THE VARIATIONAL BICOMPLEX FOR HYPERBOLIC SECOND ORDER SCALAR PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

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Abstract

The variational bicomplex associates to any partial differential equation a set of cohomology groups $H^{r,s}$ whose elements correspond to the classical ($s = 0$) and higher-degree or contact form valued ($s \geq 1$) conservation laws of arbitrary differential order for the given equation. In this paper we study the variational bicomplex for scalar, second order partial differential equations in the plane, with an emphasis on the classification of higher degree conservation laws for hyperbolic equations. This higher degree cohomology underlies such important geometric properties as Darboux integrability (at arbitrary order), the existence of variational principles and the existence of intermediate integrals.

In the spirit of E. Cartan's method of equivalence we construct, using the classical method of Laplace for solving linear hyperbolic equations, an adapted coframe (called the Laplace adapted coframe) on the infinite prolongation of the equation whose structure functions give rise to two sequences of relative invariants (called the generalized Laplace invariants) for the given partial differential equation. The non-vanishing of these invariants obstruct the existence of contact forms which are invariants for the characteristic vector fields for the equation.

We prove a structure theorem for the conservation laws in $H^{1,s}$ which characterizes these conservation laws in terms of contact form valued solutions to the adjoint of the linearization of the given equation. This enables us to prove that if neither of the two sequences of generalized Laplace invariants terminates, then $H^{1,s} = 0$ for all $s \geq 3$. When at least one of the sequences of Laplace invariants terminates, the cohomology groups $H^{1,s}$, $s \geq 3$, are shown to be generated from contact forms which are invariants for the characteristic vector fields.

As an application to the problem of geometric integrability, we prove that the two sequences of generalized Laplace invariants must terminate for Darboux integrable equations and that the cohomology groups $H^{1,s}$ for such equations are infinite dimensional for all $s \geq 0$.

§**1. Introduction.** Over the past two decades, there has been a great resurgence in the differential geometric investigation of partial differential equations, a subject pioneered and quite extensively developed around the turn of the century by Lie, Darboux, Goursat, Janet, E. Cartan, Vessiot and others. Topics of interest today include the systematic computation of (generalized) symmetries and conservation laws, classical and non-classical methods of reduction, Hamiltonian and bi-Hamiltonian structures, the inverse problem of the calculus of variations, Bäcklund transformations, Painlevé properties, and equivalence problems. Much of this activity is directed towards the goal of achieving a better understanding of the phenomena of complete integrability for partial differential equations. The purpose of this paper is to carry out a systematic study of some of these issues for second order hyperbolic scalar partial differential equations

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1.1}$$

in two independent variables x and y and one dependent variable u. We shall focus, in particular, on the calculation of conservation laws.

A *classical conservation law* for the partial differential equation (1.1) is a one form

$$\omega = M\,dx + N\,dy,$$

where M and N are functions of x, y, u and the derivatives of u up to finite order, such that

$$D\omega = (D_x N - D_y M)\,dx \wedge dy = 0$$

by virtue of the equation (1.1) and its differential consequences. Here D_x and D_y denote the total derivatives with respect to the independent variables x and y, and we have used the symbol D to emphasize that the exterior derivative of the form ω is computed in terms of these total derivatives. For instance, the one form $u\,dx + u_x\,dy$ is a conservation law for the heat equation $u_y = u_{xx}$, the form $(\frac{1}{2}u_{xx}^2 - \frac{1}{8}u_x^4)\,dx + \frac{1}{2}u_x^2 \cos u\,dy$ is a conservation law for the sine-Gordon equation $u_{xy} = \sin u$, and $(u_{xx} - \frac{1}{2}u_x^2)\,dx$ is a conservation law for the Liouville equation $u_{xy} = e^u$. A classical conservation law $\omega = M dx + N dy$ is said to be *trivial* if there exists a function f of the variables x, y, u and the derivatives of u up to finite order, such that

$$Df = (D_x f)\,dx + (D_y f)\,dy = \omega,$$

again by virtue of the differential equation (1.1) and its derivatives. It is important to be able to recognize trivial conservation laws for a given differential equation.

It is well-known that the classical conservation laws are fundamental tools for establishing the existence and regularity of solutions for partial differential equations. In this paper, we shall also be interested in *higher degree* or *form valued conservation laws*, perhaps the simplest examples of which are given by the forms

$$du_y \wedge dy, \quad du_y \wedge du_{yy} \wedge dy, \quad du_y \wedge du_{yy} \wedge du_{yyy} \wedge dy, \quad \ldots$$

which are all D closed for the wave equation $u_{xy} = 0$. These higher degree conservation laws reflect properties of partial differential equations that cannot be detected by classical conservation laws alone. For example, it is well-known that the sine-Gordon and Liouville equations both admit infinitely many classical conservation laws. However, the Liouville equation is integrable by the method of Darboux while the sine-Gordon equation is not. This is reflected at the level of higher degree conservation laws by the fact that the sine-Gordon equation has no conserved forms of degree 4 and higher whereas the Liouville equation has infinitely many conserved forms of arbitrary degree. Higher degree conservation laws are also central to the theory of Hamiltonian systems. Indeed, the symplectic two form $\Omega = \sum_i^n dp_i \wedge dq^i$ is D closed as a consequence of Hamilton's equations $\dot{q}^i = \dfrac{\partial H}{\partial p_i}$

and $\dot{p}_i = -\dfrac{\partial H}{\partial q^i}$, where $H = H(q^i, p_i)$. These higher degree conservation laws are likely to prove useful in studying the properties of families of solutions of partial differential equations in much the same way as integral invariants like the Cartan two form are used to study the properties of families of solutions to (Hamiltonian) ordinary differential equations. Finally, we note that higher degree conservation laws play a pivotal role in the solution to the classical inverse problem of the calculus of variations [*7*] and in recent generalizations of the inverse problem [*3*].

A natural geometric framework for the study of classical or higher degree conservation laws is provided by the variational bicomplex associated to the given system of differential equations. The variational bicomplex is a double complex of differential forms on the infinite jet bundle of the space of independent and dependent variables. It was originally introduced by Tulczyjew [*39*] to geometrically define the Euler-Lagrange operator and to solve the inverse problem of the calculus of variations. The variational bicomplex has since found a broad range of applications [*1*], [*27*], [*36*], [*37*], [*42*]. One particularly important application arises when the variational bicomplex is constrained by pullback to the infinite prolongation manifold of a partial differential equation. In this setting, the horizontal cocycles in the bottom row of the bicomplex correspond to the classical conservation laws and the horizontal cocycles in the higher rows give rise to the higher degree form valued conservation laws. Moreover, the horizontal cohomology of this constrained variational bicomplex precisely characterizes all non-trivial conservation laws.

One of the first general results concerning the horizontal cohomology of the variational bicomplex associated to a differential equation was obtained by Vinogradov [*42*]. As a corollary to his general theorems, it follows that if a system of partial differential equations in n independent variables is of Cauchy-Kovalevskaya type, then the horizontal cohomology of the associated bicomplex is trivial in horizontal degrees $\leq n-2$. Recently, Tsujishita [*38*], Zharinov [*43*], Bryant and Griffiths [*13*], and Barnich, Brandt and Henneaux [*11*] have generalized Vinogradov's result by obtaining a lower bound on the horizontal degree of the non-trivial cohomology of the variational bicomplex for arbitrary (in particular, overdetermined) systems of involutive differential equations. It is noteworthy that Bryant and Griffiths obtain their result within the context of the characteristic cohomology for exterior differential systems and that they compute this lower bound from the Cartan characters of the differential system. These results are nicely illustrated in the case of the Frobenius integrability conditions for a codimension one foliation – the Godbillon-Vey invariant appears in this context as a conserved three form.

The literature on conservation laws for specific differential equations is, needless to say, vast indeed and cannot be reviewed in detail here. For linear scalar differential equations in more than two independent variables we do however cite the work of Shapovalov and Shirokov [*34*] who completed the work initiated by Delong [*17*] to characterize all the higher order conservation laws for the higher dimensional Laplace's equation and wave equation. For completely integrable equations like the KdV equation there are many schemes for generating infinite sequences of non-trivial conservation laws and it is usually not too difficult to show that the infinite sequence so constructed, together with a few exceptional conservation laws of low order, yield all possible classical conservation laws for the given equation. The recent interesting work of Khorkova [*26*] shows that most of the well known integrable evolution equations have few, if any, higher degree conserved forms and those which do exist can be computed without difficulty. It seems fair to say then, that the horizontal cohomology of the variational bicomplex of the standard integrable evolution equations is well understood even though complete calculations do not explicitly appear in the literature. The general complexities involved in computing conservation laws for higher order non-linear partial differential equations not of evolutionary type are well illustrated by the work of Duzhin and Tsujishita [*18*] and Olver [*29*] on the BBM equation, of Olver [*30*], [*31*] on the equations of elasticity, of Anderson and Torre [*8*] on the vacuum Einstein equations and Barnich, Brandt and Henneaux [*11*] on the Yang-Mills equations.

There is also considerable research on the problem of classifying partial differential equations according to the number of conservation laws they admit. Bryant and Griffiths [*14*] have made a detailed study of this problem for second order scalar parabolic equations in the plane. They prove, in particular, that if (1.1) is a parabolic equation which admits four classical conservation laws, then it is contact-equivalent to a linear equation and hence admits infinitely many classical conservation laws. For general evolution equations, Mikhailov, Shabat and Sokolov [*28*] have developed an extensive enumeration of the evolution equations which admit infinitely many conservation laws. In the more complicated case of hyperbolic equations, Zhiber and Shabat [*44*] have proved that the only f-Gordon equations

$$u_{xy} = f(u), \tag{1.2}$$

which possess infinitely many classical conservation laws are equivalent to (allowing for complex changes of coordinates) one of the following equations

$$u_{xy} = 0, \quad u_{xy} = e^u, \quad u_{xy} = e^u + e^{-u}, \quad u_{xy} = e^u + e^{-2u}. \tag{1.3}$$

It is interesting to observe that their solution to this classification problem depends crucially on Lie's classification of Darboux integrable equations of the type (1.2). Recently, Bryant, Griffiths and Hsu [*15*] have undertaken a general study of the classical conservation laws, or characteristic cohomology, for general hyperbolic exterior differential systems. In particular, they have completed a systematic study of the classical conservation laws for certain classes of hyperbolic exterior differential systems corresponding to a quasi–linear hyperbolic pair of first-order partial differential equations, with emphasis on the equations admitting a maximum number of first order classical conservation laws. (We should warn the reader that in the terminology of Bryant, Griffiths and Hsu, a classical conservation law, or a conservation law of level zero, is one which is of order less than or equal to that of the equation. For us, a classical conservation law can have *arbitrary* order.) They have also given a solution to the inverse problem of the calculus of variations for such equations and proved an existence theorem for the initial value problem for Darboux integrable equations. E. Cartan's method of equivalence is an essential tool used throughout their work. Recently, Gardner and Kamran [*21*] have extensively studied the characteristic systems of second order scalar hyperbolic equations (1.1), with applications to the Cartan equivalence problem under classical contact transformations.

In this paper, we embark upon the task of computing and analyzing the structure of the horizontal cohomology of the variational bicomplex for scalar second order partial differential equations in the plane especially as it relates to the property of Darboux integrability. We will place no *a priori* restrictions on the order or the degree of the conservation laws considered by working with the full variational bicomplex on the infinite prolongation of (1.1). This will enable us to capture conservation laws of arbitrary order and degree. Our entire approach hinges upon the introduction of moving coframes for the variational bicomplex which are adapted to the geometry of the differential equation (1.1). Thus we work very much in the spirit of E. Cartan's method of equivalence. The moving coframe we construct for any scalar second order hyperbolic equation in the plane is inspired by the classical method of Laplace for the closed form integration of certain linear hyperbolic equations

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0.$$

This coframe is constructed by applying a generalization of the classical Laplace transformation to the contact form valued linearization of (1.1). We call the coframe obtained by this procedure the *Laplace adapted coframe*. Indeed, certain structure functions derived from the structure equations for the Laplace adapted coframe define relative invariants for (1.1) which, in the case of linear equations, coincide exactly with the well-known Laplace invariants

$$h = \frac{\partial a}{\partial x} + ab - c, \quad k = \frac{\partial b}{\partial y} + ab - c$$

We call these new invariants the *generalized Laplace invariants*. It is straightforward to construct higher degree conservation laws for second order hyperbolic equations if we are given contact forms which are invariant (in the sense that they Lie differentiate to zero) with respect to one of the characteristic vector fields for (1.1). It is therefore to be expected that these characteristic invariant contact forms will play a pivotal role in the study of the variational bicomplex for second order partial differential equations in the plane. With the Laplace adapted coframe at our disposal, it is easy to see that the non-vanishing of the generalized Laplace invariants is the primary obstruction to the existence of characteristic invariant contact forms. This, in turn, enables us to prove that the non-vanishing of the generalized Laplace invariants also obstructs the existence of higher degree conservation laws. This accounts, for example, for the difference between the sine-Gordon and Liouville equations in this regard. We are also able to prove that *all the non-trivial higher degree conservation laws are generated precisely in this manner from the characteristic invariant contact forms.* Our proof is in two main steps. We first put the conservation law in "quasi–normal" form by applying repeated integration by parts. In this quasi–normal form, it becomes apparent that the non-trivial conservation laws are characterized by the solutions of the *adjoint* of the contact form valued linearization of (1.1). We then solve explicitly the adjoint equation by means of the generalized Laplace transform. The closed form solution we obtain is a contact form which is directly expressed in terms of the invariant elements of the Laplace adapted coframe and the generalized Laplace invariants.

Our Laplace adapted coframe is also well suited for the analysis of *Darboux integrable* equations. By using Goursat's results [*23*] we produce an algorithm for explicitly constructing infinitely many characteristic invariant contact forms and *non-trivial* conservation laws for any Darboux integrable equation. Consequently, it follows that *for any Darboux integrable equation, the generalized Laplace invariants must eventually vanish.* Altogether, these results underscore the importance of enhancing the calculus of the variational bicomplex through the introduction of moving frames and the need to work on infinite order jet spaces in order to fully apprehend the geometry of differential equations.

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§2. The variational bicomplex for scalar, second order partial differential equations in the plane. In this section we briefly summarize the basic aspects of the differential calculus on infinite jet bundles, we define the *free* variational bicomplex for such spaces and we construct the constrainted variational bicomplex associated to the scalar, second order partial differential equation in the plane

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{2.1}$$

To define the free variational bicomplex, let $\pi\colon E \to M$ be a fiber bundle over a connected base manifold M and let $\pi^k_M\colon J^k(E) \to M$ be the bundle of k-jets of local sections of E. For a point $p \in M$, the fiber $(\pi^k_M)^{-1}(p)$ consists of equivalence classes, denoted by $j^k(s)(p)$, of local sections s of E. If U and U' are two neighborhoods of p in M and if $s\colon U \to E$ and $s'\colon U' \to E$ are two local sections of E, then s and s' are equivalent local sections at p if their partial derivatives to order k agree at p. If the dimension of M is n and that of E is $m+n$, then on E we shall use local adapted coordinates $\pi\colon (x^i, u^\alpha) \to (x^i)$, where $i = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$. The induced local coordinates on $J^k(E)$ are $(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_{i_1 i_2 \ldots i_k})$ where $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$, and where

$$u^\alpha_{i_1}(j^k(s)(p)) = \frac{\partial s^\alpha}{\partial x^{i_1}}(p) \qquad \text{and} \qquad u^\alpha_{i_1 i_2}(j^k(s)(p)) = \frac{\partial^2 s^\alpha}{\partial x^{i_1}\partial x^{i_2}}(p),$$

and so on. For any $k \ge l$, there are natural projections $\pi^k_l\colon J^k(E) \to J^l(E)$. The infinite jet bundle $\pi^\infty_M\colon J^\infty(E) \to M$ is similarly defined as equivalence classes, now denoted by $j^\infty(s)(p)$, of infinite jets of local sections and, likewise, there are projections $\pi^\infty_l\colon J^\infty(E) \to J^l(E)$. Any local section $s\colon U \to E$ of the bundle π lifts to a unique section $j^\infty(s)\colon U \to J^\infty(E)$.

By definition, a smooth real-valued function $f\colon J^\infty(E) \to \mathbf{R}$ is one which factors through some finite-order jet bundle $J^k(E)$, that is, there is a function $f_0\colon J^k(E) \to \mathbf{R}$ such that $f = f_0 \circ \pi^\infty_k$. The minimum such k is called the *order* of f. In local coordinates we shall often write $f = f(x, u^{(k)})$ to indicate that f is a function of the k-jet coordinates x^i, u^α, $u^\alpha_{i_1}$, $\ldots$, $u^\alpha_{i_1 i_2 \ldots i_k}$.

A vector field X on $J^\infty(E)$ is a derivation on the ring of smooth functions on $J^\infty(E)$. In local coordinates, we have that

$$X = A^i \frac{\partial}{\partial x^i} + \sum_{\substack{k=0 \\ 1 \le i_1 \le \cdots \le i_k \le n}}^{\infty} B^\alpha_{i_1 i_2 \cdots i_k} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \cdots i_k}}, \tag{2.2}$$

where the coefficients A^i and $B^\alpha_{i_1 i_2 \cdots i_k}$ are locally defined functions on a coordinate neighborhood of $J^\infty(E)$. Since a smooth function f on $J^\infty(E)$ is of finite order, $X(f)$ will always reduce to a finite sum. Denote by $\Omega^p(J^\infty(E))$ the smooth differential p forms on $J^\infty(E)$. In local coordinates, a p form is a finite linear combination of terms

$$a(x, u^{(k)})\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge du^{\alpha_1}_{j_1 j_2 \cdots j_{p_1}} \wedge \cdots \wedge du^{\alpha_s}_{k_1 k_2 \cdots k_{p_s}},$$

where $r + s = p$. A differential form ω on $J^\infty(E)$ is called a contact form if, for every local section s of E, $[j^\infty(s)]^*(\omega) = 0$. The contact ideal $\mathcal{C}(J^\infty(E))$ is generated, as an ideal in $\Omega^*(J^\infty(E))$, by the contact one forms

$$\theta^\alpha_{i_1 i_2 \cdots i_k} = du^\alpha_{i_1 i_2 \cdots i_k} - u^\alpha_{i_1 i_2 \cdots i_k j} dx^j \tag{2.3}$$

for all $k = 0, 1, \ldots$. Let $\mathcal{C}^s(J^\infty(E))$ denote the s-th exterior product of the contact ideal $\mathcal{C}(J^\infty(E))$.

A vector field X on $J^\infty(E)$ such that $X \lrcorner\, \omega = 0$ for any contact one form ω, is called a total vector field. If the vector field (2.2) is a total vector field, then $X = A^j\, D_j$, where

$$D_j = \frac{\partial}{\partial x^j} + u^\alpha_j \frac{\partial}{\partial u^\alpha} + u^\alpha_{i_1 j} \frac{\partial}{\partial u^\alpha_{i_1}} + u^\alpha_{i_1 i_2 j} \frac{\partial}{\partial u^\alpha_{i_1 i_2}} + \cdots . \tag{2.4}$$

The Lie bracket of two total vector fields $X = A^i D_i$ and $Y = B^i D_i$ is the total vector field

$$[X, Y] = (A^j D_j B^i - B^j D_j A^i)\, D_i. \tag{2.5}$$

To define the variational bicomplex, we bi-grade the forms on $J^\infty(E)$ as follows. A p form ω on $J^\infty(E)$ is said to be of type (r, s), where $r + s = p$, if

$$\omega(X_1, X_2, \ldots, X_p) = 0,$$

whenever either (i) more that s of the vector fields X_1, X_2, ... , X_p are π^∞_M vertical, or (ii) more that r of the vector fields X_1, X_2, ... , X_p are total vector fields. Denote the space of type (r, s) forms on $J^\infty(E)$ by $\Omega^{r,s}(J^\infty(E))$. In local coordinates, a type (r, s) form is a finite sum of terms

$$a(x, u^{(k)})\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge \theta^{\alpha_1}_{j_1 j_2 \cdots j_{p_1}} \wedge \cdots \wedge \theta^{\alpha_s}_{k_1 k_2 \cdots k_{p_s}}.$$

We evidently have the direct sum decomposition $\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E))$. We emphasize that this decomposition is *not* possible on any of the finite dimensional jet bundles $J^k(E)$. For any non-negative integers r and s such that $r + s = p$, let $\pi^{r,s} \colon \Omega^p(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$ be the projection map. The exterior derivative is a map

$$d \colon \Omega^{r,s}(J^\infty(E)) \to \Omega^{r+1,s}(J^\infty(E)) \oplus \Omega^{r,s+1}(J^\infty(E))$$

and hence d splits into horizontal and vertical differentials

$$d = d_H + d_V,$$

where, for $\omega \in \Omega^{r,s}(J^\infty(E))$, $d_H(\omega) = \pi^{r+1,s}(d\,\omega)$ and $d_V(\omega) = \pi^{r,s+1}(d\,\omega)$. Since $d^2 = 0$, we have that $d_H^2 = 0$, $d_H d_V = -d_V d_H$, and $d_V^2 = 0$. The *free variational bicomplex* for the fiber bundle $\pi : E \to M$ is the double complex $(\Omega^{*,*}(J^\infty(E)), d_H, d_V)$ of differential forms on the infinite jet bundle $J^\infty(E)$ of E.

If X is a total vector field on $J^\infty(E)$ and $\omega \in \Omega^{r,s}(J^\infty(E))$, then we define $X(\omega) \in \Omega^{r,s}(J^\infty(E))$ to be the projected Lie derivative $X(\omega) = \pi^{r,s}(\mathcal{L}_X \omega)$. It is not difficult to prove that

$$X(\omega) = X \lrcorner\, d_H(\omega) + d_H(X \lrcorner\, \omega). \tag{2.6}$$

In particular, we have that $D_j \theta^\alpha_{i_1 i_2 \cdots i_k} = \theta^\alpha_{i_1 i_2 \cdots i_k j}$. Equation (2.6) also implies that for any total vector field X,

$$X(d_H \omega) = d_H X(\omega). \tag{2.7}$$

We shall also need the following properties of the projected Lie derivative. See [*1*] for details.

Proposition 2.1. *Let $\omega \in \Omega^{r,s}(J^\infty(E))$. If X and Y are total vector fields on $J^\infty(E)$ and Z is any π^∞_M vertical vector field on $J^\infty(E)$, then*

$$X(Y(\omega)) - Y(X(\omega)) = [X,Y](\omega) \tag{2.8}$$

and

$$Z \lrcorner X(\omega) = [Z,X] \lrcorner \omega + X(Z \lrcorner \omega). \tag{2.9}$$

We now define the (constrained) variational bicomplex for scalar, second order partial differential equations in the plane. Our considerations are primarily of a local nature and so we take for E the trivial bundle $\pi\colon U \times I \to U$, where U is a open connected subset of the plane $\mathbf{R}^2$ and I is an open interval in $\mathbf{R}$. Coordinates for E are $\pi\colon (x,y,u) \to (x,y)$ and the coordinates on $J^\infty(E)$ are now denoted by

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots, u_{x^iy^j}, \ldots).$$

For the 2-jets of u we shall often use the classical Monge notation $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, and $t = u_{yy}$. A given second-order partial differential equation defines a locus

$$F(x, y, u, p, q, r, s, t) = 0 \tag{2.10}$$

in $J^2(E)$. We let $\mathcal{R}^2$ denote a open, connected and contractible subset of this locus. We assume that F is smooth in a neighborhood of $\mathcal{R}^2$ and that

$$(F_r, F_s, F_t) \neq 0 \tag{2.11}$$

on $\mathcal{R}^2$. We shall also assume that the 7 dimensional manifold $\mathcal{R}^2$ fibers over the domain U. By performing a linear change of independent variables if necessary, we can assume that (2.10) can always be solved for $r = u_{xx}$ so that our equation manifold $\mathcal{R}^2$ is defined by as a subset of the locus of the equation

$$r + f(x, y, u, p, q, s, t) = 0. \tag{2.12}$$

Additional restrictions on the set $\mathcal{R}^2$ will be made in the next section. Let $\iota\colon \mathcal{R}^2 \hookrightarrow J^2(E)$ be the inclusion map.

The successive prolongations of $\mathcal{R}^2$ are defined by the total derivatives of (2.10), for example,

$$\mathcal{R}^3 = \{ j^3(s)(p) \,|\, j^2(s)(p) \in \mathcal{R}^2 \text{ and } (D_xF)(j^3(s)(p)) = (D_yF)(j^3(s)(p)) = 0 \}.$$

Due to the condition (2.11), each prolongation $\iota\colon \mathcal{R}^k \hookrightarrow J^k(E)$ is a submanifold of dimension $2k+3$ which fibers over $\mathcal{R}^{k-1}$. Let $\iota\colon \mathcal{R}^\infty \hookrightarrow J^\infty(E)$ be the infinite prolongation of $\mathcal{R}^2$ with projection maps $\pi^\infty_k\colon \mathcal{R}^\infty \to \mathcal{R}^k$ and $\pi^\infty_U\colon \mathcal{R}^\infty \to U$. Let $\mathcal{C}(\mathcal{R}^\infty)$ denote the pullback of the contact ideal on $J^\infty(E)$ to $\mathcal{R}^\infty$, $\mathcal{C}(\mathcal{R}^\infty) = \iota^*[\mathcal{C}(J^\infty(E))]$, and let $\mathcal{R}$ be the triple

$$\mathcal{R} = \{ \mathcal{R}^\infty, \pi^\infty_U, \mathcal{C}(\mathcal{R}^\infty) \}.$$

Local solutions to (2.10) are in one-to-one correspondence with local solutions to $\mathcal{R}$, that is, sections σ of $\pi^\infty_U\colon \mathcal{R}^\infty \to U$ which satisfy $\sigma^*\mathcal{C}(\mathcal{R}^\infty) = 0$. Accordingly we shall frequently identify the original differential equation (2.10) with the triple $\mathcal{R}$.

As before, a total vector field on $\mathcal{R}^\infty$ is a vector field X such that $X \lrcorner \omega = 0$ for any one form $\omega \in \mathcal{C}(\mathcal{R}^\infty)$. The differential forms on $\mathcal{R}^\infty$ can now be bi-graded by horizontal and vertical degree just as in the free variational bicomplex on $J^\infty(E)$.

Definition 2.2. *The variational bicomplex for* $\mathcal{R} = \{\, \mathcal{R}^\infty, \pi_U^\infty, \mathcal{C}(\mathcal{R}^\infty) \,\}$ *is the pull-back of the free variational bicomplex* $(\Omega^{*,*}(J^\infty(E)), d_H, d_V)$ *to* $\mathcal{R}^\infty$*:*

$$\begin{array}{ccccccccc}
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega^{0,2}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{1,2}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{2,2}(\mathcal{R}^\infty) \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega^{0,1}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{1,1}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{2,1}(\mathcal{R}^\infty) \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{1,0}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{2,0}(\mathcal{R}^\infty).
\end{array}$$

With the differential equation written in the form (2.12), the natural coordinates for $\mathcal{R}^\infty$ are

$$(x, y, u, u_x, u_y, u_{xy}, u_{yy}, \ldots, u_{xy^{k-1}}, u_{y^k}, \ldots) \tag{2.13}$$

and a basis for the contact ideal on $\mathcal{R}^\infty$ is

$$\{\, \theta, \theta_x, \theta_y, \theta_{xy}, \theta_{yy}, \ldots, \theta_{xy^{k-1}}, \theta_{y^k}, \ldots \,\}, \tag{2.14}$$

where $\theta = du - u_x\, dx - u_y\, dy$, $\theta_{xy^{k-1}} = du_{xy^{k-1}} + (D_y^{k-1} f)\, dx - u_{xy^k}\, dy$, and $\theta_{y^k} = du_{y^k} - u_{xy^k} dx - u_{y^{k+1}}\, dy$.

The primary object of our study is the space of type $(1, s)$ conservation laws

$$E_1^{1,s} = H^{1,s}(\mathcal{R}^\infty, d_H) = \frac{\{\, \omega \in \Omega^{1,s}(\mathcal{R}^\infty) \mid d_H\omega = 0 \,\}}{\{\, \omega \in \Omega^{1,s}(\mathcal{R}^\infty) \mid \omega = d_H\eta, \eta \in \Omega^{0,s}(\mathcal{R}^\infty) \,\}}.$$

Let $\omega = dx \wedge M + dy \wedge N$ be a representative of a cohomology class in $H^{1,s}(\mathcal{R}^\infty, d_H)$ so that $d_H\omega = 0$. When $s = 0$, M and N are functions on $\mathcal{R}^\infty$ and ω is called a *classical conservation law* for $\mathcal{R}$. Observe that if $\sigma\colon U \to \mathcal{R}^\infty$ is a solution to $\mathcal{R}$, then for any smooth closed curve $\gamma \in U$, the line integral $\displaystyle\int_\gamma \sigma^*(\omega)$ is constant under smooth deformations of γ. When $s \geq 1$, the coefficients M and N are contact forms of type $(0, s)$ and we call ω a *contact form-valued conservation law.* The space of type $(1, s)$ consevation laws is preserved by the projected pull-backs of generalized contact transformations [*1*].

§3. Characteristic vector fields, universal linearizations, and the first adaptation of the coframe on $\mathcal{R}^\infty$. The characteristic equation for the second order partial differential equation

$$F(x, y, u, p, q, r, s, t) = 0 \tag{3.1}$$

is the quadratic equation

$$F_r\lambda^2 - F_s\lambda\mu + F_t\mu^2 = 0. \tag{3.2}$$

Let $\sigma = j^2(s)(x, y)$ be a point in $J^2(E)$ satisfying (3.1). We say that (3.1) is of *hyperbolic, parabolic, or elliptic type* at the point σ according to whether the discriminant of (3.2), namely $\mathcal{D}(\sigma) = F_rF_t - \frac{1}{4}F_s^2$, is negative, zero, or positive at σ. It can be shown (see, for example, Gardner [*20*]) that the type of the equation (3.1) at any fixed point σ is invariant under classical contact transformations. We now assume that that every point of $\mathcal{R}^2$ is of hyperbolic type.

Let $\mathcal{R}^\infty$ be the infinite prolongation of $\mathcal{R}^2$ and let $(\Omega^{*,*}(\mathcal{R}^\infty), d_H, d_V)$ be the variational bicomplex associated to (3.1). In this section we introduce a basis for the distribution of total vector fields on $\mathcal{R}^\infty$ which is adapted to the characteristics of the given hyperbolic differential equation. We then use this basis for the total vector fields to construction our first adapted coframe for the differential forms on $\mathcal{R}^\infty$. To begin, we consider an arbitrary basis for the total vector fields on $\mathcal{R}^\infty$, say

$$X = m_x D_x + m_y D_y \qquad \text{and} \qquad Y = n_x D_x + n_y D_y, \tag{3.3}$$

where m_x, m_y, n_x and n_y are smooth real-valued functions on $\mathcal{R}^\infty$ satisfying

$$\delta = m_x n_y - m_y n_x \neq 0 \tag{3.4}$$

at each point of $\mathcal{R}^\infty$. We shall write the Lie bracket of X and Y as

$$[X, Y] = P\, X + Q\, Y. \tag{3.5}$$

We introduce the type $(1, 0)$ forms σ and τ dual to X and Y, characterized by

$$\sigma(X) = 1, \quad \sigma(Y) = 0, \quad \tau(X) = 0, \quad \tau(Y) = 1.$$

Dual to (3.5) are the d_H structure equations

$$d_H\sigma = -P\,\sigma \wedge \tau \quad \text{and} \quad d_H\tau = -Q\,\sigma \wedge \tau. \tag{3.6}$$

If $\omega \in \Omega^{r,s}(\mathcal{R}^\infty)$ is any type (r, s) form, then

$$d_H\omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega) \tag{3.7}$$

where the total vector fields X and Y act on ω by projected Lie differentiation. In particular, if $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ is given by

$$\omega = \sigma \wedge M + \tau \wedge N,$$

where M and N are type $(0, s)$ forms on $\mathcal{R}^\infty$, then

$$d_H\omega = \sigma \wedge \tau \wedge\, [X(N) - Y(M) - PM - QN]. \tag{3.8}$$

Because we assume the equation (3.1) to be hyperbolic, we may choose $(\mu, \lambda) = (m_x, m_y)$ and $(\mu, \lambda) = (n_x, n_y)$ to be two distinct (non-proportional), real roots of the characteristic equation (3.2). Whence we have the factorization $(m_x\lambda - m_y\mu)(n_x\lambda - n_y\mu) = \kappa(F_r\lambda^2 - F_s\lambda\mu + F_t\mu^2)$, where κ is some non-vanishing function on $\mathcal{R}^\infty$ and thus, at each point on the equation manifold $\mathcal{R}^\infty$, the relations

$$m_x n_x = \kappa F_r, \qquad m_y n_x + m_x n_y = \kappa F_s, \qquad m_y n_y = \kappa F_t \tag{3.9}$$

hold. We take for our basis for the space of total vector fields on $\mathcal{R}^\infty$ the *characteristic total vector fields*

$$X = m_x D_x + m_y D_y \qquad \text{and} \qquad Y = n_x D_x + n_y D_y. \tag{3.10}$$

Of course, these characteristic vector fields are not uniquely specified as each may be independently scaled. The next proposition, which is easily verified, shows that it is possible in many cases to choose these scale factors so as to obtain vector fields X and Y that commute.

Proposition 3.1. *Let $\mathcal{R}$ be a hyperbolic equation with characteristic vector fields X and Y. If I and J are non-trivial X and Y invariant functions respectively, then the characteristic vector fields $\widetilde{X} = \dfrac{1}{X(J)}\, X$ and $\widetilde{Y} = \dfrac{1}{Y(I)}\, Y$ commute.*

Restricted to the equation manifold $\mathcal{R}^\infty$, the contact forms θ, θ_x, θ_y, θ_{xx}, θ_{xy} and θ_{yy} are not independent but are related by the equation $d_V F = 0$, that is,

$$F_r\,\theta_{xx} + F_s\,\theta_{xy} + F_t\,\theta_{yy} + F_p\,\theta_x + F_q\,\theta_y + F_u\,\theta = 0. \tag{3.11}$$

We call this equation the *universal linearization* of the original partial differential equation (3.1). This equation will play a central role in all that we do.

Definition 3.2. *The universal linearization operator associated to the scalar, second order hyperbolic equation* $\mathcal{R}$ *is the total differential operator*

$$\mathcal{L}\colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty),$$

defined for $\omega \in \Omega^{0,s}(\mathcal{R}^\infty)$, *by*

$$\mathcal{L}(\omega) = XY(\omega) + AX(\omega) + BY(\omega) + C\omega, \tag{3.12}$$

where X *and* Y *are the characteristic vector fields associated to* $\mathcal{R}$ *and the coefficients* A, B *and* C *are defined by*

$$\begin{aligned} A &= A_0 - \frac{Y(\varrho)}{\varrho}, \qquad B = B_0 - \frac{X(\varrho)}{\varrho}, \\ C &= C_0 - \frac{XY(\varrho)}{\varrho} - A_0\frac{X(\varrho)}{\varrho} - B_0\frac{Y(\varrho)}{\varrho} + 2\frac{X(\varrho)Y(\varrho)}{\varrho^2}, \end{aligned} \tag{3.13}$$

where

$$A_0 = \frac{1}{\delta}[(\kappa F_p - X(n_x))n_y - (\kappa F_q - X(n_y))n_x], \tag{3.14a}$$

$$B_0 = \frac{1}{\delta}[-(\kappa F_p - X(n_x))m_y + (\kappa F_q - X(n_y))m_x], \tag{3.14b}$$

$$C_0 = \kappa F_u. \tag{3.14c}$$

With $\Theta = \varrho\theta$, *the identity* $d_V F = 0$ *on* $\mathcal{R}^\infty$ *becomes*

$$\mathcal{L}(\Theta) = XY(\Theta) + AX(\Theta) + BY(\Theta) + C\Theta = 0. \tag{3.15}$$

Next we define contact forms $\xi_k = X^k(\Theta)$ and $\eta_k = Y^k(\Theta)$. A form $\omega \in \Omega^p(\mathcal{R}^\infty)$ is said to be of *adapted order* k if it lies in the exterior algebra generated, over the smooth functions on $\mathcal{R}^\infty$, by the one forms $\{\, \sigma, \tau, \Theta, \xi_1, \eta_1, \dots, \xi_k, \eta_k \,\}$ and k is minimal. Note that a form of adapted order k may not factor through $\mathcal{R}^{k+1}$, that is, its adapted order may be different from its order as a form on $\mathcal{R}^\infty$. The adapted order of a form is invariant under a classical contact transformation on $\mathcal{R}^\infty$. A simple induction proves the following lemma.

Lemma 3.3. *For* $k \geq 1$, *the contact forms* $Y(\xi_k)$ *and* $X(\eta_k)$, *when restricted to* $\mathcal{R}^\infty$, *have adapted order* $\leq k$.

Theorem 3.4. *Let* $\mathcal{R}$ *be a hyperbolic partial differential equation. A coframe on the equation manifold* $\mathcal{R}^\infty$ *is given by the 1 forms*

$$\{\, \sigma, \tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \dots, \xi_k, \eta_k, \dots \,\}. \tag{3.16}$$

The d_H *structure equations for this coframe are*

$$d_H\sigma = -P\,\sigma\wedge\tau, \qquad d_H\tau = -Q\,\sigma\wedge\tau, \qquad d_H\Theta = \sigma\wedge\xi_1 + \tau\wedge\eta_1 \tag{3.17a}$$

$$d_H\xi_k = \sigma\wedge\xi_{k+1} + \tau\wedge\mu_k, \quad \text{and} \quad d_H\eta_k = \sigma\wedge\nu_k + \tau\wedge\eta_{k+1}, \tag{3.17b}$$

where μ_k *and* ν_k *are contact forms of adapted order* $\leq k$.

We call (3.16) the *characteristic coframe* for the hyperbolic equation $\mathcal{R}$.

As we have already noted, we are free to rescale the characteristic vector fields X and Y and the zero-th order contact form Θ. For a given, arbitrary rescaling

$$X' = mX, \qquad Y' = nY, \qquad \text{and} \qquad \Theta' = l\,\Theta, \tag{3.18}$$

where m, n and l are non-vanishing functions on $\mathcal{R}^\infty$, let

$$X'Y'(\Theta') + A'X'(\Theta') + B'Y'(\Theta') + C'\Theta' = 0 \tag{3.19}$$

be the universal linearization of $\mathcal{R}$ in terms of X', Y' and Θ'.

Proposition 3.5. *Under the rescaling (3.18) the coefficients of the universal linearizations (3.15) and (3.19) are related by*

$$A = \frac{A'}{n} + \frac{Y(l)}{l}, \qquad B = \frac{B'}{m} + \frac{X(n)}{n} + \frac{X(l)}{l}, \quad \text{and} \tag{3.20a}$$

$$C = \frac{C'}{mn} + \frac{X(l)}{nl} A' + \frac{Y(l)}{ml} B' + \frac{X(n)Y(l)}{nl} + \frac{XY(l)}{l}. \tag{3.20b}$$

§**4. The generalized Laplace transform and the Laplace adapted coframe.** The classical Laplace transform provides a means for the exact integration of linear hyperbolic equations

$$u_{xy} + a(x,y)\, u_x + b(x,y)\, u_y + c(x,y)\, u = 0. \tag{4.1}$$

In this section we extend the theory of the classical Laplace transform so as to apply to hyperbolic total differential operators $\mathcal{F} \colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$, where

$$\mathcal{F}(\omega) = XY(\omega) + A\, X(\omega) + B\, Y(\omega) + C\, \omega, \tag{4.2}$$

defined on the equation manifold $\mathcal{R}^\infty$ of a second order hyperbolic equation with characteristics X and Y.

The basic properties of the classical Laplace transformation, as given in Darboux [*16*] (Book 4, Chapter 2), Forsyth [*19*](Chapter 8, pp. 39–158), or Goursat [*22*](Chapter 5, pp. 1–39), readily extend to (4.2) even though the characteristic vector fields X and Y do not commute. With the commutator of X and Y given by (3.5), we note that $\mathcal{F}$ can also be expressed in the equivalent form

$$\mathcal{F}(\omega) = YX(\omega) + D\, X(\omega) + E\, Y(\omega) + G\, \omega, \tag{4.3}$$

where

$$D = A + P, \qquad E = B + Q, \qquad \text{and} \qquad G = C. \tag{4.4}$$

There are two transforms associated with the hyperbolic operator $\mathcal{F}$, one for each characteristic vector field X and Y. To define the $\mathcal{Y}$ transform, associated to the vector field Y, we first define the first order total differential operator

$$\mathcal{Y}_{\mathcal{F}} \colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$$

by

$$\mathcal{Y}_{\mathcal{F}}(\omega) = Y(\omega) + A\, \omega. \tag{4.5}$$

The form $\eta = \mathcal{Y}_{\mathcal{F}}(\omega)$ is called the $\mathcal{Y}$ Laplace transform of the contact form ω associated to the operator $\mathcal{F}$. We shall simply write $\mathcal{Y}(\omega)$ for $\mathcal{Y}_{\mathcal{F}}(\omega)$ when there is no ambiguity as to the choice of operator $\mathcal{F}$. Then, in terms of η, the operator $\mathcal{F}(\omega)$ becomes

$$\mathcal{F}(\omega) = X(\eta) + B\, \eta - H\, \omega, \tag{4.6}$$

where

$$H = H(\mathcal{F}) = X(A) + AB - C. \tag{4.7}$$

Notice that $H(\mathcal{F})$ coincides formally with the classical Laplace invariant

$$h = \frac{\partial a}{\partial x} + ab - c$$

for (4.1). We call $H(\mathcal{F})$ the *generalized Laplace invariant* for the total differential operator $\mathcal{F}$.

If $H(\mathcal{F}) \neq 0$, then we may construct a new second order total differential operator, called the $\mathcal{Y}$ Laplace transform of $\mathcal{F}$ and denoted by

$$\mathcal{Y}(\mathcal{F})\colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty),$$

as follows. If ω satisfies $\mathcal{F}(\omega) = 0$, then

$$X(\eta) + B\,\eta - H\,\omega = 0. \tag{4.8}$$

Applying the total vector field Y to (4.8) and using the commutator formula (3.5), we obtain

$$XY(\eta) - P\,X(\eta) + (B - Q)\,Y(\eta) + (Y(B) - H)\,\eta + (AH - Y(H))\,\omega = 0. \tag{4.9}$$

Since $H \neq 0$, we can solve for ω in (4.8) and substitute the result into (4.9) to arrive at the equation $\mathcal{Y}(\mathcal{F})(\eta) = 0$, where

$$[\mathcal{Y}(\mathcal{F})](\eta) = XY(\eta) + \mathcal{Y}(A)\,X(\eta) + \mathcal{Y}(B)\,Y(\eta) + \mathcal{Y}(C)\,\eta, \tag{4.10a}$$

and where

$$\begin{aligned} &\mathcal{Y}(A) = A - \frac{Y(H)}{H} - P, \qquad \mathcal{Y}(B) = B - Q, \qquad \text{and} \\ &\mathcal{Y}(C) = C - X(A) - B\frac{Y(H)}{H} + Y(B). \end{aligned} \tag{4.10b}$$

The operator $\mathcal{Y}(\mathcal{F})$ is called the $\mathcal{Y}$ Laplace transform of $\mathcal{F}$. We emphasize that $\mathcal{Y}_\mathcal{F}$ is always defined for any total differential operator $\mathcal{F}$ but that $\mathcal{Y}(\mathcal{F})$ is defined only for operators $\mathcal{F}$ with non-vanishing generalized Laplace invariant $H(\mathcal{F})$.

Similarly, the $\mathcal{X}$ transform of the operator $\mathcal{F}$ is given by

$$\mathcal{X}_\mathcal{F}\colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty),$$

where

$$\mathcal{X}_\mathcal{F}(\omega) = X(\omega) + E\,\omega. \tag{4.11}$$

When $K(\mathcal{F}) = Y(E) + ED - G \neq 0$, we define the $\mathcal{X}$ Laplace transform of $\mathcal{F}$ by

$$[\mathcal{X}(\mathcal{F})](\xi) = YX(\xi) + \mathcal{X}(D)\,X(\xi) + \mathcal{X}(E)\,Y(\xi) + \mathcal{X}(G)\,\xi = 0, \tag{4.12a}$$

where

$$\begin{aligned} &\mathcal{X}(D) = D + P, \qquad \mathcal{X}(E) = E - \frac{X(K)}{K} + Q, \qquad \text{and} \\ &\mathcal{X}(G) = G - Y(E) - D\frac{X(K)}{K} + X(D). \end{aligned} \tag{4.12b}$$

Our next proposition shows that if $H(\mathcal{F}) \neq 0$, then the solutions to $\mathcal{F}(\omega) = 0$ and $[\mathcal{Y}(\mathcal{F})](\eta) = 0$ are in bijective correspondence.

Proposition 4.1. *For any hyperbolic second order total differential operator $\mathcal{F}$ and for any forms $\omega, \eta \in \Omega^{0,s}(\mathcal{R}^\infty)$,*

$$[\mathcal{X}_{\mathcal{Y}(\mathcal{F})} \circ \mathcal{Y}_\mathcal{F}](\omega) = H(\mathcal{F})\,\omega + \mathcal{F}(\omega) \tag{4.13}$$

and, provided $H(\mathcal{F}) \neq 0$,

$$\mathcal{Y}_\mathcal{F}[\frac{1}{H(\mathcal{F})}\mathcal{X}_{\mathcal{Y}(\mathcal{F})}(\eta)] = \eta + \frac{1}{H(\mathcal{F})}[\mathcal{Y}(\mathcal{F})](\eta). \tag{4.14}$$

If $H(\mathcal{F}) \neq 0$ and $\eta = \mathcal{Y}_{\mathcal{F}}(\omega)$, then

$$\mathcal{F}(\omega) = 0 \quad \textit{implies} \quad [\mathcal{Y}(\mathcal{F})](\eta) = 0$$

and if $\tilde{\omega} = \dfrac{1}{H(\mathcal{F})}\mathcal{X}_{\mathcal{Y}(\mathcal{F})}(\eta)$ then

$$[\mathcal{Y}(\mathcal{F})](\eta) = 0 \quad \textit{implies} \quad \mathcal{F}(\tilde{\omega}) = 0.$$

Similar identities hold true for $K(\mathcal{F})$ and $\mathcal{X}(\mathcal{F})$.

We remark that even when $H(\mathcal{F}) = 0$, in which case the operator $\mathcal{Y}(\mathcal{F})$ is not defined the coefficient of Y in $\mathcal{Y}(\mathcal{F})$, namely $B - Q$, is defined and therefore the $\mathcal{X}_{\mathcal{Y}(\mathcal{F})}$ transformed contact form $\mathcal{X}_{\mathcal{Y}(\mathcal{F})}(\eta)$ is defined. Equation (4.13) remains valid, that is,

$$[\mathcal{X}_{\mathcal{Y}(\mathcal{F})} \circ \mathcal{Y}_{\mathcal{F}}](\omega) = \mathcal{F}(\omega) \quad \text{when} \quad H(\mathcal{F}) = 0 \tag{4.15}$$

but the transform $\mathcal{Y}_{\mathcal{F}}$, acting on solutions to the equation $\mathcal{F}(\omega) = 0$, is no longer invertible.

We now consider successive applications of the $\mathcal{Y}$ and $\mathcal{X}$ Laplace transforms. Let $H_0 = H(\mathcal{F})$. If $H_0 \neq 0$, then we may define the transformed operator $\mathcal{Y}(\mathcal{F})$ and we let $H_1 = H(\mathcal{Y}(\mathcal{F}))$. Then, provided

$$H_0 \neq 0, \quad H_1 \neq 0, \quad \dots, H_{i-1} = H(\mathcal{Y}^{i-1}(\mathcal{F})) \neq 0,$$

we define the i-th $\mathcal{Y}$ generalized Laplace invariant of the operator $\mathcal{F}$ to be

$$H_i = H(\mathcal{Y}^i(\mathcal{F})).$$

The first integer p for which

$$H_p = 0$$

is called the $\mathcal{Y}$ *Laplace index* $\operatorname{ind}_{\mathcal{Y}}(\mathcal{F})$ of the operator $\mathcal{F}$. If $H_i \neq 0$ for all i, then we say that $\operatorname{ind}_{\mathcal{Y}}(\mathcal{F}) = \infty$. Similarly, we write $K_0 = K(\mathcal{F})$, and $K_j = K(\mathcal{X}^j(\mathcal{F}))$ and we define $\operatorname{ind}_{\mathcal{X}}(\mathcal{F})$ to be the first integer q for which $K_q = 0$.

The Laplace adapted coframe for $\mathcal{R}^\infty$ is constructed as follows. Let $\mathcal{L}\colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ be the second order total differential operator

$$\mathcal{L} = XY + AX + BY + C$$

defining the universal linearization (3.12) of the hyperbolic equation $\mathcal{R}$. Thus, if Θ is the contact form $\varrho\theta$, then

$$\mathcal{L}(\Theta) = 0. \tag{4.16}$$

We now write $\operatorname{ind}(\mathcal{Y}) = \operatorname{ind}_{\mathcal{Y}}(\mathcal{L})$ and $\operatorname{ind}(\mathcal{X}) = \operatorname{ind}_{\mathcal{X}}(\mathcal{L})$ and call $\operatorname{ind}(\mathcal{Y})$ and $\operatorname{ind}(\mathcal{X})$ the *Laplace indices* of the second order hyperbolic equation $\mathcal{R}$. For $0 \leq i \leq \operatorname{ind}(\mathcal{Y})$ and $0 \leq j \leq \operatorname{ind}(\mathcal{X})$, the functions

$$H_i = H(\mathcal{Y}^i(\mathcal{L})) \quad \text{and} \quad K_j = K(\mathcal{X}^j(\mathcal{L}))$$

are called the *generalized Laplace invariants* of $\mathcal{R}$. Recall that $H_p = 0$ when $\operatorname{ind}(\mathcal{Y}) = p$ and that $K_q = 0$ when $\operatorname{ind}(\mathcal{Y}) = q$.

If $\operatorname{ind}(\mathcal{Y}) = p$, then we define

$$\eta_1 = \mathcal{Y}_{\mathcal{L}}(\Theta) = Y(\Theta) + A\,\Theta, \tag{4.17a}$$

$$\eta_i = \mathcal{Y}_{[\mathcal{Y}^{i-1}(\mathcal{L})]}(\eta_{i-1}) = Y(\eta_{i-1}) + A_{i-1}\eta_{i-1}, \quad i = 2, \dots, p+1 \tag{4.17b}$$

$$\eta_{p+i} = Y(\eta_{p+i-1}) \quad \text{for } i \geq 2, \tag{4.17c}$$

where A_{i-1} is the coefficient of X in the operator $\mathcal{Y}^{i-1}(\mathcal{L})$. If $\operatorname{ind}(\mathcal{Y}) = \infty$, then the contact one forms η_i are defined by (4.17b) for all $i \geq 2$. It follows from (4.16) and (4.8) that η_1 satisfies the identity

$$X(\eta_1) + B\,\eta_1 - H_0\Theta = 0. \tag{4.18}$$

Furthermore, if $H_0 \neq 0$, Proposition 4.1 implies that η_1 also satisfies the total differential equation

$$[\mathcal{Y}(\mathcal{L})](\eta_1) = XY(\eta_1) + A_1X(\eta_1) + B_1Y(\eta_1) + C_1\eta_1 = 0.$$

We deduce, by a simply induction argument, that for each $i = 2, 3, \ldots, p+1$,

$$X(\eta_i) + B_{i-1}\,\eta_i - H_{i-1}\,\eta_{i-1} = 0 \tag{4.19}$$

and that for each $i = 1, 2, \ldots, p$,

$$[\mathcal{Y}^i(\mathcal{L})](\eta_i) = XY(\eta_i) + A_i\,X(\eta_i) + B_i\,Y(\eta_i) + C_i\,\eta_i = 0. \tag{4.20}$$

The forms ξ_j are similarly defined in terms of the $\mathcal{X}$ generalized Laplace transforms. If $\operatorname{ind}(\mathcal{X}) = q$, we let

$$\xi_1 = \mathcal{X}_{\mathcal{L}}(\Theta) = X(\Theta) + E\,\Theta, \tag{4.21a}$$

$$\xi_j = \mathcal{X}_{[\mathcal{X}^{i-1}(\mathcal{L})]}(\xi_{j-1}) = X(\xi_{j-1}) + E_{j-1}\,\xi_{j-1}, \quad j = 2, \ldots, q+1 \tag{4.21b}$$

$$\xi_{q+j} = X(\xi_{q+j-1}) \quad \text{for } j \geq 2, \tag{4.21c}$$

where E_{j-1} is the Y coefficient of the operator $\mathcal{X}^{j-1}(\mathcal{L})$. If $\operatorname{ind}(\mathcal{X}) = \infty$, then the contact one forms ξ_j are defined by (4.21b) for all $j \geq 2$.

The d_H structure equations for the Laplace adapted coframe are readily determined from equations (3.7) and (4.17)–(4.20).

Proposition 4.2. *Suppose that $0 \leq p = \operatorname{ind}(\mathcal{Y}) < \infty$ and $0 \leq q = \operatorname{ind}(\mathcal{X}) < \infty$. The d_H structure equations for the Laplace adapted coframe for the hyperbolic equation $\mathcal{R}$ are given by*

$$d_H\sigma = -P\,\sigma \wedge \tau, \qquad d_H\tau = -Q\,\sigma \wedge \tau, \tag{4.22a}$$

$$d_H(\Theta) = \sigma \wedge (\xi_1 - E\,\Theta) + \tau \wedge (\eta_1 - A\,\Theta), \tag{4.22b}$$

and

$$d_H\eta_1 = \sigma \wedge (-B\,\eta_1 + H_0\Theta) + \tau \wedge (\eta_2 - A_1\eta_1), \tag{4.23a}$$

$$d_H\eta_i = \sigma \wedge (-B_{i-1}\eta_i + H_{i-1}\eta_{i-1}) + \tau \wedge (\eta_{i+1} - A_i\eta_i) \qquad 2 \leq i \leq p, \tag{4.23b}$$

$$d_H\eta_{p+1} = \sigma \wedge (-B_p\eta_{p+1}) + \tau \wedge \eta_{p+2}, \tag{4.23c}$$

$$d_H\eta_{p+i} = \sigma \wedge \nu_{p+i} + \tau \wedge \eta_{p+i+1}, \qquad i \geq 2, \tag{4.23d}$$

where ν_{p+i} is a contact one form such that

$$\nu_{p+i} \equiv [(i-1)Q - B_p]\,\eta_{p+i} \qquad \operatorname{mod}\,\{\,\eta_{p+1}, \ldots, \eta_{p+i-1}\,\}. \tag{4.24}$$

If $\operatorname{ind}(\mathcal{Y}) = \infty$, then the structure equations (4.23b) remain valid for all $i \geq 2$.

Similar structure equations hold for the forms ξ_j, $j \geq 1$.

We now investigate the behaviour of the generalized Laplace invariants and the Laplace adapted coframe under classical contact transformations. Let $\mathcal{R}'$ be another scalar second order hyperbolic equation and let $\Phi\colon \mathcal{R} \to \mathcal{R}'$ be a classical contact transformation. Let X' and Y' be the characteristic vector fields for $\mathcal{R}'$ and let

$$X'Y'(\Theta') + A'X'(\Theta') + B'Y'(\Theta') + C'\Theta' = 0$$

be the universal linearization on $\mathcal{R}'^{\infty}$. The characteristic vector fields X' and Y' and the coefficients A', B' and C' are related to X and Y and the coefficients A, B and C by (3.18) and (3.20). Let H'_0, H'_1, H'_2, … and K'_0, K'_1, K'_2, … be the generalized Laplace invariants of the hyperbolic equation $\mathcal{R}'$, let $p' = \operatorname{ind}(\mathcal{Y}')$ and $q' = \operatorname{ind}(\mathcal{X}')$ be the Laplace indices of $\mathcal{R}'$ and let $\{\,\sigma', \tau', \Theta', \xi'_1, \eta'_1, \xi'_2, \eta'_2, \ldots\,\}$ be the Laplace adapted coframe associated to the hyperbolic equation $\mathcal{R}'$.

Theorem 4.3. *Let* $\Phi\colon \mathcal{R}^\infty \to \mathcal{R}'^\infty$ *be a classical contact transformation with*

$$X' = mX, \qquad Y' = nY, \qquad \text{and} \qquad \Theta' = l\Theta, \tag{4.25}$$

and suppose that $p = \operatorname{ind}(\mathcal{Y}) < \infty$ *and* $q = \operatorname{ind}(\mathcal{X}) < \infty$*. Then the Laplace indices of* $\mathcal{R}$ *and* $\mathcal{R}'$ *coincide, that is,* $p' = p$ *and* $q' = q$*; the generalized Laplace invariants are related by*

$$H_i' = mn\, H_i \qquad \text{and} \qquad K_j' = mn\, K_j; \tag{4.26}$$

and the Laplace adapted coframes are related by

$$\sigma' = \frac{1}{m}\,\sigma \qquad \text{and} \qquad \tau' = \frac{1}{n}\,\tau, \tag{4.27}$$

and

$$\eta_i' = n^i l\, \eta_i \qquad \text{for } 1 \le i \le p+1, \tag{4.28a}$$

$$\eta_{p+i}' \equiv n^{p+i} l\, \eta_{p+i} \qquad \text{mod } \{\, \eta_{p+1}, \dots, \eta_{p+i-1} \,\} \qquad \text{for } 2 \le i \le \infty \tag{4.28b}$$

and also

$$\xi_j' = m^j l\, \xi_j \qquad \text{for } 1 \le j \le q+1, \tag{4.29a}$$

$$\xi_{q+j}' \equiv m^{q+j} l\, \xi_{q+j} \qquad \text{mod } \{\, \xi_{q+1}, \dots, \xi_{q+j-1} \,\} \qquad \text{for } 2 \le j \le \infty. \tag{4.29b}$$

When $p = \infty$ *or* $q = \infty$*, equations (4.28a) or (4.29a) are valid for all* $i \ge 1$ *or all* $j \ge 1$*.*

Proof. A straight-forward but lengthy induction argument, beginning with the transformation laws (3.20) and using the defining recursion relations (4.10) determines the transformation laws for the coefficients of the operators $\mathcal{Y}^i(\mathcal{L})$ and $\mathcal{X}^i(\mathcal{L})$. With these in hand (4.26) follows immediately. We also prove (4.28a) by induction. ∎

Finally, we need to express the d_H structure equations in terms of the Lie brackets of the characteristic vector fields X, Y and the vertical vector fields U, V^1, W^1, V^2, W^2, … dual to the contact forms in our Laplace adapted coframe:

$$\begin{aligned} \Theta(U) &= 1, & \eta_i(U) &= 0, & \xi_j(U) &= 0, \\ \Theta(V^h) &= 0, & \eta_i(V^h) &= \delta_i^h, & \xi_j(V^h) &= 0 \\ \Theta(W^k) &= 0, & \eta_i(W^k) &= 0, & \xi_j(W^k) &= \delta_j^k. \end{aligned} \tag{4.30}$$

Proposition 4.4. *Let* $\mathcal{R}$ *be a second order hyperbolic equation and suppose that* $0 \le p = \operatorname{ind}(\mathcal{Y}) \le \infty$ *and* $0 \le q = \operatorname{ind}(\mathcal{X}) \le \infty$*. Then the following congruences hold for the Lie brackets of the characteristic vector* X *with the vertical vector fields* U, V^i, W^j*:*

$$[X, U] \equiv EU - H_0 V^1 \qquad \text{mod } \{\, X, Y \,\}, \tag{4.32a}$$

$$[X, V^1] \equiv BV^1 - H_1 V^2 \qquad \text{mod } \{\, X, Y \,\}, \tag{4.32b}$$

$$[X, V^i] \equiv B_{i-1} V^i - H_i V^{i+1} \qquad \text{mod } \{\, X, Y \,\}, \qquad 2 \le i \le p, \tag{4.32c}$$

$$[X, V^{p+i}] \equiv [B_p - (i-1)Q] V^{p+i} \qquad \text{mod } \{\, X, Y, V^{p+i+1}, \dots \}, \qquad 1 \le i \le \infty, \tag{4.32d}$$

$$[X, W^1] \equiv -U + E_1 W^1 \qquad \text{mod } \{\, X, Y \,\}, \tag{4.32e}$$

$$[X, W^j] \equiv -W^{j-1} + E_j W^j \qquad \text{mod } \{\, X, Y \,\}, \qquad 2 \le j \le q, \tag{4.32f}$$

$$[X, W^{q+j}] \equiv -W^{q+j-1} \qquad \text{mod } \{\, X, Y \,\}, \qquad 1 \le j \le \infty. \tag{4.32g}$$

The congruences for the Lie brackets of the characteristic vector field Y and the vertical vector fields U, V^i, W^j are similar.

Proof. If Z is any total vector field, V any π^∞_M vertical vector field on $\mathcal{R}^\infty$ and ω any contact one form, then $\omega(Z) = 0$, $(d_V\omega)(Z, V) = 0$ and

$$(d_H\omega)(Z, V) = (d\omega)(Z, V) = Z(\omega(V)) - V(\omega(Z)) - \omega([Z, V]) = Z(\omega(V)) - \omega([Z, V]).$$

In particular, if $\omega(V)$ is a constant, then $(d_H\omega)(Z, V) = -\omega([Z, V])$. We use this formula repeatedly, together with the d_H structure equations obtained in Proposition 4.2 to derive (4.32). ∎

We now examine the existence of characteristic invariant forms. If Z is a total vector field on the equation manifold $\mathcal{R}^\infty$, then a function f on $\mathcal{R}^\infty$ is said to be Z invariant if $Z(f) = 0$. Likewise, a type $(0, s)$ contact form ω is called a Z invariant contact form if $Z(\omega) = Z \lrcorner\, d_H\omega = 0$. More generally, ω is a relative Z invariant contact form if there is a function λ on $\mathcal{R}^\infty$ such that

$$Z(\omega) = \lambda\, \omega.$$

We give a partial characterization of the X and Y relative invariant contact forms, where X and Y are the characteristic vector fields for a hyperbolic differential equation $\mathcal{R}$. In particular, we prove that there are no X relative invariant contact forms unless $\operatorname{ind}(\mathcal{Y})$ is finite and there are no Y invariant contact forms unless $\operatorname{ind}(\mathcal{X})$ is finite.

Given a subring $\mathcal{M}$ of $C^\infty(\mathcal{R}^\infty)$ and a collection of one forms $\{\,\omega_1, \omega_2, \ldots, \omega_i, \ldots\}$ on $\mathcal{R}^\infty$, let

$$\Omega^s_{\mathcal{M}}(\omega_1, \omega_2, \ldots, \omega_i, \ldots)$$

be the $\mathcal{M}$ module of s forms generated by $\omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_s}$. When $\mathcal{M} = C^\infty(\mathcal{R}^\infty)$, we simply denote this module of forms by

$$\Omega^s(\omega_1, \omega_2, \ldots, \omega_i, \ldots).$$

Theorem 4.5. *Let $\mathcal{R}$ be a hyperbolic equation with characteristic vector fields X and Y, Laplace indices $\operatorname{ind}(\mathcal{Y}) = p$ and $\operatorname{ind}(\mathcal{X}) = q$ and Laplace adapted coframe $\{\,\sigma, \tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \ldots\}$. Let $s \geq 1$.*

(i) If $\omega \in \Omega^{0,s}(\mathcal{R}^\infty)$ is a relative X invariant form, then

$$\omega \in \Omega^s(\eta_{p+1}, \eta_{p+2}, \ldots). \tag{4.33}$$

If $\operatorname{ind}(\mathcal{Y}) = \infty$, then there are no non-zero relative X invariant type $(0, s)$ forms.

(ii) If $\omega \in \Omega^{0,s}(\mathcal{R}^\infty)$ is a relative Y invariant form, then

$$\omega \in \Omega^s(\xi_{q+1}, \xi_{q+2}, \ldots). \tag{4.34}$$

If $\operatorname{ind}(\mathcal{X}) = \infty$, then there are no non-zero relative Y invariant type $(0, s)$ forms.

Proof. Suppose that ω is a relative X invariant type $(0, s)$ form of adapted order k, that is, $\omega \in \Omega^s(\Theta, \xi_1, \eta_1, \ldots, \xi_k, \eta_k)$ and

$$X(\omega) = \lambda\, \omega. \tag{4.35}$$

We first prove that

$$\omega \in \Omega^s(\eta_1, \eta_2, \ldots, \eta_k). \tag{4.36}$$

Let U, V^i, W^j be the vertical vector fields (4.30) dual to our Laplace adapted basis. By assumption, we have that $W^{k+1} \lrcorner\, \omega = 0$. We interior evaluate equation (4.35) with the vector field W^{k+1}. Since, by (2.9),

$$W^{k+1} \lrcorner\, (X(\omega)) = X(W^{k+1} \lrcorner\, \omega) - [X, W^{k+1}] \lrcorner\, \omega$$

we deduce that

$$[X, W^{k+1}] \lrcorner\, \omega = 0. \tag{4.37}$$

In view of the structure equations (4.32), we have that

$$[X,W^1] \equiv -U \qquad \text{mod}\ \{\, X, Y, W^1 \,\}$$

and

$$[X,W^l] \equiv -W^{l-1} \qquad \text{mod}\ \{\, X, Y, W^l \,\} \qquad l \geq 2$$

and consequently (4.37) reduces to $W^k \mathbin{\lrcorner} \omega = 0$. Next we hook (4.35) with W^k to deduce that $W^{k-1} \mathbin{\lrcorner} \omega = 0$. This process continues until we interior evaluate with W^1 to find that $U \mathbin{\lrcorner} \omega = 0$. This proves that if ω is a relative X invariant contact form of adapted order k, then

$$U \mathbin{\lrcorner} \omega = W^1 \mathbin{\lrcorner} \omega = W^2 \mathbin{\lrcorner} \omega = \cdots = W^k \mathbin{\lrcorner} \omega = 0$$

and this implies (4.36).

To complete the proof of (i), we successively hook (4.35) with U, V^1, ... , V^{p-1} and make use of the structure equations

$$\begin{aligned} &[X,U] \equiv EU - H_0 V^1 \qquad \text{mod}\ \{\, X, Y \,\}, \\ &[X,V^1] \equiv BV^1 - H_1 V^2 \qquad \text{mod}\ \{\, X, Y \,\}, \qquad \text{and} \\ &[X,V^i] \equiv B_{i-1} V^i - H_i V^{i+1} \qquad \text{mod}\ \{\, X, Y \,\} \qquad 2 \leq i \leq p-1. \end{aligned}$$

Since $U \mathbin{\lrcorner} \omega = 0$, the interior product of (4.35) with U yields $H_0 V^1 \mathbin{\lrcorner} \omega = 0$ and therefore, because $H_0 \neq 0$, $V^1 \mathbin{\lrcorner} \omega = 0$. Next we interior evaluate with V^1 to conclude that $V^2 \mathbin{\lrcorner} \omega = 0$ and so on until we find that

$$V^1 \mathbin{\lrcorner} \omega = V^2 \mathbin{\lrcorner} \omega = \cdots = V^p \mathbin{\lrcorner} \omega = 0. \tag{4.38}$$

The interior product of (4.35) with V_p gives no condition on ω since $H_p = 0$. Equations (4.36) and (4.38) prove (4.33). If $\operatorname{ind}(\mathcal{Y}) = \infty$, then this argument shows that $\omega = 0$. The proof of (ii) is similar. ∎

§**5. Structure theorem for type $(1,s)$ conservation laws, $s \geq 1$.** In this section we shall use some basic properties of the free variational bicomplex on E and the Laplace adapted coframe to prove some general structure theorems for the form-valued conservation laws of a second order hyperbolic equation $\mathcal{R}$. Recall that the universal linearization of $\mathcal{R}$ is the total differential operator

$$\mathcal{L} = XY + AX + BY + C, \tag{5.1}$$

defined by (3.12). The adjoint of $\mathcal{L}$ is, by definition, the unique total differential operator

$$\mathcal{L}^* \colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$$

such that, for every for every $\rho \in \Omega^{0,s}(\mathcal{R}^\infty)$ and every $\omega \in \Omega^{0,s'}(\mathcal{R}^\infty)$,

$$[\rho \wedge \mathcal{L}(\omega) - \mathcal{L}^*(\rho) \wedge \omega] \wedge \sigma \wedge \tau = d_H \gamma, \tag{5.2}$$

for some $\gamma \in \Omega^{1,s+s'}(\mathcal{R}^\infty)$. Since the adjoints of the first order total differential operators X and Y are $X^* = -X + Q$ and $Y^* = -Y - P$, it is easy to see that

$$\mathcal{L}^*(\rho) = XY(\rho) + A^* X(\rho) + B^* Y(\rho) + C^* \rho, \tag{5.3a}$$

where

$$\begin{aligned} &A^* = -A, \qquad B^* = -B - 2Q, \qquad \text{and} \\ &C^* = -X(A) - Y(B+Q) + C - AB + (A-P)(B+Q). \end{aligned} \tag{5.3b}$$

Let $\{\, \sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \dots \,\}$ be the Laplace adapted coframe introduced in section 4. With respect to this coframe define, for each $s \geq 1$, a map

$$\Psi \colon \Omega^{0,s-1}(\mathcal{R}^\infty) \to \Omega^{1,s}(\mathcal{R}^\infty)$$

by

$$\Psi(\rho) = \frac{1}{2}\, \sigma \wedge [\Theta \wedge \psi_1 - \xi_1 \wedge \rho] + \frac{1}{2}\, \tau \wedge [\Theta \wedge \psi_2 + \eta_1 \wedge \rho], \tag{5.4a}$$

where

$$\psi_1 = X(\rho) - (B+Q)\, \rho \quad \text{and} \quad \psi_2 = -Y(\rho) + A\, \rho. \tag{5.4b}$$

Theorem 5.1. *Let $s \geq 1$ and let $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ be a d_H closed form. Then there exists contact forms*

$$\rho \in \Omega^{0,s-1}(\mathcal{R}^\infty) \quad \text{and} \quad \gamma \in \Omega^{0,s}(\mathcal{R}^\infty)$$

such that ω is given by

$$\omega = \Psi(\rho) + d_H\gamma, \tag{5.5}$$

and where ρ satisfies the adjoint equation

$$XY(\rho) + A^* X(\rho) + B^* Y(\rho) + C^* \rho = 0. \tag{5.6}$$

When $s = 1$, the function ρ and the type $(0,1)$ form γ are unique.

We call the type $(1, s-1)$ form ρ a *generating form* for the conservation law ω with respect to the given Laplace adapted coframe. When $s = 1$, the function ρ coincides with Olver's [*32*] notion of a characteristic function for a conservation law. Theorem 5.1 implies that

$$H^{1,1}(\mathcal{R}^\infty) \cong \{\, \rho \in C^\infty(\mathcal{R}^\infty) \,|\, \mathcal{L}^*(\rho) = 0 \,\}.$$

This result is well-known [*36*], [*42*].

Before turning to the proof of this theorem, a number of remarks are in order. First, we note that

$$d_H(\frac{1}{2}\,\Theta \wedge \rho) = \frac{1}{2}\,\sigma \wedge [\xi_1 \wedge \rho + \Theta \wedge \psi_1] + \frac{1}{2}\,\tau \wedge [\eta_1 \wedge \rho - \Theta \wedge \psi_2]$$

and therefore we can re-express (5.5) in either of the asymmetric forms

$$\omega = \sigma \wedge \Theta \wedge \psi_1 + \tau \wedge \eta_1 \wedge \rho + d_H\gamma' \tag{5.7}$$

or

$$\omega = -\sigma \wedge \xi_1 \wedge \rho + \tau \wedge \Theta \wedge \psi_2 + d_H\gamma'', \tag{5.8}$$

where $\gamma' = \gamma - \frac{1}{2}\Theta \wedge \rho$ and $\gamma'' = \gamma + \frac{1}{2}\Theta \wedge \rho$. These equivalent forms of (5.5) are often easier to compute with.

Second, by using (3.8), the d_H structure equations given by Proposition 4.2, and (5.3), we compute the horizontal exterior derivative of (5.5), or equivalently (5.8), to be

$$d_H\omega = \sigma \wedge \tau \wedge\, [X(\Theta \wedge \psi_2) + Y(\xi_1 \wedge \rho) + P\,\xi_1 \wedge \rho - Q\,\Theta \wedge \psi_2\,] = -\sigma \wedge \tau \wedge \Theta \wedge \mathcal{L}^*(\rho).$$

Consequently, the form ω is d_H closed by virtue of the fact that ρ satisfies the adjoint equation $\mathcal{L}^*(\rho) = 0$. However, the converse is generally not true, that is, if ω is of the form (5.5) and is d_H closed, then ρ need *not* satisfy the adjoint equation if $s \geq 1$. Consider, for example, Liouville's equation $u_{xy} = e^u$. If α is any D_y invariant form, then it is easy to see that

$$\rho = D_x\alpha + u_x\alpha \tag{5.9}$$

is a solution to the adjoint equation $D_{xy}\rho = e^u\rho$ and thus

$$\omega = \Psi(\rho) = \frac{1}{2}\,dx \wedge [\theta \wedge D_x\rho - \theta_x \wedge \rho] + \frac{1}{2}\,dy \wedge [-\theta \wedge D_y\rho + \theta_y \wedge \rho]$$

is d_H closed. The form

$$\omega' = d_V\omega = \Psi(\rho'),$$

where $\rho' = d_V\rho$, is evidently also d_H closed, ω' is of the form (5.8) but, as one can easily check, the form ρ' does not satisfy the adjoint equation. Nevertheless, Theorem 5.1 asserts that for every cohomology class $[\omega] \in H^{1,s}(\mathcal{R}^\infty)$, there is a solution ρ to the adjoint equation $\mathcal{L}^*(\rho) = 0$ such that

$$[\omega] = [\Psi(\rho)]$$

and hence the map Ψ is surjective in cohomology. We can therefore modify ρ', say to ρ'', where now ρ'' satisfies the adjoint equation and the form $\Psi(\rho'')$ is cohomologous to ω'. In fact, with $\alpha' = d_V\alpha$ and $\rho'' = D_x\alpha' + u_x\alpha'$ we can write $\rho' = \rho'' + \theta_x \wedge \alpha'$ and therefore

$$\begin{aligned}\omega' &= \Psi(\rho'') + \frac{1}{2}\,[dx \wedge \theta \wedge D_x(\theta_x \wedge \alpha') + dy \wedge \theta_y \wedge \theta_x \wedge \alpha'] \\ &= \Psi(\rho'') + d_H(\frac{1}{2}\,\theta \wedge \theta_x \wedge \alpha').\end{aligned}$$

In particular, we can conclude from Theorem 5.1 that *if there are no non-zero type* $(1, s-1)$ *solutions* ρ *to the adjoint equation* $\mathcal{L}^*(\rho) = 0$, *then* $H^{1,s}(\mathcal{R}^\infty) = 0$.

Third, for $s > 1$, there are non-zero solutions ρ to the adjoint equation for which the form $\Psi(\rho)$ is exact. In other words, for $s > 1$, Ψ is not injective as a map from the space of solutions of the adjoint equation into $H^{1,s}(\mathcal{R}^\infty)$. Again this is illustrated by Liouville's equation. For any D_y invariant, type $s-1$ form β, put $\alpha = D_x(\xi_3 \wedge \beta)$, where $\xi_2 = \theta_{xx} - u_x\theta_x$ and $\xi_3 = D_x\xi_2$, and define ρ by (5.9). Then α is D_y invariant and ρ satisfies the adjoint equation for Liouville's equation. But with

$$\gamma = -\theta_x \wedge \alpha + \xi_2 \wedge \xi_3 \wedge \beta,$$

we find that $D_x\gamma = -\theta_x \wedge \rho$ and $D_y\gamma = -\theta \wedge D_y(\rho)$. Hence $d_H\gamma = \Psi(\rho) + \frac{1}{2}\,d_H(\theta \wedge \rho)$ and therefore $\Psi(\rho)$ is exact on $\mathcal{R}^\infty$.

We begin the proof of Theorem 5.1 in general by characterizing those forms on $J^\infty(E)$ which lie in the kernel of the inclusion map

$$\iota\colon \mathcal{R}^\infty \to J^\infty(E).$$

Recall that $\mathcal{R}^\infty$ is the infinitely prolonged equation manifold defined by the second order equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

Let $F^{ij} = (D_x)^i(D_y)^jF$. Standard arguments from elementary manifold theory prove the following lemma. See [5] for details.

Lemma 5.2. *If* $\omega \in \Omega^p(J^\infty(E))$ *satisfies* $\iota^*\omega = 0$, *then*

$$\omega = \sum_{i,j=0}^{k} \alpha_{ij}F^{ij} + \sum_{i,j=0}^{k} \beta_{ij} \wedge d_V(F^{ij}), \tag{5.10}$$

where $\alpha_{ij} \in \Omega^p(J^\infty(E))$ *and* $\beta_{ij} \in \Omega^{p-1}(J^\infty(E))$.

For the proof of Theorem 5.1, we shall also need the transformation properties of the form $\Psi(\rho)$.

Lemma 5.3. *Under the change of frame* $X' = mX$, $Y' = nY$ *and* $\Theta' = l\Theta$, *the form* $\Psi(\rho)$ *transforms by*

$$\Psi'(\rho') = \Psi(\rho), \qquad \text{where} \quad \rho' = \frac{1}{l}\rho. \tag{5.11}$$

Proof. This is a simple calculation based upon the transformation formulas (3.20), (4.27), (4.28) and (4.29). ∎

Proof of Theorem 5.1. Let $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ be a d_H closed form on $\mathcal{R}^\infty$. We first prove that there are forms

$$\tilde\omega \in \Omega^{1,s}(J^\infty(E)), \quad \tilde\zeta \in \Omega^{0,s}(J^\infty(E)) \quad \text{and} \quad \tilde\rho \in \Omega^{0,s-1}(J^\infty(E))$$

such that

$$\iota^*(\tilde\omega) = \omega \tag{5.12a}$$

and

$$d_H\tilde{\omega} = dx \wedge dy \wedge [\, F\,\tilde{\zeta} + (d_V F) \wedge \tilde{\rho}\,]. \tag{5.12b}$$

Indeed, since we can extend the natural coordinates (2.13) and the natural coframe (2.14) on $\mathcal{R}^\infty$ to the natural coordinate system and the natural coframe on $J^\infty(E)$, we can always chose a type $(1, s)$ form $\tilde{\omega}_0$ on $J^\infty(E)$ such that $\iota^*\tilde{\omega}_0 = \omega$. Since d_H commutes with ι^*, $\iota^*(d_H\tilde{\omega}_0) = 0$ and therefore, by Lemma 5.2,

$$d_H\tilde{\omega}_0 = dx \wedge dy \wedge [\sum_{i,j=0}^{k} (D_x^i D_y^j F)\,\alpha_{ij} + \sum_{i,j=0}^{k} d_V(D_x^i D_y^j F) \wedge \beta_{ij}],$$

where $\alpha_{ij} \in \Omega^{0,s}(J^\infty(E))$ and $\beta_{ij} \in \Omega^{s-1}(J^\infty(E))$. By repeated "integration by parts", beginning with the highest derivative terms $D_x^k D_y^k F$ and $d_V D_x^k D_y^k F$ we can re-write this equation as (5.12b), where $\tilde{\omega}$ differs from $\tilde{\omega}_0$ by terms depending linearly on F, $d_V F$ and their total derivatives. These terms all vanish on $\mathcal{R}^\infty$ and so $\iota^*(\tilde{\omega}) = \iota^*(\omega_0) = \omega$. This proves (5.12).

Let $\rho_c = \frac{1}{s}\iota^*(\tilde{\rho})$. We now prove that ρ_c satisfies the adjoint equation in the coordinate frame on $\mathcal{R}^\infty$, namely,

$$\mathcal{L}_c^*(\rho_c) = \frac{\partial F}{\partial u}\,\rho_c - D_i(\frac{\partial F}{\partial u_i}\,\rho_c) + D_{ij}(\frac{\partial F}{\partial u_{ij}}\,\rho_c) = 0, \tag{5.13}$$

and that ω can be expressed as

$$\omega = \Psi_c(\rho_c) + d_H\gamma, \tag{5.14a}$$

where, with $\nu_j = \frac{\partial}{\partial x^j} \lrcorner\, (dx \wedge dy)$,

$$\Psi_c(\rho_c) = \nu_j \wedge \theta \wedge [\frac{\partial F}{\partial u_j}\,\rho_c - D_i(\frac{\partial F}{\partial u_{ij}}\,\rho_c)] + \nu_j \wedge \theta_i \wedge [\frac{\partial F}{\partial u_{ij}}\,\rho_c]. \tag{5.14b}$$

To establish (5.14), we recall [*1*] that the interior Euler-Lagrange operator

$$J\colon \Omega^{2,s}(J^\infty(E)) \to \Omega^{1,s-1}(J^\infty(E))$$

is defined by

$$J(\tilde{\alpha}) = \frac{\partial}{\partial u} \lrcorner\, \tilde{\alpha} - D_i(\frac{\partial}{\partial u_i} \lrcorner\, \tilde{\alpha}) + D_{ij}(\frac{\partial}{\partial u_{ij}} \lrcorner\, \tilde{\alpha}) + \cdots .$$

It is not difficult to prove that for any type $(1, s)$ form $\tilde{\beta}$ on $J^\infty(E)$, $J(d_H\tilde{\beta}) = 0$. We apply J to (5.12b). Since

$$\frac{\partial}{\partial u_I} \lrcorner\, [F\tilde{\zeta} + (d_V F) \wedge \tilde{\rho}] = F\frac{\partial}{\partial u_I} \lrcorner\, \tilde{\zeta} + \frac{\partial F}{\partial u_I}\tilde{\rho} - (d_V F) \wedge \frac{\partial}{\partial u_I} \lrcorner\, \tilde{\rho}$$

we arrive at the identity

$$\frac{\partial F}{\partial u}\,\tilde{\rho} - D_i(\frac{\partial F}{\partial u_i}\,\tilde{\rho}) + D_{ij}(\frac{\partial F}{\partial u_{ij}}\,\tilde{\rho}) + \{\,*\,\} = 0 \qquad \text{on } J^\infty(E),$$

where $\{\,*\,\}$ denotes terms depending linearly on F, $d_V F$ and their total derivatives. Restricted to $\mathcal{R}^\infty$, this equation reduces to the adjoint equation (5.13).

To prove (5.14), we use the horizontal homotopy operators $h_H^{r,s}\colon \Omega^{r,s}(J^\infty(E)) \to \Omega^{r-1,s}(J^\infty(E))$. In particular, if $\tilde{\omega} = dx \wedge dy \wedge M$ is a type $(2, s)$ form, then

$$h_H^{2,s}(\tilde{\omega}) = \frac{1}{s}\nu_j \wedge \theta \wedge [(\frac{\partial}{\partial u_j} \lrcorner\, M) - D_i(\frac{\partial}{\partial u_{ij}} \lrcorner\, M)] + \frac{1}{s}\nu_j \wedge \theta_i \wedge (\frac{\partial}{\partial u_{ij}} \lrcorner\, M) + \{\,*\,\}, \tag{5.15}$$

where $\{*\}$ depends on the interior products $\dfrac{\partial}{\partial u_I} \lrcorner M$ for $|I| \geq 3$.

The homotopy operators $h_H^{r,s}$ satisfy the identity (see [1])

$$\tilde{\omega} = h_H^{2,s}(d_H\tilde{\omega}) + d_H h_H^{1,s}(\tilde{\omega}) \tag{5.16}$$

for every type $(1,s)$ form $\tilde{\omega}$ on $J^\infty(E)$. With $d_H\tilde{\omega}$ given by (5.12b) we compute, using (5.15),

$$h_H^{2,s}(d_H\tilde{\omega}) = \frac{1}{s}\nu_i \wedge \theta \wedge [\frac{\partial F}{\partial u_i}\,\tilde{\rho} - D_j(\frac{\partial F}{\partial u_{ij}}\,\tilde{\rho})] + \frac{1}{s}\nu_i \wedge \theta_j \wedge \frac{\partial F}{\partial u_{ij}}\,\tilde{\rho} + \{*\},$$

where $\{*\}$ again denotes terms depending linearly on F, $d_V F$ and their total derivatives. The pull-back of (5.16) to the equation manifold therefore becomes (5.14).

It remains to re-write (5.13) and (5.14) in the Laplace adapted coframe. We do this in two steps. Let $\mathcal{L}_0$ be the universal linearization defined by (3.12) with $\varrho = 1$. Since $\mathcal{L}_0 = \kappa\mathcal{L}_c$, it is a simple matter to see that $\mathcal{L}_0^*(\rho_0) = \delta\,\mathcal{L}_c^*(\rho_c)$ where $\rho_0 = \dfrac{\delta}{\kappa}\rho_c$. Since $\mathcal{L}_c^*(\rho_c) = 0$, this implies that $\mathcal{L}_0^*(\rho_0) = 0$. Next we prove that

$$\Psi_0(\rho_0) = \Psi_c(\rho_c), \tag{5.17}$$

where $\Psi_0(\rho_0)$ is the form

$$\Psi_0(\rho_0) = \frac{1}{2}\,\sigma \wedge [\theta \wedge \psi_1^0 - \xi_1 \wedge \rho_0] + \frac{1}{2}\,\tau \wedge [\theta \wedge \psi_2^0 + \eta_1 \wedge \rho_0], \tag{5.18a}$$

$$\psi_1 = X(\rho_0) - (B_0 + Q)\,\rho_0 \quad \text{and} \quad \psi_2 = -Y(\rho_0) + A_0\,\rho_0, \tag{5.18b}$$

and where A_0 and B_0 are given by (3.14.)

Lemma 5.3 states that $\Psi_0(\rho_0)$ is independent of the scaling of the characteristic vector fields X and Y so that to prove (5.17) we may let, for simplicity,

$$X = D_x + mD_y \qquad \text{and} \qquad Y = D_x + nD_y.$$

An lengthy but nevertheless elementary calculation then establishes (5.17).

To complete the proof of the theorem, we let $\rho = \dfrac{1}{\lambda}\,\rho_0$. Since $\mathcal{L}^* = \lambda \cdot \mathcal{L}_0^* \cdot \dfrac{1}{\lambda}$, it follows that $\mathcal{L}^*(\rho) = 0$. We use Lemma 5.3 to conclude that $\Psi_0(\rho_0) = \Psi(\rho)$. The combination of this equation, (5.14) and (5.17) proves (5.5). ∎

Theorem 5.4. *Let $\mathcal{R}$ be a second order hyperbolic equation and suppose* $\mathrm{ind}\,(\mathcal{Y}) = \infty$ *and* $\mathrm{ind}\,(\mathcal{X}) = \infty$, *that is, $H_i \neq 0$ and $K_j \neq 0$ for all $i \geq 0$ and $j \geq 0$. Then, for all $s \geq 3$, all type $(1,s)$ conservation laws are trivial,*

$$H^{1,s}(\mathcal{R}^\infty) = 0.$$

Proof. According to Theorem 5.1, it suffices to prove that there are no non-zero type $(0, s-1)$ solutions ρ to the adjoint equation

$$XY(\rho) + A^*X(\rho) + B^*Y(\rho) + C^*\rho = 0. \tag{5.19}$$

It will be convenient to re-write this second order total differential equation as the first order system

$$X(\rho) = (Q + B)\,\rho + \psi_1, \tag{5.20a}$$

$$Y(\psi_1) = H_0\rho + (A - P)\psi_1. \tag{5.20b}$$

Suppose, in order to obtain a contradiction, that ρ is a non-zero solution to (5.20) of adapted order k. Since ρ is a contact form of degree ≥ 2, the order of ρ is necessarily $k \geq 1$. Then either

$V^k \lrcorner\, \rho \neq 0$ or $W^k \lrcorner\, \rho \neq 0$, where V^k and W^k are the π^∞_M vertical vector fields (4.30) dual to the Laplace adapted coframe. Assume that $V^k \lrcorner\, \rho \neq 0$. Since $V^{k+1} \lrcorner\, \rho = 0$ and $V^{k+2} \lrcorner\, \rho = 0$, we find, using (2.9) and (4.32), that

$$V^{k+1} \lrcorner\, X(\rho) = X(V^{k+1} \lrcorner\, \rho) - [\,X, V^{k+1}\,] \lrcorner\, \rho = -[\,X, V^{k+1}\,] \lrcorner\, \rho$$
$$= -B_k(V^{k+1} \lrcorner\, \rho) + H_{k+1}(V^{k+2} \lrcorner\, \rho) = 0,$$

and hence the interior product of (5.20a) with V^{k+1} leads to $V^{k+1} \lrcorner\, \psi_1 = 0$. Similarly, the interior product of (5.20b) with V^{k+1} yields $V^k \lrcorner\, \psi_1 = 0$. Finally, we hook (5.20a) with V^k to deduce that

$$X(V^k \lrcorner\, \rho) + [-B_{k-1} - (Q + B)](V^k \lrcorner\, \rho) = 0. \tag{5.21}$$

This implies that $V^k \lrcorner\, \rho$ is a non-zero relative X invariant contact form. But $\operatorname{ind}(\mathcal{Y}) = \infty$ and this contradicts Theorem 4.5. ∎

Example 5.5. If $ff'' - (f')^2 \neq 0$, then the f-Gordon equation $u_{xy} = f(u)$ has no non-trivial type $(1, s)$ conservation laws for $s \geq 3$. ∎

Example 5.6. Consider the Klein-Gordon equation $u_{xy} = u$. It is easily checked that for all $i \geq 0$, $H_i = K_i = 1$ and therefore $H^{1,s}(\mathcal{R}^\infty) = 0$ for all $s \geq 3$. We show that $H^{1,2}(\mathcal{R}^\infty)$ is infinite dimensional. With $X = D_x$ and $Y = D_y$, the universal linearization is the self-adjoint operator $\mathcal{L} = XY - 1$. For any $k \geq 0$, the one form $\rho_k = -\theta_{x^k}$ satisfies $\mathcal{L}^*(\rho_k) = 0$. From Theorem 5.1 and equation (5.8) we know that the type $(1, s)$ forms defined by

$$\omega_k = -\sigma \wedge \xi_1 \wedge \rho_k + \tau \wedge \theta \wedge \psi_2 = dx \wedge \theta_x \wedge \theta_{x^k} + dy \wedge \theta \wedge \theta_{x^{k-1}}$$

are conservation laws for the Klein-Gordon equation. These forms are d_H exact when k is odd and define non-trivial elements of $H^{1,2}(\mathcal{R}^\infty)$ when k is even. For example, when $k = 7$, we find that

$$d_H(\theta_x \wedge \theta_{x^6} - \theta_{x^2} \wedge \theta_{x^5} + \theta_{x^3} \wedge \theta_{x^4}) = \omega_7.$$

With $k = 2m$, it is easy to prove that ω_{2m} is not exact by contradiction.

This example again illustrates the fact there are non-trivial solutions to the adjoint equation which yield trivial conservation laws, a problem that contributes to the general difficulty of completely determining the spaces $H^{1,2}(\mathcal{R}^\infty)$. ∎

Next we consider hyperbolic equations $\mathcal{R}$ for which at least one of the Laplace indices, say $\operatorname{ind}(\mathcal{Y})$, is finite, that is

$$H_0 \neq 0, \quad H_1 \neq 0, \quad \dots \quad H_{p-1} \neq 0, \quad H_p = 0.$$

We begin with the simple observation that with $H_p = 0$,

$$X(\eta_{p+1}) = -B_p \eta_{p+1} = (-B + pQ)\,\eta_{p+1}$$

and consequently, if $\alpha \in \Omega^{0,s-1}(\mathcal{R}^\infty)$ is a relative X invariant form satisfying

$$X(\alpha) = [B - (p-1)Q]\,\alpha,$$

then the type $(1, s)$ form

$$\omega_1 = \tau \wedge \eta_{p+1} \wedge \alpha \tag{5.22}$$

is d_H closed,

$$d_H\omega_1 = \sigma \wedge \tau\,[X(\eta_{p+1}) \wedge \alpha + \eta_{p+1} \wedge X(\alpha) - Q\,\eta_{p+1} \wedge \alpha]$$
$$= \sigma \wedge \tau \wedge \eta_{p+1} \wedge [X(\alpha) - ((B - (p-1)Q))\,\alpha] = 0.$$

Likewise, if $\operatorname{ind}(\mathcal{X}) = q$ and if β satisfies $Y(\beta) = [A + qP]\,\beta$ then the form

$$\omega_2 = \sigma \wedge \xi_{q+1} \wedge \beta \tag{5.23}$$

is d_H closed. Our next result states that for, $s \geq 3$, every d_H closed form is cohomologous to forms of these two types, that is, every conservation law of type $(1, s)$, $s \geq 3$, is essentially constructed from contact forms which are relative invariants for the characteristics of the hyperbolic equation.

Theorem 5.7. *Let $\mathcal{R}$ be a second order hyperbolic equation. Suppose* $\operatorname{ind}(\mathcal{Y}) = p$ *and* $\operatorname{ind}(\mathcal{X}) = q$. *Then, for $s \geq 3$, every d_H closed form $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ may be written as*

$$\omega = \sigma \wedge \xi_{q+1} \wedge \beta + \tau \wedge \eta_{p+1} \wedge \alpha + d_H\gamma, \tag{5.24}$$

where $\gamma \in \Omega^{0,s}(\mathcal{R}^\infty)$ and α and β satisfy

$$X(\alpha) + [-B + (p-1)Q]\,\alpha = 0, \qquad \alpha \in \Omega^{s-1}(\eta_{p+1}, \eta_{p+2}, \dots), \tag{5.25}$$

$$Y(\beta) - [-A + qP]\,\beta = 0, \qquad \beta \in \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots). \tag{5.26}$$

If $\operatorname{ind}(\mathcal{X}) = \infty$, *then (5.24) remains valid with $\beta = 0$.*

We break the proof of Theorem 5.7 into two steps. The first step is to re-write the d_H closed forms ω_1 and ω_2 in the normal form (5.5). The second step is to find the general solution to the equation $\mathcal{L}^*(\rho) = 0$ by successive applications of the $\mathcal{X}$ Laplace transforms. We defer the details of this proof until section 7.

§**6. The adapted coframe for Darboux integrable equations.** A much studied class of second order hyperbolic equations in the plane are those equations integrable by the method of Darboux. In this section, we briefly recount a few salient aspects of this geometric integration method, we explicitly construct invariant contact forms for Darboux integrable equations and we use these invariant forms to obtain one final adaptation of the coframe on the prolonged equation manifold. All the necessary conditions for the existence of invariant contact forms described in Theorem 4.5 must therefore be satisfied by all Darboux integrable equations. In this manner we thus obtain new, *closed form* necessary conditions for a given hyperbolic equation to be Darboux integrable.

To describe the Darboux method of integration, let $\mathcal{R}$ be the second order hyperbolic differential equation determined by

$$F(x, y, u, p, q, r, s, t) = 0 \tag{6.1}$$

and let X and Y be distinct characteristic total vector fields for $\mathcal{R}$.

Definition 6.1. *A second order scalar hyperbolic equation $\mathcal{R}$ is Darboux integrable if there exists two functionally independent real-valued X invariant functions I and $\widetilde{I}$ on $\mathcal{R}^\infty$ and two functionally independent real-valued Y invariant functions J and $\widetilde{J}$ on $\mathcal{R}^\infty$, that is,*

$$X(I) = X(\widetilde{I}) = 0 \qquad \text{and} \qquad Y(J) = Y(\widetilde{J}) = 0$$

and $d\,I \wedge d\,\widetilde{I} \neq 0$ and $d\,J \wedge d\,\widetilde{J} \neq 0$. The equation $\mathcal{R}$ is semi-Darboux integrable if there exists two functionally independent invariants for either of the characteristic total vector fields.

The essence of the method of Darboux lies in the fact that the three equations (6.1),

$$\widetilde{I} = \phi(I) \qquad \text{and} \qquad \widetilde{J} = \psi(J), \tag{6.2}$$

where ϕ and ψ are arbitrary functions, determine a completely integrable Pfaffian system whose integral manifolds describe the general solution to (6.1). This completely integrable system is obtained by pulling back the contact system on $\mathcal{R}^l$ to the submanifolds defined by (6.2), where l is the maximum of the orders of I, $\widetilde{I}$, J and $\widetilde{J}$. The method of Darboux is the culmination of a series of classical geometric integration methods, starting with the methods of Laplace and Ampère. Although generically, a second order scalar hyperbolic equation will not be integrable by the method of Darboux, the general problem of classifying Darboux integrable equations seems practically intractable. One of the most interesting partial classifications of Darboux integrable equations is given by Vessiot [*40*],[*41*] where equations of the type

$$s = f(x, y, u, p, q)$$

which are Darboux integrable at order 2 are explicitly determined.

Goursat's analysis of the method of Darboux begins with a detailed study of the partial differential equations for X and Y invariant functions on $\mathcal{R}^\infty$. Quite remarkably, he is able to completely characterize the possible number of functionally independent invariants at any given order. In order to summarize the results of Goursat's analysis we define the *characteristic Pfaffian systems* of order k by

$$C_k(X) = \Omega^1(\tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \dots, \xi_k, \eta_k)$$

and

$$C_k(Y) = \Omega^1(\sigma, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \dots, \xi_k, \eta_k),$$

where the contact one forms Θ, ξ_1, η_1, ξ_2, $\eta_2, \dots,$ denote the elements of the Laplace adapted coframe on $\mathcal{R}^\infty$ (although for the purposes of the present discussion the characteristic coframe can equally well be used). The characteristic systems of infinite order are similarly defined by

$$C(X) = \Omega^1(\tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \dots)$$

and

$$C(Y) = \Omega^1(\sigma, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \dots).$$

From the identity

$$d\,I = X(I)\,\sigma + Y(I)\,\tau + d_V I$$

we deduce that I is an X invariant function on $\mathcal{R}^\infty$ of order k if and only if

$$d\,I \in C_k(X).$$

Now recall [*12*] that to any Pfaffian system $\mathcal{I}$ on a manifold M one can associate a flag of Pfaffian subsystems called the *derived flag*

$$\cdots \subset \mathcal{I}^{(i)} \subset \cdots \subset \mathcal{I}^{(2)} \subset \mathcal{I}^{(1)} \subset \mathcal{I}.$$

The i-th derived Pfaffian system $\mathcal{I}^{(i)}$ is defined inductively by the short exact sequence

$$0 \longrightarrow \mathcal{I}^{(i+1)} \xrightarrow{\ j\ } \mathcal{I}^{(i)} \xrightarrow{\ \delta_i\ } d\mathcal{I}^{(i)} \mod \mathcal{I}^{(i)} \longrightarrow 0,$$

where j denotes the inclusion map and δ_i is the composition of the exterior derivative d with the canonical projection onto the quotient modulo the ideal generated by $\mathcal{I}^{(i)}$. The derived flag always stabilizes at the maximal completely integrable subsystem of $\mathcal{I}$, denoted by $\mathcal{I}^{(\infty)}$. The dimensions of $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$ coincide with the number of functionally independent X and Y invariant functions of order $\leq k$ and we can therefore describe Goursat's results in terms of the dimensions of these Pfaffian systems.

Lemma 6.2. *Let $\mathcal{R}$ be a second order hyperbolic equation with characteristic vector fields X and Y and let*

$$\{\,\sigma, \tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \dots\}$$

be either the characteristic or Laplace adapted coframe on $\mathcal{R}^\infty$. Then for any $k \geq 1$, $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$ satisfy

$$C_k^{(\infty)}(X) \subset \Omega^1(\tau, \Theta, \xi_1, \eta_1, \eta_2, \dots, \eta_k)$$

and

$$C_k^{(\infty)}(Y) \subset \Omega^1(\sigma, \Theta, \eta_1, \xi_1, \xi_2, \dots, \xi_k).$$

Proof. Since

$$d_H\xi_j \equiv \sigma \wedge \xi_{j+1} \qquad \mathrm{mod}\ \{\, \tau, \Theta, \xi_1, \eta_1, \dots, \xi_j, \eta_j \,\}$$

it follows that

$$d\,\xi_j \not\equiv 0 \qquad \mathrm{mod}\ \{\, \tau, \Theta, \xi_1, \eta_1, \dots, \xi_j, \eta_j \,\} \tag{6.3}$$

and therefore, in particular, $\xi_k \notin C_k^{(1)}(X)$. Equation (6.3), with $j = k-1$, now implies that $\xi_{k-1} \notin C_k^{(2)}(X)$ and so on. This argument does not allow one to eliminate ξ_1 since $d\tau$ may contain $\sigma \wedge \xi_2$ ∎

As a corollary to this lemma we note that if I_k and J_k are X and Y invariant functions of order k, then

$$d_V I_k \equiv a_k \eta_k \qquad \mathrm{mod}\ \{\, \Theta, \xi_1, \eta_1, \dots \eta_{k-1} \,\} \tag{6.4a}$$

and

$$d_V J_k \equiv b_k \xi_k \qquad \mathrm{mod}\ \{\, \Theta, \eta_1, \xi_1, \dots, \xi_{k-1} \,\} \tag{6.4b}$$

where $a_k \neq 0$ and $b_k \neq 0$.

Given a Darboux integrable equation, we know that for some $k \geq 1$ the dimension of $\mathcal{C}_k^{(\infty)}(X)$ will be at least two. Let m be the *minimum* integer for which

$$\dim C_m^{(\infty)}(X) \neq 0.$$

According to Goursat, for minimum m, $\dim C_m^{(\infty)}(X)$ must be either 1,2 or 3.

Case I. In this first case,

$$\dim C_m^{(\infty)}(X) = 1$$

and therefore the second X invariant function whose existence is insured by the definition of Darboux integrability must have order strictly larger that m. Thus there is a integer $n > m$ such that

$$\dim C_m^{(\infty)}(X) = 1, \quad \dim C_{m+1}^{(\infty)}(X) = 1, \quad \dots \quad \dim C_{n-1}^{(\infty)}(X) = 1,$$

and

$$\dim C_n^{(\infty)}(X) \geq 2.$$

It is proved in Goursat that in fact

$$\dim C_n^{(\infty)}(X) = 2, \quad \dim C_{n+1}^{(\infty)}(X) = 3, \quad \dots \quad \dim C_{n+i}^{(\infty)} = 2 + i, \dots .$$

In other words there is a sequence of functionally independent X invariant functions

$$I_m, \quad I_n, \quad I_{n+1}, \quad I_{n+2}, \dots \tag{6.5}$$

of order m, n, $n+1, \dots$ such any X invariant function may be expressed in terms of these.

Case II. In this second case,

$$\dim C_m^{(\infty)}(X) = 2.$$

It is shown that this can happen only if $m = 1$ or $m = 2$. We refer to these two possibilities as Case IIa and Case IIb respectively. In Case IIa it is shown that

$$\dim C_2^{(\infty)}(X) = 3, \quad \dim C_3^{(\infty)}(X) = 4, \quad \dots \quad \dim C_{1+i}^{(\infty)} = 2 + i, \dots .$$

Thus there is a sequence of functional independent invariants

$$I_1', \quad I_1, \quad I_2, \quad I_3, \dots , \tag{6.6}$$

where I_1' and I_1 are of order one and I_j is of order j. Any X invariant function may be expressed in terms of these. In Case IIb there are no invariants of order one, two functionally independent invariants of order 2 and moreover, it is proved that

$$\dim C_3^{(\infty)}(X) = 3, \quad \dim C_4^{(\infty)}(X) = 4, \quad \dots \quad \dim C_{2+i}^{(\infty)} = 2 + i, \dots .$$

In this case any X invariant function is obtained from a sequence of functionally independent invariants

$$I_2', \quad I_2, \quad I_3, \quad I_4, \dots , \tag{6.7}$$

where I_2' and I_2 are of order 2 and I_j is of order j.

Case III. In this final case we have

$$\dim C_m^{(\infty)} = 3$$

and it proved by Goursat that this can happen only if $m = 2$ in which case

$$\dim C_3^{(\infty)}(X) = 4, \quad \dim C_5^{(\infty)}(X) = 5, \quad \dots \quad \dim C_{2+i}^{(\infty)} = 3 + i, \dots .$$

There are no X invariant functions of order less than two and there is a complete sequence of functionally independent invariants

$$I_2'', I_2', I_2, I_3, I_4, I_5 \dots \tag{6.8}$$

where I_2'', I_2' and I_2 are of order 2 and I_j is of order j.

We define the *Darboux index* $\mathrm{Darb}\,(X)$ to be the smallest integer k such that

$$\dim C_k^{(\infty)}(X) \geq 2. \tag{6.9}$$

In Case I, we have that $\mathrm{Darb}\,(X) = n$, in Case IIa we have $\mathrm{Darb}\,(X) = 1$ and in Cases IIb and III we have $\mathrm{Darb}\,(X) = 2$.

Likewise, for a Darboux integrable equation, we have four possible configurations for $C_k^{(\infty)}(Y)$ and we define $\mathrm{Darb}\,(Y)$ to be the smallest integer k such that

$$\dim C_k^{(\infty)}(Y) \geq 2. \tag{6.10}$$

The infinite sequences of functionally independent invariants obtained in each of the cases I, IIa, IIb and III may be generated by successive differentiations. Indeed, on account of Proposition 3.1, with I and J of *minimal* possible order, we may choose the characteristic total vector fields X and Y to commute. If $\widetilde{I}$ is an X invariant function then $Y(\widetilde{I})$ is also an X invariant function. It is proved in Goursat that the sequence of invariants (6.5), (6.6), (6.7) and (6.8) may be generated in this fashion. In particular, in Case III, the invariants I_2'', I_2' and I_2 may be chosen so that

$$I_2 = Y(I_2'). \tag{6.11}$$

Our goal now is to construct what we shall call the Darboux adapted coframe for hyperbolic equations integrable by the method of Darboux. We begin with the following general construction of invariant $(0, 1)$ forms.

Proposition 6.3. *Let $\mathcal{R}$ be a hyperbolic equation for which there exist commuting characteristics X and Y. If I, J and K are X invariant functions such that*

$$K = Y(J) \quad \text{and} \quad Y(I) = 1,$$

then the $(0, 1)$ form

$$\omega = d_V J - K d_V I \tag{6.12}$$

is X invariant.

Proof. Using the fact that I, J and K are X invariant we compute

$$\begin{aligned} d_H\omega &= -d_V d_H J - d_H K \wedge d_V I + K d_V d_H I \\ &= -d_V(K\,\tau) - Y(K)\tau \wedge d_V I + K\, d_V \tau \\ &= \tau \wedge [d_V K - Y(K) d_V(I)]. \end{aligned}$$

Since $X(\omega) = X \lrcorner\, d_H(\omega)$, this implies that $X(\omega) = 0$. ∎

In particular, as a corollary to Theorem 4.5 we obtain the following necessary conditions for a hyperbolic equation to be Darboux integrable.

Corollary 6.4. *Let $\mathcal{R}$ be a second order hyperbolic equation. If $\mathcal{R}$ is Darboux integrable, then the Laplace indices* $\operatorname{ind}(\mathcal{X})$ *and* $\operatorname{ind}(\mathcal{Y})$ *are finite.*

The construction of the Darboux adapted coframe proceeds according to whether the characteristic systems $C(X)$ and $C(Y)$ belong to Case I, Case IIa, Case IIb or Case III. We focus our attention on $C(X)$ – the constructions for $C(Y)$ are similar.

Case I. Here we assume that $\operatorname{Darb}(X) = n$ and the a generating set for the X invariant functions on $\mathcal{R}$ is given by

$$I_m, \quad I_n, \quad I_{n+1} = Y(I_n), \quad I_{n+2} = Y(I_{n+1}), \dots$$

where $m < n$ and $Y(I_m) = 1$. By Proposition 6.3 the contact one forms

$$\alpha_n = d_V I_n - I_{n+1}\, d_V I_m, \quad \alpha_{n+1} = d_V I_{n+1} - I_{n+2}\, d_V I_m, \dots \tag{6.13}$$

are X invariant. Their exterior derivatives are easily computed.

Proposition 6.5. *For all $i \geq 0$, the invariant contact one forms α_{n+i} satisfy*

$$d\,\alpha_{n+i} = d\,I_m \wedge \alpha_{n+i+1}. \tag{6.14}$$

CaseIIa. In this case $\operatorname{Darb}(X) = 1$ and a generating set for the X invariant functions is given by

$$I_1', \quad I_1, \quad I_2 = Y(I_1), \quad I_3 = Y(I_2), \dots ,$$

where $Y(I_1') = 1$. We apply Proposition 6.3 to obtain the X invariant contact one forms

$$\alpha_1 = d_V I_1 - I_2\, d_V I_1', \quad \alpha_2 = d_V I_2 - I_3\, d_V I_1', \dots$$

For $i \geq 2$ it follows from (6.4) that

$$\alpha_i \equiv a_i \eta_i \qquad \text{mod } \{\, \Theta, \eta_1, \eta_2, \dots, \eta_{i-1} \,\},$$

where a_i is non-zero. We also have that

$$\alpha_1 \equiv a_1 \eta_1 \qquad \text{mod } \{\, \Theta \,\},$$

where a_1 is non-zero. Indeed, if $a_1 = 0$ then because

$$d_H \Theta \equiv \sigma \wedge \xi_1 + \tau \wedge \eta_1 \qquad \text{mod } \{\, \Theta \,\}$$

we would have that

$$d_H \alpha_1 \equiv 0 \qquad \text{mod } \{\, \Theta, \eta_1, \xi_1 \,\}.$$

But in view of the structure equation

$$d_H \alpha_1 = \tau \wedge \alpha_2 \equiv a_2 \tau \wedge \eta_2 \qquad \text{mod } \{\, \Theta, \eta_1, \xi_1 \,\}$$

this contradicts the fact the α_2 has adapted order 2. We have necessarily

$$\operatorname{ind}(\mathcal{Y}) = 0$$

so that the initial generalized Laplace invariant H_0 must vanish. The part of the Darboux adapted coframe associated to $C(X)$ is given by

$$\{\, \alpha_1, \alpha_2, \alpha_3, \dots \,\}$$

and, as before, the d structure equations are

$$d\,\alpha_i = dI_1' \wedge \alpha_{i+1}. \tag{6.15}$$

CaseIIb. In this case $\mathrm{Darb}\,(X) = 2$ and the X invariant functions are generated by

$$I_2', \quad I_2, \quad I_3 = Y(I_2), \quad I_4 = Y(I_3), \ldots$$

where $Y(I_2') = 1$. Again we apply Proposition 6.3 to obtain X invariant forms

$$\alpha_2 = d_V I_2 - I_3\, d_V I_2', \quad \alpha_3 = d_V I_3 - I_4\, d_V I_2', \ldots$$

which satisfy, for $i \geq 3$,

$$\alpha_i \equiv a_i \eta_i \qquad \mathrm{mod}\ \{\, \Theta, \eta_1, \ldots, \eta_{i-1}\,\},$$

where a_i is non-zero. A similar argument as presented in Case IIa shows that

$$\alpha_2 \equiv a_2 \eta_2 \qquad \mathrm{mod}\ \{\, \Theta, \eta_1\,\},$$

where a_2 is non-zero. By Theorem 4.5, we have that

$$\mathrm{ind}\,(\mathcal{Y}) \leq 1,$$

and the part of the Darboux adapted coframe which arises from $C(X)$ is

$$\{\, \eta_1, \alpha_2, \alpha_3, \ldots\,\}.$$

When $\mathrm{ind}\,(\mathcal{Y}) = 1$, η_1 is the $(0,1)$ form $\mathcal{Y}(\Theta)$ defined by (4.17) and when $\mathrm{ind}\,(\mathcal{Y}) = 0$, η_1 is an X invariant multiple of $\mathcal{Y}(\Theta)$. For $i \geq 2$, the structure equations (6.15) hold.

Case III. We have that $\mathrm{Darb}\,(X) = 2$ and the X invariant functions are generated by

$$I_2'', \quad I_2', \quad I_2, \quad I_3 = Y(I_2), \quad I_4 = Y(I_3), \ldots .$$

We apply Proposition 6.3 to obtain the X invariant forms

$$\alpha_1 = d_V I_2' - I_2\, d_V I_2'', \quad \alpha_2 = d_V I_2 - I_3\, d_V I_2'', \quad \alpha_3 = d_V I_3 - I_4\, d_V I_2'', \ldots .$$

For $i \geq 3$, equations (4.33) and (6.4) show that

$$\alpha_i \equiv a_i \eta_i \qquad \mathrm{mod}\ \{\, \Theta, \eta_1, \ldots \eta_{i-1}\,\},$$

where a_i is non-zero. The argument used in Cases IIa and IIb may be applied to show that

$$\alpha_2 \equiv a_2 \eta_2 \qquad \mathrm{mod}\ \{\, \Theta, \eta_1\,\},$$

with $a_2 \neq 0$. Finally, even though I_2'', I_2' and I_2 are of order two, the structure equations

$$d_H \alpha_1 = \tau \wedge \alpha_2 \equiv a_2 \tau \wedge \eta_2 \qquad \mathrm{mod}\ \{\, \Theta, \eta_1\,\}$$

imply that α_1 must be of order one, that is,

$$\alpha_1 \equiv a_1 \eta_1 \qquad \mathrm{mod}\ \{\, \Theta\,\},$$

where a_1 is non-zero. By Theorem 4.5, this implies that the

$$\mathrm{ind}\,(\mathcal{Y}) = 0$$

and the initial generalized Laplace invariant H_0 vanishes. The Darboux adapted coframe associated to $C(X)$ is

$$\{\, \alpha_1, \alpha_2, \alpha_3, \ldots\,\}$$

and the structure equations are as before.

It is a simple matter to explicitly construct infinitely many conservation laws for Darboux integrable equations.

Theorem 6.6. *If $\mathcal{R}$ is a Darboux integrable equation, then there are infinitely conservation laws of type $(1,s)$ for all $s \geq 0$.*

Proof. Recall that when X and Y commute, the dual horizontal forms σ and τ are d_H closed. Thus, if I is any X invariant function and J is any Y invariant function then the type $(1,0)$ forms

$$\alpha = I\,\tau \quad \text{and} \quad \beta = J\,\sigma$$

are evidently d_H closed. They define non-trivial conservation laws provided $I \neq Y(I')$ and $J \neq X(J')$ for invariants I' and J'. To construct type $(1,s)$ conservation laws, we first observe that if $\widetilde{\omega}$ is any d closed $s+1$ form on $\mathcal{R}^\infty$ with vanishing type $(2, s-1)$ component then its type $(1,s)$ component is d_H closed. In particular, if $I_1, I_2, \dots, I_{s+1}$ and $J_1, J_2, \dots, J_{s+1}$ are X and Y invariant functions, then the forms

$$\alpha = \pi^{1,s}(dI_1 \wedge dI_2 \wedge \cdots \wedge dI_{s+1})$$

and

$$\beta = \pi^{1,s}(dJ_1 \wedge dJ_2 \wedge \cdots \wedge dJ_{s+1})$$

are d_H closed forms of type $(1,s)$ on $\mathcal{R}^\infty$. For appropriate choices of the invariants I_i and J_j, these conservation laws will be non-trivial. In fact, we can use a construction due to Olver [*33*] to make this more precise. Fix $k \geq p+s$. We show that for any non-zero form

$$\eta \in \Omega^{s-2}_{\mathcal{I}}(\alpha_{p+1}, \alpha_{p+2}, \dots, \alpha_{k-1}),$$

where $\mathcal{I}$ is the ring of X invariant functions, the type $(1,s)$ form

$$\omega = \tau \wedge \alpha_{k+1} \wedge \alpha_k \wedge \eta$$

is d_H closed but not d_H exact. The form ω is d_H closed because τ is d_H closed and because $\alpha_{k+1} \wedge \alpha_k \wedge \eta$ is X invariant. To show that ω is not d_H exact, we proceed by contradiction and suppose that there is a $(0,s)$ form γ such that $d_H\gamma = \omega$. Since

$$d_H\gamma = \sigma \wedge X(\gamma) + \tau \wedge Y(\gamma),$$

we immediately deduce that γ is X invariant and that

$$Y(\gamma) = \alpha_{k+1} \wedge \alpha_k \wedge \eta. \tag{6.16}$$

This implies that $\gamma \in \Omega^s_{\mathcal{I}}(\alpha_{p+1}, \alpha_{p+2}, \dots)$ Moreover since the adapted order of ω is $k+1$, it is easy to argue that the adapted order of γ must be k. We can therefore express γ in the form

$$\gamma = \alpha_k \wedge \beta + \delta,$$

where both β and δ are X invariant forms of adapted order $\leq k-1$. But if we substitute this expression for γ into equation (6.16) we deduce that $\beta = \alpha_k \wedge \eta$ which implies that β is of order k. This contradiction proves that ω is not exact. ∎

We conclude this section with some specific examples taken from Goursat [*23*]. For the equation $s = pu$, the characteristic vector fields are $X = D_x$ and $Y = D_y$, the Laplace invariants are $H_0 = 0$, $K_0 = K_1 = p$ and $K_2 = 0$, the X invariants are $I = y$ and $I' = q - u^2/2$ and the Y invariants are $J = x$ and $J' = \dfrac{3r^2 - 2u_{xxx}p}{2p^2}$. For the equation $rs = p$ we find that the characteristic vector fields are $X = D_x + p/s^2 D_y$ and $Y = D_x$, $H_0 = 0$,

$$K_0 = -2\frac{p}{s^4}u_{xyy} + 6\frac{p^2}{s^6}u_{xyy}^2 - 2\frac{p^2}{s^5}u_{xyyy},$$

K_1 is a rather complicated fifth order function, and $K_2 = 0$. The invariants are $I = s - x$, $I' = s^2/p$ and $I'' = y - p/s$ and $J = y$, $J' = u_{yyy} + s^2 u_{xyy}/p$. Finally, for the equation $3rt^3 + 1 = 0$, the characteristics are $X_\pm = D_x \pm 1/t^2 D_y$, the Laplace invariants are

$$H_{\pm,0} = \pm 2\frac{u_{xyy}u_{yyy}}{t^4} + 6\frac{u_{yyy}^2}{t^6} - 2\frac{u_{yyyy}}{t^5} \mp 2\frac{u_{xyyy}}{t^3},$$

$H_{\pm,1} = 0$, and the invariants are $I_\pm = s \pm 1/t$ and $I'_\pm = xI_\pm - q$. In [4], the derived flags for the characteristic Pfaffian systems for the equation $rs = p$ are determined in terms of the Laplace adapted coframe from which the invariants are readily computed.

§7. Proof of Theorem 5.7. Let $\mathcal{F}\colon \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$,

$$\mathcal{F}(\omega) = XY(\omega) + A\,X(\omega) + B\,Y(\omega) + C\,\omega \tag{7.1}$$

be a hyperbolic total differential operator defined on the equation manifold $\mathcal{R}^\infty$ of a second order hyperbolic equation with characteristics X and Y. Let $\mathcal{F}^*$ be the adjoint of $\mathcal{F}$, as defined by (5.3). We begin by reviewing some relations between the Laplace transforms of $\mathcal{F}$ and $\mathcal{F}^*$.

With the commutator of X and Y given by (3.5), we note that $\mathcal{F}$ and $\mathcal{F}^*$ can also be expressed in the equivalent form

$$\mathcal{F}(\omega) = YX(\omega) + D\,X(\omega) + E\,Y(\omega) + G\,\omega, \tag{7.2}$$

and

$$\mathcal{F}^*(\rho) = YX(\rho) + D^*X(\rho) + E^*Y(\rho) + G^*\rho, \tag{7.3}$$

where

$$\begin{aligned} &D^* = P - A = 2P - D, \qquad E^* = -B - Q = -E \qquad \text{and} \\ &G^* = C^* = -X(D-P) - Y(E) + G - DE + (D-P)(E+Q). \end{aligned} \tag{7.4}$$

The next four propositions summarize the basic relations between the Laplace invariants of $\mathcal{F}$ and the invariants of $\mathcal{F}^*$. All are proved by simple computation. See [5] for details. Similar results are obtained in [*19*].

Proposition 7.1. *The Laplace invariants of $\mathcal{F}$ and $\mathcal{F}^*$ are related by*

$$H(\mathcal{F}^*) = K(\mathcal{F}) \qquad \text{and} \qquad K(\mathcal{F}^*) = H(\mathcal{F}). \tag{7.5}$$

If $H(\mathcal{F}) \neq 0$, then

$$K(\mathcal{Y}(\mathcal{F})) = H(\mathcal{F}) \qquad \text{and} \qquad H(\mathcal{Y}(\mathcal{F})) = K(\mathcal{X}(\mathcal{F}^*)). \tag{7.6}$$

If $K(\mathcal{F}) \neq 0$, then

$$H(\mathcal{X}(\mathcal{F})) = K(\mathcal{F}) \qquad \text{and} \qquad K(\mathcal{X}(\mathcal{F})) = H(\mathcal{Y}(\mathcal{F}^*)). \tag{7.7}$$

Corollary 7.2. *If $K(\mathcal{F}) \neq 0$, then*

$$H(\mathcal{X}(\mathcal{F})) = K(\mathcal{Y}(\mathcal{F}^*)) \tag{7.8}$$

and if $H(\mathcal{F}) \neq 0$, then

$$H(\mathcal{X}(\mathcal{F}^*)) = K(\mathcal{Y}(\mathcal{F})). \tag{7.9}$$

Now let l be a non-zero real valued function on $\mathcal{R}^\infty$. We define the conjugate of the operator $\mathcal{F}$ by l to be

$$\widetilde{\mathcal{F}}_l = \frac{1}{l}\cdot \mathcal{F}\cdot l\,.$$

The following result shows that the generalized Laplace invariants are absolute invariants under conjugation and that the generalized Laplace transforms commute with conjugation.

Proposition 7.3. *The Laplace transforms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}_l$ are related by*

$$H(\mathcal{F}) = H(\widetilde{\mathcal{F}}_l) \qquad \text{and} \qquad K(\mathcal{F}) = K(\widetilde{\mathcal{F}}_l) \tag{7.10}$$

and moreover, provided $H(\mathcal{F}) \neq 0$ and $K(\mathcal{F}) \neq 0$ respectively,

$$\mathcal{Y}(\widetilde{\mathcal{F}}_l) = \frac{1}{l}\cdot \mathcal{Y}(\mathcal{F})\cdot l \qquad \text{and} \qquad \mathcal{X}(\widetilde{\mathcal{F}}_l) = \frac{1}{l}\cdot \mathcal{X}(\mathcal{F})\cdot l. \tag{7.11}$$

Proposition 7.4. *If $H(\mathcal{F}) \neq 0$, then $K(\mathcal{F}^*) \neq 0$ and*

$$[\mathcal{Y}(\mathcal{F})]^* = \frac{1}{H(\mathcal{F})} \cdot \mathcal{X}(\mathcal{F}^*) \cdot H(\mathcal{F}) \tag{7.12}$$

and, if $K(\mathcal{F}) \neq 0$, then $H(\mathcal{F}^*) \neq 0$ and

$$[\mathcal{X}(\mathcal{F})]^* = \frac{1}{K(\mathcal{F})} \cdot \mathcal{Y}(\mathcal{F}^*) \cdot K(\mathcal{F}). \tag{7.13}$$

Proposition 7.5. *If $\operatorname{ind}_{\mathcal{Y}}(\mathcal{F}) = p$, then for all $i \leq p$*

$$H(\mathcal{Y}^i(\mathcal{F})) = K(\mathcal{X}^i(\mathcal{F}^*)) \tag{7.14}$$

and hence

$$\operatorname{ind}_{\mathcal{Y}}\mathcal{F} = \operatorname{ind}_{\mathcal{X}}(\mathcal{F}^*). \tag{7.15}$$

Similarly, if $\operatorname{ind}_{\mathcal{X}}(\mathcal{F}) = q$, then for all $j \leq q$

$$H(\mathcal{X}^j(\mathcal{F})) = K(\mathcal{Y}^j(\mathcal{F}^*)) \tag{7.16}$$

and therefore

$$\operatorname{ind}_{\mathcal{X}}(\mathcal{F}) = \operatorname{ind}_{\mathcal{Y}}(\mathcal{F}^*). \tag{7.17}$$

We shall need with some formulas for the coefficients of the iterates of the Laplace transforms. Suppose that $\operatorname{ind}\mathcal{Y} = p$ and $\operatorname{ind}\mathcal{X} = q$. Write

$$\mathcal{Y}^i(\mathcal{F}) = XY + A_iX + B_iY + C_i, \qquad i = 1, 2, \ldots, p$$

$$\mathcal{X}^j(\mathcal{F}) = YX + D_jX + E_jY + G_j, \qquad j = 1, 2, \ldots, q$$

$$\mathcal{Y}^j(\mathcal{F}^*) = XY + A_j^*X + B_j^*Y + C_j^*, \qquad j = 1, 2, \ldots, q$$

and

$$\mathcal{X}^i(\mathcal{F}^*) = YX + D_i^*X + E_i^*Y + G_i^*, \qquad i = 1, 2, \ldots, p.$$

Then simple induction arguments prove that

$$A_i = A - Y(\log H_0H_1 \cdots H_{i-1}) - iP, \qquad B_i = B - iQ, \tag{7.20}$$

$$E_j = B - X(\log K_0K_1 \cdots K_{j-1}) + (j+1)Q \quad D_j = A + (j+1)P. \tag{7.21}$$

It then follows from Proposition 7.5 that

$$D_i^* = -A + (i+1)P \quad \text{and} \quad E_i^* = -B - X(\log H_0H_1 \cdots H_{i-1}) + (i-1)Q. \tag{7.22}$$

We now turn to the proof of Theorem 5.7. By Proposition 7.5, we have that $K_i^* = K(\mathcal{X}^i(\mathcal{L}^*)) = H(\mathcal{Y}^i(\mathcal{L})) = H_i$ for all $i \leq p$ and hence

$$K_0^* \neq 0, \quad K_1^* \neq 0, \quad \ldots \quad K_{p-1}^* \neq 0, \quad K_p^* = 0.$$

Consequently, we can iteratively construct the $\mathcal{X}$ Laplace transforms of the operator $\mathcal{L}^*$. Define, for notational convenience, $\mathcal{F}^i = \mathcal{X}^i(\mathcal{L}^*)$, for $i = 1, 2, \ldots, p$,

$$\mathbf{H}_i = H_0H_1 \cdots H_{i-1},$$

and

$$\mathcal{H}_p \colon \Omega^{0,s-1}(\mathcal{R}^\infty) \to \Omega^{0,s-1}(\mathcal{R}^\infty)$$

by

$$\mathcal{H}_p(\alpha) = \frac{1}{H_0}\,\mathcal{Y}_{\mathcal{F}^1}[\frac{1}{H_1}\,\mathcal{Y}_{\mathcal{F}^2}[\cdots[\frac{1}{H_{p-1}}\,\mathcal{Y}_{\mathcal{F}^p}(\mathbf{H}_p\alpha)]\cdots]. \tag{7.23}$$

Similarly, if $\operatorname{ind}(\mathcal{X}) = q$, we define

$$\mathcal{K}_q \colon \Omega^{0,s-1}(\mathcal{R}^\infty) \to \Omega^{0,s-1}(\mathcal{R}^\infty)$$

by

$$\mathcal{K}_q(\beta) = \frac{1}{K_0}\,\mathcal{X}_{\mathcal{Y}(\mathcal{L}^*)}[\frac{1}{H_1}\,\mathcal{X}_{\mathcal{Y}^2(\mathcal{L}^*)}[\cdots[\frac{1}{K_{q-1}}\,\mathcal{X}_{\mathcal{Y}^q(\mathcal{L}^*)}(K_0K_1\cdots K_{q-1}\beta)]\cdots]. \tag{7.24}$$

We shall prove the following two theorems from which Theorem 5.7 follows immediately.

Theorem 7.6. *Let $s \geq 1$. Suppose* ind$(\mathcal{Y}) = p$ *and $\alpha \in \Omega^{0,s-1}(\mathcal{R}^\infty)$ satisfies (5.25). Then the form $\rho_\alpha = \mathcal{H}_p(\alpha)$ satisfies the adjoint equation $\mathcal{L}^*(\rho_\alpha) = 0$ and there is a $\gamma \in \Omega^{0,s}(\mathcal{R}^\infty)$ such that*

$$(-1)^{p+1}\tau \wedge \eta_{p+1} \wedge \alpha = \Psi(\rho_\alpha) + d_H(\gamma). \tag{7.25}$$

If ind$(\mathcal{X}) = q$ *and $\beta \in \Omega^{0,s-1}(\mathcal{R}^\infty)$ satisfies (5.26), then the form $\rho_\beta = \mathcal{K}_q(\beta)$ satisfies the adjoint equation $\mathcal{L}^*(\rho_\beta) = 0$ and there is a $\gamma' \in \Omega^{0,s}(\mathcal{R}^\infty)$ such that*

$$(-1)^{q+1}\sigma \wedge \xi_{q+1} \wedge \beta = \Psi(\rho_\beta) + d_H(\gamma'). \tag{7.26}$$

Theorem 7.7. *Suppose* ind$(\mathcal{Y}) = p$ *and* ind$(\mathcal{X}) = q$. *Let $\rho \in \Omega^{0,s-1}(\mathcal{R}^\infty)$ be a solution to the adjoint equation $\mathcal{L}^*(\rho) = 0$. If $s \geq 3$, then ρ may be expressed in the form*

$$\rho = \mathcal{H}_p(\alpha) + \mathcal{K}_q(\beta), \tag{7.27}$$

where α and β satisfy (5.25) and (5.26) respectively. If ind$(\mathcal{X}) = \infty$, *then (7.27) remains valid with $\beta = 0$.*

Proof of Theorem 5.7. Write $\alpha \sim \beta$ if α and β are two d_H closed, cohomologous forms. If $d_H\omega = 0$, then there is a form ρ satisfying $\mathcal{L}^*(\rho) = 0$ such that $\omega \sim \Psi(\rho)$. But by Theorem 7.7, $\rho = \mathcal{H}_p(\alpha) + \mathcal{K}_q(\beta)$ and therefore, by Theorem 7.6 (and ignoring the factors of $(-1)^{p+1}$ and $(-1)^{q+1}$),

$$\omega \sim \Psi(\rho) = \Psi(H_p(\alpha)) + \Psi(K_q(\beta)) \sim \sigma \wedge \xi_{q+1} \wedge \beta + \tau \wedge \eta_{p+1} \wedge \alpha.$$

∎

The proofs of both Theorem 7.6 and Theorem 7.7 depend upon the following lemma.

Lemma 7.8. *Suppose* ind$(\mathcal{Y}) = p$ *and $s \geq 1$. Let $\rho \in \Omega^{0,s-1}(\mathcal{R}^\infty)$ and define*

$$\rho_1 = \mathcal{X}_{\mathcal{L}^*}(\rho), \quad \rho_2 = \mathcal{X}_{\mathcal{F}^1}(\rho_1), \quad \dots \quad \rho_p = \mathcal{X}_{\mathcal{F}^{p-1}}(\rho_{p-1}) \quad \text{and} \quad \alpha = \frac{1}{\mathbf{H}_p}\rho_p.$$

If ρ is a solution to the adjoint equation $\mathcal{L}^*(\rho) = 0$, then ρ may be expressed in the form

$$\rho = \mathcal{H}_p(\alpha), \tag{7.28}$$

where α satisfies the first order system of total differential equations

$$X(\alpha) = [B - (p-1)Q]\,\alpha + \gamma, \tag{7.29a}$$

$$Y(\gamma) = [A - (p+1)P - Y(\log \mathbf{H}_p)]\,\gamma. \tag{7.29b}$$

Conversely, if (α, γ) are type $(0, s-1)$ solutions to (7.29), then $\rho_\alpha = \mathcal{H}_p(\alpha)$ satisfies $\mathcal{L}^*(\rho_\alpha) = 0$.

Proof. In effect, the system (7.29) is the result of p successive applications of the $\mathcal{X}$ Laplace transform to $\mathcal{L}^*(\rho) = 0$. The key point to observe is that, as a consequence of the fact that $H_p = 0$, the system (7.29) partially decouples. The form γ is a relative Y invariant form and γ then determines α up to a relative X invariant form.

If $\mathcal{L}^*(\rho) = 0$, then by Proposition 4.1 the forms ρ_i satisfy

$$\mathcal{F}^1(\rho_1) = 0, \quad \mathcal{F}^2(\rho_2) = 0, \quad \dots \quad \mathcal{F}^p(\rho_p) = 0$$

and

$$\mathcal{Y}_{\mathcal{F}^1}(\rho_1) = H_0\rho, \quad \mathcal{Y}_{\mathcal{F}^2}(\rho_2) = H_1\rho_1, \quad \dots \quad \mathcal{Y}_{\mathcal{F}^p}(\rho_p) = H_{p-1}\rho_{p-1}.$$

The composition of this last set of equations gives $\rho = \mathcal{H}_p(\alpha)$, where $\alpha = \dfrac{1}{\mathbf{H}_p}\,\rho_p$. The equations (7.29) are equivalent to $\mathcal{F}^p(\rho_p) = 0$. Indeed, if we set

$$\rho_{p+1} = \mathcal{X}_{\mathcal{F}^p}(\rho_p), \tag{7.30}$$

then, because $H_p = K_p^* = 0$, the equation $\mathcal{F}^p(\rho_p) = 0$ holds if and only if

$$\mathcal{Y}_{\mathcal{F}^{p+1}}(\rho_{p+1}) = 0. \tag{7.31}$$

Written out in full, equations (7.30) and (7.31) become (7.29a) and (7.29b) respectively.

Conversely, suppose that (α, γ) are a pair of type $(0, s-1)$ forms which satisfy (7.29). Then it is easy to check that the form $\alpha_p = \mathbf{H}_p\alpha$ satisfies $\mathcal{F}^p(\alpha_p) = 0$. Accordingly, if we define a sequence of forms by

$$\begin{aligned}
\alpha_{p-1} &= \frac{1}{H_{p-1}}\mathcal{Y}_{\mathcal{F}^p}(\alpha_p) = \frac{1}{H_{p-1}}\,[Y(\alpha_p) + (-A + pP)\alpha_p], \quad \dots \\
\alpha_{i-1} &= \frac{1}{H_{i-1}}\mathcal{Y}_{\mathcal{F}^i}(\alpha_i) = \frac{1}{H_{i-1}}\,[Y(\alpha_i) + (-A + iP)\,\alpha_i], \quad \dots \\
\rho_\alpha &= \frac{1}{H_0}\mathcal{Y}_{\mathcal{L}^*}(\alpha_1) = \frac{1}{H_0}\,[Y(\alpha_1) + (-A + P)\,\alpha_1],
\end{aligned} \tag{7.32a}$$

then it follows from Proposition 4.1, that

$$\mathcal{F}^{p-1}(\alpha_{p-1}) = 0, \quad \dots \quad \mathcal{F}^1(\alpha_1) = 0, \quad \mathcal{L}^*(\rho_\alpha) = 0. \qquad \blacksquare$$

Proof of Theorem 7.6. We first note that the forms α_i also satisfy

$$\begin{aligned}
\alpha_p &= \mathcal{X}_{\mathcal{F}^{p-1}}(\alpha_{p-1}) = X(\alpha_{p-1}) + E_{p-1}^*\alpha_{p-1} \\
&= X(\alpha_{p-1}) + [-B - X(\log \mathbf{H}_{p-1}) + (p-2)Q]\,\alpha_{p-1}, \quad \dots \\
\alpha_{i+1} &= \mathcal{X}_{\mathcal{F}^i}(\alpha_i) = X(\alpha_i) + E_i^*\alpha_i \\
&= X(\alpha_i) + [-B - X(\log \mathbf{H}_i) + (i-1)Q]\,\alpha_i, \quad \dots \\
\alpha_1 &= \mathcal{X}_{\mathcal{L}^*}(\rho_\alpha) = X(\rho_\alpha) + [-B - Q]\,\rho_\alpha.
\end{aligned} \tag{7.32b}$$

Define, $\epsilon_0 = \dfrac{1}{2}\Theta \wedge \rho_\alpha$ and for $i = 1, 2, \dots, p$,

$$\epsilon_i = \frac{1}{\mathbf{H}_i}\eta_i \wedge \alpha_i$$

Recall from (5.7) that

$$\begin{aligned}
\Psi(\rho_\alpha) &= \sigma \wedge \Theta \wedge [X(\rho_\alpha) - (B + Q)\rho_\alpha] + \tau \wedge \eta_1 \wedge \rho_\alpha + d_H(-\frac{1}{2}\Theta \wedge \rho_\alpha) \\
&= \sigma \wedge \Theta \wedge \alpha_1 + \tau \wedge \eta_1 \wedge \rho_\alpha + d_H(-\frac{1}{2}\Theta \wedge \rho_\alpha).
\end{aligned} \tag{7.33}$$

Then we compute, for $i = 2, 3, \dots, p-1$, using (7.32b),

$$X(\epsilon_i) = \frac{1}{\mathbf{H}_i}\,[-X(\log \mathbf{H}_i)\,\eta_i \wedge \alpha_i + X(\eta_i) \wedge \alpha_i + \eta_i \wedge X(\alpha_i)] = \frac{1}{\mathbf{H}_{i-1}}\,\eta_{i-1} \wedge \alpha_i + \frac{1}{\mathbf{H}_i}\,\eta_i \wedge \alpha_{i+1}$$

and, using (7.32a),

$$Y(\epsilon_i) = \frac{1}{\mathbf{H}_i}[-Y(\log \mathbf{H}_i)\, \eta_i \wedge \alpha_i + Y(\eta_i) \wedge \alpha_i + \eta_i \wedge Y(\alpha_i)] = \frac{1}{\mathbf{H}_i}\, \eta_{i+1} \wedge \alpha_i + \frac{1}{\mathbf{H}_{i-1}}\, \eta_i \wedge \alpha_{i-1}.$$

Together, these formulas yield, for $i = 2,\, 3,\, \dots\ p-1$,

$$d_H \epsilon_i = \frac{1}{\mathbf{H}_{i-1}}\, [\sigma \wedge \eta_{i-1} \wedge \alpha_i + \tau \wedge \eta_i \wedge \alpha_{i-1}] + \frac{1}{\mathbf{H}_i}\, [\sigma \wedge \eta_i \wedge \alpha_{i+1} + \tau \wedge \eta_{i+1} \wedge \alpha_i].$$

These same computations show that

$$d_H(\epsilon_1) = \sigma \wedge \Theta \wedge \alpha_1 + \tau \wedge \eta_1 \wedge \rho_\alpha + \frac{1}{H_0}\, [\sigma \wedge \eta_1 \wedge \alpha_2 + \tau \wedge \eta_2 \wedge \alpha_1],$$

and

$$d_H(\epsilon_p) = \frac{1}{\mathbf{H}_{p-1}}[\sigma \wedge \eta_{p-1} \wedge \alpha_p + \tau \wedge \eta_p \wedge \alpha_{p-1}] + \tau \wedge \eta_{p+1} \wedge \alpha.$$

If we set

$$\epsilon = \epsilon_0 - \epsilon_1 + \epsilon_2 + \cdots + (-1)^p \epsilon_p$$

we find that

$$d_H \epsilon = d_H \epsilon_0 - d_H \epsilon_1 + d_H \epsilon_2 + \cdots + (-1)^p d_H \epsilon_p = -\Psi(\rho_\alpha) + (-1)^p \tau \wedge \eta_{p+1} \wedge \alpha.$$

and Theorem 7.6 is proved. ∎

We need the following lemma for the proof of Theorem 7.7.

Lemma 7.9. *Suppose* $\operatorname{ind}(\mathcal{Y}) = p$ *and* $\operatorname{ind}(\mathcal{X}) = q$*. If* $s \geq 3$*, then any form* α *satisfying (7.29) can be uniquely decomposed as*

$$\alpha = \alpha_0 + \gamma_0, \tag{7.34}$$

where α_0 *satisfies*

$$X(\alpha_0) + [-B + (p-1)Q]\, \alpha_0 = 0, \qquad \alpha_0 \in \Omega^{s-1}(\eta_{p+1}, \eta_{p+2}, \dots), \tag{7.35}$$

and where

$$\gamma_0 \in \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots). \tag{7.36}$$

If $\operatorname{ind}(\mathcal{X}) = \infty$*, then (7.34) holds true with* $\gamma_0 = 0$*.*

Proof. Consider the following $C^\infty(\mathcal{R}^\infty)$ modules of differential forms

$$\mathcal{U}^i = \Omega^i(\eta_{p+1}, \eta_{p+2}, \eta_{p+3}, \dots)$$

and

$$\mathcal{V}^j = \Omega^j(\Theta, \eta_1, \eta_2, \dots \eta_p, \xi_1, \xi_2, \dots).$$

By appealing to the d_H structure equations given by Proposition 4.2, it is not difficult to check, because $H_p = 0$, that these spaces are all X invariant, that is,

$$X(\mathcal{U}^i) \subset \mathcal{U}^i \quad \text{and} \quad X(\mathcal{V}^j) \subset \mathcal{V}^j.$$

For $s \geq 2$, we have the direct sum decomposition

$$\Omega^{s-1}(\mathcal{R}^\infty) = \bigoplus_{i+j=s-1} \mathcal{U}^i \otimes \mathcal{V}^j. \tag{7.37}$$

Since γ is a relative Y invariant, we can use Theorem 4.5 to conclude that

$$\gamma \in \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots) \subset \mathcal{V}^{s-1}.$$

Now write

$$\alpha = \beta_{s-1} + \sum_{i=1}^{s-2} \alpha_i \wedge \beta_{s-1-i} + \alpha_{s-1},$$

where $\alpha_i \in \mathcal{U}^i$ and $\beta_j \in \mathcal{V}^j$. We substitute this representation of α into (7.29a) to find that

$$\gamma = X(\beta_{s-1}) + X(\alpha_1) \wedge \beta_{s-2} + \alpha_1 \wedge X(\beta_{s-2}) + \cdots + X(\alpha_{s-1})$$
$$+ [-B + (p-1)Q]\,(\beta_{s-1} + \alpha_1 \wedge \beta_{s-2} + \cdots + \alpha_{s-1}).$$

By virtue of the direct sum decomposition (7.37), we conclude that

$$\gamma = X(\gamma_0) + [-B + (p-1)Q]\,\gamma_0, \tag{7.38}$$

where

$$\gamma_0 = \beta_{s-1} \in \Omega^{s-1}(\Theta, \eta_1, \eta_2, \dots \eta_p, \xi_1, \xi_2, \dots). \tag{7.39}$$

A comparison of (7.38) and (7.29) shows that the form $\alpha_0 = \alpha - \gamma_0$ satisfies

$$X(\alpha_0) + [-B + (p-1)Q]\,\alpha_0 = 0 \tag{7.40}$$

and is therefore a relative X invariant. This proves (7.35).

We return to (7.38) to prove (7.36). The interior product of (7.38) with V^p gives

$$X(V^p \mathbin{\lrcorner} \gamma_0) + 2[-B + (p-1)Q](V^p \mathbin{\lrcorner} \gamma_0) = 0$$

and hence

$$V^p \mathbin{\lrcorner} \gamma_0 \in \Omega^{s-2}(\eta_{p+1}, \eta_{p+2}, \dots).$$

But, by definition, we have

$$V^p \mathbin{\lrcorner} \gamma_0 \in \Omega^{s-2}(\Theta, \eta_1, \eta_2, \dots \eta_p, \xi_1, \xi_2, \dots)$$

and therefore, because $s - 2 \geq 1$, $V^p \mathbin{\lrcorner} \gamma_0 = 0$. We hook (7.38) in succession with V^{p-1}, V^{p-2}, … , V^1, U, W^1, W^2, … , W^q. At each step we find that the forms $V^{p-1} \mathbin{\lrcorner} \gamma_0$, $V^{p-2} \mathbin{\lrcorner} \gamma_0$, … , $W^q \mathbin{\lrcorner} \gamma_0$ are relative X invariants and must therefore vanish. This proves (7.36). ∎

Proof of Theorem 7.7. If $\operatorname{ind}(\mathcal{X}) = \infty$, then the combination of (7.28) and (7.34) (with $\gamma_0 = 0$) immediately leads to (7.27).

Suppose then that $\operatorname{ind}(\mathcal{X}) = q$. Then these equations now imply that

$$\rho = \mathcal{H}_p(\alpha_0) + \delta, \tag{7.41}$$

where $\delta = \mathcal{H}_p(\gamma_0)$. Since

$$Y(\Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots)) \subset \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots))$$

it follows that

$$\mathcal{Y}_{\mathcal{X}^i(\mathcal{L}^*)}(\Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots)) \subset \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots))$$

and thus

$$\mathcal{H}_p(\Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots)) \subset \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots)).$$

This shows that $\delta \in \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \dots)$. But because $\operatorname{ind}(\mathcal{X})$ is finite we can also solve the adjoint equation $\mathcal{L}^*(\rho) = 0$ by successive applications of the $\mathcal{Y}$ Laplace transform to deduce that

$$\rho = \tilde{\delta} + \mathcal{K}_q(\beta_0) \tag{7.42}$$

where β_0 satisfies (5.26) and $\tilde{\delta} \in \Omega^{s-1}(\eta_{p+1}, \eta_{p+2}, \dots,)$. A comparison of (7.41) and (7.42) implies that $\delta = \mathcal{K}_q(\beta_0)$ and $\tilde{\delta} = \mathcal{H}_p(\alpha_0)$. This proves (7.27). ∎

§**8 Concluding Remarks.** In this paper we established, in Theorem 5.1, a normal form for all type $(1, s)$, $s \geq 1$, conservation laws for second order, scalar, hyperbolic equations in the plane. The methods used in deriving this normal form are quite general and have wide application in addition to those given here. In [*6*], we repeat the analysis of sections 5 and 7 for the case of parabolic equations in the plane and we prove that for such equations $H^{1,s} = 0$ for all $s \geq 2$ unless the so-called Goursat invariant for such equations vanishes. In this case there are infinitely many conservations laws for all s. The normal form for the conservation laws for the BBM equation is easily obtained and can be used to dramatically simplify Olver's classification of the conservation laws for this equation [*29*]. Similar normal form methods have been used by Bryant, Griffiths, and Hsu [*14*], [*15*] in the study of conservation laws for parabolic equations in the plane and for special classes of first order hyperbolic systems in 2 independent and 2 dependent variables. Normal forms for lower degree conservation laws, that is, conservation laws of type (r, s) where $r < n - 1$, $s \geq 1$ and where n is the number of independent variables have been obtained for many of the equations of classical field theory [*9*] and used to give elementary proofs of the results found in [*11*]. Most recently, these normal form techniques have proved useful in studying asymptotic conservation laws in general relativity [*10*].

As a simple consequence of Theorem 5.1, we showed that unless one of the sequences of generalized Laplace invariants vanishes, then all type $(1, s)$ conservation laws are trivial for $s \geq 3$. We also proved that if a second order equation in the plane is semi-Darboux integrable, then it admits infinitely many form-valued conservation laws of type $(1, s)$ for all $s \geq 0$ and we used our normal form theorem to explicitly classify all form-valued conservation laws for semi-Darboux integrable and Darboux integrable equations of type $(1, s)$, $s \geq 3$. It is an open problem to extend our results to the case $s = 1$ and $s = 2$. There is currently much interest in generalizing the method of Darboux to higher order equations and to equations with more independent and dependent variables. A careful study of the examples found in Forsyth [*19*] makes apparent the fact that equations which, in the appropriate sense, are semi-Darboux integrable will always admit infinitely many contact form-valued conservation laws of high degree. Thus it would appear, quite generally, that the non-triviality of the cohomology groups $H^{1,s}(\mathcal{R}^\infty)$ for large s is closely tied to the existence of geometric integration methods.

In the course of proving these results we used the classical method of Laplace to construct a certain coframe, called the Laplace adapted coframe, for the infinitely prolonged equation manifold $\mathcal{R}^\infty$ of a second order, scalar, hyperbolic equation in the plane. Certain coefficients H_i and K_j in the structure equations for this coframe are easily seen to be relative invariants of $\mathcal{R}^\infty$ which reduce to the sequence of classical Laplace invariants in the case of linear equations. As an immediate consequence of our analysis of higher degree conservation laws, we found that the sequence of generalized Laplace invariants H_i and K_j must terminate for Darboux integrable equations. Recently, Sokolov and Ziber [*35*] established the converse for the equation

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{8.1}$$

by noting that if $H_p = 0$ and $K_q = 0$, then the non-zero invariants H_i and K_j satisfy the 2 dimensional Toda lattice equations. In [*4*], the full structure equations for the Laplace adapted coframe are obtained and used to completely determined the derived flag structures for the characteristic Pfaffian systems $\mathcal{C}_k(X)$ and $\mathcal{C}_k(Y)$. From this analysis it then follows that any second order, scalar, hyperbolic equation in the plane is Darboux integrable if and only if the sequence of generalized Laplace invariants terminates and one can determine precisely the orders at which the characteristic invariants arise. This analysis also shows how, as a practical matter, the Laplace adapted coframe can be used to explicitly find the characteristic invariants.

The Laplace invariants have a number of other interesting applications [*24*], [*25*]. It can be shown that the equation $F = 0$ admits a complete first order intermediate integral $V(x, y, u, u_x, u_y, a, b) = 0$ depending on 2 arbitrary constants a and b if and only if $H_0 = 0$ or $K_0 = 0$. The equation possesses a general first order intermediate integral $I = \phi(I')$ depending on 1 arbitrary function ϕ if and only if $H_0 = 0$ and $M_\tau = 0$, or $K_0 = 0$ and $M_\sigma = 0$, where M_τ and M_σ are the so-called Monge-Ampére invariants. It is also possible to uniquely classify a number of simple equations in terms of their Laplace invariants. For example, the equation (8.1) is contact equivalent to the wave equation

$u_{xy} = 0$, to $u_{xy} = u$, or to either $u_{xy} = e^u$ or $u_{xy} = \dfrac{2u}{(x+y)^2}$ if $H_0 = K_0 = 0$, if $H_0 = H_1 = K_0 \neq 0$, or if $H_0 = K_0 \neq 0$ and $H_1 = 0$, respectively. It is interesting to note that another invariant which arises in the study of $H^{1,2}(\mathcal{R}^\infty)$ can be used to distinguish these latter 2 equations.

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