Introduction to differential forms

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1 1-forms

A differential 1-form (or simply a differential or a 1-form) on an open subset of $\mathbb{R}^2$ is an expression $F(x,y)dx + G(x,y)dy$ where F, G are $\mathbb{R}$-valued functions on the open set. A very important example of a differential is given as follows: If $f(x,y)$ is C^1 $\mathbb{R}$-valued function on an open set U, then its total differential (or exterior derivative) is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial g}dy$$

It is a differential on U.

In a similar fashion, a differential 1-form on an open subset of $\mathbb{R}^3$ is an expression $F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$ where F, G, H are $\mathbb{R}$-valued functions on the open set. If $f(x,y,z)$ is a C^1 function on this set, then its total differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

At this stage, it is worth pointing out that a differential form is very similar to a vector field. In fact, we can set up a correspondence:

$$F\mathbf{i} + G\mathbf{j} + H\mathbf{k} \leftrightarrow Fdx + Gdy + Hdz$$

Under this set up, the gradient ∇f corresponds to df. Thus it might seem that all we are doing is writing the previous concepts in a funny notation. However, the notation is very suggestive and ultimately quite powerful. Suppose that that x, y, z depend on some parameter t, and f depends on x, y, z, then the chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Thus the formula for df can be obtained by canceling dt.

2 Exactness in $\mathbb{R}^2$

Suppose that $Fdx + Gdy$ is a differential on $\mathbb{R}^2$ with C^1 coefficients. We will say that it is *exact* if one can find a C^2 function $f(x,y)$ with $df = Fdx + Gdy$

Most differential forms are not exact. To see why, note that the above equation is equivalent to

$$F = \frac{\partial f}{\partial x},\ G = \frac{\partial f}{\partial y}.$$

Therefore if f exists then

$$\frac{\partial F}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial G}{\partial x}$$

But this equation would fail for most examples such as ydx. We will call a differential *closed* if $\frac{\partial F}{\partial y}$ and $\frac{\partial G}{\partial x}$ are equal. So we have just shown that if a differential is to be exact, then it had better be closed.

Exactness is a very important concept. You've probably already encountered it in the context of differential equations. Given an equation

$$\frac{dy}{dx} = F(x, y)$$

we can rewrite it as

$$Fdx - dy = 0$$

If the differential on the left is exact and equal to say, df, then the curves $f(x, y) = c$ give solutions to this equation.

These concepts arise in physics. For example given a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ representing a force, one would like find a function $P(x, y)$ called the potential energy, such that $\mathbf{F} = -\nabla P$. The force is called *conservative* (see section 11) if it has a potential energy function. In terms of differential forms, $\mathbf{F}$ is conservative precisely when $F_1dx + F_2dy$ is exact.

3 Line integrals

Now comes the real question. Given a differential $Fdx + Gdy$, when is it exact? Or equivalently, how can we tell whether a force is conservative or not? Checking that it's closed is easy, and as we've seen, if a differential is not closed, then it can't be exact. The amazing thing is that the converse statement is often (although not always) true:

THEOREM 4 *If $F(x, y)dx + G(x, y)dy$ is a closed form on all of $\mathbb{R}^2$ with C^1 coefficients, then it is exact.*

To prove this, we would need solve the equation $df = Fdx + Gdy$. In other words, we need to undo the effect of d and this should clearly involve some kind of integration process. To define this, we first have to choose a piecewise C^1 parametric curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$. In other words, we let x and y depend on some parameter t running from a to b.

DEFINITION 5

$$\int_{\mathbf{c}} Fdx + Gdy = \int_a^b [F(x(t), y(t))\frac{dx}{dt} + G(x(t), y(t))\frac{dy}{dt}]dt$$

Although we've done everything at once, it is often easier, in practice, to do this in steps. First change the variables from x and y to expresions in t, then replace dx by $\frac{dx}{dt}dt$ etc. Then integrate with respect to t.

While we're at it, we can also define a line integral in $\mathbb{R}^3$. Suppose that $Fdx + Gdy + Hdz$ is a differential form with C^1 coeffients. Let $c : [a, b] \to \mathbb{R}^3$ be a piecewise C^1 parametric curve, then

DEFINITION 6

$$\int_{\mathbf{c}} Fdx+Gdy+Hdz = \int_a^b [F(x(t), y(t), z(t))\frac{dx}{dt}+G(x(t), y(t), z(t))\frac{dy}{dt}+H(x(t), y(t), z(t))\frac{dz}{dt}]dt$$

[Many examples done in class.]

The notion of exactness extends to $\mathbb{R}^3$ automatically: a form is exact if it equals df for a C^2 function. One of the most important properties of exactness is its path independence:

PROPOSITION 7 *If ω is exact and $\mathbf{c}_1$ and $\mathbf{c}_2$ are two parametrized curves with the same endpoints (or more acurately the same starting point and ending point), then*

$$\int_{\mathbf{c}_1} \omega = \int_{\mathbf{c}_2} \omega$$

It's quite easy to see why this works. If $\omega = df$ and $\mathbf{c}_1 : [a, b] \to \mathbb{R}^3$ then

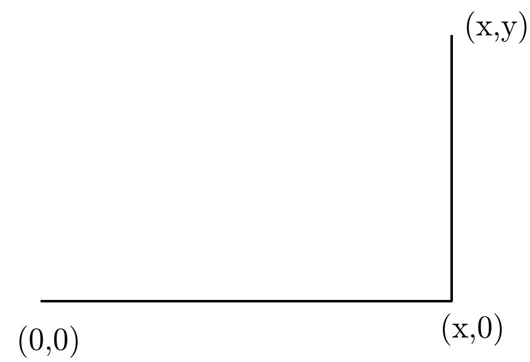
$$\int_{\mathbf{c}_1} df = \int_a^b \frac{df}{dt}dt$$

by the chain rule. Now the fundamental theorem of calculus shows that the last integral equals $f(\mathbf{c}_1(b)) - f(\mathbf{c}_1(a))$, which is to say the value of f at the endpoint minus its value at the starting point. A similar calculation shows that the integral over $\mathbf{c}_2$ gives same answer.

Now we can describe the basic idea for proving theorem 4. If $Fdx + Gdy$ is a closed form on $\mathbb{R}^2$, set

$$f(x, y) = \int_{\mathbf{c}} Fdx + Gdy$$

where the curve is indicated below:



Then $df = Fdx + Gdy$. [Details in class, see also Marsden and Tromba].

The theorem will generally fail if we replace $\mathbb{R}^2$ by a subset. For example,

$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is C^1 1-form on the open set $\{(x,y) \,|\, (x,y) \neq (0,0)\}$ which is closed but not exact.

8 Work

Line integrals have many important uses. One very direct application in physics comes from the idea of work. If you pick up a rock off the ground, or perhaps roll it up a ramp, it takes energy. The energy expended is called work. If you're moving the rock in straight line for a short distance, then the displacement can be represented by a vector $\mathbf{d} = (\Delta x, \Delta y, \Delta z)$ and the force of gravity by a vector $\mathbf{F} = (F_1, F_2, F3)$. Then the work done is simply

$$-\mathbf{F} \cdot \mathbf{d} = -(F_1\Delta x + F_2\Delta y + F_3\Delta z).$$

On the other hand, if you decide to shoot a rocket up into space, then you would have to take into account that the trajectory $\mathbf{c}$ may not be straight nor can the force $\mathbf{F}$ be assumed to be constant (it's a vector field). However as the notation suggests, for the work we would now need to calculate the integral

$$-\int_{\mathbf{c}} F_1 dx + F_2 dy + F_3 dz$$

One often writes this as

$$-\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

(think of $d\mathbf{s}$ as the "vector" (dx, dy, dz).)

9 2-forms

Recall that the cross product is an operation on vector fields satisfying:

$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ (anticommutative law)

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ (distributive law)

Geometrically $\mathbf{u} \times \mathbf{v}$ represents the vector whose length is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ with direction determined by right hand rule.

We'll introduce an operation on 1-forms called the *wedge product* (written as $\wedge$) which is analogous to the cross product. One important difference is that while the cross product of two vectors is again a vector, the wedge product results a new kind of expression called a 2-form. The wedge product will be both anticommutative and distributive like the cross product:

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$

A typical 2-form looks like this:

$$F(x, y, z)dx \wedge dy + G(x, y, z)dy \wedge dz + H(x, y, z)dz \wedge dx$$

where F, G and H are functions defined on an open subset of $\mathbb{R}^3$. The real significance of 2-forms will come later when we do surface integrals. A 2-form will be an expression that can be integrated over a surface in the same way that a 1-form can be integrated over a curve.

10 "d" of a 1-form and the curl

Given a 1-form $F(x, y, z)dx + G(x, y, z)dy + H(x, y, z)dz$. We want to define its derivative $d\omega$ which will be a 2-form. The rules we use to evaluate it are:

$$d(\alpha + \beta) = d\alpha + d\beta$$

$$d(f\alpha) = (df) \wedge \alpha + f d\alpha$$

$$d(dx) = d(dy) = d(dz) = 0$$

where α and β are 1-forms and f is a function. Putting these together yields a formula

$$d(Fdx + Gdy + Hdz) = (G_x - F_y)dx \wedge dy + (H_y - G_z)dy \wedge dz + (F_z - H_x)dz \wedge dx$$

where $F_x = \frac{\partial F}{\partial x}$ and so on.

A 2-form can be converted to a vector field by replacing $dx \wedge dy$ by $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, $dy \wedge dz$ by $\mathbf{i} = \mathbf{j} \times \mathbf{k}$ and $dz \wedge dx$ by $\mathbf{j} = \mathbf{k} \times \mathbf{i}$. If we start with a vector field $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$, replace it by a 1-form $Fdx + Gdy + Hdz$, apply d, then convert it back to a vector field, we end up with the curl of $\mathbf{V}$

$$\nabla \times \mathbf{V} = (H_y - G_z)\mathbf{i} + (G_x - F_y)\mathbf{k} + (F_z - H_x)\mathbf{j}$$

(In practice, one often writes this as a determinant

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial dx} & \frac{\partial}{\partial dy} & \frac{\partial}{\partial dz} \\ F & G & H \end{vmatrix}.$$

)

11 Exactness in $\mathbb{R}^3$ and conservation of energy

A C^1 1-form $\omega = Fdx + Gdy + Hdz$ is called exact if there is a C^2 function (called a potential) such that $\omega = df$. ω is called closed if $d\omega = 0$, or equivalently if

$$F_y = G_x,\ F_z = H_x,\ G_z = H_y$$

Then exact 1-forms are closed.

THEOREM 12 *If $\omega = Fdx + Gdy + Hdz$ is a closed form on $\mathbb{R}^3$ with C^1 coefficients, then ω is exact. In fact if $f(x_0, y_o, z_0) = \int_C \omega$, where C is any piecewise C^1 curve connecting $(0,0,0)$ to (x_0, y_0, z_0), then $df = \omega$.*

This can be rephrased in the language of vector fields. If $\mathbf{F} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$ is C^1 vector field representing a force, then it is called conservative if there is a C^2 real valued function P, called potential energy, such that $\mathbf{F} = -\nabla P$. The theorem implies that a force $\mathbf{F}$, which is C^1 on all of $\mathbb{R}^3$, is conservative if and only if $\nabla \times \mathbf{F} = 0$. $P(x, y, z)$ is given the work done by moving a particle of unit mass along a path connecting $(0,0,0)$ to (x, y, z).

To appreciate the importance of this concept, recall from physics that the kinetic energy of a particle of constant mass m and velocity

$$\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$$

is

$$K = \frac{1}{2}m||\mathbf{v}||^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}.$$

Also one of Newton's laws says

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}$$

If $\mathbf{F}$ is conservative, then we can replace it by $-\nabla P$ above, move it to the other side, and then dot both sides by $\mathbf{v}$ to obtain

$$m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} + \mathbf{v}\cdot\nabla P = 0$$

which simplifies[1] to

$$\frac{d}{dt}(K + P) = 0.$$

This implies that the total energy $K + P$ is constant.

[1] This takes a bit of work that I'm leaving as an exercise. It's probably easier to work backwards. You'll need the product rule for dot products and the chain rule.

13 "d" of a 2-form and divergence

A 3-form is simply an expression $f(x, y, z)dx \wedge dy \wedge dz$. These are things that will eventually get integrated over solid regions. The important thing for the present is an operation which takes 2-forms to 3-forms once again denoted by "d".

$$d(F dy \wedge dz + G dz \wedge dx + H dx \wedge dy) = (F_x + G_y + H_z) dx \wedge dy \wedge dz$$

It's probably easier to understand the pattern after converting the above 2-form to the vector field $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$. Then the coeffiecient of $dx \wedge dy \wedge dz$ is the divergence

$$\nabla \cdot \mathbf{V} = F_x + G_y + H_z$$

So far we've applied d to functions to obtain 1-forms, and then to 1-forms to get 2-forms, and finally to 2-forms. The real power of this notation is contained in the following simple-looking formula

PROPOSITION 14 $d^2 = 0$

What this means is that given a C^2 real valued function defined on an open subset of $\mathbb{R}^3$, then $d(df) = 0$, and given a 1-form $\omega = F dx + G dy + H dz$ with C^2 coefficents defined on an open subset of $\mathbb{R}^3$, $d(d\omega) = 0$. Both of these are quite easy to check:

$$d(df) = (f_{yx} - f_{xy}) dx \wedge dy + (f_{zy} - f_{yz}) dy \wedge dz + (f_{xz} - f_{zx}) dz \wedge dx = 0$$

$$d(d\omega) = [G_{xz} - F_{yz} + H_{yx} - G_{zx} + F_{zy} - H_{xy}] dx \wedge dy \wedge dz = 0$$

In terms of standard vector notation this is equivalent to

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \mathbf{V}) = 0$$

Introduction to differential forms II

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14 Parameterized Surfaces

Recall that a parameterized curve is a C^1 function from a interval $[a,b] \subset \mathbb{R}^1$ to $\mathbb{R}^3$. If we replace the interval by subset of the plane $\mathbb{R}^2$, we get a parameterized surface. Let's look at a few of examples

1) The upper half sphere of radius 1 centered at the origin can be parameterized using cartesian coordinates

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{1-u^2-v^2} \\ u^2+v^2 \leq 1 \end{cases}$$

2) The upper half sphere can be parameterized using spherical coordinates

$$\begin{cases} x = \sin(\phi)\cos(\theta) \\ y = \sin(\phi)\sin(\theta) \\ z = \cos(\phi) \\ 0 \leq \phi \leq \pi/2,\ 0 \leq \theta < 2\pi \end{cases}$$

3) The upper half sphere can be parameterized using cylindrical coordinates

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = \sqrt{1-r^2} \\ 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi \end{cases}$$

An orientation on a curve is a choice of a direction for the curve. For a surface an orientation is a choice of "up" or "down". The easist way to make this precise is to view an orientation as a choice of (an upward or outward pointing) unit normal vector field $\mathbf{n}$ on S. A parameterized surface S

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \\ (u,v) \in D \end{cases}$$

is called *smooth* provided that f, g, h are C^1, the function that they define from $D \to \mathbb{R}^3$ is one to one, and the tangent vector fields

$$\mathbf{T}_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$$

$$\mathbf{T}_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$$

are linearly independent. In this case, once we pick an ordering of the variables (say u first, v second) an orientation is determined by the normal

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{||\mathbf{T}_u \times \mathbf{T}_v||}$$

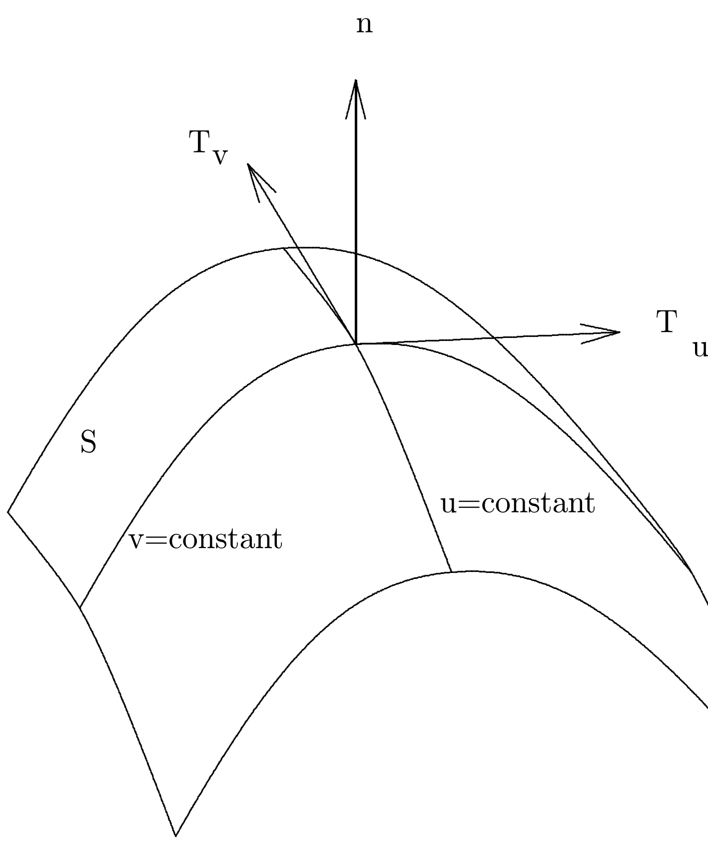

FIGURE 1

If we look at the examples given earlier. (1) is smooth. However there is a slight problem with our examples (2) and (3). Here $\mathbf{T}_\theta = 0$, when $\phi = 0$ in example (2) and when $r = 0$ in example (3). To deal with scenario, we will consider a surface smooth if there is at least one smooth parameterization for it.

15 Surface Integrals

Let S be a smooth parameterized surface

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \\ (u,v) \in D \end{cases}$$

with orientation corresponding to the ordering u, v. The symbols dx etc. can be converted to the new coordinates as follows

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$dx \wedge dy = (\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv) \wedge (\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv)$$

$$= (\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u})du \wedge dv = \frac{\partial(x,y)}{\partial(u,v)}du \wedge dv$$

In this way, it is possible to convert any 2-form ω to uv-coordinates.

DEFINITION 16 *The integral of a 2-form on S is given by*

$$\int\int_S F dx \wedge dy + G dy \wedge dz + H dz \wedge dx = \int\int_D [F\frac{\partial(x,y)}{\partial(u,v)} + G\frac{\partial(y,z)}{\partial(u,v)} + H\frac{\partial(z,x)}{\partial(u,v)}]dudv$$

In practice, the integral of a 2-form can be calculated by first converting it to the form $f(u,v)du \wedge dv$, and then evaluating $\int\int_D f(u,v)\, dudv$.

Let S be the upper half sphere of radius 1 oriented with the upward normal parameterized using spherical coordinates, we get

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(\phi,\theta)}d\phi \wedge d\theta = \cos(\phi)\sin(\phi)d\phi \wedge d\theta$$

So

$$\int\int_S dx \wedge dy = \int_0^{2\pi}\int_0^{\pi/2} \cos(\phi)\sin(\phi)d\phi d\theta = \pi$$

On the other hand if use the same surface parameterized using cylindrical coordinates

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(r,\theta)}dr \wedge d\theta r dr \wedge d\theta$$

Then

$$\int\int_S dx \wedge dy = \int_0^{2\pi} \int_0^1 r dr d\theta = \pi$$

which leads to the same answer as one would hope. The general result is:

THEOREM 17 *Suppose that a oriented surface S has two different smooth C^1 parameterizations, then for any 2-form ω, the expression for the integrals of ω calculated with respect to both parameterizations agree.*

(This theorem needs to be applied to the half sphere with the point $(0,0,1)$ removed in the above examples.) Complicated surfaces may be divided up into nonoverlapping patches which can be parameterized seperately. Then to integrate a 2-form, one would have to sum up the integrals over each patch.

18 Flux

In many situations arising in physics, one needs to integrate a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ over a surface. The resulting quantity is often called a flux. We will simply define this integral, which is usually written as $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ or $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$, to mean

$$\int\int_S F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

It is probally easier to view this as a two step process, first convert $\mathbf{F}$ to a 2-form as follows:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy,$$

then integrate. Earler, we learned how to convert a vector field to a 1-form:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1 dx + F_2 dy + F_3 dz$$

To complete the triangle, we can convert a 1-form to a 2-form and back via:

$$F_1 dx + F_2 dy + F_3 dz \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

This operation is usually denoted by $*$.

As a typical example, consider a fluid such as air or water. Associated to this, there is a scalar field $\rho(x,y,z)$ which measures the density, and a vector field $\mathbf{v}$ which measures the velocity of the flow (e.g. the wind velocity). Then the rate at which the fluid passes through a surface S is given by the flux integral $\int\int_S \rho\mathbf{v} \cdot d\mathbf{S}$

19 Green's and Stokes' Theorems

Let C be a closed C^1 curve in $\mathbb{R}^2$ and D be the interior of C. D is an example of a surface with a boundary C. In this case the surface lies flat in the plane, but more general examples can be constructed by letting S be a parameterized surface

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \\ (u,v) \in D \subset \mathbb{R}^2 \end{cases}$$

then the image of C in $\mathbb{R}^3$ will be the boundary of S. For example, the boundary of the upper half sphere S

$$\begin{cases} x = \sin(\phi)\cos(\theta) \\ y = \sin(\phi)\sin(\theta) \\ z = \cos(\phi) \\ 0 \le \phi \le \pi/2,\ 0 \le \theta < 2\pi \end{cases}$$

is the circle C given by

$$x = \cos(\theta),\ y = \sin(\theta),\ z = 0,\ 0 \le \theta \le 2\pi$$

In what follows, we will need to match up the orientation of S and its boundary curve. This will be done by the *right hand rule:* if the fingers of the right hand point in the direction of C, then the direction of the thumb should be "up".

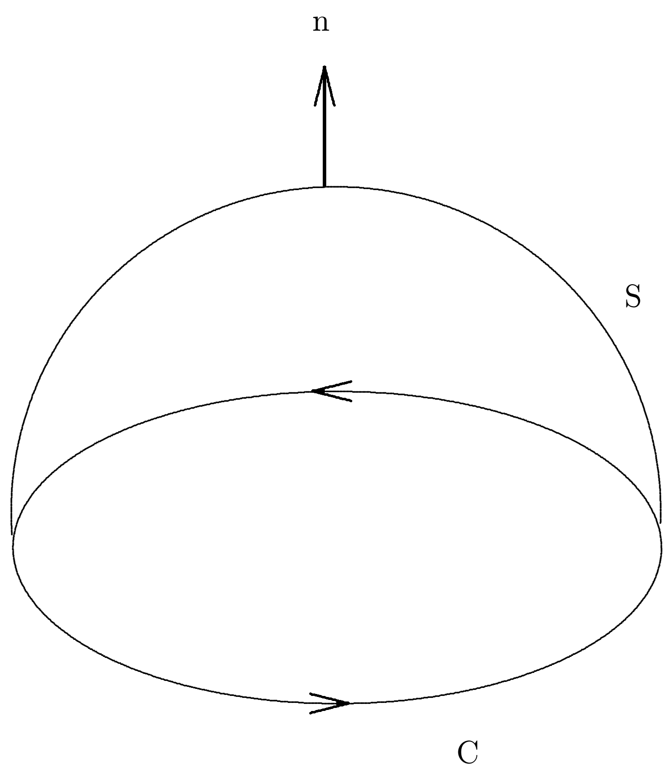

FIGURE 2

Stoke's theorem is really the fundamental theorem of calculus of surface integrals.

THEOREM 20 (Stokes' theorem) *Let S be an oriented smooth surface with smooth boundary curve C. If C is oriented using the right hand rule, then for any C^1 1-form ω on $\mathbb{R}^3$*

$$\int\int_S d\omega = \int_C \omega$$

If the surface lies in the plane, it is possible make this very explicit:

THEOREM 21 (Green's theorem) *Let C be a closed C^1 curve in $\mathbb{R}^2$ oriented counterclockwise and D be the interior of C. If $P(x, y)$ and $Q(x, y)$ are both C^1 functions then*

$$\int_C Pdx + Qdy = \int\int_D (\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x})dxdy$$

In vector notation, Stokes' theorem is written as

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{F}$ is a C^1-vector field. In physics, there a two fundamental vector fields, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. They're governed by Maxwell's equations, one of which is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where t is time. If we integrate both sides over S, apply Stokes' theorem and simplify, we obtain Faraday's law of induction:

$$\int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int\int_S \mathbf{B} \cdot \mathbf{n}\, dS$$

To get a sense of what this says, imagine that C is wire loop and that we are dragging a magnet through it. This action will induce an electric current; the left hand integral is precisely the induced voltage and the right side is related to the strength of the magnet and the rate at which it is being dragged through.

22 Triple integrals and the divergence theorem

A 3-form is an expression $f(x, y, z)dx \wedge dy \wedge dz$. Given a solid region $V \subset \mathbb{R}^3$, we define

$$\int\int\int_V f(x, y, z)dx \wedge dy \wedge dz = \int\int\int_V f(x, y, z)dxdydz$$

THEOREM 23 (Divergence theorem) *Let V be the interior of a smooth closed surface S oriented with the outward pointing normal. If ω is a C^1 2-form on $\mathbb{R}^3$ then*

$$\int\int\int_V d\omega = \int\int_S \omega$$

In standard vector notation, this reads

$$\int\int\int_V \nabla \cdot \mathbf{F}\, dV = \int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{F}$ is a C^1 vector field.

As an application, consider a fluid with density ρ and velocity $\mathbf{v}$. If S is the boundary of a solid region V with outward pointing normal $\mathbf{n}$, then the flux $\int\int_S \rho\mathbf{v} \cdot \mathbf{n}\, dS$ is the rate at which matter flows out of V. In other words, it is minus the rate at which matter flows in, and this equals $-\partial/\partial t \int\int\int_V \rho dV$. On the other hand, by the divergence theorem, the above double integral equals $\int\int\int_S \nabla \cdot (\rho\mathbf{v})\, dV$. Setting these equal and subtracting yields

$$\int\int\int_V [\nabla \cdot (\rho\mathbf{v}) + \frac{\partial \rho}{\partial t}]\, dV = 0.$$

The only way this can hold for all possible regions V is that the integrand

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0$$

This is one of the basic laws of fluid mechanics.

24 Beyond $\mathbb{R}^3$

It is possible to do calculus in $\mathbb{R}^n$ with $n > 3$. Here the language of differential forms comes into its own. While it would be impossible to talk about the curl of a vector field in, say, $\mathbb{R}^4$, the derivative of a 1-form or 2-form presents no problems; we simply apply the rules we've already learned. Integration over higher dimensional "surfaces" or *manifolds* can be defined, and there is an analogue of Stokes' theorem in this setting.

As exotic as all of this sounds, there are applications of these ideas outside of mathematics. For example, in relativity theory one needs to treat the electric $\mathbf{E} = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k}$ and magnetic fields $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ as part of a single "field" on space-time. In mathematical terms, we can take space-time to be $\mathbb{R}^4$ - with the fourth coordinate as time t. The electromagnetic field can be represented by a 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dz + B_3 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$$

If we compute dF using the analogues of the rules we've learned:

$$dF = (\frac{\partial B_3}{\partial x} dx + \frac{\partial B_3}{\partial y} dy + \frac{\partial B_3}{\partial z} dz + \frac{\partial B_3}{\partial t} dt) \wedge dx \wedge dy + \ldots$$

$$= (\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}) dx \wedge dy \wedge dz + (\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t}) dx \wedge dy \wedge dt + \ldots$$

Two of Maxwell's equations can be expressed very succintly in this language as $dF = 0$.

For more informations, see the books "Differential forms and applications to the physical sciences" by H. Flanders and "Calculus on manifolds" by M. Spivak.