

INTRODUCTION TO THE VARIATIONAL BICOMPLEX

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§1. Introduction.

The variational bicomplex was first introduced in the mid 1970's as a means of studying the inverse problem of the calculus of variations. This is the problem of characterizing those differential equations which are the Euler-Lagrange equations for a classical, unconstrained variational problem. Since then, the variational bicomplex has emerged as an effective means for studying other formal, differential-geometric aspects of the calculus of variations. Moreover, it has been shown that the basic variational bicomplex constructed to solve the inverse problem can be modified in various ways and that the cohomology groups associated with these modified bicomplexes are relevant to many topics in geometry, mathematical physics and differential equations. The purpose of this paper is to review the general construction of the variational bicomplex, to describe some of its the basic properties, and to survey some recent results.

We begin by returning to the genesis of our subject — the inverse problem of the calculus of variations. For the purposes of this introduction, we need not formulate this problem in its full generality. Accordingly, let us consider variational problems for a single function u of three independent variables x, y, z . Given a compact region \mathcal{W} in \mathbf{R}^3 with smooth boundary $\partial\mathcal{W}$ and a first order Lagrangian

$$L = L(x, y, z, u, u_x, u_y, u_z),$$

we seek those smooth real-valued functions

$$u: \mathcal{W} \rightarrow \mathbf{R}$$

with prescribed values on $\partial\mathcal{W}$ which minimize the functional

$$F[u] = \int_{\mathcal{W}} L(x, y, z, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) dx dy dz. \quad (1.1)$$

The derivation of necessary conditions for a minimum u are based upon the *first variational formula*: if u_ϵ is a smooth one parameter family of smooth functions on \mathcal{W} , then

$$\left. \frac{d}{d\epsilon} L[u_\epsilon] \right|_{\epsilon=0} = E(L)[u] v + \text{Div } V[u]. \quad (1.2)$$

In this equation, $v = \left. \frac{du_\epsilon}{d\epsilon} \right|_{\epsilon=0}$, the Euler-Lagrange operator $E(L)$ is

$$E(L) = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} - D_y \frac{\partial L}{\partial u_y} - D_z \frac{\partial L}{\partial u_z},$$

and V is the vector field

$$V = \left(\frac{\partial L}{\partial u_x}v\right)\hat{\mathbf{i}} + \left(\frac{\partial L}{\partial u_y}v\right)\hat{\mathbf{j}} + \left(\frac{\partial L}{\partial u_z}v\right)\hat{\mathbf{k}}.$$

The total derivative operators D_x , D_y and D_z are defined, for example, by

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \cdots .$$

The total divergence operator Div in (1.2) is defined in terms of the total derivatives D_x , D_y and D_z . In equation (1.2) and in what follows, we shall write $L[u]$, $V[u]$ and so on to indicate that L is a function of the independent variables x , y , z , the dependent variable u and its derivatives to some finite (in this case first) order.

With the first variational formula in hand it is not difficult to establish that if u is a local minimum for (1.1), then u is a solution to the Euler-Lagrange equations

$$E(L)[u] = 0.$$

The first variational formula plays a central role in our subject and its importance should be emphasized. Here we note that:

- (i) the first variational formula (1.2) holds for Lagrangians of any order although the local expressions for the vector field V become rather complicated.
- (ii) equation (1.2) uniquely characterizes the Euler-Lagrange operator; that is, if $T[u]$ is any differential operator and

$$\left.\frac{d}{d\epsilon}L[u_\epsilon]\right|_{\epsilon=0} = T[u]v + \text{Div} W[u] \tag{1.3}$$

then for some vector field $W[u]$, $T[u] = E(L)[u]$.

- (iii) As an immediate consequence of (ii), it follows that the Euler-Lagrange operator annihilates divergences. Indeed, if $L[u] = \text{Div} W[u]$, then

$$\left.\frac{d}{d\epsilon}L[u_\epsilon]\right|_{\epsilon=0} = \text{Div}\left(\left.\frac{d}{d\epsilon}W[u_\epsilon]\right|_{\epsilon=0}\right)$$

and hence, by uniqueness, $E(L) = 0$. This proves that

$$E(\text{Div}W)[u] = 0. \tag{1.4}$$

We can also use the first variational formula to determine when a given differential operator $T[u]$ is the Euler-Lagrange operator for some Lagrangian $L[u]$. This is the simplest version of the inverse problem of the calculus of variations. If $T[u] = E(L)[u]$, then the first variational formula (1.2) becomes (1.3). To this equation we apply the Euler-Lagrange operator E to deduce, by virtue of (1.4), that

$$E\left(\frac{d}{d\epsilon}L[u_\epsilon]\Big|_{\epsilon=0}\right) = E(T[u]v). \quad (1.5)$$

But it is a straightforward matter to check that

$$E\left(\frac{d}{d\epsilon}L[u_\epsilon]\Big|_{\epsilon=0}\right) = \frac{d}{d\epsilon}E(L)[u_\epsilon]\Big|_{\epsilon=0} = \frac{d}{d\epsilon}T[u_\epsilon]\Big|_{\epsilon=0}$$

and hence we can re-write (1.5) as

$$\left(\frac{d}{d\epsilon}T[u_\epsilon]\Big|_{\epsilon=0}\right) = E(T[u]v)$$

where, once again, $v = v(x, y, z) = \frac{du_\epsilon}{d\epsilon}\Big|_{\epsilon=0}$. Let us define a linear operator $H(T)[v]$ acting on functions $v: \mathcal{W} \rightarrow \mathbf{R}$ by

$$H(T)[v] = \frac{d}{d\epsilon}T[u_\epsilon]\Big|_{\epsilon=0} - E(T[u]v).$$

We conclude that if $T[u]$ is an Euler-Lagrange operator, then

$$H(T) = 0. \quad (1.6)$$

This is called the *Helmholtz equation*. For instance, if T is a second order operator

$$T = T(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, \dots, u_{zz})$$

then the condition (1.6) is equivalent to three differential conditions, the first being

$$\frac{\partial T}{\partial u_x} = D_x\left[\frac{\partial T}{\partial u_{xx}}\right] + \frac{1}{2}D_y\left[\frac{\partial T}{\partial u_{xy}}\right] + \frac{1}{2}D_z\left[\frac{\partial T}{\partial u_{xz}}\right]$$

and the remaining two being similar equations for $\frac{\partial T}{\partial u_y}$ and $\frac{\partial T}{\partial u_z}$.

We can summarize our discussion to this point as follows. Let $\mathcal{F}[u]$ denote the space of smooth functions in the variables $\{x, y, z, u, u_x, u_y, u_z, u_{xx}, \dots\}$, let $\mathcal{V}[u]$ be the space of vector fields on \mathbf{R}^3 with coefficients in $\mathcal{F}[u]$ and let $\mathcal{D}[u]$ be the space of linear differential operators, also with coefficients in $\mathcal{F}[u]$. Let Grad, Curl, and Div be the total gradient, curl and divergence operators on \mathbf{R}^3 defined in terms of the total derivative operators D_x , D_y and D_z . Then, in summary, the sequence of spaces and maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathcal{F}[u] & \xrightarrow{\text{Grad}} & \mathcal{V}[u] & \xrightarrow{\text{Curl}} & \mathcal{V}[u] & \xrightarrow{\text{Div}} & \mathcal{F}[u] \\ & & & & & & \xrightarrow{E} & \mathcal{F}[u] & \xrightarrow{H} & \mathcal{D}[u] & \end{array} \quad (1.7)$$

is a cochain complex — the composition of successive maps is zero. One of the maps in (1.7) is the Euler-Lagrange operator E and for this reason we call this sequence the *Euler-Lagrange complex*. As we shall see, this complex is exact and this fact solves the simplest version of the inverse problem of the calculus of variations. A differential operator $T \in \mathcal{F}[u]$ is an Euler-Lagrange operator if and only if the Helmholtz equation $H(T) = 0$ is satisfied.

Some of the primary purposes of the variational bicomplex can now be described. First, the variational bicomplex provides us with a straightforward, differential geometric generalization of our model complex (1.7). We begin with a fiber bundle $\pi: E \rightarrow M$. For the problem at hand the coordinates on M are the independent variables and the fiber coordinates are the dependent variables. Then the objects of interest — Lagrangians, partial differential equations, currents, *etc.* — are all realized as various types of differential forms on the infinite jet bundle $J^\infty(E)$ of E . The calculus of vector fields and forms on $J^\infty(E)$, along with the fundamental operations of vector field prolongation and “integration by parts” (which we shall make precise in §2) provide a powerful formal variational calculus. This variational calculus also plays an important role in the theory of symmetry group methods for differential equations and in the calculation of conservation laws. We sketch the construction of the variational bicomplex in §2.

It is not readily apparent how to continue the complex (1.7) by defining another operator whose domain is the space of linear operators $\mathcal{D}[u]$ and whose kernel is the image of the Helmholtz operator. The second immediate use of the variational bicomplex is to construct, again in a remarkably straightforward manner, the full continuation of the Euler-Lagrange complex.

Thirdly, in situations where the Euler-Lagrange complex is not exact, the variational bicomplex provides us with the powerful algebraic apparatus of spectral sequences as a general approach to computing its cohomology. Depending upon the

situation, these homological methods are combined with techniques from algebraic topology and global analysis, invariant theory and exterior differential systems. Some of the examples and theorems presented in §3, §4 and §5 illustrate why the cohomology of the Euler-Lagrange complex is of such interest.

The basic references for our subject are Anderson [2], Krasil'schik [15], Olver [20], Saunders [21], Tulczyjew [27], Tsujishita [24], [25] and Vinogradov [29].

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§2. The Variational Bicomplex.

The first step towards a complete definition of the variational bicomplex is a description of the mathematical data from which it is constructed. This data varies with the application at hand; however, for most situations the following is prescribed:

- (i) a fiber bundle $\pi: E \rightarrow M$;
- (ii) a transformation group G on E ; and
- (iii) a set of differential equations or even differential inequalities \mathcal{R} on the local sections of E .

From the fiber bundle $\pi: E \rightarrow M$ we can construct the basic variational bicomplex. The transformation group G and the differential relations \mathcal{R} are used to enhance this basic construction.

Given the fiber bundle $\pi: E \rightarrow M$, we first construct the infinite jet bundle

$$\pi_M^\infty: J^\infty(E) \rightarrow M$$

of jets of local sections of M . If $x \in M$, then the fiber $(\pi_M^\infty)^{-1}(x)$ in $J^\infty(E)$ consists of equivalence classes, denoted by $j^\infty(s)(x)$, of local sections s on E . If V_1 and V_2 are two open neighborhoods of x in M and if

$$s_1: V_1 \rightarrow E \quad \text{and} \quad s_2: V_2 \rightarrow E$$

are local sections, then s_1 and s_2 are equivalent local sections if their partial derivatives to all orders agree at x . If the dimension of M is n and that of E is $m + n$, then on E we can use adapted local coordinates

$$\pi: (x^i, u^\alpha) \rightarrow (x^i),$$

where $i = 1, 2, \dots, n$ and $\alpha = 1, 2, \dots, m$. The induced coordinates on $J^\infty(E)$ are then

$$(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, \dots),$$

where

$$u_i^\alpha(j^\infty(s)(x)) = \frac{\partial s^\alpha}{\partial x^i}(x), \quad u_{ij}^\alpha(j^\infty(s)(x)) = \frac{\partial^2 s^\alpha}{\partial x^i \partial x^j}(x),$$

and so on.

It is important to recognize that the proper setting for our theory is the infinite jet bundle and not some finite jet bundle $J^k(E)$ of fixed order k . There are a number of technical reasons for this which will emerge shortly. There are also important pragmatic considerations. For example, in classifying generalized symmetries, conservation laws or integral invariants, the differential order of these sought-after quantities is not known *a priori* and thus these problems are best formulated in terms of the infinite jet bundle.

Denote by $\Omega^p(J^\infty(E))$ the differential p forms on $J^\infty(E)$.

Definition 2.1. A differential form ω on $J^\infty(E)$ is called a contact form if, for every local section s of E ,

$$[j^\infty(s)]^*(\omega) = 0.$$

The set of all contact forms on $J^\infty(E)$ define a differential ideal $\mathcal{C}(J^\infty(E))$ in the ring $\Omega^*(J^\infty(E))$ of all differential forms on $J^\infty(E)$. This ideal is generated locally by the contact one forms

$$\theta_{i_1 i_2 \dots i_k}^\alpha = du_{i_1 i_2 \dots i_k}^\alpha - u_{i_1 i_2 \dots i_k j}^\alpha dx^j$$

for all $k = 0, 1, 2, \dots$. The exterior derivative of these contact one forms is given by

$$d\theta_{i_1 i_2 \dots i_k}^\alpha = dx^j \wedge \theta_{i_1 i_2 \dots i_k j}^\alpha. \quad (2.1)$$

A local basis for the full exterior algebra $\Omega^*(J^\infty(E))$ is given by the forms

$$dx^i, \theta^\alpha, \theta_i^\alpha, \theta_{ij}^\alpha, \dots$$

It is often very advantageous in specific applications to introduce adapted basis for the ideal of contact forms; this is one way of introducing the method of moving frames into the calculus of variations.

The important concepts of prolonged vector fields and total vector fields may be defined in terms of the contact ideal.

Definition 2.2. Let X be a vector field on E . Then there is a unique vector field on $J^\infty(E)$, called the prolongation of X and denoted by $\text{pr } X$, such that

- (i) X and $\text{pr } X$ agree on functions on E , and
- (ii) $\text{pr } X$ preserves the contact ideal: $\mathcal{L}_{\text{pr } X} \mathcal{C}(J^\infty(E)) \subset \mathcal{C}(J^\infty(E))$.

Definition 2.3. Let X be a vector field on M . Then there is a unique vector field on $J^\infty(E)$, called the total vector field of X and denoted by $\text{tot } X$, such that

- (i) X and $\text{tot } X$ agree on functions on M , and
- (ii) $\text{tot } X$ annihilates all contact one forms, that is, if ω is a contact one form, then $\text{tot } X \lrcorner \omega = 0$.

There are other more geometric and less formal definitions of $\text{pr } X$ and $\text{tot } X$ but the definitions given here have the advantage that they apply equally well to *generalized* vector fields on E and on M . Explicit local formulas for $\text{pr } X$ can be found in any text on symmetry group methods for differential equations. See, for example, Olver [20], p. 113. If a vector field X on M is given locally by

$$X = X^i \frac{\partial}{\partial x^i}, \quad \text{then} \quad \text{tot } X = X^i D_i,$$

where D_i is the total derivative vector field

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots. \quad (2.2)$$

Thus Definition 2.3 provides with an intrinsic definition of the total derivative operators used in the introduction. Note that the vector fields $\text{tot } X$ are not defined on any finite jet bundle.

To define the variational bicomplex, we bi-grade the forms on $J^\infty(E)$ as follows.

Definition 2.4. A p form ω on $J^\infty(E)$ is said to be of type (r, s) , where $r + s = p$, if at each point $\sigma = j^\infty(s)$ of $J^\infty(E)$,

$$\omega(X_1, X_2, \dots, X_p) = 0$$

whenever either

- (i) more than s of the vectors X_1, X_2, \dots, X_p are π_M^∞ vertical, or
- (ii) more than r of the vectors X_1, X_2, \dots, X_p annihilate all contact one forms.

Denote the space of type (r, s) forms on $J^\infty(E)$ by $\Omega^{r,s}(J^\infty(E))$.

In local coordinates, a type (r, s) form is a sum of terms of the form

$$f[u] dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge \theta_{J_1}^{\alpha_1} \wedge \cdots \wedge \theta_{J_s}^{\alpha_s},$$

where the coefficient $f[u]$ is a function of the coordinates x^i , u^α , and finitely many derivatives.

We have the direct sum decomposition

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)).$$

It should be emphasized that this decomposition is not possible on any finite dimensional jet bundle. Let $\pi^{r,s}$ be the projection from $\Omega^p(J^\infty(E))$ to $\Omega^{r,s}(J^\infty(E))$, where $p = r + s$.

The exterior derivative

$$d: \Omega^p(J^\infty(E)) \rightarrow \Omega^{p+1}(J^\infty(E))$$

now splits into horizontal and vertical differentials

$$d = d_H + d_V,$$

where

$$d_H: \Omega^{r,s}(J^\infty(E)) \rightarrow \Omega^{r+1,s}(J^\infty(E))$$

and

$$d_V: \Omega^{r,s}(J^\infty(E)) \rightarrow \Omega^{r,s+1}(J^\infty(E)).$$

Since $d^2 = 0$, we have that

$$d_H^2 = 0, \quad d_H d_V = -d_V d_H, \quad d_V^2 = 0.$$

In local coordinates the horizontal and vertical differentials of a function $f[u]$ are

$$d_H f = \left[\frac{\partial f}{\partial x^i} + u_i^\alpha \frac{\partial f}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial f}{\partial u_j^\alpha} + \cdots \right] dx^i = (D_i f) dx^i$$

and

$$d_V f = \frac{\partial f}{\partial u^\alpha} \theta^\alpha + \frac{\partial f}{\partial u_i^\alpha} \theta_i^\alpha + \cdots .$$

We also have that

$$d_H(dx^i) = 0 \quad \text{and} \quad d_V(dx^i) = 0$$

and, by virtue of (2.1),

$$d_H \theta_I^\alpha = dx^j \wedge \theta_{Ij}^\alpha \quad \text{and} \quad d_V \theta_I^\alpha = 0.$$

Thus the familiar process of total differentiation determines the horizontal differential d_H . The vertical differential d_V can be viewed as a general “infinitesimal field variation”. Specifically, if s_ϵ is a one parameter family of local sections of E and f is a smooth function on $J^\infty(E)$, then

$$\left. \frac{d}{d\epsilon} f(j^\infty(s_\epsilon)) \right|_{\epsilon=0} = \text{pr } X \lrcorner (d_V f), \quad (2.3)$$

where X is the vertical vector field

$$X = \left[\left. \frac{ds_\epsilon^\alpha}{d\epsilon} \right|_{\epsilon=0} \right] \frac{\partial}{\partial u^\alpha}.$$

Definition 2.5. *The variational bicomplex for the fiber bundle $\pi: E \rightarrow M$ is the double complex $(\Omega^{*,*}(J^\infty(E)), d_H, d_V)$ of differential forms on the infinite jet bundle $J^\infty(E)$ of E :*

$$\begin{array}{ccccccccccc}
& & \uparrow d_V & & & & & & & & \uparrow d_V \\
0 & \longrightarrow & \Omega^{0,3} & & & \dots & & & & & \Omega^{n,3} \\
& & \uparrow d_V & & \uparrow & & \uparrow & & \uparrow & & \uparrow d_V \\
0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \Omega^{2,2} & \xrightarrow{d_H} & \dots & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} \\
& & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V \\
0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \Omega^{2,1} & \xrightarrow{d_H} & \dots & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} \\
& & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \Omega^{2,0} & \xrightarrow{d_H} & \dots & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} \\
& & & & & & & & & & & & & & (2.4)
\end{array}$$

The bottom edge of this complex (with $n = 3$) formalizes the first four terms of our model sequence (1.7). In particular, a form $\lambda \in \Omega^{n,0}(J^\infty(E))$ is a Lagrangian

for a variational problem on E — the corresponding functional F is defined on smooth sections $s: M \rightarrow E$ by

$$F[s] = \int_M (j^\infty(s))^*(\lambda).$$

In local coordinates the form λ assumes the form

$$\lambda = L(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, \dots) dx^1 \wedge dx^2 \wedge \dots \wedge dx^n.$$

The *Euler-Lagrange form* for λ is the type $(n, 1)$ form defined by

$$E(\lambda) = E_\alpha(L) \theta^\alpha \wedge dx^1 \wedge dx^2 \wedge \dots \wedge dx^n,$$

where $E_\alpha(L)$ are the components of the classical Euler-Lagrange operator

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i \frac{\partial L}{\partial u_i^\alpha} + D_{ij} \frac{\partial L}{\partial u_{ij}^\alpha} - \dots \quad (2.5)$$

Forms of type (n, s) are automatically d_H closed but they are not always, even locally, d_H exact. (Contrast to the ordinary exterior derivative d on manifolds.) To characterize those forms of maximum horizontal degree which are d_H exact we define, for $s \geq 1$, a co-augmentation map

$$I: \Omega^{n,s}(J^\infty(E)) \rightarrow \Omega^{n,s}(J^\infty(E))$$

in local coordinates by

$$I(\omega) = \frac{1}{s} \theta^\alpha \wedge \left[\left(\frac{\partial}{\partial u^\alpha} \lrcorner \omega \right) - D_i \left(\frac{\partial}{\partial u_i^\alpha} \lrcorner \omega \right) + D_{ij} \left(\frac{\partial}{\partial u_{ij}^\alpha} \lrcorner \omega \right) - \dots \right]. \quad (2.6)$$

Because of its formal similarities with the Euler-Lagrange operator, the map I is called the *interior Euler operator* (the partial differentiations in (2.5) are replaced in (2.6) by interior products). The following theorem shows that this operator I might also justifiably be called an “integration by parts” operator.

Theorem 2.6. *The map I enjoys the following properties.*

(i) If $\eta \in \Omega^{n-1,s}(J^\infty(E))$, then

$$I(d_H \eta) = 0. \quad (2.7)$$

(ii) If $\omega \in \Omega^{n,s}(J^\infty(E))$, then there locally exists a form $\eta \in \Omega^{n-1,s}(J^\infty(E))$ such that

$$\omega = I(\omega) + d_H(\eta). \quad (2.8)$$

(iii) The operator I is a projection operator, that is

$$I^2 = I. \quad (2.9)$$

(iv) If $\lambda \in \Omega^{n,0}(J^\infty(E))$ is a Lagrangian, then

$$E(\lambda) = I(d_V \lambda). \quad (2.10)$$

(v) If Δ is a type $(n, 1)$ form of the type

$$\Delta = \Delta_\alpha[u] \theta^\alpha \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \quad (2.11)$$

then the differential operator with components Δ_α satisfies the Helmholtz conditions (1.6) of the inverse problem of the calculus of variations if and only if

$$I(d_V \Delta) = 0. \quad (2.12)$$

These properties are all easily proved [2]. Property (ii) holds globally but this is a less trivial result which we shall discuss in the next section. Let

$$\mathcal{F}^s(J^\infty(E)) = \text{im} \{I: \Omega^{n,s}(J^\infty(E)) \rightarrow \Omega^{n,s}(J^\infty(E))\}.$$

A form $\omega \in \Omega^{n,s}(J^\infty(E))$ lies in $\mathcal{F}^s(J^\infty(E))$ if and only if $I(\omega) = \omega$. The space $\mathcal{F}^1(J^\infty(E))$ consists precisely of those forms which are given locally by (2.11) — these forms are called source forms by Takens [22].

Finally, let us define another differential

$$\delta_V: \mathcal{F}^s(J^\infty(E)) \rightarrow \mathcal{F}^{s+1}(J^\infty(E))$$

by

$$\delta_V(\omega) = I(d_V(\omega)).$$

A simple calculation shows that $\delta_V^2 = 0$. Then the *augmented variational bicomplex* on $J^\infty(E)$ is the double complex

$$\begin{array}{ccccccc}
& & \uparrow d_V & & & \uparrow d_V & \uparrow \delta_V \\
0 & \longrightarrow & \Omega^{0,3} & \dots & \Omega^{n,3} & \xrightarrow{I} & \mathcal{F}^3 \longrightarrow 0 \\
& & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \dots & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} & \xrightarrow{I} & \mathcal{F}^2 \longrightarrow 0 \\
& & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \dots & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} & \xrightarrow{I} & \mathcal{F}^1 \longrightarrow 0 \\
& & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \dots & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0}
\end{array} \tag{2.13}$$

Definition 2.6. *The Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$ for the fiber bundle $\pi: E \rightarrow M$ is the edge complex of the augmented variational bicomplex on $J^\infty(E)$:*

$$\begin{aligned}
0 & \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \dots \\
& \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \mathcal{F}^3 \xrightarrow{\delta_V} \dots
\end{aligned} \tag{2.14}$$

This complex is the full and proper generalization of the model sequence (1.7) which we introduced in section one.

With the operations pr , tot , d_H , d_V , I , and $\pi^{r,s}$ in hand, as well as the usual operations on vectors fields and forms (bracket, hook, Lie differentiation, ...) one can develop a very effective variational calculus. One result used repeatedly in this calculus is the fact that if X is a vertical vector field on E and ω is any type (r, s) form, then

$$\text{pr } X \lrcorner d_H(\omega) = -d_H(\text{pr } X \lrcorner \omega). \tag{2.15}$$

From this result it then follows that

$$\mathcal{L}_{\text{pr } X}(\omega) = d_V(\text{pr } X \lrcorner \omega) + \text{pr } X \lrcorner d_V(\omega). \tag{2.16}$$

This formula generalizes (2.3).

Theorem 2.7. *Let E be the trivial bundle $\mathbf{R}^{n+m} \rightarrow \mathbf{R}^n$. Then the rows and columns of the augmented variational bicomplex (2.13) are exact. The Euler-Lagrange complex (2.14) is also exact.*

One way to prove the exactness of (2.13) is to construct homotopy operators. The homotopy operators for the columns of the variational bicomplex can be easily constructed from (2.16), as applied to the radial, vertical vector field

$$X = u^\alpha \frac{\partial}{\partial u^\alpha}.$$

The homotopy operators for the horizontal differential d_H have been constructed by Anderson [2], [3], Tulczyjew [27] and Olver [20] and are considerably more complicated. Let us simply remark here that this complexity is borne of necessity since the usual de Rham type homotopy operators cannot work. We illustrate this fact by observing that the vector field $V = (u_{xx}, u_{xy}, u_{xz})$ on \mathbf{R}^3 is Curl free and is in fact the Grad of the function $f[u] = u_x$. But f cannot be reconstructed from V using the standard line integral formula from the vector calculus. In fact, the horizontal homotopy operators for (2.16) are more closely related to the algebraic homotopy operators associated to certain Koszul complexes. An alternative proof of the horizontal exactness of (2.13), valid in the important special case where the coefficient functions are polynomials, is given using the transform method described by Professor L. Dickey elsewhere in these Proceedings.

We remark that if $\omega \in \Omega^{r,s}(J^\infty(E))$ and $d_H\omega = 0$ if $r < n$ or $I(\omega) = 0$ if $r = n$, then by local exactness there is a type $(r-1, s)$ form η such that $\omega = d_H\eta$. But if the differential order of ω is k , Theorem (2.7) says nothing about the differential order of η . In [2] a rather precise method of undetermined coefficients is presented for solving the equation $\omega = d_H\eta$ for η . This method implies, among other things, that if ω is of order k , then the order of η need not be more than k .

In summary, given the fiber bundle $\pi: E \rightarrow M$, that is, the first item on the list of mathematic data given at the beginning of this section, we can construct in a canonical fashion the infinite jet bundle $J^\infty(E)$, the variational bicomplex of differential forms $\Omega^{*,*}(J^\infty(E))$ and its edge complex, the Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$. At this point, the cohomology groups of the variational bicomplex and the Euler-Lagrange complex are well understood and will be described in the next section. The remaining data, namely the transformation group G and the differential relations \mathcal{R} are used to enhance this basic step-up.

The role of the group G is easily described. In many situations we are interested only in currents, Lagrangians, differential equations (that is, source forms)

etc. with certain prescribed symmetries. It is then natural to restrict our attention to the variational bicomplex $(\Omega_G^{*,*}(J^\infty(E)), d_H, d_V)$ and the Euler-Lagrange complex $\mathcal{E}_G^*(J^\infty(E))$ of G invariant forms. For example, if Δ is a source form which is invariant under the group G and if Δ is the Euler-Lagrange form for some Lagrangian λ , that is if $\Delta = E(\lambda)$, one can ask whether Δ is the Euler-Lagrange form of a G invariant Lagrangian. This is the G invariant version of the inverse problem to the calculus of variations. The obstructions to finding G invariant Lagrangians are given by the elements of

$$H^{n+1}(\mathcal{E}_G^*(J^\infty(E))) = \frac{\{\text{locally variational, } G \text{ invariant source forms}\}}{\{\text{source forms of } G \text{ invariant Lagrangians}\}}.$$

Of particular interest to differential geometry and classical field theory are situations where E is some product of tensor bundles over M and the group G includes the diffeomorphism group of the base manifold M . Although the cohomology of the invariant variational bicomplex has been computed in some special cases, which we shall summarize in §4, it is fair to say that there are few, if any, general results. This can be a difficult problem.

The differential relations \mathcal{R} may represent open conditions on the jets of local sections of E or they may represent systems of differential equations. These equations may be classical deterministic (well-posed) systems or they may be the kind of under-determined or over-determined systems that are often encountered in differential geometry. We prolong \mathcal{R} to a set of differential equations \mathcal{R}^∞ on $J^\infty(E)$ and then restrict (or pullback) the variational bicomplex on $J^\infty(E)$ to \mathcal{R}^∞ . One immediate consequence of this construction is that the cohomology group $H^{n-1}(\mathcal{E}^*(\mathcal{R}))$ can now be identified with the vector space of conservation laws for \mathcal{R} . Tsujishita [23] has emphasized the role that the variational bicomplex can play in determining deformation invariants on the solution space of \mathcal{R} and in detecting obstructions to Gromov's h principle for differential equations. Other interpretations and applications of the cohomology of the variational bicomplex for differential equations continue to arise. In §5 we shall discuss a few recent developments in this area.

Because of the ability to make these modifications to the free variational bicomplex (2.4) on $J^\infty(E)$, a surprising diversity of phenomena from geometry and topology, differential equations, and mathematical physics, including many topics *not* directly related to the calculus of variations, can be studied in terms of the cohomological properties of the variational bicomplex. The basic goals of current research in this area is to develop better general methods for computing the cohomology of these variational bicomplexes and to seek new interpretations and applications of these cohomology classes.

§3. Global Properties of the Free Variational Bicomplex.

The most important result concerning the variational bicomplex for the fiber bundle $\pi: E \rightarrow M$, in the absence of a group action G on E and a system of differential equations \mathcal{R} , is the following.

Theorem 3.1. *For any fiber bundle $\pi: E \rightarrow M$, the interior rows of the augmented variational bicomplex*

$$\begin{aligned} 0 \longrightarrow \Omega^{0,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{1,s}(J^\infty(E)) \xrightarrow{d_H} \Omega^{2,s}(J^\infty(E)) \xrightarrow{d_H} \dots \\ \xrightarrow{d_H} \Omega^{n,s}(J^\infty(E)) \xrightarrow{I} \mathcal{F}^s(J^\infty(E)) \longrightarrow 0, \end{aligned} \quad (3.1)$$

where $s \geq 1$, are exact.

This theorem is easily proved using standard techniques from global analysis together with the fact that the interior rows of the variational bicomplex are locally exact. One proof based upon partition of unity arguments and the generalized Mayer-Vietoris sequence (see Bott and Tu [7]) is given in [2].

Corollary 3.2. *There always exists a global first variational formula for any variational problem on E . If $\lambda \in \Omega^{n,0}(J^\infty(E))$ is any Lagrangian, then there exists a type $(n-1,1)$ form η such that*

$$d_V \lambda = E(\lambda) + d_H(\eta). \quad (3.2)$$

If X is a vertical vector field on E , then there exists a type $(n-1,0)$ form σ such that

$$\mathcal{L}_{\text{pr } X} \lambda = \text{pr } X \lrcorner E(\lambda) + d_H \sigma. \quad (3.3)$$

We have presented enough of the variational calculus to be able to prove this corollary. From the properties of the interior Euler operator I listed in Theorem (2.6) we have that $I(d_V \lambda) = E(\lambda)$ and $I(E(\lambda)) = E(\lambda)$ and hence $I(d_V \lambda - E(\lambda)) = 0$. By the exactness of (3.1), with $s = 1$, there is a form $\eta \in \Omega^{n-1,1}(J^\infty(E))$ such that (3.2) holds. Equation (3.3) now follows immediately from (2.15), (2.16) and (3.2).

Equation (3.3) is a full and proper differential-geometric formulation of the first variational formula (1.2) given in the introduction.

As Kuperschmidt [17] and Krupka [16] have emphasized, this Corollary truly requires the use of Theorem 3.1. The local first variational formula for an arbitrary order Lagrangian gives a local formula for η in terms of the Lagrangian and its

derivatives but this local formula does not patch together under changes of coordinates (except in special circumstances such as $n = 1$, where η is unique) to give a globally well-defined form η .

One can generalize equation (3.3) to the case where X is an arbitrary generalized vector field on E . This then furnishes us with a global Noether's theorem. Every global generalized symmetry of the Lagrangian λ gives rise to a conserved global $n - 1$ form σ but once again it needs be remarked that the usual local formula for this conserved form σ does not necessarily patch together to give the global result.

The general theory of Cartan forms in the calculus of variations (Krupka [15]) can also be readily developed using the variational bicomplex. Let us simply remark here that the n form $\Theta = \lambda - \eta$ provides us with a global Cartan form for the Lagrangian λ .

Now define a map Ψ from the de Rham complex $\Omega^*(J^\infty(E))$ to the Euler-Lagrange complex $\mathcal{E}^*(J^\infty(E))$ by projection and, if necessary, by integrating by parts:

$$\Psi(\omega) = \begin{cases} \pi^{p,0}(\omega) & \text{for } p \leq n \\ I(\pi^{n,p-n}(\omega)) & \text{for } p \geq n. \end{cases} \quad (3.4)$$

A standard exercise in homological algebra then proves the following.

Theorem 3.4. *The map Ψ induces an isomorphism in cohomology:*

$$\Psi^*: H^p(\Omega^*(J^\infty(E))) \rightarrow H^p(\mathcal{E}^*(J^\infty(E))).$$

Since E is a strong deformation retract of $J^\infty(E)$, the de Rham cohomology of these spaces coincide and we have the following solution to the global inverse problem of the calculus of variations cite3, [23].

Corollary 3.5. *The obstructions to the construction of global Lagrangians for source forms which satisfy the Helmholtz conditions lie in $H^{n+1}(E)$.*

EXAMPLE 3.6. Suppose that M and N are compact oriented n manifolds and that

$$h = h_{\alpha\beta}(u) du^\alpha \otimes du^\beta$$

is a Riemannian metric on N . Let $E : M \times N \rightarrow M$. Then the volume form

$$\nu = \sqrt{\det h} du^1 \wedge du^2 \cdots \wedge du^n$$

on N pulls back to a closed form on E which represents a nontrivial cohomology class in $H^n(E)$. The associated Lagrangian $\lambda = \Psi(\nu)$ on $J^\infty(E)$ is found to be

$$\lambda = \sqrt{\det h} \det \left[\frac{\partial u^\alpha}{\partial x^j} \right] dx^1 \wedge dx^2 \cdots \wedge dx^n. \quad (3.5)$$

Our general theory implies that $E(\lambda) = 0$. (It is an amusing exercise to verify this directly for any metric h .) Since ν is not exact on E , the Lagrangian λ is not a global divergence and represents a non-trivial cohomology class in the Euler-Lagrange complex. The corresponding fundamental integral, defined on sections of E , or equivalently on maps $s: M \rightarrow N$, is

$$F[s] = \int_M (j^\infty(s))^*(\lambda) = \int_M s^*(\nu),$$

and coincides, apart from a numerical factor, with the topological degree of the map s . This example illustrates how cohomology classes in $H^n(\mathcal{E}^*(J^\infty(E)))$ may lead to topological invariants for the sections s of E . Other familiar invariants such as the rotation index for regular closed curves in the plane and Gauss' linking number for non-intersecting space curves in \mathbf{R}^3 may be uncovered in this way.

Consider now the special case where M is the two sphere S^2 and N is the two torus $S^1 \times S^1$. Let $\nu = du \wedge dv$, where (u, v) are the standard angular coordinates on N . The Lagrangian (3.5) becomes

$$\lambda = (u_x v_y - u_y v_x) dx \wedge dy \tag{3.6}$$

and, on sections s on E ,

$$(j^\infty(s))^*(\lambda) = s^*(du \wedge dv) = s^*(du) \wedge s^*(dv).$$

But $s^*(du)$ is a closed one form on S^2 and is hence exact on S^2 . If we write

$$s^*(\alpha) = df,$$

where f is a smooth real-valued function on S^2 , then

$$(j^\infty(s))^*(\lambda) = d(fs^*(\beta)).$$

This proves that λ is exact on all sections s of E . This example underscores an important point — that the cohomology of the variational bicomplex is *local* cohomology. Even though the Lagrangian (3.6) is an exact two form on all sections, λ nevertheless defines a nontrivial cohomology class in the Euler-Lagrange complex because it cannot be expressed as the horizontal derivative of a one form whose value on a section s can be computed pointwise from the jets of s .

EXAMPLE 3.7. Let U be an open region in \mathbf{R}^3 , with coordinates (u^1, u^2, u^3) and let $E: \mathbf{R} \times U \rightarrow \mathbf{R}$. Let ω be a closed two form on U pulled back to E and let

$$\hat{\omega} = I(\pi^{1,1}(\omega)) = (a^1\theta^1 + a^2\theta^2 + a^3\theta^3) dx.$$

Then the system of ordinary differential equations

$$\ddot{u}^\alpha = a^\alpha(u^\beta, \dot{u}^\beta) \tag{3.7}$$

always satisfies the Helmholtz conditions. Equation (3.7) admits a global variational principle if and only if the form ω is exact on U . In particular, if $U = \mathbf{R}^3 - \{(0, 0, 0)\}$ and

$$\omega = \frac{u^1 du^2 \wedge du^3 - u^2 du^1 \wedge du^3 + u^3 du^1 \wedge du^2}{[(u^1)^2 + (u^2)^2 + (u^3)^2]^{\frac{3}{2}}}$$

then (3.7) becomes, in vector notation,

$$\ddot{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|} \mathbf{u} \times \dot{\mathbf{u}}.$$

These are the Lorentz force equations for the motion of a charged particle in the field of a magnetic monopole. No global variational principle exists on $J^\infty(E)$. (Although one may be constructed if the electromagnetic field is re-interpreted as a connection on S^3 .)

Other applications and examples of cohomology classes for the free variational bicomplex will be found in [2].

§4. Cohomology Groups for Some Invariant Variational Bicomplexes.

Given a transformation group G acting on the fiber bundle $\pi: E \rightarrow M$, we now turn to the problem of computing the cohomology of the G invariant Euler-Lagrange complex $\mathcal{E}_G^*(J^\infty(E))$. Unlike the case of the free Euler-Lagrange complex there are no general results in this area comparable to Theorem 3.4 and one must, at least for the present, address each individual situation directly.

The methods of homological algebra provide us with an overall approach. First, we try to show that the interior rows of the invariant augmented variational bicomplex

$$\begin{aligned} 0 \longrightarrow \Omega_G^{0,s}(J^\infty(E)) \xrightarrow{d_H} \Omega_G^{1,s}(J^\infty(E)) \xrightarrow{d_H} \Omega_G^{2,s}(J^\infty(E)) \xrightarrow{d_H} \dots \\ \xrightarrow{d_H} \Omega_G^{n,s}(J^\infty(E)) \xrightarrow{I} \mathcal{F}_G^s(J^\infty(E)) \longrightarrow 0, \end{aligned} \tag{4.1}$$

are exact. The methods used to prove Theorem 3.4 cannot be used directly for, in general, the local homotopy operators are not G invariant and, in addition, the partition of unity arguments which are essential to the generalized Mayer-Vietoris principle fail. Still, the exactness of (4.1) can often be established by constructing G invariant homotopy operators. Indeed, I know of no situation where (4.1) fails to be exact. An obvious open problem therefore is to establish the exactness of the interior rows of the G invariant variational bicomplex in some degree of generality or, alternatively, to give examples of group actions where exactness fails.

Once the exactness of (4.1) is established, we have that

$$H^p(\Omega_G^*(J^\infty(E))) \cong H^p(\mathcal{E}_G^*(J^\infty(E)))$$

and the problem now becomes that of computing the cohomology of the G invariant de Rham complex on $J^\infty(E)$. This can be done by using the second spectral sequence for the variational bicomplex — that is, we first compute the d_V cohomology. This calculation seems to be the heart of the matter. The subsequent terms in the spectral sequence are then computed, usually without much difficulty, to arrive at $H^*(\mathcal{E}_G^*(J^\infty(E)))$.

We now give four examples. With the exception of the first example, the results presented here are all new. Details will appear elsewhere.

Example. *The Inverse Problem of the Calculus of Variations for Autonomous Ordinary Differential Equations.*

Let N be a manifold, let E be the trivial bundle $\mathbf{R} \times N \rightarrow \mathbf{R}$, and let G be the group of translations in the base. A form ω on $J^\infty(E)$ is G invariant if and only if the coefficients of ω do not explicitly depend upon the independent variable x . In particular, a source form $\Delta \in \mathcal{F}_G^1(J^\infty(E))$ defines an autonomous system of ordinary differential equations.

Theorem 4.1. *The cohomology of the G invariant Euler-Lagrange complex is*

$$H^p(\mathcal{E}_G^*(J^\infty(E))) \cong H^{p-1}(N) \oplus H^p(N).$$

This theorem was first established by Tulczyjew [28]. Another proof, which illustrates well the use of homological methods in our subject, can be briefly sketched as follows. Since $n = 1$, the variational bicomplex consists of just two columns. Any double complex with just two columns always leads to a long exact sequence called the Wang sequence (see McCleary [18]). In the present context the Wang sequence is

$$\dots H_{d_V}^{0,p-1} \xrightarrow{d_H} H_{d_V}^{1,p-1} \xrightarrow{i} H^p(\mathcal{E}_G^*) \xrightarrow{\pi} H_{d_V}^{0,p} \xrightarrow{d_H} H_{d_V}^{1,p} \dots \quad (4.2)$$

It is not too difficult to prove that

$$H_{d_v}^{1,p-1}(\Omega_G^{*,*}(J^\infty(E))) \cong H^{p-1}(N) \quad \text{and} \quad H^{0,p}(\Omega_G^{*,*}(J^\infty(E))) \cong H^p(N)$$

and that the induced map d_H in (4.2) is zero. The result now follows from the exactness of (4.2).

For example, with $N = \mathbf{R}^2 - \{(0,0)\}$, the system of equations

$$\ddot{u} = -\frac{v}{u^2 + v^2} \quad \text{and} \quad \ddot{v} = \frac{u}{u^2 + v^2}$$

are the Euler-Lagrange equations for the x dependent Lagrangian

$$L = -\frac{1}{2}(\dot{u}^2 + \dot{v}^2) + x \frac{v\dot{u} - u\dot{v}}{u^2 + v^2}$$

but there does not exist a G invariant Lagrangian for this system.

Example. *The Natural Variational Bicomplex for Regular Plane Curves.*

For this example we let E be the trivial bundle $\mathbf{R} \times \mathbf{R}^2 \rightarrow \mathbf{R}$ and we let \mathcal{R} be the open subset of $J^\infty(E)$ consisting of jets $(x, u, v, \dot{u}, \dot{v}, \ddot{u}, \ddot{v}, \dots)$ with $\dot{u}^2 + \dot{v}^2 \neq 0$. Sections s of E with $j^\infty(s) \in \mathcal{R}$ define regular (immersed) plane curves. The natural variational bicomplex for regular plane curves is the variational bicomplex on \mathcal{R} consisting of those forms which are invariant under the group G of arbitrary orientation preserving diffeomorphisms of the base \mathbf{R} (reparametrizations of the curve) and Euclidean motions in the fiber. For example, Lagrangians $\lambda \in \Omega_G^{1,0}(\mathcal{R})$ assume the form

$$\lambda = L(\kappa, \kappa', \kappa'', \dots) ds, \tag{4.3}$$

where κ is the curvature of the curve and a prime denotes differentiation with respect to arc-length s .

Theorem 4.2. *The cohomology of the natural Euler-Lagrange complex $H^p(\mathcal{E}_G^*(\mathcal{R}))$ is one dimensional for $p = 0, 1, 2$, two dimensional for $p = 3$, one dimensional for $p = 4, 5, 6$, and zero dimensional for $p > 6$.*

Explicit generators for these cohomology classes can be given. For $p = 1$, the cohomology is generated by the Lagrangian

$$\lambda = \kappa ds, \tag{4.4}$$

a result first proved by Cheung [8] using the Griffiths formalism [14] for the calculus of variations. For $p = 2$, the cohomology is generated by the source form

$$\Delta = (\dot{v}du - \dot{u}dv) \wedge dx$$

for which we have the Lagrangian

$$\lambda = \frac{1}{2}(u\dot{v} - v\dot{u}) dx$$

but no Lagrangian of the form (4.3).

The cohomology of the natural variational bicomplex for space curves has also been computed [2]. Note that the functional defined by the Lagrangian (4.4) determines, at least in the case of closed curves, the rotation index of the curve. The interpretation of other cohomology classes in these Euler-Lagrange complexes for plane and space curves as topological invariants is under current investigation.

Example. *The Natural Variational Bicomplex for Riemannian Structures.*

Let M be a manifold and let $Q \rightarrow M$ be the fiber bundle of positive-definite quadratic forms on the tangent bundle of M . A section g of Q gives a choice of Riemannian metric on M . Let G be the diffeomorphism group on M . The G invariant variational bicomplex on $J^\infty(Q)$ is called the natural variational bicomplex for Riemannian structures. Lagrangians in this bicomplex include those defined by complete contractions of indices of polynomials in the curvature tensor and its covariant derivatives.

Theorem 4.3. *The interior rows of the augmented natural variational bicomplex for Riemannian structures are exact. The vertical cohomology groups are generated by the forms*

$$P(\omega) \wedge Q(\gamma),$$

where P and Q are $\mathfrak{gl}(n)$ invariant polynomials and ω and γ are the matrices of one and two forms

$$\omega_j^i = g^{ih} d_V g_{jh} \quad \text{and} \quad \gamma_j^i = dx^h \wedge d_V \Gamma_{jh}^i,$$

where Γ_{jh}^i are the Christoffel symbols of the metric g .

With these results in hand the cohomology $H^p(\mathcal{E}_G^*(J^\infty(Q)))$ of the natural Euler-Lagrange complex can be computed. In particular, for $p \leq n$ the cohomology is generated by the Pontryagin forms and for $p = n+1$ by the Euler-Lagrange forms of the Chern-Simons Lagrangians. Thus Theorem 4.3 provides us with generalizations of the work of Gilkey [12] and Anderson [1].

Example. *Gelfand Fuks Cohomology*

Let E be the trivial bundle $\mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ and let \mathcal{R} be the open set in $J^\infty(E)$ defined by $\det(u_i^\alpha) \neq 0$. A section s of E with $j^\infty(s) \in \mathcal{R}$ corresponds to a local diffeomorphism of \mathbf{R}^n . Let G be the diffeomorphism group of the base space \mathbf{R}^n . Then it is not too difficult to prove *a priori* that the cohomology of the G invariant Euler-Lagrange complex $\mathcal{E}_G^*(J^\infty(E))$ is isomorphic to the Gelfand-Fuks cohomology of formal vector fields on \mathbf{R}^n . The variational bicomplex thereby naturally provides us with a method to compute this Gelfand-Fuks cohomology.

Finally, we remark that there seems to be close similarities between the cohomology of certain G invariant variational bicomplexes and BRST cohomology in classical field theory although the precise relationships between these cohomology groups have yet to be uncovered.

§5. The Variational Bicomplex for Differential Equations.

Every system of k -th order differential equations on E defines a subbundle $\mathcal{R} \rightarrow M$ of the finite jet bundle $J^k(E) \rightarrow M$. A solution to the differential equation is a section $s: M \rightarrow E$ such that $j^k(s) \in \mathcal{R}$. If we differentiate the equations defining \mathcal{R} we obtain a new system of equations $\mathcal{R}^1 \rightarrow M$ called the first prolongation of \mathcal{R} . This process can be repeated. A basic assumption in this geometric theory of differential equations is that for each $l = 0, 1, 2, \dots$, the prolongation \mathcal{R}^{l+1} fibers over \mathcal{R}^l . This assumption always holds for systems of equations of Cauchy-Kowalevsky type. The integrability criteria of Goldschmidt [6] enables us to check this assumption. When this assumption holds we can pull the variational bicomplex on $J^\infty(E)$ back to the infinite prolongation \mathcal{R}^∞ of \mathcal{R} and so define the variational bicomplex $(\Omega^{*,*}(\mathcal{R}^\infty), d_H, d_V)$ of the differential equations \mathcal{R} .

For example, with $E: \mathbf{R}^2 \times \mathbf{R} \rightarrow \mathbf{R}^2$ and \mathcal{R} an evolution equation defined by

$$u_t = K(t, x, u, u_x, u_{xx}, \dots),$$

forms $\alpha \in \Omega^{1,0}(\mathcal{R}^\infty)$ and $\beta \in \Omega^{0,1}(\mathcal{R}^\infty)$ assume the form

$$\alpha = A[u] dx + B[u] dy$$

and

$$\beta = A[u] \theta + B[u] \theta_x + C[u] \theta_{xx} + \dots,$$

where the coefficients $A[u]$, $B[u]$, $C[u]$ are functions of t , x , u , and “spatial” derivatives u_x , u_{xx} , \dots only. The local formulas for the vertical differential d_V remain unchanged (and so the columns of this variational bicomplex remain exact) but now

$$\begin{aligned} d_H u &= K[u] dt + u_x dx, \\ d_H \theta &= d_H(d_V u) = dt \wedge (d_V K[u]) + dx \wedge \theta_x, \\ d_H u_x &= D_x(K[u]) dt + u_{xx} dx, \end{aligned}$$

and so on. The rows of the variational bicomplex are no longer even locally exact. In particular the elements of $H_{d_H}^{n-1,0}(\Omega^{*,*}(\mathcal{R}^\infty))$ are represented by type $(n-1, 0)$ forms (*ie.* currents) which are d_H closed by virtue of the equations and are therefore the classical differential conservation laws of \mathcal{R} . In the case of a system of ordinary differential equations ($n=1$) the elements of $H^{0,0}(\mathcal{R}^\infty)$ are precisely the first integrals of the system.

The first general result concerning the d_H cohomology of the variational bicomplex for differential equations is the Vinogradov two-line theorem (Vinogradov [29]), a restricted case of which we state as follows.

Theorem 5.1. *Suppose \mathcal{R} is a system of equations of Cauchy-Kowalevsky type. Then for all s*

$$H_{d_H}^{r,s}(\Omega^{*,*}(\mathcal{R}^\infty)) = 0 \quad \text{for } r < n.$$

For completely integrable scalar evolution equations, $H_{d_H}^{n-1,0}(\Omega^{*,*}(\mathcal{R}))$ is infinite dimensional. There is an algorithm for directly computing all the conservation laws for \mathcal{R} which is very similar to that used to compute the generalized symmetries of \mathcal{R} . See Vinogradov [30]. The effectiveness of this algorithm is still problematic if one considers the complexities involved in computing the conservation laws for the BBM equation (Duzhin [10], Olver [19]). To find all the conservation laws for a given system of equations, it seems that this algorithm works best in conjunction with other methods such as that provided by the theory of bi-Hamiltonian systems.

If $\omega \in \Omega^{n-1,0}(\mathcal{R}^\infty)$ is a conserved form, then $d_V \omega \in \Omega^{n-1,1}(\mathcal{R}^\infty)$ is also d_H closed and therefore defines an element of $H_{d_H}^{n-1,1}(\mathcal{R}^\infty)$. If the form $d_V \omega$ is d_H exact, then it may be that there is a d closed form $\tilde{\omega} \in \Omega^{n-1}(\mathcal{R})$ such that $\pi^{n-1,0}(\tilde{\omega}) = \omega$. Conservation laws for \mathcal{R} which arise in this fashion from the topology of \mathcal{R} (*ie.* from the total de Rham complex $(\Omega^*(\mathcal{R}), d)$) are called *rigid conservation laws*. For example, for the equation $u_x^2 + u_y^2 = 1$ the conserved form

$$\omega = \pi^{1,0}(-u_x du_y + u_y du_x) = (-u_x u_{xy} + u_y u_{xy}) dx + (-u_x u_{yy} + u_y u_{yy}) dy$$

is rigid. Likewise the one form $a dx + b dy$, where a and b are constants, is a rigid conservation law for Laplace's equation $u_{xx} + u_{yy} = 0$ on the torus. Thus, in some sense, the essential conservation laws for \mathcal{R} are those defined by classes $[\omega] \in H_{d_H}^{n-1,0}(\mathcal{R}^\infty)$ for which the class $[d_V \omega] \in H_{d_H}^{n-1,1}(\mathcal{R}^\infty)$ is non-trivial.

Elements $[\omega] \in H_{d_H}^{n-1,2}(\mathcal{R}^\infty)$ detect the possible existence of variational principles for \mathcal{R} . Indeed, if the system of equations \mathcal{R} are the Euler-Lagrange equations for some Lagrangian λ , then the first variational formula (3.1), when pulled back to \mathcal{R}^∞ , becomes

$$d_V \lambda = d_H \eta$$

and thus the form

$$\omega = d_V(\eta)$$

is a d_H closed type $(n-1, 2)$ form. Zuckerman [31] calls ω the universal conserved current for λ . It follows from the Vinogradov two-line theorem that ω cannot be d_H exact for otherwise the Lagrangian λ would be trivial. Thus, if $H_{d_H}^{n-1,2}(\mathcal{R}^\infty) = 0$, the equations \mathcal{R} do not admit a variational formulation. This approach to the inverse problem of the calculus of variations, based upon the variational bicomplex for \mathcal{R} is pursued in detail in [5] for the case of ordinary differential equations. It is remarkable that this approach leads directly to the fundamental equations of J. Douglas [9]. It allows us to generalize Douglas' work to the case of higher order systems and it also provides us with a natural setting for the use of exterior differential system techniques. One fact that quickly emerges from this approach is that significant differences exist between the inverse problem for second order systems and higher systems. For example, the most general Lagrangian for the second order system

$$u_{xx} = 0 \quad \text{and} \quad v_{xx} = 0$$

depends upon 2 arbitrary functions of 3 variables whereas the most general Lagrangian for the fourth order system

$$u_{xxxx} = 0 \quad \text{and} \quad v_{xxxx} = 0$$

involves just 3 arbitrary constants.

Recently, it has been observed [4] that the existence of non-trivial cohomology classes in $H_{d_H}^{n-1,s}(\mathcal{R}^\infty)$ for $s \geq 3$ is closely related to the applicability of Darboux's method of integration [13]. It is apparent that equations integrable by this method, such as the Liouville equation

$$u_{xy} = e^u,$$

can have infinite dimensional cohomology $H_{d_H}^{1,s}(\mathcal{R}^\infty)$ for each $s \geq 3$. Conversely, scalar equations in more than two independent variables are not integrable by this method and do not admit any non-trivial higher degree cohomology classes.

Finally, Tsujishita [26] has generalized the Vinogradov theorem to prove the following result. Let E and F_0 be vector bundles over M . If $P: J^k(E) \rightarrow F_0$ is a linear differential operator, then we denote its infinite prolongation by $\text{pr } P: J^\infty(E) \rightarrow J^\infty(F_0)$. A result of Goldschmidt (see [6], Chapter Ten) asserts that there always exists a sequence of vector bundles F_i and linear differential operators Q_i such that the complex

$$J^\infty(E) \xrightarrow{\text{pr } P} J^\infty(F_0) \xrightarrow{\text{pr } Q_1} J^\infty(F_1) \xrightarrow{\text{pr } Q_2} \dots \xrightarrow{Q_{l-1}} J^\infty(F_l) \longrightarrow 0 \quad (5.1)$$

is formally exact.

Theorem 5.2. *Let E and F_0 be vector bundles over M and let $P: J^k(E) \rightarrow F_0$ be a linear differential operator. If (5.1) holds, then the horizontal cohomology groups $H_{d_H}^{r,s}(\Omega^{*,*}(\mathcal{R}))$ for the variational bicomplex for the equation $P[u] = 0$ vanish for $r < n - l - 1$.*

Further general remarks on the variational bicomplex for differential equations can be found in the excellent review article [25]. In [11], Fuchs, Gabrielov and Gel'fand used the variational bicomplex to study the integrability equations for foliations. They introduced the *difference bicomplex* and proved that the canonical map from the difference complex to the variational bicomplex is a homotopy equivalence in certain situations. Many of the important ideas in this paper have yet to fully developed.

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