SYMMETRY REDUCTION OF VARIATIONAL BICOMPLEXES AND THE PRINCIPLE OF SYMMETRIC CRITICALITY

Ian M. Anderson
Department of Mathematics
Utah State University
Logan, Utah 84322
United States[1]

and

Mark E. Fels
School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455
United States[2]

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ABSTRACT

Consider a system of differential equations $\Delta = 0$ which is invariant under a Lie group G of point transformations acting on the space E of independent and dependent variables. By a method due to Lie, the G invariant solutions of these differential equations are found by solving a reduced system of differential equations $\overline{\Delta} = 0$ on the space $\overline{E}$ of invariants of G. In this paper we explore the relationship between the G invariant conservation laws and variational principles for the system of equations $\Delta = 0$ and the conservation laws and variational principles for the reduced equations $\overline{\Delta} = 0$. This problem translates into one of constructing a certain cochain map $\varrho_{\mathcal{X}}$ between the G invariant variational bicomplex for the infinite jet space on E and the free variational bicomplex for $\overline{E}$. We prove that such a cochain map exists locally if and only if the relative Lie algebra cohomology condition

$$H^q(\Gamma, \Gamma_0(e_0)) \neq 0$$

is satisfied, where q is the orbit dimension of G, Γ the Lie algebra of vector fields on E which generate the infinitesimal action of G, and $\Gamma_0(e_0)$ the linear isotropy subalgebra of Γ at e_0. As a simple consequence we prove that the vanishing of $H^q(\Gamma, \Gamma_0(e_0))$ is the only local obstruction to Palais' principle of symmetric criticality.

Key Words and Phrases: variational bicomplexes, group invariant solutions to differential equations, symmetry reduction, unimodular Lie groups

§**1. Introduction.** When a system of partial differential equations is invariant under a Lie group G of point transformations, one can construct a reduced system of partial (or ordinary) differential equations whose local solutions correspond precisely to the local G invariant solutions of the original system of equations. This simple and effective construction, pioneered by Lie himself, is still one of the most powerful and extensively used techniques for finding exact solutions of many non-linear partial differential equations [*4*], [*15*], [*23*]. In this paper we explore the relationship between the G invariant conservation laws and variational principles for a system of partial differential equations and the conservation laws and variational principles of the reduced system of equations for the G invariant solutions. Since conservation laws and variational principles for differential equations are naturally represented by certain cohomology classes in the variational bicomplex associated to the given system of partial differential equations [*2*], [*22*] our problem translates into that of constructing a cochain map between certain variational bicomplexes. Remarkably, and contrary to the usual spirit of symmetry reduction, these cochain maps do not always exist, even locally. We are able to completely classify the local obstructions to the existence of such cochain maps.

To better describe our main results, let

$$\Delta(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots) = 0 \tag{1.1}$$

be a system of k-th order partial differential equations for m unknown functions u^α of n independent variables x^i. The first, second and higher order partial derivatives of u^α with respect to x^i, x^j, x^k, $\dots$ are denoted by u^α_i, u^α_{ij}, u^α_{ijk}, $\dots$. A classical conservation law for (1.1) is a current (that is, a vector density)

$$V = V^k(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)\frac{\partial}{\partial x^k} \tag{1.2}$$

whose total divergence

$$\operatorname{Div} V = \frac{dV^k}{dx^k} = 0 \qquad \text{on solutions to } \Delta = 0. \tag{1.3}$$

For the purposes of this paper, it will be convenient to express the conservation law (1.2) as an $(n-1)$-form

$$\omega = \sum_{k=1}^{n} (-1)^k V^k\, dx^1 \wedge dx^2 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n, \tag{1.4}$$

in which case (1.3) can be re-expressed in terms of the total exterior derivative of ω as

$$D\omega = \frac{dV^k}{dx^k} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = 0 \qquad \text{on solutions to } \Delta = 0.$$

We are also interested in variational principles for (1.1). The equations (1.1) are variational if there is an action functional

$$I[u] = \int_{\mathcal{D}} L(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)\, dx^1 dx^2 \dots dx^n, \tag{1.5}$$

where $\mathcal{D}$ is a region in $\mathbf{R}^n$, whose Euler-Lagrange equations

$$E_\alpha(L) = \frac{\delta L}{\delta u^\alpha} = \frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i}\frac{\partial L}{\partial u^\alpha_i} + \frac{d}{dx^i}\frac{d}{dx^j}\frac{\partial L}{\partial u^\alpha_{ij}} - \cdots \tag{1.6}$$

coincide with (1.1). We call the function L the Lagrangian function for (1.1) and the n-form

$$\lambda = L(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n \tag{1.7}$$

the associated Lagrangian n-form. It is both natural and very convenient for the theory of symmetry reduction presented in this paper to treat conservation laws and variational principles as differential forms of degrees $n-1$ and n respectively.

Suppose now that G is a p dimensional, connected Lie transformation group acting on the space E of independent and dependent variables (x^i, u^α) and that G is a symmetry group for $\Delta = 0$, that is, the action of G on E maps the graph of any solution to (1.1) to the graph of another solution. For simplicity, we make several assumptions about the action of G. We suppose that there is an induced action of G on the space M of independent variables, that the orbits of G all have fixed dimension q and that the action of G commutes with the projection $\pi\colon (x^i, u^\alpha) \to (x^i)$. Granted these assumptions, we can construct new independent coordinates $(\widehat{x}^i, y^r)$, $i = 1, 2, \dots, q$, and $r = 1, 2, \dots, n-q$ and new dependent coordinates (v^α), $\alpha = 1, 2, \dots, m$ such that the functions (y^r, v^α) are all G invariant. The functions (y^r, v^α) also serve as local coordinates for the quotient space $\overline{E}$ of E by the orbits of G. By substituting $u^\alpha = v^\alpha(y^r)$ into (1.1), we obtain the reduced differential equations

$$\overline{\Delta}\,(y^r, v^\alpha, v^\alpha_r, v^\alpha_{rs}, \dots) = 0 \tag{1.8}$$

for the G invariant solutions to (1.1). For details of this construction, as well as a wide variety of examples, see [*4*], [*15*], and [*23*]. We remark that the variables $\widehat{x}^i$ are referred to as the *parametric variables* and the derivatives v^α_i, v^α_{ij}, v^α_{ir}, … , which do not appear in (1.8), are referred to as the *parametric derivatives.*

The *associated infinitesimal Lie transformation group* Γ is the real vector space of vector fields on E whose flows coincide with the action of the 1 parameter subgroups of G on E.

We now suppose that the conservation law ω for (1.1), given by (1.4), and the Lagrangian form λ for (1.1), given by (1.7), are invariant under the Lie group G where G is again a symmetry group for (1.1). The forms ω and λ are G invariant, if for every vector field

$$X = \sum_{i=1}^{n} A^l(x^i)\frac{\partial}{\partial x^l} + \sum_{\beta=1}^{m} B^\beta(x^i, u^\alpha)\frac{\partial}{\partial u^\beta}$$

in Γ, we have

$$\mathcal{L}_{\mathrm{pr}\,X}\omega = 0 \qquad \text{and} \qquad \mathcal{L}_{\mathrm{pr}\,X}\lambda = 0,$$

where $\mathrm{pr}\,X$ is the prolongation of X to the jet spaces of E (see equation 2.23). In terms of the components V^k and L of ω and λ, these symmetry conditions are

$$\mathrm{pr}\,X(V^k) - \frac{\partial A^k}{\partial x^l}V^l + \frac{\partial A^l}{\partial x^l}V^k = 0 \qquad \text{and} \qquad \mathrm{pr}\,X(L) + \frac{\partial A^l}{\partial x^l}L = 0.$$

The general philosophy underlying the principle of symmetry reduction ([*9*], [*14*]) is that G invariant objects associated to E ought to descend, in some canonical fashion, to objects on the quotient space $\overline{E}$. Accordingly, it is natural to ask if the G invariant conservation law ω and the G

invariant Lagrangian λ can be used to construct a conservation law ϖ and a variational principle $\overline{\lambda}$ for the reduced equation (1.8). In this paper we shall determine precisely those infinitesimal group actions Γ for which this is always the case.

The construction of the reduced conservation law ϖ and the reduced Lagrangian $\overline{\lambda}$ can be described roughly as follows. First, we re-write the original forms ω and λ in terms of the G adapted variables $(\widehat{x}^i, y^r, v^\alpha)$ as

$$\omega = \big[\sum_{r=1}^{n-q} \overline{V}^r\, dy^1 \wedge \cdots \widehat{dy^r} \wedge \cdots \wedge dy^{n-q}\big] d\widehat{x}^1 \wedge d\widehat{x}^2 \wedge \cdots \wedge d\widehat{x}^q$$
$$+ \big[\sum_{i=1}^{q} \overline{W}^i\, d\widehat{x}^1 \wedge \cdots \widehat{d\widehat{x}^i} \wedge \cdots \wedge dx^q\big] dy^1 \wedge dy^2 \wedge \cdots \wedge dy^{n-q}$$

and

$$\lambda = \big[\overline{L}\, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^{n-q}\big] d\widehat{x}^1 \wedge d\widehat{x}^2 \wedge \cdots \wedge d\widehat{x}^q,$$

where the coefficients $\overline{V}^r$, $\overline{W}^i$ and $\overline{L}$ are functions of the G adapted coordinates

$$\widehat{x}^i, y^r, v^\alpha, v_i^\alpha, v_r^\alpha, v_{ij}^\alpha, v_{ir}^\alpha, v_{rs}^\alpha \dots .$$

One might suspect that the coefficients of $d\widehat{x}^1 \wedge d\widehat{x}^2 \wedge \cdots \wedge d\widehat{x}^q$ in ω and λ would yield the conservation law and variational principle for the reduced equations but such is not the case. A slight modification of this construction is needed and, to this end, we shall introduce a certain system of linear homogeneous differential equations

$$\Xi(\widehat{x}^i, y^r, J_0, \frac{\partial J_0}{\partial \widehat{x}^i}, \frac{\partial J_0}{\partial y^r}) = 0 \tag{1.9}$$

for a scalar function $J_0 = J_0(\widehat{x}^r, y^r)$. The system of equations (1.9) will be explicitly constructed in sections 3 and 4 of the paper. For now, we simply remark that this system is readily determined from the generators X_a, $a = 1, 2, \dots, p$, of the infinitesimal transformation group Γ. We then define the reduced conservation law ϖ and $\overline{\lambda}$ by

$$\varpi = J_0 \sum_{r=1}^{n-q} \overline{V}^r_{\rm inv}\, dy^1 \wedge \cdots \wedge \widehat{dy^r} \wedge \cdots \wedge dy^{n-q} \tag{1.10}$$

and

$$\lambda = J_0\, \overline{L}_{\rm inv}\, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^{n-q}, \tag{1.11}$$

where $\overline{V}^r_{\rm inv}$ and $\overline{L}_{\rm inv}$ indicate that the coefficients $\overline{V}^r$ and $\overline{L}$ are evaluated with all the parametric derivatives v_i^α, v_{ij}^α, v_{ir}^α, $\dots$, set to zero. Our main result implies that *if J_0 is any solution to (1.9), then for any G invariant system of equations $\Delta = 0$ and any G invariant conservation law ω, the form ϖ, as given by (1.11), will always be a conservation law for the reduced equations $\overline{\Delta} = 0$. Likewise, if λ is a G invariant Lagrangian for $\Delta = 0$, then $\overline{\lambda}$ will always be a Lagrangian for $\overline{\Delta} = 0$.* Moreover, we show that the integrability conditions which obstruct the existence of solutions to

(1.9) may be explicitly described solely in terms of the structure constants C^a_{bc} of Γ. For example, in the case when the action of G on E is free (so that the dimensions of the orbits of G agree with the dimension of G), the integrability conditions for (1.9) will be shown to be

$$\sum_{a=1}^{p} C^a_{ac} = 0. \tag{1.12}$$

This condition is equivalent to the existence of a bi-invariant volume form on G — such Lie groups are said to be unimodular.

Before turning to some elementary examples of symmetry reduction of conservation laws and variational principles, it is helpful to present the reduction formulas (1.10) and (1.11) in a more invariant geometric context. Recall that a q multi-vector $\mathcal{X}$ on the manifold M is simply an alternating type $(q, 0)$ tensor field, which we may write in local coordinates as

$$\mathcal{X} = X^{i_1 i_2 \cdots i_q} \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_q}}. \tag{1.13}$$

Then, if ω is any p-form on M, given locally by

$$\omega = a_{j_1 j_2 \cdots j_p}\, dx^{j_1} \wedge dx^{j_2} \wedge \cdots dx^{j_p},$$

we can define the $(p - q)$-form $\omega(\mathcal{X})$ by

$$\begin{aligned} \omega(\mathcal{X}) &= X^{i_1 i_2 \cdots i_q}\, \omega(\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_q}}) \\ &= (X^{i_1 i_2 \cdots i_q}\, a_{i_1 i_2 \cdots i_q j_{q+1} \cdots j_p})\, dx^{j_{q+1}} \wedge dx^{j_{q+2}} \wedge \cdots \wedge dx^{j_p}. \end{aligned} \tag{1.14}$$

Given a Lie algebra Γ of vector fields on M, we say that *the multi-vector $\mathcal{X}$ is a q-chain on* Γ if we can express (1.13) in the form

$$\mathcal{X} = J^{a_1 a_2 \cdots a_q} X_{a_1} \wedge X_{a_2} \wedge \cdots \wedge X_{a_q}, \tag{1.15}$$

where the vectors X_{a_1}, X_{a_2}, $\ldots$ X_{a_p} belong to Γ. In terms of the adapted variables $(\widehat{x}^i, y^r)$ we can write every vector field $X \in \Gamma$ in the form

$$X = \sum_{i=1}^{q} X^i(\widehat{x}^j, y^r) \frac{\partial}{\partial \widehat{x}^i}$$

and hence every q-chain on Γ, where q is the orbit dimension of G, assumes the form

$$\mathcal{X} = J_0 \frac{\partial}{\partial \widehat{x}^1} \wedge \frac{\partial}{\partial \widehat{x}^2} \wedge \cdots \wedge \frac{\partial}{\partial \widehat{x}^q}. \tag{1.16}$$

Under these circumstances, the reduced forms $\overline{\omega}$ and $\overline{\lambda}$ are simply given by

$$\overline{\omega} = [\omega(\mathcal{X})]_{\text{inv}} \qquad \text{and} \qquad \overline{\lambda} = [\lambda(\mathcal{X})]_{\text{inv}}.$$

In order that this construction to be meaningful, that is, in order for $\overline{\omega}$ and $\overline{\lambda}$ to be independent of the parametric variables $\widehat{x}^i$, it is necessary that $\overline{\omega}(\mathcal{X})$ and $\overline{\lambda}(\mathcal{X})$ be G invariant and this, in turn, requires that the q-chain $\mathcal{X}$ itself be G invariant. This accounts for some but not all of the differential conditions (1.9) for J_0. We shall find that not all group actions admit such an invariant q-chain (see Proposition 3.6) but even the existence of an invariant chain $\mathcal{X}$ is not sufficient to insure that $\omega(\mathcal{X})$ will always be a conservation law.

The issues we wish to address in this paper can be nicely illustrated by Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{1.17}$$

Consider first the symmetry group SO(3) with infinitesimal generators

$$X = z\partial_y - y\partial_z, \qquad Y = -z\partial_x + x\partial_z, \qquad Z = y\partial_x - x\partial_y. \tag{1.18}$$

The conservation law $V_1 = (u_x, u_y, u_z)$, or equivalently

$$\omega_1 = u_z dx \wedge dy - u_y dx \wedge dz + u_x dy \wedge dz,$$

and the Lagrangian

$$\lambda = \frac{1}{2}\,(u_x^2 + u_y^2 + u_z^2)\, dx \wedge dy \wedge dz \tag{1.19}$$

are both invariant under the prolonged action of SO(3). Under the change of coordinates

$$\widehat{x} = x, \quad \widehat{y} = y, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad v = u, \tag{1.20}$$

where r and u are SO(3) invariant functions on E, the derivatives of u transform as

$$u_x = v_{\hat{x}} + \frac{\widehat{x}}{r} v_r, \quad u_y = v_{\hat{y}} + \frac{\widehat{y}}{r} v_r, \quad u_z = \frac{z}{r} v_r \tag{1.21}$$

and

$$u_{xx} = v_{\hat{x}\hat{x}} + \frac{2\widehat{x}}{r} v_{\hat{x}r} + \frac{\widehat{x}^2}{r^2} v_{rr} + \frac{r^2 - x^2}{r^3} v_r,$$

$$u_{yy} = v_{\hat{y}\hat{y}} + \frac{2\widehat{y}}{r} v_{\hat{y}r} + \frac{\widehat{y}^2}{r^2} v_{rr} + \frac{r^2 - y^2}{r^3} v_r, \qquad v_{zz} = \frac{z^2}{r^2} v_{rr} + \frac{r^2 - z^2}{r^3} v_r.$$

In these coordinates, Laplace's equation becomes

$$v_{\hat{x}\hat{x}} + v_{\hat{y}\hat{y}} + v_{rr} + \frac{2\widehat{x}}{r} v_{\hat{x}r} + \frac{2\widehat{y}}{r} v_{\hat{y}r} + \frac{2}{r} v_r = 0$$

so that, by setting $v_{\hat{x}} = v_{\hat{y}} = 0$, we obtain the well-known reduced equation

$$v_{rr} + \frac{2}{r} v_r = 0 \tag{1.22}$$

whose solutions characterize the rotationally invariant solutions of Laplace's equation in 3 dimensions. Let $\mathcal{X}$ be the SO(3) invariant 2-chain

$$\mathcal{X} = \frac{r}{z} X \wedge Y = r(z\,\partial_x \wedge \partial_y - y\,\partial_x \wedge \partial_z + x\,\partial_y \wedge \partial_z). \tag{1.23}$$

Then the evaluation of ω_1 and λ on $\mathcal{X}$ leads to the SO(3) invariant forms

$$\omega_1(\mathcal{X}) = r(xu_x + yu_y + zu_z)$$

and

$$\lambda(\mathcal{X}) = \frac{1}{2} r(u_x^2 + u_y^2 + u_z^2)(x\,dx + y\,dy + z\,dz)$$

which reduce, on substituting from (1.21) and setting $v_{\hat{x}} = v_{\hat{y}} = 0$, to the forms

$$\overline{\omega}_1 = r^2 v_r \qquad \text{and} \qquad \overline{\lambda} = \frac{1}{2} r^2 v_r^2\, dr.$$

The form $\overline{\omega}_1$ is a conservation law (or first integral) for (1.22) while equation (1.22) is the Euler-Lagrange equation for $\overline{\lambda}$. Other SO(3) invariant conservation laws for (1.17) include

$$\omega_2 = \frac{u}{r^3}\,[z\,dx \wedge dy - ydx\, \wedge dz + x\,dy \wedge dz] + \frac{1}{r}\omega_1$$

and

$$\begin{aligned} \omega_3 = &[\frac{1}{2} z(u_z^2 - u_x^2 - u_y^2) + u_z(\frac{u}{2} + yu_y + xu_x)]\, dx \wedge dy \\ &-[\frac{1}{2} y(u_y^2 - u_x^2 - u_z^2) + u_y(\frac{u}{2} + xu_x + zu_z)]\, dx \wedge dz \\ &+[\frac{1}{2} x(u_x^2 - u_y^2 - u_z^2) + u_x(\frac{u}{2} + yu_y + zu_z)]\, dy \wedge dz \end{aligned} \tag{1.24}$$

and these lead, by this same procedure, to the first integrals

$$\overline{\omega}_2 = rv_r + v \quad \text{and} \quad \overline{\omega}_3 = \frac{1}{2}(r^3 v_r^2 + r^2 v v_r)$$

for (1.22). Our general theory will show that if ω is any SO(3) invariant conservation law (depending on derivatives of any order) for any system of partial differential equations with rotational symmetry, then $[\omega(\mathcal{X})]_{\rm inv}$ is always a conservation law for the reduced equations. Moreover, the chain (1.23) is essentially the only SO(3) invariant chain that can be used in this fashion. It is important to acknowledge the fact that the chain $\mathcal{X}$ is constructed over the Lie algebra (1.18) with coefficients depending on the independent variables. We shall see that similar results also hold for the standard action of SO(n) on $\mathbf{R}^n$, where the appropriate invariant $(n-1)$-chain (see Example 5.3) is

$$\mathcal{X} = r^{n-2} \sum_{i=1}^{n} (-1)^{(i+1)} x^i \partial_{x^1} \wedge \cdots \wedge \widehat{\partial_{x^i}} \wedge \cdots \wedge \partial_{x^n}. \tag{1.25}$$

As a practical application, we can also conclude that the reduced equations for any system of Euler-Lagrange equations derivable from an SO(n) invariant Lagrangian can be more efficiently derived (in the sense that fewer derivatives in terms the adapted invariant coordinates need to be explicitly determined) by first reducing the Lagrangian.

However, now consider the 2 dimensional solvable group G of translations and scalings generated by

$$T = \partial_y, \quad \text{and} \quad S = x\partial_x + y\partial_y + z\partial_z - \frac{1}{2} u\partial_u.$$

This group is also a symmetry group of Laplace's equation. The two form

$$\omega = [\frac{1}{2}\, x(u_x^2 + u_y^2 - u_z^2) + z u_x u_z]\, dx \wedge dy + u_y(x u_z - z u_x)\, dx \wedge dz$$
$$+ [\frac{1}{2}\, z(u_x^2 - u_y^2 - u_z^2) - x u_x u_z]\, dy \wedge dz.$$

is a G invariant conservation law for Laplace's equation and the Lagrangian (1.19) is again G invariant. In this example the G invariant functions on E are $w = \dfrac{z}{x}$ and $v = u\sqrt{x}$ and from the change of variables

$$x = \widehat{x}, \quad y = \widehat{y}, \quad z = \widehat{x} w, \quad u = \frac{v}{\sqrt{x}}$$

we compute

$$u_x = \frac{1}{\widehat{x}^{1/2}}\, v_{\hat{x}} - \frac{w}{\widehat{x}^{3/2}}\, v_w - \frac{1}{2\widehat{x}^{3/2}}\, v, \quad u_y = \frac{1}{\widehat{x}^{1/2}}\, v_{\hat{y}}, \quad u_z = \frac{1}{\widehat{x}^{3/2}}\, v_w.$$

The reduced equation for the G invariant solutions of Laplace's equation is now

$$(w^2 + 1) v_{ww} + 3 w v_w + \frac{3}{4} v = 0. \tag{1.26}$$

The most general G invariant 2-chain in the Lie algebra generated by the vector fields T and S is

$$\mathcal{X} = x f(w, v)\, T \wedge S$$

and we compute the reductions of ω and λ to be

$$\overline{\omega} = \frac{1}{2} f(v, w)[(w^2 + 1)^2 v^2 + w(w^2 + 1) v v_w + \frac{1}{4}(w^2 - 1) v^2]$$

and

$$\overline{\lambda} = \frac{1}{2} f(v, w)[(w^2 + 1) v_w^2 + w v v_w + \frac{1}{4} v^2]\, dw.$$

Direct computation shows there is *no* possible nonzero choice of the function f such that $\overline{\omega}$ will be a conservation law for (1.26) or that $\overline{\lambda}$ will be a Lagrangian for (1.26). Our general theory will assert this to be the case for any freely acting, two dimensional, non-abelian transformation group G – there does not exist an invariant 2-chain (even with derivative dependent coefficients) that drops conservation laws for an arbitrary partial differential equation with symmetry group G to conservation laws for the reduced equation. The integrability conditions (1.12) for the equations (1.9) which determine the admissible chains $\mathcal{X}$ are not satisfied. For such groups one cannot correctly determine the reduced equations as the Euler-Lagrange equations of a reduced Lagrangian. In fact we shall give an example (Example 5.5) of an Euler-Lagrange equation whose reduction admits no local variational principle whatsoever.

These foregoing conclusions concerning the existence of variational principles for the reduced equations are closely related to Palais' *principle of symmetric criticality* [*18*], [*19*]. To explain this connection, let us formulate this principle, at least for classical variational problems, within the context of Lie's theory of symmetry reduction. Suppose, then, that the differential equations (1.1) are the Euler-Lagrange equations for the fundamental integral (1.5) and that the associated Lagrangian form (1.7) is G invariant. In terms of the new independent variables $(\widehat{x}^i, y^r)$ and dependent variables (v^α), we can express this functional as

$$I[v] = \int_{\overline{\mathcal{R}}} \overline{L}(\widehat{x}^i, y^r, v^\alpha, v^\alpha_i, v^\alpha_r, v^\alpha_{ij}, v^\alpha_{ir}, v^\alpha_{rs}, \dots) dy^1 \cdots dy^{n-q} d\widehat{x}^1 \cdots d\widehat{x}^q.$$

The *symmetric variations* of $I[v]$ are given by $\delta v^\alpha = h^\alpha(y^r)$ and the *transverse variations* by $\delta v^\alpha = h^\alpha(\widehat{x}^i)$. The principle of symmetric criticality then asserts that if the G invariant functions $v^\alpha = v^\alpha(y^r)$ are local extrema with respect to the symmetric variations, that is,

$$\frac{d}{d\epsilon} I[v + \epsilon h] = 0 \qquad \text{for all} \quad h^\alpha = h^\alpha(y^r), \tag{1.27}$$

then they are local extrema with respect to all variations

$$\frac{d}{d\epsilon} I[v + \epsilon h] = 0 \qquad \text{for all} \quad h^\alpha = h^\alpha(\widehat{x}^i, y^r)$$

and are therefore solutions to the original Euler-Lagrange equations (1.1). In particular, if the principle of symmetric criticality is to hold, then (1.27) must yield the reduced Euler-Lagrange equations. It then also follows that (1.27) implies that all the transverse variations vanish, that is,

$$\frac{d}{d\epsilon} I[v + \epsilon h] = 0 \qquad \text{for all} \quad h^\alpha = h^\alpha(\widehat{x}^i). \tag{1.28}$$

We can re-formulate the principle of symmetric criticality in the language of local differential geometry as follows. For a Lagrangian n-form

$$\lambda = L(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

we define the associated Euler-Lagrange $(n+1)$-form by

$$E(\lambda) = E_\alpha(L)\, du^\alpha \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \tag{1.29}$$

where $E_\alpha(L)$ are the Euler-Lagrange expressions (1.6). Let G be a p dimensional transformation group acting on E with q dimensional orbits. Then *the principle of symmetric criticality asserts that there exists a G invariant q-chain (1.15) such that for all G invariant Lagrangians*

$$E([\lambda(\mathcal{X})]_{\text{inv}}) = [E(\lambda)(\mathcal{X})]_{\text{inv}}. \tag{1.30}$$

The left-hand side of this equation corresponds to the symmetric variations of λ around a G invariant section of E while the right-hand side of this equation is simply the reduction of the original Euler-Lagrange equations for such G invariant sections. Indeed, if the parametric coordinates $(\widehat{x}^i)$ are chosen such that

$$\mathcal{X} = J_0(\widehat{x}^i)\, \partial_{\hat{x}^1} \wedge \partial_{\hat{x}^2} \wedge \cdots \wedge \partial_{\hat{x}^q}$$

then, in these coordinates, (1.30) becomes the rather remarkable identity

$$\frac{\delta}{\delta v^\alpha}\big[\, L|_{v_i^\alpha = v_{ir}^\alpha = \cdots = 0}\big] = \big[\frac{\delta}{\delta v^\alpha} L\big]\, |_{v_i^\alpha = v_{ir}^\alpha = \cdots = 0}\,. \tag{1.31}$$

The paper is organized as follows. Given $\pi\colon E \to M$, the fiber bundle of independent and dependent variables, let $\overline{\pi}\colon \overline{E} \to \overline{M}$ be the quotient of E by the orbits of G. In section two we quickly review the definition of the variational bicomplex $(\Omega^{*,*}(J^\infty(E)), d_H, d_V)$ on the infinite jet bundle $\pi^\infty_M\colon J^\infty(E) \to M$ and the definition of the space $\pi\colon \mathrm{Inv}^\infty_G(E) \to M$ of jets of G invariant sections of E. We recall how certain G invariant objects associated to E and $J^\infty(E)$ drop to $\overline{E}$ and $J^\infty(\overline{E})$. Denote by $\Omega^{r,s}_{\mathrm{pr}\,G}(J^\infty(E))$ the type (r,s) forms on $J^\infty(E)$ which are invariant under the prolonged action of G on $J^\infty(E)$. Our invariant conservation laws ω and Lagrangians λ are elements of $\Omega^{n-1,0}_{\mathrm{pr}\,G}(J^\infty(E))$ and $\Omega^{n,0}_{\mathrm{pr}\,G}(J^\infty(E))$ while the reduced conservation laws $\overline{\omega}$ and $\overline{\lambda}$ belong to $\Omega^{n-q-1,0}(J^\infty(\overline{E}))$ and $\Omega^{n-q,0}(J^\infty(\overline{E}))$.

We carefully show how forms ω which are in $\Omega^{r,s}_{\mathrm{pr}\,G}(J^\infty(E))$ and which are annihilated by the total vector fields $\mathrm{tot}\,X$, where $X \in \Gamma$, drop to uniquely determined forms $\varrho(\omega) = \omega_{\mathrm{inv}}$ in $\Omega^{r,s}(J^\infty(\overline{E}))$. This rigorously defines the map

$$\varrho_{\mathcal{X}}\colon \Omega^{r,s}_{\mathrm{pr}\,G}(J^\infty(E)) \to \Omega^{r-q,s}(J^\infty(\overline{E}))$$

by

$$\varrho_{\mathcal{X}}(\omega) = (-1)^{q(r+s)}\varrho(\omega(\mathcal{X})) = (-1)^{q(r+s)}[\omega(\mathcal{X})]_{\mathrm{inv}} \tag{1.32}$$

which we have already considered in the foregoing examples. In section 3 we state certain existence theorems for G invariant total vector fields on $J^\infty(E)$ which are smooth in the neighborhood of a point $\sigma \in \mathrm{Inv}^\infty_G(E)$. Such points are never regular points of $\mathrm{pr}\,G$ and so Tresse's theorem ([*16*], [*17*]) for the existence of a complete set of G invariant total vector fields does not apply. We also identify the obstructions to the existence of G invariant q-chains in $\mathrm{Tot}\,\Gamma$. In section 4 we solve the problem of determining when a q-chain $\mathcal{X}$ exists such that $\varrho_{\mathcal{X}}$ is a d_H cochain map. When the action of G is free, that is, when the orbit dimension of G is maximal, we find that such a chain exists if and only if the Lie group G is unimodular. When the orbit dimensions of G satisfy $q < p$, the problem becomes considerably more complex but with the existence theorems of section 3 in hand we are still able to find all obstructions to the local existence of a cochain map $\varrho_{\mathcal{X}}$. To formally state our main result let

$$\Gamma_0(e_0) = \{\, X \in \Gamma \mid X(e_0) = 0 \,\}$$

be the linear isotropy subalgebra of Γ at $e_0 \in E$ and let $H^*(\Gamma, \Gamma_0(e_0))$ be the Lie algebra cohomology of Γ relative to the subalgebra $\Gamma_0(e)$ (see Lemma 4.5 for details).

Theorem 1.1. *Suppose that the orbits of G have dimension q. Then for any $e_0 \in E$, there is an open neighborhood U in E around e_0 and a G invariant chain*

$$\mathcal{X} = J\,\mathrm{tot}\,X_1 \wedge \mathrm{tot}\,X_2 \wedge \cdots \wedge \mathrm{tot}\,X_q, \tag{1.33}$$

where J is a non-zero function on $\pi(U) \subset M$, such that (1.32) is a d_H cochain map if and only if

$$H^q(\Gamma, \Gamma_0(e)) \neq 0. \tag{1.34}$$

Given the cochain map (1.32), it is then simple to check that if $\omega \in \Omega^{n-1,0}_{\mathrm{pr}\,G}(J^\infty(E))$ is a conservation law for a system of differential equations $\Delta = 0$ for the local sections of E, then $\varrho_{\mathcal{X}}(\omega)$ is a conservation law for the reduced equations $\overline{\Delta} = 0$ on $\overline{E}$. Also, because $\mathcal{X}$ is a chain with coefficients in $C^\infty(M)$, we can easily obtain the following corollary.

Corollary 1.2. *The principle of symmetric criticality holds, that is, there is a G invariant q-chain (1.33) such that for every G invariant Lagrangian form λ*

$$E(\varrho_{\mathcal{X}}\lambda) = \varrho_{\mathcal{X}} E(\lambda)$$

if and only if $H^q(\Gamma, \Gamma_0(e_0)) \neq 0$.

Many of the calculations in this paper have been checked using HELMHOLTZ, a general purpose *Maple* package for the variational calculus developed at Utah State University by C. Hillyard [*11*].

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§**2. Preliminaries.** Let $\pi\colon E \to M$ be an $n+m$ dimensional fiber bundle over an n dimensional base manifold M and let G be a p dimensional Lie transformation group acting on E. In this section we quickly recall the relevant background material pertaining to the differential calculus on the infinite jet bundle $\pi^\infty_M\colon J^\infty(E) \to M$. For details, see [*2*], [*12*], [*22*]. We also review some of the basic constructions of symmetry reduction, as applied to the prolonged group action of G on $J^\infty(E)$. In particular we prove, with the appropriate technical assumptions on the action of G, that differential forms on $J^\infty(E)$ which are invariant under the prolonged action of G and which are annihilated by the infinitesimal generators of G uniquely determine forms on the jet space $J^\infty(\overline{E})$, where $\overline{\pi}\colon \overline{E} \to \overline{M}$ is the quotient of $\pi\colon E \to M$ by the orbits of G.

If s is a smooth section of π defined in a neighborhood of a point p_0 of M, we let $j^\infty(s)$ denote the local section of $J^\infty(E)$ determined by the infinite jet of s. Adapted local coordinates on E are $\pi\colon (x^i, u^\alpha) \to (x^i)$, where $i = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$ and the induced local coordinates on $J^\infty(E)$ are

$$(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots) \tag{2.1}$$

so that, for instance, $u^\alpha_i\big(j^\infty(s)(p)\big) = \dfrac{\partial s^\alpha}{\partial x^i}(p)$.

A vector X_σ at a point $\sigma = j^\infty(s)(p)$ of $J^\infty(E)$ is called a *total vector* at σ if for every real-valued function f defined on a neighborhood of σ in $J^\infty(E)$

$$X_\sigma(f) = [(\pi^\infty_M)_*(X_\sigma)](f \circ j^\infty(s)). \tag{2.2}$$

The subbundle $\mathrm{Tot}(J^\infty(E))$ of the tangent bundle $T(J^\infty(E))$ consisting of total vectors is spanned locally by the total derivative vectors

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots \qquad i = 1, 2, \ldots, n. \tag{2.3}$$

Sections of $\operatorname{Tot}(J^\infty(E))$ are called total vector fields on $J^\infty(E)$. It is easy to prove that the distribution $\operatorname{Tot}(J^\infty(E))$ is integrable. Indeed, if $R = R^i\, D_i$ and $S = S^i\, D_i$ are two total vector fields, where the components R^i and S^i are functions of the coordinates (2.1), then the Lie bracket $[\,R, S\,]$ is the total vector field

$$[\,R, S\,] = \{R^i(D_i S^j) - S^i(D_i R^j)\}\, D_j.$$

Thus $\operatorname{Tot}(J^\infty(E))$ defines a flat connection on the bundle $\pi^\infty_M\colon J^\infty(E) \to M$ and therefore the tangent bundle of $J^\infty(E)$ splits into a direct sum of total vectors and π^∞_M vertical vectors,

$$T(J^\infty(E)) = \operatorname{Tot}(J^\infty(E)) \oplus \operatorname{Vert}(J^\infty(E)).$$

We denote the horizontal projection of a vector X on $J^\infty(E)$ into $\operatorname{Tot}(J^\infty(E))$ by $\operatorname{tot} X$. If X is a vector field on M, then (2.2) can be used to define a total vector field on $J^\infty(E)$ which we also denote by $\operatorname{tot} X$, that is, $(\operatorname{tot} X)_\sigma(f) = X_p(f \circ j^\infty(s))$.

The flat connection on Tot also gives rise to a direct sum decomposition of the de Rham complex on $J^\infty(E)$ into subspaces $\Omega^{r,s}(J^\infty(E))$ of horizontal degree r and vertical degree s. A form $\omega \in \Omega^k(J^\infty(E))$ belongs to $\Omega^{r,s}(J^\infty(E))$, where $k = r + s$, if

$$\omega(X_1, X_2, \dots, X_k) = 0$$

whenever either more than r of the vectors $X_1, X_2, \dots, X_k$ are horizontal, or more than s of the vectors $X_1, X_2, \dots, X_k$ are π^∞_M vertical. We write

$$d = d_H + d_V$$

for the induced splitting of the exterior derivative into horizontal and vertical components and so obtain the *variational bicomplex* on $J^\infty(E)$:

$$\begin{array}{ccccccccccccc}
 & & & & \uparrow{\scriptstyle d_V} & & & & & & & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega^{0,3} & & & & \cdots & & & & \Omega^{n,3} \\
 & & & & \uparrow{\scriptstyle d_V} & & & & & & & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0}\,.
\end{array} \tag{2.4}$$

Locally, a form $\omega \in \Omega^{r,s}(J^\infty(E))$ is a sum of wedge products of an r–fold wedge product of the horizontal differentials dx^i together with an s–fold wedge product of the contact one forms

$$\theta^\alpha_{j_1 j_2 \cdots j_k} = du^\alpha_{j_1 j_2 \cdots j_k} - u^\alpha_{j_1 j_2 \cdots j_k l}\, dx^l,$$

with coefficients which are locally defined functions on $J^\infty(E)$. If $\omega \in \Omega^{r,s}(J^\infty(E))$ and R_1, R_2, $\dots$, R_{r+1} are total vector fields on $J^\infty(E)$, then $d_H\omega \in \Omega^{r+1,s}(J^\infty(E))$ and

$$(d_H\omega)(R_1, R_2, \dots, R_{r+1}) = \sum_{i=1}^{r+1}(-1)^{i+1} R_i[\omega(R_1, \dots, \widehat{R}_i, \dots, R_{r+1})] \qquad (2.5)$$

$$+ \sum_{1\le i<j\le r+1} (-1)^{i+j}\omega([\,R_i, R_j\,], R_1, \dots \widehat{R}_i, \dots \widehat{R}_j, \dots, R_{r+1}).$$

Within the context of the variational bicomplex, the first variational formula in the calculus of variations states that for any Lagrangian form $\lambda = L\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ in $\Omega^{n,0}(J^\infty(E))$, there is a form $\eta \in \Omega^{n-1,1}(J^\infty(E))$ such that

$$d_V\lambda = E(\lambda) + d_H\eta, \qquad (2.6)$$

where $E(\lambda)$ is the Euler-Lagrange form of λ defined by

$$E(\lambda) = E_\alpha(L)\, \theta^\alpha \wedge dx^1 \wedge dx^2 \wedge \cdots dx^n \qquad (2.7)$$

and

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i \frac{\partial L}{\partial u^\alpha_i} + D_{ij}\frac{\partial L}{\partial u^\alpha_{ij}} - \cdots \ .$$

If X is a projectable vector field on E, then the local flow of X preserves the sections of E. The tangent vector field of the induced flow on $J^\infty(E)$ is called the infinite prolongation of X and is denoted by $\operatorname{pr} X$. For projectable vector fields, Lie differentiation of a form $\omega \in \Omega^{r,s}(J^\infty(E))$ with respect to $\operatorname{pr} X$ preserves bi-degree. If X and Y are projectable vector fields on E, then

$$[\,\operatorname{pr} X, \operatorname{pr} Y\,] = \operatorname{pr}[\,X, Y\,] \qquad (2.8)$$

and

$$[\,\operatorname{pr} X, \operatorname{tot} Y\,] = [\,\operatorname{tot} X, \operatorname{tot} Y\,] = \operatorname{tot}[\,X, Y\,]. \qquad (2.9)$$

Furthermore, if $R = R^i D_i$ is any total vector field on $J^\infty(E)$, and if the projectable vector field X is given locally by

$$X = A^i(x^j)\frac{\partial}{\partial x^i} + B^\alpha(x^j, u^\beta)\frac{\partial}{\partial u^\alpha}, \qquad (2.10)$$

then the bracket $[\,\operatorname{pr} X, R\,]$ is the total vector field

$$[\operatorname{pr} X, R] = \{\operatorname{pr} X(R^i) - R(A^i)\}D_i. \qquad (2.11)$$

If R is a total vector field on $J^\infty(E)$ and $\nu \in \Omega^{n,0}(J^\infty(E))$ is a generalized (that is, derivative dependent) volume form on M, then $\operatorname{Div}_\nu R$, the total divergence of R with respect to ν, is defined by

$$(\operatorname{Div}_\nu R)\, \nu = d_H(R \lrcorner\, \nu). \qquad (2.12)$$

It is a simple matter to check that for any function f on $J^\infty(E)$

$$\mathrm{Div}_\nu(fR) = f\,\mathrm{Div}_\nu R + R(f) \tag{2.13}$$

and that for any projectable vector field X on E

$$[\mathrm{Div}_\nu(\mathrm{tot}\,X)]\,\nu = \mathcal{L}_{\mathrm{pr}\,X}\nu. \tag{2.14}$$

If ν is a G invariant volume form, then (2.14) implies that for any vector field $X \in \Gamma$,

$$\mathrm{Div}_\nu\,\mathrm{tot}\,X = 0. \tag{2.15}$$

Now let G be a p dimensional Lie transformation group acting on E. We assume that there is an induced action of G on the base manifold M, that G acts transversally to the fibers of E and that G acts regularly and effectively on both E and M with orbits of dimension $q < n$. The quotient spaces $\overline{E} = E/G$ and $\overline{M} = M/G$ are then smooth manifolds (removing non-Hausdorff points from $\overline{E}$ and $\overline{M}$ if necessary) of dimension $n+m-q$ and $n-q$ respectively, the projection maps $\pi_{\overline{E}} \colon E \to \overline{E}$ and $\pi_{\overline{M}} \colon M \to \overline{M}$ are smooth maps, and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\pi_{\overline{E}}} & \overline{E} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \bar{p}i} \\ M & \xrightarrow{\pi_{\overline{M}}} & \bar{M} \end{array}$$

commutes. Let Γ be the p dimensional Lie algebra of infinitesimal generators of the action of G on E. Every vector field X in Γ projects under π to a vector field $\pi_* X \in \pi_*\Gamma$ on M. The dimension of the integrable distribution defined by Γ is q and this distribution is transverse to the fibers of π. The kernels of the projection maps $(\pi_{\overline{E}})_* \colon T_e(E) \to T_{\bar{e}}(\overline{E})$ and $(\pi_{\overline{M}})_* \colon T_p(M) \to T_{\bar{p}}(\overline{M})$ are precisely the vector spaces $\Gamma_e = \{\, X_e \in \Gamma \mid X \in \Gamma \,\}$ and $(\pi)_*(\Gamma_e)$, where $p = \pi(e)$. The Frobenius theorem guarantees the existence of local coordinates $(\widehat{x}^i, y^r)$ on M and $(\widehat{x}^i, y^r, v^\alpha)$ on E, where $i = 1, 2, \ldots,$ q and $r = 1, 2, \ldots, n-q$, such that the orbits of G are given by the coordinate planes $y^r =$ const. and $v^\alpha =$ const. . We call the coordinates $(\widehat{x}^i, y^r, v^\alpha)$ G adapted local coordinates on E. In these coordinates every vector field $X \in \Gamma$ assumes the form

$$X = \sum_{j=1}^{q} X^j(\widehat{x}^i, y^r)\,\frac{\partial}{\partial \widehat{x}^j}. \tag{2.16}$$

The prolongation of the action of G on E to $J^\infty(E)$ is denoted by $\mathrm{pr}\,G$ and the associated Lie algebra of vector fields on $J^\infty(E)$ by $\mathrm{pr}\,\Gamma$. A total vector field R on $J^\infty(E)$ is said to be G invariant if

$$[\,\mathrm{pr}\,X, R\,] = 0 \qquad \text{for all } X \in \Gamma.$$

These invariant total vector fields play a distinguished role in the study of the differential invariants of G since if f is a differential invariant of order k, $R(f)$ is a new differential invariant of order $k+1$. These invariant total vector fields are also central to our construction of the cochain map $\varrho_{\mathcal{X}}$.

Recall that a local section s of E is locally G invariant if for all $g \in G$ sufficiently close to the identity $g \cdot [s(g^{-1} \cdot p)] = s(p)$. The jet space of G invariant local sections of E is the bundle $\mathrm{Inv}^\infty_G(E) \to M$ defined by

$$\mathrm{Inv}^\infty_G(E) = \{\sigma \in J^\infty(E) \,|\, \sigma = j^\infty(s)(p), \text{where } s \text{ is a locally } G \text{ invariant section}\}. \tag{2.17}$$

In the local coordinates $(\widehat{x}^i, y^r, v^\alpha)$ adapted to G, the locally G invariant sections of E are given by $v^\alpha = f^\alpha(y^r)$ and therefore

$$\mathrm{Inv}^\infty_G(E) = \{\, \sigma = (\widehat{x}^i, y^r, v^\alpha, v^\alpha_i, v^\alpha_r, v^\alpha_{ij}, v^\alpha_{ir}, v^\alpha_{rs}, \dots\,) \,|\, v^\alpha_i = 0, v^\alpha_{ij} = 0, v^\alpha_{ir} = 0, \dots\}. \tag{2.18}$$

There is a canonical inclusion

$$\iota \colon \mathrm{Inv}^\infty_G(E) \to J^\infty(E). \tag{2.19}$$

The action of $\mathrm{pr}\, G$ restricts to an action on $\mathrm{Inv}^\infty_G(E)$ which is evidently trivial on the fibers of $\mathrm{Inv}^\infty_G(E)$ and which commutes with ι,

$$\iota(\mathrm{pr}\, g \cdot \sigma) = \mathrm{pr}\, g \cdot \iota(\sigma). \tag{2.20}$$

This leads to the following fundamental observation.

Lemma 2.1. *Let $\sigma = j^\infty(s)(p_0) \in \mathrm{Inv}^\infty_G(E)$. If X is a vector field in Γ, then*

$$(\mathrm{pr}\, X)_\sigma = (\mathrm{tot}\, X)_\sigma \tag{2.21}$$

and

$$\dim(\mathrm{pr}\, \Gamma)_\sigma = \dim(\mathrm{Tot}\, \Gamma)_\sigma = \dim \Gamma_{p_0} = q. \tag{2.22}$$

Proof. Let $\sigma = j^\infty(s)(p_0)$, where s is a locally G invariant section of E. Let ϕ_t be the local flow of X and let $\phi_{0,t}$ be the projected flow of X on M. Let f be a smooth real-valued function defined in the neighborhood of σ in $J^\infty(E)$. Then from the definition of the prolongation of X, the fact that s is G invariant, and definition (2.2), we have

$$\begin{aligned}(\mathrm{pr}\, X)_\sigma(f) &= \frac{d}{dt}\big\{\, f[j^\infty(\phi_t \circ s \circ \phi^{-1}_{0,t})(\phi_{0,t}(p_0))]\big\}\big|_{t=0} = \frac{d}{dt}\{\, f(j^\infty(s)(\phi_{0,t}(p_0)))\}\big|_{t=0} \\ &= ((\pi_M)_* X_e)(f \circ j^\infty(s)) = (\mathrm{tot}\, X)_\sigma(f).\end{aligned}$$

It is also easy to prove this proposition using the well-known coordinate formula for $\mathrm{pr}\, X$, namely

$$\mathrm{pr}\, X = \mathrm{pr}\, X_{\mathrm{ev}} + \mathrm{tot}\, X, \tag{2.23}$$

where X_{ev} is the first order generalized vertical vector field (called the evolutionary form of X) which, for X given by (2.10), is defined by

$$X_{\mathrm{ev}} = (B^\alpha - A^i u^\alpha_i)\frac{\partial}{\partial u^\alpha}. \qquad \blacksquare$$

Olver ([*15*], pp. 228–239) makes a detailed study of the space $\mathrm{Inv}^\infty_G(E)$ and proves, among other things, that the converse of Lemma 2.1 is also true. Note that in the local G adapted coordinates $(\widehat{x}^i, y^r, v^\alpha)$ and with X given by (2.16), (2.21) becomes

$$(\mathrm{pr}\, X)_\sigma = \sum_{j=1}^{q} X^j \, [\frac{\partial}{\partial \widehat{x}^j} + v^\alpha_j \frac{\partial}{v^\alpha} + v^\alpha_{jk} \frac{\partial}{v^\alpha_k} + v^\alpha_{jr} \frac{\partial}{v^\alpha_r} + \cdots] \Big|_\sigma = \sum_{j=1}^{q} X^j \, \frac{\partial}{\partial \widehat{x}^j} \Big|_\sigma .$$

We emphasize that Lemma 2.1 implies that when the action of G is not free the rank of the distribution $\mathrm{pr}\,\Gamma$ is not maximal on $\mathrm{Inv}^\infty_G(E)$.

If $s\colon M \to E$ is a G invariant section of π, then there is a unique section $\bar{s}$ of $\overline{\pi}$ satisfying

$$\bar{s}(\pi_{\overline{M}}(p)) = \pi_{\overline{E}}(s(p)). \tag{2.24}$$

This correspondence induces a projection map

$$\Pi\colon \mathrm{Inv}^\infty_G(E) \to J^\infty(\overline{E}) \tag{2.25}$$

defined by

$$\Pi\big(j^\infty(s)(p)\big) = j^\infty(\bar{s})(\pi_{\overline{M}}(p)).$$

Conversely, since G acts transversally to π, every section $\bar{s}$ of $\overline{E}$ determines a G invariant section s of E by (2.24). It is also clear that for all $\mathrm{pr}\, g \in \mathrm{pr}\, G$ and $\sigma \in \mathrm{Inv}^\infty_G(E)$, $\Pi(\mathrm{pr}\, g \cdot \sigma) = \Pi(\sigma)$, and thus Π induces an diffeomorphism

$$\mathrm{Inv}^\infty_G(E)/\,\mathrm{pr}\, G \cong J^\infty(\overline{E}).$$

It is important to recognize that it is *not* in general true that $J^\infty(\overline{E})$ is the quotient of $J^\infty(E)$ by the orbits of the prolonged group action $\mathrm{pr}\, G$. In particular, there is no canonical projection map from $J^\infty(E)$ to $J^\infty(\overline{E})$.

Example 2.2. Consider the 3 dimensional group acting on $(x, y, u) \to (x, y)$ with infinitesimal generators

$$\frac{\partial}{\partial x}, \qquad x\frac{\partial}{\partial x}, \qquad y\frac{\partial}{\partial x}.$$

Then, on the one hand, the differential invariants on $J^1(E)$ are just y and u and these serve as local coordinates for the quotient $J^1(E)/\,\mathrm{pr}\, G$ while, on the other hand, the local coordinates for $J^1(\overline{E})$ are (y, u, u_y). Therefore there can be no projection from $J^1(E)/\,\mathrm{pr}\, G$ to $J^1(\overline{E})$. ∎

We now describe the correspondence between G invariant objects attached to $\pi\colon E \to M$ and the associated objects on $\overline{\pi}\colon \overline{E} \to \overline{M}$. We call this correspondence the reduction map for the action of G on E and we denote it by ϱ. If $s\colon M \to E$ is a G invariant section of π, then $\bar{s} = \varrho(s)$ is the unique section of $\overline{\pi}$ satisfying (2.24). If X is a G invariant vector field on E, then $\overline{X} = \varrho(X)$ is the $(\pi_{\overline{E}})_*$ associated vector field on $\overline{E}$. It is easy to check that there is a one-to-one correspondence between G invariant π_M vertical vector fields on E and $\pi_{\overline{M}}$ vertical vector fields on $\overline{E}$. Note that any vertical vector field on E of the form

$$Y = b^\alpha(y^r, v^\beta)\frac{\partial}{\partial v^\alpha} \tag{2.26}$$

is G invariant. A G invariant form ω on E which satisfies $X \lrcorner\, \omega = 0$ for every vector field $X \in \Gamma$ is said to be G *basic*. If ω is G basic, then there is a unique form $\overline{\omega} = \varrho(\omega)$ on $\overline{E}$ such that

$$\omega = (\pi_{\overline{E}})^*(\overline{\omega}). \tag{2.27}$$

This observation is central to the general theory of symplectic reduction [*9*], [*14*].

We extend this correspondence ϱ to certain G invariant objects on $J^\infty(E)$. If $f \colon J^\infty(E) \to \mathbf{R}$ is a G invariant function, then there is a unique function $\bar{f} \colon J^\infty(\overline{E}) \to \mathbf{R}$ satisfying $\bar{f}(\overline{\sigma}) = f(\sigma)$, where $\overline{\sigma} \in J^\infty(\overline{E})$, $\sigma \in \mathrm{Inv}^\infty_G(E)$ and $\Pi(\sigma) = \overline{\sigma}$. We let $\varrho(f) = \bar{f}$. More generally, let $\Omega^{r,s}_{\mathrm{pr}\,G}(J^\infty(E))$ be the forms of type (r,s) on $J^\infty(E)$ which are invariant under the prolonged action of G. Since we have assumed that G is connected, $\omega \in \Omega^{r,s}_{\mathrm{pr}\,G}(J^\infty(E))$ if and only if

$$\mathcal{L}_{\mathrm{pr}\,X}\omega = 0 \qquad \text{for all } X \in \Gamma.$$

Proposition 2.3. *(i) Let R be a G invariant total vector field on $J^\infty(E)$. Then there exists a unique total vector field $\overline{R}$ on $J^\infty(\overline{E})$ such that for any $\sigma \in \mathrm{Inv}^\infty_G(E)$ and any G invariant function $f \colon M \to \mathbf{R}$*

$$\overline{R}(\bar{f})(\overline{\sigma}) = R(f)(\sigma), \tag{2.28}$$

where $\bar{f} = \varrho(f)$ and $\overline{\sigma} = \Pi(\sigma)$.
(ii) Let $\omega \in \Omega^{r,s}_{\mathrm{pr}\,G}(J^\infty(E))$ be a G invariant form such that

$$\mathrm{tot}\, X \lrcorner\, \omega = 0 \qquad \textit{for all } X \textit{ in } \Gamma. \tag{2.29}$$

Then there exists a unique form $\overline{\omega} \in \Omega^{r,s}(J^\infty(\overline{E}))$ such that

$$\iota^*(\omega) = \Pi^*(\overline{\omega}). \tag{2.30}$$

We denote the total vector field $\overline{R}$ by $\varrho(R)$ and the form $\overline{\omega}$ by $\varrho(\omega)$.

Proof. (i) Let $\bar{f}$ be a smooth real-valued function on $\overline{M}$ and let $\overline{\sigma}$ be a point in $J^\infty(\overline{E})$. Let $f \colon M \to \mathbf{R}$ be a G invariant function such that $\varrho(f) = \bar{f}$ and let $\sigma \in \mathrm{Inv}^\infty_G(E)$ be such that $\Pi(\sigma) = \overline{\sigma}$. Define

$$\overline{R}_{\overline{\sigma}}(\bar{f}) = R(f)(\sigma).$$

It is easy to check that this is well-defined, that is, independent of the choice of σ and that $\overline{R}_{\overline{\sigma}}$ is a derivation on $C^\infty(\overline{M})$ which varies smoothly with $\overline{\sigma}$. Such derivations uniquely define total vector fields on $J^\infty(\overline{E})$.

The fact that we cannot directly pull-back real-valued functions on $J^\infty(\overline{E})$ to functions on $J^\infty(E)$ points to the difficulty in mapping arbitrary G invariant vector fields on $J^\infty(E)$ to vector fields on $J^\infty(\overline{E})$ but the fact that we can pull-back functions on $\overline{M}$ to G invariant functions on M allows us to map G invariant total vector fields on $J^\infty(E)$ to total vector fields on $J^\infty(\overline{E})$.

(ii) Let $\widetilde{\omega} = \iota^*(\omega)$ be the pullback of ω by the inclusion map (2.19) to the space $\mathrm{Inv}^\infty_G(E)$. By (2.20), the form $\widetilde{\omega}$ is invariant under the prolonged action of G on $\mathrm{Inv}^\infty_G(E)$. In addition, by (2.29) and Proposition 2.1, $\widetilde{\omega}$ is annihilated by the infinitesimal generators of $\mathrm{pr}\,G$ on $\mathrm{Inv}^\infty_G(E)$.

We now deduce, from the standard reduction arguments, that there is a form $\overline{\omega}$ on the quotient $\operatorname{Inv}^\infty_G(E)/\operatorname{pr} G \cong J^\infty(\overline{E})$ such that $\widetilde{\omega} = \Pi^*(\omega)$. ∎

It is useful to have some local coordinate expressions for the maps we have introduced. The inclusion map ι in (2.19) and the projection map Π in (2.25) are given by

$$\iota((\widehat{x}^i, y^r, v^\alpha, v^\alpha_r, v^\alpha_{rs}, \dots)) = (\widehat{x}^i, y^r, v^\alpha, 0, v^\alpha_r, 0, 0, v^\alpha_{rs}, \dots)$$

and

$$\Pi((\widehat{x}^i, y^r, v^\alpha, v^\alpha_r, v^\alpha_{rs}, \dots)) = (y^r, v^\alpha, v^\alpha_r, v^\alpha_{rs}, \dots).$$

If

$$f = f(\widehat{x}^i, y^r, v^\alpha, v^\alpha_i, v^\alpha_r, v^\alpha_{ij}, v^\alpha_{ir}, v^\alpha_{rs}, \dots)$$

is a G invariant function on $J^\infty(E)$, then the reduction $\varrho(f)$ is

$$\varrho(f)(y^r, v^\alpha, v^\alpha_r, v^\alpha_{rs}, \dots) = f(\widehat{x}^i, y^r, v^\alpha, 0, v^\alpha_r, 0, 0, v^\alpha_{rs}, \dots).$$

The invariance of f implies that the right-hand side of this equation does not depend on $\widehat{x}^i$. If

$$R = \sum_{i=1}^{q} A^i D_{\hat{x}^i} + \sum_{r=1}^{n-q} B^r D_{y^r}$$

is a G invariant total vector field on $J^\infty(E)$, then the component functions B^r are G invariant functions of the variables (2.1), and

$$\varrho(R) = \sum_{r=1}^{n-q} \varrho(B^r) D_{y^r}.$$

A type $(l, 0)$ form ω such that $\operatorname{tot}(X) \lrcorner\, \omega = 0$ for all $X \in \Gamma$ is given locally by

$$\omega = A_{r_1 r_2 \cdots r_l}\, dy^{r_1} \wedge dy^{r_2} \wedge \cdots \wedge dy^{r_l}.$$

If ω is G invariant, then the coefficients $A_{r_1 r_2 \dots r_l}$ are G invariant functions of the jet coordinates (2.1) and

$$\varrho(\omega) = \varrho(A_{r_1 r_2 \cdots r_l})\, dy^{r_1} \wedge dy^{r_2} \wedge \cdots \wedge dy^{r_l}.$$

If, for example, the contact one form

$$\omega = A^i_\alpha\, \theta^\alpha_i + B^r_\alpha\, \theta^\alpha_r + C^{ij}_\alpha\, \theta^\alpha_{ij} + D^{ir}_\alpha\, \theta^\alpha_{ir} + E^{rs}_\alpha\, \theta^\alpha_{rs},$$

is G invariant, then its reduction is given by

$$\varrho(\omega) = \varrho(B^r_\alpha)\, \theta^\alpha_r + \varrho(E^{rs}_\alpha)\, \theta^\alpha_{rs}.$$

The general formula for the symmetry reduction of any contact form is similar.

Unlike the situation in reduction of Hamiltonian systems, the ϱ map from forms on $J^\infty(E)$ to forms on $J^\infty(\overline{E})$ is not by itself enough to construct the required cochain map between the corresponding variational bicomplexes. As the examples in the introduction show we must introduce a shift in horizontal degree by the orbit dimensions of G. To this end, we introduce $\Lambda_q(\operatorname{Tot}\Gamma, C^\infty(J^\infty(E)))$ *the space of q-chains in the Lie algebra* $\operatorname{Tot}\Gamma$ *with coefficients in* $C^\infty(J^\infty(E))$. A chain $\mathcal{X} \in \Lambda_q(\operatorname{Tot}\Gamma, C^\infty(J^\infty(E)))$ assumes the form

$$\mathcal{X} = J^{a_1 a_2 \cdots a_q} \operatorname{tot} X_{a_1} \wedge \operatorname{tot} X_{a_2} \wedge \cdots \wedge \operatorname{tot} X_{a_q}, \tag{2.31}$$

where the vectors $X_{a_i} \in \Gamma$. If $\omega \in \Omega^{r,s}_{\operatorname{pr} G}(J^\infty(E))$ and $\mathcal{X}$ is the q-chain (2.31), then we define the type $(r-q, s)$ form $\omega(\mathcal{X})$ by

$$\begin{aligned}\omega(\mathcal{X})(Y_1, Y_2, \dots, Y_t) &= \omega(\mathcal{X}, Y_1, Y_2, \dots, Y_t)\\ &= J^{a_1 a_2 \cdots a_q}\, \omega(\operatorname{tot} X_{a_1}, \operatorname{tot} X_{a_2}, \operatorname{tot} X_{a_q}, Y_1, Y_2, \dots, Y_t),\end{aligned}$$

where Y_1, Y_2, $\dots$, Y_t are vector fields on $J^\infty(E)$ and $t = r + s - q$. It is clear that $\omega(\mathcal{X})$ satisfies (2.29). For $\omega(\mathcal{X})$ to be G invariant for all G invariant ω, it is necessary and sufficient that the q-chain $\mathcal{X}$ be G invariant, that is, for all $X \in \Gamma$

$$\begin{aligned}\mathcal{L}_{\operatorname{pr} X}\mathcal{X} &= (\operatorname{pr} X\, J^{a_1 a_2 \cdots a_q}) \operatorname{tot} X_{a_1} \wedge \operatorname{tot} X_{a_2} \wedge \cdots \wedge \operatorname{tot} X_{a_q}\\ &\quad + \sum_{b=1}^{q} J^{a_1 \cdots b \cdots a_q} \operatorname{tot} X_{a_1} \wedge \cdots \wedge (\operatorname{tot}[\,X, X_b]) \wedge \cdots \wedge \operatorname{tot} X_{a_q}\\ &= 0.\end{aligned} \tag{2.32}$$

When ω and $\mathcal{X}$ are G invariant, we can apply Proposition 2.3 to deduce that there is a unique form $\overline{\omega}_{\mathcal{X}} \in \Omega^{r,s}(J^\infty(\overline{E}))$ such that $\iota^*(\omega(\mathcal{X})) = \Pi^*(\overline{\omega}_{\mathcal{X}})$. This, at last, allows us to define the map

$$\varrho_{\mathcal{X}} \colon \Omega^{r,s}_{\operatorname{pr} G}(J^\infty(E)) \to \Omega^{r,s}(J^\infty(\overline{E}))$$

by

$$\varrho_{\mathcal{X}}(\omega) = (-1)^{q(r+s)} \varrho(\omega(\mathcal{X})).$$

In section 4 we shall determine when $\varrho_{\mathcal{X}}$ is a d_H cochain map between variational bicomplexes.

Finally, we show how the map $\varrho_{\mathcal{X}}$, when it is known to be a d_H cochain map, leads to the principle of symmetric criticality. Let $\lambda \in \Omega^{n,0}_{\operatorname{pr} G}(J^\infty(E))$ be a G invariant Lagrangian. If f is a real-valued function on M and X is a vector field on M, then it is not difficult to show that

$$d_V f = 0 \quad \text{and} \quad d_V(\operatorname{tot} X \lrcorner\, \omega) = -\operatorname{tot} X \lrcorner\, d_V\omega$$

It follows if $\mathcal{X}$ is a q-chain with coefficients $J^{a_1 a_2 \cdots a_q}$ in $C^\infty(M)$, then

$$d_V(\varrho_{\mathcal{X}}\omega) = \varrho_{\mathcal{X}}(d_V\omega).$$

As is well-known, the Euler-Lagrange form $E(\lambda)$ of a G invariant Lagrangian λ is always G invariant. Furthermore, when λ is invariant it follows from the results announced in [3] that the type $(n-1, 1)$

form η in (2.6) may be chosen to be G invariant. We can therefore apply $\varrho_{\mathcal{X}}$ to each term in the first variational formula to arrive at

$$d_V(\varrho_{\mathcal{X}}(\lambda)) = \varrho_{\mathcal{X}}(E(\lambda)) + d_H[\varrho_{\mathcal{X}}(\eta)].$$

We compare this equation to the first variational formula for $\varrho_{\mathcal{X}}(\lambda)$, namely

$$d_V(\varrho_{\mathcal{X}}(\lambda)) = E(\varrho_{\mathcal{X}}\lambda) + d_H\eta',$$

to deduce that

$$E(\varrho_{\mathcal{X}}\lambda) = \varrho_{\mathcal{X}}E(\lambda),$$

as required.

§**3. Invariant total vector fields, horizontal forms, and q-chains.** Let G be a connected, p dimensional Lie transformation group acting transversally, projectably and regularly on the fiber bundle $\pi\colon E \to M$ with q dimensional orbits. The Lie algebra of vector fields on E defined by the infinitesimal action of G is denoted by Γ and the prolongations of G and Γ to $J^\infty(E)$ are denoted $\operatorname{pr} G$ and $\operatorname{pr}\Gamma$. In this section we state a number of existence theorems for G invariant total vector fields, horizontal forms, and q-chains on $J^\infty(E)$. We shall prove just one of these existence theorems — the others are established by similar arguments which are presented in greater detail and generality in Fels and Olver [*7*] and Olver [*16*].

Recall that $\operatorname{Tot}(J^\infty(E))$ is the horizontal distribution of total vector fields on $J^\infty(E)$ and that if R is a total vector field then R can be expressed locally as

$$R = A^j(x, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \dots)D_j,$$

where D_j is the total derivative (2.3) with respect to x^i. We say that a total vector field R has coefficients in M if there is a vector field R^0 on M such that $R = \operatorname{tot} R^0$, that is, if the coefficients A^i of R are functions of the base variables x^i alone. A total vector field R on $J^\infty(E)$ is G invariant if

$$[\,R, \operatorname{pr} X\,] = 0 \qquad \text{for all } X \in \Gamma.$$

It follows from (2.9) that if R^0 is a G invariant vector field on M, that is

$$[\,R^0, \pi_* X\,] = 0 \qquad \text{for all } X \in \Gamma,$$

then $\operatorname{tot} R^0$ is a G invariant vector field on $J^\infty(E)$.

To put the existence theorems that we shall need in the next section into proper perspective, it is helpful to first review a fundamental result of Tresse ([*16*],[*17*], [*21*]). A point $\sigma \in J^\infty(E)$ if called a regular point of $\operatorname{pr} G$ if the dimension of the vector space $(\operatorname{pr}\Gamma)_\sigma$ is the maximum p.

Theorem 3.1. *Let $\sigma \in J^\infty(E)$ be a regular point of $\operatorname{pr} G$. Then there is an open neighborhood $\mathcal{U} \subset J^\infty(E)$ of σ and a G invariant basis R_1, R_2, … , R_n for $\operatorname{Tot}\mathcal{U}$. Furthermore, the set of regular points of G is a dense set in $J^\infty(E)$.*

Evidently, the horizontal forms $\sigma^j \in \Omega^{1,0}(J^\infty(E))$ which are dual to the total vector fields R_i ($\sigma^j(R_i) = \delta^j_i$) are G invariant. The generalized volume form $\nu = \sigma^1 \wedge \sigma^2 \wedge \cdots \wedge \sigma^n$ is also invariant.

As important and as useful as Tresse's theorem is, it nevertheless fails, for our purposes, to adequately describe the space of invariant total vector fields. First we shall need to establish the existence of the invariant total vector fields in the distribution $\operatorname{Tot}\Gamma$. Secondly, and more critically, we shall need information about the space of invariant total vector fields at the points $\sigma \in \operatorname{Inv}^\infty_G(E)$ and such points are never regular points when $q < p$. Indeed, by Lemma 2.1, $\operatorname{pr} X_\sigma = \operatorname{tot} X_\sigma$ for all $X \in \Gamma$ and $\sigma = j^\infty(s)(p_0) \in \operatorname{Inv}^\infty_G(E)$ and therefore

$$\dim(\operatorname{pr}\Gamma)_\sigma = \dim(\operatorname{Tot}\Gamma)_\sigma = \dim(\Gamma_{s(p_0)}) = q.$$

In the case where the group acts freely on E (and hence on M) we have the following refinements of Tresse's theorem ([*16*], p. 71).

Proposition 3.2. *Suppose that G acts freely on E. Let p_0 be any point of M and let X_0 be any vector in $\pi(\Gamma)_{p_0}$. Then there exists an open neighborhood U of p_0 and a G invariant vector field R^0 defined on U such that $R^0(p_0) = X_0$. The total vector field $R = \operatorname{tot} R^0$ is G invariant.*

Proposition 3.3. *Suppose that G acts freely on E. Let p_0 be any point in M and let $\overline{R}^0$ be any vector field on $\overline{M}$. Then there is an open neighborhood U of p_0 in M and a G invariant vector field R^0 defined on U such that $\varrho(R^0) = \overline{R}^0$. The total vector field $R = \operatorname{tot} R^0$ is G invariant and $\varrho(R) = \operatorname{tot}\overline{R}^0$.*

When the orbits of G are of maximal dimension these two Propositions imply there is a local frame for M consisting of G invariant vector fields

$$\{\, S^0_1, S^0_2, \ldots, S^0_p, R^0_1, R^0_2 \ldots, R^0_{n-p} \,\} \tag{3.1}$$

such that

(i) $\varrho(S^0_i) = 0$, and

(ii) $\overline{R}^0_j = \varrho(R^0_j)$, $j = 1, 2, \ldots, n-p$ is a local frame for $\overline{M}$.

We say that the *frame (3.1) is adapted to the reduction map ϱ*. Note that the horizontal forms dual to this adapted basis are G invariant one forms on M. The associated total vector fields $S_i = \operatorname{tot} S^0_i$ and $R_j = \operatorname{tot} R^0_j$ define a local G invariant basis for $\operatorname{Tot}(J^\infty(E))$. Thus, in the case of a free action, Tresse's theorem can be established without having to consider the prolongations of G to the jet spaces of E. Moreover, it is important to emphasize that we can find a full complement of invariant vector fields S^0_i in the distribution $\pi_*\Gamma$.

We turn now to the analogue of Propositions 3.2 and 3.3 in the case $q < p$. In order to proceed, it is convenient to introduce a basis for the Lie algebra Γ which is adapted to the linear isotropy subalgebra

$$\Gamma_0(e_0) = \{\, X \in \Gamma \mid X(e_0) = 0 \,\}$$

at a fixed point $e_0 \in E$. The dimension of $\Gamma_0(e_0)$ is $p - q$. In a neighborhood of e_0, we pick a basis $X_1, X_2, \ldots X_p$ for Γ such that the first q vectors $X_1, X_2, \ldots, X_q$ generate the distribution spanned by Γ while the last $p - q$ vectors $X_{q+1}, X_{q+2}, \ldots, X_p$ are a basis for $\Gamma_0(e_0)$. We write

$$X_a = \sum_{i=1}^{q} \varphi_a^i X_i, \qquad a = 1, 2, \ldots, p, \tag{3.2}$$

where the φ_a^i are smooth functions on M which satisfy $\phi_j^i = \delta_j^i$ and $\phi_\epsilon^i(\pi(e_0)) = 0$ for $i, j = 1, 2, \ldots, q$ and $\epsilon = q+1, q+2, \ldots, p$.

We may further introduce auxiliary projectable vector fields $Y_1,\, Y_2,\, \ldots\,,\, Y_{n-q}$ on E such that such that the n vector fields

$$\pi_* X_1, \ldots, \pi_* X_q,\, \pi_* Y_1, \ldots, \pi_* Y_{n-q} \tag{3.3}$$

determine a local frame on M. In terms of this local frame we define structure functions Γ^i_{ab}, ξ^i_{ar} and ζ^s_{ar} by

$$[\, X_a, X_b\,] = \sum_{i=1}^{q} \Gamma^i_{ab} X_i \tag{3.4a}$$

and

$$[\, \pi_* X_a, \pi_* Y_r] = \sum_{i=1}^{q} \xi^i_{ar} \pi_* X_i + \sum_{s=1}^{n-q} \zeta^s_{ar} \pi_* Y_s. \tag{3.4b}$$

We emphasis that these structure functions are all functions of the independent variables only. Note that $\Gamma^i_{ab} = \sum\limits_{c=1}^{p} C^c_{ab} \varphi^i_c$. It will be also convenient to define

$$A^i_{aj} = \Gamma^i_{aj} - \sum_{k=1}^{q} \varphi^k_a \Gamma^i_{kj}, \qquad B_a = \sum_{j=1}^{q} \Gamma^j_{aj}, \qquad E^i_{ar} = \xi^i_{ar} - \sum_{k=1}^{q} \varphi^k_a \xi^i_{kr}. \tag{3.5}$$

Note that A^i_{aj} and E^i_{ar} both vanish for $1 \leq a \leq q$ and that

$$B_a(\pi(e_0)) = \sum_{j=1}^{q} C^j_{aj} \quad \text{and} \quad A^i_{\epsilon j}(\pi(e_0)) = \Gamma^i_{\epsilon j}(\pi(e_0)) = C^i_{\epsilon j} \tag{3.6}$$

for $a = 1,\, 2\, \ldots\,,\, p$, for $i, j = 1,\, 2,\, \ldots\,,\, q$ and for $\epsilon = q+1,\, q+2,\, \ldots\ p$.

The functions A^i_{aj} and E^i_{ar} can also be expressed in terms of the derivatives of φ^i_a by using

$$\sum_{i=1}^{q} \Gamma^i_{aj} X_i = [\, \sum_{i=1}^{q} \varphi^i_a X_i, X_j\,]$$

and

$$\sum_{j=1}^{q} \xi^j_{ar} \operatorname{tot} X_j + \sum_{s=1}^{n-q} \zeta^s_{ar} \operatorname{tot} Y_s = [\, \sum_{j=1}^{q} \varphi^j_a \operatorname{tot} X_j, \operatorname{tot} Y_r\,]$$

to arrive at

$$A^i_{aj} = -X_j(\varphi^i_a) \quad \text{and} \quad E^i_{ar} = -Y_r(\varphi^i_a). \tag{3.7}$$

In G adapted coordinates $(\widehat{x}^i, y^r, v^\alpha)$ we may chose

$$Y_r = \frac{\partial}{\partial y^r}.$$

In these coordinates and with this choice of Y_r we shall take the liberty to identify the vector fields X_a and Y_j with their projections $\pi_* X_a$ and $\pi_* Y_r$ to M.

Lemma 3.4. *Let $\sigma_0 \in \mathrm{Inv}^\infty_G(E)$ and let R be any G invariant total vector field which is smooth in a neighborhood $\mathcal{U} \subset J^\infty(E)$ of σ_0. Let R be given in terms of the frame (3.3) by*

$$R = \sum_{i=1}^{q} g^i \operatorname{tot} X_i + \sum_{r=1}^{n-q} h^r \operatorname{tot} Y_r$$

and let $\mathcal{I} = \mathcal{U} \cap \mathrm{Inv}^\infty_G(E)$. Then for each point $\sigma = j^\infty(s)(p) \in \mathcal{I}$,

$$\sum_{j=1}^{q} A^i_{aj}(p) g^j(\sigma) + \sum_{r=1}^{n-q} E^i_{ar}(p) h^r(\sigma) = 0. \tag{3.8}$$

Proof. The $\operatorname{tot} X_i$ components of invariance condition $[\operatorname{pr} X_a, R] = 0$ implies that

$$\operatorname{pr} X_a(g^i) + \sum_{j=1}^{q} \Gamma^i_{aj} g^j + \sum_{r=1}^{n-q} \xi^i_{ar} h^r = 0 \tag{3.9}$$

and consequently, in view of Lemma 2.1,

$$[\operatorname{tot} X_a(g^i)](\sigma) + \sum_{j=1}^{q} \Gamma^i_{aj}(p) g^j(\sigma) + \sum_{r=1}^{n-q} \xi^i_{ar}(p) h^r(\sigma) = 0. \tag{3.10}$$

We then use the relationship $\operatorname{tot} X_a = \sum_{i=1}^{q} \varphi^i_a \operatorname{tot} X_i$ to eliminate the expression $[\operatorname{tot} X_a(g^i)](\sigma)$ from this equation and thereby arrive at (3.8). ∎

In order to describe the converse to Lemma 3.4 we define, for each $p \in M$, linear maps on the vector space $\mathbf{R}^q$ of q-tuples $v = [v^1, v^2, \ldots, v^q]$ by

$$A_p(v) = [\sum_{j=1}^{q} A^i_{aj}(p) v^j\,] \quad \text{and} \quad B_p(v) = \sum_{j=1}^{q} B_j(p) v^j \tag{3.11}$$

and a linear map on the space $\mathbf{R}^{n-q}$ of $(n-q)$-tuples $w = [w^1, w^2, \ldots w^{n-q}]$ by

$$Z_p(w) = [\sum_{r=1}^{n-q} \xi^i_{ar} w^r\,].$$

Proposition 3.5. *Suppose the orbits of G have dimension q. Let $p_0 \in M$, let (v, w) be any n-tuple satisfying the linear equations*

$$A_{p_0}(v) + Z_{p_0}(w) = 0 \tag{3.12}$$

and let

$$X_0 = \sum_{i=1}^{q} v^i \pi_* X_i + \sum_{r=1}^{n-q} w^r \pi_* Y_r.$$

Then there exists an open neighborhood U of p_0 and a G invariant vector field R^0 defined on U such that $R^0(p_0) = X_0(p_0)$. The total vector field $R = \operatorname{tot} R^0$ is G invariant and for any $\sigma \in (\pi^\infty_M)^{-1}(p_0)$,

$$R(\sigma) = \sum_{i=1}^{q} v^i \operatorname{tot} X_i(\sigma) + \sum_{r=1}^{n-q} w^r \operatorname{tot} Y_r(\sigma).$$

For a proof of this theorem see Fels and Olver [7].

Thus, unlike the case of a free group action, it may not in general be possible to construct a local G invariant frame on M. Nevertheless, if we let

$$\delta_1 = \dim \ker A_{p_0}$$

and

$$\delta_2 = \dim\{ w \in \mathbf{R}^{n-q} \mid Z_{p_0}(w) \in \operatorname{im} A_{p_0} \}$$

then it follows directly from Proposition 3.5 that there are locally defined, pointwise linearly independent G invariant vector fields

$$\{ S^0_1, S^0_2, \ldots, S^0_{\delta_1}, R^0_1, R^0_2 \ldots, R^0_{\delta_2} \} \tag{3.13}$$

on M such that

(i) $\varrho(S^0_i) = 0$, and

(ii) $\overline{R}^0_s = \varrho(R^0_s)$, $s = 1, 2, \ldots, \delta_2$ are pointwise linearly independent vector fields on $\overline{M}$.

Set $S_i = \operatorname{tot} S^0_i$ and $R_s = \operatorname{tot} R^0_s$. If $\sigma_0 \in (\pi^\infty_M)^{-1}(U)$ and σ_0 is a regular point of $J^\infty(E)$, then in a neighborhood of σ_0 we can find, by a simple application of Tresse's theorem, $\delta_3 = n - \delta_1 - \delta_2$ total G invariant vector fields $P_1, P_2, \ldots, P_{\delta_3}$ such that

$$\{ S_1, \ldots, S_{\delta_1}, R_1, \ldots, R_{\delta_2}, P_1, \ldots, P_{\delta_3} \} \tag{3.14}$$

form a local G invariant basis for $\operatorname{Tot} J^\infty(E)$. As one approaches the set $\operatorname{Inv}^\infty_G(E)$ of invariant jets the total vector fields P_k either become singular, that is, they are not defined on $\operatorname{Inv}^\infty_G(E)$ or they become linearly dependent on the total vector fields S_i, R_s. In section 5, we construct some explicit examples of the invariant basis (3.14).

Finally, we turn to the question of the existence of G invariant q-chains in $\operatorname{Tot}\Gamma$. Since the orbits of G have dimension q, the space of q-chains $\mathcal{X}$ in $\operatorname{Tot}\Gamma$ is one dimensional and we may write

$$\mathcal{X} = J\mathcal{X}_0 \qquad \text{where} \qquad \mathcal{X}_0 = \operatorname{tot} X_1 \wedge \operatorname{tot} X_2 \wedge \cdots \wedge \operatorname{tot} X_q,$$

where $X_1, X_2, \ldots, X_q$ generate Γ and where J is a function on $J^\infty(E)$. In order for the associated map $\varrho_{\mathcal{X}}$ to be locally well-defined and non-zero, we require that J be a smooth, non-zero function in the neighborhood of $\operatorname{Inv}^\infty_G(E)$. Our next proposition shows, in analogy with Lemma 3.4 and Proposition 3.5 that there are algebraic obstructions to the existence of a G invariant q-chain on $J^\infty(E)$ but that if these obstructions vanish then a G invariant chain can actually be found with coefficients defined on neighborhoods of M.

Proposition 3.6. *The q-chain*

$$\mathcal{X} = J \operatorname{tot} X_1 \wedge \operatorname{tot} X_2 \wedge \cdots \operatorname{tot} X_q, \tag{3.15}$$

is G invariant if and only if J satisfies

$$\operatorname{pr} X_a(J) + B_a J = 0. \tag{3.16}$$

If this equation admits a solution which is smooth and non-zero on a neighborhood $\mathcal{U}$ in $J^\infty(E)$, then

$$tr\, A(p) = \sum_{i=1}^{q} A^i_{ai}(p) = 0 \tag{3.17}$$

for all points $p \in \pi^\infty_0(\mathcal{U})$. Conversely, if (3.17) holds in a neighborhood of p, then there is a smooth non-zero function J, defined on a neighborhood U_0 of p in M, satisfying (3.16). The solution to (3.16) is unique up to multiplication by a differential invariant of Γ.

Example 3.7. As a consequence of the Jacobi identity, the integrability conditions for (3.16) are always satisfied so that by the Frobenius theorem, solutions to (3.16) always exist in the neighborhood of any regular point of G. For example, for the two dimensional solvable group G generated by $X = \partial_x$ and $Y = x\partial_x$ acting on $(x, y, u) \to (x)$, where $x > 0$, it is easy to see that $\mathcal{X} = \dfrac{1}{u_x} X$ is a G invariant 1-chain. But the G invariant sections of E are given by $u = f(y)$ and hence $\mathcal{X}$ is singular on $\operatorname{Inv}^\infty_G(E)$. For this transformation group (3.16) reduces to

$$\operatorname{pr} X(J) = 0 \qquad \text{and} \qquad \operatorname{pr} Y(J) + J = 0 \tag{3.18}$$

and (3.17) does not hold. Equations (3.17) also fails to hold for the Lie algebra spanned by $X = \partial_x$, $Y = \partial_y$ and $Z = -x\partial_x + 2y\partial_y$. The chain $\mathcal{X} = u_x\, \partial_x \wedge \partial_y$ with coefficients on $J^1(E)$ is everywhere smooth but this chain vanishes on $\operatorname{Inv}^\infty_G(E)$. ∎

Proof of Proposition 3.6. We let

$$\mathcal{X}^i_0 = (-1)^{i+1} \operatorname{tot} X_1 \wedge \cdots \widehat{\operatorname{tot} X_i} \wedge \cdots \operatorname{tot} X_q$$

and note that

$$\operatorname{tot} X_j \wedge \mathcal{X}^i_0 = \delta^i_j \mathcal{X}_0. \tag{3.19}$$

Then we find, on account of (2.2), (2.5) and (3.5) that

$$\begin{aligned} \mathcal{L}_{\operatorname{pr} X_a} \mathcal{X}_0 &= \sum_{i=1}^{q} [\, \operatorname{pr} X_a, \operatorname{tot} X_i\,] \wedge \mathcal{X}^i_0 = \sum_{i,j=1}^{q} \Gamma^j_{ai} \operatorname{tot} X_j \wedge \mathcal{X}^i_0 = [\sum_{j=1}^{q} \Gamma^j_{aj}\,]\, \mathcal{X}_0 \\ &= B_a \mathcal{X}_0 \end{aligned} \tag{3.20}$$

and hence

$$\mathcal{L}_{\operatorname{pr} X_a} \mathcal{X} = [\operatorname{pr} X_a(J) + B_a J]\, \mathcal{X}_0.$$

This proves the first statement of the proposition.

Let $\sigma_0 \in \mathrm{Inv}_G^\infty(E)$ and suppose J is a smooth non-zero solution to (3.16) in a neighborhood $\mathcal{U}$ of σ_0. Then, on account of Lemma 2.1,

$$\begin{aligned} J(\sigma_0)B_a(p) &= -[\mathrm{pr}\, X_a(J)](\sigma_0) = -[\mathrm{tot}\, X_a(J)](\sigma_0) \\ &= -\sum_{i=1}^{q} \varphi_a^i(p)[\mathrm{tot}\, X_i(J)](\sigma_0) = J(\sigma_0)\sum_{i=1}^{q} \varphi_a^i B_i(p) \end{aligned}$$

and (3.17) follows from the definitions in (3.5).

To prove the converse we suppose that J is a locally defined function on M. We first check that the integrability conditions for the system

$$X_j(J) + B_j J = 0 \qquad j = 1, 2, \ldots, q, \tag{3.21}$$

where $X_j = \pi_*(X_j)$, are satisfied on any neighborhood on which (3.17) holds. (We should, to be precise, write $(\pi_* X_j)(J)$ in place of $X_j(J)$ in (3.21) but this abuse of notation causes no problems.) To this end, we first observe that by virtue of (3.7), (3.17) is equivalent to

$$\sum_{i=1}^{q} X_i(\varphi_a^i) = 0 \quad \text{or} \quad \sum_{i=1}^{q} X_i(\Gamma_{ab}^i) = 0. \tag{3.22}$$

We now expand the Jacobi identity

$$[\, X_i, [\, X_j, X_k\,]\,] + [\, X_j, [\, X_k, X_i\,]\,] + [\, X_k, [\, X_i, X_j\,]\,] = 0$$

to arrive at

$$X_i(\Gamma_{jk}^l) + X_j(\Gamma_{ki}^l) + X_k(\Gamma_{ij}^l) + \Gamma_{jk}^m \Gamma_{im}^l + \Gamma_{ki}^m \Gamma_{jm}^l + \Gamma_{ij}^m \Gamma_{km}^l = 0.$$

In this equation we trace on l, k and substitute from (3.22) to find

$$X_i(B_j) - X_j(B_i) - \Gamma_{ij}^k B_k = 0,$$

which are the integrability conditions for (3.21).

The existence of solutions to (3.21) now follows from a simple extension of the usual Frobenius theorem. One introduces an additional variable z and defines

$$\widetilde{X}_j = X_j + B_j z \frac{\partial}{\partial z} \qquad \text{and} \qquad \widetilde{J} = Jz$$

We then have the structure equations

$$[\, \widetilde{X}_i, \widetilde{X}_j] = \sum_{k=1}^{q} \Gamma_{ij}^k \widetilde{X}_k$$

and the differential equations (3.21) become

$$\widetilde{X}_j \widetilde{J} = 0 \qquad j = 1, 2, \ldots, q$$

The q vector fields $\widetilde{X}_j$ are pointwise independent independent and one can then proceed to rectify the vector fields $\widetilde{X}_j$ (maintaining the linearity in the variable z) in the usual manner. See, for example, Olver [*16*] pp.411–415 for further details.

Finally, by (3.2) and (3.17), we deduce that

$$X_a(J) + B_a J = \sum_{i=1}^{q} \varphi_a^i [X_i(J) + B_i J] = 0$$

and (3.16) holds. ∎

We remark that when the action of G is free, $q = p$ and the algebraic condition (3.17) is vacuous and hence invariant chains, with coefficients on M, always exist. Furthermore, when $q = p$, equation (3.20) becomes

$$\mathcal{L}_{\mathrm{pr}\, X_a} \mathcal{X}_0 = [\, \sum_{b=1}^{p} C_{ab}^b \,]\, \mathcal{X}_0. \tag{3.23}$$

We also mention that the same arguments used in the proof of Proposition 3.6 can be adapted to prove Proposition 3.5.

§**4. Cochain maps to reduced variational bicomplexes.** As in the previous sections we let G be a connected, p dimensional Lie transformation group acting transversally, projectably and regularly on the fiber bundle $\pi\colon E \to M$. The Lie algebra of vector fields on E defined by the infinitesimal action of G is denoted by Γ. The orbits of G are q dimensional. We let $\overline{\pi}\colon \overline{E} \to \overline{M}$ be the quotient of E by the action of G. In this section we establish conditions under which there exists a q-chain

$$\mathcal{X} \in \Lambda_q(\mathrm{Tot}\,\Gamma, C^\infty(J^\infty(E)))$$

such that the map

$$\varrho_{\mathcal{X}}(\omega) = (-1)^{q(r+s)} \varrho(\omega(\mathcal{X})), \tag{4.1}$$

where ρ is the reduction map defined by Proposition 2.3, is a d_H cochain map from the rows of the G invariant variational bicomplex on $J^\infty(E)$ to the rows of the variational bicomplex on $J^\infty(\overline{E})$.

We begin with the case where G acts freely on E. Here the technical difficulties which arise for more general transformation groups vanish and we can present a reasonably elementary analysis. Recall that a Lie group is said to be unimodular (Helgason[*10*], p. 88) if it admits a bi-invariant volume form. Compact groups, semi-simple groups and nilpotent groups are all unimodular. In terms of the structure constants C_{ab}^c of G, the Lie group G is unimodular if and only if

$$\sum_{b=1}^{p} C_{ab}^b = 0. \tag{4.2}$$

We also call any Lie algebra which satisfies (4.2) unimodular. It is easy to see that a p dimensional Lie algebra is unimodular if and only if if its p-th homology group satisfies $H_p(\mathbf{g}) \neq 0$ (Fuks [*8*] p. 27).

Let $X_1, X_2, \ldots, X_p$ be vector fields on E which define a basis for Γ.

Theorem 4.1. *Suppose that the action of G on E is free. Then there exists a G invariant p-chain $\mathcal{X} \in \Lambda_p(\operatorname{Tot}\Gamma, C^\infty(J^\infty(E)))$ defining a cochain map*

$$\varrho_{\mathcal{X}}\colon \Omega^{r,s}_{\operatorname{pr}G}(J^\infty(E)) \longrightarrow \Omega^{r-p,s}(J^\infty(\overline{E})),$$

according to (4.1), if and only if the Lie group G is unimodular. In this case the map $\varrho_{\mathcal{X}}$ is uniquely determined, up to a multiplicative constant, by the chain

$$\mathcal{X}_0 = \operatorname{tot} X_1 \wedge \operatorname{tot} X_2 \wedge \cdots \wedge \operatorname{tot} X_p. \tag{4.3}$$

As a cautionary note we remark that the chain $\mathcal{X}$ is itself not unique up to a constant even if $\varrho_{\mathcal{X}}$ is. Indeed, if $\mathcal{X}$ defines a cochain map $\varrho_{\mathcal{X}}$ and $\mathcal{X}' = K\mathcal{X}$, where K is a non-zero G differential invariant on $J^\infty(E)$, which is a constant k on $\operatorname{Inv}^\infty_G(E)$ (that is, $\varrho(K)$ is constant on $J^\infty(\overline{E})$), then clearly

$$\varrho_{\mathcal{X}'} = k\varrho_{\mathcal{X}}. \tag{4.4}$$

For example, if G is the one-dimensional group of translations in x, acting on the bundle $(x, y, u) \to (x, y)$, then $K = u_x^2 + 1$ is a non-vanishing differential invariant which is constant on $\operatorname{Inv}^\infty_G(E)$.

Proof of Theorem 4.1. If G is unimodular, then equation (3.23) implies that $\mathcal{X}_0$ is a G invariant chain. To prove that $\varrho_{\mathcal{X}}$ is a cochain map we show that for all G invariant forms $\omega \in \Omega^{r,s}_{\operatorname{pr}G}(J^\infty(E))$

$$\varrho[(d_H\omega)(\mathcal{X}_0)] = (-1)^p d_H[\varrho(\omega(\mathcal{X}_0))]. \tag{4.5}$$

To this end, let $\overline{R}_1, \overline{R}_2, \ldots, \overline{R}_{r-p+1}$ be arbitrary total vector fields on $J^\infty(\overline{E})$ with coefficients in $C^\infty(\overline{M})$. By Proposition 3.3, there are invariant total vector fields R_j, defined on a neighborhood of $J^\infty(E)$, such that $\varrho(R_j) = \overline{R}_j$. Let $\mathcal{R} = R_1 \wedge R_2 \wedge \cdots \wedge R_{r-p+1}$ and $\overline{\mathcal{R}} = \varrho(\mathcal{R})$. We evaluate (4.5) on the chain $\overline{\mathcal{R}}$. Since

$$\varrho[(d_H\omega)(\mathcal{X}_0)](\overline{\mathcal{R}}) = \varrho[(d_H\omega)(\mathcal{X}_0 \wedge \mathcal{R})]$$

and, because ϱ commutes with d and preserves bi-degree,

$$d_H[\varrho(\omega(\mathcal{X}_0))](\overline{\mathcal{R}}) = \varrho[d_H(\omega(\mathcal{X}_0))(\mathcal{R})]$$

we deduce that (4.5) is equivalent to

$$\varrho\big[(d_H\omega)(\mathcal{X}_0 \wedge \mathcal{R}) - (-1)^p[d_H(\omega(\mathcal{X}_0))](\mathcal{R})\big] = 0. \tag{4.6}$$

To establish (4.6), let

$$\begin{aligned}
\mathcal{X}_0^b &= (-1)^{b+1} \operatorname{tot} X_1 \wedge \cdots \widehat{\operatorname{tot} X_b} \wedge \cdots \operatorname{tot} X_p \\
\mathcal{X}_0^{ab} &= (-1)^{a+b} \operatorname{tot} X_1 \wedge \cdots \widehat{\operatorname{tot} X_a} \wedge \cdots \widehat{\operatorname{tot} X_b} \cdots \operatorname{tot} X_p, \\
\mathcal{R}^s &= (-1)^{s+p+1} R_1 \wedge R_2 \wedge \cdots \widehat{R_s} \wedge \cdots R_{r-p+1}, \qquad \text{and} \\
\mathcal{R}^{st} &= (-1)^{s+t} R_1 \wedge R_2 \wedge \cdots \widehat{R_s} \cdots \widehat{R_t} \wedge \cdots R_{r-p+1}
\end{aligned}$$

and note that

$$\operatorname{tot} X_c \wedge \mathcal{X}_0^b = \delta_c^b \mathcal{X}_0 \quad \text{and} \quad \operatorname{tot} X_c \wedge \mathcal{X}^{ab} = \delta_c^a \mathcal{X}^b - \delta_c^b \mathcal{X}^a. \tag{4.7}$$

Similar identities are satisfied by $\mathcal{R}^s$ and $\mathcal{R}^{st}$. Using (2.5), we compute

$$\begin{aligned}
&(d_H\omega)(\mathcal{X}_0 \wedge \mathcal{R}) - (-1)^p[d_H(\omega(\mathcal{X}_0))](\mathcal{R}) \\
&= \sum_{a=1}^{p} \operatorname{tot} X_a[\omega(\mathcal{X}_0^a \wedge \mathcal{R})] + \sum_{s=1}^{r-p+1} R_s[\omega(\mathcal{X}_0 \wedge \mathcal{R}^s)] \\
&+ \sum_{1\le a<b\le p} \omega([\operatorname{tot} X_a, \operatorname{tot} X_b] \wedge \mathcal{X}_0^{ab} \wedge \mathcal{R}) + \sum_{a=1}^{p}\sum_{s=1}^{r-p+1} \omega([\operatorname{tot} X_a, R_s] \wedge \mathcal{X}_0^a \wedge \mathcal{R}^s) \\
&+ \sum_{1\le s<t\le r-p+1} \omega([R_s, R_t] \wedge \mathcal{X}_0 \wedge \mathcal{R}^{st}) - (-1)^p[d_H(\omega(\mathcal{X}_0))](\mathcal{R}).
\end{aligned} \tag{4.8}$$

The second and last sums combine to cancel with $[d_H(\omega(\mathcal{X}_0))](\mathcal{R})$. We continue by expanding the first sum to obtain

$$\begin{aligned}
&\sum_{a=1}^{p} \operatorname{tot} X_a[\omega(\mathcal{X}_0^a \wedge \mathcal{R})] = \sum_{a=1}^{p} (\mathcal{L}_{\operatorname{tot} X_a}\omega)(\mathcal{X}_0^a \wedge \mathcal{R}) \\
&\quad + \sum_{a=1}^{p}\sum_{b=1}^{p} \omega([\operatorname{tot} X_a, \operatorname{tot} X_b] \wedge \mathcal{X}_0^{ab} \wedge \mathcal{R}) + \sum_{a=1}^{p}\sum_{s=1}^{r-p+1} \omega([\operatorname{tot} X_a, R_s] \wedge \mathcal{X}_0^a \wedge \mathcal{R}^s).
\end{aligned} \tag{4.9}$$

Now note that by the unimodular assumption (4.2)

$$\sum_{a,b=1}^{p} [\operatorname{tot} X_a, \operatorname{tot} X_b] \wedge \mathcal{X}_0^{ab} = \sum_{a,b,c=1}^{p} C_{ab}^c \operatorname{tot} X_c \wedge \mathcal{X}_0^{ab} = 2 \sum_{a,b=1}^{p} C_{ab}^b \mathcal{X}_0^a = 0. \tag{4.10}$$

The prolongation formula (2.23)

$$\operatorname{pr} X_a = \operatorname{tot} X_a + (X_a)_{\mathrm{ev}} \tag{4.11}$$

and the G invariance of R, implies that

$$[\operatorname{tot} X_a, R_s] = -[(X_a)_{\mathrm{ev}}, R_s]. \tag{4.12}$$

The combination of (4.8) – (4.12) simplifies to

$$\begin{aligned}
&(d_H\omega)(\mathcal{X}_0 \wedge \mathcal{R}) - (-1)^p[d_H(\omega(\mathcal{X}_0))](\mathcal{R}) \\
&= -\sum_{a=1}^{p} [\mathcal{L}_{\operatorname{pr}(X_a)_{\mathrm{ev}}}\omega](\mathcal{X}_0^a \wedge \mathcal{R}) - 2\sum_{a=1}^{p}\sum_{s=1}^{r-p+1} \omega([(X_a)_{\mathrm{ev}}, R_s] \wedge \mathcal{X}_0^a \wedge \mathcal{R}^s).
\end{aligned}$$

But $\operatorname{pr}(X_a)_{\mathrm{ev}}$ vanishes on $\operatorname{Inv}_G^\infty(E)$ and consequently we arrive at (4.6) and the proof of (4.5) is complete.

We have already seen in the second example presented in the introduction that $\varrho_{\mathcal{X}}$ fails to define a d_H cochain map if the unimodular condition (4.2) is not satisfied. To see the necessity of the unimodular condition, choose a G invariant total vector field

$$S = \sum_{a=1}^{p} f^a \operatorname{tot} X_a$$

in $\operatorname{Tot}\Gamma$. By Proposition 3.2, such total vector fields exists. For any point p_0 in M, the coefficients f^a are smooth functions on an open neighborhood U of p_0 in M which can be arbitrarily prescribed at p_0. The G invariance of S implies that these coefficients satisfy

$$X_b(f^a) + \sum_{c=1}^{p} f^c C^a_{bc} = 0. \tag{4.13}$$

Next choose a G invariant volume form ν on M, the existence of which is assured by Propositions 3.2 and 3.3. Define a G invariant $(n-1)$-form by $\omega = S \lrcorner\, \nu$. Obviously $\omega(\mathcal{X}_0) = 0$ and so, in order that (4.5) hold, we must have

$$\varrho[(d_H\omega)(\mathcal{X}_0)] = 0. \tag{4.14}$$

But by (2.12)–(2.15) and (4.13) we find that

$$d_H\omega = (\operatorname{Div}_\nu S)\, \nu = \sum_{a=1}^{p} [X_a(f^a) + f^a \operatorname{Div}_\nu \operatorname{tot} X_a]\, \nu = \sum_{a,b=1}^{p} f^a C^b_{ab}\, \nu.$$

Since the coefficients f^a can be arbitrarily prescribed at any point p_0 of M, (4.14) holds if and only if G is unimodular.

Finally, to prove the uniqueness of $\varrho_{\mathcal{X}}$, we first observe that if $\mathcal{X}$ is any invariant p-chain with coefficients in $J^\infty(E)$, then $\mathcal{X}$ is a multiple of $\mathcal{X}_0$ by a differential invariant I,

$$\mathcal{X} = I\mathcal{X}_0.$$

Then, for any $\operatorname{pr} G$ invariant form $\omega \in \Omega^{n-1,s}_{\operatorname{pr} G}(J^\infty(E))$,

$$d_H[\varrho(\omega(\mathcal{X}))] = \varrho(I)\, d_H[\varrho(\omega(\mathcal{X}_0))] + [d_H\varrho(I)] \wedge \varrho(\omega(\mathcal{X}_0))$$

so that, on account of (4.5), $\mathcal{X}$ satisfies $\varrho[(d_H\omega)(\mathcal{X})] = (-1)^p d_H[\varrho(\omega(\mathcal{X}))]$ if and only if

$$d_H[\varrho(I)] \wedge \varrho(\omega(\mathcal{X}_0)) = 0. \tag{4.15}$$

This equation implies that

$$d_H[\varrho(I)] = 0 \tag{4.16}$$

and therefore $\varrho(I)$ must be a constant. This proves that $\varrho_{\mathcal{X}}$ is unique up to a multiplicative constant. To pass from (4.15) to (4.16), let $(x^i, y^r, v^\alpha) \to (x^i, y^r)$ be local coordinates on E adapted to G (here we write simply x^i for the parametric variable $\widehat{x}^i$), let $\overline{R} = \dfrac{\partial}{\partial y^r}$ and let, by virtue of Proposition

3.3, R be any (locally defined) invariant total vector field such that $\varrho(R) = \overline{R}$. Then $\omega = R \lrcorner\, \nu$ is a G invariant form and

$$\varrho(\omega(\mathcal{X}_0)) = K\, dy^1 \wedge \cdots \wedge \widehat{dy^r} \wedge \cdots \wedge dy^{n-p}$$

for some non-zero, locally defined, function K on $J^\infty(\overline{E})$. Equation (4.15) therefore implies that $d_H(\varrho I)$ does not involve the differential dy^r. Since this is true for each $r = 1, 2, \ldots, n-p$ and on each coordinate neighborhood of $\overline{E}$, (4.16) follows. ∎

Now we consider the case where the action of G is not free, so that $q < p$. We assume that the necessary condition (3.17) for the existence of an invariant q-chain holds and that $\mathcal{X}$ is an invariant q-chain. Let $\alpha \in \Omega^{q,0}(J^\infty(E))$ be any non-zero horizontal q form such that

$$\alpha(\mathcal{X}) = 1. \tag{4.17}$$

The form α need *not* be G invariant. Let $\overline{\mu}$ be any generalized volume form on $\overline{M}$ and let $\mu \in \Omega^{n-q,0}_{\mathrm{pr}\, G}(J^\infty(E))$ be any horizontal G invariant $(n-q)$-form satisfying $\varrho(\mu) = \overline{\mu}$. The form $\nu = \alpha \wedge \mu$ is a generalized volume form on M and it is not difficult to check that the form ν is $\mathrm{pr}\, G$ invariant. Henceforth we shall work in local G adapted coordinates (x^i, y^r, v^α). We can write

$$X_i = \sum_{j=1}^{q} X_i^j(x^k, y^r)\frac{\partial}{\partial x^j} \qquad \text{and} \qquad \mathcal{X}_0 = \det(X^i_j)\partial_{x^1} \wedge \partial_{x^2} \wedge \cdots \wedge \partial_{x^q}$$

and therefore

$$\alpha = \frac{1}{\det[X^i_j]\, J}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^q \qquad \mathrm{mod}\ dy^r.$$

The general form for μ is

$$\mu = K\, \mu_0, \quad \text{where} \qquad \mu_0 = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^{n-q}, \tag{4.18}$$

and where K is a non-vanishing differential invariant and this leads to the invariant volume form

$$\nu = \frac{K}{\det[X^i_j]\, J}\, dx^1 \wedge \cdots \wedge dx^q \wedge dy^1 \wedge \cdots \wedge dy^{n-q}. \tag{4.19}$$

We now take $Y_r = \dfrac{\partial}{\partial y^r}$ so that (3.7) becomes $E^i_{ar} = -\dfrac{\partial \varphi^i_a}{\partial y^r}$.

Theorem 4.2. *Suppose the orbits of G have dimension q. A G invariant q-chain $\mathcal{X}$ determines a cochain map $\varrho_{\mathcal{X}}$ if and only if for every G invariant total vector field R*

$$\varrho(\mathrm{Div}_\nu\, R) = \mathrm{Div}_{\overline{\mu}}\, \varrho(R). \tag{4.20}$$

Before turning to the proof of this theorem, we note that (4.20) does not depend upon the choice of the form $\overline{\mu}$. If $\nu_0 = \alpha \wedge \mu_0$, then $\nu = K\nu_0$ and therefore $\mathrm{Div}_\nu\, R = K\, \mathrm{Div}_{\nu_0}\, R + R(K)$ and the independence of (4.20) on μ easily follows.

Proof of Theorem 4.2. We first check that (4.20) implies

$$\varrho_{\mathcal{X}}(d_H\omega) = d_H(\varrho_{\mathcal{X}}(\omega)) \tag{4.21}$$

for all G invariant forms ω of type $(n-1,0)$. Every such form ω can be expressed as $\omega = R \lrcorner\, \nu$ for some total vector field R. Because ω and ν are G invariant, the same is true of R and we can compute, using (2.12) and (4.20),

$$\begin{aligned}\varrho_{\mathcal{X}}(d_H\omega) &= \varrho_{\mathcal{X}}[(\operatorname{Div}_\nu R)\,\nu] = (-1)^{qn}\varrho(\operatorname{Div}_\nu R)\,\varrho(\nu(\mathcal{X}))\\ &= (-1)^{qn}(\operatorname{Div}_{\overline{\mu}}\overline{R})\,\overline{\mu} = (-1)^{qn} d_H(\overline{R} \lrcorner\, \overline{\mu})\\ &= (-1)^{q(n-1)} d_H[\varrho(\omega(\mathcal{X}))] = d_H(\varrho_{\mathcal{X}}(\omega)).\end{aligned}$$

For forms ω of type $(n-1,s)$, let $Z_1, Z_2, \ldots, Z_s$ be G invariant π_M vertical vector fields on E (see (2.26)), let $\overline{Z}_i = \varrho(Z_i)$, and let

$$\mathcal{Z} = \operatorname{pr} Z_1 \wedge \operatorname{pr} Z_2 \wedge \cdots \wedge \operatorname{pr} Z_s \quad \text{and} \quad \overline{\mathcal{Z}} = \varrho(\mathcal{Z}).$$

Then $\omega(\mathcal{Z})$ is a G invariant form of type $(n-1,0)$ and hence, by what we have just established,

$$\varrho_{\mathcal{X}}[d_H(\omega(\mathcal{Z}))] = d_H[\varrho_{\mathcal{X}}(\omega(\mathcal{Z}))]. \tag{4.22}$$

For any form $\beta \in \Omega^{r,s}(J^\infty(E))$ it is easy to prove that

$$d_H(\operatorname{pr} Z_i \lrcorner\, \beta) = -\operatorname{pr} Z_i \lrcorner\, (d_H\beta).$$

For any G invariant form β satisfying $\operatorname{tot} X \lrcorner\, \beta = 0$ for all $X \in \Gamma$, it is also easy to check that

$$\varrho(\operatorname{pr} Z_i \lrcorner\, \beta) = \operatorname{pr} \overline{Z}_i \lrcorner\, \varrho(\beta).$$

Together, these two commutation rules allow us to re-write (4.22) as

$$[\varrho_{\mathcal{X}}(d_H\omega)](\overline{\mathcal{Z}}) = [d_H(\varrho_{\mathcal{X}}(\omega))](\overline{\mathcal{Z}})$$

which, owing to the arbitrariness of $\mathcal{Z}$, suffices to prove (4.21) for forms of type $(n-1,s)$.

Finally, if ω is a type (r,s) G invariant form, where $q \le r \le n-2$, then $\widetilde{\omega} = \omega \wedge dy^I$, where $dy^I = dy^{r_1} \wedge dy^{r_2} \wedge \cdots dy^{n-r-1}$, is a G invariant form of type $(n-1,s)$. By what we have already proved

$$\varrho_{\mathcal{X}}(d_H\widetilde{\omega}) = d_H(\varrho_{\mathcal{X}}(\widetilde{\omega})).$$

But wedging with dy^I commutes (apart from sign) with interior evaluation with $\mathcal{X}$, with the horizontal derivative d_H and with the reduction map ϱ so that this last equation can be re-expressed as

$$[\varrho_{\mathcal{X}}(d_H\omega)] \wedge dy^I = [d_H(\varrho_{\mathcal{X}}(\omega))] \wedge dy^I.$$

Since the $n-r-1$ forms dy^I can be freely chosen, this proves (4.21) for forms of type (r,s). ∎

In order to apply Theorem 4.2, it is convenient to have a coordinate formula for (4.20). If

$$R = \sum_{i=1}^{q} g^i \operatorname{tot} X_i + W, \qquad \text{where} \qquad W = \sum_{r=1}^{n-q} h^r D_{y^r} \tag{4.23}$$

is a G invariant total vector field on $J^\infty(E)$ and if

$$\mu = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^{n-q},$$

(that is, $K = 1$ in (4.19)) then we define

$$(\operatorname{Div}^y_\mu W)(\sigma) = \sum_{r=1}^{n-q} [D_{y^r} h^r](\sigma)$$

where $\sigma \in J^\infty(E)$. Observe that if $\sigma \in \operatorname{Inv}^\infty_G(E)$ and $\overline{\sigma} = \Pi(\sigma) \in J^\infty(\overline{E})$, then

$$(\operatorname{Div}_{\overline{\mu}}\, \varrho(R))(\overline{\sigma}) = (\operatorname{Div}^y_\mu W)(\sigma)$$

and consequently we can express (4.20) as

$$(\operatorname{Div}_\nu R)(\sigma) = (\operatorname{Div}^y_\mu W)(\sigma) \tag{4.24}$$

for all $\sigma \in \operatorname{Inv}^\infty_G(E)$.

Lemma 4.3. *Let $\sigma_0 \in \operatorname{Inv}^\infty_G(E)$, let $\mathcal{U}$ be an open neighborhood of σ_0 in $J^\infty(E)$ and let $\mathcal{I} = \mathcal{U} \cap \operatorname{Inv}^\infty_G(E)$. If R is a G invariant total vector field on $\mathcal{U}$ given by (4.23), then for all points $\sigma \in \mathcal{I}$,*

$$(\operatorname{Div}_\nu R)(\sigma) - (\operatorname{Div}^y_\mu W)(\sigma) = \frac{1}{J(\sigma)} \big[- \sum_{r=1}^{n-q} h^r (D_{y^r} J) + \sum_{i=1}^{q} g^i B_i J \big](\sigma). \tag{4.25}$$

Proof. Since the volume form ν is G invariant, we compute

$$[\operatorname{Div}_\nu R](\sigma) = \sum_{i=1}^{q} [\operatorname{tot} X_i\, g^i](\sigma) + \sum_{r=1}^{n-q} [D_{y^r}(h^r)](\sigma) + \sum_{r=1}^{n-q} h^r(\sigma) \operatorname{Div}_\nu(D_{y^r})(p), \tag{4.26}$$

where $p = \pi^\infty_M(\sigma)$. On account of equations (3.10) and (3.5) we find that

$$\begin{aligned}\sum_{i=1}^{q} [\operatorname{tot} X_i\, g^i](\sigma) &= -\sum_{i=1}^{q} \big[\sum_{j=1}^{q} \Gamma^i_{ij}(p) g^j(\sigma) + \sum_{r=1}^{n-q} \xi^i_{ir}(p) h^r(\sigma) \big] \\ &= \sum_{i=1}^{q} B_i(p) g^i(\sigma) - \sum_{r=1}^{n-q} \xi_r(p) h^r(\sigma), \end{aligned} \tag{4.27}$$

where we have let $\xi_r = \sum_{i=1}^{q} \xi^i_{ir}$. It is not difficult to prove that

$$\frac{\partial}{\partial y^r} \det(X^i_j) = -\det(X^i_j) \xi_r$$

and hence

$$\mathrm{Div}_\nu(D_{y^r}) = D_{y^r}[\frac{1}{\det(X^i_j)J}]\det(X^i_j)J = \xi_r - \frac{1}{J}\frac{\partial J}{\partial y^r}. \tag{4.28}$$

We substitute (4.27) and (4.28) into (4.26) to arrive at (4.25) and the Lemma is established. ∎

We now turn to the analogue of the unimodular conditions for the non-free case $q < p$. As we saw in the proof of Theorem 4.1, these conditions arise from the simple observation that if S is a G invariant total vector field in the distribution $\mathrm{Tot}\,\Gamma$, then obviously $\varrho(S) = 0$ and consequently for (4.20) to hold, it is necessary that $\varrho(\mathrm{Div}_\nu S) = 0$. For each $p_0 \in M$, we have defined (see (3.11)) linear maps on the vector space $\mathbf{R}^q$ of q-tuples $v = [v^1, v^2, \ldots, v^q]$ by

$$A_{p_0}(v) = [\sum_{j=1}^q A^i_{aj}(p_0)v^j\,] \quad \text{and} \quad B_{p_0}(v) = \sum_{j=1}^q B_j(p_0)v^j.$$

Proposition 4.4. *Let $\sigma_0 \in \mathrm{Inv}^\infty_G(E)$ and let S be any G invariant total vector field which lies in the distribution $\mathrm{Tot}\,\Gamma$ and which is smooth on an open neighborhood $\mathcal{U}$ of σ_0 in $J^\infty(E)$. Let $\mathcal{I} = \mathcal{U} \cap \mathrm{Inv}^\infty_G(E)$. Then*

$$(\mathrm{Div}_\nu\, S)(\sigma) = 0 \tag{4.29}$$

for all $\sigma \in \mathcal{I}$ if and only if

$$\ker A_p \subset \ker B_p \tag{4.30}$$

for each point $p \in \pi^\infty_M(\mathcal{I}) \subset M$.

Proof. Suppose (4.30) holds. If $S = \sum_{i=1}^q f^i\,\mathrm{tot}\,X_i$ is G invariant, then by Lemma 3.4 the coefficients f^i satisfy

$$\sum_{j=1}^q A^i_{aj}(p)f^j(\sigma) = 0$$

for each point $\sigma \in \mathcal{I}$ and $p = \pi^\infty_M(\sigma)$. Hence we conclude from (4.30) that,

$$\sum_{j=1}^q B_j(p)f^j(\sigma) = 0.$$

This, together with (4.25) (with $W = 0$), implies that

$$\mathrm{Div}_\nu(S)(\sigma) = \sum_{j=1}^q B_j(p)f^j(\sigma) = 0. \tag{4.31}$$

Conversely, suppose that (4.29) is true. Let $v = [v^1, v^2, \ldots, v^q]$ be any q-tuple such that $A_{p_0}(v) = 0$. By Proposition 3.4 there is a G invariant vector field

$$S^0 = \sum_{i=1}^q f^i X_i$$

defined on a neighborhood U of p_0 in M such that $f^i(p_0) = v^i$. Let $S = \operatorname{tot} S^0$. Then (4.31) implies that

$$\sum_{i=1}^{q} B_i(p_0) v^i = \sum_{i=1}^{q} B_i(p_0) f^i(p_0) = \operatorname{Div}_\nu(S)(\sigma)$$

for any point $\sigma \in \operatorname{Inv}^\infty_G(E) \cap (\pi^\infty_M)^{-1}(U)$ and hence $v \in \ker B_{p_0}$. ∎

The algebraic conditions $\operatorname{tr} A(p) = 0$ and $\ker A_p \subset \ker B_p$ arising in Propositions 3.6 and 4.4 have a natural interpretation in terms of the Lie algebra cohomology of Γ. To describe this result, we briefly recall the relevant definitions. Let $\mathbf{g}$ be a p dimensional Lie algebra and let $\mathbf{h}$ be a $p - q$ dimensional subalgebra. Let $X_1,\, X_2,\, \ldots\,,\, X_p$ be a basis for $\mathbf{g}$ such that $X_{q+1},\, X_{q+2},\, \ldots\,,\, X_p$ is a basis for $\mathbf{h}$. We have

$$[\, X_a, X_b\,] = \sum_{c=1}^{p} C^c_{ab} X_c$$

for $a, b = 1,\, 2,\, \ldots\,,\, p$, so that $C^i_{\delta\epsilon} = 0$ for $\delta, \epsilon = q+1,\, q+2,\, \ldots\; p$ and $i = 1,\, 2,\, \ldots\; q$. The 1 forms α^a dual to the vectors X_a satisfy

$$d\,\alpha^a = -\frac{1}{2} C^a_{bc} \alpha^b \wedge \alpha^c \qquad \text{and} \qquad \mathcal{L}_{X_b}(\alpha^a) = -\sum_{c=1}^{p} C^a_{bc} \alpha^c.$$

Let $A^k(\mathbf{g})$ be the alternating, multi-linear k forms on $\mathbf{g}$. The vector space of k-cochains on $\mathbf{g}$ relative to $\mathbf{h}$ is then defined by

$$A^k(\mathbf{g}, \mathbf{h}) = \{\, \omega \in A^k(\mathbf{g}) \,|\, X \lrcorner\, \omega = 0 \quad \text{and} \quad \mathcal{L}_X \omega = 0 \quad \text{for all } X \in \mathbf{h}\,\}.$$

The k-th order cocycles, coboundaries and cohomology of $\mathbf{g}$ relative to $\mathbf{h}$ are defined by the vector spaces

$$Z^k(\mathbf{g}, \mathbf{h}) = \{\, \omega \in A^k(\mathbf{g}, \mathbf{h}) \,|\, d\,\omega = 0\,\},$$

$$B^k(\mathbf{g}, \mathbf{h}) = \{\, \omega \in A^k(\mathbf{g}, \mathbf{h}) \,|\, \omega = d\,\eta \quad \text{for } \eta \in A^{k-1}(\mathbf{g}, \mathbf{h})\,\},$$

and

$$H^k(\mathbf{g}, \mathbf{h}) = Z^k(\mathbf{g}, \mathbf{h}) \big/ B^k(\mathbf{g}, \mathbf{h}).$$

Lemma 4.5. *Let* $\mathbf{g}$ *be a* p *dimensional Lie algebra and* $\mathbf{h}$ *a* $p - q$ *dimensional subalgebra. Then*

$$H^q(\mathbf{g}, \mathbf{h}) \neq 0$$

if and only if

[*i*] $\displaystyle\sum_{i=1}^{q} C^i_{\epsilon i} = 0$ *for* $\epsilon = q+1,\, q+2,\, \ldots\,,\, p$*; and*

[*ii*] *every* q *tuple* $(v^1, v^2, \ldots, v^q) \in \mathbf{R}^q$ *satisfying*

$$\sum_{i=1}^{q} v^i C^j_{\epsilon i} = 0 \quad \textit{for all} \quad \epsilon = q+1, q+2, \ldots p, \quad j = 1, 2, \ldots, q \tag{4.32}$$

also satisfies

$$\sum_{i,j=1}^{q} v^i C_{ij}^j = 0. \tag{4.33}$$

Proof. We shall prove (a) that $[i]$ holds if and only if $Z^q(\mathbf{g},\mathbf{h}) \neq 0$; and (b) given that $[i]$ holds, condition $[ii]$ is true if and only if $B^q(\mathbf{g},\mathbf{h}) = 0$. Since $A^q(\mathbf{g},\mathbf{h})$ is at most 1 dimensional, this proves the lemma.

Since any form in $A^q(\mathbf{g})$ which is annihilated by all the vectors X_ϵ, $\epsilon = q+1, q+2, \dots, p$, is a scalar multiple of

$$\nu = \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^q$$

and since

$$\mathcal{L}_{X_\epsilon}\nu = \sum_{i=1}^{q} (\mathcal{L}_{X_\epsilon}\alpha^i) \wedge (X_i \lrcorner\, \nu) = \sum_{i=1}^{q} C_{\epsilon i}^i\, \nu$$

statement (a) is immediate. Note that $[i]$ also implies that the form ν is closed.

If $\eta \in A^{q-1}(\mathbf{g})$ is annihilated by the vectors X_ϵ, then we may write

$$\eta = \sum_{i=1}^{q} v^i \nu_i, \qquad \text{where } \nu_i = X_i \lrcorner\, \nu.$$

Granted that $[i]$ holds, we compute

$$\mathcal{L}_{X_\epsilon}\eta = \sum_{i=1}^{q} v^i [\, X_\epsilon, X_i\,] \lrcorner\, \nu = \sum_{i,j=1}^{q} v^i C_{\epsilon i}^j\, \nu_j$$

and so the form $\eta \in A^{q-1}(\mathbf{g},\mathbf{h})$ if and only if the coefficients (v^i) satisfy the homogeneous equations (4.32). We then compute

$$d\,\eta = \sum_{i=1}^{q} v^i d(X_i \lrcorner\, \nu) = \sum_{i=1}^{q} v^i \mathcal{L}_{X_i}\nu = \sum_{i,j=1}^{q} v^i C_{ij}^j \nu.$$

Consequently, if $[ii]$ holds, then $d\,\eta = 0$ and $B^q(\mathbf{g},\mathbf{h}) = 0$. Conversely, if $[ii]$ does not hold, then there is a tuple (v^i) satisfying (4.32) but not (4.33). The associated form $\eta \in A^{q-1}(\mathbf{g},\mathbf{h})$ but $d\,\eta \neq 0$ and hence $B^q(\mathbf{g},\mathbf{h}) \neq 0$. ∎

Remark 4.6. It is easy to check that a vector

$$X = \sum_{i=1}^{q} v^i X_i + \sum_{\epsilon=q+1}^{p} w^\epsilon X_\epsilon$$

belongs to the normalizer $n(\mathbf{h})$ of $\mathbf{h}$ if and only if the q tuple (v^i) satisfies (4.32). Thus condition $[ii]$ of the Lemma always holds when $n(\mathbf{h}) = \mathbf{h}$.

Let $e_0 \in E$ and $p_0 = \pi(e_0)$. If we choose a basis for Γ adapted to the linear isotropy algebra $\Gamma_0(e_0)$ as in section 3, then it follows directly from (3.6) that the condition $\mathrm{tr}A(p_0) = 0$ is equivalent to the condition $[i]$ of Lemma 4.5 and that $\ker A(p_0) \subset \ker B(p_0)$ is equivalent to condition $[ii]$. In short, the conditions $\mathrm{tr}A(p_0) = 0$ and $\ker A(p_0) \subset \ker B(p_0)$ hold if and only if $H^q(\Gamma, \Gamma_0(e_0)) \neq 0$.

We are now is a position to establish the main result of this paper.

Theorem 4.7. *Suppose that the orbits of G have dimension q. Let $e_0 \in E$. Then there is an open neighborhood U in E around e_0 and a G invariant chain*

$$\mathcal{X} = J \operatorname{tot} X_1 \wedge \operatorname{tot} X_2 \wedge \cdots \wedge \operatorname{tot} X_q,$$

where J is a non-zero function on $\pi(U) \subset M$, such that

$$\varrho_{\mathcal{X}} \colon \Omega^{r,s}_{\operatorname{pr} G}(J^\infty(E)) \to \Omega^{r-q,s}(J^\infty(\overline{E})).$$

is a d_H cochain map if and only if

$$H^q(\Gamma, \Gamma_0(e_0)) \neq 0. \tag{4.34}$$

Proof. The necessity of (4.34) has been established in Propositions 3.6 and 4.4, and Lemma 4.5.

Conversely, assume that (4.34) holds so that the conclusions of Propositions 3.6 and 4.4 hold. Let

$$\mathcal{W} = \dim\{\, w \in \mathbf{R}^{n-q} \mid E_p(w) \in \operatorname{im} A_p \,\}$$

and

$$\delta_1 = \dim \ker A_p \qquad \text{and} \qquad \delta_2 = \dim \mathcal{W}.$$

We have tacitly assumed that we are working on a neighborhood of p_0 where δ_1 and δ_2 are constant.

By Proposition 3.5, there exist G invariant vector fields

$$S^0_1, S^0_2, \ldots, S^0_{\delta_1} \qquad \text{and} \qquad R^0_1, R^0_2, \ldots, R^0_{\delta_2} \tag{4.35}$$

defined locally on M such that the S^0_k, $1 \leq k \leq \delta_1$, lie in the distribution Γ and the R^0_s, $1 \leq s \leq \delta_2$, project under ϱ to linearly independent vector fields on $\overline{M}$. Every G invariant vector field on M may be expressed as a linear combination of the vector fields S^0_k and R^0_s. Because we are using the G adapted coordinates (x^i, y^r, v^α), we can write these latter vector fields in the form

$$R^0_s = \sum_{i=1}^q V^i_s X_i + W_s,$$

where the V^i_s are locally defined functions on M and $W_s = \displaystyle\sum_{r=1}^{n-q} W^r_s(y) \frac{\partial}{\partial y^r}$. The coefficients $W^r_s(y)$ do not depend upon the variables x^j because the vector fields R^0_s are G invariant. Note that $\varrho(R^0_s) = W_s$. By Lemma 3.4, the coefficients V^i_s and W^r_s satisfy

$$\sum_{j=1}^q A^i_{aj} V^j_s + \sum_{r=1}^{n-q} E^i_{ar} W^r_s = 0. \tag{4.36}$$

We have already seen that the condition that the chain

$$\mathcal{X} = J \operatorname{tot} X_1 \wedge \operatorname{tot} X_2 \wedge \cdots \wedge \operatorname{tot} X_q.$$

be G invariant leads to the differential equations

$$X_j(J) + B_j J = 0 \qquad j = 1, 2, \ldots q, \tag{4.37}$$

where B_j is defined by (3.5). If we apply Theorem 4.2 to the vector fields $R_s = \operatorname{tot} R_s^0$ and use (4.25) and (4.24), we obtain the following δ_2 additional differential equations for J,

$$W_s(J) - \sum_{i=1}^{q} V_s^i B_i J = 0. \tag{4.38}$$

To prove the theorem we must therefore show that

(A) if J is any locally defined function on M which solves (4.37) and (4.38), then $\mathcal{X}$ is a G invariant chain and (4.20) holds for every G invariant vector field R; and

(B) the integrability conditions for the system (4.37), (4.38) are satisfied. The vector fields X_j and W_s are pointwise independent and therefore, by the Frobenius theorem, we are assured that solutions to (4.37) and (4.38) exist.

We remark that (4.37) and (4.38) define the system of equations (1.9) introduced in section 1.

We have already established in Proposition 3.6 that (4.34) together with (4.37) guarantee that $\mathcal{X}$ is G invariant. Let

$$R = \sum_{i=1}^{q} g^i \operatorname{tot} X_i + \sum_{r=1}^{n-q} h^r D_{y^r}$$

be any G invariant total vector field defined in open neighborhood $\mathcal{U}$ of a point $\sigma_0 \in \operatorname{Inv}_G^\infty(E)$. Then for each point $\sigma = j^\infty(s)(p) \in \mathcal{U} \cap \operatorname{Inv}_G^\infty(E)$, we know that

$$\sum_{j=1}^{q} A_{aj}^i(p) g^j(\sigma) + \sum_{r=1}^{n-q} E_{ar}^i(p) h^r(\sigma) = 0. \tag{4.39}$$

But the $n-q$ tuples $[W_s^1(p), W_s^2(p), \ldots, W_s^{n-q}(p)]$, $s = 1,2, \ldots, \delta_2$, span the space of $n-q$ tuples $[h^r]$ such that $E_p(h)$ lies in the image of A_p. Therefore there exists functions λ^s on $\mathcal{U} \cap \operatorname{Inv}_G^\infty(E)$ such that, for $\sigma \in \mathcal{U} \cap \operatorname{Inv}_G^\infty(E)$,

$$h^r(\sigma) = \sum_{s=1}^{\delta_2} \lambda^s(\sigma) W_s^r(p). \tag{4.40}$$

We now substitute (4.40) back into (4.39) to conclude that

$$g^i(\sigma) = \sum_{s=1}^{\delta_2} \lambda^s(\sigma) V_s^i(p) + \kappa^i(\sigma), \tag{4.41}$$

where $\sum_{j=1}^{q} A_{aj}^i \kappa^j = 0$. To check the divergence condition (4.25), we compute

$$\begin{aligned} \operatorname{Div}_\nu R(\sigma) - \operatorname{Div}_\mu^y W(\sigma) &= \frac{1}{J(\sigma)} \Big[\sum_{r=1}^{n-q} -h^r D_{y^r} J + \sum_{i=1}^{q} g^i B_i J\Big](\sigma) \\ &= \frac{1}{J} \sum_{s=1}^{\delta_2} [-\sum_{r=1}^{n-q} W_s^r D_{y^r} J + \sum_{i=1}^{q} B_i V_s^i J] \lambda^s + \frac{1}{J} \sum_{i=1}^{q} B_i \kappa^i. \end{aligned} \tag{4.42}$$

The first term in (4.42) vanishes by virtue of (4.38) and the second term is zero on account of hypothesis [ii] of Lemma 4.5 (that is, (4.30)). The proof of (A) is complete.

The integrability conditions for (4.37) and (4.38) can be checked by direct, lengthy computations. It is simpler, however, to first note that (4.37) and (4.38) are equivalent to

$$\mathrm{Div}_\nu X_i = 0 \tag{4.43}$$

and

$$\mathrm{Div}_\nu R^0_s = \mathrm{Div}^y_\mu W_s. \tag{4.44}$$

Since

$$X_i \,\mathrm{Div}_\nu X_j - X_j \,\mathrm{Div}_\nu X_i = \mathrm{Div}_\nu[\, X_i, X_j],$$

$$X_i \,\mathrm{Div}_\nu R^0_s - R^0_s \,\mathrm{Div}_\nu X_i = \mathrm{Div}_\nu[\, X_i, R^0_s\,], \qquad \text{and}$$

$$R^0_s \,\mathrm{Div}_\nu R^0_t - R^0_t \,\mathrm{Div}\, R^0_s = \mathrm{Div}_\nu[\, R^0_s, R^0_t\,],$$

the integrability conditions for (4.43) and (4.44) are

$$\mathrm{Div}_\nu[\, X_i, X_j\,] = 0, \tag{4.45}$$

$$X_i \,\mathrm{Div}^y_\mu W_s = \mathrm{Div}_\nu[\, X_i, R^0_s\,] \qquad \text{and} \tag{4.46}$$

$$R^0_s(\mathrm{Div}^y_\mu W_t) - R^0_t(\mathrm{Div}^y_\mu W_s) = \mathrm{Div}_\nu[\, R^0_s, R^0_t\,], \tag{4.47}$$

where $i, j = 1, 2, \ldots, q$ and $s, t = 1, 2, \ldots, \delta_2$.

Since $[\, X_i, X_j\,] = \sum_{l=1}^{q} \Gamma^l_{ij} X_l$, we can use (3.7) and hypothesis [i] to find that

$$\begin{aligned}\mathrm{Div}_\nu[\, X_i, X_j\,] &= \mathrm{Div}_\nu\Big(\sum_{l=1}^{q} \Gamma^l_{ij} X_l\Big) = \sum_{l=1}^{q} X_l(\Gamma^l_{ij}) \\ &= \sum_{l=1}^{q} X_l\Big(\sum_{c=1}^{p} C^c_{ij}\phi^l_c\Big) = -\sum_{c=1}^{p}\sum_{l=1}^{q} C^c_{ij} A^l_{cl} = 0\end{aligned}$$

and the first set of integrability conditions (4.45) hold. Because R^0_s is a G invariant vector field, $[X_i, R^0_s] = 0$. Since

$$\mathrm{Div}^y_\mu W_s = \sum_{r=1}^{n-q} \frac{\partial}{\partial y^r} W^r_s(y)$$

we conclude that

$$X_i(\mathrm{Div}^y_\mu W_s) = 0 \tag{4.48}$$

and (4.46) is verified.

Finally, equation (4.48) allows us to re-write (4.47) as

$$\mathrm{Div}_\nu[\, R^0_s, R^0_t\,] = \mathrm{Div}^y_\mu[\, W_s, W_t\,]. \tag{4.49}$$

To check this integrability condition we use the fact that $[\,R^0_s, R^0_t\,]$ is a G invariant vector field on M to deduce that

$$[\,R^0_s, R^0_t\,] = \sum_{k=1}^{\delta_1} a^k_{st} S^0_k + \sum_{u=1}^{\delta_2} b^u_{st} R^0_u,$$

where the coefficients a^i_{st} and b^u_{st} are functions of the invariant coordinates y^r only. It is easy to check that the coefficients b^u_{st} are the structure functions for the vector fields W_s, that is,

$$[W_s, W_t] = \sum_{u=1}^{\delta_2} b^u_{st} W_u. \tag{4.50}$$

Consequently we can now compute

$$\mathrm{Div}_\nu[\,R^0_s, R^0_t\,] = \sum_{k=1}^{\delta_1} S^0_k(a^i_{st}) + \sum_{k=1}^{\delta_1} a^i_{st}\,\mathrm{Div}_\nu\, S^0_k + \sum_{u=1}^{\delta_2} R_u(b^u_{st}) + \sum_{u=1}^{\delta_2} b^u_{st}\,\mathrm{Div}_\nu\, R_u.$$

The first sum vanishes since the vector fields S^0_k lie in Γ and the coefficients a^k_{st} are G invariant. The second sum vanishes by Proposition 4.4 and hypothesis [ii]. The third sum reduces, on account of (4.48) and the fact that the functions b^u_{st} are invariants, to

$$\sum_{u=1}^{\delta_2} R_u b^u_{st} = \sum_{u=1}^{\delta_2} W_u b^u_{st}$$

and therefore, by (4.50),

$$\mathrm{Div}_\nu[\,R^0_s, R^0_t\,] = \sum_{u=1}^{\delta_2} W_u(b^u_{st}) + \sum_{u=1}^{\delta_2} b^u_{st}\,\mathrm{Div}_\mu\, W_u = \mathrm{Div}^y_\mu[W_s, W_t].$$

This proves (4.49) and the integrability conditions for (4.37) and (4.38) are satisfied. ∎

Remark 4.8. Let G be a compact, connected, Lie group acting on E and let $H(e_0)$ be the isotropy subgroup of G at $e_0 \in E$. Let $H_0(e_0)$ be the connected component of the identity. Then $H_0(e_0)$ is a closed subgroup of G and therefore, as is well-known ([5]),

$$H^k_{\text{de Rham}}(G/H_0) = H^k(\Gamma, \Gamma_0(e_0))$$

where Γ and $\Gamma_0(e_0)$ are the Lie algebras of G and $H_0(e_0)$. In particular, suppose X_1, X_2, … X_p are a basis for the left invariant vector fields on G such that the vectors X_{q+1}, X_{q+2}, … X_p are tangent to H_0. If the forms α^1, α^2, … , α^p are the dual one forms on G, then because G is unimodular, it follows that

$$\nu = \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^q$$

is H_0 invariant and therefore ν drops to a non-zero volume form $\bar\nu$ on G/H_0. This shows that

$$H^q(\Gamma, \Gamma_0(e_0)) \neq 0$$

and accordingly there is always a locally defined cochain map $\varrho_{\mathcal{X}}$ from the G invariant variational bicomplex on E to the reduced variational bicomplex on $\overline{E} = E/G$.

More generally, suppose that $(\mathbf{g}, \mathbf{h})$ and $(\mathbf{g}', \mathbf{h}')$ are two real Lie algebra, subalgebra pairs which have isomorphic complexications, that is, there is an isomorphism

$$\psi \colon \mathbf{g} \otimes \mathbf{C} \to \mathbf{g}' \otimes \mathbf{C}$$

of complex Lie algebras which restricts to an isomorphism

$$\psi \colon \mathbf{h} \otimes \mathbf{C} \to \mathbf{h}' \otimes \mathbf{C}.$$

Then the complexications of the relative cohomologies are isomorphic,

$$\begin{aligned} H^*(\mathbf{g}, \mathbf{h}) \otimes \mathbf{C} &\cong H^*(\mathbf{g} \otimes \mathbf{C}, \mathbf{h} \otimes \mathbf{C}) \cong H^*(\mathbf{g}' \otimes \mathbf{C}, \mathbf{h}' \otimes \mathbf{C}) \\ &\cong H^*(\mathbf{g}', \mathbf{h}') \otimes \mathbf{C}. \end{aligned}$$

It then follows that if the complexification of the pair $(\mathbf{g}, \mathbf{h})$ admits compatible compact real forms $(\mathbf{g}', \mathbf{h}')$, then

$$H^q(\mathbf{g}, \mathbf{h}) \neq 0.$$

For example, if $\mathbf{so}(n, 1)$ is the Lie algebra of the Lorentz group, then the pairs

$$(\mathbf{so}(n,1), \mathbf{so}(n-1,1)) \quad \text{and} \quad (\mathbf{so}(n+1), \mathbf{so}(n))$$

have the same relative cohomology since they both complexify to $(\mathbf{so}(n+1, \mathbf{C}), \mathbf{so}(n, \mathbf{C}))$ and thus

$$H^{n+1}(\mathbf{so}(n,1), \mathbf{so}(n-1,1)) \cong H^{n+1}(\mathbf{so}(n+1), \mathbf{so}(n)) = H^{n+1}_{\text{de Rham}}(S^{n+1}) \cong \mathbf{R}.$$

In exactly the same manner we deduce that

$$H^{n+1}(\mathbf{so}(n,1), \mathbf{so}(n)) \neq 0.$$

Thus for the standard action of the Lorentz group $\mathbf{so}(n, 1)$ on $M = \mathbf{R}^{n+1}$, a cochain map to the reduced variational bicomplex always exists in a neighborhood of any point not on the light cone.

As another simple example, consider the real lie algebras $\mathbf{sl}(2, \mathbf{R})$ and $\mathbf{su}(2)$ with basis

$$a_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad a_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$e_1 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

respectively, and with structure equations

$$[a_0, a_\pm] = \pm a_\pm, \qquad [a_+, a_-] = 2a_0, \qquad \text{and} \qquad [e_i, e_j] = \epsilon_{ijk} e_k.$$

The formulas

$$e_1 = -\frac{i}{2}(a_+ + a_-), \quad e_2 = -\frac{1}{2}(a_+ - a_-), \quad e_3 = -ia_0$$

define a Lie algebra isomorphism between the complexifications of $\mathbf{sl}(2, \mathbf{R})$ and $\mathbf{su}(2)$ and hence, on the one hand,

$$H^2(\mathbf{sl}(2, \mathbf{R}), \langle a_0 \rangle) \cong H^2(\mathbf{su}(2), \langle e_3 \rangle) \neq 0.$$

On the other hand, the pair $(\mathbf{sl}(2, \mathbf{R}), \langle a_+ \rangle)$ is not complex isomorphic to a pair $(\mathbf{su}(2), \mathbf{h})$ for any real subalgebra $\mathbf{h}$ of $\mathbf{su}(2)$ and indeed, by direct computation, one has that

$$H^2(\mathbf{sl}(2, \mathbf{R}), \langle a_+ \rangle) = 0.$$

We conclude that if $\mathbf{sl}(2, \mathbf{R})$ is the symmetry algebra under consideration, then a cochain map $\varrho_{\mathcal{X}}$ to the reduced variational bicomplex exists if, for instance, the isotropy subalgebra is $\langle a_0 \rangle$ but $\varrho_{\mathcal{X}}$ does not exist in case the isotropy subalgebra is $\langle a_+ \rangle$.

Remark 4.9. In closing we remark that the defining equations (4.37) and (4.38) for the chain $\mathcal{X}$ can be determined in a completely algebraic manner without first having to explicitly construct the invariant vector fields (4.35). Indeed, if we let $\overset{0}{W}_s = [\overset{0}{W}{}^r_s]$ be any basis for the linear space $\mathcal{W}$ and let $\overset{0}{V}{}^j_s$ be such that

$$\sum_{j=1}^{q} A^i_{aj} \overset{0}{V}{}^j_s + \sum_{r=1}^{n-q} E^i_{ar} \overset{0}{W}{}^r_s = 0, \tag{4.51}$$

where $s = 1, 2, \ldots, \delta_2$, then, just as with (4.40) and (4.41), we can write

$$W^r_s(p) = \sum_{t=1}^{\delta_2} \lambda^t_s(p) \overset{0}{W}{}^r_t(p)$$

and

$$V^i_s(p) = \sum_{t=1}^{\delta_2} \lambda^t_s(p) \overset{0}{V}{}^i_t(p) + \kappa^i_s(p).$$

The matrix λ^t_s is invertible and it is easy to see that (4.38) is equivalent to

$$\sum_{r=1}^{n-q} \overset{0}{W}{}^r_s \frac{\partial J}{\partial y^r} - \sum_{i=1}^{q} \overset{0}{V}{}^i_s B_i J = 0. \tag{4.52}$$

Furthermore, these equations can be expressed, not in terms of the preferred vector fields $\dfrac{\partial}{\partial y^r}$ but in terms of the arbitrary frame (3.3).

§**5. Examples.** We begin with an illustration of Theorem 4.1. We consider a slight variation of the Lie transformation group considered in [*24*] (see Table II, Case 8).

Example 5.1. Consider the Lagrangian

$$\lambda = \frac{1}{2}(-u_t^2 + u_x^2 + u_y^2 + u_z^2 - u^4)\, dt \wedge dx \wedge dy \wedge dz. \tag{5.1}$$

The associated Euler-Lagrange equation is

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} - 2u^3 = 0. \tag{5.2}$$

This equation possesses the 3 dimensional symmetry algebra

$$X_1 = (t+z)\partial_t + x\partial_x + y\partial_y + (t+z)\partial_z - u\partial_u,$$
$$X_2 = -x\partial_t - (t+z)\partial_x + x\partial_z, \qquad X_3 = \partial_y$$

whose commutators are

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = -X_3, \qquad [X_2, X_3] = 0. \tag{5.3}$$

This symmetry algebra also preserves the Lagrangian (5.1). To avoid degeneracy in the symmetry reduction of (5.2), we let

$$V = \partial_t - \partial_z$$

and modify X_1 to

$$X_1^\epsilon = X_1 + \epsilon V.$$

Since V commutes with X_1, X_2 and X_3, the structure equations (5.3) are unchanged. Thus X_1^ϵ, X_2, X_3 generate a 3 dimensional unimodular Lie algebra with 3 dimensional orbits. In accordance with Propositions 3.2 and 3.3 we have the following invariant basis for $\operatorname{Tot}(J^\infty(E))$:

$$S_1 = \operatorname{tot} X_1 - \frac{x}{t+z}\operatorname{tot} X_2 - y \operatorname{tot} X_3, \qquad S_2 = \frac{1}{\sqrt{t+z}}\operatorname{tot} X_2,$$
$$S_3 = \sqrt{t+z}\operatorname{tot} X_3, \qquad R_1 = D_t - D_z.$$

Our invariant adapted coordinates for this example are

$$r = \frac{x^2 + z^2 - t^2 + \epsilon(z+t)\log(z+t)}{z+t}, \quad \widehat{x} = x, \quad \widehat{y} = y, \quad \widehat{z} = z, \quad v = \sqrt{t+z}\, u$$

and the reduced equation for the invariant solutions of (5.2) is

$$2\epsilon \frac{d^2 v}{dr^2} + v^3 = 0. \tag{5.4}$$

By Theorem 4.1, the 3 chain

$$\mathcal{X} = \operatorname{tot} X_1 \wedge \operatorname{tot} X_2 \wedge \operatorname{tot} X_3$$

is invariant and defines a cochain map $\varrho_{\mathcal{X}}$. Thus the Lagrangian (5.1) reduces to the Lagrangian

$$\varrho_{\mathcal{X}}(\lambda) = [v v_r + \frac{1}{2}v^4 - 2\epsilon(v_r)^2]\, dr.$$

for (5.4) and the G invariant conservation law

$$\begin{aligned}\omega =&(u_t^2 - u_x^2 - u_y^2 + u^4 - 2u_t u_z)dt \wedge dx \wedge dy + 2u_y(u_t - u_z)\, dt \wedge dx \wedge dz \\ &- 2u_x(u_t - u_z)\, dt \wedge dy \wedge dz - (u_t^2 + u_x^2 + u_y^2 + u_z^2 - u^4 - 2u_t u_z)\, dt \wedge dx \wedge dy\end{aligned} \tag{5.5}$$

gives rise to the first integral

$$\varrho(\omega(\mathcal{X})) = -8\epsilon(v_r)^2 - 2v^4. \tag{5.6}$$

We hope in the future to examine the possibility of systematically lifting conservation laws for the reduced equation to G invariant conservation laws for the full equations so that, for example, given the conservation law (5.6) for (5.4) one arrives at the conservation law (5.5) for (5.2).

One curious aspect of our symmetry reduction procedure for invariant variational bicomplexes is that the cochain map $\varrho_{\mathcal{X}}$ may exist for a given group G but not for some of the subgroups of G. Take, for instance, the two dimensional subgroup generated by the vectors X_1^ϵ and X_3. The invariants of this group are

$$r = \frac{e^{z-t}}{x^{2\epsilon}}, \qquad s = \frac{t+z}{x^2}, \qquad v = xu$$

and the reduced equation is

$$r^2 v_{rr} + s(1 - 2\epsilon r)v_{rs} + s^2\epsilon^2 v_{ss} + \epsilon s(\epsilon - \frac{3}{2})v_s + \frac{5}{2} r v_r + \frac{1}{2}(v + v^3) = 0. \tag{5.7}$$

One can apply the results of [*1*] to deduce that (5.7) is not an Euler-Lagrange equation for any Lagrangian

$$\lambda = L(r, s, v, v_r, v_s, v_{rr}, v_{rs}, \dots)\, dr \wedge ds.$$

Compare with Olver [*15*], exercise 4.13, where it claimed that the reduction of a system of Euler-Lagrange equations by a group of variational symmetries is always variational. ∎

Example 5.2. In the study of exact solutions of the Einstein field equations in general relativity, the principle of symmetric criticality is often used to derive the reduced field equations. When the space-time is spatially homogeneous, that is, admits a 3 parameter simply transitive (or free) group of motions it follows immediately from our general theory the principle is valid if and only if the group is unimodular. These same conclusions were obtained, by lengthy computation, by MacCullum and Taub [*13*]. We intend to present further applications of our results to classical field theory in a subsequent paper. ∎

Example 5.3. Here we study symmetry reduction for the standard action of the special orthogonal group SO(n) on $\mathbf{R}^n$. The standard infinitesimal generators are

$$X_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$$

with structure equations

$$[X_{ij}, X_{hk}] = \delta_{jh} X_{ik} - \delta_{ih} X_{jk} - \delta_{jk} X_{ih} + \delta_{ik} X_{jh}. \tag{5.8}$$

The orbits of SO(n) are $n-1$ dimensional, the isotropy subgroup of any point $p_0 \neq 0$ is SO($n-1$) and since

$$H^n(\mathbf{so}(n), \mathbf{so}(n-1)) \cong H^n_{\text{de Rham}}(S^n) \cong \mathbf{R}$$

we conclude that a cochain map $\varrho_{\mathcal{X}}$ to the reduced variational bicomplex always exists. In fact, the $n-1$ chain

$$\mathcal{X} = r^{n-2}\mathcal{X}^0 \qquad \text{where} \qquad \mathcal{X}^0 = \sum_{i=1}^{n} (-1)^{i+1} x^i \partial_{x^1} \wedge \cdots \widehat{\partial_{x^i}} \wedge \cdots \partial_{x^n} \tag{5.9}$$

and where $r = \sqrt{(x^1)^2 + (x^2)^2 + \cdots (x^n)^2}$ defines a cochain map $\varrho_{\mathcal{X}}$. We conclude that rotationally invariant conservation laws and variational principles are always preserved under symmetry reduction.

To arrive at the chain (5.9), we first remark that the $n-1$ chain $\mathcal{X}_0$ is already SO(n) invariant and consequently the most general invariant $n-1$ chain, with coefficients on $\mathbf{R}^n$, is

$$\mathcal{X} = f(r)\mathcal{X}_0. \tag{5.10}$$

We can determine the function $f(r)$ by applying Theorem 4.2 to the invariant total vector field

$$R = rD_r = \sum_{i=1}^{n} x^i D_i. \tag{5.11}$$

We chose (see (4.17) and (4.18))

$$\nu_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

$$\alpha = \frac{1}{r^2 f(r)} R \lrcorner\, \nu_0 = \frac{1}{r^2 f(r)} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

$$\mu = \sum_{i=1}^{n} x^i dx^i, \qquad \text{and} \qquad \nu = \alpha \wedge \mu = \frac{(-1)^{n-1}}{f(r)} \nu_0.$$

Then

$$\operatorname{Div}_\nu R = R\Big(\frac{1}{f(r)}\Big) f(r) + \operatorname{Div}_{\nu_0} R = -\frac{r}{f(r)} \frac{df(r)}{dr} + n$$

and

$$\operatorname{Div}_\mu \overline{R} = 2$$

so that (4.20) leads to the differential equation

$$\frac{d\,f}{d\,r} - (n-2)\frac{f}{r} = 0 \tag{5.12}$$

and hence $f(r) = kr^{n-2}$.

We can also arrive at equation (5.12) from (4.51) and (4.52). We take as our basis for the distribution spanned by SO(n) the vector fields

$$X_j = X_{1j} = x^1 \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^1} \qquad \text{for } j = 2, 3, \dots n.$$

For the remainder of this example we restrict the range of the indices i, j, k, ... to 2, 3, ..., n. Repeated indices are summed from 2 to n. Our preferred coordinates are r, x^2, ..., x^n so that $x^1 = \sqrt{r^2 - s^2}$, where $s^2 = (x^2)^2 + \cdots (x^n)^2$. We then write the chain (5.10) in the terms of the generators X_2, X_3, ..., X_n as

$$\mathcal{X} = J \operatorname{tot} X_2 \wedge \operatorname{tot} X_3 \wedge \cdots X_n, \qquad \text{where} \qquad J = \frac{f(r)}{(r^2 - s^2)^{(n-2)/2}}.$$

It is not difficult to check that $\overset{0}{V}{}^k = \dfrac{x^k}{x^1 r}$ and $\overset{0}{W} = 1$ solve (4.51) in which case (4.52) becomes

$$\frac{d\,J}{d\,r} - \frac{(n-2)s^2}{(x^1)^2 r} J = 0.$$

But this equation reduces to (5.12) and we are done.

In the special case $n = 3$ we can explicitly construct the invariant frame for $\mathrm{Tot}(J^\infty(E))$ adapted to ϱ. We have the standard infinitesimal generators (1.21). From Olver [*16*] (pp.167–168) we obtain the invariant total vector fields

$$\begin{aligned} R_1 &= x D_x + y D_y + z D_z, \\ R_2 &= u_x D_x + u_y D_y + u_z D_z, \quad \text{and} \\ R_3 &= (y u_z - z u_y) D_x + (z u_x - x u_z) D_y + (x u_y - y u_x) D_z. \end{aligned}$$

We rewrite the invariant vector fields R_1, R_2, R_3 in terms of the coordinates (1.20) to find

$$\begin{aligned} R_1 &= \bar{x} D_{\bar{x}} + \bar{y} D_{\bar{y}} + r D_r, \\ R_2 &= \frac{1}{r}[(\bar{x} v_r + r v_{\bar{x}}) D_{\bar{x}} + (\bar{y} v_r + r v_{\bar{y}} D_{\bar{y}} + (\bar{x} v_{\bar{x}} + \bar{y} v_{\bar{y}} + r v_r) D_r)], \\ R_3 &= \sqrt{r^2 - \bar{x}^2 - \bar{y}^2}(v_{\bar{x}} D_{\bar{y}} - v_{\bar{y}} D_{\bar{x}}). \end{aligned}$$

Because the coefficient of D_r in R_2 is a differential invariant, we can define invariant total vector fields

$$\begin{aligned} S_1 &= r^2 R_2 - (\bar{x} v_{\bar{x}} + \bar{y} v_{\bar{y}} + r v_r) R_1 \\ &= [(r^2 - \bar{x}^2) v_{\bar{x}} - \bar{x}\bar{y} v_{\bar{y}}] D_{\bar{x}} + [(r^2 - \bar{y}^2) v_{\bar{y}} - \bar{x}\bar{y} v_{\bar{x}}] D_{\bar{y}} \end{aligned}$$

and $S_2 = R_3$. Both S_1 and S_2 vanish on $\mathrm{Inv}^\infty_G(E)$, as required by Lemma 3.4, so that S_1, S_2 and R_1 define an invariant frame for $\mathrm{Tot}\, J^\infty(E)$ adapted to the reduction ϱ.

As an example of symmetry reduction using (5.9) consider Laplace's equation in $\mathbf{R}^n$

$$\frac{\partial^2 u}{(\partial x^1)^2} + \frac{\partial^2 u}{(\partial x^2)^2} + \cdots + \frac{\partial^2 u}{(\partial x^n)^2} = 0. \tag{5.13}$$

The SO(n) reduced equation is

$$v_{rr} + \frac{n-1}{r} v_r = 0$$

which can be integrated twice to obtain

$$v_r = (2-n)\frac{c_1}{r^{n-1}} \quad \text{and} \quad v = \frac{c_1}{r^{n-2}} + c_2, \tag{5.14}$$

where c_1 and c_2 are constants. It is not difficult to check that

$$\omega_1 = \sum_{i=1}^n u_i \nu_0^i \qquad \text{and} \qquad \omega_2 = \sum_{i=1}^n \frac{u}{r^n} x^i \nu_0^i + \frac{1}{n-2}\frac{1}{r^{n-2}}\,\omega_1,$$

where $\nu_0^i = \partial_{x^i} \lrcorner\, \nu$, are both $\mathrm{pr}\,\mathrm{SO}(n)$ invariant conservation laws for (5.13). By evaluating these forms on the chain (5.9) we obtain the first integrals (5.14), namely,

$$I_1 = \varrho(\omega_1(\mathcal{X})) = r^{n-1} v_r \qquad \text{and} \qquad I_2 = \varrho(\omega_2(\mathcal{X})) = \frac{r}{n-2} v_r + v.$$

Finally, the standard Lagrangian for Laplace's equation

$$\lambda = \frac{1}{2}((u_{x^1}^2 + u_{x^2}^2 + \cdots + u_{x^n}^2)\,\nu_0$$

reduces to

$$\overline{\lambda} = \varrho(\lambda(\mathcal{X})) = \frac{1}{2} r^{n-1} v_r^2\, dr.$$

In view of Remark 3.6, similar considerations apply to the symmetry reductions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 u}{(\partial x^i)^2} = 0$$

by the Lorentz group $\mathbf{so}(n,1)$.

Example 5.4. Some of the technical issues arising in Theorem 4.7 are nicely illustrated by the adjoint representation of the Lie algebras $\Gamma = A_{4,5}$ taken from the classification of low dimensional Lie algebras found in [*20*](p. 988). The basis for these algebras is

$$X_1 = x^1 \partial_{x^4}, \quad X_2 = a x^2 \partial_{x^4}, \quad X_3 = b x^3 \partial_{x^4}, \quad \text{and}$$
$$X_4 = -x^1 \partial_{x^1} - a x^2 \partial_{x^2} - b x^3 \partial_{x^3},$$

where $a \neq 0$ and $b \neq 0$ are constants. We take for the total space $E = \mathbf{R}^5$ with local coordinates (x^1, x^2, x^3, x^4, u). The orbit dimension (at points other than the x^4 axis) is 2. We first compute

$H^2(\Gamma, \Gamma_0)$. Then we examine the structure of the space of invariant total differential operators within the context of Propositions 3.2 and 3.3.

Let $p_0 = (c_1, c_2, c_3, c_4)$ be a point in $\mathbf{R}^4$ and suppose that $c_1 \neq 0$, $c_2 \neq 0$, $c_3 \neq 0$. Then with respect to the basis $Y_1 = X_1$, $Y_2 = X_4$,

$$Y_3 = c_1 X_2 - a c_2 X_1, \quad \text{and} \quad Y_4 = c_1 X_3 - b c_3 X_1$$

for Γ we have that $\{Y_3, Y_4\}$ is a basis for Γ_0. The dual one forms α^1, α^2, α^3, α^4 satisfy the structure equations

$$d\,\alpha^1 = -\alpha^1 \wedge \alpha^2 + a(a-1)c_2\, \alpha^2 \wedge \alpha^3 + b(b-1)c_3\, \alpha^2 \wedge \alpha^4$$

and

$$d\,\alpha^2 = 0, \quad d\,\alpha^3 = a\,\alpha^2 \wedge \alpha^3, \quad d\,\alpha^4 = b\,\alpha^2 \wedge \alpha^4.$$

The form $\nu = \alpha^1 \wedge \alpha^2$ is d closed and hence $Z^2(\Gamma, \Gamma_0(p_0)) = \langle \alpha^1 \wedge \alpha^2 \rangle$. Let $\eta = k_1 \alpha^1 + k_2 \alpha^2$. Then η is Γ_0 invariant if and only if $a = 1$ and $b = 1$ in which case $d\,\eta = -k_2 \alpha^1 \wedge \alpha^2$ and we conclude that

$$H^2(\Gamma, \Gamma_0(p_0)) = \begin{cases} 0, & \text{if } a = b = 1 \\ \mathbf{R}, & \text{otherwise.} \end{cases}$$

This allows us to conclude that the cochain map $\varrho_{\mathcal{X}}$ always exists in the neighborhood of the point p_0 except when $a = b = 1$.

To find the invariant total differential operator, we first observe that two independent invariant functions are $I_1 = (x^1)^a / x^2$ and $I_2 = (x^1)^b / x^3$. We change coordinates to $y^1 = I_1$, $y^2 = I_2$, $\hat{x}^1 = x^1$ and $\hat{x}^2 = x^4$ and relabel the basis vectors to obtain (after dropping the accents on x^1 and x^2)

$$X_1 = -x^1 \partial_{x^1}, \quad X_2 = x^1 \partial_{x^2}, \quad X_3 = \frac{a(x^1)^a}{y^1} \partial_{x^2}, \quad X_4 = \frac{b(x^1)^b}{y^2} \partial_{x^2}.$$

In these variables the invariant jet space $\mathrm{Inv}^\infty_G(E) \subset J^\infty(E)$ is described by

$$u_{x^1} = 0, \quad u_{x^2} = 0, \quad u_{x^1 y^1} = 0, \quad u_{x^1 y^2} = 0 \ldots .$$

It is easy to check that

$$\mathcal{X} = -\frac{1}{x^1} X_1 \wedge X_2 = x^1 \partial_{x^1} \wedge \partial_{x^2} \quad \text{and} \quad \nu = \frac{1}{x^1}\, dx^1 \wedge dx^2 \wedge dy^1 \wedge dy^2$$

are invariant.

The non-zero structure relations (3.4) are

$$[\, X_2, X_1 \,] = -X_2, \quad [\, X_3, X_1 \,] = \frac{a^2 (x^1)^{a-1}}{y^1} X_2, \quad [\, X_4, X_1 \,] = \frac{b^2 (x^1)^{b-1}}{y^2} X_2,$$

and

$$[\, X_3, \partial_{y^1} \,] = \frac{a(x^1)^{a-1}}{(y^1)^2} X_2, \qquad [\, X_4, \partial_{y^2} \,] = \frac{b(x^1)^{b-1}}{(y^2)^2} X_2.$$

We also have

$$\varphi_3^2 = \frac{(x^1)^{a-1}}{y^1}, \qquad \varphi_4^2 = \frac{b(x^1)^{b-1}}{y^2}.$$

We compute the algebraic conditions (3.8) to be

$$-g^1(a-1) + \frac{h^1}{y^1} = 0 \qquad -g^1(b-1) + \frac{h^2}{y^2} = 0. \tag{5.15}$$

These equations show that if $S = g^1 \operatorname{tot} X_1 + g^2 \operatorname{tot} X_2$ is a smooth invariant total vector field, then $g^1(\sigma) = 0$ and if $R = g^1 \operatorname{tot} X_1 + g^2 \operatorname{tot} X_2 + h^1 D_{y^1} + h^2 D_{y^2}$ is invariant then $(b-1)y^2 h^1(\sigma) = (a-1)y^1 h^2(\sigma)$, for all points $\sigma \in \operatorname{Inv}^\infty_G(E)$. Consistent with these constraints we find the invariant frame

$$S_1 = D_{x^2}, \qquad R_1 = (a-1)y^1 D_{y^1} + (b-1)y^2 D_{y^2} + x^1 D_{x^1} + x^2 D_{x^2},$$

$$P_1 = x^1 u_{x^2} D_{x^1} - x^1 u_{x^1} D_{x^2}, \qquad P_2 = -u_{y^2} D_{x^2} + u_{x^2} D_{y^2}.$$

For general values of a and b, Theorem 4.7 holds. This insures that the divergence $\operatorname{Div}_\nu S_1$ vanishes on $\operatorname{Inv}^\infty_G(E)$. We still have the invariant chain

$$\mathcal{X} = J(y^1, y^2) x^1 \partial_{x^1} \wedge \partial_{x^2} \tag{5.16}$$

and invariant volume form

$$\nu = \frac{1}{J(y^1, y^2)x^1}\, dx^1 \wedge dx^2 \wedge dy^1 \wedge dy^2$$

and we set $\mu = dy^1 \wedge dy^2$. Then the equation

$$\varrho(\operatorname{Div}_\nu R_1) = \operatorname{Div}_{\overline{\mu}}(\varrho(R_1)) \tag{5.17}$$

leads to the single partial differential equation

$$(a-1)y^1 \frac{\partial J}{\partial y^1} + (b-1)y^2 \frac{\partial J}{\partial y^2} = J, \tag{5.18}$$

the general solution to which is

$$J = (y^1)^{1/(a-1)} f\left(\frac{(y^1)^{b-1}}{(y^2)^{a-1}}\right).$$

Equation (5.18) can also be derived directly from (4.51) and (4.52). Equation (5.17) holds identical with R_1 replaced by R_2.

In the special case $a = 1$, $b = 1$, we have the invariant frame

$$S_1 = D_{x^2}, \qquad S_2 = -x^1 D_{x^1} - x^2 D_{x^2},$$

$$P_1 = u_{x^2} D_{y^1} - u_{y^1} D_{x^2}, \qquad P_2 = -u_{y^2} D_{x^2} + u_{x^2} D_{y^2},$$

but Theorem 4.7 now implies that no cochain map exists. Indeed

$$\omega = -J S_2 \lrcorner\, \nu = dx^2 \wedge dy^1 \wedge dy^2 - \frac{x^2}{x^1} dx^1 \wedge dy^1 \wedge dy^2$$

is G invariant, $\omega(J x^1 \partial_{x^1} \wedge \partial_{\partial x^2})$ vanishes but

$$d_H(\omega)(J x^1 \partial_{x^1} \wedge \partial_{\partial x^2}) = J dy^1 \wedge dy^2$$

and hence no cochain map $\varrho_{\mathcal{X}}$ exists. Note that both P_1 and P_2 project to zero under ϱ as required by (5.15). ∎

EXAMPLE 5.5. To re-enforce the necessity of the unimodular condition in the case where the group action is free (Theorem 4.1), consider the bi-harmonic equation

$$u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{xxzz} + 2u_{yyzz} = 0 \tag{5.19}$$

and the two dimensional non-abelian symmetry algebra

$$Y = \partial_y \quad \text{and} \quad S = x\partial_x + y\partial_y + z\partial_z + \frac{1}{2}u\partial_u. \tag{5.20}$$

The G adapted coordinates are

$$\widehat{x} = x, \quad \widehat{y} = y, \quad s = \frac{z}{x}, \quad v = \frac{u}{\sqrt{x}}.$$

One computes

$$\begin{aligned} u_{xx} &= \frac{1}{4\widehat{x}^{3/2}}(4\widehat{x}^2 v_{\hat{x}\hat{x}} + 4s^2 v_{ss} - 8\widehat{x} s v_{\hat{x}s} + 4\widehat{x} v_{\hat{x}} + 4s v_s - v), \\ u_{yy} &= \sqrt{x} v_{\hat{y}\hat{y}}, \qquad u_{zz} = \frac{1}{\widehat{x}^{3/2}} v_{ss} \end{aligned}$$

and the reduced equation is

$$(s^2+1)^2 v_{ssss} + 10s(s^2+1)v_{sss} + \frac{15}{2}(3s^2+1)v_{ss} + \frac{15}{2}s v_s - \frac{15}{16}v = 0. \tag{5.21}$$

The original equation is derivable from the invariant Lagrangian

$$\lambda = \frac{1}{2}(u_{xx}^2 + u_{yy}^2 + u_{zz}^2 + 2u_{xy}^2 + 2u_{xz}^2 + 2u_{yz}^2)\, dx \wedge dy \wedge dz$$

but the reduced equations admits no variational principle

$$\tilde{\lambda} = L(s, v, v_s, v_{ss})\, ds$$

at all. To see this, we apply a recent result [*6*] which implies that a fourth order linear equation $v_{ssss} = f(s, v, v_s, v_{ss}, v_{sss})$ is variational if and only if the invariant

$$I = \frac{\partial f}{\partial v_s} + \frac{1}{2}\frac{d^2}{ds^2}\frac{\partial f}{\partial v_{sss}} - \frac{1}{2}\frac{d}{ds}\frac{\partial f}{\partial v_{ss}} - \frac{3}{4}\frac{\partial f}{\partial v_{sss}}\frac{d}{ds}\frac{\partial f}{\partial v_{sss}} + \frac{1}{2}\frac{\partial f}{\partial v_{ss}}\frac{\partial f}{\partial v_{sss}} + \frac{1}{8}\Big(\frac{\partial f}{\partial v_{sss}}\Big)^3$$

vanishes. For (5.21) this invariant is

$$I = \frac{25}{2}\frac{s^2(s^2+10s+1)}{(s^2+1)^3}$$

and hence (5.21) is not variational. ∎

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