GROUP INVARIANT SOLUTIONS WITHOUT TRANSVERSALITY

Ian M. Anderson
Department of Mathematics
Utah State University
Logan, Utah 84322

Mark E. Fels
Department of Mathematics
Utah State University
Logan, Utah 84322

Charles G. Torre
Department of Physics
Utah State University
Logan, Utah 84322

Abstract. We present a generalization of Lie's method for finding the group invariant solutions to a system of partial differential equations. Our generalization relaxes the standard transversality assumption and encompasses the common situation where the reduced differential equations for the group invariant solutions involve both fewer dependent and independent variables. The theoretical basis for our method is provided by a general existence theorem for the invariant sections, both local and global, of a bundle on which a finite dimensional Lie group acts. A simple and natural extension of our characterization of invariant sections leads to an intrinsic characterization of the reduced equations for the group invariant solutions for a system of differential equations. The characterization of both the invariant sections and the reduced equations are summarized schematically by the kinematic and dynamic reduction diagrams and are illustrated by a number of examples from fluid mechanics, harmonic maps, and general relativity. This work also provides the theoretical foundations for a further detailed study of the reduced equations for group invariant solutions.

Keywords. Lie symmetry reduction, group invariant solutions, kinematic reduction diagram, dynamic reduction diagram.

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1. Introduction. Lie's method of symmetry reduction for finding the group invariant solutions to partial differential equations is widely recognized as one of the most general and effective methods for obtaining exact solutions of non-linear partial differential equations. In recent years Lie's method has been described in a number of excellent texts and survey articles (see, for example, Bluman and Kumei [*10*], Olver [*29*], Stephani [*36*], Winternitz [*41*]) and has been systematically applied to differential equations arising in a broad spectrum of disciplines (see, for example, Ibragimov [*23*] or Rogers and Shadwick [*34*]). It came, therefore, as quite a surprise to the present authors that Lie's method, as it is conventionally described, does not provide an appropriate theoretical framework for the derivation of such celebrated invariant solutions as the Schwarzschild solution of the vacuum Einstein equations, the instanton and monopole solutions in Yang-Mills theory or the Veronese map for the harmonic map equations. The primary objectives of this paper are to focus attention on this deficiency in the literature on Lie's method, to describe the elementary steps needed to correct this problem, and to give a precise formulation of the reduced differential equations for the group invariant solutions which arise from this generalization of Lie's method.

A second impetus for the present article is to provide the foundations for a systematic study of the interplay between the formal geometric properties of a system of differential equations, such as the conservation laws, symmetries, Hamiltonian structures, variational principles, local solvability, formal integrability and so on, and those same properties of the reduced equations for the group invariant solutions. Two problems merit special attention. First, one can interpret the principle of symmetric criticality [*32*], [*33*] as the problem of determining those group actions for which the reduced equations of a system of Euler-Lagrange equations are derivable from a canonically defined Lagrangian. Our previous work [*2*] on this problem, and the closely related problem of reduction of conservation laws, was cast entirely within the context of transverse group actions. Therefore, in order to extend our results to include the reductions that one encounters in field theory and differential geometry, one needs the more general description of Lie symmetry reduction obtained here. Secondly, there do not appear to be any general theorems in the literature which insure the local existence of group invariant solutions to differential equations; however, as one step in this direction the results presented here can be used to determine when a system of differential equations of Cauchy-Kovalevskaya type remain of Cauchy-Kovalevskaya type under reduction [*4*].

We begin by quickly reviewing the salient steps of Lie's method and then comparing Lie's method with the standard derivation of the Schwarzschild solution of the vacuum Einstein equations. This will clearly demonstrate the difficulties with the classical Lie approach. In section 3 we describe, in detail, a general method for characterizing the group invariant sections of a given bundle. In section 4 the reduced equations for the group invariant solutions are constructed in the case where reduction in both the number of independent and dependent variables can occur. We define the residual symmetry group of the reduced equations in section 5. In section 6 we illustrate, at some length, these results with a variety of examples. In the appendix we briefly outline some of the technical issues underlying the general theory of Lie symmetry reduction for the group invariant solutions of differential equations.

2. Lie's Method for Group Invariant Solutions. Consider a system of second-order partial differential equations

$$\Delta_\beta(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}) = 0 \tag{2.1}$$

for the m unknown functions u^α, $\alpha = 1, \ldots, m$, as functions of the n independent variables x^i, $i = 1, \ldots, n$. As usual, u^α_i and u^α_{ij} denote the first and second order partial derivatives of the functions u^α. We have assumed that the equations (2.1) are second-order and that the number of equations coincides with the number of unknown functions strictly for the sake of simplicity. A fundamental feature of Lie's entire approach to symmetry reduction of differential equations, and one that contributes greatly to its broad applicability, is that the Lie algebra of infinitesimal symmetries of a system of differential equations can be systematically and readily determined. We are not so much concerned with this aspect of Lie's work and accordingly assume that the symmetry algebra of (2.1) is given. Now let Γ be a finite dimensional Lie subalgebra of the symmetry algebra of (2.1), generated by vector fields

$$V_a = \xi^i_a(x^j)\frac{\partial}{\partial x^i} + \eta^\alpha_a(x^j, u^\beta)\frac{\partial}{\partial u^\alpha}, \tag{2.2}$$

where $a = 1, \ldots, p$. A map $s\colon \mathbf{R}^n \to \mathbf{R}^m$ given by $u^\alpha = s^\alpha(x^i)$ is said to be invariant under the Lie algebra Γ if the graph is invariant under the local flows of the vector fields (2.2). One finds this to be the case if and only if the functions $s^\alpha(x^i)$ satisfy the ***infinitesimal invariance equations***

$$\xi^i_a(x^j)\frac{\partial s^\alpha}{\partial x^i} = \eta^\alpha_a(x^j, s^\beta(x^j)) \tag{2.3}$$

for all $a = 1, 2, \ldots, p$. The method of Lie symmetry reduction consists of explicitly solving the infinitesimal invariance equations (2.3) and substituting the solutions of (2.3) into (2.1) to derive the reduced equations for the Γ invariant solutions.

In order to solve (2.3) it is customarily assumed (see, for example, Olver [*29*], Ovsiannikov [*30*], or Winternitz [*41*]) that the rank of the matrix $\left[\xi^i_a(x^j)\right]$ is constant, say q, and that the Lie algebra of vector fields satisfies the ***local transversality condition***

$$\operatorname{rank}[\xi^i_a(x^j)] = \operatorname{rank}[\xi^i_a(x^j), \eta^\alpha_a(x^j, u^\alpha)]. \tag{2.4}$$

Granted (2.4), it then follows that there exist local coordinates

$$\tilde{x}^r = \tilde{x}^r(x^j), \quad \hat{x}^k = \hat{x}^k(x^j) \quad \text{and} \quad v^\alpha = v^\alpha(x^j, u^\beta), \tag{2.5}$$

on the space of independent and dependent variables, where $r = 1, \ldots, n-q$, $k = 1, \ldots, q$, and $\alpha = 1, \ldots, m$, such that, in these new coordinates, the vector fields V_a take the form

$$V_a = \sum_{l=1}^{q} \hat{\xi}_a^l(\tilde{x}^r, \hat{x}^k) \frac{\partial}{\partial \hat{x}^l}. \tag{2.6}$$

The coordinate functions $\tilde{x}^r$ and v^α are the infinitesimal invariants for the Lie algebra of vector fields Γ. In these coordinates the infinitesimal invariance equations (2.3) for $v^\alpha = v^\alpha(\tilde{x}^r, \hat{x}^k)$ can be explicitly integrated to give $v^\alpha = v^\alpha(\tilde{x}^r)$, where the $v^\alpha(\tilde{x}^r)$ are arbitrary smooth functions. One now inverts the relations (2.5) to find that the explicit solutions to (2.3) are given by

$$s^\alpha(\tilde{x}^r, \hat{x}^k) = u^\alpha(\tilde{x}^r, \hat{x}^k, v^\alpha(\tilde{x}^r)). \tag{2.7}$$

Finally one substitutes (2.7) into the differential equations (2.1) to arrive at the reduced system of differential equations

$$\tilde{\Delta}_\beta(\tilde{x}^r, v^\alpha, v_r^\alpha, v_{rs}^\alpha) = 0. \tag{2.8}$$

Every solution of (2.8) therefore determines, by (2.7), a solution of (2.1) which also satisfies the invariance condition (2.3). In many applications of Lie reduction one picks the Lie algebra of vector fields (2.2) so that $q = n-1$ in which case there is only one independent invariant $\tilde{x}$ on M and (2.8) is a system of ordinary differential equations.

For the vacuum Einstein equations the independent variables x^i, $i = 0, \ldots, 3$, are the local coordinates on a 4 dimensional spacetime, the dependent variables are the 10 components g_{ij} of the spacetime metric and the differential equations (2.1) are given by the vanishing of the Einstein tensor $G^{ij} = 0$. In the case of the spherically symmetric, stationary solutions to the vacuum Einstein equations the relevant infinitesimal symmetry generators on spacetime are $V_0 = \dfrac{\partial}{\partial x^0}$,

$$V_1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}, \qquad V_2 = -x^3 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^3} \quad \text{and} \quad V_3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

and the symmetry conditions, as represented by the Killing equations $\mathcal{L}_{V_a} g_{ij} = 0$, lead to the familiar ansatz (in spherical coordinates)

$$ds^2 = A(r)dt^2 + B(r)dtdr + C(r)dr^2 + D(r)(d\phi^2 + \sin(\phi)^2 d\theta^2). \tag{2.9}$$

The substitution of (2.9) into the field equations leads to a system of ODE whose general solution leads to the Schwarzschild solution to the vacuum Einstein field equations.

What happens if we attempt to derive the Schwarzschild solution using the classical Lie ansatz (2.7)? To begin, it is necessary to lift the vector fields V_a to the space of independent and dependent variables in order to account for the induced action of the infinitesimal spacetime transformations on the components of the metric. These lifted vector fields are $\widehat{V}_0 = V_0$ and

$$\widehat{V}_k = V_k - 2\frac{\partial V_k^l}{\partial x^i} g_{lj}\, \frac{\partial}{\partial g_{ij}}. \tag{2.10}$$

In terms of these lifted vector fields, the infinitesimal invariance equations (2.3) then coincide exactly with the Killing equations. However, (2.7) cannot possibly coincide with (2.9) since the latter contains only 4 arbitrary functions $A(r)$, $B(r)$, $C(r)$, $D(r)$ whereas (2.7) would imply that the general stationary rotationally invariant metric depends upon 10 arbitrary functions of r. This discrepancy is easily accounted for — in this example

$$\operatorname{rank}\big[V_0, V_1, V_2, V_3\big] = 3 \quad \text{while} \quad \operatorname{rank}\big[\hat{V}_0, \hat{V}_1, \hat{V}_2, \hat{V}_3\big] = 4,$$

and hence *the local transversality condition (2.4) does not hold*. Indeed, whenever the local transversality condition fails, the general solution to the infinitesimal invariance equation will depend upon fewer arbitrary functions than the original number of dependent variables. The reduced differential equations will be a system of equations with both fewer independent and dependent variables.

We remark that in many of the exhaustive classifications of invariant solutions using Lie reduction either the number of independent variables is 2 and hence, typically, the number of vector fields V_a is one, or there is just a single dependent variable and (2.1) is a scalar partial differential equation. In either circumstance the local transversality condition is normally satisfied and the ansatz (2.7) gives the correct solution to the infinitesimal invariance equation (2.3). However, once the number of independent and dependent variables exceed these minimal thresholds, as is the case in most physical field theories, the local transversality condition is likely to fail.

3. An Existence Theorem for Invariant Sections. Let M be an n-dimensional manifold and $\pi\colon E \to M$ a bundle over M. In our applications to Lie symmetry reduction the manifold M serves as the space of independent variables and the bundle E plays the role of the total space of independent and dependent variables. We refer to points of M with local coordinates (x^i) and to points of E with local coordinates (x^i, u^α), for which the projection map π is given by $\pi(x^i, u^\alpha) = (x^i)$. In many applications E either is a trivial bundle $E = M \times N$, a vector bundle over M, or a fiber bundle over M with finite dimensional structure group. However, for the purposes of this paper one need only suppose that π is a smooth submersion. We let $E_x = \pi^{-1}(x)$ denote the fiber of E over the point $x \in M$.

Now let G be a finite dimensional Lie group which acts smoothly on E. We assume that G acts projectably on E in the sense that the action of each element of G is a fiber preserving transformation on E — if p, q lie in a common fiber, then so do $g \cdot p$ and $g \cdot q$. Consequently, there is a smooth induced action of G on M. The action of G on the space of sections of E is then given by

$$(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x). \tag{3.1}$$

for each smooth section $s\colon M \to E$.

A section s is invariant if $g \cdot s = s$ for all $g \in G$. More generally, we have the following definition.

Definition 3.1. *Let G be a smooth projectable group action on the bundle $\pi\colon E \to M$ and let $U \subset M$ be open. Then a smooth section $s\colon U \to E$ is G* ***invariant****, if for all $x \in U$ and $g \in G$ such that $g \cdot x \in U$,*

$$s(g \cdot x) = g \cdot s(x). \tag{3.2}$$

Let Γ be the Lie algebra of vector fields on E which are the infinitesimal generators for the action of G on E. Since the action of G is assumed projectable, any basis V_a, $a = 1, \ldots, p$ assumes the local coordinate form (2.2). If g_t is a one-parameter subgroup of G with associated infinitesimal generator V_a on E, then by differentiating the invariance condition $s(g_t \cdot x) = g_t \cdot s(x)$ one finds that the component functions $s^\alpha(x^i)$ satisfy the infinitesimal invariance condition (2.3). If s is globally defined on all of M and if G is connected, then the infinitesimal invariance criterion (2.3) implies (3.2). This may not be true if G is not connected or if s is only defined on a proper open subset of M.

For the purposes of finding group invariant solutions of differential equations, we shall take the group G to be a symmetry group of the given system of differential equations. The task at hand is to explicitly identify the space of G invariant sections of E with sections of an auxiliary bundle $\pi_{\tilde{\kappa}_G} : \tilde{\kappa}_G(E) \to \widetilde{M}$ and to construct the differential equations for the G invariant sections as a reduced system of differential equations on the sections of $\pi_{\tilde{\kappa}_G} : \tilde{\kappa}_G(E) \to \widetilde{M}$.

Our characterization of the G invariant sections of E is based upon the following key observation. Suppose that $p \in E$ and that there is a G invariant section $s : U \to E$ with $s(x) = p$, where $x \in U$. Let $G_x = \{\, g \in G \,|\, g \cdot x = x \,\}$ be the ***isotropy subgroup of*** G ***at*** x. Then, for every $g \in G_x$, we compute

$$g \cdot p = g \cdot s(x) = s(g \cdot x) = s(x) = p. \tag{3.3}$$

This equation shows that the isotropy subgroup G_x constrains the admissible values that an invariant section can assume at the point x. Accordingly, we define the ***kinematic bundle*** $\kappa_G(E)$ ***for the action of*** G ***on*** E by

$$\kappa_G(E) = \bigcup_{x \in M} \kappa_{G,x}(E)$$

where

$$\kappa_{G,x}(E) = \big\{\, p \in E_x \,|\, g\cdot p = p \quad \text{for all} \quad g \in G_x \,\big\}. \tag{3.4}$$

It is easy to check that $\kappa_G(E)$ is a G invariant subset of E and therefore the action of G restricts to an action on $\kappa_G(E)$.

Let $\widetilde{M} = M/G$ and $\tilde{\kappa}_G(E) = \kappa_G(E)/G$ be the quotient spaces for the actions of G on M and $\kappa_G(E)$. We define the ***kinematic reduction diagram for the action of*** G ***on*** E to be the commutative diagram

$$\begin{array}{ccccc}
\tilde{\kappa}_G(E) & \xleftarrow{\ \mathfrak{q}_{\kappa_G}\ } & \kappa_G(E) & \xrightarrow{\ \iota\ } & E \\
\Big\downarrow{\scriptstyle \pi_{\tilde{\kappa}_G}} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
\widetilde{M} & \xleftarrow{\ \mathfrak{q}_M\ } & M & \xrightarrow{\ \text{id}\ } & M.
\end{array} \tag{3.5}$$

In this diagram ι is the inclusion map of the kinematic bundle $\kappa_G(E)$ into E, $\text{id}\colon M \to M$ is the identity map, the maps $\mathfrak{q}_M$ and $\mathfrak{q}_{\kappa_G}$ are the projection maps to the quotient spaces and $\pi_{\tilde{\kappa}_G}$ is the surjective map induced by π. The next lemma summarizes two of the key properties of the kinematic reduction diagram.

Lemma 3.2. *Let G act projectably on E.*

[i] *Let $p \in \kappa_G(E)$ and $g \in G$. If $\pi(g\cdot p) = \pi(p)$, then $g \cdot p = p$.*

[ii] *If $\tilde{p} \in \tilde{\kappa}_G(E)$ and $x \in M$ satisfy $\pi_{\tilde{\kappa}}(\tilde{p}) = \mathfrak{q}_M(x)$, then there is a unique point $p \in \kappa_G(E)$ such that $\mathfrak{q}_{\kappa_G}(p) = \tilde{p}$ and $\pi(p) = x$.*

Proof. **[i]** Let $x = \pi(p)$. If $\pi(g\cdot p) = \pi(p)$, then $g \cdot x = x$ and therefore, since $p \in \kappa_{G,x}(E)$, we conclude that $g \cdot p = p$.

[ii] Since $\mathfrak{q}_{\kappa_G}\colon \kappa_G(E) \to \tilde{\kappa}_G(E)$ is surjective, there is a point $p_0 \in \kappa_G(E)$ which projects to $\tilde{p}$. Let $x_0 = \pi(p_0)$. Then $\mathfrak{q}_M(x_0) = \mathfrak{q}_M(x)$ and hence, by definition of the quotient map $\mathfrak{q}_M$, there is a $g \in G$ such that $g\cdot x_0 = x$. The point $p = g \cdot p_0$ projects under $\mathfrak{q}_{\kappa_G}$ to $\tilde{p}$ and to x under π so that the existence of the point p is established. Suppose p_1 and p_2 are two points in $\kappa_G(E)$ which project to $\tilde{p}$ and x under $\mathfrak{q}_{\kappa_G}$ and π respectively. Then p_1 and p_2 belong to the same fiber $\kappa_{G,x}(E)$ and are related by a group element $g \in G$, that is, $g\cdot p_1 = p_2$. Since $\pi(p_1) = \pi(p_2)$, it follows that $\pi(g\cdot p_1) = \pi(p_1)$. Since $p_1 \in \kappa_{G,x}(E)$, we infer from **[i]** that $g \cdot p_1 = p_1$ and therefore $p_1 = p_2$. ∎

This simple lemma immediately implies that every local section $\tilde{s} : \widetilde{U} \to \tilde{\kappa}_G(E)$, where $\widetilde{U}$ is an open subset of $\widetilde{M}$, uniquely determines a G-invariant section $s\colon U \to \kappa_G(E)$, where $U = \mathfrak{q}_M^{-1}(\widetilde{U})$, such that

$$\mathfrak{q}_{\kappa_G}(s(x)) = \tilde{s}(\mathfrak{q}_M(x)). \tag{3.6}$$

To insure that this correspondence between the G invariant sections of E and the sections of $\tilde{\kappa}_G(E)$ extends to a correspondence between smooth sections it suffices to insure that $\pi_{\tilde{\kappa}_G}\colon \tilde{\kappa}_G(E) \to \widetilde{M}$ is a smooth bundle.

Theorem 3.3. (Existence Theorem for G invariant sections) *Suppose that E admits a kinematic reduction diagram (3.5) such that $\kappa_G(E)$ is an imbedded subbundle of E, the quotient spaces $\widetilde{M}$ and $\tilde{\kappa}_G(E)$ are smooth manifolds, and $\pi_{\kappa_G}\colon \tilde{\kappa}_G(E) \to \widetilde{M}$ is a bundle.*

Let $\widetilde{U}$ be any open set in $\widetilde{M}$ and let $U = \mathfrak{q}_M^{-1}(\widetilde{U})$. Then (3.6) defines a one-to-one correspondence between the G invariant smooth sections $s\colon U \to E$ and the smooth sections $\tilde{s}\colon \widetilde{U} \to \tilde{\kappa}_G(E)$.

We can describe the kinematic reduction diagram in local coordinates as follows. Since $\pi_{\tilde{\kappa}_G}\colon \tilde{\kappa}_G(E) \to \widetilde{M}$ is a bundle we begin with local coordinates $\pi_{\tilde{\kappa}_G}\colon (\tilde{x}^r, v^a) \to (\tilde{x}^r)$ for $\tilde{\kappa}_G(E)$, where $r = 1, \dots, \dim \widetilde{M}$ and a ranges from 1 to the fiber dimension of $\tilde{\kappa}_G(E)$. Since $\mathfrak{q}_M\colon M \to \widetilde{M}$ is a submersion, we can use the coordinates $\tilde{x}^r$ as part of a local coordinate system $(\tilde{x}^r, \hat{x}^k)$ on M. Here $k = 1, \dots, \dim M - \dim \widetilde{M}$ and, for fixed values of $\tilde{x}^r$, the points $(\tilde{x}^r, \hat{x}^k)$ all lie on a common G orbit. As a consequence of Lemma 3.2**[ii]** one can prove that $\mathfrak{q}_{\kappa_G}$ restricts to a diffeomorphism between the fibers of $\kappa_G(E)$ and $\tilde{\kappa}_G(E)$ and hence one can use $(\tilde{x}^r, \hat{x}^k, v^a)$ as a system of local coordinates on $\kappa_G(E)$. Finally, let $(\tilde{x}^r, \hat{x}^k, u^\alpha) \to (\tilde{x}^r, \hat{x}^k)$ be a system of local coordinates on E. Since $\kappa_G(E)$ is an imbedded sub-bundle of E, the inclusion map $\iota\colon \kappa_G(E) \to E$ assumes the form

$$\iota(\tilde{x}^r, \hat{x}^k, v^a) = (\tilde{x}^r, \hat{x}^k, \iota^\alpha(\tilde{x}^r, \hat{x}^k, v^a)), \tag{3.7}$$

where the rank of the Jacobian matrix $\left[\dfrac{\partial \iota^\alpha}{\partial v^a}\right]$ is maximal. In these coordinates the kinematic G reduction diagram (3.5) becomes

$$\begin{array}{ccccc} (\tilde{x}^r, v^a) & \xleftarrow{\mathfrak{q}_{\kappa_G}} & (\tilde{x}^r, \hat{x}^k, v^a) & \xrightarrow{\iota} & (\tilde{x}^r, \hat{x}^k, \iota^\alpha(\tilde{x}^r, \hat{x}^k, v^a)) \\ {\scriptstyle \pi_{\tilde{\kappa}_G}}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ (\tilde{x}^r) & \xleftarrow{\mathfrak{q}_M} & (\tilde{x}^r, \hat{x}^k) & \xrightarrow{\text{id}} & (\tilde{x}^r, \hat{x}^k). \end{array} \tag{3.8}$$

These coordinates are readily constructed in most applications. If $v^a = \tilde{s}^a(\tilde{x}^r)$ is a local section of $\tilde{\kappa}_G(E)$, then the corresponding G invariant section of E is given by

$$s^\alpha(\tilde{x}^r, \hat{x}^k) = \iota^\alpha(\tilde{x}^r, \hat{x}^k, \tilde{s}^a(\tilde{x}^r)). \tag{3.9}$$

Notice that when ι is the identity map, (3.9) reduces to (2.7). *The formula (3.9) is the full and proper generalization of the classical Lie prescription (2.7) for infinitesimally invariant sections of transverse actions.*

In general the fiber dimension of $\kappa_G(E)$ will be less than that of E, while the fiber dimension of $\tilde{\kappa}_G(E)$ is always the same as that of $\kappa_G(E)$. Thus, in our description of the G invariant sections of E, ***fiber reduction***, or reduction in the number of dependent variables, occurs in the right square of the diagram (3.5) while ***base reduction***, or reduction in the number of independent variables, occurs in the left square of (3.5).

We now consider the case of an infinitesimal group action on E, defined directly by a p-dimensional Lie algebra Γ of vector fields (2.2). These vector fields need not be the infinitesimal generators of a global action of a Lie group G on E. If the rank of the coefficient matrix $[\xi^i_a(x^j)]$ is q, then there are locally defined functions $\phi^a_\epsilon(x^j)$, where $\epsilon = 1, \ldots, p-q$, such that

$$\sum_{a=1}^{p} \phi^a_\epsilon(x^j)\xi^i_a(x^j) = 0.$$

Consequently, if we multiply the infinitesimal invariance equation (2.3) by the functions $\phi^a_\epsilon(x^j)$ and sum on $a = 1, \ldots, p$, we find that the invariant sections $s^\alpha(x^j)$ are constrained by the algebraic equations

$$\sum_{a=1}^{p} \phi^a_\epsilon(x^j)\eta^\alpha_a(x^j, s^\beta(x^j)) = 0. \tag{3.10}$$

These conditions are the infinitesimal counterparts to equations (3.3) and accordingly we define the the ***infinitesimal kinematic bundle*** $\kappa_\Gamma(E) = \bigcup_{x\in M} \kappa_{\Gamma,x}(E)$, where

$$\begin{aligned} \kappa_{\Gamma,x}(E) &= \Big\{\, (x^j, u^\beta) \in E_x \mid \sum_{a=1}^{p} \phi^a_\epsilon(x^j)\eta^\alpha_a(x^j, u^\beta) = 0 \,\Big\} \\ &= \Big\{\, p \in E_x \mid Z(p) = 0 \quad \text{for all } Z \in \Gamma \text{ such that } \pi_*(Z(p)) = 0 \,\Big\}. \end{aligned} \tag{3.11}$$

In most applications the algebraic conditions defining $\kappa_\Gamma(E)$ are easily solved. The Lie algebra of vector fields Γ restricts to a Lie algebra of vector fields on $\kappa_\Gamma(E)$ which now satisfies the infinitesimal transversality condition (2.4). One then arrives at (3.8) as a local coordinate description of the infinitesimal kinematic diagram for Γ, where the coordinates $(\tilde{x}^r, v^a)$ are now the infinitesimal invariants for the action of Γ on $\kappa_\Gamma(E)$.

It is not difficult to show that $\kappa_{G,x}(E) \subset \kappa_{\Gamma,x}(E)$, with equality holding whenever the isotropy group G_x is connected.

In the case where E is a vector bundle, the infinitesimal kinematic bundle appears in Fels and Olver [*16*]. For applications of the kinematic bundle to the classification of invariant tensors and spinors see [*6*] and [*7*].

4. Reduced Differential Equations for Group Invariant Solutions. Let G be a Lie group acting projectably on the bundle $\pi\colon E \to M$ and let $\Delta = 0$ be a system of G invariant differential equations for the sections of E. In order to describe geometrically the reduced equations $\tilde{\Delta} = 0$ for the G invariant solutions to $\Delta = 0$ we first formalize the definition of a system of differential equations.

To this end, let $\pi^k\colon J^k(E) \to M$ be the k-th order jet bundle of $\pi\colon E \to M$. A point $\sigma = j^k(s)(x)$ in $J^k(E)$ represents the values of a local section s and all its derivatives to order k at the point $x \in M$. Since G acts naturally on the space of sections of E by (3.1), the action of G on E can be lifted (or prolonged) to an action on $J^k(E)$ by setting

$$g\cdot\sigma = j^k(g\cdot s)(g\cdot x), \quad \text{where } \sigma = j^k(s)(x)$$

Now let $\pi\colon \mathcal{D} \to J^k(E)$ be a vector bundle over $J^k(E)$ and suppose that the Lie group acts projectably on $\mathcal{D}$ in a manner which covers the action of G on $J^k(E)$. A ***differential operator*** is a section $\Delta : J^k(E) \to \mathcal{D}$. The differential operator Δ is G invariant if it is invariant in the sense of Definition 3.1, that is,

$$g\cdot\Delta(\sigma) = \Delta(g\cdot\sigma)$$

for all $g \in G$ and all points $\sigma \in J^k(E)$. A section s of E defined on an open set $U \subset M$ is a solution to the differential equations $\Delta = 0$ if $\Delta(j^k(s)(x)) = 0$ for all $x \in U$.

Typically, the bundle $\mathcal{D} \to J^k(E)$ is defined as the pullback bundle of a vector bundle V (on which G acts) over E or M by the projections $\pi^k\colon J^k(E) \to E$ or $\pi^k_M\colon J^k(E) \to M$ and the action of G on $\mathcal{D}$ is the action jointly induced from $J^k(E)$ and V.

Our goal now is to construct a bundle $\tilde{\mathcal{D}} \to J^k(\tilde\kappa_G(E))$ and a differential operator $\tilde\Delta\colon J^k(\tilde\kappa_G(E)) \to \tilde{\mathcal{D}}$ such that the correspondence (3.6) restricts to a 1-1 correspondence between the G invariant solutions of $\Delta = 0$ and the solutions of $\tilde\Delta = 0$.

One might anticipate that the required bundle $\tilde{\mathcal{D}} \to J^k(\tilde\kappa_G(E))$ can be constructed by a direct application of kinematic reduction to $\mathcal{D} \to J^k(E)$. However, one can readily check that the quotient space of $J^k(E)$ by the prolonged action of G does *not* in general coincide with the jet space $J^k(\tilde\kappa_G(E))$ so that the kinematic reduction diagram for the action of G on $\mathcal{D}$ will not lead to a bundle over $J^k(\tilde\kappa_G(E))$. For example, if G is the group acting on $M \times \mathbf{R} \to M$ by rotations in the base $M = \mathbf{R}^2 - \{(0,0)\}$, then $J^2(E)/G$ is a 7 dimensional manifold whereas $J^2(\tilde\kappa_G(E))$ is 4 dimensional. This difficulty is easily circumvented by introducing the ***bundle of invariant*** $\boldsymbol{k}$***-jets***

$$\operatorname{Inv}^k(E) = \{\,\sigma \in J^k(E) \mid \sigma = j^k(s)(x_0), \text{ where s is a } G \text{ invariant section defined in a neighborhood of } x_0\,\}. \tag{4.1}$$

This bundle is studied in Olver [*29*] although the importance of these invariant jet spaces to the general theory of symmetry reduction of differential equations is not as widely acknowledged in the literature as it should be.

The quotient space $\mathrm{Inv}^k(E)/G$ coincides with the jet space $J^k(\tilde{\kappa}_G(E))$. We let $\mathcal{D}_{\mathrm{Inv}} \to \mathrm{Inv}^k(E)$ be the restriction of $\mathcal{D}$ to the bundle of invariant k-jets and to this we now apply our reduction procedure to arrive at the ***dynamic reduction diagram***

$$
\begin{array}{ccccccc}
\tilde{\kappa}_G(\mathcal{D}_{\mathrm{Inv}}) & \xleftarrow{\ \mathfrak{q}\ } & \kappa_G(\mathcal{D}_{\mathrm{Inv}}) & \xrightarrow{\ \iota\ } & \mathcal{D}_{\mathrm{Inv}} & \xrightarrow{\ \iota_{\mathrm{Inv}}\ } & \mathcal{D} \\
{\scriptstyle\tilde{\pi}}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\
J^k(\tilde{\kappa}_G(E)) & \xleftarrow{\ \mathfrak{q}_{\mathrm{Inv}}\ } & \mathrm{Inv}^k(E) & \xrightarrow{\ \mathrm{id}\ } & \mathrm{Inv}^k(E) & \xrightarrow{\ \iota^k\ } & J^k(E).
\end{array} \tag{4.2}
$$

Theorem 3.3 insures that there is a one-to-one correspondence between the G invariant sections of $\mathcal{D}_{\mathrm{Inv}} \to \mathrm{Inv}^k(E)$ and the sections of $\tilde{\kappa}_G(\mathcal{D}_{\mathrm{Inv}}) \to J^k(\tilde{\kappa}_G(E))$. Any G invariant differential operator $\Delta\colon J^k(E) \to \mathcal{D}$ restricts to a G invariant differential operator $\Delta_{\mathrm{Inv}}\colon \mathrm{Inv}^k(E) \to \mathcal{D}_{\mathrm{Inv}}$ and thus determines a differential operator $\tilde{\Delta}\colon J^k(\tilde{\kappa}_G(E)) \to \tilde{\kappa}_G(\mathcal{D})$. This is the reduced differential operator whose solutions describe the G invariant solutions for the original operator Δ.

To describe diagram (4.2) in local coordinates, we begin with the coordinate description (3.8) of the kinematic reduction diagram and we let

$$(\tilde{x}^r, \hat{x}^k, u^\alpha, u^\alpha_r, u^\alpha_k, u^\alpha_{rs}, u^\alpha_{rk}, u^\alpha_{kl}, \dots)$$

denote the standard jet coordinates on $J^k(E)$. Since the invariant sections are parameterized by functions $v^a = v^a(\tilde{x}^r)$, coordinates for $\mathrm{Inv}^k(E)$ are

$$(\tilde{x}^r, \hat{x}^k, v^a, v^a_r, v^a_{rs}, \dots)$$

In accordance with (3.9), the inclusion map

$$\iota\colon \mathrm{Inv}^k(E) \to J^k(E)$$

is given by

$$\iota(\tilde{x}^r, \hat{x}^k, v^a, v^a_r, v^a_{rs}, \dots) = (\tilde{x}^r, \hat{x}^k, u^\alpha, u^\alpha_r, u^\alpha_k, u^\alpha_{rs}, u^\alpha_{rk}, u^\alpha_{kl}, \dots), \tag{4.3}$$

where by a formal application of the chain rule,

$$u^\alpha = \iota^\alpha(\tilde{x}^r, \hat{x}^i, v^a), \qquad u^\alpha_r = \frac{\partial \iota^\alpha}{\partial \tilde{x}^r} + \frac{\partial \iota^\alpha}{\partial v^a} v^a_r, \qquad u^\alpha_k = \frac{\partial \iota^\alpha}{\partial \hat{x}^k},$$

$$u^\alpha_{rs} = \frac{\partial^2 \iota^\alpha}{\partial \tilde{x}^r \partial \tilde{x}^s} + \frac{\partial^2 \iota^\alpha}{\partial v^a \partial \tilde{x}^s} v^a_r + \frac{\partial^2 \iota^\alpha}{\partial v^a \partial \tilde{x}^r} v^a_s + \frac{\partial^2 \iota^\alpha}{\partial v^a \partial v^b} v^a_r v^b_s + \frac{\partial \iota^\alpha}{\partial v^a} v^a_{rs},$$

and so on. The quotient map

$$\mathfrak{q}_{\mathrm{Inv}}\colon \mathrm{Inv}^k(E) \to J^k(\tilde{\kappa}_G(E))$$

is given simply by

$$\mathfrak{q}_{\text{Inv}}(\tilde{x}^r, \hat{x}^k, v^a, v^a_r, v^a_{rs}, \dots) = (\tilde{x}^r, v^a, v^a_r, v^a_{rs}, \dots).$$

Next let $\mathbf{f}^A$ be a local frame field for the vector bundle $\mathcal{D}$. The differential operator $\Delta\colon J^k(E) \to \mathcal{D}$ can be written in terms of the standard coordinates on $J^k(E)$ and in this local frame by

$$\Delta = \Delta_A(\tilde{x}^r, \hat{x}^k, u^\alpha, u^\alpha_r, u^\alpha_k, u^\alpha_{rs}, u^\alpha_{rk}, u^\alpha_{kl}, \dots)\, \mathbf{f}^A. \tag{4.4}$$

The restriction of Δ to $\text{Inv}^k(E)$ defines the section $\Delta_{\text{Inv}}\colon \text{Inv}^k(E) \to \mathcal{D}_{\text{Inv}}$ by

$$\Delta_{\text{Inv}} = \Delta_{\text{Inv},A}(\tilde{x}^r, \hat{x}^k, v^a, v^a_r, v^a_{rs}, \dots)\, \mathbf{f}^A, \tag{4.5}$$

where the component functions $\Delta_{\text{Inv},A}(\tilde{x}^r, \hat{x}^k, v^a, v^a_r, v^a_{rs}, \dots)$ are defined as the composition of the maps (4.3) and the component maps Δ_A. Since Δ is a G invariant differential operator, Δ_{Inv} is a G invariant differential operator and hence necessarily factors through the kinematic bundle $\kappa_G(\mathcal{D}_{\text{Inv}})$,

$$\Delta_{\text{Inv}}\colon \text{Inv}^k(E) \to \kappa_G(\mathcal{D}_{\text{Inv}}).$$

Our general existence theory for invariant sections implies that we can also find a locally defined, G invariant frame $\mathbf{f}^Q_{\text{Inv}}$ for $\kappa_G(\mathcal{D}_{\text{Inv}})$. The inclusion map $\kappa_G(\mathcal{D}_{\text{Inv}}) \to \mathcal{D}_{\text{Inv}}$ is represented by writing each vector $\mathbf{f}^Q_{\text{Inv}}$ as a linear combination of the vectors $\mathbf{f}^A$,

$$\mathbf{f}^Q_{\text{Inv}} = M^Q_A \mathbf{f}^A,$$

where the coefficients M^Q_A are functions on $\text{Inv}^k(E)$. The invariant operator Δ_{Inv} can be expressed as

$$\Delta_{\text{Inv}} = \Delta_{\text{Inv},Q}(\tilde{x}^r, \hat{x}^k, v^a, v^a_r, v^a_{rs}, \dots)\, \mathbf{f}^Q_{\text{Inv}}.$$

Finally, the G invariant frame $\mathbf{f}^Q_{\text{Inv}}$ determines a frame $\tilde{\mathbf{f}}^Q$ on $\tilde{\kappa}_G(E)$, the invariance of Δ implies that the component functions $\Delta_{\text{Inv},Q}$ are necessarily independent of the parametric variables $\hat{x}^k$, that is,

$$\tilde{\Delta}_Q(\tilde{x}^r, v^a, v^a_r, v^a_{rs}, \dots) = \Delta_{\text{Inv},Q}(\tilde{x}^r, \hat{x}^k,\ v^a, v^a_r, v^a_{rs}, \dots)$$

and the reduced differential operator is

$$\tilde{\Delta} = \tilde{\Delta}_Q(\tilde{x}^r, v^a, v^a_r, v^a_{rs}, \dots)\, \tilde{\mathbf{f}}^Q.$$

At first sight, this general framework may appear to be rather cumbersome and overly complicated. However, as we shall see in examples, every square in the dynamic reduction diagram (4.2) actually corresponds to the individual steps that one performs in practice.

5. The Automorphism Group of the Kinematic Bundle. Let $\mathfrak{G}$ be the full group of projectable symmetries on E for a given system of differential equations on $J^k(E)$ and let $G \subset \mathfrak{G}$ be a fixed subgroup for which the group invariant solutions are sought. It is commonly noted (again, within the context of reduction with transversality) that $\operatorname{Nor}(G, \mathfrak{G})$, the normalizer of G in $\mathfrak{G}$, preserves the space of invariant sections and that $\operatorname{Nor}(G, \mathfrak{G})/G$ is a symmetry group of the reduced equations. However, because this is a purely algebraic construction which does not take into account the action of $\mathfrak{G}$ on E, this construction may not yield the largest possible residual symmetry group or may result in a residual group which does not act effectively on $\tilde{\kappa}_G(E)$. These difficulties are easily resolved. We let $\mathcal{O}_p(G)$ denote orbit of G through a point $p \in E$.

Definition 5.1. *Let $\mathfrak{G}$ be a group of fiber-preserving transformations acting on $\pi\colon E \to M$ and let G be a subgroup of $\mathfrak{G}$. Assume that E admits a kinematic reduction diagram (3.5) for the action of G on E.*

[i] *The* ***automorphism group*** $\tilde{\mathfrak{G}}$ *for the kinematic bundle $\pi\colon \kappa_G(E) \to M$ is the subgroup of $\mathfrak{G}$ which stabilizes the set of all the G orbits in $\kappa_G(E)$, that is,*

$$\tilde{\mathfrak{G}} = \left\{ a \in \mathfrak{G} \mid a \cdot \mathcal{O}_p(G) = \mathcal{O}_{a \cdot p}(G) \text{ and } a^{-1} \cdot \mathcal{O}_p(G) = \mathcal{O}_{a^{-1} \cdot p}(G) \text{ for all } p \in \kappa_G(E) \right\}. \tag{5.1}$$

[ii] *The* ***global isotropy subgroup*** *of $\mathfrak{G}$, as it acts on the space of G orbits of $\kappa_G(E)$, is*

$$\tilde{\mathfrak{G}}^* = \left\{ a \in \mathfrak{G} \mid a \cdot \mathcal{O}_p(G) = \mathcal{O}_p(G) \quad \text{for all} \quad p \in \kappa_G(E) \right\}. \tag{5.2}$$

[iii] *The* ***residual symmetry group*** *is* $\tilde{\mathfrak{G}}_{\text{eff}} = \tilde{\mathfrak{G}}/\tilde{\mathfrak{G}}^*$.

The key property of $\tilde{\mathfrak{G}}^*$ is that it is the largest subgroup of $\mathfrak{G}$ with exactly the same reduction diagram and invariant sections as G. This is an important interpretation of the group $\tilde{\mathfrak{G}}^*$ — from the viewpoint of kinematic reduction, one should generally replace the group G by the group $\tilde{\mathfrak{G}}^*$. For computational purposes, it is often advantageous to use the fact that $\tilde{\mathfrak{G}}^*$ fixes every G invariant section of E. It is not difficult to check that $\operatorname{Nor}(\tilde{\mathfrak{G}}^*, \mathfrak{G}) = \tilde{\mathfrak{G}}$, that the quotient group $\tilde{\mathfrak{G}}_{\text{eff}} = \tilde{\mathfrak{G}}/\tilde{\mathfrak{G}}^*$ acts effectively and projectably on the reduced bundle $\tilde{\kappa}_G(E) \to \widetilde{M}$ and that, if $\mathfrak{G}$ is a symmetry group of of a differential operator Δ, then $\tilde{\mathfrak{G}}_{\text{eff}}$ is always a symmetry group of the reduced differential operator $\tilde{\Delta}$.

Similarly, if $\mathcal{G}$ is a Lie algebra of projectable vector fields on E and $\Gamma \subset \mathcal{G}$, we define the ***infinitesimal automorphism algebra*** of $\kappa_\Gamma(E)$ as the Lie subalgebra of vector fields given by

$$\tilde{\mathcal{G}} = \left\{ Y \in \mathcal{G} \mid [\, Z, Y\,]_p \in \operatorname{span}(\Gamma)(p) \quad \text{for all } p \in \kappa_\Gamma(E) \text{ and all } Z \in \Gamma \right\}, \tag{5.3}$$

and the associated isotropy subalgebra for $\kappa_\Gamma(E)$

$$\tilde{\mathcal{G}}^* = \big\{ Y \in \mathcal{G} \,|\, Y_p \in \operatorname{span}(\Gamma)(p) \quad \text{for all } p \in \kappa_\Gamma(E) \,\big\}. \tag{5.4}$$

When $\mathfrak{G}$ is a finite dimensional Lie group and $\mathcal{G} = \Gamma(\mathfrak{G})$, then it is readily checked that $\tilde{\mathcal{G}} = \Gamma(\tilde{\mathfrak{G}})$ and $\tilde{\mathcal{G}}^* = \Gamma(\tilde{\mathfrak{G}}^*)$.

Since the automorphism group $\tilde{\mathfrak{G}}$ acts on the k-jets of invariant sections $\operatorname{Inv}^k_G(E)$, this group also plays an important role in dynamic reduction. Specifically, let us suppose that $\mathfrak{G}$ acts on the vector bundle $\mathcal{D} \to J^k(E)$ and that $\Delta\colon J^k(E) \to \mathcal{D}$ is a $\mathfrak{G}$ invariant section. Then $\Delta_{\text{Inv}}\colon \operatorname{Inv}^k(E) \to \mathcal{D}_{\text{Inv}}$ is always invariant under the action of $\tilde{\mathfrak{G}}$ and accordingly the operator Δ_{Inv} always factors through the kinematic bundle for the action of $\tilde{\mathfrak{G}}$ on $\mathcal{D}_{\text{Inv}}$, where for $\sigma \in \operatorname{Inv}^k(E)$,

$$\kappa_{\tilde{\mathfrak{G}},\sigma}(\mathcal{D}_{\text{Inv}}) = \big\{\, \Delta \in \mathcal{D}_{\text{Inv},\sigma} \,|\, g\cdot\Delta = \Delta \quad \text{for all} \quad g \in \tilde{\mathfrak{G}}_\sigma \big\}.$$

We note that

$$\kappa_{\tilde{\mathfrak{G}}}(\mathcal{D}_{\text{Inv}}) \subset \kappa_G(\mathcal{D}_{\text{Inv}})$$

and consequently one can refine the dynamic reduction diagram from (4.2) to

$$\begin{array}{ccccccc}
\tilde{\kappa}_{\tilde{\mathfrak{G}}}(\mathcal{D}_{\text{Inv}}) & \xleftarrow{\ \mathfrak{q}\ } & \kappa_{\tilde{\mathfrak{G}}}(\mathcal{D}_{\text{Inv}}) & \xrightarrow{\ \iota\ } & \mathcal{D}_{\text{Inv}} & \xrightarrow{\ \iota_{\text{Inv}}\ } & \mathcal{D} \\
\pi\big\downarrow & & \pi\big\downarrow & & \pi\big\downarrow & & \big\downarrow\pi \\
J^k(\tilde{\kappa}_G(E)) & \xleftarrow{\ \mathfrak{q}_{\text{Inv}}\ } & \operatorname{Inv}^k(E) & \xrightarrow{\ \text{id}\ } & \operatorname{Inv}^k(E) & \xrightarrow{\ \iota^k\ } & J^k(E),
\end{array}$$

where the quotient maps to the left are still by the action of G.

Given the actions of G on $\pi\colon E \to M$ and also $\mathfrak{G}$ on $\mathcal{D} \to J^k(E)$, it sometimes happens that

$$\kappa_{\tilde{\mathfrak{G}},\sigma}(\mathcal{D}_{\text{Inv}}) = 0. \tag{5.5}$$

In this case every G invariant section of E is automatically a solution to $\Delta = 0$ for every $\mathfrak{G}$ invariant operator $\Delta\colon J^k(E) \to \mathcal{D}$ — such sections are called ***universal solutions.*** Previous work on this subject (see Bleecker [*8*], [*9*], Gaeta and Morando [*19*]) have emphasized a variational approach which, from the viewpoint of the dynamic reduction diagram and the automorphism group of the kinematic reduction diagram, may not always be necessary.

6. Examples. In this section we find the kinematic and dynamic reduction diagrams for the group invariant solutions for some well-known systems of differential equations in applied mathematics, differential geometry, and mathematical physics. We begin by deriving the rotationally invariant solutions of the Euler equations for incompressible fluid flow. As noted by Olver [*29*] (p. 199), these solutions cannot be obtained by the classical Lie ansatz. The general theory of symmetry reduction without transversality leads to some interesting new classification problems for group invariant solutions which we briefly illustrate by presenting another reduction of the Euler equations.

In our second set of examples we consider reductions of the harmonic map equations. We show the classic Veronese map from $S^2 \to S^4$ is an example of a universal solution. In Example 5.4 we consider another symmetry reduction of the harmonic map equation which nicely illustrates the construction of the reduced kinematic space for quotient manifolds $\widetilde{M}$ with boundary.

In our third set of examples, the Schwarzschild and plane wave solutions of the vacuum Einstein equations are re-examined in the context of symmetry reduction without transversality. We demonstrate the importance of the automorphism group in understanding the geometric properties of the kinematic bundle and, as well, qualitative features of the reduced equations.

Finally, some elementary examples from mechanics are used to demonstrate the basic differences between symmetry reduction for group invariant solutions and symplectic reduction of Hamiltonian systems.

Although space does not permit us to do so, the kinematic and dynamic reduction diagrams are also nicely illustrated by symmetry reduction of the Yang-Mills equations as found, for example, in [*22*], [*25*], [*27*]. In particular, it is interesting to note that the invariance properties of the classical instanton solution to the Yang-Mills equations (Jackiw and Rebbi [*24*]) imply that it is a universal solution in the sense of equation (5.5).

EULER EQUATIONS FOR INCOMPRESSIBLE FLUID FLOW

The Euler equations are a system of 4 first order equations in 4 independent and dependent variables. The underlying bundle E for these equations is the trivial bundle $\mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^4$ with coordinates $(t, \mathbf{x}, \mathbf{u}, p) \to (t, \mathbf{x})$, where $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{u} = (u^1, u^2, u^3)$ and the equations are

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0. \tag{6.1}$$

The full symmetry group $\mathfrak{G}$ of the Euler equations is well-known (see, for example, [*23*], [*29*], [*34*])

Example 6.1. ROTATIONALLY INVARIANT SOLUTIONS OF THE EULER EQUATIONS. The symmetry group of the Euler equations contains the group $G = \mathbf{SO}(3)$, acting on E, by

$$R \cdot (t, \mathbf{x}, \mathbf{u}, p) = (t, R \cdot \mathbf{x}, R \cdot \mathbf{u}, p) = (t, R^i_j x^j, R^i_j u^j, p), \tag{6.2}$$

for $R = (R^i_j) \in \mathbf{SO}(3)$. To insure that the action of G on the base $\mathbf{R}^4$ is regular we restrict to the open set $M \subset \mathbf{R}^4$ where $||\mathbf{x}|| \neq 0$. The infinitesimal generators for this action are

$$V_k = \varepsilon_{kij} x^i \frac{\partial}{\partial x^j} + \varepsilon_{kij} u^i \frac{\partial}{\partial u^j}. \tag{6.3}$$

We first construct the kinematic reduction diagram for this action. For a given point $x_0 = (t_0, \mathbf{x}_0) \in M$, the isotropy subgroup G_{x_0} for the action of G on M is the subgroup $\mathbf{SO}(2)_{\mathbf{x}_0} \subset \mathbf{SO}(3)$ which fixes the vector $\mathbf{x}_0$ in $\mathbf{R}^3$. Since the only vectors invariant under all rotations about a given axis of rotation are vectors along the axis of rotation, we deduce that for $x_0 \in M$,

$$\begin{aligned}\kappa_{G,x_0}(E) &= \{\, (t_0, \mathbf{x}_0, \mathbf{u}, p) \,|\, R \cdot \mathbf{u} = \mathbf{u} \quad \text{for all} \quad R \in \mathbf{SO}(2)_{\mathbf{x}_0} \ \}\\ &= \{\, (t_0, \mathbf{x}_0, \mathbf{u}, p) \,|\, \mathbf{u} = A\mathbf{x}_0 \quad \text{for some} \quad A \in \mathbf{R}\,\}.\end{aligned}$$

The same conclusion can be obtained by infinitesimal considerations. Indeed, the infinitesimal isotropy vector field at x_0 for the action on M is

$$Z = x_0^k \varepsilon_{kij} x^i \frac{\partial}{\partial x^j}$$

and therefore, if $(t, \mathbf{x}, \mathbf{u}, p) \in \kappa_{G,x}(E)$, we must have, by (3.11),

$$x^k \varepsilon_{kij} u^i \frac{\partial}{\partial u^j} = 0.$$

This implies that $\mathbf{x} \times \mathbf{u} = 0$ and so $\mathbf{u}$ is parallel to $\mathbf{x}$.

Either way, we conclude that $\kappa_G(E)$ is a two dimensional trivial bundle $(t, \mathbf{x}, A, B) \to (t, \mathbf{x})$, where the inclusion map $\iota\colon \kappa_G(E) \to E$ is

$$\iota(t, \mathbf{x}, A, B) = (t, \mathbf{x}, \mathbf{u}, p), \qquad \text{where} \qquad \mathbf{u} = A\mathbf{x} \quad \text{and} \qquad p = B.$$

The invariants for the action of G on M are t and $r = \sqrt{x^2 + y^2 + z^2}$ so that the kinematic reduction diagram for the action of $\mathbf{SO}(3)$ on E is

$$\begin{array}{ccccc} (t, r, A, B) & \xleftarrow{\mathfrak{q}_{\kappa_G}} & (t, \mathbf{x}, A, B) & \xrightarrow{\ \iota\ } & (t, \mathbf{x}, \mathbf{u}, p) \\ {\scriptstyle \pi_{\tilde\kappa_G}}\downarrow & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ (t, r) & \xleftarrow{\mathfrak{q}_M} & (t, \mathbf{x}) & \xrightarrow[\text{id}]{} & (t, \mathbf{x}). \end{array} \tag{6.4}$$

In accordance with equation (3.9), each section $A = A(t, r)$ and $B = B(t, r)$ of $\tilde\kappa_G(E)$ determines the rotationally invariant section

$$\mathbf{u} = A(r, t)\, \mathbf{x} \quad \text{and} \quad p = B(r, t) \tag{6.5}$$

of E.

The computation of the reduced equations for the rotationally invariant solutions to the Euler equations now proceeds as follows. From (6.5) we compute

$$u^i_t = A_t x^i, \quad u^i_j = A\delta^i_j + A_r \frac{x^i x_j}{r} \quad \text{and} \quad p_i = B_r \frac{x^i}{r} \tag{6.6}$$

so that the Euler equations (6.1) become

$$A_t x^i + A x^j \big(A\delta^i_j + A_r \frac{x^i x_j}{r}\big) = -B_r \frac{x^i}{r} \quad \text{and} \quad 3A + rA_r = 0 \tag{6.7}$$

which simplify to the differential equations

$$A_t + A(A + rA_r) = -\frac{B_r}{r} \quad \text{and} \quad 3A + rA_r = 0 \tag{6.8}$$

on $J^1(\tilde{\kappa}_G(E))$. These equations are readily integrated to give

$$A = \frac{a}{r^3} \quad \text{and} \quad B = \frac{\dot a}{r} - \frac{a^2}{2r^4} + b \tag{6.9}$$

for arbitrary functions $a(t)$ and $b(t)$ and the rotationally invariant solutions to the Euler equations are

$$\mathbf{u} = \frac{a}{r^3}\mathbf{x} \quad \text{and} \quad p = \frac{\dot a}{r} - \frac{a^2}{2r^4} + b.$$

We note that for the Lie algebra of vector fields (6.3), the matrix on the right side of (2.4), namely

$$\begin{bmatrix} 0 & -x^3 & x^2 & 0 & -u^3 & u^2 \\ x^3 & 0 & -x^1 & u^3 & 0 & -u^1 \\ -x^2 & x^1 & 0 & -u^2 & u^1 & 0 \end{bmatrix}$$

has full rank 3 whereas the matrix on the left side of (2.4), consisting of the first three columns of (2.4), has rank 2. The local transversality condition (2.4) fails and the solution (6.9) to the Euler equations *cannot* be obtained using the classical Lie prescription.

To describe the derivation of the reduced equations in the context of invariant differential operators and the dynamic reduction diagram we introduce the bundle $\mathcal{D} = J^1(E) \times \mathbf{R}^3 \times \mathbf{R}$ with sections $\dfrac{\partial}{\partial u^i} \otimes dt$ and dt and define the differential operator Δ on $\mathcal{D}$ by

$$\Delta = [u^i_t + u^k u^i_k + \delta^{ij} p_j]\,\frac{\partial}{\partial u^i} \otimes dt + [u^i_i]\,dt. \tag{6.10}$$

This operator is invariant under the full symmetry group of the Euler equations. The induced action of $G = \mathbf{SO}(3)$ on $J^1(E)$ is given by

$$R \cdot (t, x^i, u^i, p, u^i_j, p_j) = (t, R^i_r x^r, R^i_r u^r, p, R^i_r R^s_j u^r_s, R^r_s p_r) \quad \text{where } R \in \mathbf{SO}(3).$$

Coordinates for the bundle of invariant jets $\text{Inv}^1(E)$ are $(t, x^i, A, A_t, A_r, B, B_t, B_r)$ and (6.6) defines the inclusion map $\iota : \text{Inv}^1(E) \to J^1(E)$. A basis for the G invariant sections of $\mathcal{D}_{\text{Inv}} \to \text{Inv}^1(E)$ is given by

$$\mathbf{f}^1 = x^i \frac{\partial}{\partial u^i} \otimes dt \quad \text{and} \quad \mathbf{f}^2 = dt.$$

Let $\tilde{\mathbf{f}}^1$ and $\tilde{\mathbf{f}}^2$ be the corresponding sections of $\tilde{\kappa}_G(\mathcal{D}_{\text{Inv}})$.

We are now ready to work though the dynamic reduction diagram (4.2), starting with the Euler operator as a section $\Delta\colon J^1(E) \to \mathcal{D}$. Restricted to the invariant jet bundle $\text{Inv}^1(E)$, Δ becomes

$$\Delta_{\text{Inv}} = [A_t x^i + A x^j \big(A\delta^i_j + A_r \frac{x^i x_j}{r}\big) + B_r \frac{x^i}{r}]\, \frac{\partial}{\partial u^i} \otimes\, dt + [\delta^j_i (A\delta^i_j + A_r \frac{x^i x_j}{r})]\, dt.$$

Restricting Δ to $\text{Inv}^1(E)$ is precisely the first step one takes in practice in computing the reduced equations and corresponds to the right most square in the dynamic reduction diagram.

Next, because Δ_{Inv} is G invariant it is necessarily a linear combination of the two invariant sections $\mathbf{f}^1$ and $\mathbf{f}^2$ and therefore factors though the kinematic bundle $\kappa_G(\mathcal{D}_{\text{Inv}})$. This means we can write Δ_{Inv} as a section of $\kappa_G(\mathcal{D}_{\text{Inv}})$, namely,

$$\Delta_{\text{Inv}} = [A_t + A\big(A + A_r \frac{x^j x_j}{r}\big) + \frac{1}{r} B_r]\, \mathbf{f}^1 + [3A + A_r(\delta^j_i \frac{x^i x_j}{r})]\, \mathbf{f}^2$$

This corresponds to the center commutative square in the dynamic reduction diagram (4.2) and coincides with the fact that equation (6.7) contained a common factor x^i – a common factor which insures that the time evolution equation for $\mathbf{u}$ reduces to a single time evolution equation for A.

Finally, as a G invariant section of $\kappa_G(\mathcal{D}_{\text{Inv}})$, a bundle on which G always acts transversally, we are assured that Δ_{Inv} descends to a differential operator $\tilde{\Delta}$ on the bundle $J^1(\tilde{\kappa}_G(E))$. In this example this implies that the independent variables (t, x^i) appear only though the invariants for the action of G on M, in this case t and r, and so

$$\tilde{\Delta} = [A_t + A(A + rA_r) + \frac{B_r}{r}]\, \tilde{\mathbf{f}}^1 + [3A + rA_r]\, \tilde{\mathbf{f}}^2. \qquad \blacksquare$$

Example 6.2. A new reduction of the Euler equations. It is possible to give a complete classification of all possible symmetry reductions of the Euler equations (6.1) to a system of ordinary differential equations in three or fewer dependent variables [*17*]. A number of authors have obtained complete lists of reductions of various differential equations (see, for example, [*14*],[*18*], [*21*], [*42*]) but this particular classification of reductions of the Euler equations may be the first such classification of group invariant solutions which explicitly requires non-trivial isotropy in the group action on the space of independent variables. There are too many cases to list the results of this classification here, but we do present one more reduction of the Euler equations, one which does not seem to appear elsewhere in the literature.

For this example it will be convenient to write $\mathbf{x} = (x, y, z)$ and $\mathbf{u} = (u, v, w)$. The infinitesimal generators for the group action are $\Gamma = \{\, V_0, V_1, V_2 = V_{x,\alpha} + V_{y,\beta}, V_3 = V_{y,\alpha} - V_{x,\beta}\,\}$, where

$$V_0 = x\partial_x + y\partial_y + z\partial_z + u\partial_u + v\partial_v + w\partial_w + 2p\partial_p, \qquad V_1 = y\partial_x - x\partial_y + v\partial_u - u\partial_v,$$

$$V_{x,\alpha} = \alpha\partial_x + \dot{\alpha}\partial_u - x\ddot{\alpha}\partial_p, \qquad \text{and} \qquad V_{y,\beta} = \beta\partial_y + \dot{\beta}\partial_v - y\ddot{\beta}\partial_p,$$

Here $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are such that $\alpha\ddot{\beta} - \ddot{\alpha}\beta = 0$, or equivalently,

$$\alpha\dot{\beta} - \beta\dot{\alpha} = c = \text{constant}. \tag{6.11}$$

This condition insures that $[V_2, V_3] = 0$ so that Γ is indeed a finite dimensional Lie algebra of vector fields. In order that Γ have constant rank on the base space, we assume that $xy\alpha \neq 0$ or $yz\beta \neq 0$.

The horizontal components of V_2 and V_3 are given by

$$\begin{bmatrix} V_2^M \\ V_3^M \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix}, \quad \text{so that} \quad \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} = \frac{1}{\delta} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} V_2^M \\ V_3^M \end{bmatrix},$$

where $\delta = \alpha^2 + \beta^2$, and therefore at the point $(t_0, \mathbf{x}_0)$, the horizontal components of the vector field

$$Z = V_1 - y_0 \frac{\alpha(t_0)V_2 - \beta(t_0)V_3}{\delta(t_0)} + x_0 \frac{\beta(t_0)V_2 + \alpha(t_0)V_3}{\delta(t_0)} \tag{6.12}$$

vanish. The isotropy condition (3.11) defining the fiber of the kinematic bundle $\kappa_{\Gamma,x}(E)$ leads, from the coefficients of ∂_u, ∂_v and ∂_p, to the relations

$$v = \frac{y\alpha - x\beta}{\delta}\dot{\alpha} + \frac{x\alpha + y\beta}{\delta}\dot{\beta} \quad \text{and} \quad u = \frac{x\alpha + y\beta}{\delta}\dot{\alpha} + \frac{x\beta - y\alpha}{\delta}\dot{\beta}. \tag{6.13}$$

We therefore conclude that the kinematic bundle has fiber dimension 2 with fiber coordinates w and p. However, these coordinates are not invariant under the action of Γ on $\kappa_\Gamma(E)$ and cannot be used in the local coordinate description (3.8) of the kinematic reduction diagram. Restricted to $\kappa_\Gamma(E)$, the vector fields V_i become

$$V_0' = x\partial_x + y\partial_y + z\partial_z + w\partial_w + 2p\partial_p, \qquad V_1' = y\partial_x - x\partial_y,$$

$$V_2' = \alpha\partial_x + \beta\partial_y - (\ddot{\alpha}x + \ddot{\beta}y)\partial_p, \quad \text{and} \qquad V_3' = -\beta\partial_x + \alpha\partial_y - (-\ddot{\beta}x + \ddot{\alpha}y)\partial_p.$$

Note that these restricted vector fields now satisfy the infinitesimal transversality condition (2.4). Invariants for this action are t,

$$A = \frac{w}{z} \quad \text{and} \quad B = \big(2p + \frac{\alpha\ddot{\alpha} + \beta\ddot{\beta}}{\delta}(x^2 + y^2)\big)/z^2. \tag{6.14}$$

To verify that B satisfies $V_2'(B) = V_3'(B) = 0$ one must use $\alpha\ddot{\beta} = \ddot{\alpha}\beta$. The kinematic reduction diagram for the action of Γ on E is therefore

$$\begin{array}{ccccc}
(t, A, B) & \xleftarrow{\ \mathfrak{q}_{\kappa_\Gamma}\ } & (t, \mathbf{x}, A, B) & \xrightarrow{\ \iota\ } & (t, \mathbf{x}, \mathbf{u}, p) \\
\Big\downarrow{\scriptstyle\tilde{\pi}} & & \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\
(t) & \xleftarrow[\ \mathfrak{q}_M\ ]{} & (t, \mathbf{x}) & \xrightarrow[\ \text{id}\ ]{} & (t, \mathbf{x}),
\end{array}$$

where the inclusion map ι is defined by (6.13) and the solutions to (6.14) for w and p. The general invariant section is then, on putting $\sigma = \ln\delta$,

$$u = x\frac{\dot\sigma}{2} - y\frac{c}{\delta}, \qquad v = x\frac{c}{\delta} + y\frac{\dot\sigma}{2},$$

$$w = zA(t), \qquad p = -\frac{\alpha\ddot\alpha + \beta\ddot\beta}{2\delta}\,(x^2+y^2) + \frac{z^2}{2}B(t).$$

Note that the u and v components are uniquely determined from the isotropy conditions (6.12) and (6.13) and that the arbitrary functions $A(t)$ and $B(t)$ defining these invariant sections appear only in the w and p components.

We now turn to the dynamic reduction diagram. Since we are treating the Euler equations as the section (6.10) of the tensor bundle $\mathcal{D}$ we can anticipate the form of Δ_{Inv} by computing the Γ invariant tensors of the form

$$T = P\,\partial_u \otimes d\,t + Q\,\partial_v \otimes d\,t + R\,\partial_w \otimes d\,t + S d\,t. \tag{6.15}$$

The isotropy condition $\mathcal{L}_Z T = 0$ at x_0, where Z is defined by (6.12), shows immediately that $P = Q = 0$ from which it follows that

$$\mathbf{f}^1 = z\frac{\partial}{\partial w}\otimes\, dt \quad \text{and} \quad \mathbf{f}^2 = d\,t$$

are a basis for the Γ invariant fields of the type (6.15). This calculation shows that the $\partial_u \otimes dt$ and $\partial_v \otimes dt$ components of the reduced Euler equations must vanish identically and, consistent with this conclusion, one readily computes

$$\begin{aligned}\Delta_{\text{Inv}} &= \Big[\frac{\ddot\sigma}{2} + (\frac{\dot\sigma}{2})^2 - \big(\frac{c}{\delta}\big)^2 - \frac{(\alpha\ddot\alpha + \beta\ddot\beta)}{\delta}\Big]\,[x\frac{\partial}{\partial u}\otimes dt + y\frac{\partial}{\partial v}\otimes dt] + \big(\dot A + A^2 + B\big)\mathbf{f}^1 + \big(\dot\sigma + A\big)\mathbf{f}^2\\ &= \big(\dot A + A^2 + B\big)\mathbf{f}^1 + \big(\dot\sigma + A\big)\mathbf{f}^2.\end{aligned}$$

Thus, the reduced differential equations are

$$\dot A + A^2 + B = 0 \quad \text{and} \quad \dot\sigma + A = 0$$

which determine A and B algebraically. In conclusion, for each choice of α and β there is precisely *one* Γ invariant solution to the Euler equations given by

$$u = x\frac{\alpha\dot\alpha + \beta\dot\beta}{\alpha^2+\beta^2} - y\frac{\alpha\dot\beta - \dot\alpha\beta}{\alpha^2+\beta^2}, \qquad v = x\frac{\alpha\dot\beta - \dot\alpha\beta}{\alpha^2+\beta^2} + y\frac{\alpha\dot\alpha + \beta\dot\beta}{\alpha^2+\beta^2}, \qquad w = -2z\frac{\alpha\dot\alpha + \beta\dot\beta}{\alpha^2+\beta^2},$$

$$p = -\frac{1}{2}(x^2+y^2)\frac{\alpha\ddot\alpha + \beta\ddot\beta}{\alpha^2+\beta^2} + z^2\left(\frac{\alpha\ddot\alpha + \beta\ddot\beta}{\alpha^2+\beta^2} + \big(\frac{\alpha\dot\beta - \beta\dot\alpha}{\alpha^2+\beta^2}\big)^2 - 3\big(\frac{\alpha\dot\alpha + \beta\dot\beta}{\alpha^2+\beta^2}\big)^2\right). \qquad \blacksquare$$

HARMONIC MAPS

For our next examples we look at two well-known reductions of the harmonic map equation for maps between spheres. For these examples the bundle E is $S^n \times S^m \to S^n$ which we realize as a subset of $\mathbf{R}^{n+1} \times \mathbf{R}^{m+1}$ by

$$E = \{ (\mathbf{x}, \mathbf{u}) \in \mathbf{R}^{n+1} \times \mathbf{R}^{m+1} \,|\, \mathbf{x} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{u} = 1 \}.$$

Let G be a Lie subgroup of $\mathbf{SO}(n+1)$, let $\rho\colon G \to \mathbf{SO}(m+1)$ be a Lie group homomorphism and define the action of G on E by

$$R \cdot (\mathbf{x}, \mathbf{u}) = (R \cdot \mathbf{x}, \rho(R) \cdot \mathbf{u}) \quad \text{for} \quad R \in G.$$

The kinematic bundle for the G invariant sections of E has fiber

$$\kappa_{G,\mathbf{x}}(E) = \{ (\mathbf{x}, \mathbf{u}) \in E \,|\, \rho(R) \cdot \mathbf{u} = \mathbf{u} \quad \text{for all } R \in G \text{ such that } R \cdot \mathbf{x} = \mathbf{x} \}.$$

We identify the jet space $J^2(E)$ with a submanifold of $J^2(\mathbf{R}^{n+1}, \mathbf{R}^{m+1})$ by

$$J^2(E) = \{ (\mathbf{x}, \mathbf{u}, \partial_i \mathbf{u}, \partial_{ij} \mathbf{u}) \in J^2(\mathbf{R}^{n+1}, \mathbf{R}^{m+1}) \,|\, \mathbf{x}\cdot\mathbf{x} = 1, \mathbf{u}\cdot\mathbf{u} = 1, \mathbf{u}\cdot\partial_i \mathbf{u} = 0, \mathbf{u}\cdot\partial_{ij}\mathbf{u} + \partial_i \mathbf{u}\cdot\partial_j \mathbf{u} = 0 \}.$$

Since the harmonic map operator (or tension field) is a tangent vector to the target sphere S^m at each point $\sigma \in J^2(E)$, we let

$$\mathcal{D} = \{ (\sigma, \Delta) \in J^2(E) \times \mathbf{R}^{m+1} \,|\, \mathbf{u} \cdot \Delta = 0 \}. \tag{6.16}$$

By combining Proposition I.1.17 (p.19) and Lemma VII.1.2 (p.129) in Eells and Ratto [*15*], it follows that one can write the harmonic map operator $\Delta\colon J^2(E) \to \mathcal{D}$ as the map

$$\Delta(\sigma) = \left[\Delta^{\mathbf{R}^{n+1}} u^\alpha + x^i x^j u^\alpha_{ij} + n x^i u^\alpha_i - \lambda u^\alpha \right] \frac{\partial}{\partial u^\alpha}, \tag{6.17}$$

where

$$\lambda = \delta_{\alpha\beta} [\delta^{ij} u^\alpha_i u^\beta_j - x^i x^j u^\alpha_i u^\beta_j] \quad \text{and} \quad \Delta^{\mathbf{R}^{n+1}} u^\alpha = -\delta^{ij} u^\alpha_{ij}.$$

This operator is invariant under the induced action of $\mathfrak{G} = \mathbf{SO}(n+1) \times \mathbf{SO}(m+1)$ on E.

Example 6.3. HARMONIC MAPS FROM S^2 TO S^4. For our first example we take $E = S^2 \times S^4 \to S^2$ and we look for harmonic maps which are invariant under the standard action of $\mathbf{SO}(3)$ acting on S^2. It can be proved that, up to conjugation, there are three distinct group homomorphisms

$\rho\colon \mathbf{SO}(3) \to \mathbf{SO}(5)$, which lead to the following three possibilities for the infinitesimal generators of $\mathbf{SO}(5)$ acting on E.

$$\text{Case I}\quad \begin{cases} V_1 = z\partial_y - y\partial_z, \\ V_2 = x\partial_z - z\partial_x, \\ V_3 = y\partial_x - x\partial_y. \end{cases} \qquad \text{Case II}\quad \begin{cases} V_1 = z\partial_y - y\partial_z - u^2\partial_{u^3} + u^3\partial_{u^2}, \\ V_2 = x\partial_z - z\partial_x - u^3\partial_{u^1} + u^1\partial_{u^3}, \\ V_3 = y\partial_x - x\partial_y - u^1\partial_{u^2} + u^2\partial_{u^1}. \end{cases}$$

$$\text{Case III}\quad \begin{cases} V_1 = z\partial_y - y\partial_z + u^2\partial_{u^1} - u^1\partial_{u^2} + (u^4 - \sqrt{3}u^5)\partial_{u^3} - u^3\partial_{u^4} + \sqrt{3}u^3\partial_{u^5}, \\ V_2 = x\partial_z - z\partial_x - u^3\partial_{u^1} + (u^4 + \sqrt{3}u^5)\partial_{u^2} + u^1\partial_{u^3} - u^2\partial_{u^4} - \sqrt{3}u^2\partial_{u^5}, \\ V_3 = y\partial_x - x\partial_y - 2u^4\partial_{u^1} + u^3\partial_{u^2} - u^2\partial_{u^3} + 2u^1\partial_{u^4}. \end{cases}$$

In Case I, the map ρ is the constant map, and in Case II, ρ is the standard inclusion of $\mathbf{SO}(3)$ into $\mathbf{SO}(5)$. The origin of the map ρ in Case III will be discussed shortly.

Since $\mathbf{SO}(3)$ acts transitively on S^2, the orbit manifold $\widetilde{M}$ consists of a single point, the space of invariant sections is a finite dimensional manifold, and the reduced differential equations are algebraic equations. The kinematic bundles $\kappa_G(E)$ are determined in each case from the isotropy constraint

$$xV_1 + yV_2 + zV_3 = 0.$$

In Case I the action is transverse, the isotropy constraint is vacuous and the kinematic bundle is $\kappa_G(E) = S^2 \times S^4$. The invariant sections are given by

$$\Phi_I(x, y, z) = (A, B, C, D, E),$$

where $A, \ldots, E$ are constants and $A^2 + B^2 + C^2 + D^2 + E^2 = 1$. In Case II the kinematic bundle is $S^2 \times S^2$ and the invariant sections are

$$\Phi_{II}(x, y, z) = (Ax, Ay, Az, B, C),$$

where A, B, C are constants such that $A^2 + B^2 + C^2 = 1$. We take $A \neq 0$, since otherwise Φ_{II} becomes a special case of Φ_I. In Case III, $\kappa_G(E) = S^2 \times \{\pm 1\}$ and the invariant sections are

$$\Phi_{III}(x, y, z) = A\sqrt{3}\big(xy, xz, yz, \frac{1}{2}(x^2 - y^2), \frac{\sqrt{3}}{6}(x^2 + y^2 - 2z^2)\big),$$

where $A = \pm 1$.

Direct substitution into (6.17) easily shows that the maps Φ_I and Φ_{III} automatically satisfy the harmonic map equation. The map Φ_{II} is harmonic if and only if $B = C = 0$ in which case Φ_{II}

is either the identity map or the antipodal map on S^2 followed by the standard inclusion into S^4. Despite the simplicity of these conclusions, it is nevertheless instructive to look at the corresponding dynamic reduction diagrams.

In Case I, the invariant sections are constant and so

$$\mathrm{Inv}^2(E) = \{(\mathbf{x}, \mathbf{A}) \in \mathbf{R}^3 \times \mathbf{R}^5 \mid \mathbf{x} \cdot \mathbf{x} = \mathbf{A} \cdot \mathbf{A} = 1, \}$$

and

$$\mathcal{D}_{\mathrm{Inv}} = \{(\sigma, \Delta) \in \mathrm{Inv}^2(E) \times \mathbf{R}^5 \mid \mathbf{A} \cdot \Delta = 0\}.$$

The automorphism group for the kinematic bundle in this case is $\tilde{\mathfrak{G}} = \mathbf{SO}(3) \times \mathbf{SO}(5)$ which acts on $\mathcal{D}_{\mathrm{Inv}}$ by

$$(R, S) \cdot (\mathbf{x}, \mathbf{A}, \Delta) = (R \cdot \mathbf{x}, S \cdot \mathbf{A}, S \cdot \Delta) \quad \text{for } R \in \mathbf{SO}(3) \text{ and } S \in \mathbf{SO}(5).$$

The isotropy constraint for $\kappa_{\tilde{\mathfrak{G}}}(\mathcal{D}_{\mathrm{Inv}})$ forces Δ to be a multiple of $\mathbf{A}$. Hence, by the tangency condition $\mathbf{A} \cdot \Delta = 0$, we have $\Delta = 0$ and $\kappa_{\tilde{\mathfrak{G}},\sigma}(\mathcal{D}_{\mathrm{Inv}}) = 0$. This shows that the map Φ_I is harmonic by symmetry considerations alone and moreover that it is a universal solution for any operator $\Delta\colon J^k(S^2 \times S^4) \to \mathcal{D}$ with $\mathbf{SO}(3) \times \mathbf{SO}(5)$ symmetry.

In Case II, the harmonic map equations force $B = C = 0$ so that the maps Φ_{II} are not universal. Interestingly however, the standard and antipodal inclusions $S^2 \to S^4$ have a larger symmetry group, namely $\mathbf{SO}(3) \times \mathbf{SO}(2) \subset \mathfrak{G}$ and it is easily seen, using these larger symmetry groups, that the standard and antipodal inclusions are universal. It is a common phenomenon that the group invariant solutions to a system of differential equations possess a larger symmetry group than the original group used in their construction.

In Case III one finds immediately that $\kappa_{G,\sigma}(\mathcal{D}_{\mathrm{Inv}}) = 0$ and Φ_{III} is universal, again for any operator $\Delta\colon J^k(S^2 \times S^4) \to \mathcal{D}$ with $\mathbf{SO}(3) \times \mathbf{SO}(5)$ symmetry.

The map Φ_{III} is the classic Veronese map. The symmetry group defining it is based on a standard irreducible representation of $\mathbf{SO}(3)$ which readily generalizes to give harmonic maps between various spheres of higher dimension. Specifically, starting with the standard action of $\mathbf{SO}(n)$ on $V = \mathbf{R}^n$, consider the induced action on $\mathrm{Sym}^k_{\mathrm{tr}}(V)$, the space of rank k symmetric, trace-free tensors or, equivalently, on the space $W = \mathcal{H}^k(V)$ of harmonic polynomials of degree k on V. The standard metric on W is invariant under this action of $\mathbf{SO}(n)$ and in this way one obtains a Lie group monomorphism $\rho\colon \mathbf{SO}(n) \to \mathbf{SO}(N)$, where $N = \dim(W) = \binom{n+k-1}{k} - 1$. For example, the polynomials

$$u^1 = xy, \quad , u^2 = xz, \quad u^3 = yz, \quad u^4 = 1/2(x^2 - y^2), \quad u^5 = \sqrt{3}/6(x^2 + y^2 - 2z^2)$$

form an orthogonal basis for $\mathcal{H}^2(\mathbf{R}^3)$ and the action of $\mathbf{SO}(3)$ on this space determines the action of $\mathbf{SO}(3)$ on $\mathbf{R}^3 \times \mathbf{R}^5$ in Case III. For further examples see Eells and Ratto [*15*] and Toth [*38*]. ∎

Example 6.4. Harmonic Maps from S^n to S^n. A basic result of Smith [*35*] states that each element of $\pi_n(S^n) = \mathbf{Z}$ can be represented by a harmonic map (with respect to the standard metric) provided $n \leq 7$ or $n = 9$. This result, which can be established by symmetry reduction of the harmonic map equation (see Eells and Ratto [*15*] and Urakawa [*39*]), illustrates a number of interesting features. First, we see that much of the general theory which we have outlined could be extended to the case where $\widetilde{M}$ is a manifold with boundary and where the fibers of $\kappa_G(E)$ change topological type on the boundary. Secondly, we find that the invariant sections for the standard action of $G = \mathbf{SO}(n-1) \times \mathbf{SO}(2) \subset \mathbf{SO}(n+1)$ on S^n are slightly more general than those considered in [*15*] and [*39*]. However, a simple analysis of the reduced equations, based upon Noether's theorem, shows that the only solutions to the reduced equations are essentially those provided by the ansatz used by Eells and Ratto and Urakawa.

If $(R, S) \in G = \mathbf{SO}(n-1) \times \mathbf{SO}(2) \subset \mathbf{SO}(n+1)$ and

$$(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) \in E \subset (\mathbf{R}^{n-1} \times \mathbf{R}^2) \times (\mathbf{R}^{n-1} \times \mathbf{R}^2),$$

where $||\mathbf{x}||^2 + ||\mathbf{y}^2|| = 1$ and $||\mathbf{u}||^2 + ||\mathbf{v}||^2 = 1$, then the action of G on $E = S^n \times S^n$ is given by

$$(R, S)(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \Big(\begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \Big).$$

The invariants for the action of G on the base $\mathbf{R}^{n+1}$ are $r = ||\mathbf{x}||$ and $s = ||\mathbf{y}||$ which, for points $(\mathbf{x}, \mathbf{y}) \in S^n$, are related by $r^2 + s^2 = 1$, where $r \geq 0$ and $s \geq 0$. The quotient manifold $\widetilde{M} = S^n/G$ is therefore diffeomorphic to the closed interval $[0, \pi/2]$.

To describe the kinematic bundle $\kappa_G(E)$ we must consider separately those points in M for which (i) $s = 0$, (ii) $s \neq 0$ and $r \neq 0$ and (iii) $r = 0$, corresponding the left-hand boundary point, the interior points and the right-hand boundary points of $\widetilde{M}$. For $(\mathbf{x}, 0) \in S^n$, the isotropy subalgebra is $\mathbf{SO}(n-1)_{\mathbf{x}} \times \mathbf{SO}(2)$ and the fiber of the kinematic bundle consists of a pair of points

$$\kappa_{G,(\mathbf{x},0)}(E) = \{\, (\mathbf{x}, 0, \mathbf{u}, \mathbf{v}) \,|\, \mathbf{u} = \pm\mathbf{x}, \quad \text{and} \quad \mathbf{v} = 0 \,\}.$$

For points $(\mathbf{x}, \mathbf{y}) \in S^n$ with $r \neq 0$ and $s \neq 0$ the isotropy group is $\mathbf{SO}(n-1)_{\mathbf{x}} \times \{\, I \,\}$ and the fiber of the kinematic bundle is the ellipsoid of revolution

$$\kappa_{G,(\mathbf{x},\mathbf{y})}(E) = \{\, (\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) \,|\, \mathbf{u} = A\mathbf{x}, \quad \text{where} \quad r^2 A^2 + ||\mathbf{v}||^2 = 1 \}.$$

Invariant coordinates on $\kappa_{G,(\mathbf{x},\mathbf{y})}(E)$ are $A = \dfrac{\mathbf{x} \cdot \mathbf{u}}{r^2}$, $B = \dfrac{\mathbf{y} \cdot \mathbf{v}}{s^2}$ and $C = \dfrac{\mathbf{y}^\perp \cdot \mathbf{v}}{s^2}$, where $\mathbf{y}^\perp = (0, -y^2, y^1)$, subject to

$$r^2 A^2 + s^2 (B^2 + C^2) = 1. \tag{6.18}$$

The inclusion map from $\kappa_{G,(\mathbf{x},\mathbf{y})}(E)$ to $E_{(\mathbf{x},\mathbf{y})}$ is

$$\mathbf{u} = A\mathbf{x} \quad \text{and} \quad \mathbf{v} = B\mathbf{y} + C\mathbf{y}^{\perp}.$$

At the points $(0, \mathbf{y})$, the isotropy subalgebra is $\mathbf{SO}(n) \times \{ I \}$ and the fiber of the kinematic bundle is the circle

$$\kappa_{G,(0,\mathbf{y})}(E) = \{ (0, \mathbf{y}, \mathbf{u}, \mathbf{v}) \,|\, \mathbf{u} = 0 \quad \text{and} \quad ||\mathbf{v}|| = 1 \}.$$

The quotient space $\tilde{\kappa}_G(E)$ is shown in Figure 1. The G invariant sections are therefore described, as maps $\Phi\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$, by

$$\Phi(\mathbf{x}, \mathbf{y}) = A(t)\mathbf{x} + B(t)\mathbf{y} + C(t)\mathbf{y}^{\perp}, \tag{6.19}$$

where t is the smooth function of $(\mathbf{x}, \mathbf{y})$ defined by $\cos(t) = \dfrac{r}{r^2 + s^2}$ and $\sin(t) = \dfrac{s}{r^2 + s^2}$, and where $\cos^2(t)A^2(t) + \sin^2(t)(B^2(t) + C^2(t)) = 1$. The isotropy conditions at the boundary of $\widetilde{M}$ imply that the functions A, B and C are subject to the boundary conditions

$$A(0) = \pm 1, \quad B(0) = 0, \quad C(0) = 0, \quad \text{and} \quad A(\frac{\pi}{2}) = 0, \quad B(\frac{\pi}{2})^2 + C(\frac{\pi}{2})^2 = 1. \tag{6.20}$$

The invariant sections considered in [*15*] and [*39*] correspond to $C(t) = 0$. Note that the space of invariant sections (6.19) is preserved by rotations in the $\mathbf{v}$ plane, that is, rotations in the BC plane and therefore $\mathbf{SO}(2) \subset \tilde{\mathfrak{G}}_{\text{eff}}$.

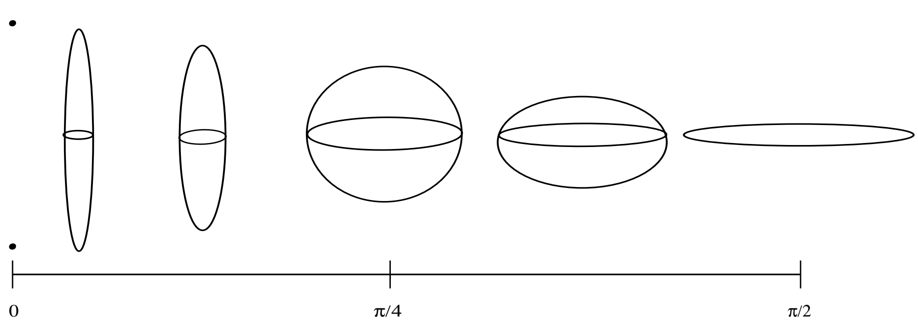

FIGURE 1. The reduced kinematic bundle for $\mathbf{SO}(n-1) \times \mathbf{SO}(2)$ invariant maps $s\colon S^n \to S^n$

By computing $\kappa_G(\mathcal{D}_{\text{Inv}})$ we deduce that the restricted harmonic operator Δ_{Inv} is of the form

$$\Delta_{\text{Inv}} = \Delta_A \left(\mathbf{x} \cdot \frac{\partial}{\partial \mathbf{u}}\right) + \Delta_B \left(\mathbf{y} \cdot \frac{\partial}{\partial \mathbf{v}}\right) + \Delta_C \left(\mathbf{y}^{\perp} \cdot \frac{\partial}{\partial \mathbf{v}^{\perp}}\right),$$

where the tangency condition (6.16) reduces to

$$r^2 A\Delta_A + s^2 B\Delta_B + s^2 C\Delta_C = 0.$$

A series of straightforward calculations, using (6.17), now shows that the coefficients of the reduced operator $\tilde{\Delta}$ are

$$\begin{aligned}
\tilde{\Delta}_A &= -\ddot{A} + \big(n\frac{\sin(t)}{\cos(t)} - \frac{\cos(t)}{\sin(t)}\big)\dot{A} + nA - \lambda A,\\
\tilde{\Delta}_B &= -\ddot{B} + \big((n-2)\frac{\sin(t)}{\cos(t)} - 3\frac{\cos(t)}{\sin(t)}\big)\dot{B} + nB - \lambda B,\\
\tilde{\Delta}_C &= -\ddot{C} + \big((n-2)\frac{\sin(t)}{\cos(t)} - 3\frac{\cos(t)}{\sin(t)}\big)\dot{C} + nC - \lambda C,
\end{aligned} \tag{6.21}$$

where

$$\begin{aligned}
\lambda = \cos^2(t)\dot{A}^2 + \sin^2(t)\big(\dot{B}^2 + \dot{C}^2\big) + 2\cos(t)\sin(t)\big(-A\dot{A} + B\dot{B} + C\dot{C}\big)\\
+ (n-1)A^2 + 2(B^2 + C^2) - \cos^2(t)A^2 - \sin^2(t)(B^2 + C^2).
\end{aligned}$$

To analyze these equations, we first invoke the principle of symmetric criticality and the formulas in [*2*] for the reduced Lagrangian to conclude that these equations are the Euler-Lagrange equations for the reduced Lagrangian

$$\tilde{L} = \frac{1}{2}\cos(t)^{n-2}\sin(t)\lambda\, dt$$

subject, of course, to the constraint (6.18). From knowledge of the automorphism group of the kinematic bundle we know that this Lagrangian is invariant under rotations in the BC plane and this leads to the first integral

$$J = \cos(t)^{n-2}\sin(t)^3(B\dot{C} - C\dot{B})$$

for (6.21). By the boundary conditions (6.20), J must vanish identically. Thus $C(t) = \mu B(t)$, for some constant μ and therefore a rotation in the $\mathbf{v}, \mathbf{v}^{\perp}$ plane will rotate the general invariant section (6.19) into the section with $C(t) = 0$. We then have $r^2A^2 + s^2B^2 = 1$ and the change of variables

$$A(t) = \frac{\cos(\phi(t))}{\cos(t)} \quad \text{and} \quad B(t) = \frac{\sin(\phi(t))}{\sin(t)}$$

converts the reduced operator (6.21) into the form found in [*15*] or [*39*]. ∎

General Relativity

We now turn to some examples of Lie symmetry reduction in general relativity which we again examine from the viewpoint of the kinematic and dynamic reduction diagrams. To study reductions of the Einstein field equations, we take the bundle E to be the bundle $Q(M)$ of quadratic forms, with Lorentz signature, on a 4-dimensional manifold M. A section of E then corresponds to a choice of Lorentz metric on M. We view the Einstein tensor

$$\Delta = G^{ij}(g_{hk}\,,g_{hk,l}\,,g_{hk,lm})\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$$

formally as a section of $\mathcal{D}\to J^2(E)$, where $\mathcal{D}$ is pullback of $V=\mathrm{Sym}^2(T(M))$ to the bundle of 2-jets $J^2(E)$. The operator Δ is invariant under the Lie pseudo-group $\mathfrak{G}$ of all local diffeomorphisms of M.

Let Div_g be the covariant divergence operator (defined by the metric connection for g) acting on (1,1) tensors,

$$\mathrm{Div}_g(S)=\nabla_i S^i_j\,dx^j.$$

The contracted Bianchi identity is $\mathrm{Div}_g\,\Delta^\flat = 0$, where $\Delta^\flat$ is the operator obtained from Δ by lowering an index with the metric.

The first point we wish to underscore with the following examples is that the kinematic reduction diagram gives a remarkably efficient means of solving the Killing equations for the determination of the invariant metrics. Secondly, we show that discrete symmetries, which will not change the dimension of the reduced spacetime $\widetilde{M}$, can lead to isotropy constraints which reduce the fiber dimension of the kinematic bundle. Thirdly, for G invariant metrics, the divergence operator Div_g is a G invariant operator to which the dynamical reduction procedure can be applied to obtain the reduction of the contracted Bianchi identities for the reduced equations. Throughout, we emphasize the importance of the residual symmetry group in analyzing the reductions of the field equations.

Finally, we remark that our conclusions in these examples are not restricted to the Einstein equations but in fact hold for any generally covariant metric field theories derivable from a variational principle.

Example 6.5. Spherically Symmetric and Stationary, Spherically Symmetric Reductions . We begin by looking at spherically symmetric solutions on the four dimensional manifold $M=\mathbf{R}\times(\mathbf{R}^3-\{\,0\,\})$, with coordinates $(x^i)=(t,x,y,z)$ for $i=0,1,2,3$. Although this is a very well-understood example, it is nevertheless instructive to consider it within the general theory of Lie symmetry reduction of differential equations. The infinitesimal generators for $G=\mathbf{SO}(3)$ are given by (2.10) and, just as in Example 6.1, we find that the infinitesimal isotropy constraint defining

$\kappa_{G,x}(E) = \kappa_{\Gamma,x}(E)$ is

$$\varepsilon_{0kij} x^k g_{li} \frac{\partial}{\partial g_{lj}} = 0,$$

or, in terms of matrices,

$$ga + a^t g = 0, \tag{6.22}$$

where

$$a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z & -y \\ 0 & -z & 0 & x \\ 0 & y & -x & 0 \end{bmatrix}$$

and $g = [\, g_{ij} \,]$. These linear equations are easily solved to give

$$g = A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + B \begin{bmatrix} 0 & x & y & z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} + C \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x^2 & xy & xz \\ 0 & xy & y^2 & yz \\ 0 & xz & yz & z^2 \end{bmatrix} + D \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.23}$$

The fiber of the kinematic bundle $\kappa_{G,x}(E)$ is therefore parameterized by four variables A, B, C, D. Since these variables are invariants for the action of G restricted to $\kappa_G(E)$ and since the invariants for the action of $\mathbf{SO}(3)$ on M are t and r, the kinematic reduction diagram for the action of $\mathbf{SO}(3)$ on the bundle of Lorentz metrics is

$$\begin{array}{ccccc} (t, r, A, B, C, D) & \xleftarrow{\mathfrak{q}_{\kappa_G}} & (x^i, A, B, C, D) & \xrightarrow{\iota} & (x^i, g_{ij}) \\ \downarrow & & \downarrow & & \downarrow \\ (t, r) & \xleftarrow{\mathfrak{q}_M} & (x^i) & \xrightarrow{\text{id}} & (x^i), \end{array} \tag{6.24}$$

where the inclusion map ι is given by (6.23).

Consequently, the most general rotationally invariant metric on M is

$$ds^2 = A(t,r)dt^2 + 2B(t,r)dt(x\,dx + y\,dy + z\,dz) + C(t,r)(x\,dx + y\,dy + z\,dz)^2 + D(t,r)(dx^2 + dy^2 + dz^2).$$

In standard spherical coordinates $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = \cos\phi$ this takes the familiar form (on re-defining the coefficients B, C and D)

$$ds^2 = A(t,r)dt^2 + B(t,r)dtdr + C(t,r)dr^2 + D(t,r)d\,\Omega^2 \tag{6.25}$$

where $d\,\Omega^2 = d\phi^2 + \sin^2\phi\, d\theta^2$.

If we enlarge the symmetry group to include time translations $V_0 = \dfrac{\partial}{\partial t}$, then the kinematic reduction diagram becomes

$$\begin{array}{ccccc} (r, A, B, C, D) & \xleftarrow{\mathfrak{q}_{\kappa_G}} & (x^i, A, B, C, D) & \xrightarrow{\iota} & (x^i, g_{ij}) \\ \downarrow & & \downarrow & & \downarrow \\ (r) & \xleftarrow{\mathfrak{q}_M} & (x^i) & \xrightarrow{\text{id}} & (x^i). \end{array} \tag{6.26}$$

At first glance there appears to be little difference between the two diagrams (6.24) and (6.26), but a computation of the automorphism groups reveals a dramatic difference in the geometry of the reduced bundles $\tilde{\kappa}_G(E)$ in (6.24) and (6.26). This difference is best explained in terms of general results on Kaluza-Klein reductions of metric theories as in, for example, Coquereaux and Jadczyk [*13*]. From our perspective, these authors show that when the action of G on M is simple in the sense that the isotropy groups G_x can all be conjugated in G to a fixed isotropy group G_{x_0}, then the reduced bundle $\tilde{\kappa}_G(E)$ is a product of three bundles over $\widetilde{M}$,

$$\tilde{\kappa}_G(E) = Q(\widetilde{M}) \oplus A(\widetilde{M}) \oplus Q_{\mathrm{Inv}}(K). \tag{6.27}$$

Here

[i] $Q(\widetilde{M})$ is the bundle of metrics on $\widetilde{M}$.

[ii] $A(\widetilde{M}) = \Lambda^1(\widetilde{M}) \otimes (P \times_H \mathfrak{h})$, where P is the principal H bundle defined as the set of points in M with isotropy group $G_{x_0}$and $H = \mathrm{Nor}(G_{x_0}, G)/G_{x_0}$.

[iii] $Q_{\mathrm{Inv}}(K)$ is the trivial bundle whose fiber consist of the G invariant metrics on the homogeneous space $K = G/G_{x_0}$.

For (6.24) one computes the residual symmetry group $\tilde{\mathfrak{G}}_{\mathrm{eff}}$ to be the diffeomorphism group of $\widetilde{M} = \mathbf{R} \times \mathbf{R}^+$ and one finds that the coefficients A, B, C transform as the components of a metric on $\widetilde{M}$ and that D is a scalar field (which one identifies as a map into the space of $\mathbf{SO}(3)$ invariant metrics on S^2). Thus, for (6.24), we find that

$$\tilde{\kappa}_G(E) = Q(\widetilde{M}) \oplus \Re,$$

where $\Re$ is a trivial line bundle over $\widetilde{M}$. By contrast, for the diagram (6.26) the automorphism group $\tilde{\mathfrak{G}}$ acts on M by

$$r \to f(r) \quad \text{and} \quad t \to \epsilon t + g(r), \tag{6.28}$$

where $f \in \mathbf{Diff}(\mathbf{R}^+)$, $g \in C^\infty(\mathbf{R})$ and $\epsilon \in \mathbf{R}^*$. Without going further into the details of the decomposition (6.27), we simply note that the variable t is now the fiber coordinate on the principle bundle P and that under the transformations (6.28) the coefficients of the metric (6.25), which are now functions of r alone, transform according to

$$A(r) \to \epsilon^2 A(f(r)), \qquad B(r) \to \epsilon[f' B(f(r)) + 2g' A(f(r))]$$
$$C(r) \to (f')^2 C(f(r)) + f'g' B(f(r)) + (g')^2 A(f(r)) \qquad D(r) \to D(f(r)).$$

Consequently, the sections of $\tilde{\kappa}_G(E)$ can be written as

$$\tilde{s}(r) = [\tilde{g}(r), \tilde{\omega}(r), \tilde{h}(r)],$$

where

$$\tilde{g}(r) = [C(r) - \frac{B(r)^2}{4A(r)}]\, dr^2, \quad \tilde{\omega}(r) = \frac{B(r)}{2A(r)} dr \otimes \frac{\partial}{\partial t}, \quad \text{and} \quad \tilde{h}(r) = A(r)dt^2 + D(r)\, d\Omega^2.$$

Here $\tilde{g}(r)$ is a metric on $\widetilde{M}$, $\tilde{\omega}(r)$ is a connection on P pulled back to $\tilde{M}$, and $\tilde{h}(r)$ is a map from $\tilde{M}$ into the G invariant metrics on $\mathbf{R} \times S^2$.

The detailed expression for the reduced operator $\tilde{\Delta}$ for the stationary, rotationally invariant metrics can be found in any introductory text on general relativity. Here we simply point out that by computing the action of G on $\mathrm{Sym}^2(TM)$, we can deduce that the reduced operator will have the form

$$\tilde{\Delta} = \tilde{\Delta}^{tt}\frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + \tilde{\Delta}^{rt}(\frac{\partial}{\partial r} \otimes \frac{\partial}{\partial t} + \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial r}) + \tilde{\Delta}^{rr}\frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \tilde{\Delta}^{\Omega}(\frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial \phi} + \frac{1}{\sin^2\phi}\frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta}),$$

where $\tilde{\Delta}^{tt}$, $\tilde{\Delta}^{rt}$, $\tilde{\Delta}^{rr}$ and $\tilde{\Delta}^{\Omega}$ are smooth functions on the 2-jets of the bundle $(r, A, B, C, D) \to (r)$. In other words, of the ten components in the field equations, the dynamic reduction diagram automatically implies that 6 of these components vanish. Moreover, the reduced operator $\tilde{\Delta}$ is constrained by the reduced Bianchi identities. Since dt and dr provide a basis for the invariant one forms on M, we know that the reduction of $\mathrm{Div}_g\, S$ is a linear combination of dt and dr,

$$\widetilde{\mathrm{Div}_g}\ S = \tilde{S}_t\, dt + \tilde{S}_r\, dr.$$

By direct computation, one finds that the dt and dr components of the reduced Bianchi identities are

$$\frac{1}{2\gamma}\frac{d}{dr}[\gamma(2A\tilde{\Delta}^{rt} + B\tilde{\Delta}^{rr})] = 0$$

and

$$\frac{1}{2}\Big(\frac{1}{\gamma}\frac{d}{dr}[\gamma(2C\tilde{\Delta}^{rr} + B\tilde{\Delta}^{rt})] - \dot{A}\tilde{\Delta}^{tt} - \dot{C}\tilde{\Delta}^{rr} - \dot{B}\tilde{\Delta}^{rt} - 2\dot{D}\tilde{\Delta}^{\Omega}\Big) = 0$$

where $\gamma = D\sqrt{\frac{1}{4}B^2 - AC}$. It follows from the first of these identities and the transformation properties of A, B, $\tilde{\Delta}^{rt}$ and $\tilde{\Delta}^{rr}$ under the residual scaling $t \to \epsilon t$ that

$$2A\tilde{\Delta}^{rt} + B\tilde{\Delta}^{rr} = 0.$$

This same identity can be derived by first observing that the principle of symmetric criticality holds for the action G and then by applying Noether's second theorem to the reduced Lagrangian with symmetry $\tilde{\mathfrak{G}}_{\mathrm{eff}}$, Consequently of the four ODE arising in the stationary, spherically symmetric reduction of the field equations one need only solve the two equations

$$\tilde{\Delta}^{tt} = 0 \quad \text{and} \quad \tilde{\Delta}^{rr} = 0.$$

The remaining two equations

$$\tilde{\Delta}^{rt} = 0 \quad \text{and} \quad \tilde{\Delta}^{\Omega} = 0$$

will automatically be satisfied (assuming $\dot{D} \neq 0$, $A \neq 0$). We stress that these conclusions actually hold true for the stationary, rotationally invariant reductions of any generally covariant metric field equations derivable from a variational principle. ∎

Example 6.6. STATIC, SPHERICALLY SYMMETRIC REDUCTIONS. A metric is *static* and spherically symmetric if, in addition to being invariant under time translations and rotations, it is invariant under time reflection. The symmetry group G now includes the transformations $t \to t + c$ and $t \to -t$ and therefore the isotropy subgroup G_{x_0} of the point $x_0 = (t_0, \mathbf{x}_0)$ now includes the reflection $t \to 2t_0 - t$. The fibers of the kinematic bundle are now constrained by (6.22) along with

$$bgb^{\mathrm{t}} = g, \quad \text{where} \quad b = \mathrm{diag}[-1, 1, 1, 1].$$

This forces $B = 0$ in (6.23) so that the fibers of the kinematic bundle are now 3 dimensional and the general invariant section is

$$ds^2 = A(r)dt^2 + C(r)dr^2 + D(r)d\,\Omega^2.$$

The automorphism group for this bundle is now $r \to f(r)$ and $t \to \epsilon t$ and the $A(\widetilde{M})$ summand in (6.27) does not appear. This example shows that while discrete symmetries will never result in a reduction of the dimension of the orbit space $\widetilde{M}$, that is, the number of independent variables, discrete symmetries can reduce the fiber dimension of the kinematic bundle, that is, the number of dependent variables.

Example 6.7. PLANE WAVES. As our next example from general relativity, we consider a class of plane wave metrics [*12*]. We take $M = \mathbf{R}^4$ with coordinates (u, v, x, y) and let $P(u)$ and $Q(u)$ be arbitrary smooth functions satisfying $P'(u) > 0$ and $Q'(u) > 0$. The symmetry group on M is the five-parameter transformation group

$$\begin{aligned} &u' = u, \qquad v' = v + \varepsilon_1 + \varepsilon_4 x + \varepsilon_5 y + 1/2\big(\varepsilon_2\varepsilon_4 + \varepsilon_3\varepsilon_5 + \varepsilon_4^2 P(u) + \varepsilon_5^2 Q(u)\big), \\ &x' = x + \varepsilon_2 + \varepsilon_4 P(u), \qquad y' = y + \varepsilon_3 + \varepsilon_5 Q(u), \end{aligned} \tag{6.29}$$

with infinitesimal generators $V_1 = \dfrac{\partial}{\partial v}$, $V_2 = \dfrac{\partial}{\partial x}$, $V_3 = \dfrac{\partial}{\partial y}$,

$$V_4 = x\frac{\partial}{\partial v} + P(u)\frac{\partial}{\partial x} \quad \text{and} \quad V_5 = y\frac{\partial}{\partial v} + Q(u)\frac{\partial}{\partial y}.$$

The only non-vanishing brackets are

$$[\,V_2, V_4\,] = V_1 \quad \text{and} \quad [\,V_3, V_5\,] = V_1$$

so that, regardless of the choice of functions P and Q, the abstract Lie algebras or groups are the same although the actions are generically different for different choices of P and Q. The coordinate function u is the only invariant and the orbits of this action are 3-dimensional. Therefore, at each point the isotropy subgroup is two dimensional and it is easily seen that, at $\mathbf{x}_0 = (u_0, v_0, x_0, y_0)$, the infinitesimal isotropy $\Gamma_{\mathbf{x}_0}$ is generated by

$$Z_1 = V_4 - x_0V_1 - P(u_0)V_2 \quad \text{and} \quad Z_2 = V_5 - y_0V_1 - Q(u_0)V_3.$$

At $\mathbf{x} \in M$ the metric components $g = [\,g_{ij}\,]$ of a G invariant metric satisfy the isotropy conditions

$$ga_1 + a_1^t g = 0 \quad \text{and} \quad ga_2 + a_2^t g = 0, \tag{6.30}$$

where

$$a_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ P'(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ Q'(u) & 0 & 0 & 0 \end{bmatrix}.$$

We find that the solutions to (6.30) are

$$g_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad g_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{P'(u)} & 0 \\ 0 & 0 & 0 & \dfrac{1}{Q'(u)} \end{bmatrix}.$$

Thus the kinematic reduction diagram is

$$\begin{array}{ccccc} (u, A, B) & \xleftarrow{\mathfrak{q}_{\kappa_G}} & (x^i, A, B) & \xrightarrow{\iota} & (x^i, g_{ij}) \\ \downarrow & & \downarrow & & \downarrow \\ (u) & \xleftarrow{\mathfrak{q}_M} & (x^i) & \xrightarrow{\text{id}} & (x^i), \end{array}$$

and the inclusion map ι sends (A, B) to $ds^2 = A du^2 + B\gamma$, where

$$\gamma = -2du\,dv + \frac{dx^2}{P'(u)} + \frac{dy^2}{Q'(u)}.$$

The most general G invariant metric is

$$ds^2 = A(u)du^2 + B(u)\gamma. \tag{6.31}$$

From the form of the most general G invariant symmetric type $\binom{2}{0}$ tensor, we are assured that the reduced field equations take the form

$$\tilde{\Delta} = \tilde{\Delta}^{vv}\frac{\partial}{\partial v}\otimes\frac{\partial}{\partial v} + \tilde{\Delta}^{\gamma}(-\frac{\partial}{\partial u}\otimes\frac{\partial}{\partial v} - \frac{\partial}{\partial v}\otimes\frac{\partial}{\partial u} + P'(u)\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial x} + Q'(u)\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial y}).$$

Every G invariant one-form is a multiple of du so that there is only one non-trivial component to the contracted Bianchi identities and, indeed, by direct computation we find that

$$\operatorname{Div}_{\tilde{g}}\tilde{\Delta}^{\flat} = \frac{d}{du}\big(B\tilde{\Delta}^{\gamma}\big)\,d\,u.$$

Since this must vanish identically, we conclude that the $\tilde{\Delta}^{\gamma}$ component of the reduced field equations is of the form

$$\tilde{\Delta}^{\gamma} = \frac{c}{B},$$

where c is a constant. Either the constant c is non-zero, in which case the reduced equations are inconsistent and there are no G invariant solutions, or else $c = 0$ and the reduced equations consist of just the single equation $\tilde{\Delta}^{vv} = 0$. For generally covariant metric theories the case $c \neq 0$ can only arise when the field equations contain a cosmological term [*37*].

It is easy to check that while the isotropy algebras $\Gamma_{\mathbf{x}_0}$ are all two-dimensional abelian subalgebras, on disjoint orbits none are conjugate under the adjoint action of G. Hence the group action (6.29) is not simple and consequently the kinematic bundle for this action need not decompose according to (6.27). Indeed, the tensor γ cannot be identified with any G invariant quadratic form on the orbits G/G_{x_0}. ∎

Example 6.8. Symplectic Reduction and Group Invariant Solutions. It is important to recognize the fundamental differences between symplectic reduction and Lie symmetry reduction for group invariant solutions of a Hamiltonian system with symmetry. Let M be an even dimensional manifold with symplectic form ω and let $H\colon M \to \mathbf{R}$ be the Hamiltonian for a dynamical system on M. For the purposes of this example, it suffices to consider reduction by a one dimensional group of Hamiltonian symmetries generated by a vector field V with associated momentum map J,

$$V \lrcorner\, \omega = d\,J. \tag{6.32}$$

In symplectic reduction the reduced space $\widehat{M}$ is obtained by **[i]** restricting to the submanifold of M defined by

$$J = \mu \equiv constant,$$

and then **[ii]** quotienting this submanifold by the action of the transformation group generated by V. Both ω and H descend to $\widehat{M}$ and the reduced equations are the associated Hamiltonian system on $\widehat{M}$. Since $\dim \widehat{M} = \dim M - 2$, the reduction in the number of dependent variables is 2. The solution to the original Hamiltonian equations are obtained from that of the reduced Hamiltonian equations by quadratures.

To compare with symmetry reduction for group invariant solutions, we transcribe Hamilton's equations into the operator-theoretic setting used to construct the kinematic and dynamic reduction diagrams. Let $E = M \times \mathbf{R} \to \mathbf{R}$ be extended phase space so that the differential operator characterizing the canonical equations is the one-form valued operator on $J^1(E)$ defined by

$$\Delta = X \lrcorner\, \omega - d\,H.$$

Here X is the total derivative operator given, in standard canonical coordinates (u^i, p_i) on M, by

$$X = \frac{d}{dt} = \frac{\partial}{\partial t} + \dot{u}^i \frac{\partial}{\partial u^i} + \dot{p}_i \frac{\partial}{\partial p_i}.$$

It is not difficult to show that if V is any vector field on M, then the prolongation of V to $J^1(E)$ satisfies $[X, \operatorname{pr}^1 V] = 0$ and therefore V is a symmetry of the operator Δ whenever V is a symmetry of ω and H.

Since V is a vertical vector field on E it is "all isotropy" and the kinematic bundle is the fixed point set for the flow of V,

$$\kappa_\Gamma(E) = \{(t, u^i, p_i) \,|\, V(u^i, p_i) = 0\,\}.$$

The dimension of $\kappa_\Gamma(E)$ therefore depends upon the choice of V and is generally less than the dimension of E by more than 2 (the decrease in the dimension in the case of symplectic reduction). In short, it is not possible to identify the fibers of the kinematic bundle with the reduced phase space $\widehat{M}$. Moreover, from (6.32), it follows that *points in $\kappa_\Gamma(E)$ always correspond to points on the singular level sets of the momentum map and, typically, to points where the level sets fail to be a manifold.* Thus the invariant solutions are problematic from the viewpoint of symplectic reduction and are subject to special treatment. See, for example, [*5*] and [*20*]. Finally, there is no guarantee that the reduced equations for the group invariant solutions possess any natural inherited Hamiltonian formulation.

We illustrate these general observations with some specific examples. First, if V is a translation symmetry of a mechanical system, then J is a linear function and symplectic reduction yields all the solutions to Hamilton's equations with a given fixed value for the first integral J. Since the vector field V never vanishes, the kinematic bundle is empty and there are *no* group invariant solutions.

Second, for the classical 3-dimensional central force problem

$$\ddot{u} = -f(\rho)u, \quad \ddot{v} = -f(\rho)v, \quad \ddot{w} = -f(\rho)w,$$

where $\rho = \sqrt{u^2 + v^2 + w^2}$, the extended phase space E is $\mathbf{R} \times \mathbf{R}^6 \to \mathbf{R}$ with coordinates

$$(t, u, v, w, p_u, p_v, p_w) \to (t),$$

the symplectic structure on phase space is $\omega = du \wedge dp_u + dv \wedge dp_v + dw \wedge dp_w$ and the Hamiltonian is $H = \frac{1}{2}(p_u^2 + p_v^2 + p_w^2) + \phi(\rho)$, where $\phi'(\rho) = \rho f(\rho)$. The vector field

$$V = -u\frac{\partial}{\partial v} + v\frac{\partial}{\partial u} - p_u\frac{\partial}{\partial p_v} + p_v\frac{\partial}{\partial p_u}$$

is a Hamiltonian symmetry.

The kinematic bundle for the V invariant sections of E is

$$\begin{array}{ccccc} (t, w, p_w) & \xleftarrow{\text{id}} & (t, w, p_w) & \xrightarrow{\iota} & (t, u, v, w, p_u, p_v, p_w) \\ \pi\downarrow & & \downarrow\pi & & \downarrow\pi \\ (t) & \xleftarrow{\text{id}} & (t) & \xrightarrow{\text{id}} & (t)\ , \end{array} \tag{6.33}$$

where $\iota(t, w, p_w) = (t, 0, 0, w, 0, 0, p_w)$, the invariant sections are of the form

$$t \to (0, 0, w(t), 0, 0, p_w(t)),$$

and the reduced differential operator for the V invariant solutions is

$$\tilde{\Delta} = (\dot{w} - p_w)\, dp_w - (\dot{p}_w + wf(|w|))\, dw.$$

Let us compare this state of affairs with that obtained by symplectic reduction based upon the Hamiltonian vector field V. The momentum map associated to this symmetry is the angular momentum

$$J = -up_v + vp_u.$$

The level sets $J = \mu$ are manifolds except for $\mu = 0$. The level set $J = 0$ is the product of a plane and a cone whose vertex is precisely the fiber of the kinematic bundle. To implement the symplectic reduction, we introduce canonical cylindrical coordinates $(r, \theta, w, p_r, p_\theta, p_w)$, where

$$u = r\cos\theta, \quad v = r\sin\theta,$$

$$p_u = p_r\cos\theta - \frac{p_\theta}{r}\sin\theta\,, \quad p_v = p_r\sin\theta + \frac{p_\theta}{r}\cos\theta\,.$$

Note that this change of coordinates fails precisely at points of the kinematic bundle. In terms of these phase space coordinates, the symplectic structure is still in canonical form $\omega = dr \wedge dp_r + d\theta \wedge dp_\theta + dw \wedge dp_w.$, the Hamiltonian is $H = \frac{1}{2}(p_r^2 + \frac{\mu^2}{r^2}p_\theta^2 + p_w^2) + \phi(\sqrt{r^2 + w^2})$, and the momentum map is $J = -p_\theta$. We can therefore describe the symplectic reduction of E by the diagram

$$\begin{array}{ccccc} (t, r, w, p_r, p_w) & \longleftarrow & (t, r, \theta, w, p_r, p_w) & \xrightarrow{\iota} & (t, r, \theta, w, p_r, p_\theta, p_w) \\ \downarrow & & \downarrow & & \downarrow \\ (t) & \longleftarrow & (t) & \longrightarrow & (t). \end{array}$$

The reduced symplectic structure is then $\hat{\omega} = dr \wedge dp_r \wedge dr + dw \wedge dp_w$, the reduced Hamiltonian is $\widehat{H} = \frac{1}{2}(p_r^2 + \frac{\mu^2}{r^2} + p_w^2) + \phi(\sqrt{r^2 + w^2})$, and the reduced equations of motion are

$$\dot{r} = p_r, \quad \dot{p}_r = -rf(\sqrt{r^2 + w^2}) + \frac{\mu^2}{r^3}, \quad \dot{w} = p_w, \quad \dot{p}_w = -wf(\sqrt{r^2 + w^2}).$$

Given a choice of μ and solutions to these reduced equations, we get a solution to the full equations via $\theta = -\mu t + const.$ ∎

7. Appendix. We summarize a few technical points concerning group actions on fiber bundles and the construction of the kinematic and dynamic reduction diagrams. For details, see [*3*].

A. Transversality and Regularity. Let G be a finite dimensional Lie group acting projectably on a bundle $\pi\colon E \to M$. We say that G acts ***transversally*** on E if, for each fixed $p \in E$ and each fixed $g \in G$, the equation

$$\pi(g \cdot p) = \pi(p) \quad \text{implies that} \quad g \cdot p = p. \tag{7.1}$$

Thus each orbit of G intersects each fiber of E exactly once. For transverse group actions the orbits of G in E are diffeomorphic to the orbits of G in M under the projection map $\pi\colon E \to M$. Projectable, transverse actions always satisfy the infinitesimal transversality condition (2.4) but the converse is easily seen to be false.

Let us say that the action of G on M is ***regular*** if the quotient space $\widetilde{M} = M/G$ is a smooth manifold and the quotient map $\mathfrak{q}_M\colon M \to \widetilde{M}$ defines M as a bundle over $\widetilde{M}$. The construction of the orbit manifold $\widetilde{M}$ is discussed in various texts, for example, [*1*], [*4*], [*29*], [*31*]. The assumption that the action of G on M is regular is a standard assumption in Lie symmetry reduction. For simplicity we suppose that $\widetilde{M}$ is a manifold without boundary but, as Example 6.4 shows, this assumption can be relaxed in applications.

The fundamental properties of transverse group actions are described in the following theorem which is proved in [*3*].

Theorem 7.1(The Regularity Theorem for Transverse Group Actions)**.** *Let G be a Lie group which acts projectably and transversally on the bundle $\pi\colon E \to M$. Suppose that G acts regularly on M.*

[i] *Then G acts regularly on E and $\widetilde{E} = E/G$ is a bundle over $\widetilde{M}$.*

[ii] *If the orbit manifold $\widetilde{M}$ is Hausdorff, then the orbit manifold $\widetilde{E}$ is also Hausdorff.*

[iii] *The bundle E can be identified with the pullback of the bundle $\tilde{\pi}\colon \widetilde{E} \to \widetilde{M}$ via the quotient map $\mathfrak{q}_M\colon M \to \widetilde{M}$.*

[iv] *Let $\widetilde{U}$ be an open set in $\widetilde{M}$ and let $U = \mathfrak{q}_M^{-1}(\widetilde{U})$. There is a one-to-one correspondence between the smooth G invariant sections of E over U and the sections of $\widetilde{E}$ over $\widetilde{U}$.*

B. Transversality and the Kinematic Bundle. Lemma 3.2 implies that the action of G on E always restricts to a transverse action on the set $\kappa_G(E)$. In fact, it is not difficult to characterize $\kappa_G(E)$ as the largest subset of E on which G acts transversally or, alternatively, as the smallest set through which all locally defined invariant sections factor. *For Lie symmetry reduction without transversality the assumption that $\kappa_G(E)$ is an imbedded subbundle of E now replaces the infinitesimal transversality condition (2.4) as the underlying hypothesis for the action of G on E* (together, of course, with the regularity of the action of G on M). In particular, the assumption that the dimension of $\kappa_{G,x}(E)$ is constant as x varies over M is clearly a necessary condition if one hopes to parameterize the space of G invariant local sections of E in terms of a fixed number of arbitrary functions. There are a variety of general results which one can apply to check whether $\kappa_G(E)$ is subbundle of E. To begin with, if $x, y \in M$ lie on the same G orbit, that is, if $y = g \cdot x$ for some $g \in G$, then it is not difficult to prove that

$$\kappa_{G,y}(E) = g \cdot \kappa_{G,x}(E).$$

By virtue of this observation it suffices to check that the restrictions of $\kappa_G(E)$ to the cross-sections of the action of G on M are subbundles. For Lie group actions G which admit slices on M, it is not difficult to establish (see [*4*]) that the kinematic bundles for the induced actions on tensor bundles of M always exist. For compact groups acting by isometries on hermitian vector bundles the existence of the kinematic bundle is established in [*11*].

Granted that $\kappa_G(E) \to M$ is a bundle, Theorem 3.3 now follows from Theorem 7.1. Theorem 7.1 also shows that there is considerable redundancy in the hypothesis of Theorem 3.3.

We emphasize that the action of G on E itself need not be regular in order to construct a smooth kinematic reduction diagram. This is well illustrated by Example 19 in Lawson [*26*] (p. 23).

C. The Bundle of Invariant Jets. The following theorem summarizes the key properties of the bundle $\mathrm{Inv}^k(E) \to M$.

Theorem 7.2. *Let G be a projectable group action on $\pi\colon E \to M$ and suppose that E admits a smooth kinematic reduction diagram (3.5).*

[i] *Then $\mathrm{Inv}^k(E)$ is a G invariant embedded submanifold of $J^k(E)$.*

[ii] *The action of G on $\mathrm{Inv}^k(E)$ is transverse and regular.*

[iii] *The quotient manifold $\mathrm{Inv}^k(E)/G$ is diffeomorphic to $J^k(\tilde{\kappa}_G(E))$ and the diagram*

$$
\begin{array}{ccccc}
J^k(\tilde{\kappa}_G(E)) & \xleftarrow{\mathfrak{q}_{\mathrm{Inv}}} & \mathrm{Inv}^k(E) & \xrightarrow{\ \iota\ } & J^k(E) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{M} & \xleftarrow{\mathfrak{q}_M} & M & \xrightarrow{\mathrm{id}} & M
\end{array}
$$

commutes.

This theorem implies that the same hypothesis on the action of G on the bundle $\pi\colon E \to M$ needed to insure that the kinematic reduction diagram is a diagram of smooth manifolds and maps also insures that the bottom row of the dynamical reduction diagram (4.2) exists. Therefore to guarantee the smoothness of the entire dynamic reduction diagram one need only assume, in addition, that $\mathcal{D}_{\mathrm{Inv}}$ is a subbundle of $\mathcal{D}$.

D. The Automorphism Group of the Kinematic Bundle. For computations of the automorphism group of the kinematic bundle it is often advantageous to use the fact that $\tilde{\mathfrak{G}}^*$ fixes every G invariant section of E, that $\tilde{\mathfrak{G}}$ preserves the space of G invariant sections and that, conversely, under very mild assumptions, these properties characterize these groups.

Theorem 7.3. *Assume that there is a G invariant section through each point of $\kappa_G(E)$. Then the group $\tilde{\mathfrak{G}}^*$ coincides with the subgroup of $\mathfrak{G}$ which fixes every invariant section of E,*

$$\tilde{\mathfrak{G}}^* = \{\, a \in \mathfrak{G} \mid a \cdot s = s \quad \textit{for all } G \textit{ invariant sections } s\colon M \to E \,\}$$

and the group $\tilde{\mathfrak{G}}$ coincides with the subgroup of $\mathfrak{G}$ which preserves the set of G invariant sections of E,

$$\tilde{\mathfrak{G}} = \{\, a \in \mathfrak{G} \mid \ a \cdot s \textit{ is } G \textit{ invariant for all } G \textit{ invariant sections } s\colon M \to E \,\}.$$

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