SYMMETRIES, CONSERVATION LAWS AND AND VARIATIONAL PRINCIPLES FOR FOR VECTOR FIELD THEORIES[1]

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Abstract. Let $T^a = 0$ be a system of differential equations for the components of a covariant vector field on $\mathbf{R}^n$. Suppose that $T^a = 0$ is invariant under the infinitesimal group Γ of translations and gauge transformations and that every element in Γ generates a conservation law for $T^a = 0$. We then prove that $T^a = 0$ necessarily arises from a variational principle provided that $n = 2$ and $T^a = 0$ is of third order, or $n \geq 3$ and $T^a = 0$ is of second order. We also show, by means of examples, that our results are sharp.

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§**1. Introduction and statement of results.** The interplay between symmetries, conservation laws, and variational principles is a rich and varied one and extends well beyond the classical Noether's theorem. In this paper we explore the relationships between symmetries, conservation laws, and variational principles for vector field theories on the Euclidean space $\mathbf{R}^n$. By a vector field theory on $\mathbf{R}^n$ we mean a classical field theory where the field is a 1 form

$$A = A_a dx^a$$

on $\mathbf{R}^n$, subject to a given system of kth order partial differential equations $T^a = 0$, where

$$T^a = T^a(x^i, A_b, A_{b,i_1}, A_{b,i_1 i_2}, \dots, A_{b,i_1 i_2 \cdots i_k}).$$

Here $A_{b,i_1 i_2 \cdots i_k}$ denotes the derivative of A_b with respect to the independent variables $x^{i_1}, x^{i_2}, \dots, x^{i_k}$.

In formulating a vector field theory on $\mathbf{R}^n$, it is customary to insist that the differential operators T^a satisfy various assumptions. Among these we consider the following.

Symmetries.

[S1] The operator T^a is invariant under the group of spatial translations

$$x^i \to x^i + a^i, \quad (a^i) \in \mathbf{R}^n.$$

[S2] The operator T^a is invariant under the infinite dimensional group of gauge transformations

$$A_a \to A_a + \frac{\partial \phi}{\partial x^a}(x), \quad \phi \in C^\infty(\mathbf{R}^n).$$

Variational Principles.

[V1] The operator T^a is the Euler-Lagrange expression

$$E^a(L_o) = \frac{\partial L_o}{\partial A_a} - D_i \frac{\partial L_o}{\partial A_{a,i}} + D_i D_j \frac{\partial L_o}{\partial A_{a,ij}} - \cdots$$

for some Lagrangian

$$L_o = L_o(x^i, A_a, A_{a,i_1}, A_{a,i_1 i_2}, \dots, A_{a,i_1 i_2 \cdots i_l}).$$

[V2] The operator T^a is the Euler-Lagrange expression $T^a = E^a(L)$ of an Lagrangian L which is invariant under the symmetry groups [S1] and [S2].

Conservation laws.

[C1] There are functions

$$t^i_j = t^i_j(x^i, A_a, A_{a,i_1}, A_{a,i_1 i_2}, \ldots, A_{a,i_1 i_2 \cdots i_l})$$

such that, for each $j = 1, 2, \ldots, n$,

$$A_{a,j} T^a = D_i(t^i_j).$$

[C2] The divergence of T^a vanishes identically,

$$D_a T^a = 0.$$

As is well known, property [V2] implies that $T^a = E^a(L)$ has the symmetries [S1] and [S2] and possesses the conservation laws [C1] and [C2] corresponding to the conservation of linear momentum and charge. The purpose of this paper is to study the extent to which the converse is true, that is, whether the symmetries [S1], [S2] and the conservation laws [C1], [C2] imply the existence of an invariant Lagrangian.

This question is answered by the following 3 theorems. These theorems are part of a program, initiated by Takens [10] and subsequently developed by Anderson and Pohjanpelto [2], [3], [9] to uncover general classes of equations for which the existence of symmetries and the associated conservation laws guarantees the existence of variational principles.

Theorem 1.1. *Suppose that the differential operator T^a has symmetries [S1], [S2] and conservation laws [C1], [C2].*

[i] *Then T^a is locally variational, that is, [V1] holds, if*

$$\begin{cases} n = 2, \text{ and } T^a \text{ is of third order,} \\ n \geq 3, \text{ and } T^a \text{ is of second order.} \end{cases}$$

[ii] *Then T^a is locally variational if the functions T^a are polynomials of degree at most n in the field variables A_a and their derivatives.*

Theorem 1.2. *Suppose that T^a has symmetries [S1], [S2] and is locally variational [V1]. Then*

$$T^a = \begin{cases} E^a(L), & \text{if } n = 2m, \\ E^a(L) + \alpha T^a_{CS}, & \text{if } n = 2m + 1, \end{cases}$$

where L is a Lagrangian with symmetries [S1], [S2], α is a constant, and T^a_{CS} is the Chern-Simons mass term

$$T^a_{CS} = \epsilon^{a b_1 c_1 b_2 c_2 \cdots b_m c_m} F_{b_1 c_1} F_{b_2 c_2} \cdots F_{b_m c_m}.$$

Here $F_{ab} = \frac{1}{2}(A_{a,b} - A_{b,a})$ stands for the field strength.

In this paper we prove Theorem 1.1 and demonstrate, by way of examples, the necessity of the various hypotheses. Results related to Theorem 1.2 have been established by Henneaux et al. [5], [6] within the context of BRST cohomology, but with the additional assumptions that T^a be a Lorentz invariant polynomial in the field strengths and their derivatives. Theorem 1.2 in the C^∞ case follows immediately from the computation of cohomology of the Euler-Lagrange complex invariant under the symmetries [S1] and [S2]. See [4].

We single out one particular set of implications as perhaps most relevant for classical field theory.

Theorem 1.3. *Suppose that $n = 4$ and that the field equations*

$$T^a(x^i, A_b, A_{b,j}, A_{b,jk}) = 0$$

are of second order. Then T^a is translationally invariant [S1], gauge invariant [S2] and satisfies the conservation of linear momentum [C1] and charge [C2] if and only if there is an invariant Lagrangian $L = L(F_{ij}, F_{ij,k})$ such that

$$T^a = E^a(L).$$

We summarize the interplay among the conditions [S1], [S2], [V1], [V2], [C1], [C2] in the following table.

Table 1

Assumptions		Conclusions
1	Invariant Lagrangian V2	Symmetries {S1,S2} and conservation laws {C1,C2}
2	Locally variational V1	Equivalence between symmetries and conservation laws, S1 <=>C1, S2 <=> C2
3	Symmetries {S1,S2} and conservation of linear momentum C1	Conservation of charge C2
4	Locally variational, symmetries {S1,S2}	$T = \begin{cases} E(L), & n = 2m, \\ E(L) + \alpha T_{CS}, & n = 2m+1, \end{cases}$ where L admits symmetries S1, S2.
5	Symmetries {S1,S2} and conservation laws C1	T second order, $n \geq 3$; T third order, $n = 2$; T polynomial of degree $\leq n$ => locally variational

Result 1 in Table 1 follows from the classical Noether's theorem [8] , and result 2 is a simple (but non-vacuous) extension of the classical Noether's theorem together with a rigorous converse (see [7]). The fact that charge conservation follows from the symmetries [S1], [S2] and the conservation of linear momentum does not seem to have been previously noted in the literature. Results 1, 2 and 3 will easily follow from the fundamental Lie derivative formula for source forms given in section 2.

In section 3 we again use the basic Lie derivative formula, together with a characterization of divergence free operators $D_a T^a = 0$, to prove Theorem 1.1, part (i). The second part of Theorem 1.1 follows from a general result obtained in [3]. We show in section 4, by means of examples, that the conclusions of Theorem 1.1 are sharp – our assumptions on the order of T^a or on the polynomial degree of T^a cannot be relaxed. We conclude with a discussion of some open problems.

§**2. Preliminaries.** In this section we collect some basic definitions and results from the formal calculus of variations that will be needed in the proof of Theorem 1.1. For more details and generalizations we refer to [1], [8], [11]. Elementary proofs of claims 1, 2 and 3 of Table 1 are given at the end this section.

Let $E = \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be the bundle of the independent and dependent variables with coordinates x^i, $i = 1, 2, \ldots, n$ and u_a, $a = 1, 2, \ldots, m$. We let $J^\infty(\mathrm{E})$ stand for the infinite jet space

$$\{(x^i, u_{a,i_1}, u_{a,i_1 i_2}, \ldots, u_{a,i_1 i_2 \cdots i_k}, \ldots)\},$$

where $u_{a,i_1}, u_{a,i_1 i_2}, \ldots, u_{a,i_1 i_2 \cdots i_k}, \ldots$ are the first, second and higher order derivatives of the u_a.

Let $I = (i_1, i_2, \ldots, i_k)$ be a multi-index of length $|I| = k$, and let $(t_1, t_2, \ldots, t_n)$ be the transpose of I, where t_l stands for the number of occurrences of the integer l amongst the entries $i_1, i_2, \ldots, i_k$ of I. Define the partial derivative operators $\partial^{a,I}$ by

$$\partial^{a,I} = \frac{t_1! t_2! \cdots t_n!}{k!} \frac{\partial}{\partial u_{a,I}},$$

and the total derivative operators D_i by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{|I| \geq 0} u_{a,Ii} \partial^{a,I}, \qquad i = 1, 2, \ldots, n. \tag{2.1}$$

We associate to a kth order differential operator on E,

$$T^a = T^a(x^i, u^{[k]}), \quad a = 1, 2, \ldots, m,$$

the source form

$$\Delta = T^a du_a \wedge \nu,$$

where $\nu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is the volume form on $\mathbf{R}^n$. Here $u^{[k]}$ stands collectively for all the derivative variables $u_{a,i_1 i_2 \cdots i_p}$, $p \leq k$. A source form $\Delta = T^a du_a \wedge \nu$ arises from a variational principle if there is a Lagrangian function

$$L = L(x^i, u^{[l]})$$

such that

$$T^a = \mathrm{E}^a(L), \qquad a = 1, 2, \ldots, m,$$

where the Euler-Lagrange operators E^a are

$$\mathrm{E}^a(L) = \sum_{|I| \geq 0} (-1)^{|I|} \mathrm{D}_I(\partial^{a,I} L). \tag{2.2}$$

Here the iterated total derivative D_I, $I = (i_1, i_2, \ldots, i_k)$, is given by

$$\mathrm{D}_I = D_{i_1} D_{i_2} \cdots D_{i_k}.$$

It will be convenient to call $\lambda = L(x^i, u^{[l]})\nu$ the Lagrangian n-form and to let $\mathrm{E}(\lambda) = \mathrm{E}^a(L)\, du_a \wedge \nu$ be the Euler-Lagrange form of λ.

It is not too difficult to see [1] that if a source form $\Delta = T^a du_a \wedge \nu$ arises from a variational principle, then the components $H_\Delta^{ab,I} = 0$ of the Helmholtz operator of Δ, defined by

$$H_\Delta^{ab,I} = \partial^{b,I} T^a - (-1)^{|I|} \mathrm{E}^{a,I}(T^b), \qquad a, b = 1, 2, \ldots, m, \quad |I| \geq 0, \tag{2.3}$$

vanish identically. In (2.3), $E^{a,I}$ stands for the higher Euler-Lagrange operators given by

$$\mathrm{E}^{a,I}(T^b) = \sum_{|J| \geq 0} (-1)^{|J|} \binom{|I|+|J|}{|I|} \mathrm{D}_J(\partial^{a,IJ} T^b).$$

Conversely, if the Helmholtz conditions of Δ are satisfied then we can construct a Lagrangian

$$L = \int_0^1 u_a T^a(x^i, tu_b, tu_{b,i_1}, tu_{b,i_1 i_2}, \ldots, tu_{b,i_1 i_2 \cdots i_k}) dt \tag{2.4}$$

for which $T^a = E^a(L)$. Hence a source form satisfying the Helmholtz conditions $H_\Delta^{ab,I} = 0$ will be called locally variational.

Next let

$$X = \Xi^i(x^j, u_b) \frac{\partial}{\partial x^i} + \Phi_a(x^j, u_b) \frac{\partial}{\partial u_a}$$

be a vector field on E. The vector field X is called projectable if the $\Xi^i = \Xi^i(x^j)$ are functions of x^j only. The prolongation $\operatorname{pr} X$ is the lift of X to a vector field on $J^\infty(\mathrm{E})$. Let

$$X_{\text{ev}} = X_{\text{ev},a}\frac{\partial}{\partial u_a},$$

where

$$X_{\text{ev},a} = (\Phi_a - \Xi^i u_{a,i})\frac{\partial}{\partial u_a}, \tag{2.5}$$

be the evolutionary form of X. Then the prolongation of X is given by

$$\operatorname{pr} X = \operatorname{tot} X + \operatorname{pr} X_{\text{ev}},$$

where

$$\operatorname{tot} X = \Xi^i D_i,$$

and

$$\operatorname{pr} X_{\text{ev}} = \sum_{|I|\geq 0} \mathrm{D}_I \partial^{a,I}(X_{\text{ev},a}).$$

By an infinitesimal transformation group Γ on E we mean a Lie subalgebra $\Gamma \subset \mathcal{X}(E)$ of all smooth vector fields on E. For $E = T^*\mathbf{R}^n \to \mathbf{R}^n$, we will, in particular, consider the semi-direct product

$$\Gamma(n) = \mathbf{t}(n) \otimes_s \mathbf{ga}(n)$$

of the infinitesimal groups of translations

$$\mathbf{t}(n) = \big\{a^i\frac{\partial}{\partial x^i} \,\big|\, (a^i) \in \mathbf{R}^n\big\}$$

and gauge transformations

$$\mathbf{ga}(n) = \big\{\phi_{,a}\frac{\partial}{\partial u_a} \,\big|\, \phi \in C^\infty(\mathbf{R}^n)\big\}.$$

It is apparent from the formula (2.4) that if a locally variational source form $\Delta = T^a du_a \wedge \nu$ is invariant under a subgroup of the group of affine transformations then the Lagrangian n-form $\lambda_o = L_o\nu$ also has these symmetries. However, formula (2.4) does not allow one to construct gauge invariant Lagrangians for gauge invariant source forms.

We say that a vector field X on E generates a conservation law for a differential operator T^a if there are functions

$$t^i = t^i(x^j, u^{[k]})$$

such that

$$X_{\mathrm{ev},a} T^a = D_i t^i. \tag{2.6}$$

Note that if (2.6) holds then, in particular, the divergence

$$D_i t^i = 0$$

vanishes on solutions of the system of differential equations $T^a = 0$.

In the next proposition H_Δ stands for the Helmholtz operator of a source form Δ acting on evolutionary vector fields $Y = Y_a \dfrac{\partial}{\partial u_a}$ on E by

$$H_\Delta(Y) = \sum_{|I|\geq 0} (\mathrm{D}_I Y_b) H_\Delta^{ab,I} du_a \wedge \nu,$$

where the components $H_\Delta^{ab,I}$ are given in (2.3). For a proof of the proposition, we refer to [2].

Proposition 2.2. *Let λ be any Lagrangian form and Δ be any source form on $J^\infty(E)$. Let X be a projectable vector field on E. Then*

i) $$E(\mathcal{L}_{\mathrm{pr}\,X}\lambda) = \mathcal{L}_{\mathrm{pr}\,X} E(\lambda), \tag{2.7}$$

ii) $$\mathcal{L}_{\mathrm{pr}\,X}\Delta = E(X_{\mathrm{ev}} \lrcorner\, \Delta) + H_\Delta(X_{\mathrm{ev}}). \tag{2.8}$$

Each term in (2.8) has a special significance. Let T^a be the differential operator associated with $\Delta = T^a \wedge du_a \nu$. By definition, a vector field X on E is a distinguished infinitesimal symmetry for T^a if Δ is invariant under X, that is, $\mathcal{L}_{\mathrm{pr}\,X}\Delta = 0$. The term $E(X_{\mathrm{ev}} \lrcorner\, \Delta)$ vanishes if and only if the vector field X generates a conservation law for the operator T^a. The term $H_\Delta(X_{\mathrm{ev}})$ vanishes whenever T^a is locally variational.

We are now in a position to prove claims 1, 2 and 3 in Table 1. For the rest of the section the bundle $E = T^*\mathbf{R}^n \to \mathbf{R}^n$ is the cotangent bundle of $\mathbf{R}^n$ with coordinates $(x^i, A_a) \to (x^i)$, where $i, a = 1, 2, \ldots, n$.

Proposition 2.3. *Suppose that a Lagrangian L on $J^\infty(T^*\mathbf{R}^n)$ has symmetries [S1] and [S2]. Then the differential operator $T^a = E^a(L)$ has symmetries [S1] and [S2] and conservation laws [C1] and [C2].*

Proof. The first conclusion immediately follows from (2.7). It thus remains to show that T^a has conservation laws [C1] and [C2].

Since the operator T^a is an Euler-Lagrange expression, the Helmholtz operator H_Δ of the source form $\Delta = T^a dA_a \wedge \nu$ vanishes identically. Thus, by equation (2.8),

$$\mathrm{E}(X_{\mathrm{ev}} \lrcorner \Delta) = 0, \qquad X \in t(n),$$

that is, T^a has conservation laws [C1].

Similarly, with $X_\phi = \phi,_a \dfrac{\partial}{\partial A_a}$, equation (2.8) yields

$$\mathrm{E}^b(\frac{\partial \phi}{\partial x^a} T^a) = 0,$$

which, after an integration by parts, becomes

$$\mathrm{E}^b(\phi D_a T^a) = 0, \qquad \text{for all smooth } \phi = \phi(x). \tag{2.9}$$

It now follows form (2.9) that the divergence $D_a T^a = c(x^i)$ is a function of x^i only. In fact, suppose that $D_a T^a = K(x^i, A^{[k]})$ is of order k and that for some $b, |J| = k$, $(\partial^{b,J} K)(x^i_o, A^{[k]}_o) \neq 0$. By (2.2), we can write (2.9) as

$$\sum_{|I|=0}^{k} (-1)^{|I|} \mathrm{D}_I \{\phi \partial^{b,I} K\} = 0.$$

Choose $\phi = \phi(x)$ such that

$$\frac{\partial^k \phi}{\partial x^J}(x_o) = 1, \qquad \frac{\partial^l \phi}{\partial x^I}(x_o) = 0, \qquad |I| = l \leq k,\, I \neq J.$$

Then at the point $(x_o, A^{[k]}_o)$,

$$\mathrm{E}^b(\phi D_a T^a)(x_o, A^{[k]}_o) = \partial^{b,J} K(x_o, A^{[k]}_o) = 0,$$

which is a contradiction.

Thus the divergence $D_a T^a = c(x^i)$ is a function of x^i only. By assumption, T^a is translationally invariant, and so $D_a T^a = c$ must be a constant. But, by the definition (2.1) of the total vector field D_a and the translational invariance of the T^a, the divergence $D_a T^a = c$ vanishes if every $A_{a,I} = 0$. Hence $c = 0$, and $D_a T^a = 0$ vanishes identically, as required. ∎

Proposition 2.4. *Suppose that a differential operator T^a on $T^*\mathbf{R}^n$ is locally variational. Then T^a has symmetries [S1] and [S2] if and only if it has conservation laws [C1] and [C2].*

Proof. We only need to note that the Helmholtz operator of $\Delta = T^a dA_a \wedge \nu$ vanishes identically. Now the first implication, that the symmetries [S1], [S2] imply the conservation laws [C1], [C2], follows from the second part of the proof of Proposition 2.3, and the converse, that the conservation laws [C1], [C2] imply symmetries [S1], [S2] immediately follows from equation (2.8). ∎

Proposition 2.5. *Suppose that a differential operator T^a on $T^*\mathbf{R}^n$ has symmetries [S1] and [S2] and conservation laws [C1]. Then T^a also has conservation laws [C2],*

$$D_a T^a = 0.$$

Proof. Let $\Delta = T^a dA_a \wedge \nu$ be the source form associated with T^a. Using the expressions (2.3) for the components $H_\Delta^{ab,I}$ of the Helmholtz operator of Δ we see that the $H_\Delta^{ab,I}$ are $\mathbf{ga}(n)$ invariant,

$$\operatorname{pr} X_\phi H^{ab,I} = 0, \qquad X_\phi \in \mathbf{ga}(n). \tag{2.10}$$

The operator T^a has symmetries [S1] and conservation laws [C1]. Thus, by (2.8), we have that

$$\sum_{|I|\geq 0} A_{b,Ij} H^{ab,I} = 0. \tag{2.11}$$

Next apply a vector field $\operatorname{pr} X_\phi \in \mathbf{ga}(n)$ to (2.11). On account of (2.10), we get

$$\sum_{|I|\geq 0} \phi_{,Ibj}\, H^{ab,I} = 0,$$

that is,

$$H_\Delta(X_\phi) = 0, \qquad X_\phi \in \mathbf{ga}(n). \tag{2.12}$$

An application of (2.8) with $X = \operatorname{pr} X_\phi$ together with (2.12) gives

$$E(\phi_{,a}\, T^a) = 0.$$

Now continue as in the second part of the proof of Proposition 2.3 to conclude that

$$D_a T^a = 0,$$

as required. ∎

§3. Proof of Theorem 1.1.

The proof of Theorem 1.1 [i] rests on the following 2 lemmas.

Proposition 3.1. *Let $\Delta = T^a(x^i, A^{[3]})dA_a \wedge \nu$ be a third order source form. Then the components $H^{ab,i}$, $H^{ab,ij}$ and $H^{ab,ijk}$ of the Helmholtz operator of Δ satisfy the integrability conditions*

$$H^{[ab],i} = D_j H^{ab,ij} - \frac{3}{2} D_j D_k H^{ab,ijk}, \qquad H^{(ab),ij} = \frac{3}{2} D_k H^{ab,ijk}. \tag{3.1}$$

Proof. For a third order operator $T^a = T^a(x^i, A^{[3]})$, formula (2.3) gives

$$\begin{aligned} H^{ab,i} &= \partial^{b,i} T^a + \partial^{a,i} T^b - 2D_j \partial^{a,ij} T^b + 3D_j D_k \partial^{a,ijk} T^b, \\ H^{ab,ij} &= \partial^{b,ij} T^a - \partial^{a,ij} T^b + 3D_k \partial^{a,ijk} T^b, \\ H^{ab,ijk} &= \partial^{b,ijk} T^a + \partial^{a,ijk} T^b. \end{aligned}$$

Now equations (3.1) can be verified by direct substitution. ∎

Lemma 3.2. *Let T^a be a kth order differential operator on $T^*\mathbf{R}^n$ such that the divergence*

$$D_a T^a = 0 \tag{3.2}$$

vanishes identically. Then the T^a are polynomials in the kth order derivative variables $A_{a,i_1 i_2 \cdots i_k}$ of degree at most $n-1$.

Proof. We first use the definition (2.1) of D_a to expand (3.2) into

$$\frac{\partial T^a}{\partial x^a} + \sum_{|I| \le k} A_{b,Ia} \partial^{b,I} T^a = 0. \tag{3.3}$$

The requirement that the terms in (3.3) involving the derivative variables $A_{b,J}$, $|J| = k+1$, vanish implies that

$$\partial^{b,(I} T^{a)} = 0, \tag{3.4}$$

for all $a, b = 1, 2, \ldots, n$ and for all $|I| = k$.

Let $X = (X_1, X_2, \ldots, X_n)$ be a covector on $\mathbf{R}^n$. We introduce partial differential operators ∂^a_X by

$$\partial^a_{k,X} = \sum_{|I|=k} X_I \partial^{a,I} = X_{i_1} X_{i_2} \cdots X_{i_k} \partial^{a, i_1 i_2 \cdots i_k}. \tag{3.5}$$

Then (3.4) is equivalent to

$$X_a \partial^b_{k,X} T^a = 0, \tag{3.6}$$

for all b and $X = (X_1, X_2, \ldots, X_n)$.

Next consider the mappings $G^{a_1 a_2 \cdots a_n}$, $1 \le a_i \le n$, given by

$$G^{a_1 a_2 \cdots a_n}(X^1, X^2, \ldots, X^n, Y) = \partial^{a_1}_{k,X^1} \partial^{a_2}_{k,X^2} \cdots \partial^{a_n}_{k,X^n} T^b Y_b.$$

Note that the T^a are polynomials in the variables $A_{a,I}$, $|I| = k$, of degree at most $n-1$ if and only if the mappings $G^{a_1 a_2 \cdots a_n}$ vanish identically for all $a_1, a_2, \ldots, a_n$. But on account of (3.6),

$$G^{a_1 a_2 \cdots a_n}(X^1, X^2, \ldots, X^n, Y) = 0,$$

whenever Y is a linear combination of the covectors X^1, X^2, ... , X^n. This implies that $G^{a_1 a_2 \cdots a_n}$ vanishes for almost all $X^1, X^2, \ldots, X^n, Y \in \mathbf{R}^{n*}$. By continuity,

$$G^{a_1 \cdots a_n} \equiv 0$$

must vanish identically. ∎

Proof of Theorem 1.1 [i] for $n \ge 3$. Let $T^a = T^a(A^{[2]})$ be a differential operator on $T^*\mathbf{R}^n$ with symmetries [S1], [S2] and conservation laws [C1], [C2]. In order to prove that T^a is locally variational we need to show that Helmholtz conditions H^{ab}, $H^{ab,i}$, $H^{ab,ij}$ of $\Delta = T^a dA_a \wedge \nu$ vanish.

First note that, with $X_\phi = \phi_{,a} \dfrac{\partial}{\partial A_a}$, equation (2.8) yields

$$\phi_{,b} H^{ab} + \phi_{,bi} H^{ab,i} + \phi_{,bij} H^{ab,ij} = 0,$$

for all smooth functions $\phi = \phi(x^i)$. Consequently, the zeroth order components of the Helmholtz operator vanish

$$H^{ab} = 0, \tag{3.7}$$

and the first order components satisfy the symmetry condition

$$H^{a(b,i)} = 0. \tag{3.8}$$

On the other hand, for second order operators, the integrability conditions (3.1) in Proposition 3.1 reduce to

$$H^{[ab],i} = D_l H^{ab,il}. \tag{3.9}$$

Suppose that the second order Helmholtz conditions $H^{ab,ij} = 0$ vanish. Then, by (3.8) and (3.9), the $H^{ab,i}$ are symmetric and skew-symmetric in overlapping pairs of indices, and therefore must vanish. This and equation (3.6) show that in order to complete the proof we only need to show that the second order Helmholtz conditions $H^{ab,ij} = 0$ of Δ vanish.

For this first note that by (3.8) and (3.9),

$$H^{ab,i} = H^{[ab],i} - H^{[bi],a} + H^{[ia],b} = D_l(H^{ab,il} - H^{bi,al} + H^{ia,bl}). \tag{3.10}$$

Equation (2.8) with $X = \dfrac{\partial}{\partial x^i}$ gives

$$A_{b,i}H^{ab} + A_{b,ij}H^{ab,j} + A_{b,ijk}H^{ab,jk} = 0,$$

which, on account of (3.7) and (3.10), becomes

$$A_{b,ij}D_l(H^{ab,jl} - H^{bj,al} + H^{ja,bl}) + A_{b,ijk}H^{ab,jk} = 0. \tag{3.11}$$

Next choose a covector $X = (X_1, X_2, \ldots, X_n)$ on $\mathbf{R}^n$, and apply the differential operator $\partial^c_{3,X}$, as defined in (3.5), on equation (3.11). Since

$$[\partial^c_{3,X}, D_l] = X_r X_s X_t \partial^{c,rs}\delta^t_l,$$

equation (3.11) becomes

$$\{A_{b,ij}(\partial^{c,rs}H^{ab,jt} - \partial^{c,rs}H^{bj,at} + \partial^{c,rs}H^{ja,bt}) + H^{ac,rs}\delta^t_i\}X_rX_sX_t = 0. \tag{3.12}$$

By Lemma 3.2 the components T^a are polynomials in the variables $A_{a,ij}$ of degree at most $n-1$, and so the components $H^{ab,ij} = \partial^{b,ij}T^a - \partial^{a,ij}T^b$ of the Helmholtz operator are polynomials in the variables $A_{a,ij}$ of degree at most $n-2$. We will conclude the proof of the Theorem by showing that equations (3.12) do not admit any non-trivial polynomial solutions $H^{ab,ij}$ in the variables $A_{a,ij}$ of degree $d \leq n-2$.

Note that without loss of generality we can assume that the $H^{ab,ij}$ satisfying equations (3.12) are homogeneous polynomials of degree $d \leq n-2$. For notational convenience we write

$$Q^{abc,jrst} = \partial^{c,rs}H^{ab,jt} - \partial^{c,rs}H^{bj,at} + \partial^{c,rs}H^{ja,bt}.$$

If $d = 0$, then it immediately follows from (3.12) that $H^{ab,ij} = 0$ vanish. Suppose next that $d \geq 1$. Choose covectors $Y^1, Y^2, \ldots, Y^d$ on $\mathbf{R}^n$, and apply the d-fold product operator $\partial^{e_1}_{2,Y^1}\partial^{e_2}_{2,Y^2}\cdots\partial^{e_d}_{2,Y^d}$ to equation (3.12). We get

$$\sum_{k=1}^{d} Y^k_i B^k + X_i N = 0, \tag{3.13}$$

where

$$B^k = Y^k_j \partial^{e_1}_{2,Y^1} \cdots \partial^{e_{\hat{k}}}_{2,Y^{\hat{k}}} \cdots \partial^{e_d}_{2,Y^d} Q^{ae_k c,jrst} X_r X_s X_t$$

and

$$N = \partial^{e_1}_{2,Y^1} \partial^{e_2}_{2,Y^2} \cdots \partial^{e_d}_{2,Y^d} H^{ac,rs} X_r X_s.$$

Suppose that $X, Y^1, Y^2, \ldots, Y^d$ are linearly independent. Then the linear relationship (3.13) amongst these vectors implies that $B^k = N = 0$, so that

$$\partial^{e_1}_{2,Y^1} \partial^{e_2}_{2,Y^2} \cdots \partial^{e_d}_{2,Y^d} H^{ac,rs} X_r X_s = 0. \tag{3.14}$$

But since $d \leq n-2$, this holds for almost all covectors $X, Y^1, Y^2, \ldots, Y^d$ on $\mathbf{R}^n$, and thus, by continuity, equation (3.14) must hold for all covectors $X, Y^1, Y^2, \ldots, Y^d$. Consequently, any d-fold derivative

$$\partial^{h_1,p_1q_1} \partial^{h_2,p_2q_2} \ldots \partial^{h_d,p_dq_d} H^{ab,ij} = 0$$

vanishes, and since $H^{ab,ij}$ is assumed to be homogeneous of degree d, $H^{ab,ij} = 0$ must vanish. ∎

Next we turn to the proof of Theorem 1.1 for $n = 2$. We let $\epsilon^{ij} = \epsilon^{[ij]}$ stand for the permutation symbol of $\mathbf{R}^2$. We need the following elementary fact from representation theory which follows from the basic identity $A^i B^j - B^i A^j = \epsilon^{ij} \det(A, B)$.

Proposition 3.3. *Suppose that the quantities* $Q^{a,i_1i_2\cdots i_k} = Q^{a,(i_1i_2\cdots i_k)}$, *where* $a, i_1, i_2, \ldots, i_k = 1, 2$, *are symmetric in the indices* $i_1, i_2, \ldots, i_k$ *and satisfy the cyclic condition*

$$Q^{(a,i_1i_2\cdots i_k)} = 0.$$

Then the $Q^{a,i_1i_2\cdots i_k}$ *can be expressed as*

$$Q^{a,i_1i_2\cdots i_k} = \epsilon^{a(i_1} R^{i_2\cdots i_k)},$$

where the quantities $R^{i_2\cdots i_k} = R^{(i_2\cdots i_k)}$ *are completely symmetric.*

Proof of Theorem 1.1 [i] for $n = 2$. Let $T^a = T^a(A^{[3]})$ be a differential operator on $J^\infty(T^*\mathbf{R}^2)$ with symmetries [S1], [S2] and conservation laws [C1], [C2]. Again, in order to prove that T^a is locally variational we need to show that the components H^{ab}, $H^{ab,i}$, $H^{ab,ij}$, $H^{ab,ijk}$ of the Helmholtz operator of $\Delta = T^a dA_a \wedge \nu$ vanish.

We start by noting that equation (2.8), with $X = \phi_a \dfrac{\partial}{\partial A_a}$, implies that

$$H^{ab} = 0, \qquad H^{a(b,i)} = 0, \qquad H^{a(b,ij)} = 0, \qquad H^{a(b,ijk)} = 0. \tag{3.15}$$

Furthermore, for a third order operator T^a formula (2.3) immediately shows that

$$H^{[ab],ijk} = 0. \tag{3.16}$$

Also, by Lemma 3.2, the components T^a are affine functions in the third order derivative variables $A_{a,ijk}$ so that

$$H^{ab,ijk} = H^{ab,ijk}(x^i, A^{[2]}).$$

Our first task is to solve equations (3.1), (3.15) and (3.16). Because $n = 2$, we shall see that these equations uniquely determine $H^{ab,i}$ and $H^{ab,ij}$ in terms of $H^{ab,ijk}$ and that the latter tensor has a particularly simple representation. In fact, since $H^{ba,ijk} = H^{ab,ijk}$, Proposition 3.2, applied twice, shows that there are functions $M^k = M^k(x^i, A^{[2]})$ such that

$$H^{ab,ijk} = \underset{\{ijk\}}{Sym}\ \epsilon^{ai}\epsilon^{bj}M^k.$$

Put

$$P^{ab,ij} = H^{ab,ij} - \frac{3}{2}D_k H^{ab,ijk} - \frac{3}{8}D_k(H^{ak,ijb} - H^{bk,ija}).$$

Then, on account of (3.1), (3.15) and (3.16), $P^{ab,ij}$ satisfies

$$P^{ab,[ij]} = 0, \qquad P^{(ab),ij} = 0, \qquad P^{a(b,ij)} = 0.$$

But when $n = 2$, it is easy to see that any tensor with these symmetries vanish identically. The first equation in (3.1) now reduces, using (3.15), to

$$H^{[ab],i} = \frac{3}{8}\mathrm{D}_{jk}(H^{ak,ijb} - H^{bk,ija}) = -\frac{3}{16}\mathrm{D}_{jk}(H^{ai,bjk} - H^{bi,ajk}).$$

As in (3.10), we solve this equation explicitly to obtain

$$H^{ab,i} = \frac{3}{8}\mathrm{D}_{jk}(H^{ab,ijk} - H^{ai,bjk}) = \frac{1}{2}\epsilon^{aj}\epsilon^{bi}\mathrm{D}_{jk}M^k.$$

Now we impose the conservation law condition C1. Equation (2.8), with $X = \dfrac{\partial}{\partial x^l}$, yields

$$A_{b,l}H^{ab} + A_{b,il}H^{ab,i} + A_{b,ijl}H^{ab,ij} + A_{b,ijkl}H^{ab,ijk} = 0.$$

Note that by (3.15), $H^{ab} = 0$. To this equation we apply the partial derivative operator $\partial^c_{4,X}$. Since $H^{ab,ijk}$ is second order, $H^{ab,ij}$ is at most third order, and since

$$[\partial^c_{4,X}, \mathrm{D}_{jk}] = 2X_{(j}D_{k)}\partial^c_{3,X} + X_jX_k\partial^c_{2,X},$$

we obtain

$$\frac{1}{2}A_{b,il}\epsilon^{bi}Y^a(\partial^c_{2,X}M(X)) + X_lY^aY^cM(X) = 0, \tag{3.17}$$

where $M(X) = M^kX_k$ and $Y^a = \epsilon^{ai}X_i$. We multiply this equation by Y^l to conclude that $\partial^c_{2,X}M(X) = 0$, in which case (3.17) reduces to $M(X) = 0$.

This proves that $H^{ab,ijk} = 0$, which in turn implies that $H^{ab,ij} = 0$ and $H^{ab,i} = 0$. Thus T^a is locally variational. ∎

§4. Examples.

We start by discussing a general construction which will used repeatedly in the examples below.

We let, as before, $E = T^*\mathbf{R}^n \to \mathbf{R}^n$ be the cotangent bundle of $\mathbf{R}^n$. In addition, we consider the bundles $\Lambda^2(T^*\mathbf{R}^n) \to \mathbf{R}^n$ with the coordinates $(x^i, f_{ab}) \to (x^i)$, where $a < b$. For notational convenience let $f_{ab} = -f_{ba}$, if $a \geq b$. Coordinates on $J^\infty(\Lambda^2(T^*\mathbf{R}^n))$ are $(x^i, f_{ab}, f_{ab,i_1}, f_{ab,i_1i_2}, \ldots)$.

Let $\Psi\colon J^\infty(T^*\mathbf{R}^n) \to J^\infty(\Lambda^2(T^*\mathbf{R}^n))$ be the mapping given by

$$\Psi(x^i, A_a, A_{a,b}, A_{a,bi_1}, \ldots) = (x^i, f_{ab}, f_{ab,i_1}, \ldots), \tag{4.1}$$

where $f_{ab} = F_{ab}$, $f_{ab,i_1} = F_{ab,i_1}, \ldots$. To each differential operator S^{ab} on $\Lambda^2(T^*\mathbf{R}^n)$, where $S^{ab} = -S^{ab}(x^i, f_{ab}, f_{ab,i_1}, \ldots)$, we associate the differential operator T^a on $T^*\mathbf{R}^n$ with the components

$$T^a = D_bS^{ab} \circ \Psi. \tag{4.2}$$

Lemma 4.1. *Let S^{ab} be a differential operator on $\Lambda^2(T^*\mathbf{R}^n)$ and suppose that S^{ab} is invariant under the infinitesimal group $\mathbf{t}(n)$ of translations and that S^{ab} admits $\mathbf{t}(n)$ conservation laws.*

Let $T^a = D_bS^{ab}$ be the associated differential operator (4.2) on $T^*\mathbf{R}^n$. Then T^a has symmetries [S1], [S2] and conservation laws [C1], [C2].

Proof. It immediately follows form the hypothesis of the Lemma and from the definition (4.2) that the differential operator T^a has symmetries [S1], [S2] and conservation laws [C2]. Thus we only need to show that T^a has $\mathbf{t}(n)$ conservation laws.

By assumption there are differential functions W^a_j on $J^\infty(\Lambda^2(T^*\mathbf{R}^n))$ such that

$$f_{ab,k}S^{ab} = D_aW^a_k, \qquad k = 1, 2, \ldots, n. \tag{4.3}$$

As is easily checked using (4.1), the mapping Ψ commutes with the total derivatives operators D_i. Thus (4.3) implies that

$$\frac{1}{2}\{(A_{a,bk} - A_{b,ak})S^{ab} \circ \Psi\} = D_a(W_k^a \circ \Psi),$$

which, upon an integration by parts, becomes

$$A_{a,k}T^a = A_{a,k}[(D_b S^{ab}) \circ \Psi] = D_a V_k^a,$$

where

$$V_k^a = -W_k^a \circ \Psi - A_{b,k}S^{ab} \circ \Psi.$$

Hence T^a has $\mathbf{t}(n)$ conservation laws, as required. ∎

Note that if S^{ab} is of order k, then T^a is of order $k+2$. A simple calculation shows that the highest order components of the Helmholtz operators of $\Delta = S^{ab} df_{ab} \wedge \nu$ and $\mathcal{T} = T^a dA_a \wedge \nu$ are related by

$$H_{\mathcal{T}}^{ab,i_1 i_2 \cdots i_{k+2}} = -2 \underset{\{i_1 i_2 \cdots i_{k+2}\}}{Sym} H_{\Delta}^{ai_1,bi_2,i_3 \cdots i_{k+2}} \circ \Psi.$$

Example 4.2. In this example we let $n = 2$. The fiber dimension of the bundle $\Lambda^2(T^*\mathbf{R}^2)$ is one, and accordingly we write $f_{12} = u$.

Consider the differential operator

$$S_\kappa = u\kappa \det(u_{ij})$$

on F, where $\kappa = \kappa(u_1, u_2)$ is any homogeneous function in u_1, u_2 of degree -4. As discussed in Example 3.17 of ref. [2], S_κ has $\mathbf{t}(2)$ symmetries and $\mathbf{t}(2)$ conservation laws. The H_Δ^i component of the Helmholtz operator of the source form $\Delta_\kappa = S_\kappa du \wedge \nu$ is

$$H_\Delta^i = 2\epsilon^{ij}\epsilon^{pq}\kappa u_p u_{jq},$$

and the second order components $H_\Delta^{ij} = 0$ vanish (see ref. [2]).

Let T_κ^a be the differential operator (4.2) on $T^*\mathbf{R}^2$ associated to S_κ. Then, by Lemma 4.1, T_κ^a admits symmetries [S1], [S2] and conservation laws [C1], [C2]. However, some straighforward manipulations show that the component $H^{11,222}$ of the Helmholtz operator of the source form $\mathcal{T}_\kappa = T_\kappa^a dA_a \wedge \nu$ is

$$H^{11,222} = \frac{1}{2} H_\Delta^2 \circ \Psi = -\epsilon^{pq}(\kappa \circ \Psi) A_{[1,2]p} A_{[1,2]1q}.$$

Consequently, the differential operator T_κ^a fails to be variational unless $\kappa = 0$, that is, $T_\kappa^a = 0$. Thus *Theorem 2.1, for $n = 2$, does not hold for fourth order differential operators.*

Example 4.3. Next we let $n = 3$. The fiber of the bundle $\Lambda^2(T^*\mathbf{R}^3)$ is 3 dimensional, and we can write

$$f^1 = f_{23}, \quad f^2 = f_{31}, \quad f^3 = f_{12}.$$

We let

$$V = \det(f^a{}_{,i}) \qquad \text{and} \qquad V^i_a = \frac{1}{2}\epsilon^{ii_1i_2}\epsilon_{aa_1a_2}f^{a_1}{}_{,i_1}f^{a_2}{}_{,i_2}.$$

Consider the differential operator

$$S_a = \lambda_i V^i_a V^{-1},$$

where λ_j, $j = 1, 2, 3$, are constant. According to Theorem 8 in ref. [9], S_a has $\mathbf{t}(3)$ symmetries and conservation laws, but S_a is locally variational only if each $\lambda_j = 0$ vanish.

Let

$$T^a = \epsilon^{abc} D_b S_c \circ \Psi,$$

be the third order differential operator on $T^*\mathbf{R}^3$ associated with S_a. By Lemma 4.1, T^a has symmetries [S1], [S2] and conservation laws [C1], [C2].

A straightforward computation shows that the $H^{11,222}$ component of the Helmholtz operator of the source form $\mathcal{T} = T^a dA_a \wedge \nu$ is

$$H^{11,222} = 2\partial^{1,222}T^1 = -\lambda_j(V^j_3 V^2_3 V^{-2}) \circ \Psi.$$

Now one can easily check that the Helmholtz condition $H^{11,222} = 0$ vanishes only if each $\lambda_j = 0$ vanish. Thus *Theorem 1.1 [i], for $n = 3$, does not hold for third order source forms.*

Example 4.4. In this final example, we let $n \geq 4$. Write $m = n(n-1)/2$ for the fiber dimension of the bundle $\Lambda^2(T^*\mathbf{R}^n)$. Let $\mathcal{E}^{a_1b_1a_2b_2\cdots a_mb_m}$ be a (non-trivial) valence $2m$ constant tensor on $\mathbf{R}^n$ with the following symmetries:

$$\begin{aligned} &\mathcal{E}^{a_1b_1\cdots b_ia_i\cdots a_mb_m} = -\mathcal{E}^{a_1b_1\cdots a_ib_i\cdots a_mb_m} \qquad \text{and} \\ &\mathcal{E}^{a_1b_1\cdots a_jb_j\cdots a_ib_i\cdots a_mb_m} = -\mathcal{E}^{a_1b_1\cdots a_ib_i\cdots a_jb_j\cdots a_mb_m}. \end{aligned} \tag{4.4}$$

One can view $\mathcal{E}^{a_1b_1a_2b_2\cdots a_mb_m}$ as the permutation symbol on $\Lambda^2(\mathbf{R}^n)$. Let S^{ab} be a differential operator on $\Lambda^2(T^*\mathbf{R}^n)$ with the components

$$S^{ab} = \mathcal{E}^{aba_1b_1\cdots a_nb_na_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}\lambda_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}f_{a_1b_1,1}\cdots f_{a_nb_n,n},$$

where the first order functions

$$\lambda_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}} = \lambda_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}(f_{ab}, f_{ab,i})$$

have the same symmetries in their indices as $\mathcal{E}^{a_1b_1a_2b_2\cdots a_mb_m}$ does. Note that

$$f_{ab,k}S^{ab} = 0, \qquad k = 1, 2, \ldots, n,$$

by the skew symmetry properties (4.4). Thus we see that S^{ab} admits $\mathbf{t}(n)$ symmetries and trivial $\mathbf{t}(n)$ conservation laws. As in the previous example, we let $T^a = T^a(A^{[3]})$ be the third order source form on $J^\infty(T^*\mathbf{R}^n)$ with the components

$$T^a = D_b S^{ab} \circ \Psi.$$

We are again able to conclude that, in general, T^a fails to be locally variational. Indeed, it is not too difficult to show that the $H^{11,222}$ component of the Helmholtz operator of $\mathcal{T} = T^a dA_a \wedge \nu$ is

$$H^{11,222} = 2\partial^{1,222}T^1 = (\partial^2_{f_{12}} S^{12}) \circ \Psi =$$

$$\mathcal{E}^{12a_1b_1\cdots a_nb_na_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}\big((\partial^2_{f_{12}}\lambda_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}) \circ \Psi\big) A_{a_1,b_1 1} \cdots A_{a_n,b_n n},$$

which, in general, does not vanish. Thus *Theorem 1.1 [i], for* $n \geq 4$, *does not hold for third order differential operators.* This example, with the $\lambda_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}$ chosen to be suitable polynomials in A_{ab}, $A_{ab,i}$, also shows that *Theorem 1.1 [ii] does not hold for polynomial differential operators of degree* $d \geq n+1$ *in the field variables and their derivatives.*

Note that the source forms in Examples 4.2 and 4.4 may be chosen to be Lorentz invariant but that these source forms will not in general possess conservation of angular momentum.

§**5. Concluding remarks.** In this paper we have shown that the assumptions of translational and gauge symmetries and conservation laws for second order vector field theories guarantee the existence of a variational principle for the field equations. An interesting, but seemingly difficult problem, would be to combine the analysis of this paper with the techniques developed in [3] to study polynomial operators $T^a = T^a(x^i, A_a, A_{a,i}, \ldots)$ which have translational, Lorentz and gauge symmetries and all the conservation laws associated with these symmetries. If non-variational operators with these properties exists at all, the results of [3] would suggest that their order must be exceedingly large. An equally interesting problem would be to extend the results of this paper to include general Yang-Mills theories. Now the differential operators under consideration would be assumed to admit the internal gauge symmetries of the theory and the associated conservation laws.

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