Superposition Formulas for Darboux Integrable Exterior Differential Systems

Ian Anderson
Dept of Math. and Stat.
Utah State University

Mark Fels
Dept of Math. and Stat.
Utah State University

Peter Vassiliou
Dept. of Math and Stat.
University of Canberra

October 18, 2007

1 Introduction

In this paper we present a far-reaching generalization of E. Vessiot's analysis [24], [25] of the Darboux integrable partial differential equations in one dependent and two independent variables. Our approach provides new insights into this classical method, uncovers the fundamental geometric invariants of Darboux integrable systems, and provides for systematic, algorithmic integration of such systems. This work is formulated within the general framework of Pfaffian exterior differential systems and, as such, has applications well beyond those currently found in the literature. In particular, our integration method is applicable to systems of hyperbolic PDE such as the Toda lattice equations, [18], [19], [21], 2 dimensional wave maps [4] and systems of overdetermined PDE such as those studied by Cartan in [10].

Central to our generalization of the method of Darboux is the novel concept of a ***superposition formula*** for an exterior differential system $\mathcal{I}$ on a manifold M. The idea is to construct associated differential systems $\hat{\mathcal{W}}$, $\check{\mathcal{W}}$, on manifolds $\hat{M}$ and $\check{M}$ and a mapping

$$\Sigma\colon \hat{M} \times \check{M} \to M \tag{1.1}$$

such that

$$\Sigma^*(\mathcal{I}) \subset \hat{\pi}^*(\hat{\mathcal{W}}) + \check{\pi}^*(\check{\mathcal{W}}). \tag{1.2}$$

Here $\hat{\pi}^*(\hat{\mathcal{W}})$ and $\check{\pi}^*(\check{\mathcal{W}})$ are the differential systems generated by the pullbacks of $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$ to the product manifold $\hat{M} \times \check{M}$ by the canonical projection maps $\hat{\pi}^*$ and $\check{\pi}^*$. It is then clear that if $\hat{\phi}\colon \hat{N} \to \hat{M}$ and $\check{\phi}\colon \check{N} \to \check{M}$ are integral manifolds for $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$, then

$$\phi = \Sigma \circ (\hat{\phi}, \check{\phi})\colon \hat{N} \times \check{N} \to M \tag{1.3}$$

is an (possibly non-immersed) integral manifold of $\mathcal{I}$.

The concept of a superposition formula for an exterior differential system becomes a practical method for finding integral manifolds if it is possible

- **[i]** to establish general sufficiency conditions for the existence of superposition formula;
- **[ii]** to establish general sufficiency conditions under which the superposition formula yields *all* local integral manifolds of $\mathcal{I}$ (in which case we say the superposition formula is surjective); and
- **[iii]** to provide an algorithmic procedure for finding the superposition formula.

These are the goals of this paper.

The following examples illustrate the concept of a superposition formula and demonstrates that the algorithm presented here can be used to exactly integrate a wide range of differential equations.

Example 1.1. Perhaps the most well-known example of a non-linear Darboux integrable equation is Liouville's equation

$$u_{xy} = e^u. \tag{1.4}$$

According to the classical Darboux method the general solution to this equation, namely

$$u = \ln \frac{2\dot{f}(x)\dot{g}(y)}{(f(x)+g(y))^2}, \tag{1.5}$$

is to be found by integrating the Frobenius system

$$u_{xx} - \frac{1}{2}u_x^2 = \phi(x), \quad u_{xy} = e^u, \quad u_{yy} - \frac{1}{2}u_y^2 = \psi(y). \tag{1.6}$$

However it is very difficult to integrate these equations (they are essential Ricatti equations) for even simple choices of the arbitrary functions $\phi(x)$ and $\psi(y)$ and we know of no way to actually arrive at (1.5) from (1.6). Goursat is certainly silent on this point. The integration method developed in this paper circumvents equations (1.6) entirely and leads directly to (1.5).

The solution (1.5) can be viewed as a superposition formula as follows. The canonical Pfaffian system I for Liouville's equation is defined on a 7-manifold M with coordinates (x, y, u, p, q, r, t) by

$$I = \{\, du - p\,dx - q\,dy, dp - r\,dx - e^u\,dy, dq - e^u\,dx - t\,dy \,\}.$$

We take $\hat{W}$ and $\check{W}$ to be the canonical differential systems on the jet space $J^3(\mathbf{R}, \mathbf{R})$, with coordinates $(x, U, \dot{U}, \ddot{U}, \dddot{U})$ and $(y, V, \dot{V}, \ddot{V}, \dddot{V})$. Then it is easy to check that the mapping $\Sigma : J^3(\mathbf{R}, \mathbf{R}) \times J^3(\mathbf{R}, \mathbf{R}) \to M$ defined by

$$\Sigma(x, U, \dot{U}, \ddot{U}, \ddot{U}, y, \dot{V}, \dot{V}, \ddot{V}, \ddot{V}) = (x, y, U, P, Q, R, T), \quad \text{where}$$

$$u = \ln \frac{2\dot{U}\dot{V}}{(U+V)^2}, \quad p = \frac{\ddot{U}}{\dot{U}} - 2\frac{\dot{U}}{U+V}, \quad q = \frac{\ddot{V}}{\dot{V}} - 2\frac{\dot{V}}{U+V},$$

$$r = \frac{\dot{U}\dddot{U} - \ddot{U}^2}{\dot{U}^2} - \frac{\ddot{2}U}{U+V} + \frac{\dot{2}U^2}{(U+V)^2}, \quad t = \frac{\dot{V}\dddot{V} - \ddot{V}^2}{\dot{V}^2} - \frac{\ddot{2}V}{U+V} + \frac{\dot{2}V^2}{(U+V)^2}.$$

satisfies (1.2) and is therefore a superposition formula for Liouville's equation. We encourage the reader to check (1.2) by showing that the pullback of each generator of I is a linear combination of the standard contact forms defining $\hat{W}$ and $\hat{W}$. The first equation defining this superposition formula gives the classical form of the solution (1.5). ∎

Example 1.2. Although it is well-known that the equations

$$u_{xy} = 2n\frac{\sqrt{u_x u_y}}{x+y} \tag{1.7}$$

are Darboux integrable for each positive integer n [21], the general solution to these equations do not seem to have appeared in the literature. We find that there is a superposition formula for these equations where $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$ are the rank $n+1$ Pfaffian systems for the under-determined ordinary differential equations

$$\frac{dX}{dx} = \frac{d^nU}{dx^n}^{\,2} \quad \text{and} \quad \frac{dY}{dy} = \frac{d^nV}{dy^n}^{\,2}. \tag{1.8}$$

Each of these systems is equivalent to $J^3(\mathbf{R},\mathbf{R})$ when $n=1$. For $n=2$ these equations are the celebrated Hilbert-Cartan equation [6], [3]. The explicit superposition formulas for (1.7) have been found and in the cases $n=1$, 2 and 3 are given by

$$u = \frac{1}{2}(X+Y) + \frac{1}{\gamma}\begin{bmatrix} U_0 \\ V_0 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} U_0 \\ V_0 \end{bmatrix},$$

$$u = \frac{1}{2}(X+Y) + \frac{1}{\gamma^3}\begin{bmatrix} U_0 \\ U_1 \\ V_0 \\ V_1 \end{bmatrix}^t\begin{bmatrix} -12 & 6 & -12 & 6 \\ 6 & -4 & 6 & -2 \\ -12 & 6 & -12 & 6 \\ 6 & -2 & 6 & -4 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ V_0 \\ V_1 \end{bmatrix}, \quad \text{and}$$

$$u = \frac{1}{2}(X+Y) + \frac{1}{\gamma^6}\begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ V_0 \\ V_1 \\ V_2 \end{bmatrix}^t\begin{bmatrix} -720 & 360 & -60 & -720 & 360 & -60 \\ 360 & -192 & 36 & 360 & -168 & 24 \\ -60 & 36 & -9 & -60 & 24 & -3 \\ -720 & 360 & -60 & -720 & 360 & -60 \\ 360 & -168 & 24 & 360 & -192 & 36 \\ -60 & 24 & -3 & -60 & 36 & -9 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ V_0 \\ V_1 \\ V_2 \end{bmatrix},$$

where $\gamma = x+y$, $U_k = \gamma^{(n-k)}\dot{U}^{(k)}$, $V_k = \gamma^{(n-k)}\dot{V}^{(k)}$.

We remark that in the classical literature, a Darboux integrable equation is said to be of ***first class*** (Goursat [15], page) if the general solution can be given explicitly in a form free of quadratures. From the perspective provided by the concept of a superposition formula, it follows that *Darboux integrable equations are of first class precisely when the differential systems $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$ are jet spaces (or products and partial prolongations thereof)* . Thus (1.7) is of first class only when $n = 1$. ∎

Example 1.3. Now consider the system of second order PDE

$$v^\alpha_{xy} = \sum_{\beta=1}^{m} a_{\alpha\beta} \exp(v^\beta), \tag{1.9}$$

where the coefficients $a_{\alpha\beta}$ are constant and the $m \times m$ matrix $[a_{\alpha\beta}]$ is invertible. We remark that under the change of variables $v^\alpha = a_{\alpha\gamma} u^\gamma$, this system becomes

$$u^\alpha_{xy} = \exp(\sum_{\beta=1}^{m} a_{\alpha\beta} u^\beta). \tag{1.10}$$

When the matrix $[a_{\alpha\beta}]$ is the Cartan matrix of simple Lie algebra, then these equations admit a Lax-pair (or zero-curvature) formulation [19]. Shabat and Ziber state that these systems are Darboux integrable (at some prolongation order) although they do not present a proof nor do they provide the solutions to any of these systems. For the A_n Toda lattice equations, we are able to use methods presented here to find the general solution for small values of n and from these computations deduce the general closed form solution for all n. For example, the A_3 Toda lattice system is

$$u^1_{xy} = \exp(2u^1 - u^2), \quad u^2_{xy} = \exp(-u^1 + 2u^2 - u^3), \quad u^3_{xy} = \exp(-u^2 + 2u^3) \tag{1.11}$$

and the general closed form solution is

$$\begin{aligned} u^1 &= \ln \frac{\det(U, \dot{U}, \ddot{U}, \dddot{U})^{1/4} \det(V, \dot{V}, \ddot{V}, \dddot{V})^{3/4}}{\det(U, V, \dot{V}, \ddot{V})}, \\ u^2 &= \ln \frac{\det(U, \dot{U}, \ddot{U}, \dddot{U})^{1/2} \det(V, \dot{V}, \ddot{V}, \dddot{V})^{1/2}}{\det(U, \dot{U}, V, \dot{V})}, \\ u^3 &= \ln \frac{\det(U, \dot{U}, \ddot{U}, \dddot{U})^{1/4} \det(V, \dot{V}, \ddot{V}, \dddot{V})^{3/4}}{\det(U, \dot{U}, \ddot{U}, V)}. \end{aligned} \tag{1.12}$$

where $U = U(x)$ and $V = V(y)$ are 4-dimensional vectors of arbitrary functions of x and y. By using the determinantal identity (or syzygy)

$$\sum_\sigma \operatorname{sign}(\sigma) \det(A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}, A_{\sigma(4)}) \det(A_{\sigma(5)}, A_6, A_7, A_8) = 0,$$

where the sum is over the cyclic permutations σ of $\{1,2,3,4,5\}$, it is a simple exercise to verify directly that (1.12) solves (1.11). ∎

Example 1.4. Cartan's 1911 paper on overdetermined systems of second order PDE in 3 independent variables and 1 dependent variable [10] clearly indicates that such systems may admit superposition formulas. As our final introductory example, we consider the overdetermined (compatible) system of linear equations for a 1 function $u = u(x,y,z)$ given by

$$\begin{aligned} u_{yz} &= \frac{u_y}{xy-z} - \frac{4du}{(xy-z)^2} \\ u_{xzz} &= \frac{4u_{xz}}{zy-z} - \frac{2u_x}{(xy-z)^2} - \frac{4yu_z}{(xy-z)^2} + \frac{4yu}{(xy-z)^2} \end{aligned} \tag{1.13}$$

Our methods show that the general solution to this system is

$$u = 6\frac{U}{(xy-z)^2} + 2\frac{U_y}{x(xy-z)} + 6\frac{V}{(xy-z^2)} + 4\frac{\dot V}{(xy-z} + \ddot V \tag{1.14}$$

where $U = U(x,y)$ and $V = V(z)$. ∎

To intrinsically describe the class of differential systems for we shall construct superposition formulas we first introduce the definition of a decomposable Pfaffian system.

Definition 1.5. *Let $\mathcal{I}$ be a rank r Pfaffian system on an n-dimensional manifold M. Then $\mathcal{I}$ is said to be **decomposable** of type $[p,q]$, where $p+q = n-r$ and $p,q \geq 2$ if about each point of M there is a coframe*

$$\theta^1,\dots,\theta^r,\ \hat\pi^1,\dots,\hat\pi^p,\ \check\pi^1,\dots,\check\pi^q$$

with $\mathcal{I} = \{\theta^i\}$ and with structure equations (taken mod $\mathcal{I}$)

$$\begin{aligned} d\,\theta^i &\equiv 0 && \textit{for } i = 1\dots r_1, \\ d\,\hat\theta^i &\equiv \hat A^i_{ab}\,\hat\pi^a \wedge \hat\pi^b && \textit{for } i = r_1+1\dots r_2, \\ d\,\hat\theta^i &\equiv \check A^i_{\alpha\beta}\,\check\pi^\alpha \wedge \check\pi^\beta && \textit{for } i = r_2+1\dots r. \end{aligned} \tag{1.15}$$

The rank $r+p$ and $r+q$ Pfaffian systems

$$\hat V = \{\theta^i, \hat\pi^a\} \quad \textit{and} \quad \check V = \{\theta^i, \check\pi^\alpha\} \tag{1.16}$$

*are called the associated **singular Pfaffian systems** for $\mathcal{I}$.*

Without loss of generality, we assume that the derived Pfaffian system $\mathcal{I}'$ is generated by the 1-forms θ^i, for $i = 1, \ldots, r_1$. This implies that the $(r_2 - r_1) \times \binom{p}{2}$ matrix $[\hat{A}^i_{ab}]$ has rank $r_2 - r_1$ and the $(r - r_2) \times \binom{q}{2}$ matrix $[\check{A}^i_{\alpha\beta}]$ has rank $r - r_2$.

The structure equations for all of the above examples satisfy (1.15). In Section 2 we make a few simple observations regarding the geometry of decomposable Pfaffian systems. In the special case $p = q = 2$, our definition of a decomposable Pfaffian system coincides with the definition of a hyperbolic Pfaffian system given in [8].

Recall that a first integral for Pfaffian system a V be a on a manifold M is real-valued function f on M for which $df \in V$. The integral manifolds of V are constrained to the level set of any first integral. Let $V^{(\infty)}$ denoted the largest integrable subbundle of V – that is, $V^{(\infty)}$ is the terminal Pfaffian system in the sequence of derived Pfaffian systems $V \supset V' \supset V'' \ldots$. Then the rank of $V^{(\infty)}$ gives the number of functionally independent first integrals for V. Classically, a scalar second order partial differential equations in the plane is Darboux integrable if the associated singular Pfaffian systems $\hat{V}$ and $\check{V}$ each admit at least 2 (functionally independent) first integrals. In order to generalize the method of Darboux we need to determine the required number functionally independent first integrals necessary to integrate a general decomposable Pfaffian system. We do this with the following definition.

Definition 1.6. *A pair of Pfaffians systems $\hat{V}$ and $\check{V}$ define a* ***Darboux pair*** *if the following conditions hold.*

[i]
$$\hat{V} + \check{V}^{(\infty)} = T^*M \quad \textit{and} \quad \check{V} + \hat{V}^{(\infty)} = T^*M. \tag{1.17}$$

[ii]
$$\hat{V}^{(\infty)} \cap \check{V}^{\infty} = \{\, 0 \,\}. \tag{1.18}$$

[iii]
$$d\omega \in \Omega^2(\hat{V}) + \Omega^2(\check{V}) \quad \textit{for all} \quad \omega \in \Omega^1(\hat{V} \cap \check{V}) \; . \tag{1.19}$$

A decomposable Pfaffian system I is ***Darboux integrable*** *if the associated singular Pfaffian systems $\hat{V}$ and $\check{V}$ define a Darboux pair.*

Property **[i]** is the critical one – it will insure that there are a sufficient number of first integrals to construct a superposition formula. Property two is a technical condition states simply that $\hat{V}$ and $\hat{V}$ share no common integrals – this condition can always be satisfied by restricting $\hat{V}$ and $\check{V}$ to the joint level sets of any common integrals. The form of the structure equations for $\hat{V} \cap \check{V}$ required by property **[iii]** is always satisfied when $\hat{V}$ and $\check{V}$ are the singular Pfaffian systems for a decomposable Pfaffian system I.

The main result of this paper can now be formulated as follows.

Theorem 1.7. *Let I be a decomposable Pfaffian system with associated singular Pfaffian systems V and V. If $\hat{V}$, $\check{V}$ form a Darboux pair, then I admits a surjective superposition formula* (1.1).

The manifolds $\hat{M}$ and $\check{M}$ for the superposition formula are simply any level sets for $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$ (subject to a mild transversality condition) and the Pfaffian systems $\hat{W}$ and $\check{W}$ are just the restrictions of I to these level sets. The construction of the superposition formula then rests upon a remarkable sequence of normalizations of the structure equations for any Darboux pair $\hat{V}$, $\check{V}$. These normalizations are performed in Sections 3 of the paper and lead to the following result.

Theorem 1.8. *Let $\hat{V}$, $\check{V}$ be a Darboux pair of Pfaffian differential systems on a manifold M. Then around each point of M there are 2 coframes*

$$\{\, \hat{\theta}^i, \hat{\pi}^a, \check{\pi}^\alpha \,\} \quad \textit{and} \quad \{\, \check{\theta}^i, \hat{\pi}^a, \check{\pi}^\alpha \,\} \tag{1.20}$$

such that

$$\hat{V} = \{\, \hat{\theta}^i, \hat{\pi}^a \,\}, \quad \hat{V}^\infty = \{\, \hat{\pi}^a \,\}, \quad \check{V} = \{\, \check{\theta}^i, \check{\pi}^a \,\}, \quad \check{V}^\infty = \{\, \check{\pi}^a \,\} \tag{1.21}$$

and with structure equations

$$\begin{aligned} d\hat{\theta}^i &= \frac{1}{2} G^i_{\alpha\beta} \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \hat{\theta}^j \wedge \hat{\theta}^k \quad \textit{and} \\ d\check{\theta}^i &= \frac{1}{2} H^i_{ab} \hat{\pi}^a \wedge \hat{\pi}^b - \frac{1}{2} C^i_{jk} \check{\theta}^j \wedge \check{\theta}^k. \end{aligned} \tag{1.22}$$

The coefficients C^i_{jk} are the structure constants of a Lie algebra whose isomorphism class is an invariant of the Darboux pair $\hat{V}$, $\check{V}$.

The fact that structure equations of the form (1.22) can be found and that they are the key to the theory of Darboux integrability can be traced back to the seminal papers of Vessiot [24], [25]. Accordingly, we shall refer to the coframes (1.20) as the ***Vessiot normal form*** for the Darboux pair $\hat{V}$, $\check{V}$. We shall call the Lie algebra defined in Theorem 1.8 the ***Vessiot algebra*** for a Darboux pair

$\check{V}$,$\hat{V}$ and denote it by $\mathfrak{vess}(\hat{V}, \check{V})$. This algebra is the fundamental invariant of any Darboux pair and for any decomposable, Darboux integrable, Pfaffian system.

Example 1.9. For Liouville's equation (1.4) the Vessiot algebra is $sl(2)$. For (1.7) the Vessiot algebra is uniquely characterized as the $2n+1$ dimensional Lie algebra which is indecomposable and 1-step nilpotent. The Vessiot algebra for the A_n Toda lattice is $sl(n+1)$. For any linear system of Darboux integrable equations the Vessiot algebra is abelian.

If G is any matrix group and U a map from $\mathbf{R}^2$ to G, then we shall see that Vessiot algebra of the equations

$$U_{xy} = U_x U^{-1} U_y \tag{1.23}$$

coincides with the Lie algebra of G. ∎

The superposition formula then follows from Theorem 1.8. Let G be a (local) group whose Lie algebra is the Vessiot algebras of the Darboux pair $\hat{V}, \check{V}$. We show that the vector fields dual to the forms $\hat{\theta}^i$ and $\check{\theta}^i$ constructed in Theorem 1.8 to define local commuting group actions $\hat{\mu} : G \times M \to M$ and $\check{\mu}\colon G \times M \to M$. with the same orbits. The final part of the proof of Theorem 1.7, found in Section 4, is to construct the superposition formula for $\mathcal{I}$ using these local actions and the multiplication formula for G. *It is a remarkable fact that the superposition formula is given by a group theoretic construction which is altogether independent of the symmetry group of the differential system.*

We then provide a number of examples. The extensive computations required by this example was done using the DifferentialGeometry package in Maple 11. Copies of the Maple worksheets detailing these computations are available from the authors. This research was supported by NSF grant DMS-0410373.

2 Decomposable Pfaffian systems

In this section we provide an algorithmic method for characterizing a decomposable Pfaffian systems I in terms of its singular integral 1-elements. We also indicate how Definition 1.5 may be extended to certain non-Pfaffian systems which are algebraically generated by 1-forms and 2-forms and we also give the definition of decomposable Pfaffian systems with independence condition.

Fix a point p in M. Then each non-zero vector $X \in \mathrm{Ann}(I)_p$ defines an integral 1-element of I and the polar equations defined by X are, by definition, the linear system of equations for $Y \in \mathrm{Ann}(I)_p$ given by

$$d\,\theta(X, Y) = 0 \quad \text{for all 1-forms } \theta \in I. \tag{2.1}$$

The vector X is said to be ***regular*** if the rank of its polar equations is maximal and ***singular*** otherwise. Let I be a decomposable Pfaffian system of type $[p, q]$ with structure equations (1.15). Let $\{\, \partial_{\theta^i}, \partial_{\hat{\pi}^a}, \partial_{\check{\pi}^\alpha} \,\}$ denote the dual frame. Then it is easy to check that any vector of the form $X_1 = t^a \partial_{\hat{\pi}^a}$ or $X_2 = s^\alpha \partial_{\check{\pi}^\alpha}$ is a singular vector and that the plane spanned by X_1, X_2 is an integral 2-plane. Thus $\mathrm{Ann}(I)$ decomposes into a direct sum of rank p and q subbundles

$$\mathrm{Ann}(I) = S_1 \oplus S_2 \tag{2.2}$$

such that:

[i] every vector in S_1 and every vector in S_2 is singular, and

[ii] every 2 plane spanned by any pair of vectors X and Y, with $X \in S_1$ and $Y \in S_2$, is an integral 2-plane.

Conversely, suppose that I is a Pfaffian system such that $\mathrm{Ann}(I)$ admits a direct sum decomposition (2.2) satisfying [i] and [ii]. In general, these conditions may not be sufficient to imply that I is decomposable but they do imply that there exists local coframes

$$\theta^1, \ldots, \theta^r,\ \hat{\pi}^1, \ldots, \hat{\pi}^p, \check{\pi}^1, \ldots, \check{\pi}^q$$

with $\mathcal{I} = \{\, \theta^i \,\}$ and with structure equations

$$d\theta^i \equiv A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta \qquad \text{mod } \mathcal{I}. \tag{2.3}$$

It now follows that if the linear maps $A = [A^i_{ab}]\colon \mathbf{R}^r \to \Lambda^2(\mathbf{R}^p)$ and $B = [B^i_{\alpha\beta}]\colon \mathbf{R}^r \to \Lambda^2(\mathbf{R}^q)$ are constant rank and satisfy

$$\operatorname{rank}(A) + \operatorname{rank}(B) = \dim(I) - \dim(I'), \tag{2.4}$$

then I is decomposable. Indeed, since (2.4) implies that

$$\begin{aligned} \dim(\ker(A) + \ker(B)) &= \dim(\ker(A)) + \dim(\ker(B)) - \dim(\ker(A) \cap \ker(B)) \\ &= 2r - \operatorname{rank}(A) - \operatorname{rank}(B) - \dim(I') = r, \end{aligned} \tag{2.5}$$

we can find r linearly independent r-dimensional column vectors $T^1, \dots, T^{r_1}, \dots, T^{r_2}, \dots, T^r$ such that

$$\begin{aligned} \ker(A) \cap \ker(B) &= \operatorname{span}\{\, T^1, \dots, T^{r_1} \,\}, \\ \ker(A) &= \operatorname{span}\{\, T^1, \dots, T^{r_1}, T^{r_1+1}, \dots, T^{r_2} \,\}, \quad \text{and} \\ \ker(B) &= \operatorname{span}\{\, T^1, \dots, T^{r_1}, T^{r_2+1}, \dots, T^r \,\}. \end{aligned}$$

The 1-forms $\tilde{\theta}^i = t^i_j \theta^j$, where $T^i = [t^i_j]$, then satisfy (1.15). Note that the Pfaffian systems (1.16) coincide with $\operatorname{Ann}(S_1)$ and $\operatorname{Ann}(S_1)$ and it for this reason that we have opted to called these systems the singular Pfaffian systems associated to the decomposition of I.

Remark 2.1. It is possible to generalize the definition of decomposable Pfaffian systems and Darboux integrability to include certain classes of more general exterior differential systems. We say that an exterior differential system $\mathcal{I}$ is decomposable of type (p, q), where $p, q \geq 2$, if there is a coframe

$$\theta^1, \dots, \theta^r, \ \hat{\pi}^1, \dots, \hat{\pi}^p, \ \check{\pi}^1, \dots, \check{\pi}^q$$

such that $\mathcal{I}$ is algebraically generated by the r 1-forms θ^i, and two forms

$$\hat{\Omega}^1, \dots, \hat{\Omega}^s \in \Omega^2(\hat{\pi}^1, \dots, \hat{\pi}^p), \quad \text{and} \quad \check{\Omega}^1, \dots, \check{\Omega}^t \in \Omega^2(\check{\pi}^1, \dots, \check{\pi}^p), \tag{2.6}$$

and with structure equations

$$d\,\theta^i = \sum_{j=1}^{s} A^i_j\, \hat{\Omega}^j + \sum_{j=1}^{t} B^i_j\, \check{\Omega}^j, \tag{2.7}$$

where, as before, $\dim(\ker A) + \dim(\ker B) = r$. At each point the vector space of integral 1-elements for such EDS is still a direct sum of subspaces of singular integral 1-elements; the associated singular Pfaffian systems $\check{V}$ and $\hat{V}$ are defined as before; and the EDS $\mathcal{I}$ is defined to be Darboux integrable if $\check{V}$ and $\hat{V}$ define a Darboux pair.

There are many examples of differential equations which can be described either by Pfaffian differential systems or by differential systems generated by 1-forms and 2-forms – with this extended definition of Darboux integrability such differential equations can satisfy the definition of Darboux integrable with respect to different encodings as an exterior differential system. The simplest example – but by no means the only example – is the wave equation $u_{xy} = 0$. Its encoding as exterior differential systems on manifold of dimensions 5, 6 and 7 by

$$I_1 = \{ du - p\,dx - q\,dy, dp \wedge dx, dq \wedge dy \} \tag{2.8a}$$

$$I_2 = \{ du - p\,dx - q\,dy, dp - r\,dx, dq \wedge dy \} \tag{2.8b}$$

$$I_3 = \{ du - p\,dx - q\,dy, dp - r\,dx, dq - t dy \} \tag{2.8c}$$

all satisfy this definition of a decomposable, Darboux integrable exterior differential system.

The associated singular Pfaffian systems are defined as before and we shall see that Theorem 1.7 is easily extended to include decomposable EDS of this type. ∎

Remark 2.2. To each decomposable exterior differential system we have associated a pair of Pfaffian systems $\check{V}$ and $\hat{V}$ and we have defined what it means for $\check{V}$ and $\hat{V}$ to define a Darboux pair. Here we remark that, conversely, starting with a Darboux pair, there is a way to construct a decomposable exterior differential system whose associated singular Pfaffian systems coincide with the given Darboux pair. Accordingly, the Darboux pair can be taken as the fundamental object of study, a viewpoint which we shall adopt in the next two sections.

This construction is as follows. Let $\mathcal{I}$ and $\mathcal{J}$ be two Pfaffian systems on a manifold M. Define a new exterior differential system $\mathcal{K} = \mathcal{I} \Cap \mathcal{J}$ to be the largest exterior differential system $\mathcal{K} \subset \mathcal{I} \cap \mathcal{J}$ with the property that every integral element of $\mathcal{K}$ is either an integral element of $\mathcal{I}$, an integral element of $\mathcal{J}$, or a direct sum of an integral element of $\mathcal{I}$ with an integral element of $\mathcal{J}$. If $\check{\mathcal{V}}$ and $\hat{\mathcal{V}}$ define a Darboux pair, then it is not difficult to show that $\check{\mathcal{V}} \Cap \hat{\mathcal{V}}$ is always a decomposable exterior differential system in the sense of Remark 2.1.

Moreover, if $\mathcal{I}$ is a decomposable exterior differential system whose associated singular systems $\check{\mathcal{V}}$ and $\hat{\mathcal{V}}$ define a Darboux pair, then $\check{\mathcal{V}} \Cap \hat{\mathcal{V}} = \mathcal{I}$. ∎

Remark 2.3. We may also extend our definition of a decomposable EDS to EDS $\mathcal{I}$ with independence condition $\omega^1 \wedge \cdots \wedge \omega^m \neq 0$. Here we demand the existence of local coframes

$$\theta^1, \ldots, \theta^r,\ \hat{\omega}^1, \ldots, \hat{\omega}^{m_1}, \hat{\pi}^1, \ldots, \hat{\pi}^p,\ \check{\omega}^1, \ldots, \check{\omega}^{m_2}, \check{\pi}^1, \ldots, \check{\pi}^q$$

with $m_1 + p \geq 2$, $m_2 + q \geq 2$,

$$\omega^1 \wedge \cdots \wedge \omega^m = \hat{\omega}^1 \wedge \cdots \wedge \hat{\omega}^{m_1} \wedge \check{\omega}^1 \wedge \cdots \wedge \check{\omega}^{m_2},$$

and replace (2.6) with the conditions

$$\hat{\Omega}^j = \sum_{\substack{1 \leq a \leq p,\\ 1 \leq b \leq m_1}} \hat{N}^i_{ab}\, \hat{\pi}^a \wedge \hat{\omega}^b \quad \text{and} \quad \check{\Omega}^j = \sum_{\substack{1 \leq \alpha \leq q,\\ 1 \leq \beta \leq m_2}} \check{N}^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\omega}^\beta.$$

The partial differential equations in all our examples may all be as treated as decomposable Pfaffian system with independence condition.

It is not difficult to check that if $\mathcal{I}$ is a decomposable Pfaffian system with independence condition, then the prolongation of $\mathcal{I}$ is also a decomposable Pfaffian system with independence condition. ∎

3 The coframe adaptations for a Darboux pair

In this section we present a series of coframe adaptations for any Darboux pair which lead to the proof of Theorem 1.8. In very special cases such as Liouville's equation one finds that $\hat{V} \cap \check{V}^{(\infty)} = \hat{V}^{(\infty)} \cap \check{V} = \phi$ so that the vector space sums appearing in (1.17) are direct sum decompositions. Under these circumstances, the initial decomposable coframe is automatically 2-adapted and the sequence of coframe adaptations can begin with the third coframe adaptation. When the vector space sums in (1.17) are not direct (as is generally the case for Darboux integrable systems in more than 1 dependent variable), one must begin with the first 2 coframe adaptations. The third and fourth coframe adaptations lead to the definition of the Vessiot algebra as a fundamental invariant for any Darboux integrable differential system. The importance of this algebra was noted by Vassiliou [23] who referred to as the tangential symmetry algebra. This algebra also points a special in Eendebak's theory [12] of pseudo-symmetries for Darboux integrable equations. At this point one also has a Lie algebra of vector fields Γ on M which structure constants define the structure constants for the Vessiot algebra. Of course, Γ determines a local transformation group on M but this transformation group will generally not be the correct transformation group for constructing the superposition formula. One final coframe adaption is needed to arrive at the proper Vessiot transformation groups for the constructing of the superposition formula – this is the coframe satisfies the structure equations given in Theorem 1.8.

The coframe adaptations 1–4 involve only differentiations and elementary linear algebra operations. For the fifth coframe adaptation and in the case where the Vessiot algebra is not semi-simple, one must integrate a linear system of total differential equations (see (3.68)) and also solve the exterior derivative equation $d(\phi) = \chi$ where χ is a closed 2-form (see (3.71)).

3.1 The zero and first adapted coframe for a Darboux pair

Let $\{\hat{V}, \check{V}\}$ be a Darboux pair on a manifold M. In addition to the properties listed in Definition 1.6, we shall assume that the derived systems $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$, as well as the intersections $\hat{V}^{(\infty)} \cap \check{V}$, $\hat{V} \cap \check{V}^{(\infty)}$ and $\hat{V} \cap \check{V}$ are all (constant rank) subbundles of T^*M.

To define our initial adapted coframe for the Darboux pair $\hat{V}, \check{V}$, we first chose tuples of independent 1-forms $\hat{\boldsymbol{\eta}}$ and $\check{\boldsymbol{\eta}}$ such that

$$\hat{V}^{(\infty)} \cap \check{V} = \operatorname{span}\{\hat{\boldsymbol{\eta}}\} \quad \text{and} \quad \hat{V} \cap \check{V}^{(\infty)} = \operatorname{span}\{\check{\boldsymbol{\eta}}\} \tag{3.1a}$$

and then chose tuples $\boldsymbol{\theta}$, $\hat{\boldsymbol{\sigma}}$ and $\check{\boldsymbol{\sigma}}$ of 1-forms such that

$$\hat{V}^{(\infty)} = \operatorname{span}\{\,\hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,\}, \quad \hat{V}^{(\infty)} = \operatorname{span}\{\,\hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,\}$$
$$\text{and} \quad \check{V} \cap \hat{V} = \operatorname{span}\{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\,\}. \tag{3.1b}$$

The sets of 1-forms $\{\,\hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,\}$, $\{\,\hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,\}$ and $\{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\,\}$ are taken to be independent. It is then not difficult to check that the totality of these 1-forms define a local coframe for T^*M and that

$$\hat{V} = \operatorname{span}\{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\,\} \quad \text{and} \quad \check{V} = \operatorname{span}\{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\}. \tag{3.1c}$$

Any local coframe $\{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\}$ satisfying (3.1) is is called a ***0-adapted coframe*** for the Darboux pair $\{\,\hat{V},\, \check{V}\}$.

We shall also need the dual definition of a 0-adapted frame. Let

$$\hat{H} = \operatorname{ann} \hat{V}, \quad \check{H} = \operatorname{ann} \check{V}, \quad \hat{H}^{(\infty)} = \operatorname{ann} \hat{V}^{(\infty)}, \quad \check{H}^{(\infty)} = \operatorname{ann} \check{V}^{(\infty)},$$
$$\text{and} \qquad K = \hat{H}^{(\infty)} \cap \check{H}^{(\infty)}. \tag{3.2}$$

If we introduce the dual basis $\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}},\, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}},\, \boldsymbol{\partial}_{\hat{\eta}},\, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}},\, \boldsymbol{\partial}_{\check{\eta}}\,\}$ to the 0-adapted coframe $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$, that is,

$$\boldsymbol{\theta}(\boldsymbol{\partial}_{\boldsymbol{\theta}}) = \mathbf{1}, \quad \hat{\boldsymbol{\sigma}}(\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}) = \mathbf{1}, \quad \hat{\boldsymbol{\eta}}(\boldsymbol{\partial}_{\hat{\eta}}) = \mathbf{1}, \quad \check{\boldsymbol{\sigma}}(\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}) = \mathbf{1}, \quad \check{\boldsymbol{\eta}}(\boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}) = \mathbf{1}, \tag{3.3}$$

with all others pairings yielding 0, then it is not difficult to check that

$$\hat{H} = \operatorname{span}\{\,\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}\,\}, \qquad \hat{H}^{(\infty)} = \operatorname{span}\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}},\, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}},\, \boldsymbol{\partial}_{\check{\eta}}\,\},$$
$$\check{H} = \operatorname{span}\{\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}\}, \qquad \check{H}^{(\infty)} = \operatorname{span}\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}},\, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}},\, \boldsymbol{\partial}_{\hat{\eta}}\,\},$$
$$\text{and} \quad K = \operatorname{span}\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}}\,\}. \tag{3.4}$$

For example, from equations (3.1c) and (3.2) we have immediately that

$$\hat{H} = \operatorname{ann} \hat{V} = \operatorname{ann} \{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\,\} = \operatorname{span}\{\,\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}\,\}.$$

Any local frame $\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}},\, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}},\, \boldsymbol{\partial}_{\hat{\eta}},\, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}},\, \boldsymbol{\partial}_{\check{\eta}}\,\}$ satisfying (3.4) is called a 0-adapted frame. Naturally, the dual to a 0-adapted frame is a 0-adapted coframe.

Lemma 3.1. *Let f be a real-valued function on M. If $X(f) = 0$ for all vector fields in $\hat{H}$, then $df \in \hat{V}^{(\infty)}$. Likewise, if $X(f) = 0$ for all vector fields in $\check{H}$, then $df \in \check{V}^{(\infty)}$.*

Proof. It suffices to note that if $X(f) = 0$ for all vector fields $X \in \hat{H}$, then $Z(f) = Z \,\lrcorner\, df = 0$ for all vector fields $Z \in \hat{H}^{(\infty)}$ and therefore $df \in \operatorname{ann} \hat{H}^{(\infty)} = \hat{V}^{(\infty)}$. ∎

Our 1st adaption of the coframe for a Darboux pair is easily constructed from complete sets of functionally independent invariants for $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$, that is, functions $\{\,\hat{\boldsymbol{I}}\,\}$ and $\{\,\check{\boldsymbol{I}}\,\}$ such that

$$\hat{V}^{(\infty)} = \operatorname{span}\{\, d\hat{\boldsymbol{I}}\,\} \quad \text{and} \quad \check{V}^{(\infty)} = \operatorname{span}\{\, d\check{\boldsymbol{I}}\,\}.$$

First, from the 1-forms $d\hat{\boldsymbol{I}}$ we select a maximal set of 1-forms $\{\, d\hat{\boldsymbol{I}}_1\,\}$ which are independent of the 1-forms $\hat{\boldsymbol{\eta}}$. These differentials give a local basis for the bundle $\hat{V}^{(\infty)}$ and accordingly we may chose

$$\hat{\boldsymbol{\sigma}} = d\hat{\boldsymbol{I}}_1 \quad \text{so that} \quad d\hat{\boldsymbol{\sigma}} = 0. \tag{3.5}$$

Next let $\{\,\hat{\boldsymbol{I}}_2\,\}$ be a complementary set of invariants to the set $\{\,\hat{\boldsymbol{I}}_1\,\}$ just chosen so that $\{\,\hat{\boldsymbol{I}}\} = \{\,\hat{\boldsymbol{I}}_1,\, \hat{\boldsymbol{I}}_2\,\}$. Because the forms $\hat{\boldsymbol{\eta}}$ belong to $\hat{V}^{\infty}$, we can write

$$\hat{\boldsymbol{\eta}} = \widehat{\mathbf{R}}_0\, d\hat{\boldsymbol{I}}_1 + \hat{\boldsymbol{S}}_0\, d\hat{\boldsymbol{I}}_2$$

Since the 1-forms $\hat{\boldsymbol{\eta}}$ are independent of the 1-forms $\hat{\boldsymbol{\sigma}}$, the coefficient matrix $\hat{\boldsymbol{S}}_0$ must be invertible and we can therefore adjust our coframe $\{\,\hat{\boldsymbol{\eta}}\,\}$ for $\hat{V}^{(\infty)} \cap \check{V}$ by setting

$$\hat{\boldsymbol{\eta}} = d\hat{\boldsymbol{I}}_2 + \widehat{\mathbf{R}}\, d\hat{\boldsymbol{I}}_1 = d\hat{\boldsymbol{I}}_2 + \widehat{\mathbf{R}}\, \hat{\boldsymbol{\sigma}}.$$

The exterior derivative of these forms is

$$d\hat{\boldsymbol{\eta}} = d\widehat{\mathbf{R}} \wedge \hat{\boldsymbol{\sigma}} = \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}(\widehat{\mathbf{R}})\, \check{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}} + \cdots$$

and therefore, on account of (1.19) and the fact that $\hat{\boldsymbol{\eta}} \in \check{V} \cap \hat{V}$, we must have $\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}(\widehat{\mathbf{R}}) = 0$. Lemma 3.1 implies that $d(\widehat{\mathbf{R}}) \in \hat{V}^{(\infty)}$ and consequently

$$d\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{E}}\, \hat{\boldsymbol{\eta}} \wedge \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{F}} \hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}. \tag{3.6}$$

Theorem 3.2. *Let* $\{\hat{V}, \check{V}\}$ *be a Darboux pair on a manifold* M*. Then about each point of* M *there exists 0-adapted coframe* $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$ *with structure equations*

$$\begin{aligned} d\check{\boldsymbol{\sigma}} &= 0, \qquad d\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{E}}\,\hat{\boldsymbol{\eta}} \wedge \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{F}}\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}, \\ d\check{\boldsymbol{\sigma}} &= 0, \qquad d\check{\boldsymbol{\eta}} = \check{\boldsymbol{E}}\,\check{\boldsymbol{\eta}} \wedge \check{\boldsymbol{\sigma}} + \check{\boldsymbol{F}}\,\check{\boldsymbol{\eta}} \wedge \check{\boldsymbol{\sigma}}, \end{aligned} \tag{3.7}$$

$$d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}} + \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}} \qquad \mathrm{mod}\ \{\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}\}. \tag{3.8}$$

We remark that the structure equations for the 1-forms $\boldsymbol{\theta} \in \hat{V} \cap \check{V}$ are a consequence of (1.19) and (3.1). A ***1-adapted coframe*** for the Darboux pair $\hat{V}, \check{V}$ is a 0-adapted coframe $\{\boldsymbol{\theta}, \hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$ satisfying the structure equations (3.7) and (3.8).

3.2 The second adapted coframe for a Darboux pair.

Let $\{\hat{V}, \check{V}\}$ be a Darboux pair and let $\{\,\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\,\}$ be a 1-adapted coframe with dual frame $\{\,\boldsymbol{\partial_\theta}, \boldsymbol{\partial_{\hat{\sigma}}}, \boldsymbol{\partial_{\hat{\eta}}}, \boldsymbol{\partial_{\check{\sigma}}}, \boldsymbol{\partial_{\check{\eta}}}\,\}$. From this point forward, we are solely interested in adjustments to the forms $\boldsymbol{\theta}$ that will simply the structure equations

$$d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}} + \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}} \mod \{\,\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}\,\}. \tag{3.9}$$

Of the "mixed" wedge products

$$\hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}, \quad \hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\eta}}, \quad \hat{\boldsymbol{\eta}} \wedge \check{\boldsymbol{\sigma}}, \quad \text{and} \quad \hat{\boldsymbol{\eta}} \wedge \check{\boldsymbol{\sigma}} \tag{3.10}$$

the products $\hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}$ are the only ones definitely not present in (3.9). We shall now show that it is possible to make an adjustment to the $\boldsymbol{\theta}$ of the form

$$\boldsymbol{\theta'} = \boldsymbol{\theta} + \boldsymbol{P}\,\hat{\boldsymbol{\eta}} + \boldsymbol{Q}\,\check{\boldsymbol{\eta}} \tag{3.11}$$

so that the structure equations for the modified $\boldsymbol{\theta'}$ are free of all the wedge products (3.10). We begin with the following simple observation.

Lemma 3.3. *If* $\{\,\boldsymbol{\partial_\theta}, \boldsymbol{\partial_{\hat{\sigma}}}, \boldsymbol{\partial_{\hat{\eta}}}, \boldsymbol{\partial_{\check{\sigma}}}, \boldsymbol{\partial_{\check{\eta}}}\,\}$ *is the dual frame to a 1-adapted coframe for the Darboux pair* $\{\,\hat{V}, \check{V}\,\}$*, then*

$$[\,\boldsymbol{\partial_{\hat{\sigma}}}, \boldsymbol{\partial_{\check{\sigma}}}\,] = 0. \tag{3.12}$$

Proof. It suffices to note that none of the structure equations for a 1-adapted coframe contain any of the wedge products $\hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}$. ∎

The construction of the second adapted coframe is completely algebraic in that only differentiations and linear algebraic operations are involved. Let $\hat{S}_a = \partial_{\hat{\sigma}^a}$ and $\hat{S}_\alpha = \partial_{\check{\sigma}^a}$ and define two sequences of vector field inductively by

$$\hat{S}_{a_1a_2\cdots a_\ell} = [\,\hat{S}_{a_1a_2\cdots a_{\ell-1}}, \hat{S}_{a_\ell},\,] \quad \text{and} \quad \check{S}_{\alpha_1\alpha_2\cdots\alpha_\ell} = [\,\check{S}_{\alpha_1\alpha_2\cdots\alpha_{\ell-1}}\,\check{S}_{\alpha_\ell}\,]. \tag{3.13}$$

The vector fields $S_{a_1a_2\cdots a_\ell}$ belong to $\hat{H}^{(\infty)}$, and in fact, by virtue of the Jacobi identity,

$$\hat{H}^{(\infty)} = \operatorname{span}\{\,\hat{S}_{a_1a_2\cdots a_\ell}\,\}_{\ell\geq 0} \quad \text{and, likewise,} \quad \check{H}^{(\infty)} = \operatorname{span}\{\,\check{S}_{\alpha_1\alpha_2\cdots\alpha_\ell}\,\}_{\ell\geq 0}. \tag{3.14}$$

A simple induction argument, based upon (3.12) and the Jacobi identity, also shows that

$$[\,\hat{S}_{a_1a_2\cdots a_k}, \check{S}_{\alpha_1\alpha_2\cdots\alpha_\ell}\,] = 0. \tag{3.15}$$

Now recall that the frame elements $\{\boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}\}$ and $\{\boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\}$ are chosen to be any complementary sets of vectors such that (3.4) hold. By virtue of (3.15) we can in fact choose iterated Lie brackets $\hat{S}_{a_1a_2\cdots a_\ell}$ and $\check{S}_{\alpha_1\alpha_2\cdots\alpha_\ell}$ which complete $\{\boldsymbol{\partial}_{\boldsymbol{\theta}},\, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}\}$ and $\{\boldsymbol{\partial}_{\boldsymbol{\theta}},\, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}\}$ to local bases for $\hat{H}^{(\infty)}$ and $\check{H}^{(\infty)}$ respectively. Denote the iterated brackets so chosen by $\{\hat{\boldsymbol{l}}'\}$ and $\{\check{\boldsymbol{l}}'\}$. This gives us a preferred 0-adapted coframe $\{\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \hat{\boldsymbol{l}}', \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \check{\boldsymbol{l}}'\}$ where, on account of (3.15),

$$[\,\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}},\, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}\,] = 0, \quad [\,\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}},\, \check{\boldsymbol{l}}'\,] = 0 \quad [\,\hat{\boldsymbol{l}}',\, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}\,] = 0, \quad [\,\hat{\boldsymbol{l}}',\, \check{\boldsymbol{l}}'\,] = 0. \tag{3.16}$$

The vectors $\{\hat{\boldsymbol{l}}'\}$ and $\{\check{\boldsymbol{l}}'\}$ can be expressed in terms of the original frame by

$$\hat{\boldsymbol{l}}' = (^*)\,\boldsymbol{\partial}_{\boldsymbol{\theta}} + (^*)\,\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}} + (^\dagger)\,\boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}} \quad \text{and} \quad \check{\boldsymbol{l}}' = (^*)\,\boldsymbol{\partial}_{\boldsymbol{\theta}} + (^*)\,\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}} + (^\dagger)\,\boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}. \tag{3.17}$$

The matrices denoted by $(^\dagger)$ are invertible from which it follows that the coframe dual to $\{\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \hat{\boldsymbol{l}}', \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \check{\boldsymbol{l}}'\}$ is a 0-adapted coframe for the Darboux pair $\hat{V}$, $\check{V}$. On account of (3.16) the structure equations for the forms $\{\boldsymbol{\theta}'\}$ are free of all the wedge products $\hat{\boldsymbol{\sigma}}' \wedge \check{\boldsymbol{\sigma}}'$, $\hat{\boldsymbol{\sigma}}' \wedge \check{\boldsymbol{\eta}}'$, $\hat{\boldsymbol{\eta}}' \wedge \check{\boldsymbol{\sigma}}'$, and $\hat{\boldsymbol{\eta}}' \wedge \check{\boldsymbol{\sigma}}'$. We can express this result by writing

$$d\boldsymbol{\theta}' \in \Omega^2(\hat{V}^{(\infty)}) + \Omega^2(\check{V}^{(\infty)}) \mod \{\boldsymbol{\theta}'\}. \tag{3.18}$$

Theorem 3.4. *Let $\{\hat{V}, \check{V}\}$ be a Darboux pair on a manifold M. Then about each point of M there exists a 0-adapted coframe $\{\boldsymbol{\theta},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\}$ with structure equations* (3.7) *and*

$$d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}} + \boldsymbol{B}\,\check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}} \qquad \mod \{\boldsymbol{\theta}\}. \tag{3.19}$$

where $\hat{\boldsymbol{\pi}}$ and $\check{\boldsymbol{\pi}}$ denote the tuples of forms $[\,\hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,]$ and $[\,\check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,]$.

Coframes which satisfy the conditions of Theorem 3.1 are call ***2-adapted coframes***.

3.3 The third adapted coframes for a Darboux pair

Written out in full, the structure equations (3.19) are

$$d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}} + \boldsymbol{B}\,\check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}} + \boldsymbol{C}\,\boldsymbol{\theta} \wedge \boldsymbol{\theta} + \boldsymbol{M}\,\hat{\boldsymbol{\pi}} \wedge \boldsymbol{\theta} + \boldsymbol{N}\,\check{\boldsymbol{\pi}} \wedge \boldsymbol{\theta}. \tag{3.20}$$

We obtain the third coframe reduction for a Darboux pair by showing that a change of frame $\boldsymbol{\theta'} = \boldsymbol{P}\,\boldsymbol{\theta}$ can be made so as to eliminate either all the wedge products $\hat{\boldsymbol{\pi}} \wedge \boldsymbol{\theta}$ *or* all the wedge products $\check{\boldsymbol{\pi}} \wedge \boldsymbol{\theta}$ in (3.20). The construction of this coframe uses another set of iterated Lie brackets.

We start with a 2-adapted coframe$\{\,\boldsymbol{\theta},\,\hat{\boldsymbol{\sigma}},\,\hat{\boldsymbol{\eta}},\,\check{\boldsymbol{\sigma}},\,\check{\boldsymbol{\eta}},\}$ and introduce the provisional frame $\{\,\boldsymbol{\theta},\,\hat{\boldsymbol{\iota}},\check{\boldsymbol{\iota}}\}$, where $\hat{\boldsymbol{\iota}} = d\hat{\boldsymbol{I}}$ and $\check{\boldsymbol{\iota}} = d\check{\boldsymbol{I}}$. This is not a 0-adapted coframe although it is the case that

$$\operatorname{span}\{\,\hat{\boldsymbol{\iota}}\,\} = \operatorname{span}\{\hat{\boldsymbol{\sigma}},\,\hat{\boldsymbol{\eta}}\} \quad \text{and} \quad \operatorname{span}\{\,\check{\boldsymbol{\iota}}\,\} = \operatorname{span}\{\check{\boldsymbol{\sigma}},\,\check{\boldsymbol{\eta}},\} \tag{3.21}$$

The structure equations for this frame are

$$\begin{gathered} d\hat{\boldsymbol{\iota}} = 0, \quad d\check{\boldsymbol{\iota}} = 0, \qquad \text{and} \\ d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\iota}} \wedge \hat{\boldsymbol{\iota}} + \boldsymbol{B}\,\check{\boldsymbol{\iota}} \wedge \check{\boldsymbol{\iota}} + \boldsymbol{C}\,\boldsymbol{\theta} \wedge \boldsymbol{\theta} + \boldsymbol{M}\,\hat{\boldsymbol{\iota}} \wedge \boldsymbol{\theta} + \boldsymbol{N}\,\check{\boldsymbol{\iota}} \wedge \boldsymbol{\theta}. \end{gathered} \tag{3.22}$$

Denote the dual to this provisional coframe by $\{\,\boldsymbol{\partial_\theta}, \hat{\boldsymbol{U}}, \check{\boldsymbol{U}}\}$. Since $\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}} = \operatorname{ann} \check{V} = \operatorname{ann}\{\boldsymbol{\theta}, \hat{\boldsymbol{\iota}}, \check{\boldsymbol{\sigma}}\} \subset \operatorname{ann}\{\boldsymbol{\theta}, \hat{\boldsymbol{\iota}}\} = \operatorname{span}\{\,\check{\boldsymbol{U}}\,\}$, we have that

$$\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}} \subset \operatorname{span}\{\,\check{\boldsymbol{U}}\,\} \quad \text{and} \quad \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}} \subset \operatorname{span}\{\,\hat{\boldsymbol{U}}\,\}. \tag{3.23}$$

From (3.22) it follows that the structure equations for the vectors fields $\hat{\boldsymbol{U}}$ and $\hat{\boldsymbol{U}}$ are

$$\begin{gathered} [\,\hat{\boldsymbol{U}},\,\hat{\boldsymbol{U}}\,] = -\boldsymbol{A}\boldsymbol{\partial_\theta}, \qquad [\,\hat{\boldsymbol{U}},\,\boldsymbol{\partial_\theta}\,] = -\boldsymbol{M}\boldsymbol{\partial_\theta}, \\ [\,\check{\boldsymbol{U}},\,\check{\boldsymbol{U}}\,] = -\boldsymbol{B}\boldsymbol{\partial_\theta}, \qquad [\,\check{\boldsymbol{U}},\,\boldsymbol{\partial_\theta}\,] = -\boldsymbol{N}\boldsymbol{\partial_\theta}, \\ \text{and} \quad [\,\check{\boldsymbol{U}},\,\hat{\boldsymbol{U}}\,] = 0. \end{gathered} \tag{3.24}$$

As before, define two sequences of vector fields inductively by

$$\hat{U}_{a_1 a_2 \cdots a_\ell} = [\,\hat{U}_{a_1 a_2 \cdots a_{\ell-1}}\,, \hat{U}_{a_\ell},\,] \quad \text{and} \quad \check{U}_{\alpha_1 \alpha_2 \cdots \alpha_\ell} = [\,\hat{U}_{\alpha_1 \alpha_2 \cdots \alpha_{\ell-1}}\,\hat{U}_{\alpha_\ell}\,]. \tag{3.25}$$

A simply induction argument, based upon the last of (3.24) and the Jacobi identity, shows that

$$[\,\hat{U}_{a_1 a_2 \cdots a_k}, \check{U}_{\alpha_1 \alpha_2 \cdots \alpha_\ell}\,] = 0 \tag{3.26}$$

On account of (3.2), (3.4) and (3.23) the iterated brackets $\hat{U}_{a_1 a_2 \cdots a_\ell}$ will span all of $\check{H}$ and therefore, by the first two equations in (3.59), we can choose a basis

$\boldsymbol{X}$ for K consisting of a linear independent set of the $\hat{U}_{a_1 a_2 \cdots a_\ell}$. Alternatively, we can choose a basis $\boldsymbol{Y}$ for K consisting of a linear independent set of the $\check{U}_{a_1 a_2 \cdots a_\ell}$. These two basis for K satisfy

$$[\,\boldsymbol{X},\, \check{\boldsymbol{U}}\,] = 0, \quad [\,\boldsymbol{X},\, \boldsymbol{Y}\,] = 0 \quad [\,\boldsymbol{Y},\, \hat{\boldsymbol{U}}\,] = 0. \tag{3.27}$$

Denote the duals of these coframes by $\{\,\boldsymbol{\theta_X},\, \hat{\boldsymbol{\iota}}\,, \hat{\boldsymbol{\iota}}\,\}$ and $\{\,\boldsymbol{\theta_Y},\, \hat{\boldsymbol{\iota}}\,, \hat{\boldsymbol{\iota}}\,\}$.

Theorem 3.5. *Let* $\{\,\hat{V}, \check{V}\}$ *be a Darboux pair on a manifold* M*. Then about each point of* M *there are two 0-adapted coframes*

$$\{\,\boldsymbol{\theta_X},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\} \quad \textit{and} \quad \{\,\boldsymbol{\theta_Y},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\}$$

with structure equations (3.7)*,*

$$d\boldsymbol{\theta_X} = \boldsymbol{A}\,\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}} + \boldsymbol{B}\,\check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}} + \boldsymbol{C}\,\boldsymbol{\theta_X} \wedge \boldsymbol{\theta_X} + \boldsymbol{M}\,\hat{\boldsymbol{\pi}} \wedge \boldsymbol{\theta_X} \tag{3.28}$$

and

$$d\boldsymbol{\theta_Y} = \boldsymbol{E}\,\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}} + \boldsymbol{F}\,\check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}} + \boldsymbol{K}\,\boldsymbol{\theta_Y} \wedge \boldsymbol{\theta_Y} + \boldsymbol{N}\,\check{\boldsymbol{\pi}} \wedge \boldsymbol{\theta_Y} \tag{3.29}$$

Moreover, $\mathrm{span}\{\,\boldsymbol{\theta_X}\} = \mathrm{span}\{\,\boldsymbol{\theta_Y}\}$ *and the vector fields* $\boldsymbol{X}$*,* $\boldsymbol{Y}$ *belonging to the dual frames* $\{\,\boldsymbol{X},\, \hat{\boldsymbol{U}},\, \check{\boldsymbol{U}}\,\}$ *and* $\{\,\boldsymbol{Y},\, \hat{\boldsymbol{U}},\, \check{\boldsymbol{U}}\,\}$ *satisfy*

$$[\,\boldsymbol{X}, \boldsymbol{Y}] = 0. \tag{3.30}$$

Coframes which satisfy the conditions of Theorem 3.5 are call ***3-adapted coframes***.

Remark 3.6. We define a involution of a Darboux pair $\hat{V}, \check{V}$ on M to be a diffeomorphism $\Phi\colon M \to M$ such that $\Phi^2 = \mathrm{id}_M$ and $\Phi^*(\hat{V}) = \check{V}$. Many examples admit such an involution in which case the structure equations (3.29) can be immediately inferred from (3.28).

3.4 The fourth adapted coframe and the Vessiot algebra for a Darboux pair.

At this point we now have two locally defined frames $\{ X_i, \hat{U}_a, \check{U}_\alpha \}$ and $\{ Y_i, \hat{U}_a, \check{U}_\alpha \}$ with corresponding dual coframes $\{ \theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha \}$ and $\{ \theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha \}$. The structure equations are (3.30). We also have

$$\theta^i_X = Q^i_j \theta^j_Y. \tag{3.31}$$

On account of the structure equations (3.28) and the fact that the vector fields X_i and Y_j commute, the identity

$$d\theta^i_X(X_j, Y_k) = X_j(\theta^i_X(Y_k)) - Y_k(\theta^i_X(X_j) - \theta^i_X([X_j, Y_k])$$

leads to

$$X_j(Q^i_k) = C^i_{j\ell} Q^\ell_k. \tag{3.32}$$

We shall use this result repeatedly in what follows – it is a consequence of the fact that the vector fields X_i and Y_j commute, a fact which is not encoded in the structure equations (3.28)and (3.29).

We next substitute (3.31) into (3.28) and equate the coefficients of $\theta^j_Y \wedge \theta^k_Y$ to deduce that

$$X_\ell(Q^i_k)Q^\ell_j - X_l(Q^i_j)Q^l_k + Q^i_l K^l_{jk} = C^i_{\ell m} Q^\ell_j Q^m_k.$$

By virtue of (3.32), this equation simplifies to

$$Q^i_l K^l_{jk} = -C^i_{lm} Q^l_j Q^m_k. \tag{3.33}$$

Also, by equating to zero the coefficients of $\check{\pi}^\alpha \wedge \theta^i_X \wedge \theta^j_X$ and $\hat{\pi}^a \wedge \theta^i_Y \wedge \theta^j_Y$ in the expansions of the equations $d^2\theta^i_X = 0$ and $d^2\theta^i_Y = 0$, we find that

$$\check{U}_a(C^i_{jk}) = 0 \quad \text{and} \quad \hat{U}_\alpha(K^i_{jk}) = 0.$$

This equations imply respectively that $dC^i_{jk} \in \Omega^1(\hat{V})$ and $dK^i_{jk} \in \Omega^1(\check{V})$ and therefore

$$C^i_{jk} \in \text{Inv}(\hat{V}) \quad \text{and} \quad K^i_{jk} \in \text{Inv}(\check{V}). \tag{3.34}$$

Our next goal is to show that the coframes θ^i_X and θ^i_Y may be adjusted so that $K^i_{jk} = -C^i_{jk}$.

Theorem 3.7. *Let $\{\hat{V}, \check{V}\}$ be a Darboux pair on M.*

[i] *Then, about each point of M, there exist local coframes $\{\,\theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha\,\}$ and $\{\,\theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha\,\}$ which are adapted to XXX and which satisfy*

$$d\theta^i_X = \frac{1}{2}A^i_{ab}\,\hat{\pi}^a\wedge\hat{\pi}^b + \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha\wedge\check{\pi}^\beta + \frac{1}{2}C^i_{jk}\theta^j_X\wedge\theta^k_X + M^i_{aj}\hat{\pi}^a\wedge\theta^j_X \tag{3.35}$$

and

$$d\theta^i_Y = \frac{1}{2}E^i_{ab}\,\hat{\pi}^a\wedge\hat{\pi}^b + \frac{1}{2}F^i_{\alpha\beta}\,\check{\pi}^\alpha\wedge\check{\pi}^\beta - \frac{1}{2}C^i_{jk}\theta^j_Y\wedge\theta^k_Y + N^i_{\alpha j}\check{\pi}^\alpha\wedge\theta^j_Y. \tag{3.36}$$

[ii] *The structure functions $C^i_{jk} = -C^i_{kj}$ are constant on M and are the structure constants of a real s-dimensional Lie algebra.*

[iii] *The isomorphism class of the Lie algebra defined by the structure constants C^i_{jk} is an invariant of the Darboux pair $\{\,\hat{V}, \check{V}\}$.*

[iv] *The coframe elements θ^i_X and θ^i_Y are related by* (3.31)*, where the coefficients Q^i_j satisfy* (3.32)*.*

Proof. We first prove **[ii]** and **[iii]**, assuming **[i]**. With $K^i_{jk} = -C^i_{jk}$ equation (3.34) implies that $C^i_{jk}\in \mathrm{Inv}(\hat{V})\cap\mathrm{Inv}(\check{V})$ and hence, by (1.18), the coefficients C^i_{jk} are constant on M. By equating to zero the coefficients of $\theta^i_X\wedge\theta^j_X\wedge\theta^k_X$ in the expansion of the equations $d^2\theta^i_X = 0$ one finds that the C^i_{jk} satisfy the Jacobi identities and are therefore the structure constants of a real s-dimensional Lie algebra.

To prove **[iii]** let $\{\,\theta'^i_X, \hat{\pi}'^a, \check{\pi}'^\alpha\,\}$ be another coframe adapted to the Darboux pair $\{\,\hat{V}, \check{V}\,\}$ with structure equations

$$d\theta'^i_X = \frac{1}{2}A'^i_{ab}\,\hat{\pi}'^a\wedge\hat{\pi}'^b + \frac{1}{2}B'^i_{\alpha\beta}\,\check{\pi}'^\alpha\wedge\check{\pi}'^\beta + \frac{1}{2}C'^i_{jk}\theta'^j\wedge\theta'^k + M'^i_{aj}\hat{\pi}'^a\wedge\theta'^j. \tag{3.37}$$

Since $\mathrm{span}\{\,\theta'^i_X\,\} = \mathrm{span}\{\,\theta^i_X\,\}$ there are functions T^i_j such that $\theta^i_X = T^i_j\theta'^j_X$. The structure equations (3.36) and (3.37) first imply that $T^i_j\in\mathrm{Inv}(\hat{V})$ and then subsequently imply that

$$C^i_{lm}T^l_jT^m_k = T^i_lC'^l_{jk}.$$

Hence the structure constants C^i_{jk} and C'^i_{jk} define the same abstract Lie algebra.

To prove **[i]** we introduce local coordinate $(\hat{I}^a, \check{I}^\alpha, z^m)$ on M, where $d\hat{I}^a = \hat{\pi}^a$ and $d\check{I}^\alpha = \check{\pi}^\alpha$. On account of (3.34), the evaluation of (3.33) at the point $(\check{I}^a, \hat{I}^\alpha, z^m)$ reads

$$Q^\ell_k(\hat{I}^a, \check{I}^\alpha, z^m)\,K^k_{ij}(\check{I}^\alpha) = -C^\ell_{hk}(\hat{I}^a)\,Q^h_i(\hat{I}^a, \check{I}^\alpha, z^m)\,Q^k_j(\hat{I}^a, \check{I}^\alpha, z^m). \tag{3.38}$$

Evaluate this equation, first at a fixed point $(\hat{I}^a_0, \check{I}^a\alpha_0, z^m_0)$ and then at the point $(\hat{I}^a, \check{I}^\alpha_0, z^m_0)$. With

$$\overset{\circ}{K}{}^k_{ij} = K^k_{ij}(\check{I}^\alpha_0), \quad \overset{\circ}{C}{}^k_{ij} = C^k_{ij}(\hat{I}^a_0), \quad \overset{\circ}{Q}{}^j_i = Q^j_i(\hat{I}^a_0, \check{I}^\alpha_0, z^m_0), \quad \text{and}$$

$$Q^j_i(\hat{I}^a) = Q^j_i(\hat{I}^a, \check{I}^\alpha_0, z^m_0)$$

the results are

$$\overset{\circ}{Q}{}^\ell_k \, \overset{\circ}{K}{}^k_{ij} = -\overset{\circ}{C}{}^\ell_{hk} \, \overset{\circ}{Q}{}^h_i \, \overset{\circ}{Q}{}^k_j \quad \text{and} \quad Q^\ell_k(\hat{I}^a) \, \overset{\circ}{K}{}^k_{ij} = -C^\ell_{hk}(\hat{I}^a)\, Q^h_i(\hat{I}^a)\, Q^k_j(\hat{I}^a). \tag{3.39}$$

It then readily follows that the matrix

$$P^i_j(\hat{I}^a) = Q^i_\ell(\hat{I}^a)(\overset{\circ}{Q}{}^{-1})^\ell_j. \tag{3.40}$$

satisfies

$$P^\ell_k(\hat{I}^a)\, \overset{\circ}{C}{}^k_{ij} = C^\ell_{hk}(\hat{I}^a)\, P^h_i(\hat{I}^a)\, P^k_j(\hat{I}^a). \tag{3.41}$$

The one forms $\overset{\circ}{\theta}{}^i_X$ defined by

$$\overset{\circ}{\theta}{}^j_X P^i_j = \theta^i_X \tag{3.42}$$

then satisfy structure equations of the required form (3.33), where the structure functions C^i_{jk} coincide with the constants $\overset{\circ}{C}{}^i_{ij}$ Finally, by evaluating (3.38) at $(\hat{I}^a_0, \check{I}^\alpha, z^m)$, it follows that

$$R^i_j(\check{I}^\alpha) = (Q^{-1}(\hat{I}^a_0, \check{I}^\alpha, z^m_0))^i_j \tag{3.43}$$

satisfies

$$-R^\ell_k(\check{I}^\alpha)\, \overset{\circ}{C}{}^k_{ij} = K^\ell_{hk}(\check{I}^\alpha)\, R^h_i(\check{I}^\alpha)\, R^k_j(\check{I}^\alpha) \tag{3.44}$$

and the one forms $\overset{\circ}{\theta}{}^i_Y$ defined by

$$\overset{\circ}{\theta}{}^j_Y R^i_j = \theta^i_Y \tag{3.45}$$

satisfy the required structure equations (3.36), again with $C^i_{jk} = \overset{\circ}{C}{}^i_{jk}$.

The proof of part **[iv]** depends upon three simple observations. We first note that (3.42) and (3.45) imply that

$$\overset{\circ}{\theta}{}^i_X = \overset{\circ}{Q}{}^i_j \overset{\circ}{\theta}{}^j_Y, \quad \text{where} \quad \overset{\circ}{Q} = P^{-1}QR. \tag{3.46}$$

Secondly, we note that if $\{\, \overset{\circ}{X}{}^i, \hat{\pi}^a, \check{\pi}^\alpha \,\}$ is the dual frame to $\{\, \overset{\circ}{\theta}{}^i_X, \hat{\pi}^a, \check{\pi}^\alpha \,\}$, then $X_i = P^j_i \overset{\circ}{X}_j$. Finally because $P^i_j = P^i_j(\hat{I}^a)$ and $R^i_j = R^i_j(\check{I}^\alpha)$, it follows that

$$\overset{\circ}{X}_j(P^i_k) = \overset{\circ}{X}_j(R^i_k) = 0. \tag{3.47}$$

A straightforward calculation based upon these three observations and equations (3.44) and (3.41) leads to the required result, namely, $\overset{\circ}{X}_j(\overset{\circ}{Q}{}^i_k) = \overset{\circ}{C}{}^i_{jl}\, \overset{\circ}{Q}{}^l_k$. ∎

Corollary 3.8. *Let* $\{\check{V}, \hat{V}\}$ *and* $\{\hat{W}, \check{W}\}$ *be Darboux pairs on manifolds* M *and* N *respectively and suppose that* $\phi\colon M \to N$ *is smooth map satisfying*

$$\phi^*(\hat{W}) \subset \check{V} \quad \textit{and} \quad \phi^*(\check{W}) \subset \hat{V}. \tag{3.48}$$

Then ϕ *induces a Lie algebra homomorphism*

$$\phi_*\colon \mathfrak{vess}(\check{V}, \hat{V}) \to \mathfrak{vess}(\hat{W}, \check{W}). \tag{3.49}$$

Proof. The proof of this corollary is similar to that of part **[iii]** of Theorem 3.7. Let $\{\theta'^s_X, \hat{\pi}'^c, \check{\pi}'^\gamma\}$ be a coframe on N adapted to the Darboux pair $\{\hat{W}, \check{W}\}$ with structure equations

$$d\theta'^s_X = \frac{1}{2}A'^s_{cd}\,\hat{\pi}'^c \wedge \hat{\pi}'^d + \frac{1}{2}B'^s_{\gamma\delta}\,\check{\pi}'^\gamma \wedge \check{\pi}'^\delta + \frac{1}{2}K^s_{rt}\theta'^r \wedge \theta'^t + M'^s_{at}\hat{\pi}'^a \wedge \theta'^j. \tag{3.50}$$

In (3.50) the indices r, s, t range from 1 to $\dim(\hat{W} \cap \check{W})$, the indices c, d range from 1 to $\dim(\hat{W}^{(\infty)})$, and the indices γ, δ range from 1 to $\dim(\check{W}^{(\infty)})$. The constants K^s_{rt} are the structure constants for the Lie algebra $\mathfrak{vess}(\hat{W}, \check{W})$. Denote the dual frame by $\{X'_r, U'_c, U'_\gamma\}$.

Let $\{\theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha\}$ be a coframe on M which is adapted to the Darboux pair $\{\hat{V}, \check{V}\}$ and which satisfies the structure equations (3.35).

The inclusions (3.48) imply that

$$\phi^*(\hat{W} \cap \check{W}) \subset \hat{V} \cap \check{V}, \quad \phi^*(\hat{W}^{(\infty)}) \subset \hat{V}^{(\infty)} \quad \text{and} \quad \phi^*(\check{W}^{(\infty)}) \subset \check{V}^{(\infty)}$$

and therefore there are functions R^s_i, S^c_a and T^γ_α on M such that

$$\phi^*(\theta'^s_X) = R^s_i\theta^i_X, \quad \phi^*(\hat{\pi}'^c) = S^c_a\hat{\pi}^a \quad \text{and} \quad \phi^*(\check{\pi}'^\gamma) = T^\gamma_\alpha\check{\pi}^\alpha.$$

We pullback (3.50) using these equations and substitute from (3.35). From the result we deduce that $R^s_i \in \mathrm{Inv}(\hat{V})$ and subsequently that $R^r_i C^i_{jk} = K^r_{st}R^s_j R^t_k$. This shows that for each (fixed) point $x \in M$ the Jacobian $\phi_*(x)$, which maps X_i to $\phi_*(x)(X_i) = R^s_i X'_s$, determines a Lie algebra homomorphism of Vessiot algebras. ∎

Remark 3.9. Let $\{\theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha\}$ and $\{\theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha\}$ be two coframes which are adapted to the Darboux pair $\{\hat{V}, \check{V}\}$ and which satisfy the structure equations (3.35) and (3.36). Then it is not difficult to check that the commutativity of the the dual vector fields X_i and Y_j is equivalent to the supposition that the change of frame (3.31) defines an automorphism of the Vessiot algebra, that is,

$$Q^i_\ell C^\ell_{jk} = C^i_{lm}Q^\ell_j Q^m_k. \tag{3.51}$$

In the next section we shall need all the derivatives of the matrix Q^i_j. The $\hat\pi^a$ and $\check\pi^\alpha$ components of dQ^i_j are easily determined by substituting (3.31) into (3.35) and comparing the result with (3.36). If we then take into account (3.32) and (3.51) we find that

$$dQ^i_j = Q^\ell_j M^i_\ell - Q^i_\ell N^\ell_j + C^i_{j\ell} Q^\ell_k \theta^j_X \tag{3.52}$$

where $M^i_j = M^i_{aj}\hat\pi^a$ and $N^i_j = N^i_{\alpha j}\check\pi^\alpha$. We note, also for future use, that

$$A^i_{ab} = Q^i_j E^j_{ab} \quad \text{and} \quad B^i_{ab} = Q^i_j F^j_{\alpha\beta}. \tag{3.53}$$

3.5 The fifth and final adapted coframe for a Darboux pair

Let $\{ \hat{V}, \check{V} \}$ be a Darboux pair on M and let $\{ \theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha \}$ and $\{ \theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha \}$ be local coframes on M which are adapted to the Darboux pair and which satisfy the structure equations (3.35) and (3.36) established in the previous section. In this section we shall prove that it is possible to define forms

$$\hat{\theta}^i = \hat{R}^i_j \theta^j_X + \phi^i_a \hat{\pi}^a \quad \text{and} \quad \check{\theta}^i = \check{R}^i_j \theta^j_Y + \psi^i_a \check{\pi}^a \tag{3.54}$$

which satisfy structure equations

$$d\hat{\theta}^i = \frac{1}{2} G^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \hat{\theta}^j \wedge \hat{\theta}^k \tag{3.55}$$

and

$$d\check{\theta}^i = \frac{1}{2} H^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b - \frac{1}{2} C^i_{jk} \check{\theta}^j \wedge \check{\theta}^k. \tag{3.56}$$

The coefficients in (3.54) will satisfy

$$\hat{R}^i_j, \phi^i_a \in \operatorname{Inv}(\hat{V}) \quad \text{and} \quad \check{R}^i_j, \psi^i_\alpha \in \operatorname{Inv}(\check{V}).$$

Note that the coframes $\{ \hat{\theta}^i, \hat{\pi}^a, \check{\pi}^\alpha \}$ and $\{ \check{\theta}^i, \hat{\pi}^a, \check{\pi}^\alpha \}$ are *not* adapted to the Darboux pair $\{ \hat{V}, \check{V} \}$ although the forms $\{ \hat{\theta}^i, \hat{\pi}^a \}$ and $\{ \check{\theta}^i, \check{\pi}^\alpha \}$ are coframes for $\hat{V}$ and $\check{V}$ respectively. Let $\{ \hat{X}_i, \hat{V}_a, \check{V}_\alpha \}$ and $\{ \check{X}_i, \hat{W}_a, \check{W}_\alpha \}$ be the corresponding dual frames on M, in particular,

$$\hat{\theta}^i(\hat{X}_j) = \delta^i_j \quad \text{and} \quad \check{\theta}^i(\check{X}_j) = \delta^i_j \tag{3.57}$$

Definition 3.10. *The infinitesimal Vessiot group actions for the Darboux pair $\hat{V}, \check{V}$ are the Lie algebras of vector fields $\hat{X}_j$ and $\hat{X}_i$. These define local group actions $\hat{\mu}\colon G \times M \to M$ and $\check{\mu}\colon G \times M \to M$.*

We shall focus on (3.55) and simply note that the proof of (3.56) is similar. Our starting point is (3.35), that is,

$$d\theta^i_X = \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \theta^j_X \wedge \theta^k_X + M^i_{aj} \hat{\pi}^a \wedge \theta^j_X. \tag{3.58}$$

Setting to zero the coefficients of $\hat{\pi}^a \wedge \check{\pi}^\alpha \wedge \theta^i_X$ and $\check{\pi}^\alpha \wedge \hat{\pi}^a \wedge \hat{\pi}^b$ in the equation $d^2\theta^i_X = 0$ we find that $\check{U}_\alpha(M^i_{aj}) = 0$ and $\check{U}_\alpha(A^i_{ab}) = 0$. These equations imply that $M^i_{aj} \in \operatorname{Inv}(\hat{V})$ and $A^i_{ab} \in \operatorname{Inv}(\hat{V})$ and therefore

$$dM^i_{aj} = \hat{U}_b(M^i_{aj})\, \hat{\pi}^b \quad \text{and} \quad dA^i_{ab} = \hat{U}_c(A^i_{ab})\, \hat{\pi}^c. \tag{3.59}$$

Bearing these two results in mind, we then deduce respectively from the coefficients of $\hat{\pi}^a \wedge \theta^i_X \wedge \theta^j_X$, $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \theta^i_X$ and $\hat{\pi}^c \wedge \hat{\pi}^a \wedge \hat{\pi}^b$ in $d^2\theta^i_X = 0$ that

$$M^\ell_{aj} C^i_{\ell k} + M^\ell_{ak} C^i_{j\ell} - M^i_{a\ell} C^\ell_{jk} = 0, \tag{3.60}$$

$$\hat{U}_a(M^i_{bj}) - \hat{U}_b(M^i_{aj}) - M^i_{a\ell} M^\ell_{bj} + M^i_{b\ell} M^\ell_{a\ell} + C^i_{\ell j} A^\ell_{ab} = 0, \quad \text{and} \tag{3.61}$$

$$\hat{U}_{[c} A^i_{ab]} - A^\ell_{[ab} M^i_{c]\ell} = 0. \tag{3.62}$$

The square brackets in (3.62) indicate skew-symmetrization over the enclosed indices.

The proof of (3.14) hinges upon a detailed analysis of equations (3.60)–(3.62). We deal first with (3.60) since it is a purely algebraic constraint. It states that for each fixed value of a, the linear transformation $M_a \colon \mathfrak{vess} \to \mathfrak{vess}$ defined by

$$M_a(X_j) = M^i_{aj} X_i$$

is a derivation or infinitesimal automorphism of the Lie algebra $\mathfrak{vess}$. Consequently, to analyze this equation we shall invoke some basis structure theory for Lie algebras. Specifically, we shall consider separately the cases where $\mathfrak{vess}$ is semi-simple, where $\mathfrak{vess}$ is abelian and where $\mathfrak{vess}$ is solvable. Then we shall make use of the Levi decomposition of $\mathfrak{vess}$ to solve the general case.

Case I. We first consider the case where the Lie algebra $\mathfrak{vess}$ is semi-simple. Here the proof of (3.14) is relatively straightforward and is based upon the fact that every derivation of a semi-simple Lie algebra is an inner derivation. In fact, because the proof of this result is constructive, we can assert that there are uniquely defined smooth functions $\phi^\ell_a \in \mathrm{Inv}(\hat{V})$ such that

$$M^i_{aj} = \phi^\ell_a C^i_{\ell j}. \tag{3.63}$$

The forms $\hat{\theta}^i$ defined by

$$\hat{\theta}^i = \theta^i_X + \phi^i_a \hat{\pi}^a \tag{3.64}$$

then satisfy structure equations of the form

$$d\hat{\theta}^i = \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \hat{\theta}^j \wedge \hat{\theta}^k. \tag{3.65}$$

For these structure equations the integrable conditions (3.61) reduce to

$$C^i_{\ell j} A^\ell_{ab} = 0.$$

Since we are assuming that $\mathfrak{vess}$ is semi-simple, the center of $\mathfrak{vess}$ is trivial and therefore this last equation implies that $A^{\ell}_{ab} = 0$. The structure equations (3.65) then reduce to the form (3.14), as desired.

Case II. Now we consider the other extreme case, namely, the case where $\mathfrak{vess}$ is abelian. Strictly speaking, we need not treat this as a separate case but the analysis here will simplify our subsequent discussion of the case where $\mathfrak{vess}$ is solvable. When $\mathfrak{vess}$ is abelian the structure equations are

$$d\theta^i_X = \frac{1}{2}A^i_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta + M^i_{aj}\hat{\pi}^a \wedge \theta^j_X, \tag{3.66}$$

equation (3.60) is vacuous and, by virtue of (3.59), (3.61) simplifies to

$$dM^i_j - M^i_\ell \wedge M^\ell_j = 0, \quad \text{where} \quad M^i_j = M^i_{aj}\hat{\pi}^a. \tag{3.67}$$

By the Frobenius theorem we deduce that there are locally defined functions $R^i_j \in \mathrm{Inv}(\hat{V})$ such that

$$d(R^i_j) + R^i_\ell\, M^\ell_j = 0 \quad \text{and} \quad \det(R^i_j) \neq 0. \tag{3.68}$$

The forms $\hat{\theta}^i_0 = R^i_j\theta^j_X$ are then easily seen to satisfy

$$d\hat{\theta}^i_0 = \frac{1}{2}\,\overset{\circ}{A}{}^i_{ab}\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}\,\overset{\circ}{B}{}^i_{\alpha\beta}\check{\pi}^\alpha \wedge \check{\pi}^\beta. \tag{3.69}$$

Since the structure coefficients $\overset{\circ}{A}{}^i_{ab} = R^i_j A^j_{ab} \in \mathrm{Inv}(\hat{V})$ it follows, either directly from (3.55) or from (3.62) (with $M^i_{a\ell} = 0$) that the 2-forms

$$\chi^i = \frac{1}{2}\,\overset{\circ}{A}{}^i_{ab}\hat{\pi}^a \wedge \hat{\pi}^b \tag{3.70}$$

are all d closed. If we pick 1-forms $\phi^i = \phi^i_a\hat{\pi}^a$, with $\phi^i_a \in \mathrm{Inv}(\hat{V})$, such that $d\phi^i = -\chi^i$, then the forms

$$\hat{\theta}^i = R^i_j\theta^j_X + \phi^i_a\hat{\pi}^a \tag{3.71}$$

will satisfy the required structure equations (3.14) with $C^i_{jk} = 0$.

Case III. Now we suppose that $\mathfrak{vess}$ is solvable. Recall that a Lie algebra $\mathfrak{g}$ is said to be p-step solvable if the derived algebras $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ satisfy

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \cdots \supset \mathfrak{g}^{(p-1)} \supset \mathfrak{g}^{(p)} = \{0\}.$$

The annihilating subspaces $\Lambda^{(i)} = \operatorname{ann}(\mathfrak{g}^{(i)})$ therefore satisfy

$$\{0\} = \Lambda^{(0)} \subset \Lambda^{(1)} \subset \Lambda^{(2)} \cdots \subset \Lambda^{(p-1)} \subset \Lambda^{(p)} = \mathfrak{g}^* \tag{3.72}$$

and, because the $\mathfrak{g}^{(i)}$ are all ideals,

$$d\Lambda^{(i)} \subset \Lambda^{(i-1)} \otimes \Lambda^{(i)}. \tag{3.73}$$

Let $s_i = \dim \Lambda^{(i)}$. If $M\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation, then a simple induction shows that $M\colon \mathfrak{g}^{(i)} \to \mathfrak{g}^{(i)}$ and therefore $M^*\colon \Lambda^{(i)} \to \Lambda^{(i)}$. Thus the matrix representing M^* with respect to a basis for $\mathfrak{g}^*$ adapted to the flag (3.72) is block upper triangular.

We apply these observations to the forms $\{\, \theta^i_X \,\}$ and the structure equations (3.58). At each point of M the forms $\{, \theta^i_X \,\}$, restricted to the vectors X_i, define a basis for $\mathfrak{vess}^*$ and consequently, by a *constant* coefficient change of basis, we can suppose that $\{\, \theta^i_X \,\}$ are adapted to the derived flag (3.72). Since the notation for this adaptation becomes rather cumbersome in the general case of a p-step Lie algebra, we shall consider the just cases where $\mathfrak{vess}$ is a 1-step or a 2-step solvable Lie algebra. These two special cases will make the nature of the general argument clear.

In the case that $\mathfrak{vess}$ is a 1-step solvable Lie algebra, we begin with a Darboux adapted coframe

$$\{\, \theta^1_1, \dots, \theta^{s_1}_1, \theta^{s_1+1}_2, \dots, \theta^{s_2}_2, \hat\pi^a, \check\pi^\alpha \,\}, \tag{3.74}$$

where

$$\operatorname{span}\{\, \theta^1_X, \dots, \theta^s_X \,\} = \operatorname{span}\{\, \theta^1_1, \dots, \theta^{s_1}_1, \theta^{s_1+1}_2, \dots, \theta^{s_2}_2 \,\} \quad \text{over } \mathbf{R},$$

and where $\{\, \theta^1_1, \dots, \theta^{s_1}_1\}$ is a basis for $\Lambda^{(1)}(\mathfrak{vess})$. In this basis the structure equations (3.58) become

$$d\theta^r_1 = \frac{1}{2} A^r_{ab}\, \hat\pi^a \wedge \hat\pi^b + \frac{1}{2} B^r_{\alpha\beta}\, \check\pi^\alpha \wedge \check\pi^\beta + M^r_{as} \hat\pi^a \wedge \theta^s_1, \quad \text{and} \tag{3.75a}$$

$$\begin{aligned} d\theta^i_2 = {} & \frac{1}{2} A^i_{ab}\, \hat\pi^a \wedge \hat\pi^b + \frac{1}{2} B^i_{\alpha\beta}\, \check\pi^\alpha \wedge \check\pi^\beta + \frac{1}{2} C^i_{rs} \theta^r_1 \wedge \theta^s_1 + C^i_{rj} \theta^r_1 \wedge \theta^j_2 \\ & + M^i_{ar} \hat\pi^a \wedge \theta^r_1 + M^i_{aj} \hat\pi^a \wedge \theta^j_2. \end{aligned} \tag{3.75b}$$

In these equations the indices r, s range from 1 to s_1 and the indices i, j range from $s_1 + 1$ to s_2. There are no quadratic θ terms in the structure equations for θ^r_1 because the $\theta^r_1 \in \Lambda^{(1)}(\mathfrak{vess})$. There are no $\hat\pi^a \wedge \theta^i_2$ terms in (3.75a) because each M_a defines a derivation for $\mathfrak{vess}$. There are no $\theta_2^i \wedge \theta_2^j$ terms in (3.75b) on account of (3.73).

Since the structure equations (3.75a) are formally identically with the structure equations (3.66) for the abelian case, we can invoke the arguments there to define new forms

$$\hat{\theta}_1^r = R_s^r \theta_1^s + \phi_a^r \hat{\pi}^a \tag{3.76}$$

so that the structure equations (3.75) reduce to

$$d\hat{\theta}_1^r = \frac{1}{2} B_{\alpha\beta}^r \, \check{\pi}^\alpha \wedge \check{\pi}^\beta, \quad \text{and} \tag{3.77a}$$

$$d\theta_2^i = \frac{1}{2}\, \mathring{A}_{ab}^i \, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}\, \mathring{B}_{\alpha\beta}^i \, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2}\, \mathring{C}_{rs}^i \hat{\theta}_1^r \wedge \hat{\theta}_1^s + \mathring{C}_{rj}^i \hat{\theta}_1^r \wedge \theta_2^j \tag{3.77b}$$
$$+ \mathring{M}_{ar}^i \hat{\pi}^a \wedge \hat{\theta}_1^r + \mathring{M}_{aj}^i \hat{\pi}^a \wedge \theta_2^j.$$

It is important to track the functional dependencies of the coefficients as we perform these frame changes. Since the coefficients A_{ab}^i, M_{ar}^i and M_{aj}^i in (3.75b) and the coefficients R_s^r and ϕ_a^r in (3.76) are all $\hat{V}$ invariants, it is easily checked that the same is true of the coefficients $\mathring{A}_{ab}^i$, $\mathring{C}_{rs}^i$, $\mathring{C}_{rj}^i$, $\mathring{M}_{ar}^i$ and $\mathring{M}_{aj}^i$ in (3.77b). The coefficients of $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \theta_2^i$ and $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \hat{\pi}^c$ in $d^2\theta_2^i = 0$ give formally the same equations as we had in the abelian case and consequently we can define

$$\hat{\theta}_2^i = R_j^i \theta_2^j + \phi_a^i \hat{\pi}^a \tag{3.78}$$

so as to eliminate the $\hat{\pi}^a \wedge \hat{\pi}^b$ and $\hat{\pi}^a \wedge \theta_2^j$ terms in (3.77b). The structure equations are now

$$d\hat{\theta}_1^r = \frac{1}{2} B_{\alpha\beta}^r \, \check{\pi}^\alpha \wedge \check{\pi}^\beta \quad \text{and} \tag{3.79a}$$

$$d\hat{\theta}_2^i = \frac{1}{2}\, \overset{1}{B}{}_{\alpha\beta}^i \, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2}\, \overset{1}{C}{}_{rs}^i \hat{\theta}_1^r \wedge \hat{\theta}_1^s + \overset{1}{C}{}_{rj}^i \hat{\theta}_1^r \wedge \hat{\theta}_2^j + \overset{1}{M}{}_{ar}^i \hat{\pi}^a \wedge \hat{\theta}_1^r. \tag{3.79b}$$

Finally, the coefficients of $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \hat{\theta}_1^r$ in $d^2\hat{\theta}_2^i = 0$ give

$$\hat{U}_a(\overset{1}{M}{}_{br}^i) - \hat{U}_b(\overset{1}{M}{}_{ar}^i) = 0$$

and therefore there are functions R_r^i such that $M_{ar}^i = \hat{U}_a(R_r^i)$. The change of frame

$$\hat{\theta}_2^i = \hat{\hat{\theta}}_2^i + R_r^i \hat{\theta}_1^r \tag{3.80}$$

then leads to the desired structure equations

$$d\hat{\theta}_1^r = \frac{1}{2} B_{\alpha\beta}^r \, \check{\pi}^\alpha \wedge \check{\pi}^\beta \quad \text{and} \tag{3.81a}$$

$$d\hat{\hat{\theta}}_2^i = \frac{1}{2}\, \overset{2}{B}{}_{\alpha\beta}^i \, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2}\, \overset{2}{C}{}_{rs}^i \hat{\theta}_1^r \wedge \hat{\theta}_1^s + \overset{2}{C}{}_{rj}^i \hat{\theta}_1^r \wedge \hat{\hat{\theta}}_2^j. \tag{3.81b}$$

At this point it is a simple matter to check that the equations $d^2\hat{\hat{\theta}}{}^i_2 = 0$ force the coefficients $\overset{2}{C}{}^i_{rs}$ and $\overset{2}{C}{}^i_{rj}$ (which belong to $\mathrm{Inv}(\hat{V})$) to be constant.

When $\mathfrak{vess}$ is a 2-step solvable algebra we make assume structure equations of the form

$$d\theta^r_1 = \frac{1}{2}A^r_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^r_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta + M^r_{as}\hat{\pi}^a \wedge \theta^s_1, \tag{3.82a}$$

$$d\theta^u_2 = \frac{1}{2}A^u_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^u_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2}C^u_{rs}\theta^r_1 \wedge \theta^s_1 + C^u_{rv}\theta^r_1 \wedge \theta^v_2 \tag{3.82b}$$
$$+ M^u_{ar}\hat{\pi}^a \wedge \theta^r_1 + M^u_{av}\hat{\pi}^a \wedge \theta^v_2, \quad \text{and}$$

$$d\theta^i_3 = \frac{1}{2}A^i_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta \tag{3.82c}$$
$$+ \frac{1}{2}C^i_{rs}\theta^r_1 \wedge \theta^s_1 + C^i_{ru}\theta^r_1 \wedge \theta^u_2 + C^i_{rj}\theta^r_1 \wedge \theta^j_3 + \frac{1}{2}C^i_{uv}\theta^u_2 \wedge \theta^v_2 + C^i_{uj}\theta^u_2 \wedge \theta^j_3$$
$$+ M^i_{ar}\hat{\pi}^a \wedge \theta^r_1 + M^i_{au}\hat{\pi}^a \wedge \theta^u_2 + M^i_{aj}\hat{\pi}^a \wedge \theta^j_3.$$

In these equations r, s range from 1 to s_1, u, v from $s_1 + 1$ to s_2, and i, j from $s_2 + 1$ to s_3. The coframe $\{\,\theta^r_1, \theta^u_2, \theta^i_3, \hat{\pi}^a, \check{\pi}^\alpha\}$ is adapted to the flag (3.72) in that

$$\Lambda^{(1)} = \mathrm{span}\{\,\theta^r_1\,\}, \quad \Lambda^{(2)} = \mathrm{span}\{\,\theta^r_1, \theta^u_2\,\} \quad \text{and} \quad \Lambda^{(3)} = \mathrm{span}\{\,\theta^r_1, \theta^u_2, \theta^i_3\,\}.$$

Note that the form of the structure equations (3.75) is preserved by changes of coframe of the form

$$\theta_1 \to R_{11}\theta_1 + \phi_1\hat{\pi}, \quad \theta_2 \to R_{12}\theta_1 + R_{22}\theta_2 + \phi_2\hat{\pi}, \quad \text{and}$$
$$\theta_3 \to R_{13}\theta_1 + R_{23}\theta_2 + R_{33}\theta_3 + \phi_3\hat{\pi},$$

where the coefficients all belong to $\mathrm{Inv}(\hat{V})$. Exactly as in the previous case of a 1-step solvable algebra, we can use such a change of coframe to reduce the structure equations to

$$d\hat{\theta}^r_1 = \frac{1}{2}B^r_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta, \tag{3.83a}$$

$$d\hat{\theta}^u_2 = \frac{1}{2}B^u_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2}C^u_{rs}\hat{\theta}^r_1 \wedge \hat{\theta}^s_1 + C^u_{rv}\hat{\theta}^r_1 \wedge \hat{\theta}^v_2, \quad \text{and} \tag{3.83b}$$

$$d\theta^i_3 = \frac{1}{2}A^i_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta. \tag{3.83c}$$
$$+ \frac{1}{2}C^i_{rs}\hat{\theta}^r_1 \wedge \hat{\theta}^s_1 + C^i_{ru}\hat{\theta}^r_1 \wedge \hat{\theta}^u_2 + C^i_{rj}\hat{\theta}^r_1 \wedge \theta^j_3 + \frac{1}{2}C^i_{uv}\hat{\theta}^u_2 \wedge \hat{\theta}^v_2 + C^i_{uj}\hat{\theta}^u_2 \wedge \theta^j_3$$
$$+ M^i_{ar}\hat{\pi}^a \wedge \hat{\theta}^r_1 + M^i_{au}\hat{\pi}^a \wedge \hat{\theta}^u_2 + M^i_{aj}\hat{\pi}^a \wedge \theta^j_3.$$

Again, as in the case **II**, a change of frame $\hat{\theta}_3 = R_{33}\theta_3 + \phi_3\hat{\pi}$ leads to the simplification of (3.83c) to

$$d\hat{\theta}^i_3 = \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha\wedge\check{\pi}^\beta + M^i_{ar}\hat{\pi}^a\wedge\hat{\theta}^r_1 + M^i_{au}\hat{\pi}^a\wedge\hat{\theta}^u_2 \qquad (3.84)$$
$$+\frac{1}{2}C^i_{rs}\hat{\theta}^r_1\wedge\hat{\theta}^s_1 + C^i_{ru}\hat{\theta}^r_1\wedge\hat{\theta}^u_2 + C^i_{rj}\hat{\theta}^r_1\wedge\hat{\theta}^j_3 + \frac{1}{2}C^i_{uv}\hat{\theta}^u_2\wedge\hat{\theta}^v_2 + C^i_{uj}\hat{\theta}^u_2\wedge\hat{\theta}^j_3.$$

Then, just in the reduction from (3.79) to (3.81), we can use a change of frame $\hat{\theta}_3 \to \hat{\theta}_3 + R_{13}\hat{\theta}_1 + R_{23}\hat{\theta}_2$ to transform (3.84) to

$$d\hat{\theta}^i_3 = \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha\wedge\check{\pi}^\beta \qquad (3.85)$$
$$+\frac{1}{2}C^i_{rs}\hat{\theta}^r_1\wedge\hat{\theta}^s_1 + C^i_{ru}\hat{\theta}^r_1\wedge\hat{\theta}^u_2 + C^i_{rj}\hat{\theta}^r_1\wedge\hat{\theta}^j_3 + \frac{1}{2}C^i_{uv}\hat{\theta}^u_2\wedge\hat{\theta}^v_2 + C^i_{uj}\hat{\theta}^u_2\wedge\hat{\theta}^j_3.$$

Equations (3.83a), (3.83b) and (3.85) give the desired result. We note once again that the coefficients C^*_{**} in (3.83c) and(3.84) are not necessarily constant but they are constant in (3.85).

The reduction of the structure equations for a general p-step solvable algebra follows this argument and can be formally established by induction on p.

Case IV: Finally, we consider the case where $\mathfrak{vess}$ is a generic Lie algebra. We recall that every real Lie algebra $\mathfrak{g}$ admits a Levi decomposition into a semi-direct sum $\mathfrak{g} = \mathfrak{s}\oplus\mathfrak{r}$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{s}$ is a semi-simple subalgebra of $\mathfrak{g}$. The radical $\mathfrak{r}$ is the unique maximal solvable ideal in $\mathfrak{g}$ – the semi-simple component $\mathfrak{s}$ in the Levi decomposition is not unique. The vector space of one forms on $\mathfrak{g}$ then decomposes according to

$$\mathfrak{g}^* = \mathrm{ann}(\mathfrak{r})\oplus\mathrm{ann}(\mathfrak{s}). \qquad (3.86)$$

The structure equations associated with the Levi decomposition then imply

$$d\,\mathrm{ann}(\mathfrak{r})\subset\Lambda^2(\mathrm{ann}(\mathfrak{r}))\quad\text{and}\quad d\,\mathrm{ann}(\mathfrak{s})\subset\mathfrak{g}^*\otimes\mathrm{ann}(\mathfrak{s}). \qquad (3.87)$$

If $M\colon\mathfrak{g}\to\mathfrak{g}$ is a derivation, then M preserves $\mathfrak{r}$ and hence $M^*\colon\mathrm{ann}(\mathfrak{r})\to\mathrm{ann}(\mathfrak{r})$.

We therefore deduce that we may reduce the structure equations (3.58), by a constant coefficient change of coframe, to the form

$$d\theta^r_1 = \frac{1}{2}A^r_{ab}\,\hat{\pi}^a\wedge\hat{\pi}^b + \frac{1}{2}B^r_{\alpha\beta}\,\check{\pi}^\alpha\wedge\check{\pi}^\beta + \frac{1}{2}C^r_{st}\theta^s_1\wedge\theta^t_1 + M^r_{as}\hat{\pi}^a\wedge\theta^s_1, \qquad (3.88a)$$

$$d\theta^i_2 = \frac{1}{2}A^i_{ab}\,\hat{\pi}^a\wedge\hat{\pi}^b + \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha\wedge\check{\pi}^\beta + \frac{1}{2}C^i_{rs}\theta^j_2\wedge\theta^k_2 + C^i_{rj}\theta^r_1\wedge\theta^j_2 \qquad (3.88b)$$
$$+ M^i_{ar}\hat{\pi}^a\wedge\theta^r_1 + M^i_{aj}\hat{\pi}^a\wedge\theta^j_2.$$

In this equations the indices r, s, t range from 1 to $s_0 = \dim(\mathrm{ann}(\mathfrak{r}))$ and i, j, k range from s_0+1 to $s = \dim(\mathfrak{g})$. The forms θ_1^r and θ_2^i are adapted to the decomposition (3.86) with $\mathrm{ann}(\mathfrak{r}) = \mathrm{span}\{\,\theta_1^r\,\}$ and $\mathrm{ann}(\mathfrak{s}) = \mathrm{span}\{\,\theta_2^i\,\}$.

Since the structure constants C^r_{st} are those for the semi-simple Lie algebra $\mathfrak{s}$ we can return to the analysis presented in Case **I** to prove that there is a change of coframe $\hat\theta_1 = \theta_1 + \phi\hat\pi$ which reduces the structure equations (3.88) to

$$d\hat\theta_1^r = \frac{1}{2}B^r_{\alpha\beta}\,\check\pi^\alpha\wedge\check\pi^\beta + \frac{1}{2}C^r_{st}\hat\theta_1^s\wedge\hat\theta_1^t, \quad \text{and} \tag{3.89a}$$

$$\begin{aligned} d\theta_2^i = {}& \frac{1}{2}A^i_{ab}\,\hat\pi^a\wedge\hat\pi^b + \frac{1}{2}B^i_{\alpha\beta}\,\check\pi^\alpha\wedge\check\pi^\beta + \frac{1}{2}C^i_{jk}\theta_2^j\wedge\theta_2^k + C^i_{rj}\hat\theta_1^r\wedge\theta_2^j \\ &+ M^i_{ar}\hat\pi^a\wedge\hat\theta_1^r + M^i_{aj}\hat\pi^a\wedge\theta_2^j. \end{aligned} \tag{3.89b}$$

The structure functions C^r_{st}, C^i_{rs} and C^i_{rj} in (3.89) are identical to the corresponding structure constants in (3.88). One now checks that the $\hat\theta_1^r$ terms in (3.89b) do not effect the arguments made in Cases **II** and **III**. Thus by a change of frame $\hat\theta_2 = R\theta_2 + \phi\hat\pi$ one can reduce (3.89) to

$$d\hat\theta_1^r = \frac{1}{2}B^r_{\alpha\beta}\,\check\pi^\alpha\wedge\check\pi^\beta + \frac{1}{2}C^r_{st}\hat\theta_1^s\wedge\hat\theta_1^t, \quad \text{and} \tag{3.90a}$$

$$\begin{aligned} d\hat\theta_2^i = {}& \frac{1}{2}B^i_{\alpha\beta}\,\check\pi^\alpha\wedge\check\pi^\beta + \frac{1}{2}C^i_{jk}\hat\theta_2^j\wedge\hat\theta_2^k + C^i_{rj}\hat\theta_1^r\wedge\theta_2^j \\ &+ M^i_{ar}\hat\pi^a\wedge\hat\theta_1^r. \end{aligned} \tag{3.90b}$$

Finally, by a now familiar computation, one sees that a change of frame $\hat\theta_2 \to \hat\theta_2 + R\hat\theta_1$ allows one to eliminate the $\hat\pi^a\wedge\hat\theta_1^r$ in (3.90b). This completes our derivation of the structure equations (3.14).

Remark 3.11. If the forms (3.54) satisfy (3.55) and (3.56), then it easy to check that coefficients $\hat R^i_j$, $\check R^i_j$, $\phi^i_a\hat\pi^a$ and $\psi^i_\alpha\check\pi^\alpha$ satisfy

$$d\hat R^i_j = -C^i_{lk}\,\hat R^\ell_j\,\phi^k - \hat R^i_\ell\,M^l_j, \quad d\check R^i_j = C^i_{lk}\,\check R^\ell_j\psi^k - \check R^i_\ell N^l_j, \tag{3.91a}$$

$$\hat R^i_\ell B^\ell_{\alpha\beta} = G^i_{\alpha\beta}, \quad \check R^i_\ell\,E^\ell_{ab} = H^i_{ab}, \quad \check R^i_\ell\,F^\ell_{\alpha\beta} = G^i_{\alpha\beta}, \tag{3.91b}$$

$$d\phi^i = \frac{1}{2}C^i_{jk}\phi^j\wedge\phi^k - \frac{1}{2}\hat R^i_\ell\,A^\ell, \quad d\psi^i = -\frac{1}{2}C^i_{jk}\psi^j\wedge\psi^k - \frac{1}{2}\check R^i_\ell\,F^\ell. \tag{3.91c}$$

where $M^\ell_j = M^\ell_{aj}\hat\pi^a$, $N^\ell_j = N^\ell_{\alpha j}\check\pi^\alpha$, $A^\ell = A^\ell_{ab},\hat\pi^a\wedge\hat\pi^b$, and $F^\ell = F^\ell_{\alpha\beta}\check\pi^\alpha\wedge\check\pi^\beta$. In addition, the matrices $\hat R^i_j$ and $\check R^i_j$ define automorphisms of the Vessiot algebra,that is

$$\hat R^i_\ell\,C^\ell_{hj} = C^i_{k\ell}\,\hat R^k_h\,\hat R^j_h,\check R^i_\ell, \quad \text{and} \quad \check R^i_\ell\,C^\ell_{hj} = C^i_{k\ell}\,\check R^k_h\,\check R^j_h, \tag{3.92}$$

A series of straightforward computations then shows that the matrix

$$\lambda = \hat{R} Q \check{R}^{-1} \tag{3.93}$$

satisfies

$$\lambda^i_\ell C^\ell_{jk} = C^i_{\ell m} \lambda^\ell_j \lambda^m_k \quad \text{and} \quad d\lambda^i_j = C^i_{\ell m}\, \lambda^m_j\, \hat{\theta}^\ell + \lambda^i_h\, C^h_{mj}\, \psi^m. \tag{3.94}$$

Finally, we determine the residual freedom in the determination of the 1-forms $\hat{\theta}^i\}$, that is, we compute the group of coframe transformations fixing the forms $\hat{\pi}^a$ and $\check{\pi}^\alpha$ and which transform the $\hat{\theta}^i$ by

$$\tilde{\theta}^i = \Lambda^i_j \hat{\theta}^i + \sigma^i, \quad \text{where} \quad \sigma^i = S^i_a \hat{\pi}^a \tag{3.95}$$

in such a manner as to preserve the form of the structure equations (3.14).

If we take the exterior derivative of (3.95) and substitute into the structure equations for $\tilde{\theta}^i$ we find, first from the $\check{\pi}^\alpha \wedge \theta^i$ and the $\hat{\pi}^a \wedge \check{\pi}^\alpha$ terms, that the coefficients Λ^i_j and S^i_a are functions of the invariants I^a alone and then that

$$d\Lambda^i_j = C^i_{\ell k}\, \Lambda^\ell_j \sigma^k \quad \Lambda^i_\ell\, C^\ell_{hj} = C^i_{k\ell}\, \Lambda^k_h\, \Lambda^j_h, \quad d\sigma^i = \frac{1}{2} C^i_{jk} \sigma^j \wedge \sigma^k. \tag{3.96}$$

To integrate these equations, let G be a local Lie group whose Lie algebra of is the Vessiot algebra $\mathfrak{vess}$ with structure constants C^i_{jk}. On G construct a coframe η^i of invariant forms with structure equations $d\eta^i = \frac{1}{2} C^i_{jk} \eta^j \wedge \eta^k$. Then there exists ([17], Proposition 1.3) a map from the space of invariants $\mathrm{Inv}(\hat{V})$ to G such that $\sigma^i = \sigma^*(\eta^i)$. Define $S(\hat{I}) = \mathrm{Ad}(\sigma(\hat{I})$. Then

$$dS^i_j = C^i_{\ell k} \sigma^\ell S^k_j \quad \text{and} \quad d(\Lambda^i_\ell (S^{(-1)})^\ell_j) = 0.$$

and hence the general solution to (3.96) is therefore

$$\sigma^i = \sigma^*(\eta^i) \quad \Lambda^i_j = \overset{o}{\Lambda}{}^i_\ell\, S^\ell_j, \quad S = \mathrm{Ad}(\sigma(\hat{I})), \tag{3.97}$$

where $\overset{o}{\Lambda}{}^i_\ell$ are the components of a constant automorphism of the Vessiot algebra. When σ is a constant map, S is constant inner automorphism so that as far as the general solution (3.97) is concerned, one can restrict the $\overset{o}{\Lambda}{}^i_\ell$ to a set of representatives of the group of outer automorphisms of the Vessiot algebra.

In particular, when the Vessiot algebra is semi-simple, the freedom in the coframe elements $\hat{\theta}$ is completely determined by the arbitrary map $\sigma(\hat{I})$ taking values in the Vessiot group whereas when the Vessiot algebra is abelian the freedom in the coframe elements $\hat{\theta}$ are characterized by an arbitrary constant matrix $\overset{o}{\Lambda}{}^i_\ell$.

Remark 3.12. The basis for the infinitesimal Vessiot transformation groups $\hat{\Gamma} = \{\, \hat{X}_i \,\}$ and $\check{\Gamma} = \{\, \hat{X}_i \,\}$ are related by $\check{X}_j = \lambda^i_j \hat{X}_i$, where λ is the matrix (3.93). Let $\hat{X}_r$, $r = 1 \dots m$ be a basis for the center of $\hat{\Gamma}$ and pick a complementary set of vectors $\hat{X}_u$ which complete the $\hat{X}_r$ to a basis for $\hat{\Gamma}$. Choose a similar basis for $\check{\Gamma}$. Then, because λ defines a Lie algebra automorphism, we have that

$$\check{X}_r = \lambda^s_r \hat{X}_s \quad \text{and} \quad \check{X}_u = \lambda^v_u \hat{X}_v + \lambda^s_u \hat{X}_s \tag{3.98}$$

and therefore, by (3.94), $d\,\lambda^r_s = 0$. Consequently we may always pick basis for the infinitesimal Vessiot transformation groups $\hat{\Gamma}$ and $\check{\Gamma}$ so that the centers coincide. In turn, this implies that *any vector in the center of either infinitesimal Vessiot transformation group is a infinitesimal symmetry of the original Pfaffian system $\mathcal{I}$.*

4 The Superposition Formula for Darboux Pairs

We start with equations (3.31) and (3.54), namely,

$$\theta^i_X = Q^i_j \theta^j_Y, \quad \hat{\theta}^i = \hat{R}^i_j \theta^j_X + \phi^i_a \hat{\pi}^a, \check{\theta}^i = \check{R}^i_j \theta^j_Y + \psi^i_\alpha \check{\pi}^\alpha \tag{4.1}$$

and define forms $\hat{\omega}^i$ and $\check{\omega}^i$ defined by

$$\hat{\omega}^i = \hat{\theta}^i + \lambda^i_j \psi^j_\alpha \check{\pi}^\alpha \quad \text{and} \quad \check{\omega}^i = \check{\theta}^i + \mu^i_j \phi^j_a \hat{\pi}^a. \tag{4.2}$$

where $\lambda = \hat{R} Q \check{R}^{-1}$ and $\mu = \lambda^{-1}$.

Lemma 4.1. *The forms $\hat{\omega}^i$ and $\check{\omega}^i$ have the same span and satisfy the structure equations*

$$d\hat{\omega}^i = \frac{1}{2} C^i_{jk} \hat{\omega}^j \wedge \hat{\omega}^k \quad \textit{and} \quad d\check{\omega}^i = -\frac{1}{2} C^i_{jk} \check{\omega}^j \wedge \check{\omega}^k. \tag{4.3}$$

Proof. To check that the forms $\hat{\omega}^i$ and $\check{\omega}^i$ have the same span we simply use the various definitions given here to calculate

$$\begin{aligned}
\hat{\omega} &= \hat{\theta} + \lambda\, \psi\, \check{\pi} = \hat{R}\, \theta_X + \phi\, \hat{\pi} + \lambda\, \psi\, \check{\pi} \\
&= \hat{R}\, Q\, \theta_Y + \phi\, \hat{\pi} + \lambda\, \psi\, \check{\pi} = (\hat{R}\, Q\, \check{R}^{-1})\, \check{R}\, \theta_Y + \phi\, \hat{\pi} + \lambda\, \psi\, \check{\pi} \\
&= \lambda\, (\check{R}\, \theta_Y + \psi\, \check{\pi} + \lambda^{-1}\, \phi\, \hat{\pi}) = \lambda\, (\check{\theta} + \mu\, \phi\, \hat{\pi}) = \lambda\, \check{\omega}.
\end{aligned} \tag{4.4}$$

We now calculate $d\hat{\omega}^i$ using (3.55), (3.91), (3.92) and (3.94) to find that

$$\begin{aligned}
d\hat{\omega}^i &= d\hat{\theta}^i + d\lambda^i_j \wedge \psi^j + \lambda^i_j \wedge d\psi^j \\
&= \frac{1}{2} G^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk}\, \hat{\theta}^j \wedge \hat{\theta}^k + (C^i_{\ell m}\, \lambda^m_j\, \hat{\theta}^\ell + \lambda^i_h\, C^h_{mj}\, \psi^m) \wedge \psi^j \\
&\quad + \lambda^i_j\, (-\frac{1}{2} C^j_{hk} \psi^h \wedge \psi^k - \frac{1}{2} \check{R}^j_\ell\, F^\ell_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta) \\
&= \frac{1}{2} C^i_{jk}\, \hat{\theta}^j \wedge \hat{\theta}^k + C^i_{\ell m}\, \lambda^m_j\, \hat{\theta}^\ell \wedge \psi^j + \frac{1}{2} \lambda^i_\ell\, C^l_{hk}\, \psi^h \wedge \psi^k = \frac{1}{2} C^i_{jk} \hat{\omega}^j \wedge \hat{\omega}^k.
\end{aligned}$$

This proves the first of (4.3). The derivation of the structure equations for $\check{\omega}^i$ is similar. ∎

One more lemma is needed before we turn to the definition and the proof of the superposition formula.

Lemma 4.2. *Let G be a Lie group and let $m\colon G\times G\to G$ be the multiplication map $z=m(z_1,z_2)=z_1z_2$. Let $\{\,\hat\omega^i\,\}$ be a left invariant coframe on G, let $\{\,\check\omega^i\,\}$ be a right invariant coframe on G, and set $\hat\omega^i=\lambda^i_j\,\check\omega^j$. Then*

$$\lambda^i_j(z_1z_2)=\lambda^i_\ell(z_2)\lambda^\ell_j(z_1)\quad\text{and}\quad m^*(\hat\omega^i(z_1z_2))=\lambda^i_j(z_2)\hat\omega^j(z_1)+\hat\omega^i(z_2). \tag{4.5}$$

Proof. We assume that $\hat\omega^i(e)=\check\omega^i(e)$, where e is the identity element of G. Since $\hat\omega^i(z)=L^*_{z^{-1}}(e)(\hat\omega^i(e))$ and $\check\omega^i(z)=R^*_{z^{-1}}(e)(\check\omega^i(e))$, it follows that $\lambda^i_j(z)$ is the matrix for the linear transformation $Ad(z)=R^*_z\circ L^*_{z^{-1}}\colon T^*_eG\to T*_e\,G$ with respect to the basis $\hat\omega^i(e)$. From this observation and the fact that R_z and L_z commute one easily arrives at the first of (4.5).

To prove the second equation in (4.5), let (X_{z_1},Y_{z_2}) be a tangent vector to $G\times G$ at the point $z=(z_1,z_2)$. We have that

$$m_*(z)(X_{z_1},Y_{z_2})=(R_{z_2})_*(X_{z_1})+(L_{z_1})_*(Y_{z_2})$$

and hence

$$\begin{aligned}[m^*(z)(\hat\omega^i)](X_{z_1},Y_{z_2})&=[\hat\omega^i(z_1z_2)]\big(m_*(z)(X_{z_1},Y_{z_2})\big)\\&=[\hat\omega^i(z_1z_2)]\big((R_{z_2})_*(X_{z_1})\big)+\hat\omega^i\big((L_{z_1})_*(Y_{z_2})\big)\\&=R^*_{z_2}(\hat\omega^i(z_1z_2))(X_{z_1})+L^*_{z_1}(\hat\omega^i(z_1z_2)(X_{z_2})\\&=\lambda^i_j(z_1z_2)\check\omega^j(z_1)(X_{z_1})+\hat\omega^i(z_2)(X_{z_2}).\end{aligned}$$

Since $\lambda^i_j(z_1z_2)\check\omega^j(z_1)=\lambda^i_\ell(z_2)\lambda^\ell_j(z_1)\check\omega^j(z_1)=\lambda^i_\ell(z_2)\hat\omega^\ell(z_1)$ this gives the required result. ∎

We are now ready to define the superposition formula for any Darboux integrable Pfaffian differential system. By way of a summary, we recall that our starting point was the local coframe $\{\,\boldsymbol\theta,\hat{\boldsymbol\eta},\hat{\boldsymbol\sigma},\check{\boldsymbol\eta},\check{\boldsymbol\sigma}\,\}$ such that

$$I=\operatorname{span}\{\boldsymbol\theta,\hat{\boldsymbol\eta},\check{\boldsymbol\eta}\},\quad \hat V=\operatorname{span}\{\boldsymbol\theta,\hat{\boldsymbol\sigma},\hat{\boldsymbol\eta},\check{\boldsymbol\eta}\},\quad \check V=\operatorname{span}\{\boldsymbol\theta,\hat{\boldsymbol\eta},\check{\boldsymbol\sigma},\check{\boldsymbol\eta}\}.$$

In our first coframe adaption we adjusted the forms $\hat{\boldsymbol\eta}$ and $\check{\boldsymbol\eta}$ to the form

$$\hat{\boldsymbol\eta}=d\hat{\boldsymbol I}_2+\hat{\boldsymbol R}\,\hat{\boldsymbol\sigma}\quad\text{and}\quad\check{\boldsymbol\eta}=d\check{\boldsymbol I}_2+\check{\boldsymbol R}\,\check{\boldsymbol\sigma}. \tag{4.6}$$

Then, for the fourth coframe adaption we defined coframes $\{\,\boldsymbol\theta_{\boldsymbol X},\hat{\boldsymbol\eta},\hat{\boldsymbol\sigma},\check{\boldsymbol\eta},\check{\boldsymbol\sigma}\,\}$ and $\{\,\boldsymbol\theta_{\boldsymbol Y},\hat{\boldsymbol\eta},\hat{\boldsymbol\sigma},\check{\boldsymbol\eta},\check{\boldsymbol\sigma}\,\}$ from which the Maurer-Cartan forms $\hat\omega^i$ are define by

$$\hat\omega^i=\hat R^i_j\,\theta^j_X+\phi^i_a\hat\pi^a-\lambda^i_k\psi^k_\alpha\check\pi^\alpha. \tag{4.7}$$

We now introduce local coordinates $[\hat{I}^a, \check{I}^\alpha, z^i]$ on M such that $\check{\pi}^a = d\check{I}^a$, $\hat{\pi}^\alpha = \hat{I}^\alpha$,

$$\hat{\omega}^i = A^i_j(z^k)dz^j \quad \text{and} \quad \check{\omega}^i = B^i_j(z^k)dz^j$$

and a local multiplication map $z^i = m^i(z_1^j, z_2^k)$ whose associated left and right Maurer-Cartan forms are the $\hat{\omega}^i$ and the $\check{\omega}^i$. We remark that in terms of these coordinates (see (3.91c) and (4.4))

$$\phi^i_a = \phi^i_a(\hat{I}) \quad \psi^i_\alpha = \psi^a_\alpha(\check{I}) \quad \text{and} \quad \lambda^i_j = \lambda^i_j(z)$$

In terms of the local coordinates $[\hat{I}^a, \check{I}^\alpha, z^i]$ on M we define a *provisional* superposition map $\Sigma\colon M \times M \to M$ by

$$\Sigma([\hat{I}^a_1, \check{I}^\alpha_1, z^i_1], [\hat{I}^b_2, \check{I}^\beta_2, z^j_2]) = [\hat{I}^b_2, \check{I}^\alpha_1, m^k(z^i_1, z^j_2)] \tag{4.8}$$

We now calculate the pullback of the generators for I, name $\boldsymbol{\theta}$, $\hat{\boldsymbol{\eta}}$ and $\check{\boldsymbol{\eta}}$, by Σ^* and prove that the resulting forms all lie in $\hat{V}_1 + \check{V}_2$ on $M \times M$. Because the coefficients $\widehat{\mathbf{R}}$ are functions of the invariants $\hat{I}^a$ alone, the 1-forms $\hat{\boldsymbol{\eta}}$ all pullback under Σ^* into $\check{V}$. Likewise the 1-forms $\hat{\boldsymbol{\eta}}$ all pullback under Σ^* into $\hat{V}$. Since the span of the 1-forms $\boldsymbol{\theta}$ coincides with that of the 1-forms θ^i_X, it suffices to calculate the pullback of the forms $R^i_j\theta^j_X$ and for this we use (4.7) and Lemma 4.2 —

$$\begin{aligned}
\Sigma^*(\hat{R}^i_j\,\theta^j_X) &= \Sigma^*\big(\hat{\omega}^i(z) - \phi^i(\hat{I}) - \lambda^i_k(z)\,\psi^k(\check{I})\big) \\
&= m^*(\hat{\omega}^i(z)) - \phi^i(\hat{I}_2) - \lambda^i_k(z_1z_2)\,\psi^k(\check{I}_1)) \\
&= \lambda^i_j(z_2)\,\hat{\omega}^j(z_1) + \hat{\omega}^i(z_2) - \phi^i(\hat{I}_2) - \lambda^i_j(z_1z_2)\psi^j(\check{I}_1)) \\
&= \lambda^i_j(z_2)\,\big(\hat{\omega}^j(z_1) - \lambda^j_k(z_1)\psi^k(\check{I}_1)\,\big) + \lambda^i_j(z_2)\,\big(\hat{\omega}^j(z_2) - \mu^j_k(z_2)\,\phi^k(\hat{I}_2)\,\big) \\
&= \lambda^i_j(z_2)\,\hat{\theta}^j(z_1, \check{I}_1) + \lambda^i_j(z_2)\,\check{\theta}^j(z_2, \hat{I}_2) \in \hat{V}_1 + \check{V}_2,
\end{aligned} \tag{4.9}$$

as required.

It should be emphasize that this gives a superposition formula for any Darboux pair although in general this formula will not yield *all* the local integral manifolds for $\hat{V} \cap \check{V}$. We now show that if I is a decomposable Pfaffian then (4.8) defines a superposition formula which locally generates all the integral manifolds of I from the integral manifold of $\hat{V}$ and $\check{V}$.

To prove this subjectivity, we use the concept of an integrable extension of Pfaffian system I on M [7]. This is a Pfaffian system J on a manifold N

together with a submersion $\varphi\colon N \to M$ such that $\varphi^*(I) \subset J$ and such that the quotient EDS $\mathcal{J}/\mathcal{I}$ is a completely integrable Pfaffian system. Under these conditions the map φ is guaranteed to be a local surjection from the integral manifolds of J to the integral elements of I. Indeed, let $P \subset M$ be a integral manifold of I defined in a neighborhood of $x \in M$. Choose a point $y \in N$ such that $\varphi(y) = x$. Then $\tilde{P} = \varphi^{-1}(P)$ is a submanifold on N containing y and the restriction of J to $\tilde{P}$ is a completely integrable Pfaffian system $\tilde{J}$. By the Frobenius theorem (applied to $\tilde{J}$ as a Pfaffian system on $\tilde{P}$) there is locally a unique integral manifold Q for $\tilde{J}$ through y. The manifold Q is then an integral manifold of J which projects by φ to the original integral manifold P on some open neighborhood of x.

Remark 4.3. If $J^{(\infty)} \neq \{0\}$, let J_0 be the restriction of J to any fixed integral manifold $\iota\colon N_0 \to N$ of $J^{(\infty)}$. Then J_0 is an integral extension of I provided

$$J^{(\infty)} \cap \varphi^*(I) = \{0\} \quad \text{at all points of } N_0 \tag{4.10}$$

This condition insures that $\varphi_0 = \varphi \circ \iota$ is a submersion. The structure equation that are required for J to be in integral extension of I by φ pullback by ι to show that J_0 is an integral extension of I by φ_0.

Thus, to prove the surjectivity of Σ^*, it suffices to show that the Pfaffian system $J = \hat{V}_1 + \check{V}_2$ on $M \times M$ is an integral extension for I. Granted the I is decomposable, we split the 2-adapted coframe $\{\, \boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}} \,\}$ to

$$I = \operatorname{span}\{\, \boldsymbol{\theta_A}, \boldsymbol{\theta_B}, \boldsymbol{\theta_C}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}} \,\} \tag{4.11}$$

with structure equations

$$d\boldsymbol{\theta_A} \equiv \boldsymbol{A}\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}, \quad d\boldsymbol{\theta_B} \equiv \boldsymbol{B}\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}, \quad d\boldsymbol{\theta_C} \equiv 0 \mod \{\, \boldsymbol{\theta_A}, \boldsymbol{\theta_B}, \boldsymbol{\theta_C} \,\}. \tag{4.12}$$

The forms generating J on $M \times M$ are then

$$\{\, \boldsymbol{\theta_{A}}_1, \boldsymbol{\theta_{B}}_1, \boldsymbol{\theta_{C}}_1, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\eta}}_1, \check{\boldsymbol{\eta}}_1, \boldsymbol{\theta_{A}}_2, \boldsymbol{\theta_{B}}_2, \boldsymbol{\theta_{C}}_2, \hat{\boldsymbol{\eta}}_2, \check{\boldsymbol{\sigma}}_2, \check{\boldsymbol{\eta}}_2 \,\}$$

while the pullbacks of our coframe on M by Σ are

$$\begin{aligned}
\Sigma^*(\boldsymbol{R\theta}) &= \boldsymbol{\lambda}_2\hat{\boldsymbol{\theta}}_1 + \boldsymbol{\lambda}_2\check{\boldsymbol{\theta}}_2 \\
\Sigma^*(\hat{\boldsymbol{\pi}}) &= \Sigma^*([\, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}} \,] = \ [\, \hat{\boldsymbol{\sigma}}_2, \hat{\boldsymbol{\eta}}_2 \,] = \hat{\boldsymbol{\pi}}_2 \quad \text{and} \\
\Sigma^*(\check{\boldsymbol{\pi}}) &= \Sigma^*([\, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}} \,] = \ [\, \check{\boldsymbol{\sigma}}_1, \check{\boldsymbol{\eta}}_1 \,] = \check{\boldsymbol{\pi}}_1.
\end{aligned}$$

We may therefore take the subspace

$$F = \mathrm{span}\{\boldsymbol{\theta}_{\boldsymbol{A}2}, \boldsymbol{\theta}_{\boldsymbol{B}1}, \boldsymbol{\theta}_{\boldsymbol{C}1}, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\eta}}_1, \check{\boldsymbol{\sigma}}_2, \check{\boldsymbol{\eta}}_2\}.$$

for the complement of $\Sigma^*(I)$ in J To show that J is an integrable extension of I we check that these generators for F are closed, module F and modulo the 1-forms and 2-forms in the differential systems $\Sigma^*(\mathcal{I})$, that is, modulo J and the 2-forms in $\Sigma^*(\mathcal{I})$. The forms $\hat{\boldsymbol{\sigma}}_1$ and $\hat{\boldsymbol{\sigma}}_2$ are closed; the forms $\hat{\boldsymbol{\eta}}_1$ and $\check{\boldsymbol{\eta}}_2$ are closed modulo $\hat{\boldsymbol{\sigma}}_1$ and $\hat{\boldsymbol{\sigma}}_2$; and the forms $\boldsymbol{\theta}_{\boldsymbol{C}1}$ are closed modulo $\{\boldsymbol{\theta}_{\boldsymbol{A}1}, \boldsymbol{\theta}_{\boldsymbol{B}1}, \boldsymbol{\theta}_{\boldsymbol{C}1}\}$. We also have

$$d\boldsymbol{\theta}_{\boldsymbol{A}2} \equiv \boldsymbol{A}\hat{\sigma}_2 \wedge \hat{\sigma}_2 \equiv \Sigma^*(d\boldsymbol{\theta}_{\boldsymbol{A}}) \qquad \mathrm{mod}\ \{\boldsymbol{\theta}_{\boldsymbol{A}2}, \boldsymbol{\theta}_{\boldsymbol{B}2}, \boldsymbol{\theta}_{\boldsymbol{C}2}\},$$
$$d\boldsymbol{\theta}_{\boldsymbol{B}1} \equiv \boldsymbol{B}\check{\sigma}_1 \wedge \check{\sigma}_2 \equiv \Sigma^*(d\boldsymbol{\theta}_{\boldsymbol{B}}) \qquad \mathrm{mod}\ \{\boldsymbol{\theta}_{\boldsymbol{A}1}, \boldsymbol{\theta}_{\boldsymbol{B}1}, \boldsymbol{\theta}_{\boldsymbol{C}1}\}.$$

and therefore J is an integral extension of I.

One final issue regarding the domain of the superposition formula Σ needs to be addressed. Since $(\hat{V} + \check{V})^\infty = \hat{V}^{(\infty)} + \check{V}^{(\infty)}$, Remark 4.3 implies that we can restrict Σ to a map

$$\Sigma : \hat{M} \times \check{M} \to M \tag{4.13}$$

where $\hat{M}$ is a level set of $\hat{V}^{(\infty)}$ ($\check{I}_1$= constant) and $\check{M}$ is a level set of $\check{V}^{(\infty)}$ ($\check{I}_2$= constant). The Pfaffians systems $\hat{W}$ on $\hat{M}$ and $\check{W}$ on $\check{M}$ are the pullbacks of $\hat{V}$ and $\check{V}$. In fact, $\hat{W}$ and $\check{W}$ coincide with the pullbacks of the initial Pfaffian system I. Then Σ is a superposition for I with respect the Pfaffian systems $\hat{W}$ and $\check{W}$ and the proof of our main result, Theorem 1.8, is complete.

We conclude this section by presenting a geometric intepretation of the superposition formula in terms of the general theory of symmetry reduction of exterior differential systems [1]. To this end, let us summarize the results of the analysis in sections 3 and 4 as follows. Let I be a decomposable Pfaffian system on a manifold M for which the associated singular Pfaffian systems $\hat{V}$ and $\check{V}$ define a Darboux pair. We have constructed, though the coframe adaptions of section 3, a local Lie group G and local (left) group actions

$$\hat{\mu}, \check{\mu} : G \times M \to M. \tag{4.14}$$

For the sake of simplicity, let us suppose that G is a Lie group and that these actions are globally defined. The actions $\hat{\mu}$, $\check{\mu}$ have the following properties.

[i] The actions $\hat{\mu}$ and $\check{\mu}$ are free actions with the same orbits.

[ii] The actions $\hat{\mu}$ and $\check{\mu}$ commute.

[iii] The actions $\hat{\mu}$ and $\check{\mu}$ are symmetry groups of $\hat{V}$ and $\check{V}$, respectively.

[iv] Each integral manifold of $\hat{V}^\infty$ or $\check{V}^\infty$ is fixed by the actions $\hat{\mu}$ and $\check{\mu}$.

We have defined $\hat{M}$ and $\check{M}$ to be fixed integral manifolds of $\hat{V}^\infty$ and $\check{V}^\infty$ and $\hat{W}$ and $\check{W}$ to be the restrictions of $\hat{V}$ and $\check{V}$ to these integral manifolds. Properties **[i]** – **[iv]**, together with (3.57), imply that

[v] The actions $\hat{\mu}$ and $\check{\mu}$ restrict to free actions on $\hat{M}$ and $\check{M}$ and are symmetries of $\hat{W}$ and $\check{W}$. Consequently $\hat{M}$ and $\check{M}$ are principle G bundles.

[vi] The the vector-fields $\hat{X}_i$ and $\check{X}_i$ are the fundamental vector fields for the action of the structure G on $\hat{M}$ and $\check{M}$. The forms $\hat{\theta}^i$ and $\check{\theta}^i$ define connection forms on $\hat{M}$ and $\check{M}$. The curvature two forms for these connections are determined from the final coframe adaptions (3.55) and(3.56).

With respect to the local coordinates $[\hat{I}, \check{I}, z]$ on M, the coordinates on $\hat{M}$ are $[\check{I}, z]$ and the coordinates on $\check{M}$ are $[hI, z]$. The actions $\hat{\mu}$ and $\check{\mu}$ are given by

$$\hat{\mu}(g, [\check{I}, z]) = [\check{I}, zg^{-1}] \quad \check{\mu}(g, [\hat{I}, z]) = [\check{I}, gz] \tag{4.15}$$

[vii] The superposition formula, as a map $\Sigma\colon \hat{M} \times \check{M} \to M$ is invariant with respect to the diagonal action of G on $\hat{M} \times \check{M}$, that is,

$$\Sigma(\hat{\mu}(g, [\check{I}, z]), \check{\mu}(g, [\hat{I}, z])) = \Sigma([\check{I}, z]), [\hat{I}, z]).) \tag{4.16}$$

Therefore Σ can be identified with the quotient map from $\hat{M} \times \check{M}$ to $[\hat{M} \times \check{M}]/G$ for the diagonal action of the Vessiot group G on the product $\hat{M} \times \check{M}$.

[viii] Finally, the fundamental property of the superposition formula, namely (1.2), can be re-intepretated as the fact that the orginal Pfaffian system I is the symmetry reduction of the direct sum of $\hat{W}$ and $\check{W}$ by G. In summary, we have established the following general result.

Every decomposable, Darboux integrable exterior differential system is the symmetry reduction of the differential system generated by a pair of G invariant connections on the product of two principle G bundles.

5 Examples

Example 5.1. For our first example, let G be a n-parameter matrix group and, for the mapping $(x, y) \to U(x, y) \in G$ we consider the system of differential equations

$$U_{xy} = U_x U^{-1} U_y \tag{5.1}$$

The general solution to these equations is well-known to be $U(x, y) = A(x)B(y)$, with $A(x), B(y) \in G$. In the case when U is a 1×1 matrix, this system reduces to the wave equation $v_{xy} = 0$ under the change of variable $u = \exp(v)$. This example nicely illustrates the algorithm presented in the previous section – it leads directly to the general solution and, along the way, we calculate the Vessiot algebra of (5.1) to be the Lie algebra of G. However, this example is also very special because (5.1) admits the maximal number of first order invariants.

The Pfaffian system for (5.1) is $I = \{\Theta, \Theta^1, \Theta^2\}$, where

$$\begin{aligned} \Theta &= dU - U_x\, dx - U_y\, dy, \\ \Theta^1 &= dU_x - U_{xx}\, dx - U_x U^{-1} U_y\, dy, \\ \Theta^2 &= dU_y - U_x U^{-1} U_y\, dx - U_{yy}\, dy. \end{aligned} \tag{5.2}$$

The invariants for this systems are

$$\begin{aligned} &\hat{I}^1 = y, \quad \hat{I}^2 = U^{-1}U_y, \quad \hat{I}^3 = D_y(\hat{I}^2) = U^{-1}U_{yy} - U^{-1}U_yU^{-1}U_y, \\ &\check{I}^1 = x, \quad \check{I}^2 = U_xU^{-1}, \quad \check{I}^3 = D_x(\check{I}^2) = U_{xx}U^{-1} - U_xU^{-1}U_xU^{-1}, \end{aligned} \tag{5.3}$$

and our 0-adapted coframe for I is $\{\Theta, \hat{\eta}, d\hat{I}^1, d\hat{I}^3, \check{\eta}, d\check{I}^1, d\check{I}^3\}$, where

$$\begin{aligned} \hat{\eta} &= d\hat{I}^2 - \hat{I}^3 d\hat{I}^1 = U^{-1}\Theta^2 - U^{-1}\,\Theta\,\hat{I}^2, \quad \text{and} \\ \check{\eta} &= d\check{I}^2 - \check{I}^3 d\check{I}^1 = \Theta^1 U^{-1} - \check{I}^2\,\Theta\,U^{-1}. \end{aligned}$$

The structure equations are

$$d\Theta = d\hat{I}^1 \wedge (U\ \hat{\eta} + \Theta\,\hat{I}^2) + d\check{I}^1 \wedge (\check{\eta}\ U + \check{I}^2\Theta). \tag{5.4}$$

As such, this coframe satisfies (3.19) and is therefore 2-adapted. The next step is to eliminate either the $d\check{I}^1 \wedge (\hat{I}^2\Theta)$ or the $d\hat{I}^1 \wedge (\Theta\,\check{I}^2)$ terms in (5.4). In the more complex examples that follow, we shall invoke the methods described in Section 3 but here, by virtue of the group structure granted us in the presentation of the equations (5.1), we easily find that the forms $\Theta_X = U^{-1}\Theta$

and $\Theta_Y = \Theta\, U^{-1}$ provide us with required 4-adapted coframes with structure equations

$$d\Theta_X = d\hat{I}^1 \wedge \check{\eta} + d\check{I}^1 \wedge (U^{-1}\hat{\eta}\; U) - \Theta_X \wedge \Theta_X + d\hat{I}^1 \wedge (\Theta_X \hat{I}^2 - \hat{I}^2\Theta_X)$$
$$d\Theta_Y = d\hat{I}^1 \wedge (U\; \check{\eta}\; U^{-1}) + d\check{I}^1 \wedge \check{\eta} + \Theta_Y \wedge \Theta_Y - d\check{I}^1 \wedge (\Theta_Y \check{I}^2 - \check{I}^2\Theta_Y). \tag{5.5}$$

Since the forms Θ_X and Θ_Y are Lie algebra valued, these structure equations show that the Vessiot algebra for (5.1) is the Lie algebra of G.

The final coframe adaptation in Section 3.5 is given by

$$\hat{\Theta} = \Theta_X + \hat{I}^2 d\hat{I}^1 = U^{-1}dU - U^{-1}U_x dx$$
$$\check{\Theta} = \Theta_Y + \check{I}^2 d\check{I}^1 = dU\, U^{-1} - U_y\, U^{-1}dy \tag{5.6}$$

with structure equations

$$d\hat{\Theta} = d\check{I}^1 \wedge (U^{-1}\check{\eta}\; U) - \hat{\Theta} \wedge \hat{\Theta} \quad \text{and} \quad d\check{\Theta} = d\hat{I}^1 \wedge (U\; \hat{\eta}\; U^{-1}) + \check{\Theta} \wedge \check{\Theta}.$$

In accordance with (4.2) the Maurer-Cartan forms associated to the system of differential equations (5.1) are indeed the left and right invariant forms on G, that is,

$$\hat{\omega} = U^{-1}\, dU \quad \text{and} \quad \check{\omega} = dU\, U^{-1}. \tag{5.7}$$

With respect to coordinates x, U_1, U_{1x}, U_{1xx} on the level set $\hat{I}^a = 0$ and coordinates y, U_{2y}, U_{2yy} on the level set $\check{I}^a = 0$, the EDS $\hat{W}$ and $\check{W}$ are

$$\hat{W} = \{\, dU_1 - U_{1x}dx, dU_{1x} - U_{1xx}dx\} \quad \text{and} \quad \check{W} = \{\, dU_2 - U_{2y}dy, dU_{2y} - U_{1xx}dx\}$$

and the superposition formula is

$$U = U_1U_2,\; U_x = U_{1x}U_2,\; U_y = U_1U_{2y},\; U_{xx} = U_{1xx}U_2,\; U_{yy} = U_1U_{2yy}.$$

Example 5.2. As our first example we shall find the closed-form, general solution to

$$u_{xy} = \frac{u_x u_y}{u - x}. \tag{5.8}$$

This example is taken from Goursat's well-known classification (Equation VI) of Darboux integrable equations [16] and is simple enough that all the coframe adaptations described in the previous sections of the paper are easily computed. See also Vessiot [25] (pages 9–22) or Stomark [22] (pages 350–356).

The canonical Pfaffian system for (5.8) is $I = \{\ \alpha^1, \alpha^2, \alpha^3\ \}$, where

$$\alpha^1 = du - p\,dx - q\,dy, \quad \alpha^2 = dp - r\,dx - vpq\,dy, \quad \alpha^3 = dq - vpq\,dx - t\,dy$$

and $v = 1/(u-x)$, and the associated singular Pfaffian systems are

$$\hat{V} = \{\, \alpha^i,\ dx,\ dr - q(vr + v^2 p)\,dy\,\} \quad \text{and} \quad \check{V} = \{\, \alpha^i,\ dy,\ dt - tpv\,dy\,\}. \tag{5.9}$$

The first integrals for $\hat{V}$ and $\check{V}$ are

$$\hat{I}^1 = x, \quad \hat{I}^2 = vp, \quad \hat{I}^3 = vr + v^2 p, \quad \check{I}^1 = y, \quad \check{I}^2 = \frac{t}{q} - y \tag{5.10}$$

and we easily calculate that $\hat{V} \cap \check{V}^{(\infty)} = \{\,0\,\}$ and $\hat{V}^{(\infty)} \cap \check{V} = \{\,\hat{\eta}\,\}$, where

$$\hat{\eta} = d\hat{I}^2 + ((\hat{I}^2)^2 - \hat{I}^3)\,d\hat{I}^1 = v\alpha^2 - v^2 p\alpha^1. \tag{5.11}$$

A 1-adapted coframe (3.1) is therefore given by

$$\theta^1 = \alpha^1,\ \theta^2 = \alpha^3,\ \hat{\sigma}^1 = d\hat{I}^1,\ \hat{\sigma}^2 = d\hat{I}^3,\ \hat{\eta},\ \check{\sigma}^1 = d\check{I}^1,\ \check{\sigma}^2 = d\check{I}^2. \tag{5.12}$$

This coframe satisfies the structure equations (3.19) and is therefore 2-adapted. We relabel the coframes elements by $\hat{\pi}^1 = \sigma^1$, $\hat{\pi}^2 = \sigma^2$, $\hat{\pi}^3 = \hat{\eta}^1$, $\check{\pi}^1 = \check{\sigma}^1$, $\check{\pi}^2 = \check{\sigma}^2$ and calculate

$$\begin{aligned} d\pi^3 &= -2\,\hat{I}^2\,\hat{\pi}^1 \wedge \hat{\pi}^3 + \hat{\pi}^1 \wedge \hat{\pi}^2, \\ d\theta^1 &= (u-x)\,\hat{\pi}^1 \wedge \hat{\pi}^3 + \hat{I}^2\,\hat{\pi}^1 \wedge \theta^1 + \check{\pi}^1 \wedge \theta^2, \\ d\theta^2 &= \ q\,\hat{\pi}^1 \wedge \hat{\pi}^3 + q\,\check{\pi}^1 \wedge \check{\pi}^2 + \hat{I}^2\,\hat{\pi}^1 \wedge \theta^2 + \check{I}^2\,\check{\pi}^1 \wedge \theta^2. \end{aligned} \tag{5.13}$$

The next step is to eliminate the $\check{\pi}^\alpha \wedge \theta^i$ terms from (5.13). We calculate the distributions $\hat{U}$ and $\check{U}$ (see (3.21)) and their derived flags to be

$$\begin{aligned} &\hat{U} = \{\partial_{\hat{\pi}^1} + ((\hat{I}^2)^2 - \check{I}^3)\partial_{\hat{\pi}^3},\ \partial_{\hat{\pi}^3},\ \partial_{\hat{\pi}^2}\,\}, \qquad \check{U} = \{\partial_{\check{\pi}^1},\ \partial_{\check{\pi}^2}\} \\ &\hat{U}^{(\infty)} = \hat{U} \cup \{\,(u-x)\,\partial_{\theta^1} + q\partial_{\theta^2},\ \partial_{\theta^1}\}, \qquad \check{U}^{(\infty)} = \check{U} \cup \{\,q\,\partial_{\theta^2},\ q\,\partial_{\theta^1}\}. \end{aligned} \tag{5.14}$$

and then, in accordance with the algorithm given in Section 5, define

$$X_1 = (u - x)\,\partial_{\theta^1} + q\partial_{\theta^2}, \quad X_2 = \partial_{\theta^1}, \quad Y_1 = q\,\partial_{\theta^2}, \quad Y_2 = \partial_{\theta^1}, \tag{5.15}$$

The coframes dual the vector fields $\{\, X_i, \hat{U}, \check{U}\,\}$ and $\{\, Y_i, \hat{U}, \check{U}\,\}$ are the 3-adapted coframes

$$\theta^1_X = \frac{1}{q}\theta^2, \quad \theta^2_X = \theta^1 - \frac{u-x}{q}\theta^2, \quad \theta^1_Y = \frac{1}{q}\theta^2, \quad \theta^2_Y = \frac{1}{q}\theta^1 \quad \text{with} \tag{5.16}$$

$$\begin{aligned} d\theta^1_X &= \hat{\pi}^1 \wedge \hat{\pi}^3 + \check{\pi}^1 \wedge \check{\pi}^2 = d\theta^1_Y, \\ d\theta^2_X &= -(u-x)\,\check{\pi}^1 \wedge \check{\pi}^2 + \theta^1_X \wedge \theta^2_X + \check{\pi}^1 \wedge \theta^1_X + \hat{I}^2\,\hat{\pi}^1 \wedge \theta^2_X, \\ d\theta^2_Y &= \frac{u-x}{q}\,\hat{\pi}^1 \wedge \hat{\pi}^3 - \theta^1_Y \wedge \theta^2_Y + \check{\pi}^1 \wedge \theta^1_Y - \check{I}^2\,\check{\pi}^1 \wedge \theta^2_Y. \end{aligned} \tag{5.17}$$

The Vessiot algebra for (5.8) is therefore a 2 dimensional non-abelian Lie algebra.

The frame (5.16) is in fact 4-adapted and we may skip the adaptations given in Section 6 and move on to the final adaptations given in Section 7. The Vessiot algebra is 1-step solvable and the structure equations (5.17) are precisely of the form (3.75). The change of coframe $\hat{\theta}^1 = \theta^1_X + \check{I}^2\pi^1$ transforms to the structure equations (5.17) to the form (3.77). The change of coframe $\hat{\theta}^2 = \theta^2_X - x\hat{\theta}^1$ leads to the 5-adapted coframe $\{\hat{\theta}^1, \hat{\theta}^2\}$ with structure equations

$$d\hat{\theta}^1 = \check{\pi}^1 \wedge \check{\pi}^2 \quad d\hat{\theta}^2 = -u\,\check{\pi}^1 \wedge \check{\pi}^2 + \hat{\theta}^1 \wedge \hat{\theta}^2. \tag{5.18}$$

Similarly, 5Y-adapted coframe $\check{\theta}^1 = \theta^1_Y + \check{I}^2\check{\pi}^1$, $\check{\theta}^2 = \theta^2_Y - y\check{\theta}^1$ satisfies

$$d\check{\theta}^1 = \hat{\pi}^1 \wedge \hat{\pi}^3, \quad d\check{\theta}^2 = \frac{u-x-yq}{q}\hat{\pi}^1 \wedge \hat{\pi}^3 - \check{\theta}^1 \wedge \check{\theta}^2. \tag{5.19}$$

Before continuing we remark that vector fields X_1, X_2 (are given in terms of the dual vector fields $\hat{X}_1$ and $\hat{X}_2$,computed from the 5X-adapted coframe, by $X^1 = \hat{X}^1 - x\hat{X}^2$ and $X^2 = \hat{X}^2$. These vector field systems have the same orbits and structure equations but the actions are evidently different and and it is the latter action that is needed to properly construct the superposition formula.

The Maurer-Cartan forms are

$$\begin{aligned} \hat{\omega}^1 &= \hat{\theta}^1 + \check{I}^2\,\check{\pi}^1, \quad \hat{\omega}^2 = \hat{\theta}^2 - u\check{I}^2\,\check{\pi}^1, \\ \check{\omega}^1 &= \check{\theta}^1 + \hat{I}^2\,\hat{\pi}^1, \quad \check{\omega}^2 = \check{\theta}^2 + (\frac{p}{q} + yvp)\,\hat{\pi}^1. \end{aligned} \tag{5.20}$$

The forms $\hat{\omega}^1, \hat{\omega}^2$ are the left invariant forms for the matrix group $\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$, where the group coordinates a, b are defined in terms of the original coordinates by $a = q$ and $b = u - yq$.

Finally, if we introduce coordinates $y_1 = y$, $u_1 = u$, $q_1 = q$, $t_1 = t$ on the level set $\hat{M} = \{\, \hat{I}^1 = 0,\ \hat{I}^2 = 0,\ \hat{I}^3 = 0 \,\}$ and $x_2 = x$, $u_2 = u$, $p_2 = p$, $q_2 = u$, $r_2 = r$ on $\check{M} = \{\, \check{I}^1 = 0, \check{I}^2 = 0 \,\}$, then the superposition formula is

$$x = x_2, \quad \frac{p}{u-x} = \frac{p_2}{u_2 - x_2}, \quad \frac{r}{u-x} + \frac{p}{(u-x)^2} = \frac{r_2}{u_2-x_2} + \frac{p_2}{(u_2-x_2)^2}$$
$$y = y_1, \quad \frac{t}{q} = \frac{t_1}{q_1}, \quad q = q_1 q_2, \quad u - yq = u_2 - y_2 q_2 + (u_1 - y_1 q_1) q_2,$$

or, explicitly in terms of the coordinates $\{x, y, u, p, q, r, t\}$,

$$x = x_2, \quad y = y_1, \quad u = u_2 + q_2 u_1, \quad p = (1 + \frac{u_1 q_2}{u_2 - x_2}) p_2, \quad q = q_1 q_2,$$
$$r = (1 + \frac{u_1 q_2}{u_2 - x_2}) r_2 + \frac{u_1 p_2 q_2}{(u_2 - x_2)^2}, \quad t = t_1 q_2. \tag{5.21}$$

It remains to find the integral manifolds for $\hat{W}$ and $\check{W}$. Restricted to $\hat{M}$, the Pfaffian system I becomes $\hat{W} = \{\, du_1 - q_1\, dy_1, dq_1 - t_1 dy_1 \,\}$ whose integral manifolds are

$$y_1 = \beta, u_1 = f(\beta), q_1 = f'(\beta), t_1 = f''(\beta) \tag{5.22}$$

Restricted to $\check{M}$, the Pfaffian system I becomes $\check{W} = \{\, du_2 - p_2 dx_2,\ dp_2 - r_2 dx_2,\ dq_2 - \dfrac{p_2 q_2}{u_2 - x_2} dx_2 \}$. To find the integral manifolds of $\check{W}$ we calculate the second derived Pfaffian system to be $\check{W}^{(2)} = \{\, dq_2 - \dfrac{q_2}{u_2 - x_2}\, du_2 \,\}$ which leads to the equation $q_2\, du_2 - u_2\, dq_2 + x_2\, dq_2 = 0$ or

$$d\left(\frac{u_2}{q_2}\right) - x_2\, d\left(\frac{1}{q_2}\right) = 0 \quad \text{or} \quad d\left(\frac{u-x}{q}\right) + \frac{1}{q_2}\, dx_2 = 0.$$

The integral manifolds for $\check{W}$ are therefore given by

$$x_2 = \alpha \quad u = x - g(\alpha)/g'(\alpha), \quad q_2 = -1/g'(\alpha) \tag{5.23}$$

with p_2 and r_2 determined algebraically from the vanishing of the first and second forms in $\check{W}$. The substitution of (5.22) and (5.23) into the superposition formula (5.21) leads to the closed form general solution

$$u = \frac{f(y) - g(x)}{g'(x)} + x \tag{5.24}$$

for (5.8).

Example 5.3. In this example we shall construct the superposition formula for the Pfaffian system $I = \{\alpha^1, \alpha^2, \alpha^3\}$, where

$$\begin{aligned}&\alpha^1 = du - p\,dx - q\,dy, \quad \alpha^2 = dp + \frac{1}{b^3}\,(\tan b\tau - b\tau)\,dx - s\,dy,\\ &\alpha^3 = dq - s\,dx - b(b\tau + \cot b\tau)\,dy.\end{aligned} \tag{5.25}$$

This example nicely illustrates the various coframe adaptations in Sections 5 and 6 and make some surprising connection with some of Cartan's computations in the celebrated 5 variables paper [9]. The values $b = 1$, $b = \sqrt{-1}$ and the limiting value $b = 0$ give the three Pfaffian systems for the equations

$$u_{xx} = f(u_{yy})$$

which are Darboux integrable at the 2-jet level ([2], pages 373-374) and [5], pages 400–411). One easily calculates $\hat{V} \cap \check{V}^\infty = \hat{V}^\infty \cap \check{V} = \{\,0\,\}$ and that the first integrals for $\hat{V}$ and $\check{V}$ are

$$\begin{aligned}\hat{I}^1 = s + \tau, \quad \hat{I}^2 = -(x + b^2y)\hat{I}^1 + q + b^2p,\\ \check{I}^1 = s - \tau, \quad \check{I}^2 = -(x - b^2y)\check{I}^1 + q - b^2p.\end{aligned} \tag{5.26}$$

We immediately arrived at the 2-adapted coframe

$$\begin{aligned}&\hat{\pi}^1 = d\hat{I}^1, \quad \hat{\pi}^2 = d\hat{I}^2, \quad \check{\pi}^1 = d\hat{I}^1, \quad \check{\pi}^2 = d\check{I}^2, \quad \theta^1 = 2\alpha^1,\\ &\theta^2 = -b\cot b\tau\,\alpha^2 + \frac{1}{b}\tan b\tau\,\alpha^3, \quad \theta^3 = -b\cot b\tau\,\alpha^2 - \frac{1}{b}\tan b\tau\,\alpha^3,\end{aligned} \tag{5.27}$$

with structure equations

$$\begin{aligned}d\theta^1 &= (x + b^2y)\,\hat{\pi}^1 \wedge \theta^2 + \hat{\pi}^2 \wedge \theta^2 - (x - b^2y)\,\check{\pi}^1 \wedge \theta^3 - \check{\pi}^2 \wedge \theta^3,\\ d\theta^2 &= \hat{\pi}^1 \wedge \hat{\pi}^2 - b\cot 2b\tau\,\hat{\pi}^1 \wedge \theta^2 + b\csc 2b\tau\,\check{\pi}^1 \wedge \theta^3,\\ d\theta^3 &= \check{\pi}^1 \wedge \check{\pi}^2 - b\csc 2b\tau\,\hat{\pi}^1 \wedge \theta^2 + b\cot 2b\tau\,\check{\pi}^1 \wedge \theta^3.\end{aligned} \tag{5.28}$$

The diffeomorphism $\Phi(x, y, u, p, q, s, \tau) = (x, -y, u, p, -q, -s, \tau)$ is an involution of the Darboux pair for (5.28) in the sense of Remar3.6.

To compute the 3-adapted coframe θ^i_X, we simply calculate the derived flag for the 2 dimensional distribution $\hat{U} = \{\,\partial_{\hat{\pi}^1}, \partial_{\hat{\pi}^2}\,\}$ (see Section 5). From the structure equations (5.28) we find

$$\begin{aligned}&\hat{U}^{(1)} = \hat{U} \cup \{\,-\partial_{\theta^2}\,\} \quad \text{and}\\ &\hat{U}^{(2)} = \hat{U} \cup \{\,-\partial_{\theta^2}, (x + b^2y)\,\partial_{\theta^1} - b\cot 2b\tau\,\partial_{\theta^2} - b\csc 2b\tau\,\partial_{\theta^3}, \partial_{\theta^1}\,\}.\end{aligned} \tag{5.29}$$

In accordance with the algorithm in Section 5, we take $\{ X_1, X_2, X_3 \}$ to be the last 3 vectors in $\hat{U}^{(2)}$ and calculate

$$[\, X_1, X_2 \,] = b^2 X_3. \tag{5.30}$$

Therefore *the Vessiot algebra for the Pfaffian system* (5.25) *is abelian if* $b = 0$ *and nilpotent otherwise.* The 3-adapted coframe $\boldsymbol{\theta_X}$ is

$$\theta^1_X = -\theta^2 + \cos 2b\tau\, \theta^3,\ \ \theta^2_X = -\frac{1}{b} \sin 2b\tau\, \theta^3,\ \ \theta^3_X = \theta^1 + \frac{1}{b}(x + b^2 y) \sin 2b\tau\, \theta^3,$$

and the structure equations (3.28) are

$$\begin{aligned} d\theta^1_X &= -\hat{\pi}^1 \wedge \hat{\pi}^2 + \cos 2b\tau\, \check{\pi}^1 \wedge \check{\pi}^2 + b^2 \hat{\pi}^1 \wedge \theta^2_X, \\ d\theta^2_X &= -\frac{1}{b} \sin 2b\tau\, \check{\pi}^1 \wedge \check{\pi}^2 - \hat{\pi}^1 \wedge \theta^1_X, \\ d\theta^3_X &= \frac{1}{b}(x + b^2 y)\, \sin 2b\tau\, \check{\pi}^1 \wedge \check{\pi}^2 - \hat{\pi}^2 \wedge \theta^1_X - b^2 \theta^1_X \wedge \theta^2_X. \end{aligned} \tag{5.31}$$

This coframe is actually 4-adapted.

Note that we labeled the vectors in the derived flag (5.29) so that the last vector X_3 spans the derived algebra of the Vessiot algebra. By doing so we are assured that the structure equations (5.31) are of the precise form (3.75a)-(3.75b). The algorithm at this point refers back to Case I and the structure equations (3.66) as applied to just the first two equations in (5.31). The matrices M and R (see equations (3.67) and (3.68) are found to be

$$M = \begin{bmatrix} 0 & b^2 \hat{\pi}^1 \\ -\hat{\pi}^1 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \cos \hat{I}^1 & -b \sin \hat{I}^1 \\ \frac{1}{b} \sin \hat{I}^1 & \cos \hat{I}^1 \end{bmatrix}. \tag{5.32}$$

We then compute the 2-forms χ^i and the 1-forms ϕ^i (see (3.70)) to be

$$\begin{gathered} \chi^1 = -\cos b\hat{I}^1\, \hat{\pi}^1 \wedge \hat{\pi}^2, \quad \chi^2 = -\frac{1}{b} \sin b\hat{I}^1\, \hat{\pi}^1 \wedge \hat{\pi}^2, \\ \phi^1 = \hat{I}^2 \cos b\hat{I}^1\, \hat{\pi}^1, \quad \phi^2 = \frac{1}{b} \hat{I}^2 \sin b\hat{I}^1\, \hat{\pi}^1, \end{gathered} \tag{5.33}$$

so that the forms (3.76) are

$$\begin{aligned} \hat{\theta}^1_1 &= \ \cos b\hat{I}^1\, \theta^1_X - b \sin b\hat{I}^1\, \theta^2_X + \ \hat{I}^2 \cos b\hat{I}^1\, \hat{\pi}^1, \\ \hat{\theta}^2_1 &= \frac{1}{b} \sin b\hat{I}^1\, \theta^1_X + \ \cos b\hat{I}^1 \theta^2_X + \frac{1}{b} \hat{I}^2 \sin b\hat{I}^1\, \hat{\pi}^1 \end{aligned} \tag{5.34}$$

The structure equations (5.31) are now reduced to the form (3.77). The final required frame change (3.80) leads to the 5-adapted coframe

$$\hat{\theta}^1 = \hat{\theta}^1_1, \quad \hat{\theta}^2 = \hat{\theta}^2_1, \quad \hat{\theta}^3 = \theta^3_2 + \hat{I}^2 \cos b\hat{I}^1\, \hat{\theta}^1_1 + b\hat{I}^2 \sin b\hat{I}^1\, \theta^2_1 + \frac{1}{2} \hat{I}^2)^2\, \hat{\pi}^2 \tag{5.35}$$

with structure equations

$$d\hat{\theta}^1 = \cos b\check{I}^1\,\check{\pi}^1\wedge\check{\pi}^2,\quad d\hat{\theta}^2 = \frac{1}{b}\sin b\check{I}^1\,\check{\pi}^1\wedge\check{\pi}^2,$$
$$d\hat{\theta}^3 = \frac{1}{b}\big((x+b^2y)\sin 2b\tau - b\hat{I}^2\cos 2b\tau\big)\check{\pi}^1\wedge\check{\pi}^2 - b^2\hat{\theta}^1\wedge\hat{\theta}^2. \tag{5.36}$$

The forms (4.2) are

$$\hat{\omega}^1 = \hat{\theta}^1 - \check{I}^2\cos b\check{I}^1\check{\pi}^1,\quad \hat{\omega}^2 = \hat{\theta}^2 - \check{I}^2\frac{1}{b}\sin b\check{I}^1\check{\pi}^1,\quad \hat{\omega}^3 = \hat{\theta}^3 + A\check{\pi}^1, \tag{5.37}$$

where $A = \check{I}^2(\hat{I}^2\cos 2b\tau + (x+b^2y)\sin 2b\tau - \frac{1}{2}\check{I}^2)$. These forms can be written as

$$\hat{\omega}^1 = dz^1,\quad \hat{\omega}^2 = dz^2,\quad \hat{\omega}^3 = dz^3 + \frac{b^2}{2}(z^1dz^2 - z^2dz^1). \tag{5.38}$$

where

$$z^1 = \frac{1}{b^2}\left(-2x\sin bs\sin b\tau + 2b^2y\cos bs\cos b\tau - b\hat{I}^2\sin b\hat{I}^1 + b\check{I}^2\sin b\check{I}^1\right)$$
$$z^2 = \frac{1}{b^3}\left(\ 2bx\cos bs\sin b\tau +\ \ 2y\sin bs\cos b\tau\ +\ \hat{I}^2\cos b\hat{I}^1 -\ \check{I}^2\cos\check{I}^1\right) \tag{5.39}$$
$$z^3 = 2u - \frac{1}{2b^3}(\sin 2b\tau - 2b\tau\cos 2b\tau)(x^2 - b^4y^2) - 2px\cos^2 2b\tau -$$
$$2qy\sin^2 2b\tau + \frac{1}{2b}\hat{I}^2\check{I}^2\sin 2b\tau.$$

The multiplication map for which the forms (5.38) are the left invariant Maurer-Cartan forms is

$$z^1 = z_1^1 + z_2^1,\quad z^2 = z_1^2 + z_2^2,\quad z^3 = z_1^3 + z_2^3 - \frac{b^2}{2}(z_1^1z_2^2 - z_1^2z_2^1). \tag{5.40}$$

Finally, the combination of equations (5.26), (5.39), and (5.40) leads to the superposition formula

$$x = \frac{1}{2}(x_1 + x_2 + b^2(y_2 - y_1)) + \frac{\sin b(\tau_1 - \tau_2)}{\sin b(\tau_1 + \tau_2)}\xi,$$
$$y = \frac{1}{2b^2}(x_2 - x_1 + b^2(y_1 + y_2)) - \frac{\cos b(\tau_1 - \tau_2)}{b^2\cos b(\tau_1 + \tau_2)}\xi,$$
$$u = u_1 + u_2 + 2\,\frac{p_1\sin 2b\tau_2 - p_2\sin 2b\tau_1)}{\sin 2b(\tau_1 + \tau_2)}\,\xi$$
$$+\frac{1}{b^2}\left(\frac{2\tau_1\sin(2b\tau_2)^2}{\sin(2b(\tau_1+\tau_2))^2} + \frac{2\tau_2\sin(2b\tau_1)^2}{\sin(2b(\tau_1+\tau_2))^2} - \frac{\sin 2b\tau_1\sin 2b\tau_2}{b\sin 2b(\tau_1+\tau_2)}\right)\xi^2, \tag{5.41}$$
$$p = p_1 + p_2 + 2\,\frac{\tau_1\sin 2b\tau_2 - \tau_2\sin 2b\tau_1}{b^2\sin 2b(\tau_1 + \tau_2)}\,\xi,$$
$$q = -b^2p_1 + b^2p_2 - 2\frac{\tau_1\sin 2b\tau_2 + \tau_2\sin 2b\tau_1}{\sin 2b(\tau_1 + \tau_2)}\,\xi,$$
$$s = -\tau_1 + \tau_2,\quad \tau = \tau_1 + \tau_2,\quad \text{where}\quad \xi = x_2 - x_1 - b^2(y_1 + y_2).$$

The restriction of I to the level set $\hat{I}^1 = \hat{I}^2 = 0$ gives

$$\hat{W} = \{\, du - p\,dx + b^2\,dydp - 1/b^3(\tan b\tau - b\tau)dx + \tau dy \,\} \tag{5.42}$$

This is a rank 3 Pfaffian system on a 5 manifold with derived flag dimensions $[3, 2, 0]$. The equivalence problem for such systems was analyzed in detail by Cartan [9] where it is established that the fundamental invariant for such systems is a certain rank 4 symmetric tensor T in two variables. For $b = 0$, this tensor vanishes while for $b <> 0$ we find that T is the 4-th symmetric power of a 1-form. In accordance with Cartan's result the symmetry algebra of $\hat{W}$ when $b = 0$ is the 14 dimensional exceptional Lie algebra g_2 and indeed, it is not difficult to transform $\hat{W}$ to the canonical Pfaffian system for the Hilbert-Cartan equation $z' = (y'')^2$. For $b <> 0$ the symmetry algebra is the 7 dimensional solvable Lie algebra with infinitesimal generators

$$\{\, \partial_x, \partial_y, \partial_u, x\partial_x + y\partial_y + 2u\partial_u + p\partial_p, (x - b^2 y)\partial_u - D_p, Y_1, Y_2 \,\} \tag{5.43}$$

where $Y_2 = [\partial_y, Y_1]$ and

$$\begin{aligned} Y_1 =& (x + b^2 y)\big(\frac{b}{\tan(b\tau)}\partial_x - \frac{\tan(b\tau)}{b}\partial_y + \frac{2pb}{\sin(b\tau)}\partial_u + \frac{\tau}{\cos(b\tau)\sin(b\tau)}D_p\big) \\ &+ 2xyD_u + (y - \frac{x}{b^2})D_p - \partial_\tau. \end{aligned} \tag{5.44}$$

Example 5.4. It is an open problem to determine which scalar Darboux integrable equations admit generalizations wherein the dependent variable U takes values in an arbitrary non-commutative, finite dimensional algebra $\mathcal{A}$. Here are two such examples which provide us with many Darboux integrable systems amenable to the methods presented in this paper.

I. $U_{xy} = (U_x + I)\, U^{-1}\, U_y$

II. $U_{xy} = U_x\,(U - y)^{-1}\, U_y + U_y\,(U - x)^{-1}\, U_x$.

The first integrals for the singular systems (excluding x and y) and general solutions are [1]

[1] We remark that the solution to **II** given by Vessiot [25] (equation (70), page 45) in the scalar case is incorrect.

$$\textbf{I}. \quad \hat{I}^1 = (U_x + I)\,U^{-1}, \quad \hat{I}^2 = D_x(\hat{I}^1), \quad \check{I}^2 = U_y^{-1}\,U_{yy},$$
$$U = (F')^{-1}\,(-F + G),$$

$$\textbf{II}. \quad \hat{I}^1 = (U - x)^{-1}\,[\,U_{xx}\,U_x^{-1}\,(U - x) - 2U_x - I\,],$$
$$\check{I}^1 = (U - y)^{-1}\,[\,U_{yy}\,U_y^{-1}\,(U - y) - 2U_y - I\,],$$
$$U = (xF' + yF' - F - G)\,(F' + G')^{-1},$$

where $F = F(x)$ and $G = G(y)$ take values in $\mathcal{A}$. For both systems the Vessiot algebra is the tensor product of $\mathcal{A}$ with the Vessiot algebra for the corresponding scalar equation. We conjecture that all equations of Moutard type ([15] Volume II, page 250, equation 19) admit non-commutative generalizations.

Example 5.5. Some of the simplest examples of Darboux integrable systems can be constructed by the coupling of a Darboux integrable scalar equation to a linear or Moutard-type equation. As examples, we give

$$\textbf{I}. \quad u_{xy} = e^{2u}, \qquad v_{xy} = n(n+1)e^{2u}v,$$
$$\textbf{II}. \quad u_{xy} = e^{u}u_y, \quad v_{xy} + ((n-\alpha)e^{u} + \alpha u_x)v_x = 0,$$
$$\textbf{III}. \quad u_{xy} = e^{u}u_y, \quad v_{xy} - e^{u}v_y + (n+1)\,u_y v_x + (n+1)!\,e^{u}u_y = 0,$$
$$\textbf{IV}. \quad u_{xy} = e^{u}u_x \quad v_{xy} + (e^{v})_x - (nBe^{-v})_y + (n-2)B = 0,$$

where $B = e^u u_x$. The system **I** appears in [18](page 116); systems **II**–**IV** do not seem to have appeared in the literature. For each of these systems the restricted EDS $\hat{W}$ and $\check{W}$ are jet spaces for two functions of a single variable. The Vessiot algebra for **I** is the semi-direct product of $sl2$ and an Abelian Lie algebra of dimension $2n+1$ determined by the $2n+1$ dimensional irreducible representation of $sl2$. The infinitesimal Vessiot group action for **I**, restricted to $\hat{W}$ or $\check{W}$, is the action listed in the classification of vector fields systems in the plane in [14] as number 27 (where the variables x, y in [14] serve as the dependent variables for the jet spaces $\hat{W}$ and $\check{W}$. For **II** the Vessiot algebra is a semi-direct product of the 2 dimensional solvable algebra with an n-dimensional Abelian algebra. The infinitesimal Vessiot group action is number 24 in [14]. The infinitesimal Vessiot groups for **III** and **IV** have dimensions $n+3$ and $n+4$ and coincide, respectively, with numbers 25 and 26 in [14].

For $n = 1$ the general solutions to these systems are

$$\textbf{I}. \quad u = \frac{1}{2}\ln\frac{F_1'G_1'}{(F_1+G_1)^2}, \quad v = 2\frac{F_2-G_2}{F_1+G_1} - \frac{F_2'}{F_1'} + \frac{G_2'}{G_1'}$$

$$\textbf{II}. \quad u = \ln\frac{F_1'}{G_1-F_1}, \quad v = \frac{1}{F_1^{\alpha}}\big(F_2 - G_2 - (F_1-G_1)\frac{G_2''}{G_2'}\big),$$

$$\textbf{III}. \quad u = \ln\frac{F_1'}{G_1-F_1}, \quad v = \frac{F_2-G_2}{(F_1-G_1)^2} - \frac{G_2''}{(F_1-G_1)G_2'} - u,$$

$$\textbf{IV}. \quad u = \ln\frac{G_1'}{G_1-F_1}, \quad v = \ln\big(\frac{G_1'\,(F_2-G_2) - G_2'\,(F_1-G_1)\,F_1'}{(F_1'\,(F_2-G_2) - F_2'\,(F_1-G_1))\,(F_1-G_1)}\big).$$

The general solutions for arbitrary n can be obtained in closed compact form by the method of Laplace.

In addition, any non-linear Darboux integrable system can be coupled to its formal linearization to obtain another Darboux integrable system. For example, if we prolong the partial differential equations

$$\textbf{V}. \quad 3u_{xx}u_{yy}^3 + 1 = 0, \quad v_{xx} - \frac{1}{u_{yy}^4}v_{yy} = 0$$

to order 3 in the derivatives of u, we obtain a rank 8 Pfaffian system on a 14-dimensional manifold which is Darboux integrable, with 4 first integrals for each associated singular Pfaffian system. The Vessiot algebra is Abelian and of dimension 6. The two Lie algebras of vectors field dual to the forms $\hat{\theta}$ and $\check{\theta}$ for the fifth adapted coframe coincide and are given by $\{\partial_y,\ \partial_u,\ x\partial_u + \partial_{u_x},\ \partial_v,\ x\partial_v + \partial_{v_x},\ u_y\partial_v + u_{xy}\partial_{v_x} + u_{yy}\partial_{v_y} + u_{xyy}\partial_{v_{xy}} + u_{yyy}\partial_{v_{yy}}\}$. In accordance with Remark 3.12, these vector fields are also infinitesimal symmetries for **V**. In terms of the arbitrary functions $\phi(\alpha)$ and $\psi(\beta)$ appearing in the general solution to $3u_{xx}u_{yy}^3 + 1 = 0$, the general solution for v can be written as

$$v = F + G - \frac{\phi'' - \psi'' + (\beta-\alpha)\phi'''}{(\beta-\alpha)\phi''''}F' - \frac{\phi'' - \psi'' + (\beta-\alpha)\psi'''}{(\beta-\alpha)\psi''''}G', \qquad (5.45)$$

where $F = F(\alpha)$ and $G = G\beta)$.

Finally, we remark that the Laplace transformation (not the integral one, Darboux [11] , Forsyth [13], pages 45–59.) can be applied to the linear components of any of the above systems to obtain new Darboux integrability systems. As well, the linear component in any of these systems can be replaced by their formal adjoint to obtain yet other Darboux integrable systems.

Example 5.6. Although we are unaware of an explicit general proof it is generally acknowledged that the Toda lattice equations (see, for example, [18], [21]) are Darboux integrable. In this example we shall check that the B2 Toda lattice

$$u_{xy} = 2e^u - 2e^v, \quad v_{xy} = -e^u + 2e^v. \tag{5.46}$$

is Darboux integrable and find the closed form general solution. We use this example to illustrate a different computational approach based upon the symmetry reduction intepretation of the superposition formula given at the end of section 4.

This system satisfies our definition of Darboux integrable upon prolongation to 4-th order, that is, as a rank 14 Pfaffian system I on a 20 dimensional manifold. The first integrals for the singular Pfaffian system containing dx are $\hat{I}^1 = x$, $\hat{I}^2 = v_{xx} + \frac{2}{3}u_{xx} - \frac{1}{3}u_x v_x - \frac{1}{6}u_x^2 - \frac{1}{3}v_x^2$, $\hat{I}^3 = D_x\hat{I}^2$, $\hat{I}^4 = D_x\hat{I}^3$ and

$$\hat{I}^5 = u_{xxxx} + 2v_{xxxx} - 2v_xv_{xxx} - u_{xx}u_x^2 - 2u_xu_{xx}v_x + \frac{1}{8}u_x^4 + \frac{1}{2}u_{xx}^2 + \frac{1}{2}v_x^2u_x^2 - v_x^2u_{xx} - \frac{1}{2}v_{xx}u_x^2 + v_{xx}u_{xx} + \frac{1}{2}u_x^3v_x - u_xv_{xxx}. \tag{5.47}$$

For $\hat{M}$ we take the level set $\hat{I}^a = 0$ and let $\hat{W}$ be the restriction of I to $\hat{M}$. We find that $\hat{W}$ is a rank 12 Pfaffian system on a 15 manifold. The derived flag of $\hat{W}$ has dimensions $[12, 10, 8, 6, 4, 2, 1, 0]$ while the dimensions of the space of Cauchy characteristics for these derived Pfaffian sytems are [0, 2, 4, 6,8 , 10, 12, 15]. By using the invariants of the Cauchy characteristics as new coordinates, we are able to write $\hat{W}$ in the canonical form

$$\hat{W} = \{\, du_2 - \dot{u}_2du_1,\ d\dot{u}_2 - \ddot{u}_2du_1,\ \ldots,\ du_1 - u_1'dx,\ du_1' - du_1''dx,\ \ldots\} \tag{5.48}$$

(There are 7 contact forms for u_2 and 5 for u_1). In these coordinates the integral manifolds of $\hat{W}$ are given by $u_1 = F_1(x)$, $u_1' = F_1'(x)$, ... and $u_2 = F_2(F_1(x))$, $\dot{u}_2 = (\dot{F}_2)(F_1(x))$, ... In these coordinates the 10 dimensional infinitesimal Vessiot group $\hat{X}$, restricted to $\hat{M}$, takes a remarkable simple and well-known form. (See, for example Olver [20] page 473.) It is the conformal algebra $o(3,2)$ acting on the 3-dimensional space $(u_1, u_2, \dot{u}_2)$ by contact transforms. Explicitly, the generating functions for $\hat{X}$ are

$$Q = [u_2, -u_2 + u_1\dot{u}_2, -\dot{u}_2^2u_1 + 2\dot{u}_2u_2, \frac{1}{2}, \dot{u}_2, \frac{1}{2}\dot{u}_2^2, u_1, -\frac{1}{2}u_1^2\dot{u}_2^2 - 2u_2^2 + 2u_1\dot{u}_2u_2, -2u_1u_2 + \dot{u}_2u_1^2, \frac{1}{2}u_1^2]. \tag{5.49}$$

To obtain the vector field $\hat{X}_q \in \hat{X}$ corresponding to $q \in Q$, first construct the vector field $X_q^0 = -q_{\dot{u}_2}\partial_{u_1} + (q - \dot{u}_2q_{\dot{u}_2})\partial_{u_2} + (q_u + \dot{u}_2q_{u_2})\partial_{u_2}$ and then prolong

X_q^0 to a vector field on $\hat{M}$ by requiring it to be a symmetry of $\hat{W}$. This basis for $o(3,2)$ is the canonical basis where the first 2 vectors define the Cartan subalgebra, the next 4 correspond to the positive roots, and the last 4 to the negative roots.

In accordance with the remarks at the end of section 4, the superposition formula for the B2 Toda lattice can therefore be constructed from the joint differential invariants for the diagonal action of the conformal algebra $o(3,2)$. We use coordinates $[y, v_1, v_1', v_1'' \ldots, v_2, \dot{v}_2, \ddot{v}_2, \ldots]$ on $\check{M}$. To compactly describe these joint invariants we first calculate the joint differential invariants in the variables

$$\{u_1,\ u_1',\ v_1,\ v_1',\ \dot{u}_2,\ \dot{v}_2,\ \ddot{u}_2,\ \ddot{v}_2,\ \dddot{u}_2,\ \dddot{v}_2\}$$

for the 7 dimensional subalgebra of $o(3,2)$ generated by $Q[1]$, $Q[2]$, $Q[4]$, $Q[5]$, $Q[6]$, $Q[7]$, $Q[10]$ to be

$$\begin{aligned}
J_1 &= v_1'(\dddot{u}_2)^{1/3}(\dddot{v}_2)^{2/3}/(\ddot{u}_2 - \ddot{v}_2), \quad J_2 = u_1'(\dddot{u}_2)^{3/3}(\dddot{v}_2)^{1/3}/(\ddot{u}_2 - \ddot{v}_2),\\
J_3 &= (\ddot{u}_2 v_1 - \ddot{u}_2 u_1 + \dot{u}_2 - \dot{v}_2)(\dddot{u}_2)^{1/3}(\dddot{v}_2)^{2/3}/(\ddot{v}_2 - \ddot{u}_2)^2,\\
J_4 &= (\ddot{v}_2 u_1 - \ddot{v}_2 v_1 + \dot{v}_2 - \dot{u}_2)(\dddot{u})^{2/3}(\dddot{v})^{1/3}/(\ddot{v}_2 - \ddot{u}_2)^2,
\end{aligned} \tag{5.50}$$

$$\begin{aligned}
J_5 = -(&\ddot{u}_2\ddot{v}_2 v_1^2 - 2\ddot{u}_2\ddot{v}_2 v_1 u_1 + \ddot{u}_2\ddot{v}_2 u_1^2 - 2\ddot{u}_2\dot{u}_2 u_1 + 2\ddot{u}_2\dot{u}_2 v_1 + 2\ddot{u}_2 u_2\\
&- 2\ddot{u}_2 v_2 - 2\ddot{v}_2\dot{v}_2 v_1 + 2\ddot{v}_2\dot{v}_2 u_1 + 2\ddot{v}_2 v_2 - 2\ddot{v}_2 u_2\\
&+ \dot{u}_2^2 - 2\dot{u}_2\dot{v}_2 + \dot{v}_2^2)\dddot{u}_2\dddot{v}_2/(2(\ddot{v}_2 - \ddot{u}_2)^4).
\end{aligned} \tag{5.51}$$

Then, in terms of these partial invariants the (lowest) order joint differential invariants for $o(3,2)$ are

$$K_1 = -\frac{J_1 J_2 (J_3 J_4 - 2J_5)^2}{(J_3 J_4 - J_5)^2} \quad \text{and} \quad K_2 = -\frac{J_1 J_2 (J_3 J_4 - J_5)}{(J_3 J_4 - 2J_5)^2} \tag{5.52}$$

and the solutions to the B2 Toda lattice are

$$u = \ln(K_1/4) \quad \text{and} \quad v = \ln(2K_2) \quad \text{where} \tag{5.53}$$

$$u_1 = F_1(x),\ u_2 = F_2(F_1(x)),\ v_1 = G_1(y),\ v_2 = G_2(G_1(y)). \tag{5.54}$$

It is hoped that a more transparent representation of these solutions, similar to that available for the A_n Toda lattice can be be obtained.

Example 5.7. Let P denote the 2-dimensional Minkowski plane with metric $dx \odot dy$ and let N be a pseudo-Riemannian manifold with metric g. A mapping $\varphi : P \to N$ which is a solution to the Euler-Lagrange equations for the

Lagrangian

$$L = g(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y})\, dx \wedge dy \tag{5.55}$$

is called a wave map. There are precisely two inequivalent, non-flat metrics [2] (up to constant scalaring) in 2 dimensions, namely

$$g_1 = \frac{1}{e^{-u}+1}(du^2+dv^2) \quad \text{and} \quad g_2 = \frac{1}{e^{u}-1}(du^2+dv^2) \tag{5.56}$$

which define Darboux integrable wave maps at the 2-jet level (that is, without prolongation). Surprizingly , these metrics are not constant curvature. It is not difficult to check that under the change of coordinates

$$x = x - y, \quad t = x + y, \quad \theta = \arctan(\sqrt{e^u - 1}), \quad \chi = v/2 \tag{5.57}$$

the differential equations (4.13) in [4] become the wave map equations for the metric g_2. The Vessiot algebras for the wave map equations for g_1 and g_2 are $sl(2) \times R$ amd $so(3) \times R$ respectively.

The wave map equations for g_1 are

$$u_{xy} = \frac{v_x v_y - u_x u_y}{2e^u + 2}, \quad v_{xy} = -\frac{u_x v_y + u_y v_x}{2e^u + 2}. \tag{5.58}$$

The standard encoding of these equations as Pfaffian system results in a rank 6 Pfaffian system on a 12 manifold. There are 4 first integrals for each singular Pfaffian system – for the singular Pfaffian system containing dx the first integrals are $\hat{I}^1 = x$, $\hat{I}^2 = \dfrac{e^u(u_x^2 + v_x^2)}{1+e^u}$,

$$\hat{I}^3 = \frac{e^u((2u_x u_{xx} + 2v_x v_{xx})e^u + u_x^3 + u_x v_x^2 + 2u_x u_{xx} + 2v_x v_{xx})}{(1+e^u))^2}, \text{ and} \tag{5.59}$$

$$\hat{I}^4 = \frac{e^u((2u_x v_{xx} + 2v_x u_x^2 + 2v_x^3 - 2v_x u_{xx})e^u - 2v_x u_x x + v_x u_x^2 + 2u_x v_{xx} + v_x^3)}{(1+e^u))^2}.$$

After considerable computation, the superposition is obtained and we find the general solution, in terms of the four arbitrary functions $(F_1(x), F_2(x), G_1(y), G_2(y))$ to be

$$2e^u = -1 + AB\sqrt{1+A^2}\sqrt{1+B^2}\sin(F_2 - G_2) \tag{5.60}$$

[2]Note, however, that the metric g_1 is transformed to g_2 under the complex change of variable $u = v =$

and

$$
\begin{aligned}
v &= F_1(x) - F_2(x) + G_1(y) - G_2(y) \\
&+ \arctan\left(\frac{AG_1'\sqrt{1+A^2}}{A'}\right) + \arctan\left(\frac{BG_1'\sqrt{1+B^2}}{B'}\right) \\
&+ \arctan\left(\frac{AB'\cos(\Delta) + G_2'B^2\sqrt{1+A^2}\sqrt{1+B^2} + G_2'AB(1+B^2)\sin(\Delta)}{G_2'AB\sqrt{1+B^2}\cos(\Delta) - BB'\sqrt{1+A^2} - AB'\sqrt{1+B^2}\sin(\Delta)}\right) \\
&+ \arctan\left(\frac{A'B\cos(\Delta) - F_2'A^2\sqrt{1+A^2}\sqrt{1+B^2} - F_2'AB(1+A^2)\sin(\Delta)}{F_2'AB\sqrt{1+A^2}\cos(\Delta) + AA'\sqrt{1+B^2} + A'B\sqrt{1+A^2}\sin(\Delta)}\right),
\end{aligned}
\tag{5.61}
$$

where

$$
A(x) = \sqrt{\left(\frac{F_1'}{F_2'}\right)^2 - 2\frac{F_1'}{F_2'}}, \quad B(y) = \sqrt{\left(\frac{G_1'}{K'}\right)^2 - 2\frac{G_1'}{G_2'}}, \quad \Delta = F_2 - G_2. \tag{5.62}
$$

Note that

$$
F_1'(x) = F_2'(x)(1 + \sqrt{1 + A(x)^2}), \quad G_1'(y) = G_2'(y)(1 + \sqrt{1 + B(y)^2}). \tag{5.63}
$$

Example 5.8. We turn now to some examples of overdetermined systems for a single unknown function of 3 independent variable, beginning with the system

$$u_{xz} = uu_x, \quad u_{yz} = uu_y. \tag{5.64}$$

The structure equations for the canonical encoding of this system as a rank 4 Pfaffian system $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ are (modulo I), $d\alpha^1 \equiv 0$,

$$d\alpha^2 \equiv \hat{\pi}^1 \wedge \hat{\pi}^3 + \pi^2 \wedge \pi^4, \quad d\alpha^3 \equiv \hat{\pi}^1 \wedge \hat{\pi}^4 + \pi^2 \wedge \pi^5, \quad d\alpha^4 \equiv \check{\pi}^1 \wedge \check{\pi}^2, \tag{5.65}$$

where $\hat{\pi}^1 = dx$, $\hat{\pi}^2 = dy$, $\check{\pi}^1 = dz$,

$$\begin{aligned} &\hat{\pi}^3 = du_{xx} - (u_{xx}u + u_x^2)\, dz, \quad \hat{\pi}^4 = du_{xy}u - (u_{xy}u - u_y u_x)\, dz, \\ &\hat{\pi}^5 = du_{yy} - (u_{yy}u + u_y^2) dz, \quad \check{\pi}^2 = du_{zz} - (u_z + u^2)(u_x dx + u_y dy). \end{aligned} \tag{5.66}$$

The first integrals for the singular Pfaffian systems $\hat{V} = I \cup \{\hat{\pi}^1, \ldots, \hat{\pi}^5\}$ and $\check{V} = I \cup \{\check{\pi}^1, \check{\pi}^2\}$ are $\hat{I}^1 = x$, $\hat{I}^2 = y$, $\check{I}^1 = z$ and

$$\hat{I}^3 = \frac{u_y}{u_x}, \quad \hat{I}^4 = D_x \check{I}^3, \quad \hat{I}^5 = D_y \check{I}^3, \quad \check{I}^2 = u_z - u^2, \quad \check{I}^3 = D_z \check{I}^2. \tag{5.67}$$

The form $\hat{\pi}^3$ is not in $\hat{V}^\infty + \check{V}$ and therefore (5.64) is not Darboux integrable on the 2-jet. The prolongation of (5.64) defines a decomposable rank 8 Pfaffian system $\{\alpha^1, ...\alpha^8\}$ on a 16 dimensional manifold. In addition to the first integrals (5.67), we now also have

$$\hat{I}^8 = D_x^2 \hat{I}^3, \ \hat{I}^7 = D_x D_y \hat{I}^3, \ \hat{I}^8 = D_y^2 \hat{I}^3, \hat{I}^9 = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{u_x^2}, \ \check{I}^4 = D_z^2 \check{I}^2 \tag{5.68}$$

and (1.17) -(1.19) are now satisfied. A 1-adapted coframe is $\hat{\sigma}^1 = d\hat{I}^1$, $\hat{\sigma}^2 = d\hat{I}^2$, $\hat{\sigma}^6 = d\hat{I}^6$, $\hat{\sigma}^7 = d\hat{I}^7$, $\hat{\sigma}^8 = d\hat{I}^8$, $\hat{\sigma}^9 = d\hat{I}^9$, $\check{\sigma}^1 = d\check{I}^1$, $\check{\sigma}^2 = d\check{I}^4$,

$$\begin{aligned}
&\hat{\eta}^1 = \frac{1}{u_x}\alpha^3 - \frac{u_y}{u_x^2}\alpha^2 = d\hat{I}^3 - \hat{I}^4 d\hat{I}^1 - \hat{I}^5 d\hat{I}^2, \\
&\hat{\eta}^2 = \frac{1}{u_x}\alpha^6 - \frac{u_y}{u_x^2}\alpha^5 - \frac{u_{xx}}{u_x^2}\alpha^3 - \frac{u_{xy}u_x - 2u_y u_{xx}}{u_x^3}\alpha^2 = d\hat{I}^4 - \hat{I}^6 d\hat{I}^1 - \hat{I}^7 d\hat{I}^2, \\
&\hat{\eta}^3 = \frac{1}{u_x}\alpha^7 - \frac{uy}{u_x^2}\theta^6 - \frac{u_{xy}}{u_x^2}\alpha^3 - \frac{-2u_y u_{xy} + u_{yy}u_x}{u_x^3}\alpha^2 = d\check{I}^5 - \hat{I}^7 d\hat{I}^1 - \hat{I}^8 d\hat{I}^2, \\
&\check{\eta}^1 = \alpha^4 - u\alpha^1 = d\check{I}^2 - \check{I}^3 d\check{I}^1, \quad \check{\eta}^2 = \alpha^8 - u\alpha^4 - u_z\alpha^1 = d\check{I}^3 - \check{I}^4 d\check{I}^1, \\
&\theta^1 = \alpha^1 = du - u_x dx - u_y dy - u_z dz, \\
&\theta^2 = \alpha^2 = du_x - u_{xx}dx - u_{xy}dy - u_x u dz, \\
&\theta^3 = \alpha^3 = du_{xx} - u_{xxx}dx - u_{xxy}dy - (uu_{xx} + u_x^2)dz.
\end{aligned}$$

This frame is in fact 2-adapted. We next determine $\hat{U}^{(\infty)} = \hat{U} \cup \{X_1, X_2, X_3\}$ and $\check{U}^{(\infty)} = \check{U} \cup \{Y_1, Y_2, Y_3\}$, where

$$\begin{aligned} X_1 &= -u_x \hat{I}^4 \partial_{\theta^1} - (u_{xx}\hat{I}^4 + u_x \hat{I}^6)\partial_{\theta^2} - \frac{3u_{xx}^2 \hat{I}^4 + 4u_{xx}u_x\hat{I}^6 - 2u_x^2\hat{I}^4\hat{I}^9}{2u_x}\partial_{\theta^3}, \\ X_2 &= -u_x\partial_{\theta^3},\ X_3 = -u_x\partial_{\theta^1} - u_{xx}\partial_{\theta^2} - \frac{3u_{xx}^2 + 2u_x^2\hat{I}^9}{2u_x}\partial_{\theta^3},\ Y_1 = -\partial_{\theta^1}, \quad (5.69) \\ Y_2 &= u\partial_{\theta^1} + u_x\partial_{\theta^2} + u_{xx}\partial_{\theta^3},\ Y_3 = (-\frac{u^2}{2} + \check{I}^2)\partial_{\theta^1} - uu_x\partial_{\theta^2} - (u_x^2 + uu_{xx})\partial_{\theta^3} \end{aligned}$$

With respect to the base point with coordinates determined so that $u = 0$, $u_x = 1$, $u_{xx} = 0$, $\hat{I}^6 = 1$ and all other first integrals vanish, the matrices (3.40) and (3.43) are given by

$$P = \begin{bmatrix} \frac{1}{\hat{I}^6} & \frac{2\hat{I}^4\hat{I}^9}{\hat{I}^6} & -\frac{\hat{I}^4}{\hat{I}^6} \\ 0 & 1 & 0 \\ 0 & -\hat{I}^9 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & -1 & 0 \\ \check{I}^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad (5.70)$$

from which we may calculate the 4 adapted coframes

$$\begin{aligned} \theta_X^1 &= \frac{u_{xx}}{u_x^2}\theta^1 - \frac{1}{u_x}\theta^2, & \theta_Y^1 &= -\frac{u_x^2 + uu_{xx}}{u_x^3}\theta^2 + \frac{u}{u_x^2}\theta^3, \\ \theta_X^2 &= -\frac{u_{xx}^2}{u_x^3}\theta^1 + \frac{2u_{xx}}{u_x^2}\theta^2 - \frac{1}{u_x}\theta^3, & \theta_Y^2 &= \frac{u_{xx}}{u_x^2}\theta^2 - \frac{1}{u_x^2}\theta^3, \qquad (5.71) \\ \theta_X^3 &= -\frac{1}{u_x}\theta^1, & \theta_Y^3 &= -\theta^1 + \frac{u(2u_x^2 + uu_{xx})}{2u_x^3}\theta^2 - \frac{u^2}{2u_x^2}\theta^3. \end{aligned}$$

The Vessiot algebra is $sl2$. Because this algebra is semi-simple we directly determine the matrix representation of the Vessiot group from (5.71) to be

$$\lambda = \begin{bmatrix} \frac{u_x^2 - uu_{xx}}{u_x^2} & \frac{u_{xx}}{u_x^2} & \frac{u(uu_{xx} - 2u_x^2)}{2u_x^2} \\ -\frac{u}{u_x} & \frac{1}{u_x} & \frac{u^2}{2u_x} \\ -\frac{u_{xx}(uu_{xx} - 2u_x^2)}{2u_x^3} & \frac{u_{xx}^2}{2u_x^3} & \frac{(uu_{xx} - 2u_x)^2}{4u_x^3} \end{bmatrix}. \qquad (5.72)$$

To obtain the superposition formula we introduce local coordinates x, y, v, v_x, v_y, v_z, v_{xx}, v_{xy}, v_{yy}, v_{yz}, v_{xxx}, v_{xxy}, v_{xyy}, v_{yyy}, v_{yyz} for $\hat{M}$ and z, w, w_x, w_{xx}, w_z, w_{zz}, w_{zzz} for $\check{M}$. The inclusions $\hat{\iota} : \hat{M} \to M$ and $\check{\iota} \colon \check{M} \to M$ are fixed by mapping $v_I \to u_I$, $w_I \to u_I$ and requiring that $\hat{\iota}^*(\hat{I}^a) = 0$ and $\check{\iota}^*(\hat{I}^a) = \delta_6^a$. The superposition formula is then found by solving the equations

$$\hat{\iota}^*(\hat{I}^a) = \hat{I}^a, \quad \check{\iota}^*(\check{I}^a) = \check{I}^a, \quad \lambda = \hat{\iota}^*(\lambda) \cdot \check{\iota}^*(\lambda) \qquad (5.73)$$

for the coordinates of M. We find that

$$u = w - \frac{2vw_x^2}{-2w_x + vw_{xx}}. \tag{5.74}$$

Finally, we calculate the integral manifolds for $\hat{W}$ and $\check{W}$. It is immediate that $\hat{W}$ is the canonical Pfaffian system on $J^3(R, R^2)$ and hence the integral manifolds are defined by $v = V(x, y)$. The last non-zero form in the derived flag for $\check{W}$ yields the Pfaffian equation

$$dw_{xx} - \frac{w_{xx}}{w_x} - w_x^2 dz = 0$$

from which it follows that $w_x = G'(z)$ and $w_{xx} = G(z)G'(z)$. One of the remaining equations in $\check{W}$ then gives $w = G''(z)/G'(z)$ which leads, on replacing $V(x, y)$ by $-2/F(x, y)$, to the final solution

$$u(x, y, z) = \frac{G''(z)}{G'(z)} - \frac{2G'(z)}{F(x, y) + G(z)}. \tag{5.75}$$

We continue this example by considering two variations on (5.64). First we observe that to (5.64) we may add any equation of the form

$$F(x, y, \frac{u_y}{u_x}, \frac{u_y u_{xx} - u_x u_{xy}}{u_x}, \frac{u_y u_{xy} - u_x u_{tt}}{u_x^2}) = 0 \tag{5.76}$$

to obtain a rank 4 Pfaffian system $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ on an 10 dimensional manifold with structure equations $d\alpha^1 \equiv 0$,

$$d\alpha^2 \equiv \hat{\pi}^1 \wedge \hat{\pi}^2 \quad d\alpha^3 \equiv \hat{\pi}^3 \wedge \hat{\pi}^4 \quad d\alpha^4 \equiv \check{\pi}^1 \wedge \check{\pi}^2 \tag{5.77}$$

These systems are involutive with Cartan character $s_1 = 3$.

For example, consider

$$u_y u_{xy} - u_x u_{yy} = 0, \quad u_{xz} = uu_x, \quad u_{yz} = uu_y \tag{5.78}$$

The foregoing calculations can repeated, almost without modification, to arrive at same superposition formula (5.74) – the only difference is that now $\hat{W}$ is the (prolonged) canonical Pfaffian system for the equation $v_y v_{xy} - v_x v_{yy} = 0$, an equation which is itself Darboux integrable. Thus, in more complicate situations, the method of Darboux, can be used to integrate the Pfaffian systems $\hat{W}$ and $\check{W}$ and the superposition formula for the original system is given by a composition of superposition formulas. In the case of the present example, the calculation of the first integrals for $\hat{W}$ reveals that this system is contact equivalent to the wave equation and leads to parametric solution

$$x = f'(\sigma)y = f(\sigma) - \sigma f'(\sigma) + g(\tau), u = \frac{G''(z)}{G'(z)} - \frac{2G'(z)}{tau + G(z)}. \tag{5.79}$$

Our second variation of (5.64) is obtained by the differential substitution $u_x = \exp(v)$, which leads to the equations

$$v_{xz} = \exp(v) \quad \text{and} \quad v_{yzz} = v_{yz}v_z. \tag{5.80}$$

It is surprising that the canonical Pfaffian system for these equations (obtained from the contact ideal on $J^3(R, R^2)$ does not define a decomposable Pfaffian system. The following theorem resolves this difficulty.

Theorem 5.9. *The system of differential equations*

$$\begin{aligned} u_{xz} &= F(x, y, z, u, u_z, u_{yz}, u_{zz}), \\ u_{yzz} &= G(x, y, z, u, u_x, u_y, u_z, u_y z, u_{zz}, u_{yyz}, u_{zzz}) \end{aligned} \tag{5.81}$$

determines the rank 4 Pfaffian system

$$\begin{aligned} &\alpha^1 = du - u_x dx - u_y dy - u_z dz, \quad \alpha^2 = du_z - F dx - u_{yz} - u_{zz} dz, \\ &\alpha^3 = du_{yz} - D_y(F)dx - u_{yyz}dy - G dz, \\ &\alpha^4 = du_{zz} - D_z(F)dx - G dy - u_{zzz}dz, \end{aligned} \tag{5.82}$$

on an 11 dimensional manifold. If the integrability condition $D_y D_z F = D_x(G)$ holds, then this Pfaffian system is decomposable and involutive with Cartan characters $s_1 = 1$ and $s_2 = 2$

We use Theorem 5.9 to write (5.80) as a rank 4 Pfaffian system on an 11-manifold. The prolongation of this system is Darboux integrable and calculations, virtually identical to those provided for (5.80) lead directly to the general solution

$$v(x, y, z) = \ln(\frac{2F_x(x, y)G'(z)}{(2F(x, y) + G(z))^2}. \tag{5.83}$$

References

[1] I. M. Anderson and M. E. Fels, *Exterior Differential Systems with Symmetry*, Acta. Appl. Math. **87** (2005), 3–31.

[2] I. M. Anderson and M. Juráš, *Generalized Laplace Invariants and the Method of Darboux*, Duke J. Math **89** (1997), 351–375.

[3] I. M. Anderson, N. Kamran, and P. Olver, *Internal, external and generalized symmetries*, Advances in Math. **100** (1993), 53–100.

[4] B. M. Barbashov, V. V. Nesterenko, and A. M. Chervyakov, *General solutions of nonlinear equations in the geometric theory of the relativistic string*, Commun. Math Physics **84** (1982), 471–481.

[5] F. De Boer, *Application de la méthode de Darboux à l'intégration de l'équation différentielle $s = f(r,t)$.*, Archives Neerlandaises **27** (1893), 355–412.

[6] R. L. Bryant, S. S Chern, R. B Gardner, H. Goldschmidt, and P. A. Griffiths, *Essays on Exterior Differential Systems*, MSRI Publications, Vol. 18, Springer-Verlag, New York, 1991.

[7] R. L Bryant and P. A. Griffiths, *Characteristic cohomology of differential systems (I)*, Selecta Math. (N.S.) **1** (1995), 21–112.

[8] R. L. Bryant, P. A. Griffiths, and L. Hsu, *Hyperbolic exterior differential systems and their conservation laws, Parts I and II*, Selecta Math., New series **1** (1995), 21–122 and 265–323.

[9] É Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. École Norm. **3** (1910), no. 27, 109–192.

[10] ______, *Sur les systèmes en involution d'équations aux dériv'ees partielles du second ordre à une fonction inconnue de trois variable indèpendantes*, Bull Soc. Math France **39** (1911), 352-443.

[11] G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, Gauthier-Villars, Paris, 1896.

[12] P. T. Eendebak, *Contact Structures of Partial Differential Equations*, 2007.

[13] A. Forsyth, *Theory of Differential Equations, Vol 6*, Dover Press, New York, 1959.

[14] A. González-López, N. Kamran, and P. J. Olver, *Lie algebras of vector fields in the real plane*, Proc. London Math. Soc. **64** (1992), 339-368.

[15] E. Goursat, *Lecon sur l'intégration des équations aux dériées partielles du second ordre á deux variables indépendantes, Tome 1, Tome 2*, Hermann, Paris, 1897.

[16] E. E. Goursat, *Recherches sur quelques équations aux dériées partielles du second ordre*, Ann. Fac. Sci. Toulouse **1** (1899), 31–78 and 439–464.

[17] P. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. **41** (1974), 775–814.

[18] A. N. Leznov and M. V. Saleliev, *Representation theory and integration of nonlinear spherically symmetric equations to gauge theories*, Commun. Math. Phys. **74** (1980), 111-118.

[19] ______, *Two-dimensional exactly and completely integrable dynamical systems*, Commun. Math. Phys. **89** (1983), 59–75.

[20] P. J. Olver, *Equivalence, Invariants, and Symmetry* (1995).

[21] V. V. Sokolov and A. V. Ziber, *On the Darboux integrable hyperbolic equations*, Phys Lett. A **208**, 303–308.

[22] O. Stormark, *Lie's structural approach to PDE systems* **80** (2000).

[23] P. J. Vassiliou, *Vessiot structure for manifolds of* (p, q)*-hyperbolic type: Darboux integrability and symmetry*, Trans. Amer. Math. Soc. **353** (2001), 1705–1739.

[24] E. Vessiot, *Sur les équations aux dérivées partielles du second ordre, F(x,y,z,p,q,r,s,t)=0, intégrables par la méthode de Darboux*, J. Math. Pure Appl. **18** (1939), 1–61.

[25] ———, *Sur les équations aux dérivées partielles du second ordre, F(x,y,z,p,q,r,s,t)=0, intégrables par la méthode de Darboux*, J. Math. Pure Appl. **21** (1942), 1–66.