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Complete Symmetry Groups of Ordinary Differential Equations and Their Integrals: Some Basic Considerations

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The concept of the complete symmetry group of a differential equation introduced by J. Krause (1994, *J. Math. Phys.* **35**, 5734–5748) is extended to integrals of such equations. This paper is devoted to some aspects characterising complete symmetry groups. The algebras of the symmetries of both differential equations and integrals are studied in the context of equations for which the elements are represented by point or contact symmetries so that there is no ambiguity about the group. Both algebras and groups are found to be nonunique. © 2001 Academic Press

1. INTRODUCTION

In 1994 Krause [9] provided a precise and useful definition of the expression "complete symmetry group" by describing it as the group of the symmetries which completely specified a differential equation. The preciseness comes from the designated purpose, viz. that of the set of symmetries which defines the differential equation and the utility in that the group has a specific purpose, which is to specify the differential equation. He introduced the idea in the context of the Kepler problem for which in addition to the five Lie point symmetries

$$\begin{array}{ll} G_1 = \partial_t & G_3 = y\partial_z - z\partial_y \\ G_2 = 3t\partial_t + 2r\partial_r & G_4 = z\partial_x - x\partial_z \\ & G_5 = x\partial_y - y\partial_x, \end{array} \tag{1.1}$$

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where $r^2 = x^2 + y^2 + z^2$, of the three-dimensional problem described by the equation of motion

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = 0, \tag{1.2}$$

where the overdot indicates differentiation with respect to the independent variable t, requires another three symmetries, all of them nonlocal, for the equation (1.2) to be unique up to the value of the constant μ (this is always subject to rescaling). The additional nonlocal symmetries are

$$\begin{aligned} G_6 &= \left(2\int x\,dt\right)\partial_t + xr\partial_r \\ G_7 &= \left(2\int y\,dt\right)\partial_t + yr\partial_r \\ G_8 &= \left(2\int z\,dt\right)\partial_t + zr\partial_r. \end{aligned} \tag{1.3}$$

One of the problems in dealing with nonlocal symmetries is the question of the identification of the algebra. Fortunately Nucci [19] was able to reduce the problem of determining these nonlocal symmetries to one of determining point symmetries of a reduced system. Using the interactive symmetry finder of Nucci [20], Nucci and Leach [21] were able to identify the natural variables for the reduced system which turned out to be simply

$$\begin{aligned} u_1'' + u_1 &= 0 \\ u_2' &= 0 \\ u_3' &= 0, \end{aligned} \tag{1.4}$$

where the prime denotes differentiation with respect to the new independent variable, the azimuthal angle ϕ, i.e., a simple harmonic oscillator and two constants of the motion, the total angular momentum, and the z-component of the angular momentum. Apart from the symmetry, ∂_t, which is used for the reduction of order the symmetries listed in (1.1) and (1.3) are a proper subset of the Lie point symmetries of the system (1.4) which has the algebra $2A_1 \oplus sl(3, R)$, where the Abelian subalgebra is due to the second and third equations and the $sl(3, R)$ is the algebra of the first equation.

Krause introduced his concept of the complete symmetry group in the context of integrable equations of classical mechanics. Naturally there is no reason to restrict the differential equations to those of classical mechanics. Indeed there is no reason to restrict the class of equations to those which are integrable as was demonstrated by Leach et al. [12]. They demonstrated that an equation which had been shown by numerical experiments to exhibit chaos [23] still possessed a complete symmetry group. Naturally the symmetries were highly nonlocal! In this paper we wish to illustrate some of the properties of complete symmetry groups without the obfuscation of nonlocal symmetries. The equations which we treat are simple second and third order equations in the main. They are selected for their simplicity since then we are able to recognise the algebraic properties of their complete symmetry groups easily. In addition to these equations we also consider the complete symmetry groups of their first integrals thereby complementing the results presented in earlier papers by Flessas et al. [5, 6].

Before we commence the work it is just as well to mention that a complete symmetry group need not necessarily precisely specify the equation. This was already recognised by Krause [9] in the case of the Kepler problem for which the gravitational constant, μ in (1.2), remains unspecified and was found again by Leach et al. [12] in the case of their nonintegrable equation. In the case of the Kepler problem the gravitational constant can be rescaled to any desired value and in the equation treated by Leach et al. there is a constant of translation. Because of this freedom there is no way to specify precisely either equation using symmetries. However, one should note that in the case of these two equations the arbitrariness of the value of the constant does not affect the general properties of the solution. Zero is not a value for μ attainable by rescaling.

2. SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

As we wish to illustrate the properties of complete symmetry groups in the context of Lie point symmetries and in the simplest possible setting so that the properties will not be occluded by extraneous factors, we consider the equation

$$y'' = 0, \tag{2.1}$$

where now, and henceforth, the prime denotes differentiation with respect to the independent variable x, which is representative of all linear and linearisable second order equations. It has the Lie point symmetries

$$\begin{array}{ll} G_1 = \partial_y & G_5 = x\partial_x + \frac{1}{2}y\partial_y \\ G_2 = x\partial_y & G_6 = x^2\partial_x + xy\partial_y \\ G_3 = y\partial_y & G_7 = y\partial_x \\ G_4 = \partial_x & G_8 = xy\partial_x + y^2\partial_y. \end{array} \tag{2.2}$$

Our intention is to demonstrate how many of the symmetries listed in (2.2) are required to specify (2.1) completely. To determine this we consider the general second order equation

$$y'' = f(x, y, y') \tag{2.3}$$

and impose the symmetries of (2.2) in turn until we recover (2.1). Under G_1 the function f becomes $f(x, y')$. The y' is removed from f when G_2 is applied. Finally the effect of G_3 is to require that f be zero. Thus we require three symmetries to specify (2.1) completely. The Lie Brackets of the three symmetries are

$$[G_1, G_2] = 0, \quad [G_1, G_3] = G_1, \quad [G_2, G_3] = G_2, \tag{2.4}$$

which is the algebra $A_{3,3}$ in the Mubarakzynov classification [16–18], also described as $D \oplus_s T_2$, i.e., the semidirect sum of dilations and translations in the plane.

Immediately one wonders about the other symmetries and we find that the symmetries G_3, G_7, and G_8 are also sufficient to specify (2.1) completely. With a little bit of rearrangement one finds that the algebra of these three symmetries is also $A_{3,3}$. Consequently the group is the same even if the representation in the terms of an algebra is different. There still remain the $sl(2, R)$ symmetries, G_4, G_5, and G_6. The effect of these symmetries is to reduce (2.4) to the form

$$y'' = \frac{K}{y^3}, \tag{2.5}$$

which is an equation of the Ermakov–Pinney class [4, 22] which is known to be characterised by the algebra $sl(2, R)$ [10]. To recover the equation (2.1) it is necessary to add another symmetry, say G_3, to remove the nonlinearity in (2.5). There are now four elements of the algebra which is $A_1 \oplus sl(2, R)$. This is also a representation of a complete symmetry group of the original equation, (2.1). We observe that not only is it possible to have different representations of the same group to specify uniquely the differential equation but we can also have different groups. Clearly if we wish to talk about a complete symmetry group, we should mean the group of minimal dimension.

Although it is our intention to concentrate on the realisation of the complete symmetry group in terms of point symmetries, it is well to observe how nonlocal symmetries can enter the discussion. By way of example we consider the well-known equation

$$y'' + kyy' + x^3 = 0 \tag{2.6}$$

which has the two Lie point symmetries

$$G_1 = \partial_x \qquad \text{and} \qquad G_2 = -x\partial_x + y\partial_y \tag{2.7}$$

for all values of the parameter k. (For $k = 3$ (2.6) has eight Lie point symmetries and consequently is linearisable by means of a point transformation. However, the discussion here is independent of the value of k.) The two symmetries in (2.7) constrain the general second order ordinary differential equation

$$y'' = f(x, y, y') \tag{2.8}$$

to the form

$$y'' = y^3 f\left(\frac{y'}{y^2}\right), \tag{2.9}$$

where f is an arbitrary function of its argument. We can find the third symmetry needed to specify the equation precisely by assuming a symmetry of the form [3]

$$G_3 = \xi \partial_x \tag{2.10}$$

without any restriction on the functional dependence of ξ. The condition that G_3 be a symmetry of (2.6) is

$$2y''\xi' + y'\xi'' + kyy'\xi' = 0, \tag{2.11}$$

which is a linear first order equation in the variable ξ' and so is trivial to integrate. (This is the attraction of the technique!) The solution is

$$\xi = \int \frac{\exp[-\int ky\,\mathrm{d}y]}{y'^2}\mathrm{d}y, \tag{2.12}$$

up to an additive constant which gives the symmetry G_1. When the symmetry corresponding to (2.12) is applied to (2.9), the function f is constrained to be linear in its argument. Consequently we recover the equation (2.6) up to an arbitrary constant which can be rescaled.

3. THE INTEGRALS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

The first integrals of second order equations of maximal symmetry have been shown to have interesting algebraic properties [11, 14]. Again we can use the equation (2.1) as the vehicle for our discussion. It has the two fundamental first integrals

$$I_1 = xy' - y \qquad \text{and} \qquad I_2 = y' \tag{3.1}$$

with the Lie point symmetries

$$\begin{array}{ll} X_1 = x\partial_x & Y_1 = \partial_x \\ X_2 = x^2\partial_x + xy\partial_y & Y_2 = x\partial_x + y\partial_y \\ X_3 = x\partial_y & Y_3 = \partial_y \end{array} \tag{3.2}$$

respectively which have the same algebra as that given in (2.4). A third integral, the ratio of the two and so not independent, also has three symmetries with the same algebra.

We define the complete symmetry group of a first integral as the minimal number of symmetries required to specify it up to an arbitrary function of itself. If we consider the symmetries of I_1 acting upon an arbitrary function $f(x, y, y')$, the action of the first extension of X_1 gives the partial differential equation

$$x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'} = 0, \tag{3.3}$$

the characteristics of which are y and $xy' = u$ so that now $f = f(y, xy')$. When we apply X_2, we obtain

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} = 0, \tag{3.4}$$

which has the characteristic $v = u - y$. The function invariant under these two symmetries is $f = f(xy' - y)$. The action of X_3 on this function is identically zero.

One can apply the symmetries in different orders and combinations without any change of the result. Any two of the three Lie point symmetries is sufficient to specify the integral up to an arbitrary function of itself. This applies equally to I_2 and the ratio integral. In the case of I_1 the use of X_1 and X_2 (equally X_3) to specify the integral completely means that the complete symmetry group of the integral is represented by the algebra A_2 or Lie's algebra of Type III whereas, if X_2 and X_3 are used, the algebra is $2A_1$ or Lie's algebra of Type I [13, Kap 18, p. 412]. As far as these integrals are concerned, the complete symmetry group is not unique. In the case of the differential equation, (2.1), there were different representations of the same group, but here that is not the case.

We conclude our discussion of the complete symmetry group of first integrals for second order equations with an example which is not an instance of maximal symmetry. We take the equation of Emden–Fowler form

$$y'' + 2y^3 = 0, \tag{3.5}$$

which has the obvious integral

$$I = y'^2 + y^4. \tag{3.6}$$

The integral, (3.6), has the obvious point symmetry ∂_x. Although the equation (3.5) has a second, rescaling, point symmetry, this is not a symmetry of the integral. We assume a symmetry of the form

$$G = \eta\partial_y \tag{3.7}$$

and apply its first extension to (3.6) to obtain the equation

$$\frac{\eta'}{\eta} = -2\frac{y^3}{y'}, \tag{3.8}$$

which has the solution

$$\eta = \exp\left[-2\int \frac{y^3}{y'}\mathrm{d}x\right]. \tag{3.9}$$

If we consider a general function $f(x, y, y')$, the action of the first symmetry, ∂_x, gives $f = f(y, y')$. The action of the first extension of the second symmetry, (3.7) with η as given in (3.9), on this function gives

$$\exp\left[-2\int \frac{y^3}{y'}\mathrm{d}x\right]\left(\frac{\partial f}{\partial y} - 2\frac{y^3}{y'}\frac{\partial f}{\partial y'}\right) = 0. \tag{3.10}$$

The equation for the characteristic is simply

$$0 = y'\,\mathrm{d}y' + 2y^3\,\mathrm{d}y \tag{3.11}$$

and the integral follows immediately.

In (3.9) we have written the coefficient function, η, in nonlocal form. Equally we could have invoked the differential equation to write

$$\eta = \exp\left[\int \frac{y''}{y'}\mathrm{d}x\right] = y', \tag{3.12}$$

so that the nonlocal symmetry is equivalent to a generalised symmetry. When we apply the first extension of the generalised symmetry to the integral, it is necessary to take the differential equation into account. We note that there exists a third symmetry of the differential equation of the form

$$G = \left(\int \frac{\mathrm{d}x}{y'^2}\right)\partial_x, \tag{3.13}$$

which is not a symmetry of the integral. Under the conventional route of reduction of the order of (3.5) using the symmetry ∂_x this symmetry becomes local and so is a hidden symmetry of Type II [1].

We observe that it appears that the complete symmetry group of a first integral of a second order ordinary differential equation is represented by an algebra with two elements. The group is not necessarily unique.

4. THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

When one encounters equations of the third order and higher, there are differences by comparison with second order equations. Indeed it is only by the study of higher order equations that one recognises generic properties rather than peculiarities proper only to second order equations. Linear equations of higher order do not have the same number of point symmetries. A linear nth order ordinary differential equation can have $n+4$, $n+2$, or $n+1$ Lie point symmetries. The third order equation of maximal symmetry also has three contact symmetries. In this respect it is exceptional. A linear nth order ordinary differential equation cannot have $n+3$ point symmetries, but this is possible for a nonlinear equation such as the Kummer–Schwarz equation

$$2y'y''' - 3y''^2 = 0. \tag{4.1}$$

By one of those curious coincidences this equation has ten contact symmetries and so is related to the third order equation of maximal symmetry, but by a contact transformation and this does not preserve point symmetries. The contact symmetries of the third order equation of maximal symmetry, viz.

$$y''' = 0, \tag{4.2}$$

are given by

$$\begin{array}{ll} G_1 = \partial_y & G_6 = x\partial_x + y\partial_y \\ G_2 = x\partial_y & G_7 = x^2\partial_x + 2xy\partial_y \\ G_3 = x^2\partial_y & G_8 = y'\partial_x + \tfrac{1}{2}y'^2\partial_y \\ G_4 = y\partial_y & G_9 = 2(xy' - y)\partial_x + xy'^2\partial_y \\ G_5 = \partial_x & G_{10} = (x^2y' - 2xy)\partial_x + (\tfrac{1}{2}x^2y'^2 - 2y^2)\partial_y. \end{array} \tag{4.3}$$

Note that the symmetries comprise seven point symmetries and three intrinsically contact symmetries and have the algebra $sp(5)$ [2]. From our experience with the second order equation of maximal symmetry we expect that the symmetries required to specify completely (4.2) are $G_1 - G_4$.

We proceed to verify this expectation. We may write the general third order equation as

$$y''' = f(x, y, y', y''). \tag{4.4}$$

The action of G_1 on (4.4) is to remove the y, that of G_2 is to remove the y', and that of G_3 is to remove the y''. An arbitrary function of x remains. This is removed by the action of the homogeneity symmetry G_4 and so our expectation is justified. In the case of the second order equation (2.1) we saw that the non-Cartan symmetries, G_7 and G_8, with the homogeneity symmetry G_3 played the same role as the solution symmetries and the homogeneity symmetry in specifying the equation and having the same algebra. For the third order equation (4.2) the equivalent symmetries are the intrinsically contact symmetries. We find, after a certain amount of routine algebra, that following the application of the three contact symmetries, G_8, G_9, and G_{10}, (4.4) is constrained to take the form

$$y''' = f\left(x - \frac{2y}{y'}\right). \tag{4.5}$$

We note that the argument of the arbitrary function possesses the homogeneity symmetry G_4 and consequently the application of G_4 to (4.5) yields (4.2).

In the case of (2.1) the three symmetries possessing the $sl(2, R)$ algebra were insufficient to specify it and it was necessary to add a fourth symmetry, the homogeneity symmetry, to obtain the equation. For the third order equation we could expect no better and again our expectations are realised. Under the action of the symmetries G_5, G_6, and G_7 of (4.3) we find that (4.4) becomes

$$y''' = f(2yy'' - y'^2). \tag{4.6}$$

In passing we note that the argument of the arbitrary function f in (4.6) is a first integral of (4.2) and is given by the combination $2I_1I_3 - I_2^2$ of the fundamental first integrals of (4.2) listed in (5.2). This time the application of the homogeneity symmetry does not produce (4.2) but rather

$$y''' = K(2yy'' - y'^2)^{1/2} \tag{4.7}$$

in which the arbitrary function has been replaced by one which possesses the same degree of homogeneity as the left-hand side of the equation. To obtain (4.2) it is necessary to use another symmetry, say one of the solution symmetries.

We conclude our consideration of third order equations with the Kummer–Schwarz equation (4.1). The Lie point symmetries of this equation consist of two separate representations of $sl(2, R)$, viz.

$$\begin{array}{ll} G_1 = \partial_x & G_4 = \partial_y \\ G_2 = x\partial_x & G_5 = y\partial_y \\ G_3 = x^2\partial_x & G_6 = y^2\partial_y. \end{array} \tag{4.8}$$

Considering our experience with $sl(2, R)$ in the cases of (2.1) and (4.2) it comes as no surprise that after the application of the four symmetries G_1, G_2, G_4, and G_5 to (4.4) we have only obtained the generalised Kummer–Schwarz equation

$$y'y''' - ky''^2 = 0. \tag{4.9}$$

We obtain the Kummer–Schwarz equation when one of G_3 or G_6 is applied. The constant k in (4.9) cannot be rescaled by a self-similar transformation since both terms in the equation are homogeneous of degree two and minus four in y and x, respectively. To fix the value of the constant for the Kummer–Schwarz equation one of the so-called conformal symmetries is required. Since the Kummer–Schwarz equation can be obtained from (4.2) by means of a contact transformation, the number of symmetries required to specify it completely is only four. Obviously the contact symmetries of the Kummer–Schwarz equation must enter into the algebra of the complete symmetry group. With a general value for the constant, k, in (4.9) we are specifying a class of equations known as the generalised Kummer–Schwarz equation. This equation has only four point symmetries. Consequently to specify it for a specific value of k we must look to a nonlocal symmetry. This should not be surprising since (4.9) can be obtained from (4.2) by means of a nonlocal transformation. We can determine a nonlocal symmetry by the same stratagem that we used with (2.9) by the selection of a symmetry of the form of (2.10). The equation to be satisfied by ξ is

$$y'\xi''' + (3-2k)y''\xi'' = 0, \tag{4.10}$$

which is a linear first order equation in ξ'' with a particularly simple form in the case of the Kummer–Schwarz equation. When $k = 3/2$, the three symmetries $G_1 - G_3$ of one of the $sl(2, R)$ algebras follow immediately. In general we obtain the solution

$$\xi = A + Bx + C\int\int y'^{2k-3}\,\mathrm{d}x\,\mathrm{d}x, \tag{4.11}$$

which, indeed, is quite nonlocal.

5. THE INTEGRALS OF THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

We consider the equation of the third order with maximal symmetry, viz.

$$y''' = 0. \tag{5.1}$$

The three fundamental integrals of (5.1) and their symmetries are [5, 6]

$$\begin{array}{lll}
I_1 = \frac{1}{2}x^2y'' - xy' + y & I_2 = xy'' - y' & I_3 = y'' \\
 & & \\
X_1 = x\partial_y & Y_1 = \partial_y & Z_1 = \partial_y \\
X_2 = x^2\partial_y & Y_2 = x^2\partial_y & Z_2 = x\partial_y \\
X_3 = x^2\partial_x + 2xy\partial_y & Y_3 = x\partial_x + y\partial_y & Z_3 = \partial_x \\
X_4 = x\partial_x & & Z_4 = x\partial_x + 2y\partial_y.
\end{array} \tag{5.2}$$

In the case of the integrals I_1 and I_3 we find a very curious situation. First we dispense with the integral I_2 as it is straightforward. All three symmetries of the integral I_2 are required to specify it, naturally up to an arbitrary function of itself. In the case of I_2 the algebra of its complete symmetry group is $A_{3,4}$, which is better known as the algebra of the pseudo-Euclidean group $E(1,1)$, with the Lie brackets

$$[Y_1, Y_2] = 0, \qquad [Y_1, Y_3] = Y_1, \qquad [Y_2, Y_3] = -Y_2. \tag{5.3}$$

For the other two integrals one would also expect that three symmetries would be required to specify completely the integral. Consequently there are four possible combinations of the symmetries. The Lie brackets for three of those possible combinations are

$$\begin{array}{lll} [X_1, X_2] = 0, & [X_1, X_3] = X_2, & [X_2, X_3] = 0 \\ [X_1, X_2] = 0, & [X_1, X_4] = -X_1, & [X_2, X_4] = -2X_2 \\ [X_2, X_3] = 0, & [X_2, X_4] = -2X_2, & [X_3, X_4] = -X_3 \end{array} \tag{5.4}$$

in the case of I_1. The algebras are $A_{3,1}$ for (5.4a) and $A_{3,5}^{1/2}$ for (5.4b) and (5.4c) (after a modest amount of rescaling). The same structure is observed for the symmetries of I_3. The fourth combination, X_1, X_3, and X_4, does not exhibit the same behaviour. In fact the integral, I_1, is completely specified by X_1 and X_3 so that X_4 is not needed. However, the Lie bracket of the two symmetries X_1 and X_3, as is evident from (5.4a), requires X_2 to be added to the algebra. This is a situation which is different from that observed for the first integrals of (2.1). (In the case of I_3 the two symmetries are Z_2 and Z_3.) We note that the two symmetries X_3 and Z_3 are equivalent in the algebra $sl(2,R)$ and that the solution symmetry involved in both cases corresponds to the "middle" solution.

The number of Lie point symmetries which a third order linear equation, or one linearisable by means of a point transformation, can have is seven (the maximum), five, or four. The number of point symmetries which the first integrals of the equations of less than maximal symmetry possess is lower than in the case of the equations of maximal symmetry [7].

We consider two representatives of the equations of lower symmetry and examine their first integrals for the number of symmetries required to specify them. The representative equation with five Lie point symmetries is

$$y''' - y = 0. \tag{5.5}$$

Its three fundamental first integrals and their symmetries are

$$I_1 = e^{-x}(y + y' + y'') \quad I_2 = e^{\omega x}(\omega^2 y - \omega y' + y'') \quad I_3 = e^{-\omega^2 x}(-\omega y + \omega^2 y' + y'')$$

$$\begin{array}{lll} X_1 = e^{-\omega x}\partial_y & Y_1 = e^{x}\partial_y & Z_1 = e^{-\omega x}\partial_y \\ X_2 = e^{\omega^2 x}\partial_y & Y_2 = e^{\omega^2 x}\partial_y & Z_2 = e^{x}\partial_y \\ X_3 = \partial_x + y\partial_y & Y_3 = -\frac{1}{\omega}\partial_x + y\partial_y & Z_3 = \frac{1}{\omega^2}\partial_x + y\partial_y, \end{array} \tag{5.6}$$

where $\omega = (1 + i\sqrt{3})/2$ is a cube root of -1. For each case the three point symmetries are sufficient to specify completely the integral. The Lie algebra of the symmetries is $A_{3,3}$ which is the algebra of the group comprising the semi-direct product of dilations and translations, $D \oplus_s T_2$, in the plane. The first integrals and their symmetries can be transformed in a cyclic manner by setting $x \to \omega X$.

The representative equation for the class of linear equations possessing only four Lie point symmetries is [8, 3.23, p. 512]

$$y''' + y' + f(x)(y'' + y) = 0, \tag{5.7}$$

where $f(x)$ is an arbitrary function.

The three fundamental first integrals and their symmetries of (5.7) are [7]

$$\begin{aligned}
I_1 &= y' \sin x - y \cos x - z(x)(y'' + y) \exp\left(\int f(u)\mathrm{d}u\right)\\
X_1 &= \sin x \partial_y\\
X_2 &= \zeta(x)\partial_y\\
\\
I_2 &= y' \cos x + y \sin x - z(x)(y'' + y) \exp\left(\int f(u)\mathrm{d}u\right)\\
Y_1 &= \cos x \partial_y\\
Y_2 &= \zeta(x)\partial_y\\
\\
I_3 &= (y'' + y) \exp\left(\int f(u)\mathrm{d}u\right)\\
Z_1 &= \sin x \partial_y\\
Z_2 &= \cos x \partial_y,
\end{aligned} \tag{5.8}$$

where

$$\zeta(x) = \int_0^x \exp\left(-\int f(u)\mathrm{d}u\right) \sin(x - u)\mathrm{d}u \tag{5.9}$$

and

$$z(x) = \int \exp\left(-\int f(u)\mathrm{d}u\right) \sin u \,\mathrm{d}u. \tag{5.10}$$

We note that in each case the two symmetries associated with each integral are solution symmetries. For all three integrals the number of point symmetries is insufficient to specify the integral up to an arbitrary function of itself. After the action of the two symmetries in each case we obtain

$$f_1 = f_1\left(x, (y'' + y)\int e^{-\int f} \sin u - (y' \sin x - y \cos x)e^{-\int f}\right)$$
$$f_2 = f_2\left(x, (y'' + y)\int e^{-\int f} \sin u - (y' \cos x + y \sin x)e^{-\int f}\right)$$
$$f_3 = f_3(x, y'' + y), \tag{5.11}$$

respectively. The complete specification of these integrals requires the introduction of a further symmetry which, obviously, cannot be a point symmetry. The determination of the additional symmetry is not as simple as in the case above. We illustrate the procedure in the case of I_3. We assume a symmetry of the form

$$G = \xi \partial_x \tag{5.12}$$

without any assumptions on the nature of the coefficient function ξ. The requirement that G be a symmetry of I_3 produces the differential equation

$$(y'\xi)'' + (y'\xi) = 0 \tag{5.13}$$

when the differential equation (5.7) is taken into account. The solution of (5.13) is

$$y'\xi = A \sin x + B \cos x. \tag{5.14}$$

We have a choice of symmetries. Taking the first and applying it to f_3 in (5.11) we immediately recover I_3. The same applies to the other two integrals. For all three a generalised symmetry must be introduced to specify the integral completely.

We conclude this section by returning to the remarks made after (5.4) and seek to clarify the situation by reference to the fourth order equation of maximal symmetry

$$y^{iv} = 0, \tag{5.15}$$

which has the four integrals [6]

$$I_1 = \frac{1}{6}x^3 y''' - \frac{1}{2}x^2 y'' + xy' - y$$
$$I_2 = \frac{1}{2}x^2 y''' - xy'' + y'$$
$$I_3 = xy''' - y''$$
$$I_4 = y''', \tag{5.16}$$

with I_1 and I_4 having five Lie point symmetries with the same algebra $A^1_{4,9}$ and I_2 and I_3 having four symmetries and also the same algebra. In the case of I_2 and I_3 all four symmetries are required to specify the integral up to an arbitrary function of itself. The subalgebra $sl(2,R)$ is represented by the rescaling symmetry. The additional symmetry in the cases of I_1 and I_4 is one of the other elements of the $sl(2,R)$ subalgebra. The five point symmetries of I_4 are

$$\begin{aligned} &Z_1 = \partial_y && Z_4 = \partial_x \\ &Z_2 = x\partial_y && Z_5 = x\partial_x + 3y\partial_y, \\ &Z_3 = \tfrac{1}{2}x^2\partial_y && \end{aligned} \tag{5.17}$$

where $Z_1 - Z_3$ are solution symmetries and Z_4 and Z_5 come from $sl(2,R)$. In a natural generalisation of the case of the third order equation (5.1), any four of the five symmetries will specify the integral completely. However, if we commence with Z_4 the integral must be free of x and consequently the application of Z_3, since its third extension is

$$Z_3^{[3]} = \frac{1}{2}x^2\partial_y + x\partial_{y'} + \partial_{y''}, \tag{5.18}$$

immediately removes y, y', and y'' from the integral. The integral is completely specified by two symmetries. Since

$$[Z_4, Z_3] = Z_2 \qquad \text{and} \qquad [Z_4, Z_2] = Z_1, \tag{5.19}$$

the algebra can only be closed by the addition of the two solution symmetries.

6. GENERAL COMMENTS

From the results reported in the preceding sections we are now in a position to state the following theorem.

THEOREM. *All linear homogeneous nth order ordinary differential equations have the complete symmetry group represented by the algebra* $A_1 \oplus_s nA_1$.

Proof. All linear homogeneous nth order ordinary differential equations have the symmetries of the algebra $A_1 \oplus_s nA_1$ [15]. The n-dimensional Abelian subalgebra consists of the so-called solution symmetries and the one-dimensional Abelian subalgebra is the homogeneity symmetry. Thus we have

$$G_i = s_i\partial_y, \qquad i = 1, n \qquad G_{n+1} = y\partial_y, \tag{6.1}$$

where the n functions $s_i(x)$ are linearly independent solutions of the nth order ordinary differential equation.

If we represent the general nth order equation as

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right), \tag{6.2}$$

the action of the ith solution symmetry is

$$s_i^{(n)} = s_i^{(j)} \frac{\partial f}{\partial_y^{(j)}}, \tag{6.3}$$

where we imply summation over the repeated index. The associated Lagrange's system is

$$\frac{\mathrm{d}y^{(j)}}{s_i^{(j)}} = \frac{\mathrm{d}f}{s_i^{(n)}} \tag{6.4}$$

with the $n-1$ characteristics

$$u_{ij} = s_i y^{(j)} - s_i^{(j)} y \tag{6.5}$$

which are obtained from the n members of the left side of (6.4) by solving them pairwise with the first member and the nth characteristic

$$u_{in} = s_i f - s_i^{(n)} y \tag{6.6}$$

which is obtained by using the right side of (6.4) in the same way. Equation (6.2) is constrained to the form

$$y^{(n)} = \frac{s_i^{(n)} y}{s_i} + \frac{1}{s_i} g\left(x, s_i y^{(j)} - s_i^{(j)} y\right) \tag{6.7}$$

in which the function g contains the $n-1$ characteristics of (6.5).

We now apply the kth solution symmetry to (6.7). From

$$s_k^{(n)} = \frac{s_i^{(n)} s_k}{s_i} + \frac{1}{s_i} \frac{\partial_g}{\partial_{u_{ij}}} \left(s_i s_k^{(j)} - s_i^{(j)} s_k\right) \tag{6.8}$$

we obtain the associated Lagrange's system

$$\frac{\mathrm{d}u_{ij}}{s_i s_k^{(j)} - s_i^{(j)} s_k} = \frac{\mathrm{d}g}{s_i s_k^{(n)} - s_i^{(n)} s_k} \tag{6.9}$$

and so the characteristics

$$\begin{aligned} v_{ijk} &= \left(s_i s_k^{(1)} - s_i^{(1)} s_k\right) u_{ij} - \left(s_i s_k^{(j)} - s_i^{(j)} s_k\right) u_{i1} \\ v_{ij,n-1} &= \left(s_i s_k^{(1)} - s_i^{(1)} s_k\right) g - \left(s_i s_k^{(n)} - s_i^{(n)} s_k\right) u_{i1}. \end{aligned} \tag{6.10}$$

Now (6.7) has the form

$$y^{(n)} = \frac{s_i s_k^{(n)} - s_i^{(n)} s_k}{s_i s_k^{(1)} - s_i^{(1)} s_k} u_{ij} + \frac{1}{s_i s_k^{(1)} - s_i^{(1)} s_k} g(x, v_{ijk}). \tag{6.11}$$

We note that the denominators are nonzero since the s_i are linearly independent solutions of the nth order ordinary differential equation.

We continue the process of applying the solution symmetries and clearly conclude with

$$y^{(n)} = A(x)\omega + B(x), \tag{6.12}$$

where ω is linear in y and its derivatives. Finally we apply the homogeneity symmetry, $y\partial_y$, which forces $B(x)$ to be zero.

Note that in the absence of the homogeneity symmetry, G_{n+1}, one obtains a nonhomogeneous equation and in fact we have a corollary.

COROLLARY 1. *The complete symmetry group of a nonhomogeneous linear nth order ordinary differential equation consists of the $n + 1$ symmetries*

$$\begin{aligned} G_i &= s_i \partial_y, \qquad i = 1, n \\ G_{n+1} &= (y - \alpha(x))\partial_y, \end{aligned} \tag{6.13}$$

where the $s_i(x)$ are linearly independent solutions of the corresponding homogeneous equation and $\alpha(x)$ is a particular solution of the nonhomogeneous equation.

Two further corollaries are evident.

COROLLARY 2. *All linear equations have a complete symmetry group represented by an algebra comprising point symmetries only.*

COROLLARY 3. *All equations linearisable by a point transformation have a complete symmetry group represented by the algebra $A_1 \oplus_s nA_1$ of point symmetries.*

We conclude with a summary of our observations about the integrals of linear equations.

A linear nth order equation of maximal symmetry has a set of n linearly independent integrals linear in the dependent variable and its derivatives. Two of these integrals have $(n + 1)$ Lie point symmetries and the others have n Lie point symmetries. In all cases $(n - 1)$ of the symmetries are solution symmetries and the other comes from the $sl(2, R)$ subalgebra. The complete symmetry group for the $(n - 2)$ integrals comprises all of the point symmetries of the integral. In the case of the two exceptional integrals it is possible to specify the integral completely with just two symmetries, one of the solution symmetries and one of the $sl(2, R)$ symmetries. However, the algebra closes only on the addition of the remaining solution symmetries.

In the case of linear equations of less than maximal symmetry each of the set of n fundamental integrals of an equation with $(n+2)$ Lie point symmetries possesses n Lie point symmetries and is completely specified by those symmetries. For a linear equation with only $(n+1)$ Lie point symmetries each of the fundamental integrals has only $(n-1)$ Lie point symmetries and the integral can only be specified completely be the determination of another symmetry, either generalised or nonlocal.

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