

Group Invariant Solutions in Mathematical Physics and Differential Geometry

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1. Introduction.

Inspired by Galois' theory of symmetry of polynomial equations, Sophus Lie (1842-1899) developed an analogous theory of symmetry for differential equations. Besides this leading to the mathematics of Lie groups and transformation groups, Lie's theory led to an algorithmic way to find special explicit solutions to differential equations with symmetry. These special solutions are called group invariant solutions and they constitute practically every known explicit solution to the systems of non-linear partial differential equations arising in mathematical physics and differential geometry. For example, the Schwarzschild solution to the vacuum Einstein equations, and the instanton solution to the (self-dual) Yang-Mills equations are both group invariant solutions.

Today the search for group invariant solutions is still a common approach to explicitly solving non-linear partial differential equations. For instance, interesting examples of Einstein manifolds, harmonic maps and Ricci-solitons with cohomogeneity one symmetry groups can be found in [4],[7] and [6].

An excellent introductory reference to Lie's theory of group invariant solutions is [8]. However in this treatment (as in others [3],[9]) the theory is developed to mainly study scalar partial differential equations and does not apply to the standard equations of mathematical physics. In fact one cannot even reproduce the Schwarzschild solution from the techniques provided by these references. The technical assumption made in these references which precludes these famous examples is known as *transversality*. In this article, which is more or less a summary of [1], we provide a general method to find group invariant solutions for non-transverse group actions. The method also resolves the question of "how many" differential equations determine the group invariant solutions, a question which does not arise for scalar equations.

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2. Notation and Preliminaries.

Let $\pi : E \rightarrow M$ be a fibre bundle with n -dimensional base, and m -dimensional fibre. The map π in local coordinates has the form

$$\pi(x^i, u^\alpha) = (x^i) \quad i = 1 \dots n, \quad \alpha = 1 \dots m .$$

The local coordinates (x^i) on the base M play the role of the independent variables, while the local coordinates (u^α) on the fibres will be the dependent variables. The sections of E will be denoted by

$$S(E) = \{s : M \rightarrow E, \quad \pi \circ s = I_M\}$$

and $s \in S(E)$ takes the form $u^\alpha = u^\alpha(x)$ in local coordinates.

A group G acting on E is said to act **projectably** (or by fibre preserving transformations) if there exists an action of G on M such that the projection map $\pi : E \rightarrow M$ is equivariant. In other words for all $g \in G$ the diagram

$$(1) \quad \begin{array}{ccc} E & \xrightarrow{g} & E \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{g} & M \end{array}$$

commutes. The notation gp and gx denotes the action of G on E and M respectively.

Given a group G acting projectably on E (and hence acting on M) and a point $x \in M$, the **isotropy or stabilizer subgroup of x** is

$$G_x = \{g \in G \mid gx = x\} .$$

For each $x \in M$ the subgroup G_x acts on the fibre E_x . That is, given $x \in M$, $g \in G_x$, and $p \in E_x = \pi^{-1}(x)$ then as a consequence of the commutative diagram (1) we find

$$\pi(gp) = g\pi(p) = gx = x = \pi(p)$$

and hence $gp \in E_x$.

3. Examples Part I.

EXAMPLE 1. Let G be any Lie group acting on manifolds M and N . Let $E = M \times N$ be the total space in the trivial bundle $\pi : M \times N \rightarrow M$. Take the action of G on E to be the product action

$$g(\mathbf{x}, \mathbf{u}) = (g\mathbf{x}, g\mathbf{u}) \quad \mathbf{x} \in M, \quad \mathbf{u} \in N, \quad g \in G$$

which is projectable with the action of G on M being the given one.

EXAMPLE 2.A. Let M be any differentiable manifold and let $G = \text{diff}(M)$ be the group of diffeomorphisms of M , and let E be any tensor-bundle over M . The action of $\text{diff}(M)$ on M lifts to a natural action of $\text{diff}(M)$ on E , which in the case $E = TM$ and $\phi \in \text{diff}(M)$ is $\phi_* : TM \rightarrow TM$. These lifted actions are projectable.

Later on we will be interested in a special case of example 2.a, which we now list as a separate example.

EXAMPLE 2.B. The lifted action of any Lie group $G \subset \text{diff}(M)$ to any tensor-bundle over M as in example 2.a is projectable.

4. Invariant Sections.

Given a group G acting projectably on E there is an induced action of G on $S(E)$ the sections of E . Given $s \in S(E)$ and $g \in G$ then the section $g \cdot s$ is defined by

$$(g \cdot s)(x) = g s(g^{-1}x), \quad \forall x \in M.$$

DEFINITION 1. Let G act projectably on E . A section $s \in S(E)$ is *G -invariant* if $g \cdot s = s$ or

$$(2) \quad g s(x) = s(gx) .$$

Denote by $S(E)^G$ the set of G -invariant sections of E , where of course $S(E)^G$ is the set of fixed points of the action of G on $S(E)$.

EXAMPLE 1. (continued) The group G acts on each term separately in the total space E of the trivial bundle $\pi : M \times N \rightarrow M$. In this case the space $S(E)^G$ admits an alternate description. Given $\phi \in C^\infty(M, N)$, the smooth maps from M to N , we define a section $s_\phi : M \rightarrow E$ by $s_\phi(x) = (x, \phi(x))$, which identifies

$$S(E) = C^\infty(M, N) .$$

Using this identification, a section $s \in S(E)$ is invariant if and only if

$$g s_\phi(x) = (gx, g\phi(x)) = (gx, \phi(gx)) = s_\phi(gx)$$

or $\phi(gx) = g\phi(x)$. Thus the invariant sections of E are just the G -equivariant maps $\phi : M \rightarrow N$ and for this reason group invariant solutions are sometimes called equivariant [7].

A fundamental problem in the theory of group invariant solutions is how to parameterize the space $S(E)^G$. Adding the following hypothesis, which is not often valid but whose definition will be used later, makes this fairly easy.

DEFINITION 2. A projectable group action G on E , is said to be *transverse* if for all $p \in E$

$$(3) \quad \pi(gp) = \pi(p) \quad \text{implies} \quad gp = p.$$

Equivalently, G acts transversely on E if the orbits of G in E project diffeomorphically under π to the orbits of G in M . In local coordinates, if the infinitesimal generators of the action of G on E are given by

$$\Gamma = \{ \xi_a^i(\mathbf{x}) \partial_{x^i} + \phi_a^\alpha(\mathbf{x}, \mathbf{u}) \partial_{u^\alpha} \} \quad a = 1 \dots \dim G$$

and if the action is transverse, then

$$\text{rank} [\xi_a^i(\mathbf{x})] = \text{rank} [\xi_a^i(\mathbf{x}), \phi_a^\alpha(\mathbf{x}, \mathbf{u})] .$$

The invariant sections $S(E)^G$ for transverse actions are easily parameterized.

THEOREM 1. Let G be a Lie group acting projectably and transversally on the bundle $\pi : E \rightarrow M$, and regularly on M (so that M/G is a manifold). Then $S(E)^G$ is in one-to-one correspondence with the set of sections of the bundle $E/G \rightarrow M/G$.

PROOF. We will only show how a section $\tilde{s} : M/G \rightarrow E/G$ defines a section $s : M \rightarrow E$, and leave the rest as an exercise. Given $\tilde{x} \in M/G$, let $x \in M$ with $q_M(x) = \tilde{x}$, where $q_M : M \rightarrow M/G$. First we claim there exists a unique $p \in E$ such that

$$i] \ q_E(p) = \tilde{s}(\tilde{x}) \quad ii] \ \pi(p) = x ,$$

where $q_E : E \rightarrow E/G$. Suppose there exist p, p' satisfying conditions $i], ii]$. By condition $i]$ $p' = gp$ for some $g \in G$, which when used in condition $ii]$ along with transversality (3) implies $p' = p$. Therefore define $s(x) = p$ to get the appropriate section. See [2] for more details. ■

The following simple corollary demonstrates that transverse group actions are unusual, and hence most examples of group actions on bundles are not transverse.

COROLLARY 1. *If G is a group acting projectably on E , then G acts transversally if and only if for all $x \in M$, G_x acts trivially on E_x .*

For the examples we are interested in we need to find a way to parameterize the space $S(E)^G$ when the action of G on E may not be transverse. In order to do so, we use the following observation. Let $s \in S(E)^G$ be an invariant section, let $x \in M$ and let $g \in G_x$. The invariant section condition (2) at x gives

$$(4) \quad gs(x) = s(gx) = s(x)$$

with the last identity coming from $g \in G_x$. Therefore if $s \in S(E)^G$ then $s(x) \in (E_x)^{G_x}$, the fixed point set of G_x acting on E_x , which motivates the following definition.

DEFINITION 3. Let G act projectably on E and let $\kappa_x(E) = (E_x)^{G_x}$ and

$$\kappa(E) = \bigcup_{x \in M} \kappa_x(E) .$$

We call $\pi : \kappa(E) \rightarrow M$ **the kinematic bundle**.¹

A simple consequence of this definition and equation (4) is

COROLLARY 2. *Every $s \in S(E)^G$ factors through $\pi : \kappa(E) \rightarrow M$.*

Note that this corollary does not say that every section of $\kappa(E) \rightarrow M$ is an element of $S(E)^G$. However the following is true,

THEOREM 2. *The subset $\kappa(E) \subset E$ is G -invariant, G acts transversally on $\pi : \kappa(E) \rightarrow M$, and $S(E)^G = S(\kappa(E))^G$.*

PROOF. Let $p \in \kappa(E)$ with $x = \pi(p)$ and let $g \in G$, $h \in G_{gx}$. The subgroup $G_{gx} = gG_xg^{-1}$ therefore $h = gh'g^{-1}$ for some $h' \in G_x$ and

$$hgp = gh'g^{-1}gp = gh'p = gp$$

so $gp \in \kappa_{gx}(E)$. Transversality follows directly from the definition of $\kappa(E)$. ■

Theorem 2 along with Theorem 1 imply the next theorem.

¹The set $\kappa(E)$ is not necessarily a manifold, but for our discussion we will assume it is and that $\pi : \kappa(E) \rightarrow M$ is a submersion.

THEOREM 3. *Suppose $\kappa(E) \rightarrow M$ is a bundle, and that M/G is a manifold; then the space $S(E)^G$ of smooth invariant sections is in one-to-one correspondence with smooth sections of $\kappa(E)/G \rightarrow M/G$.*

The situation is neatly summarized by the diagram,

Kinematic Reduction Diagram

$$\begin{array}{ccccc}
 \tilde{\kappa}(E) & \xleftarrow{\mathfrak{q}_\kappa} & \kappa(E) & \xrightarrow{\iota} & E \\
 \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
 \widetilde{M} & \xleftarrow{\mathfrak{q}_M} & M & \xrightarrow{id} & M
 \end{array}$$

(5)

where $\tilde{\kappa}(E) = \kappa(E)/G$ and $\widetilde{M} = M/G$, and every section of $\tilde{\kappa}(E) \rightarrow \widetilde{M}$ lifts to an invariant section of $E \rightarrow M$.

It is worth emphasizing that the invariant sections of a bundle with a non-transverse group action are parameterized by the bundle $\tilde{\kappa}(E) \rightarrow \widetilde{M}$ whose base and *fibres* are typically of smaller dimension than those of $E \rightarrow M$.

5. Examples Part II - Kinematic Reduction.

EXAMPLE 1. (continued) Let $G = SO(3)$ act on $M = N = \mathbb{R}^3 - 0$ in the standard way, and let E be the total space in the trivial bundle $\pi : M \times N \rightarrow M$. The isotropy $G_{\mathbf{x}}$ at $\mathbf{x} \in M$ is $SO(2)_{\mathbf{x}}$, the rotations about the line containing \mathbf{x} and the origin. The set $\kappa_{\mathbf{x}}(E) \subset E$ is

$$\kappa_{\mathbf{x}}(E) = \{(\mathbf{x}, \mathbf{u}) \mid R\mathbf{u} = \mathbf{u}, R \in SO(2)_{\mathbf{x}}\}$$

and so $\mathbf{u} \propto \mathbf{x}$, or $\mathbf{u} = v\mathbf{x}$, $v \in \mathbb{R}$. The bundle $\kappa(E)$ is a trivial line bundle with coordinates (\mathbf{x}, v) and the inclusion $\kappa(E) \rightarrow E$ given by $(\mathbf{x}, v) \rightarrow (\mathbf{x}, \mathbf{u} = v\mathbf{x})$. The function $r = \sqrt{x^2 + y^2 + z^2}$ defines a global coordinate on M/G . A coordinate description of the kinematic reduction diagram is

$$\begin{array}{ccccc}
 (r, v) & \xleftarrow{\mathfrak{q}_\kappa} & (\mathbf{x}, v) & \xrightarrow{\iota} & (\mathbf{x}, \mathbf{u}) \\
 \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
 (r) & \xleftarrow{\mathfrak{q}_M} & (\mathbf{x}) & \xrightarrow{id} & (\mathbf{x})
 \end{array}$$

(6)

where $\iota(\mathbf{x}, v) = (\mathbf{x}, \mathbf{u} = v\mathbf{x})$. Any invariant section $v = v(r)$ of $\tilde{\kappa}(E) \rightarrow \widetilde{M}$ (the left hand side of (6)) lifts to the invariant section $u(\mathbf{x}) = v(r)\mathbf{x}$ of E .

EXAMPLE 2.B. (continued) Let $G = SO(3) \times (Z_2 \ltimes \mathbb{R})$, where $Z_2 = \{\pm 1\}$, act on $(\mathbb{R}^3 - 0) \times \mathbb{R}$ with local coordinates (x, y, z, t) by

$$(R, \epsilon, \tau)(\mathbf{x}, t) = (R\mathbf{x}, \epsilon t + \tau),$$

where $R \in SO(3)$, $\epsilon \in Z_2$ and $\tau \in \mathbb{R}$. The isotropy group at (\mathbf{x}, t_0) is $G_{(\mathbf{x}, t_0)} = SO(2)_{\mathbf{x}} \times Z_2$ where the Z_2 action is

$$(\mathbf{x}, t) \rightarrow (\mathbf{x}, \epsilon(t - t_0) + t_0) .$$

Let $E = T^*M \odot T^*M$, the bundle of symmetric tensors over M . In this example the set $\kappa_{(\mathbf{x}, t_0)}(E)$, which we wish to compute, is just the set of invariants of the symmetric tensor product of the dual of the linear isotropy representation.

In order to compute $\kappa_{(\mathbf{x}, t_0)}(E)$ we consider each term in the product group $G_{(\mathbf{x}, t_0)}$ separately. For $R \in SO(2)_{\mathbf{x}}$, the condition that $\gamma \in E_{(\mathbf{x}, t_0)}$ be an element of $\kappa_{(\mathbf{x}, t_0)}(E)$ is $R^T \gamma R = \gamma$. Since $SO(2)_{\mathbf{x}}$ is connected this condition can be written infinitesimally as

$$X^T \gamma + \gamma X = 0 \quad \text{where} \quad X = \begin{pmatrix} 0 & z & -y & 0 \\ -z & 0 & x & 0 \\ y & -x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solving these equations gives

$$\gamma = A(xdx + ydy + zdz)^2 + B(dx^2 + dy^2 + dz^2) + Cdt^2 + D(xdx + ydy + zdz) \odot dt$$

where $A, B, C, D \in \mathbb{R}$. The constraint for the Z_2 part of the isotropy is $S^T \gamma S = \gamma$ where $S = \text{diag}(0, 0, 0, \epsilon)$, which implies $D = 0$. Therefore $\kappa_{(\mathbf{x}, t_0)}(E) = (A, B, C)$ and $\kappa(E)$ is a rank 3 vector subbundle of E . The kinematic reduction diagram is

$$(7) \quad \begin{array}{ccccc} (r, A, B, C) & \xrightarrow{\mathfrak{Q}^\kappa} & (\mathbf{x}, t, A, B, C) & \xrightarrow{\iota} & (\mathbf{x}, t, \gamma) \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ (r) & \xrightarrow{\mathfrak{Q}^M} & (\mathbf{x}, t) & \xrightarrow{id} & (\mathbf{x}, t) \end{array}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and $\iota(\mathbf{x}, t, A, B, C) = (\mathbf{x}, t, \gamma = A(rdr)^2 + B(dx^2 + dy^2 + dz^2) + Cdt^2)$. Finally the sections $(A(r), B = B(r), C = C(r))$ of $\tilde{\kappa}(E)$ (the left side of diagram (7)) give rise to the G -invariant symmetric covariant 2-tensors

$$\gamma = A(r)(rdr)^2 + B(r)(dx^2 + dy^2 + dz^2) + C(r)dt^2.$$

6. The Reduced Equations.

Let $J^k(E)$ be the bundle of k -jets of sections of E and let G act projectably on E . Then G acts on $J^k(E)$ by

$$g \cdot j^k(s)(x) = j^k(g \cdot s)(gx)$$

where $j^k(s)$ is the k^{th} -jet of a section $s \in S(E)$. A k^{th} -order differential operator Δ is a section of a vector-bundle $\mathcal{D} \rightarrow J^k(E)$. A solution to the differential equations $\Delta = 0$ is a section $s : M \rightarrow E$ such that $\Delta \circ j^k(s) = 0$. Denote the solutions to Δ by the subset $S_\Delta(E) \subset S(E)$.

If G is a group acting projectably on the bundle $\mathcal{D} \rightarrow J^k(E)$, then G is a symmetry group of Δ if $\Delta : J^k(E) \rightarrow \mathcal{D}$ is a G -invariant section. It is easy to verify

LEMMA 1. *If G is a symmetry group of Δ , then for any $g \in G$ and $s \in S_\Delta(E)$, $g \cdot s \in S_\Delta(E)$.*

In other words if G is a symmetry group of Δ then the subset $S_\Delta(E) \subset S(E)$ is G -invariant. The fixed point set $S_\Delta(E)^G$ of the action of G on $S_\Delta(E)$ leads to the definition

DEFINITION 4. Let G be a symmetry group for Δ . A solution

$$s \in S_{\Delta}(E) \cap S(E)^G = S_{\Delta}(E)^G$$

is called a *G -invariant solution*.

We now determine a reduced differential operator $\tilde{\Delta}$ whose solutions determine all the G -invariant solutions to Δ . Every G -invariant solution to a differential equation is an element of $S(E)^G$, which by Theorem 3 can be identified with a section of $\tilde{\kappa}(E) \rightarrow \tilde{M}$. Therefore the reduced differential operator will be a differential operator on $J^k(\tilde{\kappa}(E))$ and define a section of some vector-bundle $\tilde{D} \rightarrow J^k(\tilde{\kappa}(E))$.

It is **not** possible to directly apply the theory developed in section 4 to find the reduced differential operator and bundle \tilde{D} for Δ . We need to introduce the so-called invariant jet space $Inv^k(E)$. The set $Inv^k(E) \subset J^k(E)$ is defined by

$$Inv^k(E) = \{ \sigma \in J^k(E), \text{ where } \sigma = j^k(s)(x) \text{ for some } s \in S(E)^G \}.$$

We list some facts about the invariant jet space.

LEMMA 2. *Let G act projectably on E , then*

- a) $Inv^k(E) \subset J^k(E)$ is G -invariant.
- b) G acts transversally on $Inv^k(E) \rightarrow M$.
- c) Every $j^k(s) : M \rightarrow J^k(E)$, for $s \in S(E)^G$ factors through $Inv^k(E)$.

LEMMA 3. *If $\kappa(E)$ is a bundle and M/G is a manifold then*

- a) $Inv^0(E)$ is diffeomorphic to $\kappa(E)$.
- b) $Inv^k(E) \rightarrow M$ is equivalent to the pullback bundle $q_M^* J^k(\tilde{\kappa}(E))$ where $q_M : M \rightarrow M/G$.

Suppose that the conditions in Lemma 3 hold, let $i : Inv^k(E) \rightarrow J^k(E)$ be the inclusion map, and let $\mathcal{D}_i \rightarrow Inv^k(E)$ be the restriction (or pullback) of the vector-bundle $\mathcal{D} \rightarrow J^k(E)$ to $Inv^k(E)$. Then we have

LEMMA 4. *The group G acts on the sub-bundle $i : Inv^k(E) \rightarrow J^k(E)$ and Δ defines an invariant section of the restricted bundle $\mathcal{D}_i \rightarrow Inv^k(E)$.*

At this point we apply the theory of section 4 to the bundle $\mathcal{D}_i \rightarrow Inv^k(E)$ to obtain the left side in the diagram

Dynamic Reduction Diagram

$$\begin{array}{ccccccc}
 \kappa(\mathcal{D}_i)/G & \xleftarrow{q} & \kappa(\mathcal{D}_i) & \xrightarrow{\iota} & \mathcal{D}_i & \xrightarrow{i} & \mathcal{D} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 J^k(\tilde{\kappa}(E)) & \xleftarrow{q} & Inv^k(E) & \xrightarrow{id} & Inv^k(E) & \xrightarrow{i} & E
 \end{array}$$

(8)

Theorem 3 states that the invariant sections of $\mathcal{D}_i \rightarrow Inv^k(E)$ are in one to one correspondence with sections of $\kappa(\mathcal{D}_i)/G \rightarrow J^k(\tilde{\kappa}(E))$. Given an invariant differential operator Δ , we call the operator $\tilde{\Delta}$ obtained through this correspondence the *reduced operator*.

7. Examples Part III - Dynamic Reduction.

EXAMPLE 1. (continued) The Euler-Equations for an ideal fluid. Let $G = SO(3)$ act on $M = N = (\mathbb{R}^3 - 0) \times \mathbb{R}$ on the $\mathbb{R}^3 - 0$ term in the standard way, and let E be the total space of the trivial bundle $\pi : M \times N \rightarrow M$. We use (\mathbf{x}, t) and (\mathbf{u}, p) as coordinates on M and N respectively. The standard coordinates on $J^1(E)$ are $(\mathbf{x}, t, \mathbf{u}, p; u^i, p_j, p_t)$, $i, j = 1, 2, 3$.

Let $\mathcal{D} = \mathbb{R}^4 \times J^1(E)$, a trivial rank 4 vector bundle. We use (Δ^i, Δ^4) , $i = 1, 2, 3$ for fibre coordinates, and take for the action of G on the fibres $R(\Delta^i, \Delta^4) = (R_j^i \Delta^j, \Delta^4)$, $R \in SO(3)$. The Euler-equations are given by the vanishing of the differential operator $\Delta : J^1(E) \rightarrow \mathcal{D}$

$$\begin{aligned}\Delta^i &= u_t^i + u^i_{,j} u^j + \delta^{ij} (p_{,j}) \quad i = 1 \dots 3 \\ \Delta^4 &= u^j_{,j}\end{aligned}$$

where δ^{ij} are the components of the three dimensional (contravariant) Euclidean metric in standard coordinates.

To find the reduced operator we first determine $Inv^1(E) \rightarrow J^1(E)$ using a slight extension of diagram (6). The invariant sections of E are easily seen to be $S(E)^G = \{\mathbf{u} = v(r, t)\mathbf{x}, p = p(r, t)\}$, where $r = \sqrt{x^2 + y^2 + z^2}$. The coordinates on $Inv^1(E)$ are then $(\mathbf{x}, t, v, p, v_r, v_t, p_r, p_t)$ and the inclusion $i : Inv^1(E) \rightarrow J^1(E)$ in coordinates is

$$\begin{aligned}i : (\mathbf{x}, t, v, p, v_r, v_t, p_r, p_t) &\rightarrow \\ (\mathbf{x}, t, \mathbf{u} = v\mathbf{x}, p, u^i_{,j} = r^{-1}v_r x^i x^j + v\delta_j^i, u_t^i = v_t x^i, p_j = r^{-1}p_r x^j, p_t = p_t) &.\end{aligned}$$

To finally determine the reduced operator we restrict Δ to $Inv^1(E)$ which gives

$$\begin{aligned}\Delta_{Inv}^i &= (v_t + v(v + rv_r) + r^{-1}p_r) x^i \\ \Delta_{Inv}^4 &= 3v + rv_r .\end{aligned}$$

From this we conclude that $\kappa(\mathcal{D}_i)$ is a rank 2 bundle, and the components of the reduced operator are

$$\tilde{\Delta} = (v_t + v(v + rv_r) + r^{-1}p_r, 3v + rv_r) .$$

EXAMPLE 2.B. (continued) The Schwarzschild solution. Here $G = SO(3) \times Z_2 \times \mathbb{R}$, $M = (\mathbb{R}^3 - 0) \times \mathbb{R}$, and $E = T^*M \odot T^*M$. Using $\Pi : J^2(E) \rightarrow M$, let $\mathcal{D} = \Pi^*E$ be the pullback bundle and let G act on \mathcal{D} in the natural way. In this example the differential operator $\Delta : J^2(E) \rightarrow \mathcal{D}$ is

$$\Delta = R_{ij} dx^i \odot dx^j$$

where R_{ij} is the Ricci-tensor considered as a function of a symmetric covariant two tensor and its first and second derivative. The equations $\Delta = 0$ are the vacuum Einstein equations, or Ricci-flat conditions.

The differential operator Δ takes values in $T^*M \odot T^*M$, so the bundle $\kappa(\mathcal{D}_i)$ will have the same structure as $\kappa(E)$ (see (7)). Thus restricting Δ to $Inv^2(E)$ we find

$$\Delta_{Inv} = \tilde{\Delta}_A (xdx + ydy + zdz)^2 + \tilde{\Delta}_B (dx^2 + dy^2 + dz^2) + \tilde{\Delta}_C dt^2 ,$$

where $\tilde{\Delta}_A, \tilde{\Delta}_B$, and $\tilde{\Delta}_C$ are the components of the reduced operator. Solving the reduced equations $\tilde{\Delta}_{A,B,C} = 0$ leads to the Schwarzschild solution.

8. Quotients with boundary.

In a number of applications, such as the cohomogeneity one reductions of the harmonic map and Einstein equations [4],[7], the quotient M/G is a manifold with boundary and consequently Theorem 3 does not hold and a general theory for the parameterization of the space $S(E)^G$ is unknown.

EXAMPLE 4. Let $G = SO(2)$, $M = N = S^2 \subset \mathbb{R}^3$, and let E be the total space in the trivial bundle $\pi : M \times N \rightarrow M$. The action will be given rotation about the z axis on both S^2 . The action is not transverse because at the poles $(0, 0, \pm 1) \in M$ we have $G_{(0,0,\pm 1)} = SO(2)$, which acts non trivially on the S^2 fibre. The isotropy condition at $(0, 0, \pm 1)$ leads to

$$(9) \quad \kappa_{(0,0,\pm 1)}(E) = (0, 0, \pm 1) .$$

Therefore any $SO(2)$ equivariant map from S^2 to S^2 must take the poles to the poles. The quotient $M/G = [-1, 1]$ and condition (9) would lead to constraints on the boundary conditions for any group invariant solution having this symmetry. See [1] for an explicit application to harmonic maps.

The above example shows that we can use the κ “functor” to obtain information about the space of invariant sections of E when M/G is a manifold with boundary. Using the projection maps $\pi^k : J^k(E) \rightarrow M$ we can construct $\kappa(J^k(E))$, where $\kappa_x(J^k(E)) = (J_x^k(E))^{G_x}$. It is clear that $Inv^k(E) \subset \kappa(J^k(E))$ and so $\kappa(J^k(E))$ constrains the k -jet of any invariant section. It is also possible to define $\kappa(J^k(E))$ inductively using the projection maps $\pi_{k-1}^k : J^k(E) \rightarrow J^{k-1}(E)$. Let $\sigma \in \kappa(J^{k-1}(E))$ then

$$(10) \quad \kappa_\sigma(J^k(E)) = ((\pi_{k-1}^k)^{-1}(\sigma))^{G_\sigma} .$$

Note that $\kappa(J^0(E)) = \kappa(E)$.

EXAMPLE 5. Let $G = SO(n)$, $M = \mathbb{R}^n$, and $E = M \times \mathbb{R}$ so $S(E)^G$ is the space of smooth $SO(n)$ -invariant functions on \mathbb{R}^n , and $M/G = [0, \infty)$. Unlike the case when M/G is a manifold, the space $S(E)^G$ cannot be identified with the smooth functions $C^\infty(M/G)$.

Let $\sigma \in J^{k-1}(E)$, with $\pi^{k-1}(\sigma) = \mathbf{0}$ the origin \mathbb{R}^n . We identify

$$(\pi_{k-1}^k)^{-1}(\sigma) = \odot^k(\mathbb{R}^n) ,$$

which represents the k^{th} order coefficients of the Taylor polynomial of a smooth-function f defined in a neighbourhood of $\mathbf{0}$. We compute $\kappa_{\mathbf{0}}(J^k(E))$ using (10). The stabilizer at $\mathbf{0}$ is $G_{\mathbf{0}} = SO(n)$ and for $\sigma \in \kappa(J^{k-1}(E))$ with $\pi^{k-1}(\sigma) = \mathbf{0}$ we have

$$\kappa_\sigma(J^k(E)) = (\odot^k \mathbb{R}^n)^{SO(n)} ,$$

where the $SO(n)$ action is the k^{th} order symmetric tensor product of the standard action of $SO(n)$ on \mathbb{R}^n . It is well known [11] that $(\odot^k \mathbb{R}^n)^{SO(n)} = 0$ for k odd. These necessary conditions are easily derived in even dimensions from the fact that if $f \in S(E)^G$ then $f(-\mathbf{x}) = f(\mathbf{x})$, or $f(r)$ is even.

For examples where these conditions can be used to guarantee smoothness see [5] or [10].

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