Proof: (Theorem stated in 2-D but holds in higher dimensions.) Consider the function $v = u + \varepsilon(x^2 + y^2)$, for any $\varepsilon > 0$. Then $\nabla^2 v = 4\varepsilon > 0$ in Ω (since $\nabla^2(x^2 + y^2) = 4$), and using the previous theorem,

$$v \leq M + \varepsilon R^2 \quad \text{in } \Omega,$$

where $u \leq M$ on $\partial\Omega$ and R is the radius of the circle containing Ω. As this holds for any ε, let $\varepsilon \to 0$ to obtain

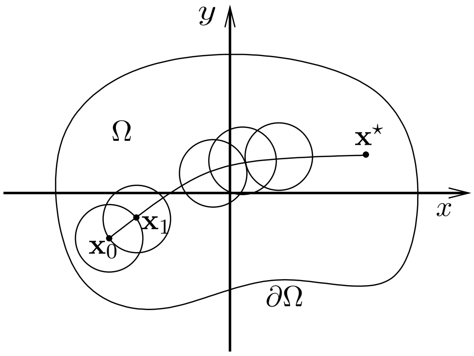
$$u \leq M \quad \text{in } \Omega,$$

i.e., if u satisfies $\nabla^2 u = 0$ in Ω, then u cannot exceed M, the maximum value of u on $\partial\Omega$.
Also, if u is a solution of $\nabla^2 u = 0$, so is $-u$. Thus, we can apply all of the above to $-u$ to get a minimum principle: if $u \geq m$ on $\partial\Omega$, then $u \geq m$ in Ω.

This theorem does not say that harmonic functions cannot attain m and M inside Ω though. We shall now progress into the strong Maximum-Minimum Principle.

Theorem: Suppose that u has the mean value property in a bounded region Ω and that u is continuous in $\bar{\Omega} = \Omega \cup \partial\Omega$. If u is not constant in Ω then u attains its maximum value on the boundary $\partial\Omega$ of Ω, not in the interior of Ω.

Proof: Since u is continuous in the closed, bounded domain $\bar{\Omega}$ then it attains its maximum M somewhere in $\bar{\Omega}$. Our aim is to show that, if u attains its maximum at an interior point of Ω, then u is constant in $\bar{\Omega}$.
Suppose $u(\mathbf{x}_0) = M$ and let $\mathbf{x}^\star$ be some other point of Ω. Join these points with a path covered by a sequence of overlapping balls, B_r.



Consider the ball with $\mathbf{x}_0$ at its center. Since u has the mean value property then

$$M = u(\mathbf{x}_0) = \frac{1}{A(r)} \int_{\Sigma_r} u \, \mathrm{d}S \leq M.$$

This equality must hold throughout this statement and $u = M$ throughout the sphere surrounding $\mathbf{x}_0$. Since the balls overlap, there is $\mathbf{x}_1$, centre of the next ball such that $u(\mathbf{x}_1) = M$; the mean value property implies that $u = M$ in this sphere also $[B_r(\mathbf{x}_1)]$. Continuing like this gives $u(\mathbf{x}^\star) = M$. Since $\mathbf{x}^\star$ is arbitrary, we conclude that $u = M$ throughout Ω, and by continuity throughout $\bar{\Omega}$. Thus if u is not a constant in Ω it can attain its maximum value only on the boundary $\partial\Omega$.

Corollary: Applying the above theorem to $-u$ establishes that if u is non constant it can attain its minimum only on $\partial\Omega$.
Also as a simple corollary, we can state the following theorem. (The proof follows immediately from the previous theorem and the weak Maximum-Minimum Principle.)

Theorem: The strong Maximum-Minimum Principle for Laplace's equation.
Let u be harmonic in $\bar{\Omega}$, i.e., solution of $\nabla^2 u = 0$ in Ω and continuous in $\bar{\Omega}$, with M and m the maximum and minimum values respectively of u on the boundary $\partial\Omega$. Then, either $m < u < M$ in Ω or else $m = u = M$ in $\bar{\Omega}$.

Note that it is important that Ω be bounded for the theorem to hold. E.g., consider $u(x, y) = \mathrm{e}^x \sin y$ with $\Omega = \{(x, y)| - \infty < x < +\infty, \, 0 < y < 2\pi\}$. Then $\nabla^2 u = 0$ and on the boundary of Ω we have $u = 0$, so that $m = M = 0$. But of course u is not identically zero in Ω.

Corollary: If $u = C$ is constant on $\partial\Omega$, then $u = C$ is constant in Ω. Armed with the above theorems we are in position to prove the uniqueness and the stability of the solution of the Dirichlet problem for Poisson's equation.
Consider the Dirichlet BVP

$$\nabla^2 u = F \text{ in } \Omega \quad \text{with} \quad u = f \text{ on } \partial\Omega$$

and suppose u_1, u_2 two solutions to the problem. Then $v = u_1 - u_2$ satisfies

$$\nabla^2 v = \nabla^2 (u_1 - u_2) = 0 \quad \text{in } \Omega, \text{ with } v = 0 \text{ on } \partial\Omega.$$

Thus, $v \equiv 0$ in Ω, i.e., $u_1 = u_2$; the solution is unique.

To establish the continuous dependence of the solution on the prescribed data (i.e., the stability of the solution) let u_1 and u_2 satisfy

$$\nabla^2 u_{\{1,2\}} = F \text{ in } \Omega \quad \text{with} \quad u_{\{1,2\}} = f_{\{1,2\}} \text{ on } \partial\Omega,$$

with $\max|f_1 - f_2| = \varepsilon$. Then $v = u_1 - u_2$ is harmonic with $v = f_1 - f_2$ on $\partial\Omega$. As before, v must have its maximum and minimum values on $\partial\Omega$; hence $|u_1 - u_2| \leq \varepsilon$ in $\bar{\Omega}$. So, the solution is stable — small changes in the boundary data lead to small changes in the solution.

We may use the Maximum-Minimum Principle to put bounds on the solution of an equation without solving it.

The strong Maximum-Minimum Principle may be extended to more general linear elliptic equations

$$\mathcal{L}[u] = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = F,$$

and, as for Poisson's equation it is possible then to prove that the solution to the Dirichlet BVP is unique and stable.

4.3 Solving Poisson Equation Using Green's Functions

We shall develop a formal representation for solutions to boundary value problems for Poisson's equation.

4.3.1 Definition of Green's Functions

Consider a general linear PDE in the form

$$\mathcal{L}(\mathbf{x})u(\mathbf{x}) = F(\mathbf{x}) \text{ in } \Omega,$$

where $\mathcal{L}(\mathbf{x})$ is a linear (self-adjoint) differential operator, $u(\mathbf{x})$ is the unknown and $F(\mathbf{x})$ is the known homogeneous term.
(Recall: $\mathcal{L}$ is self-adjoint if $\mathcal{L} = \mathcal{L}^\star$, where $\mathcal{L}^\star$ is defined by $\langle v|\mathcal{L}u\rangle = \langle \mathcal{L}^\star v|u\rangle$ and where $\langle v|u\rangle = \int v(\mathbf{x})w(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x}$ (($w(\mathbf{x})$ is the weight function).)

The solution to the equation can be written formally

$$u(\mathbf{x}) = \mathcal{L}^{-1}F(\mathbf{x}),$$

where $\mathcal{L}^{-1}$, the inverse of $\mathcal{L}$, is some integral operator. (We can expect to have $\mathcal{L}\mathcal{L}^{-1} = \mathcal{L}\mathcal{L}^{-1} = \mathcal{I}$, identity.) We define the inverse $\mathcal{L}^{-1}$ using a Green's function: let

$$u(\mathbf{x}) = \mathcal{L}^{-1}F(\mathbf{x}) = -\int_\Omega G(\mathbf{x}, \boldsymbol{\xi})F(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{4.2}$$

where $G(\mathbf{x}, \boldsymbol{\xi})$ is the Green's function associated with $\mathcal{L}$ (G is the kernel). Note that G depends on both the independent variable $\mathbf{x}$ and the new independent variable $\boldsymbol{\xi}$, over which we integrate.

Recall the Dirac δ-function (more precisely distribution or generalised function) $\delta(\mathbf{x})$ which has the properties,

$$\int_{\mathbb{R}^n} \delta(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^n} \delta(\mathbf{x} - \boldsymbol{\xi})\,h(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = h(\mathbf{x}).$$

Now, applying $\mathcal{L}$ to equation (4.2) we get

$$\mathcal{L}u(\mathbf{x}) = F(\mathbf{x}) = -\int_\Omega \mathcal{L}(\mathbf{x})G(\mathbf{x}, \boldsymbol{\xi})F(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi};$$

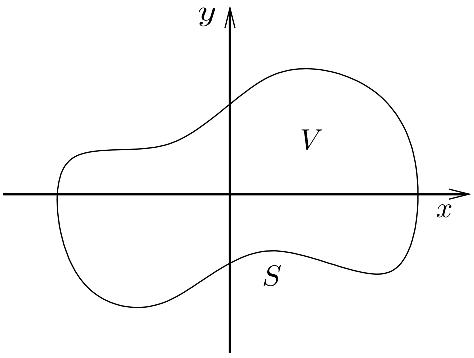
hence, the Green's function $G(\mathbf{x}, \boldsymbol{\xi})$ satisfies

$$u(\mathbf{x}) = -\int_\Omega G(\mathbf{x}, \boldsymbol{\xi})\,F(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \quad \text{with} \quad \mathcal{L}\,G(\mathbf{x}, \boldsymbol{\xi}) = -\delta(\mathbf{x} - \boldsymbol{\xi}) \text{ and } \mathbf{x}, \boldsymbol{\xi} \in \Omega.$$

4.3.2 Green's function for the Laplace Operator

Consider Poisson's equation in the open bounded region V with boundary S,

$$\nabla^2 u = F \text{ in } V. \tag{4.3}$$



Then, Green's theorem ($\mathbf{n}$ is normal to S outward from V), which states

$$\int_V \left(u\nabla^2 v - v\nabla^2 u\right) \mathrm{d}V = \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) \mathrm{d}S,$$

for any functions u and v, with $\partial h/\partial n = \mathbf{n}\cdot\nabla h$, becomes

$$\int_V u\nabla^2 v \,\mathrm{d}V = \int_V vF \,\mathrm{d}V + \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) \mathrm{d}S;$$

so, if we choose $v \equiv v(\mathbf{x}, \boldsymbol{\xi})$, singular at $\mathbf{x} = \boldsymbol{\xi}$, such that $\nabla^2 v = -\delta(\mathbf{x} - \boldsymbol{\xi})$, then u is solution of the equation

$$u(\boldsymbol{\xi}) = -\int_V vF \,\mathrm{d}V - \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) \mathrm{d}S \tag{4.4}$$

which is an integral equation since u appears in the integrand. To address this we consider another function, $w \equiv w(\mathbf{x}, \boldsymbol{\xi})$, regular at $\mathbf{x} = \boldsymbol{\xi}$, such that $\nabla^2 w = 0$ in V. Hence, apply Green's theorem to the function u and w

$$\int_S \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) \mathrm{d}S = \int_V \left(u\nabla^2 w - w\nabla^2 u\right) \mathrm{d}V = -\int_V wF \,\mathrm{d}V.$$

Combining this equation with equation (4.4) we find

$$u(\boldsymbol{\xi}) = -\int_V (v + w)F \,\mathrm{d}V - \int_S \left(u\frac{\partial}{\partial n}(v + w) - (v + w)\frac{\partial u}{\partial n}\right) \mathrm{d}S,$$

so, if we consider the <u>fundamental solution</u> of Laplace's equation, $G = v + w$, such that $\nabla^2 G = -\delta(\mathbf{x} - \boldsymbol{\xi})$ in V,

$$u(\boldsymbol{\xi}) = -\int_V GF \,\mathrm{d}V - \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) \mathrm{d}S. \tag{4.5}$$

Note that if, F, G and the solution u are sufficiently well-behaved at infinity this integral equation is also valid for unbounded regions (i.e., for exterior BVP for Poisson's equation).

The way to remove u or $\partial u/\partial n$ from the RHS of the above equation depends on the choice of boundary conditions.

Dirichlet Boundary Conditions

Here, the solution to equation (4.3) satisfies the condition $u = f$ on S. So, we choose w such that $w = -v$ on S, i.e., $G = 0$ on S, in order to eliminate $\partial u/\partial n$ form the RHS of equation (4.5). Then, the solution of the Dirichlet BVP for Poisson's equation

$$\nabla^2 u = F \text{ in } V \text{ with } u = f \text{ on } S$$

is

$$u(\boldsymbol{\xi}) = -\int_V GF\,\mathrm{d}V - \int_S f\frac{\partial G}{\partial n}\,\mathrm{d}S,$$

where $G = v + w$ (w regular at $\mathbf{x} = \boldsymbol{\xi}$) with $\nabla^2 v = -\delta(\mathbf{x} - \boldsymbol{\xi})$ and $\nabla^2 w = 0$ in V and $v + w = 0$ on S. So, the Green's function G is solution of the Dirichlet BVP

$$\begin{aligned} &\nabla^2 G = -\delta(\mathbf{x} - \boldsymbol{\xi}) \text{ in } V,\\ \text{with}\quad &G = 0 \text{ on } S. \end{aligned}$$

Neumann Boundary Conditions

Here, the solution to equation (4.3) satisfies the condition $\partial u/\partial n = f$ on S. So, we choose w such that $\partial w/\partial n = -\partial v/\partial n$ on S, i.e., $\partial G/\partial n = 0$ on S, in order to eliminate u from the RHS of equation (4.5). However, the Neumann BVP

$$\begin{aligned} &\nabla^2 G = -\delta(\mathbf{x} - \boldsymbol{\xi}) \text{ in } V,\\ \text{with}\quad &\frac{\partial G}{\partial n} = 0 \text{ on } S, \end{aligned}$$

which does not satisfy a compatibility equation, has no solution. Recall that the Neumann BVP $\nabla^2 u = F$ in V, with $\partial u/\partial n = f$ on S, is ill-posed if

$$\int_V F\mathrm{d}V \neq \int_S f\mathrm{d}S.$$

We need to alter the Green's function a little to satisfy the compatibility equation; put $\nabla^2 G = -\delta + C$, where C is a constant, then the compatibility equation for the Neumann BVP for G is

$$\int_V (-\delta + C)\,\mathrm{d}V = \int_S 0\,\mathrm{d}S = 0 \Rightarrow C = \frac{1}{\mathcal{V}},$$

where $\mathcal{V}$ is the volume of V. Now, applying Green's theorem to G and u:

$$\int_V (G\nabla^2 u - u\nabla^2 G)\,\mathrm{d}V = \int_S \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)\mathrm{d}S$$

we get

$$u(\boldsymbol{\xi}) = -\int_V GF\,\mathrm{d}V + \int_S Gf\,\mathrm{d}S + \underbrace{\frac{1}{\mathcal{V}}\int_V u\,\mathrm{d}V}_{\bar{u}}.$$

This shows that, whereas the solution of Poisson's equation with Dirichlet boundary conditions is unique, the solution of the Neumann problem is unique up to an additive constant $\bar{u}$ which is the mean value of u over Ω.
Thus, the solution of the Neumann BVP for Poisson's equation

$$\nabla^2 u = F \text{ in } V \text{ with } \frac{\partial u}{\partial n} = f \text{ on } S$$

is

$$u(\boldsymbol{\xi}) = \bar{u} - \int_V GF\,\mathrm{d}V + \int_S Gf\,\mathrm{d}S,$$

where $G = v + w$ (w regular at $\mathbf{x} = \boldsymbol{\xi}$) with $\nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi}), \nabla^2 w = 1/\mathcal{V}$ in V and $\partial w/\partial n = -\partial v/\partial n$ on S. So, the Green's function G is solution of the Neumann BVP

$$\begin{aligned} &\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi}) + \frac{1}{\mathcal{V}} \text{ in } V, \\ \text{with } &\frac{\partial G}{\partial n} = 0 \text{ on } S. \end{aligned}$$

Robin Boundary Conditions

Here, the solution to equation (4.3) satisfies the condition $\partial u/\partial n + \alpha u = f$ on S. So, we choose w such that $\partial w/\partial n + \alpha w = -\partial v/\partial n - \alpha v$ on S, i.e., $\partial G/\partial n + \alpha G = 0$ on S. Then,

$$\int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) \mathrm{d}S = \int_S \left(u\frac{\partial G}{\partial n} + G(\alpha u - f) \right) \mathrm{d}S = -\int_S Gf \,\mathrm{d}S.$$

Hence, the solution of the Robin BVP for Poisson's equation

$$\nabla^2 u = F \text{ in } V \text{ with } \frac{\partial u}{\partial n} + \alpha u = f \text{ on } S$$

is

$$u(\boldsymbol{\xi}) = -\int_V GF \,\mathrm{d}V + \int_S Gf \,\mathrm{d}S,$$

where $G = v + w$ (w regular at $\mathbf{x} = \boldsymbol{\xi}$) with $\nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi})$ and $\nabla^2 w = 0$ in V and $\partial w/\partial n + \alpha w = -\partial v/\partial n - \alpha v$ on S. So, the Green's function G is solution of the Robin BVP

$$\begin{aligned} &\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi}) \text{ in } V, \\ \text{with } &\frac{\partial G}{\partial n} + \alpha G = 0 \text{ on } S. \end{aligned}$$

Symmetry of Green's Functions

The Green's function is symmetric (i.e., $G(\mathbf{x},\boldsymbol{\xi}) = G(\boldsymbol{\xi},\mathbf{x})$). To show this, consider two Green's functions, $G_1(\mathbf{x}) \equiv G(\mathbf{x},\boldsymbol{\xi}_1)$ and $G_2(\mathbf{x}) \equiv G(\mathbf{x},\boldsymbol{\xi}_2)$, and apply Green's theorem to these,

$$\int_V \left(G_1\nabla^2 G_2 - G_2\nabla^2 G_1 \right) \mathrm{d}V = \int_S \left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n} \right) \mathrm{d}S.$$

Now, since, G_1 and G_2 are by definition Green's functions, $G_1 = G_2 = 0$ on S for Dirichlet boundary conditions, $\partial G_1/\partial n = \partial G_2/\partial n = 0$ on S for Neumann boundary conditions or $G_2\,\partial G_1/\partial n = G_1\,\partial G_2/\partial n$ on S for Robin boundary conditions, so in any case the right-hand side is equal to zero. Also, $\nabla^2 G_1 = -\delta(\mathbf{x}-\boldsymbol{\xi}_1)$, $\nabla^2 G_2 = -\delta(\mathbf{x}-\boldsymbol{\xi}_2)$ and the equation becomes

$$\int_V G(\mathbf{x},\boldsymbol{\xi}_1)\,\delta(\mathbf{x}-\boldsymbol{\xi}_2)\,\mathrm{d}V = \int_V G(\mathbf{x},\boldsymbol{\xi}_2)\,\delta(\mathbf{x}-\boldsymbol{\xi}_1)\,\mathrm{d}V,$$

$$G(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) = G(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2).$$

Nevertheless, note that for Neumann BVPs, the term $1/\mathcal{V}$ which provides the additive constant to the solution to Poisson's equation breaks the symmetry of G.

Example 1: Consider the 2-dimensional Dirichlet problem for Laplace's equation,

$$\nabla^2 u = 0 \text{ in } V, \text{ with } u = f \text{ on } S \text{ (boundary of } V).$$

Since u is harmonic in V (i.e., $\nabla^2 u = 0$) and $u = f$ on S, then Green's theorem gives

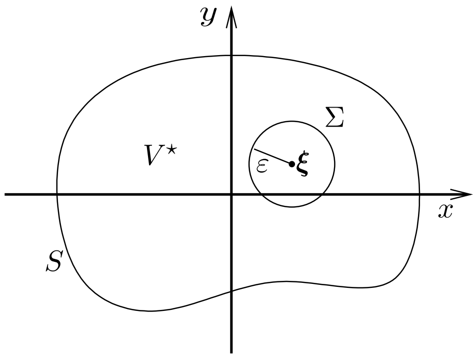
$$\int_V u\nabla^2 v \,\mathrm{d}V = \int_S \left(f\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}S.$$

Note that we have no information about $\partial u/\partial n$ on S or u in V. Suppose we choose,

$$v = -\frac{1}{4\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right),$$

then $\nabla^2 v = 0$ on V for all points except $P \equiv (x=\xi, y=\eta)$, where it is singular (undefined).
To eliminate this singularity, we 'cut this point P out' — i.e., surround P by a small circle of radius $\varepsilon = \sqrt{(x-\xi)^2+(y-\eta)^2}$ and denote the circle by Σ, whose parametric form in polar coordinates is

$$\Sigma : \{x-\xi = \varepsilon\cos\theta, y-\eta = \varepsilon\sin\theta \text{ with } \varepsilon > 0 \text{ and } \theta \in (0, 2\pi)\}.$$



Hence, $v = -1/2\pi\ \ln\varepsilon$ and $\mathrm{d}v/\mathrm{d}\varepsilon = -1/2\pi\varepsilon$ and applying Green's theorem to u and v in this new region $V^\star$ (with boundaries S and Σ), we get

$$\int_S \left(f\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S + \int_\Sigma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S = 0. \tag{4.6}$$

since $\nabla^2 u = \nabla^2 v = 0$ for all point in $V^\star$. By transforming to polar coordinates, $\mathrm{d}S = \varepsilon\mathrm{d}\theta$ and $\partial u/\partial n = -\partial u/\partial\varepsilon$ (unit normal is in the direction ε) onto Σ; then

$$\int_\Sigma v\frac{\partial u}{\partial n}\,\mathrm{d}S = \frac{\varepsilon\ln\varepsilon}{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial\varepsilon}\,\mathrm{d}\theta \to 0 \text{ as } \varepsilon \to 0,$$

and also

$$\int_\Sigma u\frac{\partial v}{\partial n}\mathrm{d}S = -\int_0^{2\pi} u\,\frac{\partial v}{\partial\varepsilon}\,\varepsilon\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} u\,\varepsilon\frac{1}{\varepsilon}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} u\,\mathrm{d}\theta \to u(\xi,\eta) \text{ as } \varepsilon \to 0,$$

and so, in the limit $\varepsilon \to 0$, equation (4.6) gives

$$u(\xi,\eta) = \int_S \left(v\frac{\partial u}{\partial n} - f\frac{\partial v}{\partial n}\right)\mathrm{d}S, \quad \text{where} \quad v = -\frac{1}{4\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right).$$

Now, consider w, such that $\nabla^2 w = 0$ in V but with w regular at $(x=\xi, y=\eta)$, and with $w = -v$ on S. Then Green's theorem gives

$$\int_V \left(u\nabla^2 w - w\nabla^2 u\right)\mathrm{d}V = \int_S \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right)\mathrm{d}S \Leftrightarrow \int_S \left(f\frac{\partial w}{\partial n} + v\frac{\partial u}{\partial n}\right)\mathrm{d}S = 0$$

since $\nabla^2 u = \nabla^2 w = 0$ in V and $w = -v$ on S. Then, subtract this equation from equation above to get

$$u(\xi,\eta) = \int_S \left(v\frac{\partial u}{\partial n} - f\frac{\partial v}{\partial n}\right)\mathrm{d}S - \int_S \left(f\frac{\partial w}{\partial n} + v\frac{\partial u}{\partial n}\right)\mathrm{d}S = -\int_S f\,\frac{\partial}{\partial n}(v+w)\,\mathrm{d}S.$$

Setting $G(x,y;\xi,\eta) = v + w$, then

$$u(\xi,\eta) = -\int_S f\,\frac{\partial G}{\partial n}\,\mathrm{d}S.$$

Such a function G then has the properties,

$$\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi}) \text{ in } V, \quad \text{with} \quad G = 0 \text{ on } S.$$

4.3.3 Free Space Green's Function

We seek a Green's function G such that,

$$G(\mathbf{x}, \boldsymbol{\xi}) = v(\mathbf{x}, \boldsymbol{\xi}) + w(\mathbf{x}, \boldsymbol{\xi}) \quad \text{where} \quad \nabla^2 v = -\delta(\mathbf{x} - \boldsymbol{\xi}) \text{ in } V.$$

How do we find the free space Green's function v defined such that $\nabla^2 v = -\delta(\mathbf{x} - \boldsymbol{\xi})$ in V? Note that it does not depend on the form of the boundary. (The function v is a "source term" and for Laplace's equation is the potential due to a point source at the point $\mathbf{x} = \boldsymbol{\xi}$.)

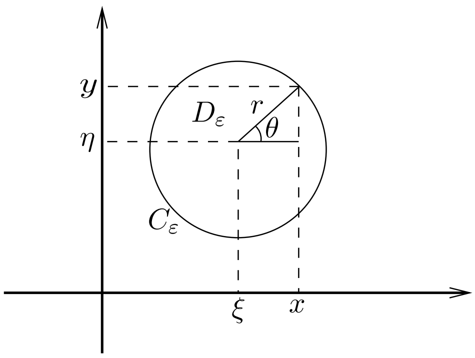
As an illustration of the method, we can derive that, in two dimensions,

$$v = -\frac{1}{4\pi} \ln\left((x-\xi)^2 + (y-\eta)^2\right),$$

as we have already seen. We move to polar coordinate around (ξ, η),

$$x - \xi = r\cos\theta \quad \& \quad y - \eta = r\sin\theta,$$

and look for a solution of Laplace's equation which is independent of θ and which is singular as $r \to 0$.



Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = 0$$

which has solution $v = B\ln r + A$ with A and B constant. Put $A = 0$ and, to determine the constant B, apply Green's theorem to v and 1 in a small disc D_r (with boundary C_r), of radius r around the origin (η, ξ),

$$\int_{C_r} \frac{\partial v}{\partial n}\,\mathrm{d}S = \int_{D_r} \nabla^2 v\,\mathrm{d}V = -\int_{D_r} \delta(\mathbf{x} - \boldsymbol{\xi})\,\mathrm{d}V = -1,$$

so we choose B to make

$$\int_{C_r} \frac{\partial v}{\partial n}\,\mathrm{d}S = -1.$$

Now, in polar coordinates, $\partial v/\partial n = \partial v/\partial r = B/r$ and $\mathrm{d}S = r\mathrm{d}\theta$ (going around circle C_r). So,

$$\int_0^{2\pi} \frac{B}{r}\,r\mathrm{d}\theta = B\int_0^{2\pi} \mathrm{d}\theta = -1 \Rightarrow B = -\frac{1}{2\pi}.$$

Hence,

$$v = -\frac{1}{2\pi}\ln r = -\frac{1}{4\pi}\ln r^2 = -\frac{1}{4\pi}\ln\left((x-\xi)^2 + (y-\eta)^2\right).$$

(We do not use the boundary condition to determined v.)

Similar (but more complicated) methods lead to the free-space Green's function v for the Laplace equation in n dimensions. In particular,

$$v(\mathbf{x},\boldsymbol{\xi}) = \begin{cases} -\dfrac{1}{2}|x-\xi|, & n=1, \\ -\dfrac{1}{4\pi}\ln\left(|\mathbf{x}-\boldsymbol{\xi}|^2\right), & n=2, \\ -\dfrac{1}{(2-n)A_n(1)}\,|\mathbf{x}-\boldsymbol{\xi}|^{2-n}, & n\geq 3, \end{cases}$$

where $\mathbf{x}$ and $\boldsymbol{\xi}$ are distinct points and $A_n(1)$ denotes the area of the unit n-sphere. We shall restrict ourselves to two dimensions for this course.

Note that Poisson's equation, $\nabla^2 u = F$, is solved in unbounded $\mathbb{R}^n$ by

$$u(\mathbf{x}) = -\int_{\mathbb{R}^n} v(\mathbf{x},\boldsymbol{\xi})\,F(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

where from equation (4.2) the free space Green's function v, defined above, serves as Green's function for the differential operator ∇^2 when no boundaries are present.

4.3.4 Method of Images

In order to solve BVPs for Poisson's equation, such as $\nabla^2 u = F$ in an open region V with some conditions on the boundary S, we seek a Green's function G such that, in V

$$G(\mathbf{x},\boldsymbol{\xi}) = v(\mathbf{x},\boldsymbol{\xi}) + w(\mathbf{x},\boldsymbol{\xi}) \quad \text{where} \quad \nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi}) \quad \text{and} \quad \nabla^2 w = 0 \text{ or } 1/\mathcal{V}(V).$$

Having found the free space Green's function v — which does not depend on the boundary conditions, and so is the same for all problems — we still need to find the function w, solution of Laplace's equation and regular in $\mathbf{x}=\boldsymbol{\xi}$, which fixes the boundary conditions (v does not satisfies the boundary conditions required for G by itself). So, we look for the function which satisfies

$$\begin{aligned} &\nabla^2 w = 0 \text{ or } 1/\mathcal{V}(V) \text{ in } V, \quad \text{(ensuring } w \text{ is regular at } (\xi,\eta)), \\ \text{with} \quad & w = -v \quad \text{(i.e., } G=0) \text{ on } S \text{ for Dirichlet boundary conditions,} \\ \text{or} \quad & \frac{\partial w}{\partial n} = -\frac{\partial v}{\partial n} \text{ (i.e., } \frac{\partial G}{\partial n} = 0) \text{ on } S \text{ for Neumann boundary conditions.} \end{aligned}$$

To obtain such a function we superpose functions with singularities at the image points of (ξ,η). (This may be regarded as adding appropriate point sources and sinks to satisfy the boundary conditions.) Note also that, since G and v are symmetric then w must be symmetric too (i.e., $w(\mathbf{x},\boldsymbol{\xi}) = w(\boldsymbol{\xi},\mathbf{x})$).

Example 1

Suppose we wish to solve the Dirichlet BVP for Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } y>0 \quad \text{with} \quad u = f(x) \text{ on } y=0.$$

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Example 2

Find the Green's function for the Dirichlet BVP

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F \text{ in the quadrant } x > 0,\ y > 0.$$

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Example 3

Consider the Neumann BVP for Laplace's equation in the upper half-plane:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } y > 0 \quad \text{with} \quad \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y} = f(x) \text{ on } y = 0.$$

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Example 4

Solve the Dirichlet problem for Laplace's equation in a disc of radius a:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \text{ in } r < a \quad \text{with } u = f(\theta) \text{ on } r = a.$$

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Example 5

Interior Neumann problem for Laplace's equation in a disc:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \text{ in } r < a,$$
$$\frac{\partial u}{\partial n} = f(\theta) \text{ on } r = a.$$

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Exercise: Exterior Neumann problem for Laplace's equation in a disc

$$u(r,\theta) = \frac{a}{2\pi} \int_0^{2\pi} \ln\left(a^2 + r^2 - 2ar\,\cos(\theta - \phi)\right) f(\phi)\,\mathrm{d}\phi, \forall r \geq a.$$

4.4 Extensions of Theory

1. Alternative to the method of images to determine the Green's function G: (a) eigenfunction method when G is expanded in terms of the basis of the eigenfunctions of the Laplacian operator; conformal mapping of the complex plane for solving 2-D problems.
2. Green's functions for more general operators.

Chapter 5

Parabolic Equations

Contents

5.1 Definitions and Properties

Unlike elliptic equations, which describe a steady state, parabolic (and hyperbolic) evolution equations describe processes that are evolving in time. For such an equation the initial state of the system is part of the auxiliary data for a well-posed problem.

The archetypal parabolic evolution equation is the "heat conduction" or "diffusion" equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{(1-dimensional)},$$

or more generally, for $\kappa > 0$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot (\kappa \, \nabla u) \\ &= \kappa \, \nabla^2 u \quad (\kappa \text{ constant}), \\ &= \kappa \, \frac{\partial^2 u}{\partial x^2} \quad \text{(1-D)}. \end{aligned}$$

Problems which are well-posed for the heat equation will be well-posed for more general parabolic equation.

5.1.1 Well-Posed Cauchy Problem (Initial Value Problem)

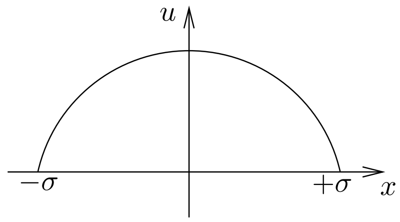
Consider $\kappa > 0$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \nabla^2 u \quad \text{in } \mathbb{R}^n, \ t > 0, \\ \text{with} \quad u &= f(\mathbf{x}) \quad \text{in } \mathbb{R}^n \text{ at } t = 0, \\ \text{and} \quad |u| &< \infty \quad \text{in } \mathbb{R}^n, \ t > 0. \end{aligned}$$

Note that we require the solution $u(\mathbf{x}, t)$ bounded in $\mathbb{R}^n$ for all t. In particular we assume that the boundedness of the smooth function u at infinity gives $\nabla u|_\infty = 0$. We also impose conditions on f:

$$\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} < \infty \Rightarrow f(\mathbf{x}) \to 0 \text{ as } |\mathbf{x}| \to \infty.$$

Sometimes $f(\mathbf{x})$ has compact support, i.e., $f(\mathbf{x}) = 0$ outside some finite region. (E.g., in 1-D, see graph hereafter.)



5.1.2 Well-Posed Initial-Boundary Value Problem

Consider an open bounded region Ω of $\mathbb{R}^n$ and $\kappa > 0$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \, \nabla^2 u \quad \text{in } \Omega, \ t > 0, \\ \text{with} \quad u &= f(\mathbf{x}) \quad \text{at } t = 0 \text{ in } \Omega, \\ \text{and} \quad \alpha \, u(\mathbf{x}, t) + \beta \, \frac{\partial u}{\partial n}(\mathbf{x}, t) &= g(\mathbf{x}, t) \quad \text{on the boundary } \partial\Omega. \end{aligned}$$

Then, $\beta = 0$ gives the Dirichlet problem, $\alpha = 0$ gives the Neumann problem ($\partial u/\partial n = 0$ on the boundary is the zero-flux condition) and $\alpha \neq 0$, $\beta \neq 0$ gives the Robin or radiation problem. (The problem can also have mixed boundary conditions.)
If Ω is not bounded (e.g., half-plane), then an additional behaviour-at-infinity condition may be needed.

5.1.3 Time Irreversibility of the Heat Equation

If the initial conditions in a well-posed initial value or initial-boundary value problem for an evolution equation are replaced by conditions on the solution at other than the initial time, the resulting problem may not be well-posed (even when the total number of auxiliary conditions is unchanged). E.g., the backward heat equation in 1-D is ill-posed; this problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \, \frac{\partial^2 u}{\partial x^2} \quad \text{in } 0 < x < l, \ 0 < t < T, \\ \text{with} \quad u &= f(x) \quad \text{at } t = T, \ x \in (0, l), \\ \text{and} \quad u(0, t) &= u(l, t) = 0 \quad \text{for } t \in (0, T), \end{aligned}$$

to find previous states $u(x, t)$, $(t < T)$ which will have evolved into the state $f(x)$, has no solution for arbitrary $f(x)$. Even when a solution exists, it does not depend continuously on the data.

The heat equation is irreversible in the mathematical sense that forward time is distinguishable from backward time (i.e., it models physical processes irreversible in the sense of the Second Law of Thermodynamics).

5.1.4 Uniqueness of Solution for the Cauchy Problem

The 1-D initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x \in \mathbb{R},\ t > 0,$$
$$\text{with} \quad u = f(x) \text{ at } t = 0\ (x \in \mathbb{R}), \nabla u|_\infty = 0,\ \text{such that} \int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x < \infty,$$

has a unique solution.

Proof:

We can prove the uniqueness of the solution of the Cauchy problem using the energy method. Suppose that u_1 and u_2 are two bounded solutions. Consider $w = u_1 - u_2$. Then w satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \quad (-\infty < x < \infty,\ t > 0),$$
$$\text{with} \quad w = 0 \quad \text{at } t = 0 \quad (-\infty < x < \infty) \quad \text{and} \quad \left.\frac{\partial w}{\partial x}\right|_\infty = 0,\ \forall t.$$

Consider the function of time

$$I(t) = \frac{1}{2}\int_{-\infty}^{\infty} w^2(x,t)\,\mathrm{d}x, \quad \text{such that} \quad I(0) = 0 \quad \text{and} \quad I(t) \geq 0\ \forall t \quad (\text{as } w^2 \geq 0),$$

which represents the energy of the function w. Then,

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial w^2}{\partial t}\,\mathrm{d}x = \int_{-\infty}^{\infty} w\,\frac{\partial w}{\partial t}\,\mathrm{d}x = \int_{-\infty}^{\infty} w\,\frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x \quad (\text{from the heat equation}),\\
&= \left[w\,\frac{\partial w}{\partial x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x \quad (\text{integration by parts}),\\
&= -\int_{-\infty}^{\infty}\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x \leq 0 \quad \text{since} \quad \left.\frac{\partial w}{\partial x}\right|_\infty = 0.
\end{aligned}$$

Then,

$$0 \leq I(t) \leq I(0) = 0,\ \forall t > 0,$$

since $\mathrm{d}I/\mathrm{d}t \leq 0$. So, $I(t) = 0$ and $w \equiv 0$ i.e., $u_1 = u_2,\ \forall t > 0$.

5.1.5 Uniqueness of Solution for Initial-Boundary Value Problem

Similarly we can make use of the energy method to prove the uniqueness of the solution of the 1-D Dirichlet or Neumann problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in } 0 < x < l,\ t > 0,\\
\text{with} \quad &u = f(x) \quad \text{at } t = 0,\ x \in (0,l), \quad (\text{initial condition}),\\
&u(0,t) = g_0(t) \quad \text{and} \quad u(l,t) = g_l(t),\ \forall t > 0 \quad (\text{Dirichlet}),\\
\text{or} \quad &\frac{\partial u}{\partial x}(0,t) = g_0(t) \quad \text{and} \quad \frac{\partial u}{\partial x}(l,t) = g_l(t),\ \forall t > 0 \quad (\text{Neumann}).
\end{aligned}$$

Suppose that u_1 and u_2 are two solutions and consider $w = u_1 - u_2$. Then w satisfies

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \quad (0 < x < l,\ t > 0),\\
\text{with} \quad &w = 0 \quad \text{at } t = 0 \quad (0 < x < l),\\
\text{and} \quad &w(0,t) = w(l,t) = 0,\ \forall t > 0 \quad (\text{Dirichlet}),\\
\text{or} \quad &\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(l,t) = 0,\ \forall t > 0 \quad (\text{Neumann}).
\end{aligned}$$

Consider the function of time

$$I(t) = \frac{1}{2}\int_0^l w^2(x,t)\,\mathrm{d}x, \quad \text{such that} \quad I(0)=0 \quad \text{and} \quad I(t)\geq 0\;\forall t \quad (\text{as } w^2 \geq 0),$$

which represents the energy of the function w. Then,

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{1}{2}\int_0^l \frac{\partial w^2}{\partial t}\,\mathrm{d}x = \int_0^l w\,\frac{\partial^2 w}{\partial x^2}\mathrm{d}x,\\ &= \left[w\,\frac{\partial w}{\partial x}\right]_0^l - \int_0^l \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x = -\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \leq 0.\end{aligned}$$

Then,

$$0 \leq I(t) \leq I(0) = 0,\ \forall t>0,$$

since $\mathrm{d}I/\mathrm{d}t \leq 0$. So $I(t)=0\;\forall t>0$ and $w\equiv 0$ and $u_1=u_2$.

5.2 Fundamental Solution of the Heat Equation

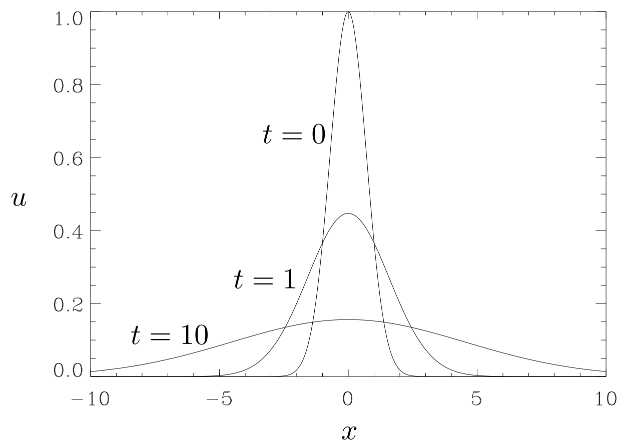
Consider the 1-D Cauchy problem:

$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{on } -\infty < x < \infty,\ t>0,\\ \text{with} \quad u=f(x) \quad \text{at } t=0 \quad (-\infty<x<\infty),\\ \text{such that} \quad \nabla u|_{\pm\infty}=0, \quad \text{and} \int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x < \infty.\end{gathered}$$

Example: To illustrate the typical behaviour of the solution of this Cauchy problem, consider the specific case where $u(x,0)=f(x)=\exp(-x^2)$. The solution is

$$u(x,t) = \frac{1}{(1+4t)^{1/2}}\exp\left(-\frac{x^2}{1+4t}\right) \quad (\text{exercise: check this}).$$

Starting with $u(x,0)=\exp(-x^2)$ at $t=0$, the solution becomes $u(x,t)\sim 1/2\sqrt{t}\,\exp(-x^2/4t)$, for t large, i.e., the amplitude of the solution scales as $1/\sqrt{t}$ and its width scales as $\sqrt{t}$.



Spreading of the Solution: The solution of the Cauchy problem for the heat equation spreads such that its integral remains constant:

$$Q(t) = \int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = \text{constant}.$$

Proof: Consider

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x \quad \text{(from equation)},$$
$$= \left[\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} = 0 \quad \text{(from conditions on } u\text{)}.$$
$$\text{So, } Q = \text{ constant.}$$

5.2.1 Integral Form of the General Solution

To find the general solution of the Cauchy problem we define the Fourier transform of $u(x,t)$ and its inverse by

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,t)\,\mathrm{e}^{-ikx}\,\mathrm{d}x,$$
$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U(k,t)\,\mathrm{e}^{ikx}\,\mathrm{d}k.$$

So, the heat equation gives,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{\partial U(k,t)}{\partial t} + k^2 U(k,t)\right] \mathrm{e}^{ikx}\,\mathrm{d}k = 0 \quad \forall x,$$

which implies that the Fourier transform $U(k,t)$ satisfies the equation

$$\frac{\partial U(k,t)}{\partial t} + k^2 U(k,t) = 0.$$

The solution of this linear equation is

$$U(k,t) = F(k)\,\mathrm{e}^{-k^2 t},$$

where $F(k)$ is the Fourier transform of the initial data, $u(x, t = 0)$,

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\,\mathrm{e}^{-ikx}\,\mathrm{d}x.$$

(This requires $\int_{-\infty}^{+\infty} |f(x)|^2\,\mathrm{d}x < \infty$.) Then, we back substitute $U(k,t)$ in the integral form of $u(x,t)$ to find:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k)\,\mathrm{e}^{-k^2 t}\,\mathrm{e}^{ikx}\,\mathrm{d}k = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(\xi)\,\mathrm{e}^{-ik\xi}\,\mathrm{d}\xi\right) \mathrm{e}^{-k^2 t}\,\mathrm{e}^{ikx}\,\mathrm{d}k,$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) \int_{-\infty}^{+\infty} \mathrm{e}^{-k^2 t}\,\mathrm{e}^{ik(x-\xi)}\,\mathrm{d}k\,\mathrm{d}\xi.$$

Now consider

$$H(x,t,\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{-k^2 t}\,\mathrm{e}^{ik(x-\xi)}\,\mathrm{d}k = \int_{-\infty}^{+\infty} \exp\left[-t\left(k - i\frac{x-\xi}{2t}\right)^2 - \frac{(x-\xi)^2}{4t}\right] \mathrm{d}k,$$

since the exponent satisfies

$$-k^2 t + ik(x-\xi) = -t\left(k^2 - ik\frac{x-\xi}{t}\right) = -t\left[\left(k - i\frac{x-\xi}{2t}\right)^2 + \frac{(x-\xi)^2}{4t^2}\right],$$

and set $k - i(x-\xi)/2t = s/\sqrt{t}$, with $\mathrm{d}k = \mathrm{d}s/\sqrt{t}$, such that

$$H(x,t,\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{-s^2} \exp\left[-\frac{(x-\xi)^2}{4t}\right] \frac{\mathrm{d}s}{\sqrt{t}} = \sqrt{\frac{\pi}{t}}\, \mathrm{e}^{-(x-\xi)^2/4t},$$

since $\displaystyle\int_{-\infty}^{+\infty} \mathrm{e}^{-s^2}\, \mathrm{d}s = \sqrt{\pi}$ (see appendix A).

So,

$$u(x,t) = \frac{1}{\sqrt{4\pi\, t}} \int_{-\infty}^{+\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] \mathrm{d}\xi = \int_{-\infty}^{+\infty} K(x-\xi,t)\, f(\xi)\, \mathrm{d}\xi,$$

where the function

$$K(x,t) = \frac{1}{\sqrt{4\pi\, t}} \exp\left(-\frac{x^2}{4t}\right)$$

is called the fundamental solution — or source function, Green's function, propagator, diffusion kernel — of the heat equation.

5.2.2 Properties of the Fundamental Solution

The function $K(x,t)$ is solution (positive) of the heat equation for $t > 0$ (check this) and has a singularity only at $x = 0,\ t = 0$:

1. $K(x,t) \to 0$ as $t \to 0^+$ with $x \neq 0$ ($K \sim O(1/\sqrt{t}\ \exp[-1/t])$),
2. $K(x,t) \to +\infty$ as $t \to 0^+$ with $x = 0$ ($K \sim O(1/\sqrt{t})$),
3. $K(x,t) \to 0$ as $t \to +\infty$ ($K \sim O(1/\sqrt{t})$),
4. $\displaystyle\int_{-\infty}^{\infty} K(x-\xi,t)\, \mathrm{d}\xi = 1$

At any time $t > 0$ (no matter how small), the solution to the initial value problem for the heat equation at an arbitrary point x depends on all of the initial data, i.e., the data propagate with an infinite speed. (As a consequence, the problem is well posed only if behaviour-at-infinity conditions are imposed.) However, the influence of the initial condition dies out very rapidly with the distance (as $\exp(-r^2)$).

5.2.3 Behaviour at large t

Suppose that the initial data have compact support — or decays to zero sufficiently quickly as $|x| \to \infty$ and that we look at the solution of the heat equation on spatial scales, x, large compared to the spatial scale of the data ξ and at t large. Thus, we assume the ordering $x^2/t \sim O(1)$ and $\xi^2/t \sim O(\varepsilon)$ where $\varepsilon \ll 1$ (so that, $x\xi/t \sim O(\varepsilon^{1/2})$). Then, the solution

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi\, t}} \int_{-\infty}^{+\infty} f(\xi)\, \mathrm{e}^{-(x-\xi)^2/4t}\, \mathrm{d}\xi = \frac{\mathrm{e}^{-x^2/4t}}{\sqrt{4\pi\, t}} \int_{-\infty}^{+\infty} f(\xi)\, \mathrm{e}^{-\xi^2/4t} \mathrm{e}^{-x\xi/2t}\, \mathrm{d}\xi, \\ &\simeq \frac{\mathrm{e}^{-x^2/4t}}{\sqrt{4\pi\, t}} \int_{-\infty}^{+\infty} f(\xi)\, \mathrm{d}\xi \simeq \frac{F(0)}{\sqrt{2\, t}} \exp\left(-\frac{x^2}{4t}\right), \end{aligned}$$

where $F(0)$ is the Fourier transform of f at $k = 0$, i.e.,

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x \Rightarrow \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = \sqrt{2\pi}\, F(0).$$

So, at large t, on large spatial scales x the solution evolves as $u \simeq u_0/\sqrt{t}\ \exp(-\eta^2)$ where u_0 is a constant and $\eta = x/\sqrt{2t}$ is the <u>diffusion variable</u>. This solution spreads and decreases as t increases.

5.3 Similarity Solution

For some equations, like the heat equation, the solution depends on a certain grouping of the independent variables rather than depending on each of the independent variables independently. Consider the heat equation in 1-D

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0,$$

and introduce the dilatation transformation

$$\xi = \varepsilon^a\, x, \quad \tau = \varepsilon^b\, t \quad \text{and} \quad w(\xi,\tau) = \varepsilon^c\, u(\varepsilon^{-a}\xi, \varepsilon^{-b}\tau),\ \varepsilon \in \mathbb{R}.$$

This change of variables gives

$$\frac{\partial u}{\partial t} = \varepsilon^{-c}\,\frac{\partial w}{\partial \tau}\,\frac{\partial \tau}{\partial t} = \varepsilon^{b-c}\,\frac{\partial w}{\partial \tau}, \quad \frac{\partial u}{\partial x} = \varepsilon^{-c}\,\frac{\partial w}{\partial \xi}\,\frac{\partial \xi}{\partial x} = \varepsilon^{a-c}\,\frac{\partial w}{\partial \xi}$$

$$\text{and} \quad \frac{\partial^2 u}{\partial x^2} = \varepsilon^{a-c}\,\frac{\partial^2 w}{\partial \xi^2}\,\frac{\partial \xi}{\partial x} = \varepsilon^{2a-c}\,\frac{\partial^2 w}{\partial \xi^2}.$$

So, the heat equation transforms into

$$\varepsilon^{b-c}\,\frac{\partial w}{\partial \tau} - \varepsilon^{2a-c} D\,\frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{i.e.,} \quad \varepsilon^{b-c}\left(\frac{\partial w}{\partial \tau} - \varepsilon^{2a-b} D\,\frac{\partial^2 w}{\partial \xi^2}\right) = 0,$$

and is invariant under the dilatation transformation (i.e., $\forall \varepsilon$) if $b = 2a$. Thus, if u solves the equation at (x,t) then $w = \varepsilon^{-c} u$ solve the equation at $(x = \varepsilon^{-a}\,\xi, t = \varepsilon^{-b}\,\tau)$.
Note also that we can build some groupings of independent variables which are invariant under this transformation, such as

$$\frac{\xi}{\tau^{a/b}} = \frac{\varepsilon^a\, x}{\left(\varepsilon^b\, t\right)^{a/b}} = \frac{x}{t^{a/b}}$$

which defines the dimensionless similarity variable $\eta(x,t) = x/\sqrt{2Dt}$, since $b = 2a$. ($\eta \to \infty$ if $x \to \infty$ or $t \to 0$ and $\eta = 0$ if $x = 0$.) Also,

$$\frac{w}{\tau^{c/b}} = \frac{\varepsilon^c\, u}{\left(\varepsilon^b\, t\right)^{c/b}} = \frac{u}{t^{c/b}} = v(\eta)$$

suggests that we look for a solution of the heat equation of the form $u = t^{c/2a}\, v(\eta)$. Indeed, since the heat equation is invariant under the dilatation transformation, then we also expect the solution to be invariant under that transformation. Hence, the partial derivatives become,

$$\frac{\partial u}{\partial t} = \frac{c}{2a}\, t^{c/2a-1}\, v(\eta) + t^{c/2a}\, v'(\eta)\,\frac{\partial \eta}{\partial t} = \frac{1}{2}\, t^{c/2a-1}\,\left(\frac{c}{a}\, v(\eta) - \eta\, v'(\eta)\right),$$

since $\partial\eta/\partial t = -x/(2t\sqrt{2Dt}) = -\eta/2t$, and

$$\frac{\partial u}{\partial x} = t^{c/2a}\, v'(\eta)\,\frac{\partial \eta}{\partial x} = \frac{t^{c/2a-1/2}}{\sqrt{2D}}\, v'(\eta), \quad \frac{\partial^2 u}{\partial x^2} = \frac{t^{c/2a-1}}{2D}\, v''(\eta).$$

Then, the heat equation reduces to an ODE

$$t^{\gamma/2-1}\,(v''(\eta) + \eta\, v'(\eta) - \gamma\, v(\eta)) = 0. \tag{5.1}$$

with $\gamma = c/a$, such that $u = t^{\gamma/2} v$ and $\eta = x/\sqrt{2Dt}$. So, we may be able to solve the heat equation through (5.1) if we can write the auxiliary conditions on u, x and t as conditions on v and η. Note that, in general, the integral transform method is able to deal with more general boundary conditions; on the other hand, looking for similarity solutions allows to solve other types of problems (e.g., weak solutions).

5.3.1 Infinite Region

Consider the problem

$$\begin{aligned}
\frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} \quad \text{on } -\infty < x < \infty,\ t > 0,\\
\text{with} \quad u &= u_0 \quad \text{at } t = 0,\ x \in \mathbb{R}^*_-, \quad u = 0 \quad \text{at } t = 0,\ x \in \mathbb{R}^*_+,\\
\text{and} \quad u &\to u_0 \text{ as } x \to -\infty, \quad u \to 0 \text{ as } x \to \infty,\ \forall t > 0.
\end{aligned}$$

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5.3.2 Semi-Infinite Region

Consider the problem

$$\begin{aligned}
\frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} \quad \text{on } 0 < x < \infty,\ t > 0,\\
\text{with} \quad u &= 0 \quad \text{at } t = 0,\ x \in \mathbb{R}^*_+,\\
\text{and} \quad \frac{\partial u}{\partial x} &= -q \quad \text{at } x = 0,\ t > 0, \quad u \to 0 \text{ as } x \to \infty,\ \forall t > 0.
\end{aligned}$$

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5.4 Maximum Principles and Comparison Theorems

Like the elliptic PDEs, the heat equation or parabolic equations of most general form satisfy a maximum-minimum principle.
Consider the Cauchy problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad -\infty < x < \infty,\ 0 \le t \le T.$$

and define the two sets V and V_T as

$$\begin{aligned}
V &= \{(x,t) \in (-\infty,+\infty) \times (0,T)\},\\
\text{and} \quad V_T &= \{(x,t) \in (-\infty,+\infty) \times (0,T]\}.
\end{aligned}$$

Lemma: Suppose

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} < 0 \quad \text{in } V \quad \text{and} \quad u(x,0) \le M,$$

then $u(x,t) < M$ in V_T.

Proof: Suppose $u(x,t)$ achieves a maximum in V, at the point (x_0,t_0). Then, at this point,

$$\frac{\partial u}{\partial t}=0,\quad \frac{\partial u}{\partial x}=0 \quad\text{and}\quad \frac{\partial^2 u}{\partial x^2}\leq 0.$$

But, $\partial^2 u/\partial x^2 \leq 0$ is contradictory with the hypothesis $\partial^2 u/\partial x^2 > \partial u/\partial t = 0$ at (x_0,t_0). Moreover, if we now suppose that the maximum occurs at $t=T$ then, at this point

$$\frac{\partial u}{\partial t}\geq 0,\quad \frac{\partial u}{\partial x}=0 \quad\text{and}\quad \frac{\partial^2 u}{\partial x^2}\leq 0,$$

which again leads to a contradiction.

5.4.1 First Maximum Principle

Suppose

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\leq 0 \quad\text{in } V \quad\text{and}\quad u(x,0)\leq M,$$

then $u(x,t)\leq M$ in V_T.

Proof: Suppose there is some point (x_0,t_0) in V_T $(0<t\leq T)$ at which $u(x_0,t_0)=M_1>M$. Put $w(x,t)=u(x,t)-(t-t_0)\,\varepsilon$ where $\varepsilon=(M_1-M)/t_0>0$. Then,

$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}=\underbrace{\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}}_{\leq 0}-\underbrace{\varepsilon}_{>0}<0 \quad\text{(in form of lemma)},$$

and by lemma,

$$\begin{aligned} w(x,t) &< \max\{w(x,0)\} \quad \text{in } V_T,\\ &< M+\varepsilon\, t_0,\\ &< M+\frac{M_1-M}{t_0}\,t_0,\end{aligned}$$

$$\Rightarrow w(x,t)<M_1 \quad \text{in } V_T.$$

But, $w(x_0,t_0)=u(x_0,t_0)-(t_0-t_0)\,\varepsilon=u(x_0,t_0)=M_1$; since $(x_0,t_0)\in V_T$ we have a contradiction.