Generalized Laplace Invariants and the Method of Darboux

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ABSTRACT

It is well-known that the scalar hyperbolic linear equation

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y) = 0$$

is solvable in closed form if and only if the associated sequence of Laplace invariants for the equation terminates. In this paper we generalize this classical result to the case of nonlinear hyperbolic equations

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

by proving that the vanishing of the generalized Laplace invariants is both necessary and sufficient for the equation to be integrable by the method of Darboux. The derived flag computations for the characteristic Pfaffian systems leading to this conclusion also provide an elementary proof of Goursat's general classification of Darboux integrable equations. We illustrate our results by studying a nonlinear wave equation introduced by Calogero and and by classifying those equations of the type $u_{xx} = f(u_{yy})$ which are Darboux integrable on at order 2.

Key Words and Phrases: hyperbolic PDE in the plane, the method of Laplace, the method of Darboux, Laplace invariants, characteristic Pfaffian systems.

§**1. Introduction.** The methods of Laplace, Ampère and Darboux for the exact integration of second order, hyperbolic equations in the plane have been studied in a number of recent papers ([*1*], [*2*], [*5*], [*7*], [*15*], [*17*]). The current activity in this classical subject is motivated, in part, by a general resurgence of interest in geometric methods for the study of differential equations and also, in part, by a desire to classify various types of completely integrable partial differential equations [*14*]. In [*1*], it was established that if a hyperbolic equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1.1}$$

is Darboux integrable, then the associated sequence of *generalized Laplace invariants* is finite. In this paper we prove, conversely, that the finiteness of the sequence of generalized Laplace invariants for (1.1) insures that this equation is integrable by the Darboux method. Our result generalizes to the case of fully nonlinear equations the well-known result that a linear equation

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0 \tag{1.2}$$

is integrable by the method of Darboux, or equivalently, by the method of Laplace, if and only if the sequence of classical Laplace invariants is finite. Our result also generalizes the recent work of Sokolov and Ziber [*15*] who consider the special class of Monge-Ampère equations

$$u_{xy} = f(x, y, u, u_x, u_y).$$

They prove, by a rather ingenious method, that the finiteness of the sequence of generalized Laplace invariants for this equation is sufficient for the successful application of the method of Darboux.

Our main result can be stated more precisely as follows. We first recall, by way of motivation, that the first terms in the sequence of Laplace invariants for the linear equation (1.2) are

$$h_0 = \frac{\partial a}{\partial x} + ab - c \quad \text{and} \quad k_0 = \frac{\partial b}{\partial y} + ab - c.$$

Now let

$$X = m_x D_x + m_y D_y \quad \text{and} \quad Y = n_x D_x + n_y D_y \tag{1.3}$$

be two distinct characteristic vector fields for (1.1). (See section 2 for details.) Then we can rewrite the formal linearization of (1.1), that is,

$$\frac{\partial F}{\partial u_{xx}}\,\varphi_{xx} + \frac{\partial F}{\partial u_{xy}}\,\varphi_{xy} + \frac{\partial F}{\partial u_{yy}}\,\varphi_{yy} + \frac{\partial F}{\partial u_x}\,\varphi_x + \frac{\partial F}{\partial u_y}\,\varphi_y + \frac{\partial F}{\partial u}\,\varphi = 0 \tag{1.4}$$

in the form

$$XY(\varphi) + AX(\varphi) + BY(\varphi) + C\varphi = 0, \tag{1.5}$$

where the coefficients A, B and C are functions of the variables x, y, u, and the derivatives of u. The first generalized Laplace invariant of the nonlinear equation (1.1) is, by definition, the analogue of the classical Laplace invariant h_0 for the linearization (1.5), namely

$$H_0 = X(A) + AB - C.$$

Suppose $H_0 \neq 0$. If we formally introduce a new variable $\eta_1 = Y(\varphi) + A\varphi$, then it is not difficult to show that η_1 satisfies a linear relation of the form

$$XY(\eta_1) + A_1X(\eta) + B_1Y(\eta_1) + C_1\eta_1 = 0$$

as a consequence of (1.5). The next generalized Laplace invariant of (1.1) is defined by

$$H_1 = X(A_1) + A_1 B_1 - C_1.$$

If $H_1 \neq 0$, then we can repeat this transformation to determine H_2 and so forth. We can also express (1.4) in the form

$$YX(\varphi) + DX(\varphi) + EY(\varphi) + G\varphi = 0$$

and define the other Laplace invariant by

$$K_0 = Y(E) + ED - G.$$

If $K_0 \neq 0$, then we can introduce a new variable $\xi_1 = X(\varphi) + E\varphi$ and define K_1 in terms of the coefficients D_1, E_1, G_1 determining the linear equation for ξ_1.

Our main result can now be stated as follows.

Theorem 1.1. *There are independent functions I and $\tilde{I}$ of the variables x, y, u and the derivatives of u to order at most $p+2$ such that $X(I) = X(\tilde{I}) = 0$ if and only $H_p = 0$ Accordingly, equation (1.1) is semi-integrable by the method of Darboux if either $H_p = 0$ or $K_q = 0$ and integrable by the method of Darboux if and only $H_p = 0$ and $K_q = 0$.*

Goursat's magnificent treatise [*9*] still provides us with the best account of the method of Darboux. At the heart of Darboux's method is the observation that for arbitrary functions ϕ and ψ the system of equations (1.1), its differential consequences, and the equations

$$\tilde{I} = \phi(I) \quad \text{and} \quad \tilde{J} = \psi(J)$$

form a completely integrable Frobenius system whose solutions describe the general solution to (1.1).

The most well-known example of a Darboux integrable equation is the Liouville equation

$$u_{xy} = e^u.$$

Here $X = D_x$, $Y = D_y$ and the generalized Laplace invariants are $H_0 = K_0 = e^u$ and $H_1 = K_1 = 0$. The X and Y invariant functions are $I = y$, $\tilde{I} = u_{yy} - 1/2u_y^2$ and $J = x$, $\tilde{J} = u_{xx} - 1/2u_x^2$ and it is a simple matter to see that the integrability conditions for the system of equations

$$u_{xx} = \frac{1}{2}u_x^2 + \psi(x), \quad u_{xy} = e^u, \quad u_{yy} = \frac{1}{2}u_y^2 + \phi(y),$$

are all satisfied and lead to the well-known solution

$$e^u = \frac{2U'V'}{(U+V)^2}\,,$$

where $U = U(x)$ and $V = V(y)$ and

$$\psi = \frac{U'''}{U'} - \frac{3}{2}\frac{U''}{(U')^2} \quad \text{and} \quad \phi = \frac{V'''}{V'} - \frac{3}{2}\frac{V''}{(V')^2}.$$

Many other examples of nonlinear Darboux integrable equations can be found in [*6*], [*8*], [*9*], [*5*], [*19*].

Our proof of Theorem 1.1 is based upon the Laplace adapted coframe introduced in [*1*]. In section 2 of this paper we quickly review the construction of this coframe and its relevant properties. In section 3 the full structure equations for this coframe are investigated. With these structure equations in hand it is a simple matter to carry out, in section 5, the derived flag computations for the characteristic Pfaffian systems for (1.1). Not only do these computations prove Theorem 1.1 but they also provide us with a simple set of contact invariant conditions which characterize Goursat's general classification of Darboux integrable equations.

In section 6 illustrate how our derived flaged computations can be used to find the characteristic invariants. We then apply our results to the nonlinear wave equation

$$u_{xy} + uu_{xx} + f(y, u_x) = 0$$

which was first solved by quadratures by Calogero [*3*] We also prove that there are at most three contact inequivalent equations of the form

$$u_{xx} = f(u_{yy})$$

which are Darboux integrable at order 2, a conclusion which differs from that obtained by Forsyth [*6*](§267). It is hoped that this work may contribute to the general problem (see Vessiot [*18*]) of classifying Darboux integrable equations.

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§**2. The Laplace adapted coframe.** Here we quickly review the construction of the Laplace adapted coframe introduced in [*1*]. Let $\pi\colon E \to M$ be the trivial bundle with local coordinates $\pi\colon (x, y, u) \to (x, y)$ and let $\pi^\infty_M\colon J^\infty(E) \to M$ be the infinite jet bundle of local sections of π. The second order equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{2.1}$$

together with all its differential consequences, defines the infinitely prolonged equation manifold $\mathcal{R}^\infty \hookrightarrow J^\infty(E)$. The characteristic equation for (2.1) is

$$\frac{\partial F}{\partial u_{xx}}\lambda^2 + \frac{\partial F}{\partial u_{xy}}\lambda\mu + \frac{\partial F}{\partial u_{yy}}\mu^2 = 0.$$

We consider only the case of hyperbolic equations (2.1) and therefore this equation has a pair of distinct (that is, non-proportional) real roots $(\mu, \lambda) = (m_x, m_y)$ and $(\lambda, \mu) = (n_x, n_y)$ which define the characteristic total vector fields

$$X = m_x D_x + m_y D_y \quad \text{and} \quad Y = n_x D_x + n_y D_y. \tag{2.2}$$

We write the Lie bracket of these total vector fields as

$$[\,X, Y\,] = PX + QY,$$

and we denote the horizontal forms (more precisely, the π^∞_M semi-basic forms) dual to X and Y by σ and τ,

$$\sigma(X) = 1, \quad \sigma(Y) = 0, \quad \tau(X) = 0, \quad \tau(Y) = 1.$$

On the prolonged equation manifold $\mathcal{R}^\infty$ the contact forms

$$\theta = du - u_x\,dx - u_y\,dy, \quad \theta_x = du_x - u_{xx}\,dx - u_{xy}\,dy, \quad \theta_y = du_y - u_{xy}\,dx - u_{yy}\,dy, \ \ldots$$

are related by the *universal linearization*

$$\frac{\partial F}{\partial u_{xx}}\,\theta_{xx} + \frac{\partial F}{\partial u_{xy}}\,\theta_{xy} + \frac{\partial F}{\partial u_{yy}}\,\theta_{yy} + \frac{\partial F}{\partial u_x}\,\theta_x + \frac{\partial F}{\partial u_y}\,\theta_y + \frac{\partial F}{\partial u}\,\theta = 0.$$

We let $\Theta = \varrho\,\theta$, where ϱ is any smooth function on $\mathcal{R}^\infty$, and we rewrite this equation, using the characteristic vector fields (2.2) as

$$XY(\Theta) + A\,X(\Theta) + B\,Y(\Theta) + C\,\Theta = 0 \tag{2.3}$$

and, equivalently, as

$$YX(\Theta) + D\,X(\Theta) + E\,Y(\Theta) + G\,\Theta = 0. \tag{2.4}$$

Explicit formulas for the coefficients A, B, ... , G are given in [*1*].

The Laplace adapted coframe on $\mathcal{R}^\infty$ is constructed by successive applications of the classical Laplace transform to (2.3) and (2.4). The first elements of this coframe are defined by

$$\eta_1 = Y(\Theta) + A\,\Theta, \qquad H_0 = X(A) + AB - C, \tag{2.5}$$

and

$$\xi_1 = X(\Theta) + E\,\Theta, \qquad K_0 = Y(E) + ED - G. \tag{2.6}$$

Then, provided $H_0 \neq 0$, $H_1 \neq 0$, ... , $H_{i-1} \neq 0$, η_i satisfies the identity

$$XY(\eta_i) + A_iX(\eta_i) + B_iY(\eta_i) + C_i\eta_i = 0. \tag{2.7}$$

We set

$$H_i = X(A_i) + A_iB_i - C_i \qquad \text{and} \qquad \eta_{i+1} = Y(\eta_i) + A_i\,\eta_i. \tag{2.8}$$

This process continues until $H_p = 0$ in which case we define

$$\eta_{p+i+1} = Y(\eta_{p+i}) \qquad \text{for all } i \geq 1. \tag{2.9}$$

Similarly, provided $K_0 \neq 0$, $K_1 \neq 0$, ... , $K_{i-1} \neq 0$, ξ_i satisfies the identity

$$YX(\xi_i) + D_iX(\xi_i) + E_iY(\xi_i) + G_i\xi_i = 0 \tag{2.10}$$

and we set

$$K_i = Y(E_i) + D_iE_i - G_i \qquad \text{and} \qquad \xi_{i+1} = X(\xi_i) + E_i\xi_i. \tag{2.11}$$

This process continues until $K_q = 0$ in which case we define

$$\xi_{q+i+1} = X(\xi_{q+i}) \qquad \text{for all } i \geq 1. \tag{2.12}$$

The forms $\{\,\sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \ldots\,\}$ define a coframe on $\mathcal{R}^\infty$ called the *Laplace adapted coframe.*

It is convenient to split the exterior derivative d on $\mathcal{R}^\infty$ into two components

$$d = d_H + d_V, \tag{2.13}$$

where

$$d_H\omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega). \tag{2.14}$$

The next proposition ([*1*]) gives the d_H structure equations for the Laplace adapted coframe.

Proposition 2.1. *Suppose that $H_p = 0$ and $K_q = 0$. The d_H structure equations for the Laplace adapted coframe for the hyperbolic equation $\mathcal{R}^\infty$ are given by*

$$d_H\sigma = -P\,\sigma\wedge\tau, \qquad d_H\tau = -Q\,\sigma\wedge\tau, \tag{2.15a}$$

$$d_H(\Theta) = \sigma\wedge(\xi_1 - E\,\Theta) + \tau\wedge(\eta_1 - A\,\Theta), \tag{2.15b}$$

and

$$d_H\eta_1 = \sigma\wedge(-B\,\eta_1 + H_0\Theta) + \tau\wedge(\eta_2 - A_1\eta_1), \tag{2.16a}$$

$$d_H\eta_i = \sigma\wedge(-B_{i-1}\eta_i + H_{i-1}\eta_{i-1}) + \tau\wedge(\eta_{i+1} - A_i\eta_i) \qquad 2\le i\le p, \tag{2.16b}$$

$$d_H\eta_{p+1} = \sigma\wedge(-B_p\eta_{p+1}) + \tau\wedge\eta_{p+2}, \tag{2.16c}$$

$$d_H\eta_{p+i} = \sigma\wedge\nu_{p+i} + \tau\wedge\eta_{p+i+1} \quad i\ge 2. \tag{2.16d}$$

In equation (2.16d) ν_{p+i} is a contact one form such that

$$\nu_{p+i} \equiv [(i-1)Q - B_p]\,\eta_{p+i} \qquad \text{mod}\ \{\,\eta_{p+1},\ldots,\eta_{p+i-1}\,\}. \tag{2.17a}$$

If $H_p \ne 0$ for all p then the structure equations (2.16b) remain valid for all $i \ge 2$ or $j \ge 2$.
Similar structure equations hold for the forms ξ_j.

In [*1*], the problem of classifying the higher degree contact-valued conservation laws for the equation $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ was studied. It was proved that if the two sequences of Laplace invariants H_i and K_j do not terminate, then this equation admits no such conserved forms. This result follows, without much difficulty, from a normal form theorem for such conservation laws and from the d_H structure equations. But it is easy to see that Darboux integrable equations always admit infinitely many higher degree form-valued conservation laws and therefore we concluded that the vanishing of the generalized Laplace invariants is neccessary for Darboux integrability. To prove that the vanishing of the Laplace invariants is sufficient one must establish the existence of the pairs of X and Y invariant functions and to this end one must compute the full d structure equations for the Laplace adapted coframe. Since we already have the d_H stucture equations, we turn to the computation of the d_V structure equations.

§**3. Structure equations for the Laplace adapted coframe.** In this section we complete the computation of the structure equations for the Laplace adapted coframe, begun in Proposition 2.1, by computing the d_V structure equations. We say that a contact form α is of adapted order k if it belongs to the span of contact forms Θ, η_i, and ξ_i, $1 \le i \le k$. If α_1, α_2, … , α_k are 1 forms on $\mathcal{R}^\infty$, then $\Omega^*(\alpha_1, \alpha_2, \ldots, \alpha_k)$ denotes the exterior algebra generated (over $C^\infty(\mathcal{R}^\infty)$) by these forms. We begin with the structure equations for the horizontal forms σ and τ.

Proposition 3.1. *The d_V structure equations for the horizontal forms σ and τ are*

$$d_V\sigma = \sigma\wedge\mu_1 + \tau\wedge\alpha \quad \text{and} \quad d_V\tau = \sigma\wedge\beta + \tau\wedge\mu_2, \tag{3.1}$$

where α, β, μ_1, μ_2 are contact one forms. The adapted order of α and β is 2. Moreover, the following relations hold:

$$d_VP = X(\alpha) - Y(\mu_1) + P\mu_2 - Q\alpha, \qquad d_VQ = X(\mu_2) - Y(\beta) + Q\mu_1 - P\beta, \tag{3.2}$$

and

$$d_V\beta = \beta\wedge(\mu_2 - \mu_1), \quad d_V\mu_2 = \alpha\wedge\beta = -d_V\mu_1, \quad d_V\alpha = \alpha\wedge(\mu_1 - \mu_2). \tag{3.3}$$

Proof. The existence of contact 1 forms α, β, μ_1, μ_2 satisfying (3.1) is immediate from the definition of d_V and the fact that σ and τ are horizontal forms. That the forms α and β have adapted order 2 now follows follows from the explicit formulas for σ and τ in terms of the characteristic roots (m_x, m_y) and (n_x, n_y) and the fact that these roots may be chosen to be second order functions on $\mathcal{R}^\infty$.

Equations (3.2) follow from integrability conditions arising from (2.15a) and (3.1). To derive (3.3) we apply d_V to (3.1). ∎

REMARK: 3.2. It is not difficult to prove ([*11*]) that the coefficient of ξ_2 in α and the coefficient of η_2 in β are relative contact invariants, denoted by M_σ and M_τ which vanish if and only if the equation is of Monge-Ampere type. ∎

Since the Laplace adapted coframe is defined inductively in terms of the characteristic vector fields, we shall need the commutation rules for X, Y and d_V.

Proposition 3.3. *Let α, β, μ_1 and μ_2 be given by (3.1). Then for any $\omega \in \Omega^*(\Theta, \eta_1, \xi_1, \dots)$*

$$d_V[X(\omega)] - X(d_V\omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega) \tag{3.4}$$

and

$$d_V[Y(\omega)] - Y(d_V\omega) = \alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega). \tag{3.5}$$

Proof. On the one hand we have, by (2.14), that

$$d_H[d_V(\omega)] = \sigma \wedge X(d_V\omega) + \tau \wedge Y(d_V\omega) \tag{3.6}$$

while, on the other hand, from (3.1), we compute that

$$\begin{aligned} d_V[d_H(\omega)] &= d_V[\sigma \wedge X(\omega) + \tau \wedge Y(\omega)] \\ &= \sigma \wedge \{\mu_1 \wedge X(\omega) - d_V[X(\omega)] + \beta \wedge Y(\omega)\} \\ &+ \tau \wedge \{\alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega) - d_V[Y(\omega)]\}. \end{aligned} \tag{3.7}$$

A comparison of (3.6) and (3.7) leads to (3.4) and (3.5). ∎

Our first approximation to the d_V structure equations follows from Proposition 3.3 by induction.

Proposition 3.4. *The Laplace adapted coframe satisfies the following congruences:*

$$d_V\Theta \equiv 0 \qquad \text{mod } \{\, \Theta \,\}; \tag{3.8}$$

$$d_V\eta_i \equiv 0 \qquad \text{mod } \{\, \xi_1, \Theta, \eta_1, \dots, \eta_i \,\} \qquad i \geq 1; \quad \textit{and} \tag{3.9}$$

$$d_V\xi_i \equiv 0 \qquad \text{mod } \{\, \eta_1, \Theta, \xi_1, \dots, \xi_i \,\} \qquad i \geq 1. \tag{3.10}$$

Proof. Equation (3.8) and equations (3.9) and (3.10), for $i = 1$, are established from (2.5), (2.6) and Proposition 3.3. We prove (3.9) for $i > 1$ by induction. Note that the d_H structure equations given in Proposition 2.1 imply that $X(\eta_j) \equiv 0, \quad \text{mod } \{\, \Theta, \eta_1, \eta_2, \dots, \eta_j \,\}$ and that $Y(\eta_j) \equiv 0$, mod $\{\, \eta_1, \eta_2, \dots, \eta_{j+1} \,\}$ for all $j \geq 1$. We also note that $Y(\Theta) = \eta_1 - A\Theta$ and $Y(\xi_1) = -D\xi_1 + K_0\Theta$.

Assume that (3.9) is true for all $i \leq j$. Then it is a simple matter to check that $Y(d_V \eta_j) \equiv 0$, mod $\{ \xi_1, \Theta, \eta_1, \ldots, \eta_{j+1} \}$. Then, for $j \leq p$, where $H_p = 0$, we use (2.8) and (3.5) to compute

$$\begin{aligned} d_V \eta_{j+1} &= d_V(Y(\eta_j) + A_j \eta_j) \\ &= \alpha \wedge X(\eta_j) + \mu_2 \wedge Y(\eta_j) + Y(d_V \eta_j) + d_V(A_j) \wedge \eta_j + A_j d_V(\eta_j) \\ &\equiv 0 \qquad \text{mod } \{ \xi_1, \Theta, \eta_1, \ldots, \eta_{j+1} \}. \end{aligned}$$

For $j \geq p+1$ these same computations, but with $A_j = 0$, remain valid. Equation (3.10) is similarly established. ∎

In [*1*] it is shown that if $\omega \in \Omega^*(\Theta, \eta_1, \xi_1, \ldots)$ is a relative X invariant contact form, that is, $X(\omega) = \lambda \omega$ and $H_p = 0$, then ω is in the exterior algebra $\Omega^*(\eta_{p+1}, \eta_{p+2}, \ldots)$. The following generalization of this result will enable use to refine the crude structure equations of Proposition 3.4. just to the degree neccessary to prove that the vanishing of the Laplace invariants are the only obstructions to Darboux integrablity.

Proposition 3.5. *Suppose $H_p = 0$. Let l be a non-negative integer, let $\omega \in \Omega^*(\Theta, \eta_1, \xi_1, \ldots)$ and suppose*

$$X(\omega) = \lambda \omega \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}. \tag{3.11}$$

Then ω decomposes uniquely into a sum

$$\omega = \omega_1 + \omega_2 \tag{3.12}$$

where $\omega_1 \equiv 0$ mod $\{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}$ and $\omega_2 \in \Omega^*(\eta_{p+l+1}, \eta_{p+l+2}, \ldots)$.

Proof. We begin with the observation that if a form

$$\omega_1 \equiv 0 \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \} \tag{3.13}$$

then it is always the case that

$$X(\omega_1) \equiv \lambda \omega_1 \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}. \tag{3.14}$$

Now decompose ω uniquely into the form $\omega = \omega_1 + \omega_2$, where ω_1 satisfies (3.13) and

$$\omega_2 \in \Omega^*(\xi_k, \xi_{k-1}, \ldots, \xi_1, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots).$$

Write

$$\omega_2 = \xi_k \wedge \gamma + \epsilon$$

where $\gamma, \epsilon \in \Omega^*(\xi_{k-1}, \ldots, \xi_1, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots)$. Then by equations (3.11) and (3.14) it follows that

$$X(\omega_2) \equiv \lambda \omega_2 \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}. \tag{3.15}$$

We compute, using the d_H structure equations and (3.15)

$$X(\omega_2) \equiv \xi_{k+1} \wedge \gamma + \delta \quad \text{mod } \{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \},$$

where $\delta \in \Omega^*(\xi_{k-1}, \ldots, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots)$. From the congruence (3.11) we now deduce that $\gamma \equiv 0$ mod $\{ \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{p+l} \}$ and therefore $\gamma = 0$. This proves that

$$\omega_2 \in \Omega^*(\xi_{k-1}, \xi_{k-2}, \ldots, \xi_1, \Theta, \eta_1, \ldots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \ldots).$$

We can obviously repeat this argument to establish that

$$\omega_2 \in \Omega^*(\eta_1, \dots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \dots).$$

This proves the theorem if $p = 0$.

Assume $p \geq 1$ and now write

$$\omega_2 = \eta_1 \wedge \gamma + \epsilon$$

where $\gamma, \epsilon \in \Omega^*(\eta_2, \dots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \dots ,)$. This time we compute

$$X(\omega_2) \equiv H_0 \Theta \wedge \gamma + \delta \quad \text{mod } \{ \eta_{p+1}, \eta_{p+2}, \dots, \eta_{p+l} \}$$

where $\delta \in \Omega^*(\eta_1, \dots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \dots)$. Since $H_0 \neq 0$, we can conclude from this congruence that $\gamma = 0$ and hence $\omega_2 \in \Omega^*(\eta_2, \dots, \eta_p \eta_{p+1}, \eta_{p+2}, \dots, \eta_{p+l})$. We can repeat this argument until $H_p = 0$. ∎

We combine Propositions 3.4 and 3.5 to arrive at the following structure equations.

Theorem 3.6. *If $H_p = 0$, then there are unique forms*

$$\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \dots, \eta_p) \tag{3.16}$$

such that

$$d_V \eta_{p+1} = \eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \tilde{\eta} \tag{3.17}$$

The form Υ satisfies

$$X(\Upsilon) \equiv -Q\Upsilon + \beta \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2} \} \tag{3.18}$$

and

$$d_V \Upsilon \equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2} \}. \tag{3.19}$$

The forms η_{p+i}, $i \geq 1$, satisfy the d_V structure equations

$$d_V \eta_{p+i} \equiv \eta_{p+i+1} \wedge \Upsilon \qquad \text{mod } \{ \eta_{p+1}, \dots, \eta_{p+i} \}. \tag{3.20}$$

Proof. Since

$$\begin{aligned} X(d_V \eta_{p+1}) &= d_V[X(\eta_{p+1})] - \mu_1 \wedge X(\eta_{p+1}) - \beta \wedge Y(\eta_{p+1}) \\ &= d_V(-B_p \eta_{p+1}) - \mu_1 \wedge (-B_p \eta_{p+1}) - \beta \wedge \eta_{p+2} \\ &= -B_p d_V(\eta_{p+1}) \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2} \} \end{aligned}$$

we can deduce from Proposition 3.5 that

$$d_V \eta_{p+1} = \eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \tilde{\eta} + \omega,$$

where $\omega \in \Omega^2(\eta_{p+3}, \eta_{p+4}, \dots)$. But then by Proposition 3.4 we must have $\omega = 0$ and $\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \dots, \eta_{p+1})$. Any terms in Υ involving η_{p+1} can be absorbed into $\tilde{\eta}$. This proves (3.17). The uniqueness of Υ is immediate.

From (3.17) and the d_H structure equations, we compute

$$d_H(d_V\eta_{p+1}) \equiv (d_H\eta_{p+2}) \wedge \Upsilon - \eta_{p+2} \wedge d_H\Upsilon + d_H\eta_{p+1} \wedge \tilde{\eta} \qquad \text{mod } \{\, \eta_{p+1} \,\}$$

$$\equiv \sigma \wedge [(Q - B_p)\eta_{p+2} \wedge \Upsilon + \eta_{p+2} \wedge X(\Upsilon)]$$

$$+ \tau \wedge [\eta_{p+2} \wedge Y(\Upsilon) + \eta_{p+2} \wedge \tilde{\eta}] \qquad \text{mod } \{\, \eta_{p+1} \,\}$$

while from (2.16c) we obtain

$$d_V(d_H\eta_{p+1}) = d_V(-B_p\sigma \wedge \eta_{p+1} + \tau \wedge \eta_{p+2})$$

$$\equiv \sigma \wedge [B_p\eta_{p+2} \wedge \Upsilon + \beta \wedge \eta_{p+2}] + \tau \wedge [\mu_2 \wedge \eta_{p+2} - d_V\eta_{p+2}] \qquad \text{mod } \{\, \eta_{p+1} \,\}.$$

The comparison of these two equations implies that

$$\eta_{p+2} \wedge [\, Q\Upsilon + X(\Upsilon) - \beta\,] \equiv 0 \qquad \text{mod } \{\, \eta_{p+1} \,\}$$

and (3.18) follows. To prove (3.19), we simply take d_V of (3.17) and use the fact that

$$d_V\eta_{p+2} = d_V[Y(\eta_{p+1})]$$

$$= Y[d_V(\eta_{p+1})] + \alpha \wedge X(\eta_{p+1}) + \mu_2 \wedge Y(\eta_{p+1})$$

$$\equiv \eta_{p+3} \wedge \Upsilon + \eta_{p+2} \wedge Y(\Upsilon) + \eta_{p+2} \wedge \tilde{\eta} + \mu_2\eta_{p+2} \qquad \text{mod } \{\, \eta_{p+1} \,\}.$$

A straightforward induction argument, based upon a calculation similar to that just given, proves (3.20).

∎

Remark 3.7. Write

$$\beta = c_0\Theta + c_1\eta_1 + c_2\eta_2 + b_1\xi_1 + b_2\xi_2. \tag{3.21}$$

By using (3.18) it is not difficult to prove that Υ is given explicitly by

$$\Upsilon = G_1\xi_1 + F_0\Theta + \sum_{i=1}^{p} F_i\eta_i \tag{3.22}$$

where $G_1 = b_2$, and

$$F_0 = -X(G_1) + (E_1 - Q)G_1 + b_1, \quad F_{i+1} = -\frac{1}{H_i}(X(F_i) - B_iF_i - c_i).$$

where $c_i = 0$ for $i \geq 3$. Note that c_2 equals the Monge-Ampére invariant M_τ. ∎

§**4. The Characteristic Pfaffian Systems and Darboux Integrability.** If I and J are functions of order k on $\mathcal{R}^\infty$ such that $X(I) = 0$ and $Y(J) = 0$, then $dI \in \mathcal{C}_k(X)$ and $dJ \in \mathcal{C}_k(Y)$, where

$$\mathcal{C}_k(X) = \Omega^1(\tau, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k)$$

and

$$\mathcal{C}_k(Y) = \Omega^1(\sigma, \Theta, \eta_1, \xi_1, \ldots, \eta_k, \xi_k).$$

We call $\mathcal{C}_k(X)$ and $\mathcal{C}_k(Y)$ the *k-th order characteristic Pfaffian systems* associated to the second order hyperbolic equation $\mathcal{R}^\infty$. The original partial differential equation (1.1) is therefore Darboux integrable if for sufficiently large k the characteristic Pfaffian systems $\mathcal{C}_k(X)$ and $\mathcal{C}_k(Y)$ each contain a completely integrable subsystem of dimension ≥ 2. We say that (1.1) is Darboux semi-integrable if either one of the characteristic Pfaffian systems contains a completely integrable subsystem if dimension ≥ 2.

Theorem 4.1. *The Pfaffian systems*

$$\mathcal{D}_{p+i}(X) = \Omega^1(\tau - \Upsilon, \eta_{p+1}, \dots, \eta_{p+i}) \quad \textit{and} \quad \mathcal{D}_{q+i}(Y) = \Omega^1(\sigma - \Xi, \xi_{q+1}, \dots, \xi_{q+i})$$

are completely integrable for $i \geq 2$ if $H_p = 0$ and $K_q = 0$ respectively.

Proof. This theorem follows directly from the structure equations for the Laplace adapted coframe and Theorem 3.6. For $i = 2$, we find that

$$\begin{aligned}
d(\tau - \Upsilon) &= d_H\tau + d_V\tau - d_H\Upsilon - d_V\Upsilon \\
&= -Q\sigma\wedge\tau + \sigma\wedge\beta + \tau\wedge\mu_2 - \sigma\wedge X(\Upsilon) - \tau\wedge Y(\Upsilon) - d_V\Upsilon \\
&\equiv -Q\sigma\wedge\Upsilon + \sigma\wedge\beta + \Upsilon\mu_2 - \sigma\wedge X(\Upsilon) - \Upsilon\wedge Y(\Upsilon) - d_V\Upsilon \quad \mod \mathcal{D}_{p+2}(X) \\
&\equiv -\sigma\wedge[X(\Upsilon) + Q\Upsilon - \beta] + \Upsilon\wedge[\mu_2 - Y(\Upsilon)] - d_V\Upsilon \quad \mod \mathcal{D}_{p+2}(X) \\
&\equiv 0 \quad \mod \mathcal{D}_{p+2}(X), \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
d(\eta_{p+1}) &= d_H\eta_{p+1} + d_V\eta_{p+1} \\
&= \sigma\wedge(-B_p\eta_{p+1}) + \tau\wedge\eta_{p+2} + \eta_{p+2}\wedge\Upsilon + \eta_{p+1}\wedge\tilde{\eta} \quad \mod \mathcal{D}_{p+2}(X) \\
&\equiv 0 \quad \mod \mathcal{D}_{p+2}(X) \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
d(\eta_{p+2}) &= d_H\eta_{p+2} + d_V\eta_{p+2} \\
&\equiv \sigma\wedge X(\eta_{p+2}) + \tau\wedge\eta_{p+3} + \eta_{p+3}\wedge\Upsilon \quad \mod \mathcal{D}_{p+2}(X) \\
&\equiv (\tau - \Upsilon)\wedge\eta_{p+3} \equiv 0 \quad \mod \mathcal{D}_{p+2}(X).
\end{aligned}$$

An elementary induction argument proves the theorem for $i > 2$. The proof for $\mathcal{D}_{q+i}(Y)$ is similar. ∎

The results of [*1*] and Theorem 4.1 combine to establish our main result.

Corollary 4.2. *A second order hyperbolic scalar equation in the plane,*

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

is Darboux integrable if and only if for some integers $p \geq 0$ and $q \geq 0$, the generalized Laplace invariants H_p and K_q vanish. In fact, there are always 3 independent X invariant functions of order at most $p + 2$ and 3 independent Y invariant functions of order at most $q + 2$.

As we mentioned in the introduction, Sokolov and Ziber [*15*] proved that the vanishing of the Laplace invariants H_p and K_q is sufficient for the Darboux integrablity of the equation

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{4.1}$$

However, their method does not appear to furnish the invariants of lowest possible order so that even for equations of the type (4.1), Corollary 4.2 is an improvement over their results.

We now recall that the space $H^{1,s}(\mathcal{R}^\infty)$ of type $(1, s)$ conservation laws for the equation $F = 0$ is the space of d_H closed $(1 + s)$-forms of the type

$$\omega = M\wedge\sigma + N\wedge\tau,$$

where M and N are contact s forms in $\Omega^*(\Theta, \eta_1, \xi_1, \dots)$, modulo d_H exact forms of this type. In [*1*] it was shown that if the Laplace invariants H_p and K_q never vanish, then $H^{1,s}(\mathcal{R}^\infty) = 0$ for all $s \geq 3$. It was also shown Darboux semi-integrable equations possess infinitely many type $(1, s)$ conservation laws for all $s \geq 0$

Corollary 4.3. *A second order hyperbolic scalar equation in the plane*

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

is Darboux semi-integrable if and only if $H^{1,s}(\mathcal{R}^\infty) \neq 0$ *for some* $s > 3$*.*

This last corollary satisfactorily resolves the problem posed by [*16*], namely, to give a criteria for Darboux integrablity of scalar second order equations in the plane in terms of the cohomology of the associated variational bicomplex. We conjecture that Corollary 4.3 generalizes to other classes of differential equations.

§**5. Goursat's Classification of Darboux Integrable Equations.** A considerable portion of Goursat's analysis of the method of Darboux [*9*](pp. 133-171) is devoted to a complete classification of the functionally independent X and Y invariant functions at any given order. It is a simple matter for us to refine the congruences (3.18) and (3.19) and thereby rederive Goursat's results. Moreover we are able to give explicit formulas for the maximally completely integral subsystems $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$ which, as a practical matter, may be quite useful in the implementation of Darboux's method.

To this end we recall that the form β, defined by (3.1), takes the form (3.21). The function $c_2 = M_\tau$ is a relative invariant under contact transforms and will play a central role in what follows. As a first step we sharpen the (3.18).

Proposition 5.1. *Let* $H_p = 0$*.*

(i) *If* $p \geq 2$*, then* $X(\Upsilon) + Q\Upsilon - \beta = 0.$

(ii) *If* $p = 1$*, then* $X(\Upsilon) + Q\Upsilon - \beta + M_\tau \eta_2 = 0.$

(iii) *If* $p = 0$*, then* $X(\Upsilon) + Q\Upsilon - \beta + c_1\eta_1 + M_\tau \eta_2 = 0.$

Proof. From (3.16) and (3.18), we have that

$$\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \eta_2, \ldots, \eta_p)$$

and

$$X(\Upsilon) + Q\Upsilon - \beta \equiv 0 \qquad \mathrm{mod}\ \{\, \eta_{p+1}, \eta_{p+2} \,\}. \tag{5.1}$$

Since $X(\eta_p) = -B_{p-1}\eta_p + H_{p-1}\eta_{p-1}$, it follows that for $p \geq 2$ there are no terms involving η_{p+1} and η_{p+2} on the left-hand side of the congruence (5.1) and hence this congruence must be an identity.

For $p = 1$, write $\Upsilon = g_1\xi_1 + f_0\theta + f_1\eta_1$ and

$$X(\Upsilon) + Q\Upsilon - \beta = k_2\eta_2 + k_3\eta_3.$$

Then we use the d_H structure equations and match coefficients to deduce that $k_2 = M_\tau$ and $k_3 = 0$. The proof of (iii) is similar, starting with $\Upsilon = g_1\xi_1 + f_0\Theta$. ∎

Next we use Proposition 5.1 to improve the conqruences (3.19).

Proposition 5.2. *Let $H_p = 0$.*

(i) *If $p \geq 2$, then* $d_V \Upsilon = \Upsilon \wedge [\mu_2 - Y(\Upsilon)]$.

(ii) *If $p = 1$, then* $d_V \Upsilon \equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \mod \{ \eta_2 \}$.

(iii) *If $p = 1$ and $M_\tau = 0$, then* $d_V \Upsilon = \Upsilon \wedge [\mu_2 - Y(\Upsilon)]$.

(iv) *If $p = 0$, then* $d_V \Upsilon \equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \mod \{ \eta_1, \eta_2 \}$.

(v) *If $p = 0$ and $M_\tau = 0$, then* $d_V \Upsilon \equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \mod \{ \eta_1 \}$.

Proof. We know that

$$X(\Upsilon) = -Q\Upsilon + \beta + \delta \quad \text{where} \quad \delta = k_1\eta_1 + k_2\eta_2 \tag{5.2}$$

where $k_1 = k_2 = 0$ if $p \geq 2$ and so on, and

$$d_V \Upsilon = \Upsilon \wedge [\mu_2 - Y(\Upsilon)] + \lambda_{p+1} \wedge \eta_{p+1} + \lambda_{p+2} \wedge \eta_{p+2}. \tag{5.3}$$

Since $\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_p)$ it follows immediately from Proposition 3.4 that

$$d_V \Upsilon \equiv 0 \qquad \text{mod } \{ \xi_1, \Theta, \eta_1, \ldots, \eta_{p'} \},$$

where $p' = \max\{ 1, p \}$, and therefore

$$\lambda_{p+1},\ \lambda_{p+2} \in \Omega^1(\xi_1, \Theta, \eta_1, \ldots, \eta_{p'}) \tag{5.4}$$

We substitute (5.2) and (5.3) into the commutator identity (3.4) to conclude, after considerable simplification, that

$$\begin{aligned} &\eta_{p+2} \wedge [X(\lambda_{p+2}) + (2Q - B_p)\lambda_{p+2}] + \\ &\eta_{p+1} \wedge [X(\lambda_{p+1}) + (Q - B_p)\lambda_{p+1} + \lambda_{p+2}\{ * \}] + \Delta = 0. \end{aligned} \tag{5.5}$$

Here Δ denotes the terms in (3.4) which arise from the term δ in (5.2), that is,

$$\Delta = d_V(\delta) - \delta \wedge [\mu_2 - Y(\Upsilon)] + \Upsilon \wedge [Y(\delta) + P\delta] - \mu_1 \wedge \delta \tag{5.6}$$

Suppose that $p \geq 2$. Then, by Proposition 5.1, $\delta = 0$, $\Delta = 0$ and (5.5) therefore implies that

$$X(\lambda_{p+2}) + (2Q - B_p)\lambda_{p+2} \equiv 0 \qquad \text{mod } \{ \eta_{p+1}, \eta_{p+2} \}.$$

We combine Theorem 3.5 with (5.4) to deduce that $\lambda_{p+2} = 0$. We return to (5.5) and repeat this argument to deduce that $\lambda_{p+1} = 0$.

The same computations apply to prove (iii) since in this case $\delta = 0$. In case (ii), we have that $\delta = M_\tau \eta_2$ and it is a simple matter to show that $\Delta \equiv 0 \mod \{ \eta_2 \}$ and we can proceed as before. For $p = 0$ and $M_\tau = 0$ we have that $\delta = c_1\eta_1$ and now it happens that $\Delta \equiv 0 \mod \{ \eta_1 \}$. We proceed as before to deduce that $\lambda_2 \equiv 0 \mod \{ \eta_1 \}$. ∎

Goursat's results can be summarized as follows.

Theorem 5.3. *Let $H_p = 0$, $\widehat{\tau} = \tau - \Upsilon$ and let l be the adapted order of $\widehat{\tau}$. Let $k \geq 1$.*

(i) *If $p \geq 2$, then* (a) $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_{p+1}, \ldots, \eta_k)$ *for $k \geq p$,*

(b) $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau})$ *if $l \geq k \leq p$,*

(c) $\mathcal{C}_k^{\infty}(X) = \{0\}$ *if $k < l$.*

(ii) *If $p = 1$, then* (a) $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_2, \ldots, \eta_k)$ *for $k \geq 2$,*

(b) $\mathcal{C}_1^{(\infty)}(X) = \Omega^1(\widehat{\tau})$ *if $M_\tau = 0$, and*

(c) $\mathcal{C}_1^{(\infty)}(X) = \{0\}$ *if $M_\tau \neq 0$.*

(iii) *If $p = 0$, then* (a) $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_1, \ldots, \eta_k)$ *for $k \geq 2$,*

(b) $\mathcal{C}_1^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_1)$ *if $M_\tau = 0$ and,*

(c) $\mathcal{C}_1^{(\infty)}(X) = \{0\}$ *if $M_\tau \neq 0$.*

Proof. We can use the refined structure equations given in Proposition 5.2 to show that Pfaffian systems listed in the theorem are all integrable. The theorem then follows immediately from the easily established fact that

$$C_k^{(\infty)}(X) \subset \Omega^1(\widehat{\tau}, \eta_{p+1}, \ldots, \eta_k). \qquad \blacksquare$$

Goursat proves that once a pair of X invariant functions are known, a new X invariant function can always be constructed from the ratio of the total derivatives of the given invariants (see also [*1*]). In case (i) there is exactly one invariant I_l of order l equal to the adapted order of $\widehat{\tau}$ and one invariant of order $p+1$. In case (ii)b there is one X invariant of order 1 and one new X invariant of order 2 (so that this case is similar to case (iii)) while in case (ii)c there are no invariants of order 1 but two invariants I_2 and I_2' of order 2. In case (iii)b there are two invariants I_1 and I_1' of order 1. In case (iii)c there are no invariants of order 1 but there are three invariants I_2, I_2' and I_2'' of order 2.

Finally, we remark that for second order elliptic equations in the plane the characteristic total vector fields are complex. A complex valued Laplace adapted coframe and complex generalized Laplace invariants can be defined [*1*]. In this situation, one needs the complex version of the Frobenius theorem [*13*], (p. 23) to determine the existence of complex valued characteristic invariant functions. But it not difficult to show that Theorem 5.3, with the obvious modifications, remains valid. See [*11*] for details.

§6. Examples.

For our first example, we consider the equation

$$u_{xx}u_{yy} = u_x$$

and we show how the results of Theorem 5.3 can be used to find the characteristic invariants. We use the notation $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$ and $t = u_{yy}$ when convenient. We find that $H_0 = 0$, $M_\tau = -2\dfrac{s}{p}$ and $K_2 = 0$ so that, according to this theorem, there are three X second order invariant functions I, I' and I'' which describe the solutions to the integrable Pfaffian system $\{\hat{\tau}, \eta_1, \eta_2\}$, one Y invariant function J of order 1 arising from the integrable 1 form $\hat{\sigma}$ and one Y invariant of order 3, which together with the invariant J solves the integrable system $\{\hat{\sigma}, \xi_3\}$. We find that

$$\hat{\tau} = dx - \frac{s^2}{p}dy, \quad \eta_1 = \theta_x = dp - \frac{p}{s}dx - s dy \quad \text{and}$$
$$\eta_2 = -\frac{p}{s^2}\theta_{xy} + \frac{1}{s}\theta_x = -\frac{p^2 u_{xyy}}{s^4}\,\hat{\tau} \; - dy + \frac{1}{s}dp - \frac{p}{s^2}ds$$

and hence

$$\eta_2 \equiv d(-y + \frac{p}{s}) \mod \hat{\tau}, \qquad \eta_1 \equiv d(\frac{p}{s^2}) \mod \{\,\hat{\tau}, \eta_2\,\}, \quad \text{and}$$
$$\hat{\tau} \equiv d(x - s) \mod \{\,\eta_1, \eta_2\,\}$$

The X invariants are thus

$$I = -y + \frac{p}{s}, \qquad I' = \frac{p}{s^2}, \qquad I'' = x - s$$

Likewise, we compute $\hat{\sigma} = \dfrac{s^2}{p}dy$ and

$$\begin{aligned}\xi_3 &= \frac{p^3}{s^6}\theta_{yyy} + \frac{p^2}{s^4}\theta_{xyy} + \frac{2u_{xyy}}{s^2}\theta_{xy} - \frac{p u_{xyy}}{s^4}\theta_x \\ &\equiv \frac{p^3}{s^6}du_{yyy} + \frac{p^2}{s^4}du_{xyy} + \frac{2p^2 u_{xyy}}{s^5}ds - \frac{p u_{xyy}}{s^4}dp \qquad \mod \hat{\sigma} \\ &\equiv \frac{p^3}{s^6}d(\frac{s^2}{p}u_{xyy} + u_{yyy}) \qquad \mod \hat{\sigma}\end{aligned}$$

to arrive at the Y invariants $J = y$ and $J' = \dfrac{s^2}{p}u_{xyy} + u_{yyy}$.

For our second example, we consider the equation

$$u_{xy} + u u_{xx} + f(y, p) = 0, \tag{6.1}$$

where, as usual, $p = u_x$. This equation was first studied in detail by Calogero [3] who proved, by *ad hoc* methods, that a general solution to (6.1) can be obtained by quadratures. Our general results can be applied to (6.1) to substantially clarify Calogero's analysis.

The characteristic vector fields for (6.1) are

$$X = uD_x + D_y \qquad \text{and} \qquad Y = D_x,$$

the commutator is $[\,X, Y\,] = -u_x Y$ and we easily see that the coefficients of the universal linearizations (2.3) and (2.4) are $A = 0$, $B = F_p$, $C = u_{xx}$ and $D = 0$, $E = F_p - u_x$ and $G = u_{xx}$. It is simple matter to directly check that

$$H_0 = -u_{xx} \qquad \text{and} \qquad H_1 = 0 \tag{6.2}$$

while, from the general recursion formula [*11*]

$$K_n = 2K_{n-1} - K_{n-2} - YX(\log K_{n-1}) - PX(\log K_{n-1}) + Y(Q) - X(P) + 2PQ,$$

where $n \geq 1$ and $K_{-1} = H_0$, it follows that

$$K_n = Y(\alpha_n) \tag{6.3}$$

where the functions $\alpha_n(y, p)$ are given recursively by

$$\alpha_0 = f_p - 2u_x, \qquad \alpha_1 = 3(f_p - u_x) + \frac{f f_{ppp} - f_{ypp}}{f_{pp} - 2}, \qquad \text{and}$$
$$\alpha_n = 2\alpha_{n-1} - \alpha_{n-2} + f_p + \big(\frac{\partial}{\partial p} - \frac{\partial}{\partial y}\big)\log\big(\frac{\partial \alpha_{n-1}}{\partial p}\big).$$

For simplicity, we now assume that $f = f(p)$. By Theorem 5.3, we are assured that there are two X invariant functions and that (6.1) is always semi-darboux integrable. Indeed, we know that $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\hat{\tau}, \eta_2)$ where

$$\hat{\tau} = dx - u\,dy + \frac{1}{r}\theta_x = \frac{1}{r}[dp + f\,dy]$$

and

$$\eta_2 = \theta_{xx} - \frac{u_{xxx}}{r}\theta_x \equiv dr - \frac{r}{f}(p + f')dp \qquad \text{mod } \hat{\tau}$$

from which it is a simple matter to arrive at invariants

$$I_1 = y + \int \frac{dp}{f} \qquad \text{and} \qquad I_2 = \frac{r}{f}\exp\big[-\int \frac{p\,dp}{f}\big]$$

if $f \neq 0$ and, for $f = 0$,

$$I_1 = p \qquad \text{and} \qquad I_2 = re^{yp}.$$

The equation

$$I_2 = \phi(I_1) \tag{6.4}$$

is a *second order general intermediate integral* for (6.1). Viewed as an ordinary differential equation for p, we can solve (6.4) by quadratures to obtain p in terms of x and y, the arbitrary function ϕ and a constant of integration $\gamma(y)$. The second derivative u_{xx} is then determined by (6.4), mixed partial u_{xy} is determined by $u_{xy} = \dfrac{d\,p}{d\,y}$ and the general solution to (6.1) therefore given by

$$u = -\frac{1}{u_{xx}}[u_{xy} + f(p)]. \tag{6.5}$$

For example, for the equation

$$u_{xy} + uu_{xy} - 2u_x(u_x + a) = 0, \tag{6.6}$$

where a is a constant, the invariants are

$$I_1 = \frac{p + a}{p}e^{2ay} \qquad \text{and} \qquad I_2 = \frac{u_{xx}}{p\sqrt{p + a}}$$

and with

$$\phi(z) = -2b\sqrt{\frac{cz + a - c}{acz}},$$

for constants b and c, the equation (6.4) becomes

$$\frac{d\,p}{d\,x} = -2bp\sqrt{\frac{p}{h(y)} + 1} \quad \text{where} \quad h(y) = \frac{ace^{2ay}}{a - c + ce^{2ay}}. \tag{6.7}$$

The integration of (6.7) leads to $p = -h(y)\text{sech}^2(bx + \gamma(y))$ in which case (6.5) becomes

$$u = -\frac{h(y)}{b}\tanh(bx + \gamma(y)) - \frac{\gamma'(y)}{b}.$$

This solution to (6.6) contains, as a special case, the solution obtained by Calagero. It is perhaps worth noting that it is not possible to find a closed form solution to (6.1) involving 2 arbitrary functions since this would imply the (6.1) is Darboux integrable. This is not the case since for (6.6) we find that

$$K_n = -(2n^2 + 7n + 6)u_{xx}. \tag{6.8}$$

Nevertheless, it is possible to find functions $f(p)$ for which (6.1) is Darboux integrable at low orders. For example, the equation

$$u_{xy} + uu_{xx} + u_x^2 = 0 \tag{6.9}$$

satisfies $H_1 = 0$ and $K_0 = 0$. The X invariants are now

$$I_1 = y - \frac{1}{p} \qquad \text{and} \qquad I_2 = \frac{r}{p^3}$$

and the Y invariants are

$$J_1 = y \qquad \text{and} \qquad J_2 = q + up$$

and hence, in accordance with the method of Darboux, we set

$$r = p^3\phi_0(y - \frac{1}{p}) = \frac{p^3}{\phi''(\alpha)} \qquad \text{and} \qquad q = -up + \psi''(\beta)$$

where $\alpha = y - \dfrac{1}{p}$ and $\beta = y$. Then, in terms of the variables α, β, x and u, the differential system

$$du - p\,dx - q\,dy = 0 \quad \text{and} \quad dp - r\,dx - s\,dy = 0$$

becomes

$$du - \frac{dx}{\beta - \alpha} + \big[\frac{u}{\beta - \alpha} - \psi''(\beta)\big]\,d\beta = 0 \quad \text{and} \quad \phi''(\alpha)\,d\alpha - \frac{dx}{\beta - \alpha} + \frac{u}{\beta - \alpha}\,d\beta = 0$$

We therefore deduce that the general solution to (6.9) is given by

$$u = \phi'(\alpha) + \psi'(\beta), \qquad x = \beta\phi'(\alpha) - \alpha\phi'(\alpha) + \phi(\alpha) + \psi(\beta), \qquad y = \beta.$$

For our third example, we classify all hyperbolic equations of the form

$$r = f(t), \tag{6.10}$$

where $f' > 0$, which are Darboux integrable on the second jet bundle. It is convenient to write

$$f'(t) = \frac{1}{g^2(t)}. \tag{6.11}$$

where $g(t) > 0$.

Proposition 6.1. *Let $H_0 \neq 0$ and $H_1 = 0$. Then the equation (6.10) is contact equivalent to one of the following three equations determined by*

$$g(t) = t^2, \tag{6.12}$$

$$\tan(\sqrt{g} - t) = \sqrt{(g)}, \qquad \text{or} \tag{6.13}$$

$$\tanh(\sqrt{g} - t) = \sqrt{g}. \tag{6.14}$$

Proof. We first note that the constant of integration arising from (6.11) in the determination of f from g can always be absorbed by the change of variables $\bar{u} = u + \lambda x^2$. We remark that (6.12) leads to $3rt^3 + 1 = 0$ which is integrated in Goursat [*9*] (example IV, p. 130). The computations of Forsyth [*6*] would seem to indicate, contrary to our conclusions, that (6.12) is the only Darboux integrable equation of the type (6.10)

An easy computation, using Maple, shows that in order for the highest order terms in H_1 to vanish, the function g must satisfy

$$4g^2(g'')^2 - 4g^2g'''g' - 4g(g')^2g'' + (g')^4 = 0. \tag{6.15}$$

Since $g \neq 0$, the only singular solution to this equation is $g' = 0$. But then $g = k$ is a constant and $H_0 = 0$ and this is a case which we have excluded. The general solution to (6.15) for $g = g(t)$ depends upon 3 arbitrary constants and so f depends upon 4 arbitrary constants. To prove the theorem, we must show that these constants can be normalized by contact transformations so as to arrive at exactly one of the three equations (6.12), (6.13) or (6.14).

We observe that under the reflection

$$\bar{x} = y, \quad \bar{y} = x, \quad \bar{u} = u \tag{6.16}$$

the equation $r = f(t)$ is transformed into $r = f^{-1}(t)$. Also, under the simple point transformations

$$\bar{x} = x, \quad \bar{y} = y, \qquad \bar{u} = u + \frac{1}{2}ay^2 \tag{6.19}$$

and

$$\bar{x} = x, \quad \bar{y} = b^2y, \quad \bar{u} = cu \tag{6.20}$$

the equation (6.10) is transformed into $\bar{r} = \bar{f}(\bar{t})$, where

$$\bar{f}(\bar{t}) = f(\bar{t} - a) \quad \text{and} \quad \bar{f}(\bar{t}) = cf(\frac{b^4}{c}\bar{t}),$$

respectively. Hence the function g transforms under (6.19) and (6.20) as

$$\bar{g}(\bar{t}) = g(\bar{t} - a) \quad \text{and} \quad \bar{g}(\bar{t}) = \frac{1}{b^2}g(\frac{b^4}{c}\bar{t}). \tag{6.21}$$

Since the condition $H_1 = 0$ is contact invariant it follows that the equation (6.15) is invariant under the 3 parameter solvable group (6.21) and is therefore solvable by quadratures. Applying Lie's method for the solution of such equations, or alternatively, by noting that

$$\frac{g'}{\sqrt{g}} + 2\frac{\sqrt{g}g''}{g'} = 4\alpha \qquad \text{and} \qquad \sqrt{g}g' - 2\frac{g^{3/2}g''}{g'} = 4\beta$$

are first integrals for (6.15), we deduce that every solution to (6.15) is a solution to the first order equation

$$g' = 2\alpha\sqrt{g} + \frac{2\beta}{\sqrt{g}} \tag{6.22}$$

for some choice of constants α and β. It is easy to check that this equation implies that $H_1 = K_1 = 0$ so that every solution to (6.22) determines a Darboux integrable equation.

Under the change of variables (6.20), the first integrals α and β transform as

$$\bar{\alpha} = \frac{b^3}{c}\alpha \qquad \text{and} \qquad \bar{\beta} = \frac{b}{c}\beta.$$

Thus according to whether (i) $\alpha \neq 0$, $\beta = 0$; (ii) $\alpha = 0$, $\beta \neq 0$; (iii) $\alpha\beta > 0$; or (iv) $\alpha\beta < 0$, we can use the point transformation (6.20) to transform g so that (i) $\alpha = 1$, $\beta = 0$; (ii) $\alpha = 0$, $\beta = 1$; (iii) $\alpha = 1$, $\beta = 1$; or (iv) $\alpha = 1$, $\beta = -1$. In case (i) we find that $g(t) = (t + k)^2$ and we use (6.19) to transform this to $g(t) = t^2$. This gives the equation $3rt^3 + 1 = 0$. In case (ii), we find that $g(t) = (3t)^{2/3}$ which yields the equation obtained from (6.12) by the reflection (6.16). In case (iii) we find that g is uniquely determined by (6.13) while (iv) implies that

$$\sqrt{g} + \frac{1}{2}\log\left|\frac{\sqrt{g} - 1}{\sqrt{g} + 1}\right| = t.$$

There are two subcases to consider here depending on whether $0 < g < 1$ or $g > 1$. In the first case we find the g satisfies (6.14) while the second case can be reduced to the first, again by the reflection (6.16).

Finally, it is not difficult to see that the Lie algebra of contact symmetries for (6.12), (6.13) and (6.14) consists solely of point symmetries. For (6.12), there is 9-dimensional algebra of point symmetries while that for (6.13) and (6.14) is 7-dimensional ([*10*](p. 215)). This shows that (6.12) is not equivalent to either (6.13) or (6.14). We have, as yet, been unable to distinguish (6.13) from (6.14) by contact invariant means. ∎

For the equation $3rt^3 + 1 = 0$, we have $X = D_x + \frac{1}{t^2}D_y$, $H_1 = 0$ and $M_\tau = \frac{t^3}{4}$ so that, by Theorem 5.3, there are two X invariants of order 2. We determine

$$\hat{\tau} = \frac{1}{2}(st + 1)dx - \frac{t}{2}dq \qquad \text{and}$$

$$\eta_2 = \frac{2}{t^4}\theta_{yy} - \frac{2}{t^2}\theta_{xy} + \frac{2}{t^5}(t^2 u_{xyy} - u_{yyy})\theta_y \equiv -\frac{2}{t^2}ds + \frac{2}{t^4}dt \qquad \text{mod } \hat{\tau}$$

$$\equiv -\frac{2}{t^2}d(s + \frac{1}{t}) \qquad \text{mod } \hat{\tau}$$

so that one invariant is $I = s + \frac{1}{t}$. Since $\hat{\tau} = \frac{t}{2}[I dx - dq]$ it follows that the other invariant is $I' = Ix - q$.

We conclude by remarking that the generalized Laplace invariants may be used in a variety of other ways. For example, Lie's well-known classification of the wave equation $u_{xy} = 0$ may be reformulated as the statement that, up to contact equivalence, the only equation with $H_0 = K_0 = 0$ is the wave equation. It is possible to give similar classification of the equations $u_{xy} = u$, $u_{xy} = e^u$ and $u_{xy} = \frac{2u}{(x+y)^2}$ in terms of the generalized Laplace invariants. These and other applications will be presented elsewhere [*11*], [*12*].

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