A REVIEW OF RESIDUES AND INTEGRATION — A PROCEDURAL APPROACH

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1. Introduction

When working with complex functions, it is best to understand exactly how they work. Of course, complex functions are rather strange and exotic, so it may be difficult to develop a good intuition. Once one has understood complex functions, their behaviour is not complicated, and the calculations are often very simple. As a result, the difficulty in working with complex functions is not in doing the calculations — these are almost all calculations that are familiar from calculus classes. What is new is that we have to check for various kinds of exotic things that might happen, such as poles, branch cuts or essential singularities. These checks are the essential part of any problem in complex variables; once the bad behaviour of a function has been explicitly described, the calculations are straightforward (if occasionally messy). Thus when a student writes up their solution to a problem, the most essential part of the solution is their description of the function. Did they check that the function was analytic? Did they find all the relevant poles? Did they miss a residue?

In order to simplify the process of checking these facts, these notes set forth a procedure for solving most problems that will be encountered in a complex variables class. Generally many options are presented, not all of which will work for any given problem. But if a student follows these steps and explains why each required condition is satisfied, they can be confident that their answer is right and that the marker can see that it is right.

It is by no means recommended that students memorize these notes: ideally students should understand the material well enough that all the procedures described make sense and could be figured out on the spot.

2. Residues

Suppose that f has an isolated singularity at z_0. Then we have seen that f has a Laurent series expansion near z_0. This means that we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

This is what we call a Laurent series. There are similar series that we do not call Laurent series. For example,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z (z - z_0)^n$$

is not a Laurent series near z_0 (unless $z_0 = 0$) because of the extraneous factor z in the series expansion.

Laurent series are of interest for a number of reasons, but the main one is this: They tell us how a function behaves in small neighborhoods of z_0. If a Laurent series contains nonzero a_n for infinitely many $n < 0$, then the function f blows up very badly at z_0. If the first nonzero coefficient is a_n, then we know f behaves like $(z - z_0)^n$ for z close to z_0. So if $n < 0$, then f blows up like $1/(z - z_0)^{-n}$; we call this having a pole of order n. If $n = 0$, then f has a nonzero value at z_0; if $n > 0$ then f has a zero at z_0, and in fact it looks locally like $(z - z_0)^n$; we call this a zero of order n. Note that at a zero of order n, the first $n - 1$ derivatives of f will also be zero.

However, the Laurent series does not tell us anything useful about the behavior of f far from z_0. To understand f elsewhere we have to use some other method; for example, we could compute a Laurent series for f at some other z_1; it will probably have completely different behavior.

For example, the Laurent series for sin at zero is

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots .$$

Looking at this series, we can tell sin has a zero of order one at z, but we cannot tell that it has one at π. Its behavior at $\pi/2$ we can examine:

$$\sin(z) = 1 - \frac{(z - \pi/2)}{2!} + \frac{(z - \pi/2)^4}{4!} - \cdots .$$

We see that sin has neither zero nor pole at $\pi/2$. But we also see that the Laurent series takes on a completely different form there. In general, to compute such a Laurent series, one needs to compute all the derivatives of sin at $\pi/2$. So computing the Laurent series is usually a laborious proposition. Even a simple function such as tan has a messy Laurent series:

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \cdots .$$

Therefore we generally look for easier ways to understand the behavior of a function near a point.

2.1. **Describing all the singularities of a function.** Frequently we need to know something about the singularities of a function: perhaps we want to integrate around a region; then we need to know what bad behavior f has inside the region and on its boundary. In principle, we would need to use something like the Cauchy-Riemann equations. In practice, generally functions are assembled from other functions we know to be analytic. So here is a procedure for determining all the problem points of f:

(1) Is f a familiar function? If so, then we know its problem points. For example, if f is Log, then we know it has singularities at 0 and on the branch cut. Be careful with functions that seem familiar such as cosh, which does have zeros in the complex plane although none of them are real.
(2) Is $f = gh$ or $f = g + h$? Then the problem points of f will be the problem points of g and the problem points of h. For example, $\text{Log}(z - i) + \text{Log}(z + i)$ has problem points $z = x + iy$ whenever $x \leq 0$ and $y = 1$ or $y = -1$.
(3) Is $f = g/h$? If so, f has problem points anywhere f and g do, plus it has additional problem points wherever $h = 0$.
(4) Is $f(z) = g(h(z))$? If so, then f will have a problem point wherever h has a problem point, but it will also have a problem point at every z for which $h(z)$ is a problem point for g. For example, $\tan(1/z)$ has a problem point at zero, and a problem point wherever $1/z = k\pi + \pi/2$ for some integer k, that is, at $z = 1/(k\pi + \pi/2)$.

If f is *not* constructed from familiar complex functions, like $f(x + iy) = x^2 + y^2$, then we have to manually check analyticity everywhere; anywhere that doesn't work is a problem point. This particular example isn't analytic anywhere.

2.2. **Describing the behavior of a function at a problem point.** Generally, once one has found the problem points of a function f, one wants to know whether they are poles, removable singularities, essential singularities, or other types of problem. The following procedure will allow us to determine what kind of problem point z_0 is. Note that it will do no harm to do this to points that are not actually a problem; they will look just like isolated removable singularities.

(1) Is z_0 really a problem point? We can check this by checking, first, that
$$\lim_{z\to z_0} f(z)$$
exists. If this limit exists (that is, really is some number; ∞ does not count as "existing" for limits), then z_0 is a removable singularity. We still need to check analyticity of f at z_0 to claim that it's not really a problem point; if z_0 is isolated, this is automatic (from the Laurent expansion). For example, $\sin(z)/z$ has a problem point at zero: it is not defined there. But the limit exists and is 1, and this is an isolated singular point, so 0 is a removable singularity. You can ignore removable singularities from this point on.

(2) Is z_0 an isolated problem point? In other words, are there other problem points close to z_0? If we have obtained a list of all the problem points of f, we can check this. For example, 0 is not an isolated problem point for Log, as there are points on the branch cut arbitrarily close to 0. 0 is also not an isolated problem point for $\tan(1/z)$. If z_0 is not isolated, there is probably not much that you can do with it except check that it is not in your region of integration.

(3) Is z_0 an essential singularity or a pole? If it's a pole, what is its order? There are a number of ways to check this.
   (a) If z_0 is a pole of order n, by definition that means that near z_0 we can describe f using the Laurent series
   $$f(z) = a_{-n}(z - z_0)^{-n} + a_{-n+1}(z - z_0)^{-n+1} + \cdots$$
   where $a_{-n} \neq 0$. If z_0 is an essential singularity, it will have infinitely many negative terms. If you can actually figure out the Laurent series for f and look at its coefficients, this answers the question. But be careful: it really needs to be a Laurent series for f. No spurious z floating around can be allowed.
   (b) Usually the Laurent series is too hard to work with. But in principle it exists, so we can use other methods to get the same answer. For example, the limit
   $$\lim_{z\to z_0}(z - z_0)^m f(z) = \lim_{z\to z_0} a_{-n}(z - z_0)^{m-n} + a_{-n+1}(z - z_0)^{m-n+1} + \cdots$$
   will be 0 if $m > n$, $a_{-n} \neq 0$ if $m = n$, and ∞ if $m < n$. So if you think z_0 is a pole of order m, you can try taking this limit; if you get 0 you guessed too large, and if you get ∞ you guessed too small. If z_0 is an essential singularity, you will *always* get ∞.
   (c) If $f = gh$ and z_0 is neither a problem point nor a zero for g, then we need only ask about h. For example, looking at $1/(z \sin z)$ near π, we see the $1/z$ is has no problems at π and is nonzero there, so we need only look at $1/\sin z$.
   (d) If $f = g/h$, and g looks locally like $(z-z_0)^k$ and h looks locally like $(z-z_0)^l$, then f looks locally like $(z-z_0)^{k-l}$. So, for example, if h has a pole of order 3 and g has a pole of order 5, then h looks like $(z - z_0)^{-3}$ and g looks like $(z - z_0)^{-5}$, so their ratio $f(z) = g(z)/h(z)$ looks like $(z - z_0)^{-2}$, and f has

a pole of order 2 at z_0. This works for zeros as well: if h has a zero of order 1 and g has a pole of order 2, then h looks like $(z - z_0)^1$ and g looks like $(z - z_0)^{-2}$, so their ratio $f(z) = g(z)/h(z)$ looks like $(z - z_0)^{-3}$, and f has a pole of order 3 at z_0. On the other hand, if g or h has an essential singularity at z_0, this method can't help.

2.3. **Computing residues.** We saw that the Laurent series of f at z_0 describes how f behaves near z_0. We can use our theorems about the independence of path for a line integral to turn any line integral into a small loop around each problem point; we can make this small loop small enough that the Laurent series tells us what will happen. If C is such a small loop and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

then we have seen that

$$\oint_C f(z)\,\mathrm{d}z = \oint_C \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n dz = \sum_{n=-\infty}^{\infty} a_n \oint_C (z - z_0)^n dz = 2\pi i a_{-1}.$$

We defined

$$\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z.$$

This means that if C is any curve, there are no poles on C, the list $\{z_1, \ldots, z_n\}$ contains all the problem points inside C, and if each z_i is isolated, then

$$\oint_C f(z)\,\mathrm{d}z = \sum_{k=1}^{n} 2\pi i \operatorname{Res}(f, z_i).$$

This means if we can compute residues, we can easily compute complex contour integrals around closed loops. Note that we must ensure that $z_1, \ldots, z_n$ include *all* the problem points inside C: if we have missed any, the answer will probably be wrong. Also, if there are poles on C, we have to do something different; see Section 3.3 for what to do if this happens.

Given a point z_0, how do we go about computing $\operatorname{Res}(f, z_0)$?

(1) If z_0 is a removable singularity or a point at which f is analytic, then $\operatorname{Res}(f, z_0) = 0$.
(2) If $f = g/(z - z_0)$ and g is analytic at z_0 and $g(z_0) \neq 0$, then

$$\operatorname{Res}(f, z_0) = g(z_0).$$

For example,

$$\operatorname{Res}\left(\frac{z}{(z-1)(z+1)}, 1\right) = \frac{1}{(1+1)} = \frac{1}{2},$$

since $\frac{z}{z+1}$ is analytic and nonzero at 1.
(3) If we know the Laurent series for f around z_0 (nowhere else will do, so this is usually hard), so that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

then $\operatorname{Res}(f, z_0) = a_{-1}$. As before, there can be no spurious zs floating around: they can only occur in exactly the way described here.

(4) If you can guess that f has a pole of order at most m at z_0, then evaluate

$$\lim_{z\to z_0} \frac{1}{(n-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z).$$

If f actually has a pole worse than m, you will get ∞. Otherwise, you will get $\text{Res}(f, z_0)$, even if you chose your m too large. You can conclude that f has a pole of order at most m, but the pole could be of a lesser order; if the residue is zero, there might possibly be no pole at all. On the other hand, if f has an essential singularity, you will get ∞ no matter what m you supply.

(5) If $f = g/h$, and g and h are analytic at z_0, and we know $g(z_0) \neq 0$ but $h(z_0) = 0$, we can compute

$$\lim_{z\to z_0} \frac{g(z)}{h'(z)}.$$

If z_0 is a simple pole of f, then this will give us $\text{Res}(f, z_0)$. If z_0 is not a simple pole, then we will get ∞, and we will need to try some other method. If we were wrong about the conditions (if $g(z_0) = 0$ or $h(z_0) \neq 0$) then we may get a *reasonable-seeming but wrong* answer: if we mistakenly apply this method to $f(z) = z/z^2$ at 0, we get the wrong answer $1/2$ rather than the right answer 1. If we mistakenly apply the method to $f(z) = (1+z)/(1-z)$ at 0, we will get the wrong answer -1 instead of the right answer 0. So it really is necessary to verify that $g(z_0) \neq 0$, $h(z_0) = 0$ and g and h are analytic at z_0. If this verification is easy, then this is the quickest way to solve the problem.

(6) You can try to explicitly calculate a line integral. If C is some curve that encircles z_0 once in the counterclockwise direction, and there are no other problem points inside C, then you can try to evaluate

$$\frac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z.$$

This is usually very difficult, but if all other methods have failed, then this is worth a try. Remember that you can choose C to be a convenient contour for the integral.

3. Evaluating integrals with residues

The motivation for constructing residues was that they gave a simple way to calculate complex contour integrals. So, obviously, if you have a complex contour integral, you can calculate it using residues. But complex contour integrals are relatively rare in physical problems. They are of interest primarily because there are a number of kinds of real integral that are most easily evaluated using complex contour integration.

3.1. **Integrals around circles.**

When studying Fourier series and in many other contexts, integrals of periodic functions become important. Moreover, it is very common that these functions depend on the variable θ in a particular way, through sines and cosines only. Often these functions will be rational functions of $\sin\theta$ and $\cos\theta$, possibly with variations such as $\sin k\theta$ and $\cos k\theta$ for some integer k. These are usually purely real integrals, over a real interval. It may seem perverse to transform such a familiar problem into one involving complex calculus, but this will allow us to apply the method of residues. So the procedure is as follows:

(1) Verify that the problem is of the correct form. It should look like

$$\int_0^{2\pi} f(\sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta, \ldots)\,\mathrm{d}\theta,$$

where f is some function in which θ does not occur (outside the specified dependences on the sin and cos). The function f should also be a complex function that is analytic on the unit circle. If the problem is not of this form, it may be possible to transform it slightly, giving such a problem; if this cannot be done, this method will be of no use.

(2) Change the problem into a complex calculus problem. To do this, write $z = e^{i\theta}$. This change of variables introduces the usual extra factor of $\frac{d\theta}{dz} = -i/z$. We can do this by noting that if $z = e^{i\theta}$ and θ is a real number, then $\sin\theta = (z - z^{-1})/2i$ and $\cos\theta = (z + z^{-1})/2$. In fact, if k is any integer, we have

$$\sin k\theta = \frac{z^k - z^{-k}}{2i} \quad \text{and}$$
$$\cos k\theta = \frac{z^k + z^{-k}}{2}.$$

So the problem becomes (remembering the extra factor due to the change of variable) to compute:

$$\oint_C f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}, \frac{z^2 - z^{-2}}{2i}, \frac{z^2 + z^{-2}}{2}, \dots\right)(-i)z^{-1}\,\mathrm{d}z,$$

where C is the unit circle in the usual counterclockwise direction.

(3) This is now a complex contour integral, so try to calculate it by using residues. If f is in fact a rational function, then the contour integral will be the integral of a rational function, and its residues probably aren't too difficult to calculate. Using these values to evaluate the integral, we will get some complex number as an answer. In the very common case where f is supposed to give real values for all θ, then we can check our answer: if it is supposed to be the integral of a real function over a real interval, the answer had better be real.

3.2. **Improper integrals on the real line.** Periodic functions do turn up from time to time, and it is nice to have some sort of way to integrate them, but in practice this kind of integral is not too common. We do, however, often deal with functions on the whole real line, and we would often like to be able to compute their integrals over the whole real line.

Of course, this doesn't work for every function you can dream up: unless the function goes to zero pretty fast, the area under its curve will very probably be infinite. But other things can go wrong too. For example, suppose we tried to compute

$$\int_{-\infty}^{\infty} \cos x\,\mathrm{d}x.$$

We can use the Fundamental Theorem of Calculus to do this: we know cos is the derivative of sin, so the answer should be something like $\sin\infty - \sin\infty$. Now, of course this makes no sense, so we need to look at some sort of limit. If we just take

$$\lim_{R\to\infty}\int_{-R}^{R} \cos x\,\mathrm{d}x,$$

for each R the answer is $\sin R - \sin(-R) = 2\sin R$, which oscillates, and so the limit does not converge. On the other hand, if we had tried to integrate sin instead, for each R we would get $-\cos R + \cos(-R) = 0$, and the limit would converge. But cos is just a shifted version of sin, so we're in trouble: changing the variable of integration, the way we do all the time, might change our answer.

The solution to this problem, for a mathematician, is to define the integral somewhat differently: you take two limits, over the top and bottom of the range, independently. Only if both converge, separately, do you say that the integral really converges. In our example, both oscillate, so the integral doesn't converge for either sin or cos.

Of course, it's more awkward to work with two limits than just one. So for computational purposes, we define the Cauchy principal value, just the simplest possible way, as

$$\lim_{R\to\infty}\int_{-R}^{R}\cos x\,\mathrm{d}x.$$

This has its pitfalls; in particular, you can't change the variable of integration without risking changing the answer. Fortunately, these pitfalls only occur for integrals that don't exist in the strict sense. Since we are not particularly interested in exactly which integrals converge and which don't, it will be enough to simply check that the integral we are interested really converges and then calculate the Cauchy principal value. This sometimes allows us to split an integral that actually converges up into two integrals that don't actually converge, but whose Cauchy principal value we can compute; their difference will then be the value of the original integral.

So, all this said, how do we actually calculate one of these integrals over the real line using residues? Well, first we have to produce a closed contour of some sort. If we integrate along the real line from some $-R$ to some R, that's part of a contour. But we must get back to where we started; since we're integrating over some new territory, away from the integral we actually want to calculate, we have to make sure this new part of the integral only contributes a small amount to the total. So we need to make sure that our function is small for large values of $|z|$. That's okay for real z, since if our function doesn't get small the integral won't converge no matter how we calculate it. But if we're going to get back to where we started using a big half-circle in the complex plane, we must ensure that the function is small for all values on this half-circle. In fact, because the length of the half-circle grows with R, the size of our function has to go down faster than $1/R$ for the integral to get smaller. Usually a complex function won't get small like this all over the place (in fact, if it was bounded and entire, we saw that it would have to be a constant). So we have to choose whether to take a half-circle in the upper half-plane to get back to the origin, or to take one in the lower half-plane.

We can put all this talk together to get a procedure for calculating integrals on the real line. Suppose we want to calculate

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

(1) Verify that f is a suitable function for this method. First of all, it must be a complex function (that is, we can feed in complex numbers, not just real ones), and it must be analytic except possibly for some isolated singularities, none of which can be on the real line.[1] Second, and more difficult, it must go to zero fast enough for large $|z|$ in one or the other half-plane. In general, this could be messy. But there are some simple tests that cover almost all the situations encountered in practice.
   (a) If f is a rational function, say $f = p/q$ for polynomials p and q, and q has no zeros on the real line, then f is suitable if $\deg q - \deg p \geq 2$, and you can

[1] In fact, it can have branch cuts, non-isolated singularities, or even regions of non-analyticity in the lower half-plane as long as we use the upper half-plane for integration, or vice versa. It must still be analytic on the real line.

use either half-plane. If this condition is not met, then f is not suitable, and in fact the integral will diverge.

(b) If $f(x) = g(x)e^{itx}$ for some real number t, then it is necessary to consider cases separately. If $t > 0$, and $g(x) = a(x)/b(x)$ for some polynomials a and b with $\deg b > \deg a$, Jordan's lemma asserts that f will be suitable in the upper half-plane. If $t < 0$, then we need to look at the lower half-plane, but the condition on g is the same. If $t = 0$, $f(x) = g(x)$, so the exponential plays no role, and we have to check the suitability of g.

(c) If $f(x) = g(x)\sin(kx)$ or $f(x) = g(x)\cos(kx)$ for some real number k, then f is almost certainly *not* suitable: we cannot directly integrate. However, we can compute this integral by solving a related problem in two ways: the first, which always works but is more difficult, is to replace the sine or cosine with its expression in terms of complex exponentials: use $\sin(x) = (e^{ix} - e^{-ix})/2i$ or $\cos(x) = (e^{ix} + e^{-ix})/2$ and break up the sum. This will be laborious but will always allow one to use the previous method. If $g(x)$ always gives a real answer when given a real input (for example, $g(x) = 1/(x^2 + 1)$ but *not* $g(x) = 1/(x + i)$) then we can write $f(x) = \operatorname{Re} g(x)e^{ikx}$ or $f(x) = \operatorname{Im} g(x)e^{ikx}$; we can then move the real-part outside the integral and apply the previous method.

(d) If $|f(z)| \leq |g(z)|$ for all z in the upper (or lower) half-plane with $|z|$ larger than some K, then it is enough to check the suitability of g. This can occasionally simplify problems but be wary of assuming, for example, that $|\sin z| \leq 1$: this is not necessarily true for z complex.

(2) Knowing that f is suitable, construct the contour of integration. This will be a large half-circle running along the real line from $-R$ to R and then going back to the start through either the upper or lower half-plane. If it goes through the lower half-plane, notice that the direction of integration is opposite to the usual direction, so each residue should be added with a coefficient of -1.

(3) Figure out what poles f has in the region of integration. You can assume R is very large, so if f has only finitely many poles, you can assume that they all lie inside the half-circle. If f has infinitely many poles, you cannot make such an assumption; the value of the contour integral will depend on R.

(4) Compute the residue of f at each pole. The contour integral is then either $2\pi i$ times the sum of these residues if we are working in the upper half-plane or $-2\pi i$ times the sum if we are working in the lower half-plane.

(5) If the value of the contour integral you get depends on R, then take the limit as $R \to \infty$; this should only occur when f has infinitely many poles in the half-plane of interest. If this limit fails to converge, you have made an error; probably f was not suitable. Unfortunately, the fact that the limit converges does not mean that f *was* suitable; you still need to check.

3.3. **What to do if a contour passes through a singularity.** Essentially, this means the problem was to compute an improper integral, and in fact it means that the improper integral cannot converge. As before, we can define a Cauchy principal value, which could be of some use. If the singularity is anything worse than a first-order pole, the Cauchy principal value does not exist. For example, it can be used to compute the Fourier transform of $\sin(x)/x$. The trick in this case is to "indent" the contour: we replace the contour by a similar contour which makes a small half-circle around the singularity. We have a theorem in the book which says that the integral around such a small half-circle around a simple pole in the counterclockwise direction is $i\pi$ times the

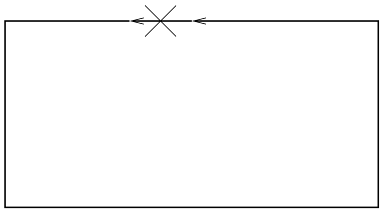

FIGURE 1. The contour C we would like to integrate around.

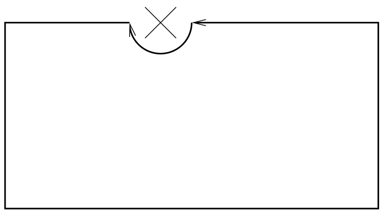

FIGURE 2. A contour C_1 that we can actually integrate around.

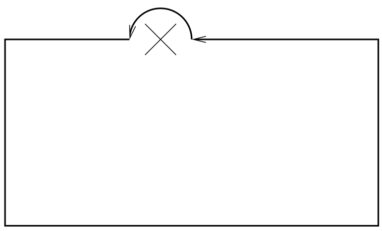

FIGURE 3. A contour C_2 that we can actually integrate around.

residue at the pole. Of course, you have a choice about which direction you indent the contour, and the integral around the small contour will change depending on which side you choose. But which side you choose will also determine whether the pole lies within the contour, and these two effects will cancel.

Suppose that we have a function f which we want to integrate around the contour C of Figure 3.3, and suppose that f has a simple pole having residue 1 at the point indicated by the "x". Then we will have to choose to integrate around one of the contours C_1 or C_2, letting the size of the indent go to zero. If we choose C_1, the integral along the small semicircle will be $-i\pi$, and the integral we actually want will be

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz - (-i\pi) = i\pi,$$

since there is no pole inside C_1. If instead we had chosen C_2, the integral around the small semicircle would be in the positive direction, giving $i\pi$. But now the contour contains a pole, so

$$\oint_C f(z)dz = \oint_{C_2} f(z)dz - i\pi = i2\pi - i\pi = i\pi.$$

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