ISSN 1652-4934

Archives of ALGA

Volume 3, 2006

- Extension of Laplace's method
- Conservation laws

ALGA Publications
BLEKINGE INSTITUTE OF TECHNOLOGY
Karlskrona, Sweden

Volume 3, November 2006 ISSN 1652-4934

Archives of ALGA

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ALGA Publications
Blekinge Institute of Technology
Karlskrona, Sweden

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Editor's preface

This volume contains four papers: two translations of classical works and two original research papers.

The first paper is the translation from French of the *Introduction* of Louise Petrén's PhD thesis defended in Lund in 1911. Louise Petrén is an interesting person in the history of Swedish mathematics not only because she was the first Swedish woman who defended PhD in mathematics, but also because she made a profound contribution to the constructive integration theory of partial differential equations in the direction initiated by Euler and continued by Laplace, Legendre, Imschenetsky, Darboux, Goursat. In her PhD thesis she extended to higher-order equations Laplace's method of integration of second-order linear hyperbolic equations with two independent variables. The *Introduction* presented here is an excellent survey of Laplace's method and its generalizations to quasi-linear partial differential equations in two independent variables. It is interesting to consider her results from the point of view of equivalence transformations and invariants of differential equations and compare with the theory of the Laplace invariants. However, L. Petrén's research was not known until recently among mathematicians working in group analysis. The aim of this translation is to fill this gap. We plan to publish the translation of the whole thesis in one of future volumes of *Archives of ALGA*. Professor Lars Haikola, Louise Petrén's grandson, provides in his preface a family background of Louise Petrén and introduces her as a person.

The second paper is the translation from German of Bessel-Hagen's paper on derivation of conservation laws in electrodynamics. It is one of the first papers dealing with an application of E. Noether's theorem on symmetry and conservation laws. It is worth noting that Bessel-Hagen applies Noether's theorem not to the first-order system of Maxwell equations but the second-order equations for the vector-potential, e.g. to the wave equations. Then the equations admit a variational formulation and have a usual Lagrangian. It is interesting to see how the author uses (in §5) the infinite part of the symmetry group to eliminate coordinates of the four-potential from the final expressions for the conservation laws.

In the third paper I prove a new theorem on derivation of conservation laws using symmetries of differential equations. The new theorem which I call *theorem on nonlocal conservation laws* is applicable, unlike Noether's theorem, to all differential equations independently on existence of usual Lagrangians. These two theorems do not exclude each other and, in the case of variational problems, provide different conservation laws for each symmetry. The nonlocal conservation theorem is applied, in particular, to derive the infinite series of local and nonlocal conservation laws for the Korteweg-de Vries equation.

The fourth paper is dedicated to derivation of conservation laws for symmetrized electromagnetic equations by using the general conservation theorem from the third paper. We obtain new conservation laws in electrodynamics. In particular, we arrive at nonlocal conservation laws depending on the values of the electric and magnetic vector fields at times t and $-t$.

Nail H. Ibragimov

Louise Petrén-Overton, min mormor

”Om jag inte får ta matematikböckerna med mig till himlen, så vill jag inte komma dit!” Detta dramatiska yttrande fällde min mormor när hon låg sjuk i scharlakansfeber som barn och hon berättade gärna om händelsen ännu nästan nittio år senare.

Min mormor – Louise Petrén, gift Overton – var inte bara min mormor, utan också min granne under tjugo år. Jag besökte henne nästan dagligen för att hjälpa henne med allt som en åldrande och alltmer orörlig människa behöver hjälp med i vardagen. Hon berättade gärna om sin barndom, sin uppväxt och sitt liv. Hennes huvud var fullständigt klart nästan intill hennes död vid 96 års ålder.

Min mormor föddes 1880 som yngsta syskon av tolv i en prästfamilj. Hennes nio bröder blev naturvetare, medicinare och jurister medan hennes två systrar fick hjälpa till i hushållet. Lillasyster behövdes inte i hushållet varför hon fick möjlighet att studera. Hon tog studentexamen som privatist 1899, filosofie kandidatexamen 1902 och filosofie licentiatexamen 1910. Hennes grundexamen bestod av matematik, mekanik, fysik och kemi. 1911 disputerade hon för doktorsgraden i matematik.

Hennes far, som alltså var kyrkoherde, hade själv disputerat på en avhandling med titeln ”Om ett vidsträktare bruk av qvadrat – och om rutetalens tabell” 1850. Möjligen var detta ämne inspirerat av att han bodde hos sin morbror – Carl Johan Hill, professor i matematik vid Lunds universitet – som vid denna tid publicerade olika typer av matematiska tabellverk.

När hon påbörjade sina studier vid Lunds universitet så fanns där 600 studenter varav 12 kvinnor. Som naturvetare var hon naturligtvis ensam kvinna och professorn i matematik frångick inte sina vanor att hälsa auditoriet med ”mina herrar” men försäkrade att han i detta inbegrep fröken Petrén. Studietiden beskrev hon som förvånansvärt torftig – det var helt enkelt för små grupper av studenter som läste matematik och mekanik. Hon tyckte också det var slöseri att professorn skulle föreläsa för enbart två personer - henne och ytterligare en manlig student. När hon en gång fick möjlighet att delta i en konferens i Stockholm hos professor Gösta Mittag-Leffler upplevde hon det som utomordentligt stimulerande att möta andra matematiker och upptäcka andra grenar av matematiken.

Lars Gårding – professor i matematik vid Lunds universitet 1953 – 85 – har i sin bok om svenska matematiker* beskrivit Louise Petréns avhandling med orden ”Louise Petrén visar en imponerande förtrogenhet med litteraturen och en nyskapande förmåga inom dess ram. Hennes framställning är också klar och vårdad”. Men han gör också klart att ämnet för avhandlingen vid denna tidpunkt egentligen redan var föråldrat.

Efter sin disputation ägnade sig mormor inte åt nya forskningsuppgifter. Det gavs heller inte mycket möjlighet åt detta. Professorn i matematik uttalade mycket tydligt att en docentur bara kunde komma ifråga för en man av det enkla skälet att en denne hade försörjningsbörda vilket kvinnan saknade. Min mormors grämelse över detta förhållande hade inte försvunnit ens när hon fyllt 95 år! Hon undervisade dock i matematik under tjugo år på deltid och arbetade en kort tid som försäkringsmatematiker men var i övrigt inte ”aktiv” matematiker. Hon gifte sig året efter disputationen, födde fyra barn och levde ett synnerligen aktivt liv utöver hemmets sysslor.

Min mormor var en synnerligen prosaisk person – en typisk exponent för det förra sekelskiftets förnuftstro. Darwinismen var det stora debattämnet och tilltron till vetenskapens möjlighet att lösa alla problem var stor. Hon var prästdotter men liksom sina nio bröder vetenskapligt skolad och ingen i syskonkretsen såg några konflikter mellan tro och förnuft. Det var en nästan osannolik tillfällighet att morfar - Ernest Overton, professor i farmakologi - faktiskt var släkt med Darwin.

Hennes prosaiska sinnelag tog sig flera uttryck. Humor och skämt såg hon mest som slöseri med tid och vid ett tillfälle i radions barndom sägs hon ha fräst ”stäng av den där dansmusiken” – när dansmusiken bestod av Sveriges nationalsång. Hon var fullständigt omusikalisk vilket visar det mytiska i att alla matematiker också är musikälskare. Jag har hört av någon av de elever hon undervisade under årens lopp att hon var en svag pedagog. Det är inte osannolikt att det förhöll sig så, ty hon var lätt ofördragsam med den som inte genast förstod. Matematik och rationalitet var för henne själv alltid något självklart.

Hennes långa liv blev ovanligt innehållsrikt. Hon var en skarp och en stark personlighet som skulle ha uppskattat att hennes avhandling nu kan nå en vidare krets.

Lars Haikola
Rektor för Blekinge Tekniska Högskola

* Lars Gårding, Matematik och matematiker. Matematiken i Sverige före 1950. Lund University Press 1996.

Louise Petrén-Overton, my grandmother

“If I am not allowed to bring the mathematics books with me to heaven, I do not want to go there!” My maternal grandmother uttered these dramatic words when as a child she was down with scarlet fever. She continued to enjoy telling the story almost ninety years later.

My grandmother-Louise Petrén, married Overton-was not only my granny but also my neighbor for nearly twenty years. I visited her almost daily to help her with everything that an aging and increasingly immobile person needs assistance with. She gladly talked about her childhood, adolescence, and adult life. Her mind was totally clear almost to the day of her death at the age of 96.

My grandmother was born in 1880 as the youngest child of twelve siblings in a clergyman’s family. Her nine brothers became scientists, medical doctors, and lawyers, while her two sisters helped with housekeeping. Since the household duties did not require the youngest sister’s involvement, she was given the opportunity to study. She received her General Certificate of Education as a privately-tutored student in 1899. In 1902 she got her BA degree in a combination of four subjects: mathematics, physics, chemistry, and mechanics. In 1910 she received her Licentiate of Philosophy and in 1911 defended her doctoral dissertation in mathematics.

Her father, a parish priest, had himself defended a doctoral dissertation in mathematics with the title ”About a more general use of square and about the table of square numbers” in 1850. It is likely that he was inspired in the choice of the subject by his maternal uncle in whose household he lived. The uncle, Carl Johan Hill, professor of mathematics at Lund University, had published diverse types of mathematical tables.

When my grandmother began her university studies, there were about 600 students at Lund University, 12 of whom were women. She was, not unexpectedly, the only woman among the science students, and the professor of mathematics did not change his habit of addressing the auditorium with “Gentlemen”; he assured his students, however, that Miss Petrén was included in this greeting.

My grandmother described her university time as surprisingly uninspiring. The group of students studying mathematics and mechanics was quite simply too small; she also thought that it was a waste of time for the professor to give lectures for only two students: herself and another, male, student. When at some point she had the opportunity to participate in a conference in Stockholm, organized by Professor Gösta Mittag-Leffler, she found it extremely stimulating to meet other mathematicians and to discover new branches of mathematics.

In his book[1] about Swedish mathematicians, Lars Gårding, Professor of mathematics at Lund University from 1953 to 1985, described Louise Petrén's dissertation as follows: "Louise Petrén shows an impressive knowledge of literature in the field and a creative talent within the framework of the discipline. Her presentation is clear and correct." But he also makes it clear that he finds the subject of her dissertation quite dated.

After her doctoral defense my grandmother did not undertake new research projects; there was hardly any opportunity to do so. The mathematics professor sternly announced that a promotion to associate professorship was only an option for a man who had to support a family, which was not a woman's duty. At 95, my grandmother was still mortified over this attitude! She taught mathematics for about twenty years on a part-time basis and worked for a shorter period as an actuarial mathematician. Apart from that, she was not an "active" mathematician. A year after her doctoral defense she got married, gave birth to four children, and led a very active life in addition to taking care of the household.

My grandmother was an extremely down-to-earth person, a typical exponent of the previous turn-of-the-century belief in reason. Darwin's theory was the big subject of debate and the trust in the power of science to solve problems was great. She was the daughter of a clergyman, but, like her nine brothers, she had an education in the sciences; none of the siblings saw a conflict between faith and reason. It was an almost improbable coincidence that my maternal grandfather, Ernest Overton, a professor of pharmacology, was Darwin's relative.

My grandmother's no-nonsense temperament manifested itself in various ways. She saw humor and jokes as a waste of time. At one point in the early years of the radio, it is said, she hissed "Turn off this dance music," the "dance music" being the Swedish national anthem. She was totally unmusical, contrary to the myth that all mathematicians are also music-lovers. One of her students told me once that she was a bad pedagogue. It is not impossible that it indeed was so, since she was easily irritated when people did not speedily comprehend what she was saying. She took it for granted that mathematics and intellect went hand in hand.

Her long life was unusually eventful. She had a distinct and strong character and she would appreciate the fact that her doctoral dissertation can now reach new readers.

Lars Haikola
Vice-Chancellor
Blekinge Institute of Technology

[1]Lars Gårding, *Mathematics and mathematicians. Mathematics in Sweden before 1950,* History of mathematics, Vol. 13, AMS, 1998. Translated from the Swedish by Lars Gårding.

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EXTENSION OF LAPLACE'S METHOD TO THE EQUATIONS $$\sum_{i=0}^{n-1} A_{1i}(x,y)\frac{\partial^{i+1}z}{\partial x \partial y^i} + \sum_{i=0}^{n} A_{0i}(x,y)\frac{\partial^i z}{\partial y^i} = 0$$

BY LOUISE PETRÉN

Translated from French by
Nail H. Ibragimov and Gunter Leguy

[Louise Petrén, Extension de la méthode de Laplace aux équations
$\sum\limits_{i=0}^{n-1} A_{1i}(x,y)\frac{\partial^{i+1}z}{\partial x \partial y^i} + \sum\limits_{i=0}^{n} A_{0i}(x,y)\frac{\partial^i z}{\partial y^i} = 0,$
Lunds Universitets Årsskrift, N.F. Afd. 2, Bd. 7, Nr. 3, p. 1-165,
Kongl. Fysiografiska Sällskapets Handlingar, N.F. Bd. 22, Nr. 3.
Imprimerie Håkan Ohlsson, Lund 1911]

Introduction

The first significant results on integration of the equation

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0, \tag{1}$$

where $a,\ b$ and c are functions of x and y, were obtained by Euler[2]. He proved that the necessary and sufficient condition that Equation (1) admits a first-order intermediate integral, is that the coefficients a, b and c satisfy one of the following two equations:

$$\frac{\partial a}{\partial x} + ab - c = 0, \tag{2}$$

$$\frac{\partial b}{\partial y} + ab - c = 0. \tag{3}$$

[2]EULER, Institutiones Calculi Integralis, T. III, Pars prima, Sectio secunda, Cap. II.

If $\dfrac{\partial a}{\partial x} + ab - c = 0$ then Equation (1) can be reduced to the form:

$$\Big(\frac{\partial}{\partial x} + b\Big)\Big(\frac{\partial z}{\partial y} + az\Big) = 0$$

and the equation admits the integral $e^{-\int ady}X$, where X is an arbitrary function of x. In the same way, if $\frac{\partial b}{\partial y} + ab - c = 0$ then Equation (1) can be reduced to the form:

$$\Big(\frac{\partial}{\partial y} + a\Big)\Big(\frac{\partial z}{\partial x} + bz\Big) = 0$$

and the equation has the integral $e^{-\int bdx}Y$, where Y is an arbitrary function of y. In both cases the integration of Equation (1) is reduced to successive integrations of two first-order linear partial differential equations, which require only quadratures.

Moreover, Euler gave examples of equations of type (1) for which one can find, in an explicit form, an integral depending on an arbitrary function and for which there are no first-order intermediate integrals. The integrals in question are written explicitly as follows:

$$\alpha_0 X^{(m)} + \alpha_1 X^{(m-1)} + \cdots + \alpha_{m-1}X' + \alpha_m X \tag{4}$$

or

$$\beta_0 Y^{(r)} + \beta_1 Y^{(r-1)} + ... + \beta_{r-1}Y' + \beta_r Y \tag{5}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha_m$, $\beta_0, \beta_1, \ldots, \beta_{r-1}, \beta_r$ are given functions of x and y, $X^{(i)}$ and $Y^{(j)}$ are the ith- and jth-order derivatives of X and Y, respectively, and m and r are certain numbers. Euler was the first who discovered integrals of this type. Recently they were called integrals in Euler's form[3].

In a paper [4] presented to the Academy of Sciences in Paris in 1773, Laplace gave a method for recognizing if Equation (1) admits an integral of the form (4) or (5). If Equation (1) admits an integral of the form (4) or (5), the integral is obtained by the method of Laplace and the integration of the Equation (1) can be reduced to integration of two first-order linear differential equations. Laplace's method consists of a repeated application of the transformations

$$z_1 = \frac{\partial z}{\partial y} + az, \tag{6}$$

$$z_{-1} = \frac{\partial z}{\partial x} + bz \tag{7}$$

[3]Le Roux is the first who gave the name integral in Euler's form to the integrals expressed linearly with the help of an arbitrary function of a characteristic variable and a certain number of derivatives of this function. Journal de Mathématiques pures et appliquées", $5^{\text{iéme}}$ série, T. IV, 1898, page 401.

[4]Oeuvres complètes de Laplace, Tome IX, pages 5-68.

and reduces the integration of Equation (1) to integration of an equation of the same type which admits a first-order intermediate integral and therefore can be immediately integrated. By applying the transformation (6) Equation (1) can be written in the form

$$\frac{\partial z_1}{\partial x} + bz_1 = hz, \qquad h = \frac{\partial a}{\partial x} + ab - c. \tag{8}$$

If $h \neq 0$, the elimination of z from Equations (6) and (8) leads to the equation

$$\frac{\partial^2 z_1}{\partial x \partial y} + a_1 \frac{\partial z_1}{\partial x} + b_1 \frac{\partial z_1}{\partial y} + c_1 z_1 = 0, \tag{9}$$

where $\quad a_1 = a - \dfrac{\partial \ln h}{\partial y}, \quad b_1 = b, \quad c_1 = c - \dfrac{\partial a}{\partial x} + \dfrac{\partial b}{\partial y} - b\dfrac{\partial \ln h}{\partial y}\cdot$

It follows from Equations (6) and (8) that if one of Equations (1) and (9) admits an integral in Euler's form, the same will be true for the other equation. The integrals for the two equations (1) and (9) are in a one-to-one correspondence, and hence one obtains from every integral of one of these equations a solution of the other equation without integration. If Equation (1) admits the integral

$$z = \alpha_0 X^{(m)} + \alpha_1 X^{(m-1)} + \ldots + \alpha_{(m-1)} X' + \alpha_m X,$$

then Equation (9) has an integral of the same type which contains derivatives of X maximum of order $m - 1$ because $\quad \dfrac{\partial \alpha_0}{\partial y} + a\alpha_0 = 0$. Therefore, if Equation (1) admits an integral of the form (4) then transformation (6) (applied m times at most) leads to an equation which is of the same type as Equation (1) and which admits an integral of the form αX with a determined function α of x and y. Hence, the general integral has been obtained by successive integration of two first-order linear differential equations. Likewise, if Equation (1) has an integral of the form (5) then, applying the transformation (7) at most r times, one can reduce the integration of Equation (1) to integration of two first-order linear differential equations. But if Equation (1) does not admit integrals of the form (4) and (5), Laplace's method is not successful.

The linear partial differential equation of second order

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz = 0 \tag{10}$$

where A, B, C, D, E, F are functions of x and y, can be reduced to an equation of type (1) by a change of independent variables if the equation admits two distinct systems of characteristics. The characteristics of Equation (10) are:

$$f(x, y) = \text{const.}, \qquad \text{where} \quad A\left(\frac{\partial f}{\partial x}\right)^2 + B\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + C\left(\frac{\partial f}{\partial y}\right)^2 = 0.$$

We also say that $f(x, y)$ is a characteristic variable. The necessary and sufficient condition that two systems of characteristics do not to coincide is $B^2 - 4AC \neq 0$. If $B^2 - 4AC = 0$, then one can reduce Equation (10) to the form

$$\frac{\partial^2 z}{\partial x^2} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0$$

by a change of independent variables; the condition for Equation (10) to admit an integral of Euler's form is $b = 0$. But for recognizing if Equation (10) is integrable by Laplace's method, it is not necessary to make a change of independent variables. Legendre[5] gave a method, similar to Laplace's method, which can be applied directly to Equation (10). Equation (10) is integrable by the Laplace (Legendre) method if and only if it admits an integral of Euler's form. The advantage of Legendre's approach is that it allows one to identify if the equation is integrable without determining the characteristics. If Equation (10) admits an integral of Euler's form, then after calculating the characteristics, one can reduce the integration of the equation to quadratures. In this way, one obtains for the general integral an expression involving an arbitrary function under the signs of quadratures which can not be removed if the equation does not admit two integrals of Euler's form.

Later Imschenetsky [6] has presented the Legendre process in a very simple form. Imschenetsky [7] also gave an extension of Laplace's method to the equations

$$Gs + Hp + K = 0 \tag{11}$$

with G, H, K depending on q, z, x, y. Here $s = \dfrac{\partial^2 z}{\partial x \partial y}, p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$.
Imschenetsky applied to Equation (11) the transformation

$$z_1 = F(q, z, x, y) \tag{12}$$

where $F(q, z, x, y)$ satisfy to the equation $G\dfrac{\partial F}{\partial z} = H\dfrac{\partial F}{\partial q}$. Equation (11) can be reduced to

$$z_1 = F(q, z, x, y), \quad p_1 = \frac{\partial z}{\partial x} - \mu K$$

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[5]Histoire de l'Académie des sciences de Paris, 1787: Mémoire sur l'intégration de quelques équations aux différences partielles, § IV, pages 319-323

[6]Grunert's Archiv der Mathematik und Physik, Bd. 54, 1872: Intégration des équations aux dérivées partielles du second ordre d'une fonction de deux variables, §. 11. Translated from Russian: Mémoire de Kasan 1868 (Kazanskij vestnik 1868).

[7]The same book, § 9.

where $\mu = \frac{\frac{\partial F}{\partial z}}{H} = \frac{\frac{\partial F}{\partial q}}{G}$, $\quad p_1 = \frac{\partial z}{\partial x}$. Let us write $\frac{\partial F}{\partial x} - \mu K = F_{1(q,z,x,y)}$. The equation

$$\frac{\partial F}{\partial z}\frac{\partial F_1}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial F_1}{\partial z} = 0$$

is the necessary and sufficient condition for Equation (11) to admit a first-order intermediate integral depending on an arbitrary function of y. If

$$\frac{\partial F}{\partial z}\frac{\partial F_1}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial F_1}{\partial z} \neq 0$$

equations

$$z_1 = F(q, z, x, y), \quad p_1 = F_1(q, z, x, y)$$

can be solved with respect to z and q, and the equations become

$$z = f(p_1, z_1, x, y), \qquad q = f_1(p_1, z_1, x, y);$$

The elimination for z between those two equations leads to equation

$$\frac{\partial f}{\partial p_1}s_1 + \frac{\partial f}{\partial z_1}q_1 + \frac{\partial f}{\partial y} - f_1 = 0 \tag{13}$$

where $s_1 = \frac{\partial^2 z_1}{\partial x \partial y}$, $\quad q_1 = \frac{\partial z_1}{\partial y}$. Equations (11) and (13) can be integrated simultaneously. If the solutions of one of these equations are obtained, the solutions to the remaining equation is obtained without integration.

Since Equation (13) is linear with respect to s_1 and q_1, there always exists an Imschenetsky transformation of the form $u = \varphi(p_1, z_1, x, y)$ which can be applied to Equation (13). But this transformation leads again to the equation under consideration, and hence it is not interesting. Imschenetsky's transformation (12) can be applied to Equation (13), if Equation (13) is linear with respect to s_1 and q_1. It is only in exceptional cases that we can apply Imschenetsky's transformation (12) several times to an Equation (11). Let us suppose that Equation (13) is linear with respect to s_1 and p_1. Then Equation (13) can admit a first-order intermediate integral which depends on an arbitrary function of y. Otherwise Imschenetsky's transformation (12) can be applied to Equation (13). One can continue this procedure until one obtains an equation admitting a first-order intermediate integral, and hence reduces the integration problem to an integration of two first-order differential equations, or arrives to an equation to which one cannot apply Imschenetsky's transformation (12). In the last case, the Imschenetsky method doesn't succeed. In this way the Imschenetsky method reduces - by a repeated application of transformation (12) - an equation (11) which does not admit any first-order intermediate integral depending on an arbitrary function of y, to a same type equation which admits an intermediate integral of this type. When Equation (11) has the form (1)[8], the Imschenetsky transformations[9] are obviously identical with the Laplace transformations.

Teixeira[10] tried to extend the Imschenetsky method to the equation

$$A\frac{\partial^2 z}{\partial x^2}+B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}+D=0$$

where A, B, C, D are functions of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, z, x$ and y.

Moutard has treated a problem which has several connections with the Laplace method. In a paper presented to the Academy of Sciences of Paris in 1870[11], Moutard

[8]Note that other equations of type (11) were integrated, before Imschenetsky, by applying transformations identical to Imschenetsky's transformations. I cite an example Liouville's equation $\frac{\partial^2 z}{\partial x\partial y}=e^{kz}$, where k is a constant, integrated by J. Liouville (see: Goursat, "Leçons sur l'intgration des quations aux drives partielles du second ordre", T. I, page 97.)

[9]It seems to me that the Imschenetsky transformations deserve to be examined closer. Indeed, if Equation (11) can be reduced by m transformations (12) to an equation which admits a first-order intermediate integral, then we obtain an integral of the form $z=\varphi(x,y,X,X',...,X^{(m)})$, where φ is a determined function of the arguments, and there exists an equation of the form

$$\Phi\Big(\frac{\partial^m+1z}{\partial y^{m+1}},\frac{\partial^m z}{\partial y^m},...,\frac{\partial z}{\partial y},z,x,y\Big)=Y$$

providing a system in involution with Equation (11). If Equation (11) has a solution of the form $z=\varphi(x,y,X,X',...,X^{(m)})$ then, by applying transformation (12) at most m times, one obtains an equation admitting an intermediate integral of the first order depending on an arbitrary function of y. It follows that the integration of Equation (11) can be reduced by the Imschenetsky method to the integration of two first-order differential equations provided that Equation (11) admits an integral of the form $z=\varphi(x,y,X,X',...,X^{(m)})$. If Equation (11) does not admit an integral of this form, the method doesn't succeed. It would be interesting to single out Equations (11) admitting a solution of the form $z=\varphi(x,y,X,X',...,X^{(m)})$; in other words, to the conditions G, H, K that guarantee that Equation (11) can be reduced with m transformations (12) to an equation admitting a first-order intermediate integral. - The Imschenetsky transformations are not so much mentioned in the literature. Goursat (Leçons, T. II, page 263) mentions these transformations, it is true, but without motivating of their use and without indicating their connection with Laplace's method. Forsyth ("Theory of differential equations", Part IV) does not mentioned them at all. J. Hagen ("Synopsis der hoeheren Mathematik", Bd III, page 425) writes about the Imschenetsky transformation as it applies to Equation (11): "His substitution requires a quadrature. In the new variables the equation takes again the original form. The substitution can be repeated until one arrives at an equation satisfying the integrability condition". All statements are wrong.

[10]Comptes rendus, T. XCIII, 1881, page 702. Paper of Belgium's academy, 3^{ime} series, T.III 1882, pages 486-498.

[11]See Comptes Rendus, T. LXX, 1870, pages 1068-1070.

has examined which of the equations

$$\frac{\partial^2 z}{\partial x \partial y} = f(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, z, x, y)$$

have the general integral of the form

$$z = F(x, y, X + X' + ... + X^m, Y + Y' + ... + Y^r)$$

where F is a given function. In his paper, Moutard has shown that the equations

$$\frac{\partial^2 z}{\partial x \partial y} = f(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, z, x, y)$$

which admit a general integral in the above space and are not reducible to Equation (1) or to the J. Liouville equation $\frac{\partial^2 z}{\partial x \partial y} = \mathrm{e}^z$ by a changing of variables, are all reducible - except for two simple cases- to the form

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\Big(A\mathrm{e}^z\Big) - \frac{\partial}{\partial y}\Big(B\mathrm{e}^{-z}\Big),$$

where A and B are functions of x and y subjected themselves to certain conditions (Moutard's theorem). The integration of the latter equation reduces to integration of the following equation of the form (1):

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{A}\frac{\partial A}{\partial x}\frac{\partial z}{\partial y} + ABz.$$

Thus, the problem is reduced to Equation (1). Then Moutard has shown how to find the most general equation of type (1) whose general integral belongs to the space indicated above. But Moutard's original manuscript has disappeared in 1871. Only the introduction[12], where the contents is indicated and the above mentioned results are formulated (but not proved[13]), and the third part[14] on a special equation

$$\frac{\partial^2 z}{\partial x \partial y} + cz = 0$$

have been published.

[12]Comptes Rendus, T. LXX, 1870, pages 834-838. Printed later in Journal de l'École polytechnique, Cahier 56, 1886, pages 1-5.

[13]Later Moutard's theorem has been prved by Cosserat in: Darboux, *Leçons sur la théorie générale des surfaces*, Part IV, pages 405-422. See also: Forsyth, *Theory of Differential Equations*, Part IV, Ch. XV. - Moutard's theorem can also be proved by means of Imschenetsky's transformations.

[14]Journal de l'École polytechnique, Cahier 45, 1878, pages 1-11

Later, Darboux continued the studies on applications of Laplace's method to Equation (1). In his very beautiful and complete studies, Darboux[15] had notably improved and simplified the method. Today Laplace's method is usually presented after Darboux[16]. In this work, I also follow the notation of Darboux.

Darboux has demonstrated the invariance properties of the functions

$$h = \frac{\partial a}{\partial x} + ab - c, \quad k = \frac{\partial b}{\partial y} + ab - c.$$

Namely, the functions h and k don't changed under the substitution $z = \lambda \overline{z}$, where λ is any function of x and y. If one makes the change of variables $x = \varphi(\overline{x})$, $y = \psi(\overline{y})$, the functions h and k are only multiplied by $\varphi'(\overline{x})\psi'(\overline{y})$.

Under the Laplace transformations of Equation (1), the invariants of the transformed equations depend only of the invariants h and k. Namely, the invariants h_1 and k_1 of Equation (9) are

$$h_1 = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y}, \quad k_1 = h;$$

and the invariants h_{-1} k_{-1} of the equation for z_{-1} are

$$k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial x \partial y}, \quad h_{-1} = k.$$

Thus, in order to identify the Equations (1) integrable by Laplace's method, there is no necessity to compute two series of equations obtain by repeated application of Laplace's transformations to Equation (1). Instead, one can calculate step by step the invariants of the transformed equations:

$$h_1, h_2, h_3, ..., k_{-1}, k_{-2}, k_{-3}, ...$$

starting with the invariants h, k and using the recurrent formulas

$$\begin{cases} h_{i+1} + h_{i-1} = 2h_i - \frac{\partial^2 \log h_i}{\partial x \partial y}, \\ k_{i+1} = h_i, \qquad i \gtrless 0. \end{cases}$$

The necessary and sufficient condition for Equation (1) to be integrable by Laplace's method is that one of the invariants in the above series vanishes. Equation (1) admits an integral of type (4), if $h_m = 0$, and an integral of type (5) if $k_{-r} = 0$.

Darboux studied in detail the equations (1) for which Laplace's method provides the general integral. He gave the general form of Equations (1) admitting an integral of

[15]Darboux, *Leçon sur la théorie générale des surfaces*, Part II., book IV., Ch II-IX, pages 23-218.

[16]See Goursat, *Leçons,* Vol. II, pages 1-39; Forsyth, *Theory of Differential Equations,* Part IV, Vol. VI, pages 39-158.

Euler's type as well as the general form of Equations (1) admitting an integral of type (4) and an integral of type (5). The latter problem was already solved by Moutard, but since his paper was lost nobody knew how he solved this problem.

Then Darboux studied the relation between the integrals of Equation (1) and of the adjoint equation

$$\frac{\partial^2 x}{\partial x \partial y} - \frac{\partial}{\partial x}(au) - \frac{\partial}{\partial y}(bu) + cu = 0 \tag{14}$$

and showed that if the general integral of one of Equations (1) and (14) is obtained by Laplace's method, then from this integral one can derive the general integral of the other equation. – R. Liouville[17] has already indicated the remarkable property of both equations (1) and (14) that they have the same invariants but in the inverse order and that both equations can be integrated simultaneously by Laplace's method; if one of the equations is reduced, by applying the transformation (6) m times, to an equation admitting an integral of the type αX, then the other equation is reduced, by applying the transformation (7) m times, to an equation admitting an integral of the type βY. – I have cited here only some of the most important results obtained by Darboux which are interesting for us because of their connections with Laplace's method.

The Laplace method was essentially simplified by using the concept of invariants, but the calculation of the invariants $h_1, h_2, h_3, ..., k_{-1}, k_{-2}, k_{-3}, ...$ can be very complicated. Therefore, one should try to find easier way to identify the Equations (1) admitting an integral of Euler's form. The most important criterion is Goursat's one.

Goursat[18] proved that if Equation (1) admits $m + 2$ integrals such that there is no linear and homogeneous relation with constant coefficients, but there exists a homogeneous and linear relation with coefficients depending on x, then Equation (1) admits a solution of the form

$$\alpha_0 X^{(m)} + \alpha_1 X^{(m-1)} + ... + \alpha_{(m-1)} X' + \alpha_m X.$$

Comparison of the method of Laplace (Legendre) for integration of Equation (10) with the general method of Darboux[19] for integration of the equation

$$F\Big[\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, z, x, y\Big] = 0 \tag{15}$$

applied to Equation (10) shows that both methods succeed equally, namely they give the general integral of Equation (10) admitting an integral of Euler's form[20]. Indeed,

[17] Journal de l'École polytechnique, Cahier 56, 1886: Formes intégrables des équations linéaires du second order, IV, pages 49-50.

[18] American Journal of Mathematics, Vol. XVIII, 1896: "Sur les équations linéaires et la méthode de Laplace", pages 347-364. Also: Goursat, "Leçons". T. II, Chap. V, pages 21-31.

[19] For the first time given in Comptes Rendus, T. LXX, 1870, pages 675-678, 746-749; Annales de l'École normale, T. VII, 1870, pages 163-173. See: Goursat, Leçons. T. II, Chap. VII. Forsyth, Theory of Differential Equations, Part IV, Ch. XVIII.

[20] Goursat, Leçons. T. II, Chap. VII, pages 174-182.

the methods of Laplace (Legendre) and Darboux, when applied to Equation (10), have both the same goal which is to find an intermediate integral depending on an arbitrary function, i.e. to find an equation containing an arbitrary function and such that all integrals of Equation (10) are also integrals of this equation. In other words, both methods apply successfully to Equation (10), if there exists an equation with an arbitrary function which furnishes a system in involution with the equation in question for all possible forms of the arbitrary function. Laplace's method leads more directly to the goal. Since the method of Darboux gives all explicit integrals[21] depending on an arbitrary function, then each explicit integral of Equation (10) depending on an arbitrary function is an integral of Euler's form.

Methods for identifying the equations (10) that can be integrated by the method of Laplace (Legendre) are important not only for Equation (10) and for the equations whose integration can be reduced to integration of an equation of the forme (10), but also for the general partial differential equation of second order when one want to know if the method of Darboux provides the general integral of Equation (15). Darboux[22] showed that if his method provides the general integral to Equation (15), then the auxiliary equation of the form (10):

$$\frac{\partial F}{\partial \frac{\partial^2 z}{\partial x^2}}\frac{\partial^2 z'}{\partial x^2}+\frac{\partial F}{\partial \frac{\partial^2 z}{\partial x\partial y}}\frac{\partial^2 z'}{\partial x\partial y}+\frac{\partial F}{\partial \frac{\partial^2 z}{\partial y^2}}\frac{\partial^2 z'}{\partial y^2}+\frac{\partial F}{\partial \frac{\partial z}{\partial x}}\frac{\partial z'}{\partial x}+\frac{\partial F}{\partial \frac{\partial z}{\partial y}}\frac{\partial z'}{\partial y}+\frac{\partial F}{\partial z}z'=0,$$

where z' is the independent variable, can be integrated by the Laplace (Legendre) method[23].

It is natural to try to extend Laplace's method and the related studies on the second order linear equations to linear equations of higher order. Two attempts have been already done in this direction.

Le Roux [24] tried to extend Laplace's method to the nth-order linear equation:

[21] An explicit integral depending of an arbitrary function has the form

$$\begin{cases} x=F_1(\alpha,f(\alpha),f'(\alpha),...,f^m(\alpha),\beta),\\ y=F_2(\alpha,f(\alpha),f'(\alpha),...,f^m(\alpha),\beta),\\ z=F_3(\alpha,f(\alpha),f'(\alpha),...,f^m(\alpha),\beta),\end{cases}$$

where f is the arbitrary function, i.e. x, y, z are expressed as determined functions of two auxiliary variables α, β, of an arbitrary function of α and a finite number of its derivatives.

[22] Darboux, Leçons, Part IV, Note XI. Also: Goursat Leçons. T. II, Note I, pages 334-336.

[23] In this way it is often possible to find which form an equation should have in order to be integrable by the Darboux method. I cite as example the equation $\frac{\partial^2 z}{\partial x\partial y}=f(z)$, considered by Lie [See Goursat, Leçons. T. II, pages 182-186.]; the problem is easier solved by comparing the equation in question with the auxiliary equation.

[24] Bulletin de la Société Mathématique de France, T. 27, 1899, pages 237-262: Extension de la méthode de Laplace aux equations linéaires aux derivées partielles d'ordre supérieur au second.

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}A_{ij}\frac{\partial^{i+j}z}{\partial x^i\partial y^j}=0, \tag{16}$$

where the coefficients A_{ij} are function of x and y. The characteristics of Equation (16) are

$$f_i(x,y)=\text{const.},\quad \text{where}\quad \frac{\partial f_i}{\partial x}+u_i\frac{\partial f_i}{\partial y}=0,\quad (i=1,2,...,n)$$

where u_i $(i=1,2,...,n)$ are the n roots of the equation

$$\sum_{i=0}^{n}(-1)^iA_{i,n-i}u_i=0.$$

The function $f_i(x,y)$ is also called a characteristic variable. The term $\dfrac{\partial^n z}{\partial x^n}$ can be eliminated by a change of independent variables. Hence, one can assume without loss of generality that $A_{n0}=0$. Then x is a characteristic variable and Equation (16) is written

$$\varphi_p\frac{\partial^p z}{\partial x_p}+\varphi_{p-1}\frac{\partial^{p-1}z}{\partial x_{p-1}}+...+\varphi_1\frac{\partial z}{\partial x}+\varphi_0 z=0, \tag{16'}$$

where $\varphi_i=\sum\limits_{j=0}^{n-i}A_{ij}\frac{\partial^j}{\partial y^j}$ $(i=0,1,...,p)$; $p\le n-1$. If Equation (16′) admits an integral of the form (4), α_0 has to be an integral of the equation $\varphi_p\alpha=0$. By assuming as an starting idea that the essential property of the Laplace transformation, corresponding to the characteristic variable x, is that it reduces integral (4) to an integral of the same type which contains the derivatives of X at most up to the order $m-1$, Le Roux considers each transformation of the type

$$\theta_1=\alpha_0\frac{\partial}{\partial y}\left(\frac{z}{\alpha_0}\right),\quad \text{where}\quad \varphi_p\alpha_0=0, \tag{17}$$

as a Laplace transformation of Equation (16′). The transformations (17) provide the simplest transformations which can be significant for the problem in question. Le Roux considers also the completely determined transformation

$$\theta=\varphi_p z \tag{18}$$

and treats it as a Laplace transformation corresponding to the characteristic variable x, because the transformation (18) is composed of transformations of type (17).

Note that if $A_{pj}=0$ $(j=1,2,...,n-p)$, the transformation (18) is reduced to $\theta=A_{po}z$, and Equation (16′) does not admit integrals of the form (4).

Likewise Le Roux considers the Laplace transformations corresponding to other systems of characteristics. Le Roux has found that the application of a transformation (17) to Equation (16′) doesn't usually lead to an equation of the same type, but to a system of several partial derivative equations of order greater than n. Repeated application of this transformation will lead to a very complicated system of equations.

Pisati[25] showed later that the equation

$$\varphi_1 \frac{\partial z}{\partial x} + \varphi_0 z = 0, \text{ where } \quad \varphi_1 = \sum_{j=0}^{n-1} A_{1j} \frac{\partial^j}{\partial y^j}; \quad \varphi_0 = \sum_{j=0}^{n} A_{0j} \frac{\partial^j}{\partial y^j}, \qquad (19)$$

is, in general, the only equation among the equations (16′) which is reduced by transformation (17) to an equation of the same type and the same order at most. Furthermore, Pisati proved that if Equation (19) admits $m+1$ integrals such that there is no linear and homogeneous relation with constant coefficients, but there exists a homogeneous and linear relation with coefficients depending on x, then by applying the transformation

$$\theta = \varphi_1 z$$

successively $m-1$ times (should be: at most $m-1$ times) one obtains an equation admitting an integral of the form αX. (Pisati excluded the case when $A_{1,n-1} = 0$, but the statement is valid for this case as wsell). For the equation

$$\frac{\partial^3 z}{\partial x \partial y^2} + A_{11} \frac{\partial^2 z}{\partial x \partial y} + A_{10} \frac{\partial z}{\partial x} + A_{02} \frac{\partial^2 z}{\partial y^2} + A_{01} \frac{\partial z}{\partial y} + A_{00} z = 0,$$

Pisati has defined the functions of the coefficients that don't change after the substitution $z = \lambda \overline{z}$. Pisati gave also the conditions under which these invariants guarantee that the equation admits one or two integrals of the form αX.

I studied Equation (19) (independently of Pisati whose paper was not known to me until 1908) and found that almost all the results obtained by Darboux for Equation (1) can be extended to Equation (19).

Equation (19) of n order satisfying the condition $A_{1,n-1} \neq 0$ has two systems of distinct characteristics, namely:

$$x = \text{const.}$$

is a system of multiple characteristics of order $n-1$ and

$$f(x, y) = \text{const.}, \quad \text{where} \quad A_{1,n-1} \frac{\partial^2 z}{\partial y^2} + A_{0,n} \frac{\partial z}{\partial y} = 0,$$

is a system of simple characteristics. If $A_{1,n-1} = 0$, then

$$x = \text{const.}$$

[25] Rendiconti del Circolo Matematico di Palermo, T. 20, 1905, pages 344-374: Sulla estensione del metodo di Laplace alle equazioni differenziali lineari di ordine qualunque con due variabili indipendenti.

is a system of multiple characteristics of the n order. In my work I have treated separately the case when all systems of characteristics coincide and the case when only $n-1$ of them coincide.

If there are two systems of distinct characteristics, we can change the independent variables so that y will be a simple characteristic variable and x will be a characteristic variable of order $n-1$. Then Equation (19) will be written:

$$\begin{cases} \sum_{i=0}^{q} \left(A_i \frac{\partial^{i+1} z}{\partial x \partial y^i} + B_i \frac{\partial^i z}{\partial y^i} \right) = 0, \\ A_q = 1, \quad q+1 = n, \end{cases} \tag{20}$$

where A_i, B_i are functions of x and y. Equation (20) is the subject of investigation in the first four chapters.

In the first chapter I give an extension of Laplace's method to Equation (20). Laplace's method, applied to Equation (20), should be a method for recognizing if Equation (20) admits an integral of Euler's form, and if yes, the method should allow one to determine this integral or these integrals. When Equation (20) admits an integral of Euler's form, the latter have to be either of type (4) or (5). According to Le Roux, there are two completely determined Laplace transformations: the transformation

$$z_q = \sum_{i=0}^{q} A_i \frac{\partial^i z}{\partial y^i} \tag{21}$$

corresponding to the system of multiple characteristics $x = \text{const.}$, and the transformation

$$\theta = \frac{\partial z}{\partial x} + B_q z, \tag{21'}$$

corresponding to the system of simple characteristics $y = \text{const.}$ - If one applies the transformation (21) to Equation (20), one obtains the results completely similar to those obtained by the transformation (6) applied to Equation (1). Hence, one can consider the transformation (21) as a Laplace transformation of Equation (20). If Equation (20) admits an integral of the form (4), then by applying transformation (21) at most m times, one obtains an equation of the same type and the same order which admits an integral in the form αX, and the order of the transformed equation can be immediately reduced by one; each integral of type (4) can be obtained by applying the transformation (21) sufficiently many times. To obtain integrals of type (4), it is necessary to integrate only first-order linear differential equations (in exceptional cases of higher order). When Equation (20) admits s distinct integrals of type (4), the integration of Equation (20) can be reduced, by repeated application of the transformation (21), to integration of an $(n-s)$-order equation of type (20) and to integration of linear differential equations. - If one applies the transformation (21′) to Equation (20), θ is defined, as a rule, by $n-1$ linear partial differential equations of order $2(n-1)$, and a repeated application of this transformation usually leads to a complicated system of equations. But I have found a linear transformation which is analogous to Laplace's transformation (7), namely:

$$z_{-q} = \sum_{i=0}^{q-1} \Big(T_i \frac{\partial^{i+1} z}{\partial x \partial y^i} + U_i \frac{\partial^i z}{\partial y^i} \Big), \tag{22}$$

where T_i and U_i are completely determined functions of the coefficients of Equation (20). Application of the transformation (22) to Equation (20) leads to an equation of the same type and the same order, provided that Equation (20) does not have the form

$$\frac{1}{\gamma} \frac{\partial}{\partial y} \Big[\gamma \sum_{i=0}^{q-1} \Big(T_i \frac{\partial^{i+1} z}{\partial x \partial y^i} + U_i \frac{\partial^i z}{\partial y^i} \Big) \Big] = 0,$$

where γ is a function of x and y. If Equation (20) has the latter form, its order can be immediately reduced by one. When Equation (20) admits an integral of type (5), one can obtain it by means of the transformation (22) and by integrating linear differential equations, usually of the first order. The transformation (22) is inverse to the transformation (21), as the the transformation (7) is inverse to the transformation (6). It is the transformation (22) which should be considered as the completely determined Laplace transformation corresponding to the characteristics system y = const. - I designate Equation (20) by (E). By repeated application of the transformation (21) to Equation (20), one obtains a series of equations of the same type and of the order n at most; these equations are denoted by $(E_1), (E_2), (E_3), ...$ Likewise, the repeated application of the transformation (22) to Equation (20) yields a series of equations of the same type and of order the n at most; these equations are denoted by $(E_{-1}), (E_{-2}), (E_{-3}), ...$ - By examining the connection existing between Equation (20) and the adjoint equation

$$\sum_{i=0}^{q} (-1)^{i+1} \Big[\frac{\partial^{i+1}}{\partial x \partial y^i} (A_i u) - \frac{\partial^i}{\partial y^i} (B_i u) \Big] = 0, \tag{23}$$

I have obtained some results containing the results known for the equations (1) and (14). The necessary and sufficient condition for Equation (20) to admit an integral of type (5) is that the adjoint Equation (23) admits q distinct integrals of type (4); if Equation (23) admits q distinct integrals

$$\alpha_{0i} X_i^{(m_i)} + \alpha_{1i} X_i^{(m_i - 1)} + ... + \alpha_{m_i - 1, i} X_i' + \alpha_{m_i i} X_i, \quad (i = 1, 2, ..., q),$$

where X_i are arbitrary functions of x, and $\alpha_{0i}, \alpha_{1i}, ..., \alpha_{m_i - 1, i}, \alpha_{m_i i}$ are determined functions of x and y, Equation (20) admits an integral of the form

$$\beta_0 Y^{(r)} + \beta_1 Y^{(r-1)} + ... + \beta_{(r-1)} Y' + \beta_r Y, \quad \text{where} \quad r = \sum_{i=1}^{q} m_i.$$

When the adjoint Equation (23) admits s distinct integrals of type (4), one can reduce the order of Equation (20) s times by repeated application of the transformation (22). If one of the equations (20) and (23) admits an integral of type (5) and consequently the other equation admits q distinct integrals of type (4), integration of Equation (20) and Equation (23) can be reduced by the Laplace method to integration of linear differential equations. There are also some other cases when integration of equations (20) and (23) can be reduced by the Laplace method to integration of linear differential equations; a necessary (but not sufficient) condition for that is that Equation (20) admits s distinct integrals of type (4) and that Equation (23) admits at least $q-s$ distinct integrals of type (4); the necessary and sufficient condition is given by Proposition 12. - When Equation (20) admits an integral of Euler's form, the adjoint Equation (23) has an intermediate integral depending on an arbitrary function. The necessary and sufficient condition for Equation (20) to admit an integral of the form (4) is that Equation (23) admits an intermediate integral of the form

$$\sum_{j=0}^{q-1}\sum_{i=0}^{m+1} a_{ij}\frac{\partial^{i+j}u}{\partial x^i \partial y^j} = X,$$

where the coefficients a_{ij} are functions of x and y; every integral of Equation (23) is also an integral of this equation. Furthermore, the necessary and sufficient condition for Equation (20) to admit an integral of form (5) is that Equation (23) admits an intermediate integral in the form

$$\sum_{i=0}^{q+r} b_i \frac{\partial^i u}{\partial y^i} = Y,$$

where the coefficients b_i are functions of x and y.

In chapter II, I define $2q$ functions of the coefficients of Equation (20) which furnish a complete system of invariants. Two equations having the same values of invariants forming a complete system can be reduce one to another to the substitution $z = \lambda \overline{z}$. I give the relations that the invariants have to satisfy in order that Equation (20) admits s distinct integrals $\alpha_i X_i, (i = 1, 2, ..., s)$, as well as the relations that the invariants have to satisfy in order that the adjoint Equation (23) admits s distinct integrals $\alpha_i X_i, (i = 1, 2, ..., s)$. Then I give the relations existing between the invariants of Equation (20) and of Equation (E_1), when they are of order n and when of order lower than n. Similar relations between the invariants of Equation (20) and of Equation (E_{-1}) are also given. Therefore there is no necessity to compose the equations $(E_1), (E_2), (E_3), ...$ and $(E_{-1}), (E_{-2}), (E_{-3}), ...$ in order to clarify if Equation (20) or (23) admits an integral of Euler's form. Instead, one can calculate the invariants of the equations $(E_1), (E_2), (E_3), ...$ and $(E_{-1}), (E_{-2}), (E_{-3}), ...$ by starting by the invariants of Equation (20), but usually the calculations will be very complicated. – In special cases it can be easy to find the values of invariant of equations

$(E_1), (E_2), ..., (E_{-1}), (E_{-2}), ...$ As an example, I consider the equation

$$\sum_{i=0}^{q}\left[a_i\frac{\partial^{i+1}z}{\partial x\partial y^i}+(b_i+ca_iy)\frac{\partial^i z}{\partial y^i}\right]=0 \quad (a_q=1; b_q=0; a_{q-1}=0; c\neq 0),$$

where a_i, b_i, c are constants. All invariants of this equation are constants. The necessary and sufficient condition that this equation admits s distinct integrals of type (4) is that the algebraic equation

$$\sum_{i=0}^{q} d_i(mc)^i=0$$

has s positive integer roots (including zero), where the coefficients d_i are completely determined functions of the invariants of the given equation and m is the unknown quantity. The necessary and sufficient condition that the adjoint equation admits s distinct integrals of type (4) is that this algebraic equation has s roots which are negative integer numbers (including zero). – As I said (page 24), Pisati generalized Goursat's criterium. I obtained more complete results. If Equation (20) admits $m+1$ integrals ζ_i $(i=1,2,...,m+1)$ between which there exists one and only one relation of the form

$$\zeta_{m+1}=\sum_{i=1}^{m} f_i(x)\zeta_i,$$

then the $m+1$ integrals determine an equation of the form

$$\sum_{i=0}^{m}\lambda_i\frac{\partial^i\zeta}{\partial y^i}=0,$$

where λ_i are functions of x and y, which provides a system in involution with Equation (20). One can always suppose that there is no relation of the form

$$\sum_{i=1}^{m} c_i f_i(x)=c_{m+1}$$

with constant coefficients c_i $(i=1,2,...,m+1)$. Two equations:

$$\sum_{i=1}^{q} A_i\frac{\partial^i\zeta}{\partial y^i}=0, \qquad \sum_{i=1}^{m}\lambda_i\frac{\partial^i\zeta}{\partial y^i}=0$$

have at least one common solution. If the above two equations have only one common solution, then Equation (20) admits an integral of the form

$$\alpha_0X^{(m-1)}+\alpha_1X^{(m-2)}+...+\alpha_{m-2}X'+\alpha_{m-1}X,$$

and this integral can be derived directly from the integrals ζ_i $(i = 1, 2, ..., m+1)$. If the equations in question have s distinct common solutions, then Equation (20) admits s distinct integrals of type (4), and these integrals are of the form

$$\alpha_{0i} X_i^{(m_i-1)} + \alpha_{1i} X_i^{(m_i-2)} + ... + \alpha_{m_i-2,i} X_i' + \alpha_{m_i-1,i} X_i \quad (i = 1, 2, ..., s),$$

where $\sum_{i=1}^{s} m_i = m.$

In chapter III, I investigate the equations (20) admitting n distinct integrals of Euler's form. The general integral of the equations in question has the form

$$z = M \begin{vmatrix} X_1 & X_1' ... X_1^{(m_1)} & X_2 & X_2' ... X_2^{(m_2)} ...... & X_q & X_q' ... X_q^{(m_q)} & Y & Y' ... Y^{(r)} \\ x_{11} & x_{11}' ... x_{11}^{(m_1)} & x_{12} & x_{12}' ... x_{12}^{(m_2)} ...... & x_{1q} & x_{1q}' ... x_{1q}^{(m_q)} & y_1 & y_1' ... y_1^{(r)} \\ x_{21} & x_{21}' ... x_{21}^{(m_1)} & x_{22} & x_{22}' ... x_{22}^{(m_2)} ...... & x_{2q} & x_{2q}' ... x_{2q}^{(m_q)} & y_2 & y_2' ... y_2^{(r)} \\ \vdots & \vdots \quad \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \quad \vdots \\ x_{h1} & x_{h1}' ... x_{h1}^{(m_1)} & x_{h2} & x_{h2}' ... x_{h2}^{(m_2)} ...... & x_{hq} & x_{hq}' ... x_{hq}^{(m_q)} & y_h & y_h' ... y_h^{(r)} \end{vmatrix} \tag{24}$$

where $h = r + q + \sum_{i=1}^{q} m_i$; x_{ij} $(i = 1, 2, ..., h;\ j = 1, 2, ..., q)$ are determined functions of y, and M is a determined function of x and y. Conversely, every expression of the form (24), where the coefficient $Y^{(r)}$ are not equal to zero, represents the general integral of an equation of type (20) and of order equal or greater than n. The necessary and sufficient condition that an expression of the form (24), where the coefficient $Y^{(r)}$ are not equal to zero, is the general integral of an $n - s$ order equation of type (20), is that there exist s (and no more than s) distinct relations of the form

$$\sum_{i=1}^{q} f_i(x) \alpha_{0i} = 0,$$

where α_{0i} are the coefficients of $X_i^{(m_i)}$ in the determinant (24). By supposing that the general integral of Equation (20) is given in the form (24), one can derive directly the general integral of equation (E_i) $(i \leqslant 0)$, at least if equation (E_i) is of order n. Moreover, the general integral of the adjoint Equation (23) can be derived from the integral (24).

At last, I study in chapter IV the equations (20) admitting an integral of type (5). The general integral of the equations in question is given by the formula

$$z = G(\theta)$$

$$\equiv M \begin{vmatrix} \theta & \alpha_1 & \frac{\partial \alpha_1}{\partial x} & \cdots & \frac{\partial^{m_1-1}\alpha_1}{\partial x^{m_1-1}} & \cdots\cdots & \alpha_q & \frac{\partial \alpha_q}{\partial x} & \cdots & \frac{\partial^{m_q-1}\alpha_q}{\partial x^{m_q-1}} \\ \frac{\partial \theta}{\partial y} & \frac{\partial \alpha_1}{\partial y} & \frac{\partial^2 \alpha_1}{\partial x \partial y} & \cdots & \frac{\partial^{m_1}\alpha_1}{\partial x^{m_1-1}\partial y} & \cdots\cdots & \frac{\partial \alpha_q}{\partial y} & \frac{\partial^2 \alpha_q}{\partial x \partial y} & \cdots & \frac{\partial^{m_q}\alpha_q}{\partial x^{m_q-1}\partial y} \\ \cdots & \cdots & \cdots\cdots & & \cdots\cdots\cdots & & \cdots & \cdots\cdots & & \cdots\cdots \\ \frac{\partial^h \theta}{\partial y^h} & \frac{\partial^h \alpha_1}{\partial y^h} & \frac{\partial^{h+1}\alpha_1}{\partial x \partial y^h} & \cdots & \frac{\partial^{h+m_1-1}\alpha_1}{\partial x^{m_1-1}\partial y^h} & \cdots\cdots & \frac{\partial^h \alpha_q}{\partial y^h} & \frac{\partial^{h+1}\alpha_q}{\partial x \partial y^h} & \cdots & \frac{\partial^{h+m_q-1}\alpha_q}{\partial x^{m_q-1}\partial y^h} \end{vmatrix}, \tag{25}$$

where $h = \sum\limits_{i=1}^{q} m_i$; $\alpha_1, \alpha_2, ..., \alpha_q$ and M are determined functions of x and y; and finally θ is the general integral of the equation

$$\begin{vmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial^2\theta}{\partial x\partial y} & \frac{\partial^3\theta}{\partial x\partial y^2} & \cdots & \frac{\partial^{q+1}\theta}{\partial x\partial y^q} \\ \alpha_1 & \frac{\partial\alpha_1}{\partial y} & \frac{\partial^2\alpha_1}{\partial y^2} & \cdots & \frac{\partial^q\alpha_1}{\partial y^q} \\ \alpha_2 & \frac{\partial\alpha_2}{\partial y} & \frac{\partial^2\alpha_2}{\partial y^2} & \cdots & \frac{\partial^q\alpha_2}{\partial y^q} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_q & \frac{\partial\alpha_q}{\partial y} & \frac{\partial^2\alpha_q}{\partial y^2} & \cdots & \frac{\partial^q\alpha_q}{\partial y^q} \end{vmatrix} = 0.$$

Conversely, each expression of the form (25), where θ is defined as above, represents the general integral of an equation of type (20) and of order not grater than n. When there exist s (and no more than s) distinct relations of the form

$$\sum_{i=1}^{q} f_i(x) G\Big(\frac{\partial^{m_i}\alpha_i}{\partial x^{m_i}}\Big) = 0,$$

the expression (25) is the general integral of an $n - s$ order equation of type (20). By supposing that the general integral of Equation (20) is given in the form (25), one can derive directly the general integral of the equation (E_i) $(i \gtrless 0)$, at least if equation (E_i) is of order n. – I could not obtain the general formula for the general integral of equations (20) admitting q distinct integrals of type (4); in consequence, I could not derive from the integral (25) the general solution of the adjoint equation (23).

In the last chapter V, I apply the Laplace method to those of Equations (19) for which all systems of characteristics coincide. I write the equations in question as

$$\sum_{i=0}^{q} A_i \frac{\partial^{i+1} z}{\partial x \partial y^i} + \sum_{i=0}^{n} B_i \frac{\partial^i z}{\partial y^i} = 0, \quad (A_q = 1;\ B_n \neq 0;\ q+1 < n). \tag{26}$$

If Equation (26) admits an integral of Euler's form, the later should be of type (4), because x is the characteristic variable. Equation (26) admits at most q distinct integrals of this type. Consequently, the general integral cannot be represented by integrals of Euler's form. Since Equation (26) has only one system of characteristics, there is only one completely determined Laplace transformation, namely, the transformation (21). If one applies the transformation (21) to Equation (26), one obtains the same results as for Equation (20). Each integral of Euler's form can be obtained by repeated application of the transformation (21); to obtain the integrals of Euler's form, it is necessary only to integrate linear differential equations of first order (higher order in an exceptional case). If Equation (26) admits s distinct integrals of type (4), the integration of Equation (26) can be reduced, by repeated application of the transformation (21), to integration of an equation of the same type and of order $n - s$ and integration of linear differential equations. – There is also one completely determined transformation

$$z_{-q} = \sum_{i=0}^{q-1} T_i \frac{\partial^{i+1} z}{\partial x \partial y^i} + \sum_{i=0}^{n-1} U_i \frac{\partial^i z}{\partial y^i} \tag{27}$$

(the inverse transformation to the transformation (21)) which reduces Equation (26) to an equation of same type and same order, provided that Equation (26) is not in the form

$$\frac{1}{\gamma} \frac{\partial}{\partial y} \Big[\gamma \Big(\sum_{i=0}^{q-1} T_i \frac{\partial^{i+1} z}{\partial x \partial y^i} + \sum_{i=0}^{n-1} U_i \frac{\partial^i z}{\partial y^i} \Big) \Big] = 0.$$

If one applies the transformation (27) to Equation (26) and the transformation (21) to the adjoint equation

$$\sum_{i=0}^{q} (-1)^{i+1} \frac{\partial^{i+1}}{\partial x \partial y^i} (A_i u) + \sum_{i=0}^{n} (-1)^i \frac{\partial^i}{\partial y^i} (B_i u) = 0, \tag{28}$$

one obtains two equations that are again adjoint. When the adjoint equation (28) admits s distinct integrals of Euler's form, one can reduce the order of Equation (26) by s by repeated application of the transformation (27).

Archives of ALGA
Volume 3, 2006

ON CONSERVATION LAWS OF ELECTRODYNAMICS

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Translated from German by
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[E. Bessel-Hagen, Über die Erhaltungssätze der Elektrodynamik,
Mathematische Annalen, 84, 1921, pp. 258–276]

Introduction

During the winter semester 1920, at the occasion of a colloquium on mathematical questions of the theory of relativity, Mr. privy council F. Klein expressed a desire that Miss Emmy Noether's theorems on invariant variational problems, proved approximately two years ago[1], should be applied to the Maxwell equations. In short the content of these theorems is the following: from the invariance of a variational problem with respect to a continuous transformation group follows a number of relations which are fulfilled identically in virtue of the differential equations of the problem. In the case of one independent variable they describe first integrals. In the case of a finite group these relations have the form of so-called "conservation laws".

The Maxwell equations are, as well known now, invariant with respect to a finite 10-parameter group, the so-called Lorentz group. It consists of the real "movements" of the four-dimensional $x,\ y,\ z,\ t$-space which leave invariant the infinitesimal deformation of the surface

$$x^2 + y^2 + z^2 - c^2t^2 = 0\ .$$

Moreover H. Bateman[2] discovered in 1909 that the Maxwell equations are invariant with respect to an even wider group of transformations, namely the group which does not change the equation

$$dx^2 + dy^2 + dz^2 - c^2\,dt^2 = 0$$

[1] Göttinger Nachrichten 1918, p. 235 et seq., in what follows quoted brifly E. Noether. See also Felix Klein, Gesammelte mathematische Abhandlungen **1**, p. 585, Berlin 1921.

[2] Proc. London Math. Soc. (2), **8**, p. 228 et seq. In this and the preceding volume of this journal there are more investigations of Bateman and Cunningham about the significance of our $\mathfrak{G}_{15}$ for physics. See also F. Klein Ges. math. Abhandl. **1**, p. 552.

and does not change the signature of four-dimensional figures[3]. If one denotes the variables x, y, z, ict by x_1, x_2, x_3, x_4, this group will correspond (ignoring the question whether the parameters are real) to the largest subgroup of the 15-parameter group of transformations of reciprocal radii, the so-called conformal group [4] in R_4. As J. L. Larmor[5] has already mentioned the Maxwell equations can be obtained from a variational problem. And since, as will be shown later, this problem is invariant with respect to the mentioned group $\mathfrak{G}_{15}$, E. Noether's theorems must supply us with 15 linearly independent electrodynamic conservation laws. The aim of this paper is to obtain these conservation laws.

The first seven of them (cf. formulae (27) a_r, a_z and b_r) are the well-known theorems about the conservation of energy, momentum and angular momentum[6]; so I don't go further in their interpretation. The three following (27 b_z) provide an exact analogy of the second center-of-mass theorem of classical mechanics and were obtained for the *electrodynamic* phenomena, to my knowledge, for the first time by A. Einstein[7] by formal integration of the Maxwell equations. Einstein claimed their validity only in the first order approximation because he didn't know at that time if the Lorentz group is appropriate for the dynamics of the theory of relativity. G. Herglotz[8] established the similar formulae for the *continuum mechanics* in terms of the theory of relativity and identified them precisely as center of mass theorems. His approach is similar to the method used in the present paper. The five other formulae (27 c, d_r and d_z) are, to the best of my knowledge, new. The future will show if they have any physical significance.

§ 1. E. Noether's theorem

First I formulate both of E. Noether's theorems in a more general form that it is written in her paper cited above. I owe this generalization to an oral communication from Emmy Noether. Consider an integral

$$I_x = \int \dots \int f(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots)\, dx, \tag{1}$$

[3]Bateman calls it "spherical wave transformations".

[4]Details on the conformal group can be found in S. Lie and G. Scheffers, Geometrie der Berührungstransformationen, Leipzig 1896, ch. 10, §1 and §2.

[5]Aether and matter, Cambridge 1900, §50, p. 83 et seq. See also F. Klein, Seminarvorträge über die Entwicklung der Mathematik im neunzehnten Jahrhundert, ch.X, vol. II, §4 (1917). (The detailed version of these lectures is available as a transcript in many mathematical departments of universities.)

[6]See also M. v. Laue, Die Relativitätstheorie **1**, 4th ed., Braunschweig 1921, §15 b-e.

[7]Ann. d. Phys. (4), **20** (1906), p. 627 et seq.

[8]Ann. d. Phys. (4), **36** (1911), p. 493 et seq.. Compare especially the formulae 96' on p. 513

taken over an arbitrary domain of the real variables $x_1,\ x_2, \ldots,\ x_n$. Here $u,\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2}, \ldots$ are abbreviations for μ real valued functions of $x_1, \ldots,\ x_n$ and their partial derivatives[9], and dx stands for $dx_1\, dx_2 \ldots\, dx_n$. A one-to-one transformation of variables

$$\begin{cases} y_i &= A_i(x, u, \frac{\partial u}{\partial x}, ...) \quad [i \ = 1, 2, ..., n], \\ v_\varrho(y) &= B_\varrho(x, u, \frac{\partial u}{\partial x}, ...) \quad [\varrho \ = 1, 2, ..., \mu] \end{cases} \tag{2}$$

considered together with its extension to the derivatives $\frac{\partial v}{\partial y},\ \frac{\partial^2 v}{\partial y^2}, \ldots$ maps (1) into

$$I_y = \int \ldots \int \bar{f}(x, v, \tfrac{\partial v}{\partial y}, \tfrac{\partial^2 v}{\partial y^2}, \ldots)\, dy,$$

where the integral is taken over the y-domain corresponding to the x-domain in (1). If the function $\bar{f}$ is identical with the function f, then I is said to be invariant under the transformation (2).

Now we consider a continuous group of transformations (2). We assume that the parameters $\varepsilon_1,\ \varepsilon_2, \ldots,\ \varepsilon_\varrho$ in the case of a finite group $\mathfrak{G}_\varrho$ [and the arbitrary functions $p^{(1)}(x),\ p^{(2)}(x), \ldots,\ p^{(\varrho)}(x)$ in the case of an infinite group $\mathfrak{G}_{\infty\varrho}$] are chosen so that the identical transformation corresponds to the values $\varepsilon = 0$ [resp. to the functions $p(x) \equiv 0,\ \frac{\partial p(x)}{\partial x} \equiv 0, \ldots$]. Then the transformation (2) takes the form

$$\begin{cases} y_i &= x_i + \Delta x_i + \ldots \quad [i \ = 1, 2, ..., n], \\ v_\varrho(y) &= u_\varrho + \Delta u_\varrho + \ldots \quad [\varrho \ = 1, 2, ..., \mu], \end{cases} \tag{3}$$

where $\Delta x_i,\ \Delta x_\varrho$ are linear[10] in ε [resp. in p and their derivatives]. If we keep in the right-hand sides of (3) only these linear members we obtain what is called by Lie *infinitesimal transformations.* The invariance of the integral I with respect to an infinitesimal transformation means that $\bar{f}$ and f differ only in terms which are at least of the second order in ε [resp. in $p,\ \frac{\partial p}{\partial x}, \ldots$].

A divergence is an expression of the form

$$\text{Div}\, A = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \ldots + \frac{\partial A_n}{\partial x_n}\,,$$

where A_i are functions of $x,\ u,\ \frac{\partial u}{\partial x}, \ldots$. The differentiations with respect to x are total, i.e. $u,\ \frac{\partial u}{\partial x}, \ldots$ are considered as functions of x.

I will say that I is "invariant up to a divergence" with respect to an infinitesimal transformation if

$$\bar{f} = f + \text{Div}\, C + \text{higher-order members}, \tag{4}$$

[9]On admissibility of complex variables, see E. Noether, p. 237, footnote 3.

[10]See E. Noether, p. 244 and p. 246, beginning of §4.

where the expression C is *linear* in ε [resp. in p, $\frac{\partial p}{\partial x}, \ldots$]. The case when C is identically zero is included in this notion[11]. The introduction of this notion is the generalization (mentioned in the beginning of this section) of Emmy Noether's original publication.

Now we can formulate E. Noether's theorems as follows.

If the integral I is invariant up to a divergence with respect to infinitesimal transformations of a finite group $\mathfrak{G}_\varrho$ then ϱ linearly independent combinations of Lagrange's expressions become divergences.

Namely, let us set

$$\delta u_i = v_i(x) - u_i(x) = \Delta u_i - \sum_\lambda \frac{\partial u_i}{\partial x_\lambda} \Delta x_\lambda \tag{5}$$

and define $A_1, \ldots, A_n$ by the equation

$$\sum \psi_i \, \delta u_i = \delta f + \mathrm{Div}\, A,$$

where ψ_i are Lagrange's expressions of the function f and $B_1, \ldots, B_n$ are defined by

$$B_i = C_i + A_i - f \Delta x_i \, . \tag{6}$$

Then one separates δu and B with respect to the several ε:

$$\begin{aligned} \delta u_i &= \varepsilon_1 \delta^{(1)} u_i + \ldots + \varepsilon_\varrho \delta^{(\varrho)} u_i, \\ B_i &= \varepsilon_1 B_i^{(1)} + \ldots + \varepsilon_\varrho B_i^{(\varrho)}, \end{aligned}$$

and the wanted divergence relations become[12]

$$\sum \psi_i \, \delta^{(1)} u_i = \mathrm{Div}\, B^{(1)}, \ldots \quad \sum \psi_i \, \delta^{(\varrho)} u_i = \mathrm{Div}\, B^{(\varrho)}. \tag{7}$$

Conversely, let us assume that for given Lagrange expressions there are suitable functions $\delta\, u$ and B such that there exist precisely ϱ linearly independent relations (7). Then one can find[13] ϱ linearly independent infinitesimal transformations leaving I invariant up to a divergence. Since the separation of B into C and $A - f\,\Delta\, x$ is possible in different ways, there are various possibilities to specify such infinitesimal transformations. However, one can easily verify that there is only one possibility of the mentioned separation such that the resulting infinitesimal transformations are independent of $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots$ whenever the functions $\delta\, u$ do not depend on $\frac{\partial^2 u}{\partial^2 x}, \frac{\partial^3 u}{\partial x^3}, \ldots$ and, in

[11] The usual invariance with respect to an infinitesimal transformation T implies the usual invariance with respect to the one-parameter group generated by T. The same statement, in general, is not true for the invariance up to a divergence. Therefore the definition of this notion must be necessarily linked to infinitesimal transformations.

[12] See E. Noether, § 2, p. 242.

[13] As it is done by E. Noether, § 3.

addition, they also do not depend on $\frac{\partial u}{\partial x}$ or dependent in a very special way[14]. If these conditions are satisfied, one can prove that the ϱ infinitesimal transformations, which are obtained, generate exactly a ϱ-parameter group.

For further applications, I provide also the expression for B_i in the case when f depends on derivatives $\frac{\partial u}{\partial x}$ of the first order only:

$$B_i = C_i - \sum_k \frac{\partial f}{\partial \frac{\partial u_k}{\partial x_i}} \Delta u_k + \sum_\lambda \Delta x_\lambda \Big(\sum_k \frac{\partial f}{\partial \frac{\partial u_k}{\partial x_i}} \frac{\partial u_k}{\partial x_\lambda} - \delta_{\lambda i} f \Big), \tag{8}$$

$$\delta_{\lambda i} = \begin{cases} 0, & \text{if} \quad \lambda \neq i, \\ 1, & \text{if} \quad \lambda = i. \end{cases}$$

The second theorem corresponds to an infinite continuous group $\mathfrak{G}_{\infty \varrho}$ and says that:

The invariance of I up to a divergence with respect to the infinitesimal transformations of $\mathfrak{G}_{\infty \varrho}$ provides ϱ linearly independent relations between ψ_i and their total derivatives with respect to x. Conversely, existence of ϱ such linearly independent relations leads to the invariance of I up to a divergence with respect to certain ϱ infinitesimal transformations with ϱ arbitrary functions.

To find the mentioned relations one writes equation (5) in the expanded form:

$$\delta u_i = \sum_{\lambda=1}^{\varrho} \Big\{ a_i^{(\lambda)}(x, u, \ldots) p^{(\lambda)}(x) + b_i^{(\lambda)}(x, u, \ldots) \frac{\partial p^{(\lambda)}}{\partial x} + \cdots + c_i^{(\lambda)}(x, u, \ldots) \frac{\partial^\sigma p^{(\lambda)}}{\partial x^\sigma} \Big\}. \tag{9}$$

Then the relations are written in the simple form[15]

$$\sum_i \Big\{ (a_i^{(\lambda)} \psi_i) - \frac{\partial}{\partial x}(b_i^{(\lambda)} \psi_i) + \ldots + (-1)^\nu \frac{\partial^\sigma}{\partial x^\sigma}(c_i^{(\lambda)} \psi_i) \Big\} = 0 \quad [\lambda = 1, 2, \ldots, \varrho]. \tag{10}$$

§ 2. Application to the n-body problem

As a first illustration of utility of E. Noether's theorem let us apply it for derivation of the known ten integrals in the n-body problem. Although the underlying idea and its

---

[14]Namely, $\delta\, u_i = \alpha_i(x, u) + \sum\limits_\lambda \beta_\lambda\,(x, u) \frac{\partial u_i}{\partial x_\lambda}$ · Then one obtains: $\Delta x_i = -\beta_i(x, u)$, $\Delta u_i = \alpha_i(x, u)$, $C_i = \frac{A_i - B_i}{f} + \beta_i(x, u)$.

[15]E. Noether, § 2, p. 243.

accomplishment are not new[16], I will give the short calculations completely in order to provide the formal analogy with the electrodynamic conservation equations listed further. The differential equations of the n-body problem are derived from the variational problem

$$\delta \overline{\underline{\int}} L \, dt = 0,$$

where the bars indicate that the variation is taken in fixed limits. The Lagrange function L has the form

$$\begin{aligned}
L = & \; T - U, \\
T = & \; \tfrac{1}{2} \sum_{i=1}^{n} m_i (\dot{x}_{i1}^2 + \dot{x}_{i2}^2 + \dot{x}_{i3}^2), \\
U = & \; -\sum \tfrac{\kappa \, m_i \, m_k}{r_{ik}}, \qquad\qquad 1 \leq i < k \leq n, \\
r_{ik} = & \; \sqrt{(x_{i1} - x_{k1})^2 + (x_{i2} - x_{k2})^2 + (x_{i3} - x_{k3})^2}, \\
\kappa = & \; \text{gravitational constant,}
\end{aligned}$$

and the x_{ik} are regarded as functions of t. Thus, we have here a *single* integral, where t replaces x and the x_{ik} take the place of the former u.

It is well known that the equations of motion in the n-body are invariant with respect to a finite ten-parameter group known the "Galilei-Newton group". Under this group, the variational problem behaves so that L is invariant with respect to some of the infinitesimal transformations and invariant up to a divergence with respect to other infinitesimal transformations of the group. Namely, the following holds:

[16]This question has been also discussed by F. Engel, Gött. Nachr. 1916, p. 270 et seq. He uses Lie's methods, however without using the advantageous circumstance that the differential equations originate from a variational problem. – The comparison should express the preference of the variational approach clearly. – One should compare the appropriate places in Jacobi's Lectures on Dynamics for the historical development of the comprehension of the meaning and relationships of the ten integrals for the equations of motion; see also the interesting remark by J. R. Schütz, published in Gött. Nachr. 1897, p. 110 et seq., as well as a survey by F. Klein in "Die Entwicklung der Mathematik im neunzehnten Jahrhundert" (The development of mathematics in the 19th century), chap. 10, A § 2 and C § 4, 1917.

$$
\begin{array}{llll}
a) & \Delta t = \tau, & \Delta x_{ik} = 0, & \\
b) & \Delta t = 0, & \Delta x_{ik} = \alpha_k, & \\
c) & \Delta t = 0, & \Delta x_{ik} = \sum\limits_{\varrho=1}^{3} \beta_{k\varrho} x_{i\varrho} & \left(\begin{array}{l} \beta_{kk} = 0 \\ \beta_{k\varrho} = -\beta_{\varrho k} \end{array} \right), \\
d) & \Delta t = 0, & \Delta x_{ik} = \gamma_k t & [k = 1, 2, 3].
\end{array}
\tag{11}
$$

One can verify that in the cases a), b), c) we have $\Delta L = 0$, whereas in the case d)

$$
\Delta L = \frac{d}{dt}\Big(\sum_{i=1}^{n} \sum_{k=1}^{3} m_i \gamma_k x_{ik} \Big) = \frac{d}{dt} C = \text{Div } C.
$$

Furthermore, the equations (5) and (8) are written

$$
\begin{aligned}
\delta x_{ik} = {} & \Delta x_{ik} - \dot{x}_{ik} \Delta t, \\
B = {} & C - \sum_{i=1}^{n} \sum_{k=1}^{3} m_i \dot{x}_{ik} \Delta x_{ik} + \Delta t (T + U),
\end{aligned}
$$

and E. Noether's divergence relations take the form:

$$
\begin{aligned}
& a) \ -\sum_{i=1}^{n} \sum_{k=1}^{3} \dot{x}_{ik} \psi_{ik} = \frac{d}{dt}(T + U), \\
& b) \ \sum_{i=1}^{n} \psi_{ik} = -\frac{d}{dt} \sum_{i=1}^{n} m_i \dot{x}_{ik} \quad [k = 1, 2, 3], \\
& c) \ \sum_{i=1}^{n} (x_{i\mu} \psi_{i\nu} - x_{i\nu} \psi_{i\mu}) = -\frac{d}{dt} \sum_{i=1}^{n} m_i (x_{i\mu} \dot{x}_{i\nu} - x_{i\nu} \dot{x}_{i\mu}) \\
& \qquad\qquad [(\mu, \nu) = (2, 3), (3, 1), (1, 2)], \\
& d) \ \sum_{i=1}^{n} t \psi_{ik} = \frac{d}{dt} \Big[\sum_{i=1}^{n} m_i x_{ik} - t \sum_{i=1}^{n} m_i \dot{x}_{ik} \Big] \quad [k = 1, 2, 3].
\end{aligned}
\tag{12}
$$

So far we have only *formal* identities which can be verified directly by substituting

$$\psi_{ik} = \frac{\partial L}{\partial x_{ik}} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}_{ik}}\Big) = \sum_{\substack{\nu=1\\ \nu\neq i}}^{n}{}' \frac{\kappa m_i m_\nu}{r_{i\nu}^3}(x_{\nu k} - x_{ik}) - \frac{d}{dt}(m_i \dot{x}_{ik}).$$

We did not use yet the equation $\delta \overline{\underline{\int}}\, L\, dt = 0$. But if we consider the *differential equations* of the n-body problem, we have to set the Lagrange expressions ψ_i equal to zero. Then the equations (12) whose left-hand sides vanish, provide us the ten known first integrals of the problem. Namely, (12 a) is the energy theorem, b) the first three center-of-mass theorems (also known as the momentum theorems), c) the three area theorems, and d) the three second center-of-mass theorems. The form of the latter differs from the usual formulation which can be obtained, however, by merely noting that according to (12 b) one has $\sum_i^n m_i \dot{x}_{ik} = c_k$, whence:

$$\sum_{i=1}^{n} m_i x_{ik} = c_k t + c_k{}' \qquad [k = 1, 2, 3]. \qquad (c_k, c_k{}' = \text{constants.}) \tag{13}$$

For us, however, the form (12 d) is useful by two reasons. First, because it shows that the second center-of-mass theorems are are properly arranged in the sequence of the other conservation equations, and second, because they give us a key for an interpretation of the similar electrodynamic relations (28).

§ 3. Survey of the notations used in what follows

Before proceeding to the electrodynamic equations I give a survey of the notations used in what follows. In general, I adopt the notation used by M. v. Laue in his book “Die Relativitätstheorie”, Vol. I, as well as his symbolism for three- and four-dimensional vector and tensor calculus (even though they are annoying), so that the reader can check all symbols which are unknown to him. The Lorentz measurement system[17] is adopted in CGS-units, c denotes, as usually, the light velocity.

Table of the notations[18]

In four dimensional notation	In three dimensional notation
x_1, x_2, x_3, x_4	$x, y, z, i\,ct$

[17]See also Encycl. d. math. Wissenschaften 5, article 13, 7 d.

[18]The comparative columns contain not all corresponding quantities because it will be confusing to include symbols that will not be used later.

In four dimensional notation	In three dimensional notation
	$\mathfrak{r} =$ vector from the origin to a fixed point in space, not to a moving particle.
Electromagnetic six tensor: $f: \quad f_{23}, f_{31}, f_{12}\,; \quad f_{14}, f_{24}, f_{34}$ $f_{ik} = -f_{ki}$	$\mathfrak{H}_x, \mathfrak{H}_y, \mathfrak{H}_z; \quad -i\mathfrak{E}_x, -i\mathfrak{E}_y, -i\mathfrak{E}_z$
The corresponding dual six tensor: $f^*: \quad f^*_{12} = f_{34}, \quad f^*_{13} = f_{42}, \quad f^*_{14} = f_{23}$ $f^*_{23} = f_{14}, \quad f^*_{24} = f_{31}, \quad f^*_{34} = f_{12}$	
Four-potential: $\varphi: \varphi_1, \varphi_2, \varphi_3, \varphi_4$	
Analogy to the Lagrangian function: $\Lambda = \frac{1}{4}\sum_{i=1}^{4}\sum_{k=1}^{4} f_{ik}^2 = \frac{1}{2}\sum_{1\le i<k\le 4} f_{ik}^2$	$\frac{1}{2}\left(\mathfrak{H}^2 - \mathfrak{E}^2\right)$
Electromagnetic energy-momentum tensor: $S_{ik} = S_{ki} = \sum_{\lambda=1}^{4} f_{i\lambda} f_{\lambda k} + \delta_{ik}\Lambda$ $\delta_{ik} = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases}$	$\begin{pmatrix} \boldsymbol{p}_{exx} & \boldsymbol{p}_{exy} & \boldsymbol{p}_{exz} & \frac{i}{c}\mathfrak{S}_{ex} \\ \boldsymbol{p}_{eyx} & \boldsymbol{p}_{eyy} & \boldsymbol{p}_{eyz} & \frac{i}{c}\mathfrak{S}_{ey} \\ \boldsymbol{p}_{ezx} & \boldsymbol{p}_{ezy} & \boldsymbol{p}_{ezz} & \frac{i}{c}\mathfrak{S}_{ez} \\ \frac{i}{c}\mathfrak{S}_{ex} & \frac{i}{c}\mathfrak{S}_{ey} & \frac{i}{c}\mathfrak{S}_{ez} & -W_e \end{pmatrix}$

In four dimensional notation	In three dimensional notation
	$\boldsymbol{p}_e =$ density of Maxwell's tensions
	$\mathfrak{S}_e = c[\mathfrak{E}, \mathfrak{H}]$
	$=$ the Poynting vector of the electromagnetic energy flow
	$W_e = \frac{1}{2}(\mathfrak{E}^2 + \mathfrak{H}^2)$
	$=$ density of the electromagnetic field energy
	$\mathfrak{g}_e = \frac{\mathfrak{S}_e}{c^2}$
	$=$ density of the electromagnetic momentum of the field
Density of the four-dimensional flow: $P_1,\ P_2,\ P_3,\ P_4$	$\frac{\varrho\mathfrak{q}_x}{c},\ \frac{\varrho\mathfrak{q}_y}{c},\ \frac{\varrho\mathfrak{q}_z}{c},\ i\varrho$
	$\varrho =$ spatial density of the electric charge
	$\mathfrak{q} =$ velocity vector of the electric charges and of their material carrier respectively
Density of the electric four-dimensional force: $F_1,\ F_2,\ F_3,\ F_4$ $F_i = \sum_k f_{ik} P_k$	Density of the field force to the charges: $\mathfrak{F} = \varrho\Big(\mathfrak{E} + \frac{1}{c}[\mathfrak{q}, \mathfrak{H}]\Big)$

In four dimensional notation	In three dimensional notation
	Density of the power of this force: $(\mathfrak{F}\,\mathfrak{q}) = \varrho\,(\mathfrak{E}\,\mathfrak{q})$
Mechanical energy-momentum tensor: $R_{ik} = R_{ki}$	$\left(\begin{array}{c\|c} [[\mathfrak{g}_m, \mathfrak{q}]] & ic\mathfrak{g}_m \\ \hline \frac{i}{c}\mathfrak{S}_m & -W_m \end{array}\right)$
	$\mathfrak{g}_m =$ density of the mechanical momentum
	$= \frac{k_0\,\mathfrak{q}}{\sqrt{1-\frac{\mathfrak{q}^2}{c^2}}}$
	$k_0 =$ mass density
	$W_m =$ density of the kinetic energy of moving matter
	$= \frac{k_0\,c^2}{\sqrt{1-\frac{\mathfrak{q}^2}{c^2}}}$
	$\mathfrak{S} = \mathfrak{q}\,W_m$
	$=$ density of the energy flow which appears due to the movement of the matter
Total energy-momentum tensor: $T_{ik} = T_{ki} = R_{ik} + S_{ik}$	$\left(\begin{array}{c\|c} \boldsymbol{p} & ic\mathfrak{g} \\ \hline \frac{i}{c}\mathfrak{S}_m & -W \end{array}\right)$
	$\boldsymbol{p} = \boldsymbol{p}_e + [[\mathfrak{g}_m, \mathfrak{q}]] =$ total tension tensor
	$\mathfrak{g} = \mathfrak{g}_e + \mathfrak{g}_m =$ total momentum density
	$\mathfrak{S} = \mathfrak{S}_e + \mathfrak{S}_m =$ total energy flow
	$W = W_e + W_m =$ total energy density

§ 4. Invariance of the Maxwell equations under the conformal group

In the two systems of Maxwell equations in vacuum:

$$I.\quad \Delta iv f^* = 0 \qquad II.\quad \Delta iv f = 0 \tag{14}$$

the first will be satisfied identically upon setting

$$f = \mathfrak{Rot}\ \varphi. \tag{15}$$

If one inserts this in II., the left-hand sides of II. will become exactly the Lagrange expressions ψ_i of the variational problem

$$\delta \overline{\int\!\!\int\!\!\int\!\!\int} \Lambda\, dx_1\, dx_2\, dx_3\, dx_4 = 0,$$

where $x_1,\ x_2,\ x_3,\ x_4$ are the independent variables and $\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4$ are unknown functions. The variation should be accomplished with fixed boundaries (so that the horizontal bars are unaltered) and fixed boundary values of φ. The integral in question remains unaltered when the variables $x_1, \ldots, x_4$ undergo an arbitrary transformation of the 15-parameter conformal group of R_4 provided that the components $\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4$ of the four-potential transform contragradiently to the differentials $dx_1,\ dx_2,\ dx_3,\ dx_4$. Since we deal only with formal operations, we don't have to worry about the question (though important from the physical point of view) if the parameters of the group are real numbers. One can easily verify the invariance of the integral by considering the condition that the quantities $f_{ik} = \frac{\partial \varphi_k}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_k}$ transform contragradiently to $dx_i\, dx_k$. This can be done by calculating the expression obtained from the expression

$$\Big(\sum_{i,k} f_{ik}^2\Big)\, dx_1\, dx_2\, dx_3\, dx_4$$

after an arbitrary linear transformation of dx. One can show that the new expression will have the same form in the new variables:

$$\Big(\sum_{i,k} \bar{f}_{ik}^2\Big)\, d\bar{x}_1\, d\bar{x}_2\, d\bar{x}_3\, d\bar{x}_4$$

if and only if the transformation maps the equation $\sum_i dx_i^2 = 0$ into the same equation, $\sum_i d\bar{x}_i^2 = 0$. These transformations form exactly the conformal group.

Along with this finite continuous group, the variational problem admits also an infinite group containing the first derivatives of an arbitrary function:

$$\overline{x}_i = x_i, \quad \overline{\varphi}_i = \varphi_i + \frac{\partial p}{\partial x_i} \qquad [i = 1, 2, 3, 4],$$

since the rotation (see 15) of a gradient vanishes identically.

§ 5. List of formal identities

According to E. Noether's theorems, 15 linearly independent linear combinations of the Lagrange expressions

$$\psi_i = \sum_k \frac{\partial f_{ik}}{\partial x_k} = \sum_k \frac{\partial}{\partial x_k}\Big(\frac{\partial \varphi_k}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_k}\Big) \tag{16}$$

will become divergences. Furthermore, one relation between ψ_i and their first-order partial derivatives must be satisfied identically.

A system of 15 linearly independent infinitesimal transformations of our $\mathfrak{G}_{15}$ is[19]:

$$\begin{aligned}
&a) \quad \Delta x_k = \alpha_k, \\
&b) \quad \Delta x_k = \sum_\varrho \beta_{k\varrho} x_\varrho \qquad \begin{pmatrix} \beta_{kk} = & 0 \\ \beta_{k\varrho} = & -\beta_{\varrho k} \end{pmatrix}, \\
&c) \quad \Delta x_k = \gamma x_k, \\
&d) \quad \Delta x_k = 2x_k \sum_\varrho \varepsilon_\varrho x_\varrho - \varepsilon_k \sum_\varrho x_\varrho^2 \qquad [k = 1,2,3,4].
\end{aligned} \tag{17}$$

The transformation (17 a) with $k = 4$ corresponds to the transformation (11 a) of the Galilei-Newton group, three other (17 a) correspond to the spatial translations (11 b). The "spatial rotations" corresponding to the parameters β_{23}, β_{31}, β_{12} in (17 b) provide the rotations (11 c) of the Galilei-Newton group, while the "time-like rotations" associated with β_{14}, β_{24}, β_{34} match (11 d) in the Galilei-Newton group with a changed direction of t-axis and fixed x, y, z-space. The formulae (17 c) and d) result from combinations of transformations of reciprocal radii. Finally, the transformation of φ contragradient to dx is simply given by the formula

$$\Delta\varphi_k = -\sum_\lambda \frac{\partial \Delta x_\lambda}{\partial x_k}\,\varphi_\lambda \qquad [k = 1,2,3,4] \tag{18}$$

and the infinitesimal transformation of the infinite continuous group by

$$\Delta x_k = 0 \qquad \Delta\varphi_k = \frac{\partial p}{\partial x_k}\,\cdot$$

Let us consider the dependence corresponding to the last transformation. According to (5) we have

$$\delta\,\varphi_i = \frac{\partial p}{\partial x_i}\,,$$

[19]See, e.g. S. Lie and F. Engel, Theorie der Transformationsgruppen **3**, Leipzig 1893, p. 281.

and comparison with (9) and (10) yields:

$$\sum_i \frac{\partial}{\partial x_i}\psi_i = 0. \tag{19}$$

In order to find the divergence relations, we apply (8) and obtain

$$B_i = -\sum_k f_{ik}\Delta\varphi_k + \sum_\lambda \Delta x_\lambda \Big\{ \sum_k f_{ik}\frac{\partial \varphi_k}{\partial x_\lambda} - \delta_{\lambda i}\Lambda \Big\}.$$

Invoking the definition of the electromagnetic energy-momentum tensor S_{ik}, we have:

$$B_i = -\sum_k f_{ik}\Delta\varphi_k + \sum_\lambda \Delta x_\lambda \Big\{ - S_{\lambda i} + \sum_k f_{ik}\frac{\partial \varphi_\lambda}{\partial x_k} \Big\}. \tag{20}$$

If we insert here the expressions for $\Delta x,\ \Delta\varphi$ given by (17) and (18), we will arrive at divergence relations, that are very long and complicated[20]. But te main lack of the relations is that they contain the components of the four-potential φ explicitly, and not only via the combinations f_{ik} having physical significance. The components of φ are only auxiliary mathematical quantities and have no autonomous real physical meaning. However, we can overcome this obstacle by the following trick. Note that one can single out from the infinite continuous group finite groups by letting the function p be not arbitrarily, but assuming that p depends on a finite number of parameters. This can be done in many different ways. The function p being specified, we add to the transformation (18) the expression $\frac{\partial p}{\partial x_k}$ and obtain an infinitesimal transformations that yield to divergence relations where φ will appear only in the combinations f_{ik}. In order to clarify how to specify p, let us make the calculations.

If we insert in (20) the following expression for $\Delta\varphi_k$:

$$-\sum_\lambda \frac{\partial \Delta x_\lambda}{\partial x_k}\,\varphi_\lambda + \frac{\partial p}{\partial x_k}\ , \tag{21}$$

we obtain:

$$\begin{aligned} B_i = &\ \textstyle\sum_k f_{ik}\big\{ \sum_\lambda \frac{\partial \Delta x_\lambda}{\partial x_k}\,\varphi_\lambda - \frac{\partial p}{\partial x_k} + \sum_\lambda \Delta x_\lambda \frac{\partial \varphi_\lambda}{\partial x_k} \big\} - \sum_\lambda S_{\lambda i}\Delta x_\lambda \\ = &\ \textstyle\sum_k f_{ik}\,\frac{\partial}{\partial x_k}\big(\sum_\lambda \varphi_\lambda \Delta x_\lambda - p \big) - \sum_\lambda S_{\lambda i}\Delta x_\lambda. \end{aligned}$$

It is clear now that the appropriate choice for p is

[20] As an example, I give here their expressions provided by (17 d):

$$\textstyle\sum_i \frac{\partial}{\partial x_i}\big\{2f_{ik}\sum_r x_r\varphi_r - 2x_ix_k\Lambda + \sum_r f_{ir}\big[2(x_k\varphi_r - x_r\varphi_k) + 2x_k\sum_s x_s\frac{\partial\varphi_r}{\partial x_s} - \frac{\partial \varphi_r}{\partial x_k}\sum_s x_s^2\big]\big\} + \frac{\partial}{\partial x_k}\big(\Lambda\sum_s x_s^2\big) = -\sum_i \psi_i\big\{2(x_k\varphi_i - x_i\varphi_k) + 2x_k\sum_s x_s\frac{\partial \varphi_i}{\partial x_s} - \frac{\partial\varphi_i}{\partial x_k}\sum_s x_s^2\big\} - 2\psi_k\sum_r x_r\varphi_r \qquad [k=1,2,3,4].$$

$$p = \sum_\lambda \varphi_\lambda \, \Delta x_\lambda \, .$$

In accordance with (5) and (21)

$$\delta \, \varphi_i = \sum_\lambda f_{i\lambda} \, \Delta x_\lambda \, ,$$

and hence the divergence relation takes the form

$$\sum_i \sum_\lambda \psi_i f_{i\lambda} \Delta x_\lambda = \sum_i \frac{\partial}{\partial x_i} \Big(\sum_\lambda S_{\lambda i} \Delta x_\lambda \Big). \tag{22}$$

Substituting the expressions (17), we finally obtain:

$$\begin{aligned}
&a) \quad \sum_i \tfrac{\partial}{\partial x_i} S_{\lambda i} = \sum_i \psi_i f_{i\lambda} \qquad\qquad [\lambda = 1,2,3,4], \\
&b) \quad \sum_i \tfrac{\partial}{\partial x_i} \Big(x_\mu S_{\nu i} - x_\nu S_{\mu i} \Big) = \sum_i \psi_i \Big(x_\mu f_{i\nu} - x_\nu f_{i\mu} \Big) \\
&\qquad\qquad [(\mu, \nu) = (1,2),(1,3),(1,4),(2,3),(2,4),(3,4)], \\
&c) \quad \sum_i \tfrac{\partial}{\partial x_i} \Big(\sum_\varrho x_\varrho S_{\varrho i} \Big) = \sum_i \psi_i \Big(\sum_\varrho x_\varrho f_{i\varrho} \Big) \, , \\
&d) \quad \sum_i \tfrac{\partial}{\partial x_i} \Big\{ 2x_\lambda \sum_\varrho x_\varrho S_{\varrho i} - S_{\lambda i} \sum_\varrho x_\varrho^2 \Big\} = \sum_i \psi_i \Big\{ 2x_\lambda \sum_\varrho x_\varrho f_{i\varrho} - f_{i\lambda} \sum_\varrho x_\varrho^2 \Big\} \\
&\qquad\qquad [\lambda = 1,2,3,4].
\end{aligned} \tag{23}$$

§ 6. Introduction of physical formulations

Until this point the discussion was only about pure formal identities which can be verified by using (16) and the expressions of S_{ik} as functions of φ obtained from the table in §3. Now we will introduce physical formulations by letting the Lagrange expressions ψ_i equal to zero for the free aether in accordance with (14 II), and equal to the related components P_i of the four-dimensional flow for regions containing a massive matter[21]. Then the identities (19) and (23) give rise to physical theorems known as "conservation laws".

In order to express completely the physical content, we should not confine ourselves to formulation of the occurrence of an electro-magnetic field in presence of a massive matter, given by the assumption $\psi_i = P_i$, but we also have to consider the interaction of a field and a massive matter by introducing the four-dimensional force

[21] For the sake of simplicity, I confine myself to the fundamental equations of the theory of electrons, i.e. the limiting case $\varepsilon = 1, \mu = 1, \sigma = 0$ of the equations for a massive matter.

$$F_k = \sum_i f_{ki} P_i \; .$$

Furthermore, due to the relativistic dynamics, the force (resp. power) density is related with the momentum (resp. energy) density of moving masses, namely:

$$\begin{aligned} \mathfrak{F} &= \tfrac{\partial \mathfrak{g}_m}{\partial t} + \mathfrak{div}[[\mathfrak{g}_m, \mathfrak{q}]], \\ (\mathfrak{F}\,\mathfrak{q}) &= \tfrac{\partial W_m}{\partial t} + \text{div}\; \mathfrak{q}\; W_m, \end{aligned}$$

or in the four-dimensional form:

$$F_k = \sum_i \tfrac{\partial}{\partial x_i} R_{ki}.$$

Then the conservation equations become

$$\begin{aligned} &a) \quad \sum_i \tfrac{\partial}{\partial x_i} T_{\lambda i} = 0 && [\lambda = 1,2,3,4], \\ &b) \quad \sum_i \tfrac{\partial}{\partial x_i} \left(x_\mu T_{\nu i} - x_\nu T_{\mu i} \right) = 0 && [(\mu,\nu) = (1,2),\ldots,(3,4)], \\ &c) \quad \sum_i \tfrac{\partial}{\partial x_i} \Big(\sum_\varrho x_\varrho T_{\varrho i} \Big) = \sum_i R_{ii} \; , \\ &d) \quad \sum_i \tfrac{\partial}{\partial x_i} \Big\{ 2x_\lambda \sum_\varrho x_\varrho T_{\varrho i} - T_{\lambda i} \sum_\varrho x_\varrho^2 \Big\} = 2x_\lambda \sum_i R_{ii} && [\lambda = 1,2,3,4], \end{aligned} \tag{24}$$

where the notation $S_{\lambda i} + R_{\lambda i} = T_{\lambda i}$ and the symmetry $R_{ik} = R_{ki}$ are used.

§ 7. Physical significance of the results

To manifest the physical meaning of Eqs. (24), we split them into spatial and time components[22] even though this destroys their beautiful symmetry. Using the notation of three-dimensional vector analysis and denoting ΣR_{ii} by K we have:

[22]Below, the indices r and z denote the spatial and time components, respectively.

$$
\begin{aligned}
&a_r) \quad \tfrac{\partial}{\partial t}\mathfrak{g} + \mathfrak{div}\, \boldsymbol{p} = 0\,, \\
&a_z) \quad \tfrac{\partial}{\partial t}W + \mathrm{div}\mathfrak{S} = 0\,, \\
&b_r) \quad \tfrac{\partial}{\partial t}[\mathfrak{r},\, \mathfrak{g}] + \mathfrak{div}\, [\mathfrak{r} \times \boldsymbol{p}] = 0^{23}\,, \\
&b_z) \quad \tfrac{\partial}{\partial t}\Big\{\mathfrak{r}\tfrac{W}{c^2} - t\,\mathfrak{g}\Big\} + \mathfrak{div}\,\Big\{\big[[\mathfrak{r}, \tfrac{\mathfrak{S}}{c^2}]\big] - t\boldsymbol{p}\Big\} = 0\,, \\
&c) \quad \tfrac{\partial}{\partial t}\,\big\{(\mathfrak{r}\,\mathfrak{g}) - W\,t\big\} + \mathrm{div}\,\big\{[\mathfrak{r}, \boldsymbol{p}] - \mathfrak{S}\,t\big\} = K\,, \\
&d_r) \quad \tfrac{\partial}{\partial t}\big\{\mathfrak{r}\,(\mathfrak{r}\,\mathfrak{g}) + \big[\mathfrak{r},\, [\mathfrak{r},\, \mathfrak{g}]\big] - 2\,\mathfrak{r}\,t\,W + c^2\,t^2\,\mathfrak{g}\big\} \\
&\qquad + \mathfrak{div}\big\{[[\mathfrak{r}, [\mathfrak{r}, \boldsymbol{p}]]] + [\mathfrak{r} \times [\mathfrak{r} \times \boldsymbol{p}]] - 2t[[\mathfrak{r},\, \mathfrak{S}]] + c^2\,t^2\,\boldsymbol{p}\big\} = 2\,\mathfrak{r}\,K\,, \\
&d_z) \quad \tfrac{\partial}{\partial t}\big\{2\,t\,(\mathfrak{r}\,\mathfrak{g}) - \tfrac{W}{c^2}\,(\mathfrak{r}^2 + c^2\,t^2)\big\} + \ \mathrm{div}\,\big\{2\,t\,[\mathfrak{r}, \boldsymbol{p}] - \mathfrak{g}(\mathfrak{r}^2 + c^2\,t^2)\big\} = 2\,t\,K.
\end{aligned}
\tag{25}
$$

In addition, Eq. (19) yields the equation of continuity equation for electricity:

$$\mathrm{div}\,(\varrho\,\mathfrak{q}) + \frac{\partial \varrho}{\partial t} = 0. \tag{26}$$

Equations (25) are often transformed from the differential into an integral form by integrating them over a three-dimensional part of the space and using the fact that the integrals of the divergence members become surface integrals. We will assume that we have a closed system of masses and of charges with a finite basis and that the components of the energy-momentum tensor decrease outwards so fast that we can neglect the surface integrals compared to the space integrals for a sufficiently large integration area B containing the masses and charges. Then Eqs. (25) $a_r)$, $a_z)$ and $b_r)$ result in the conservation of momentum, energy and angular momentum of the system:

[23] Since v. Laue has already used the symbol $[\mathfrak{r}, \boldsymbol{p}]$ for the vector with the x-component $x\boldsymbol{p}_{xx} + y\boldsymbol{p}_{xy} + z\boldsymbol{p}_{xy}$, I have taken the liberty to use the symbol $[\mathfrak{r} \times \boldsymbol{p}]$ instead of the tensor

$$\begin{pmatrix} y\,\boldsymbol{p}_{zx} - z\,\boldsymbol{p}_{yx} & y\,\boldsymbol{p}_{zy} - z\,\boldsymbol{p}_{yy} & y\,\boldsymbol{p}_{zz} - z\,\boldsymbol{p}_{yz} \\ z\,\boldsymbol{p}_{xx} - x\,\boldsymbol{p}_{zx} & z\,\boldsymbol{p}_{xy} - x\,\boldsymbol{p}_{zy} & z\,\boldsymbol{p}_{xz} - x\,\boldsymbol{p}_{zz} \\ x\,\boldsymbol{p}_{yx} - y\,\boldsymbol{p}_{xx} & x\,\boldsymbol{p}_{yy} - y\,\boldsymbol{p}_{xy} & x\,\boldsymbol{p}_{yz} - y\,\boldsymbol{p}_{xz} \end{pmatrix}.$$

By the way, the following is valid: $[\mathfrak{r},\, \mathfrak{div}\boldsymbol{p}] = \mathfrak{div}[\mathfrak{r} \times \boldsymbol{p}]$.

$$
\begin{aligned}
&a_r) \qquad \int\!\!\int_B\!\!\int \mathfrak{g}\, d\tau = \mathfrak{G} = \text{constant vector}\,, \\
&a_z) \qquad \int\!\!\int_B\!\!\int W\, d\tau = E = \text{const.}\,, \\
&b_r) \qquad \int\!\!\int_B\!\!\int [\mathfrak{r}, \mathfrak{g}]\, d\tau = \mathfrak{L} = \text{constant vector}\,.
\end{aligned}
\tag{27}
$$

In contrast, formula (25) takes at first the form

$$
\frac{\partial}{\partial t}\Big\{ \int\!\!\int_B\!\!\int \mathfrak{r}\, \frac{W}{c^2}\, d\tau - t \int\!\!\int\!\!\int \mathfrak{g}\, d\tau \Big\} = 0\,, \tag{28}
$$

whose obvious analogy to the second center of mass theorem (12 d) in the n-body problem can be noticed immediately. From point of view of the theory of relativity, mass and energy can be treated as identical quantities, namely the mass m can be regarded as energy of the value $m\, c^2$, and visa versa, any energy with a density W is equivalent to a mass density of the value $\frac{W}{c^2} = k$. Accordingly, the electromagnetic field in the free aether also obtains a "center of mass", and the quantity

$$
\int\!\!\int_B\!\!\int \mathfrak{r}\, \tfrac{W}{c^2}\, d\tau = \int\!\!\int_B\!\!\int \mathfrak{r}\, k\, d\tau
$$

is the radius vector from the origin of the coordinate system to the common center of mass of the electromagnetic field and the massive matter multiplied by the total mass $\frac{E}{c^2}$. Hence, this quantity corresponds to the expression $\sum_i m_i\, x_{ik}$ in (12 d), whereas the coefficients of t in (28) and (12 d) mean the total momentum of the system in both cases. It follows from formula (28) in combination with (27 a_r) that

$$
\int\!\!\int_B\!\!\int \mathfrak{r}\, \tfrac{W}{c^2}\, d\tau = \mathfrak{C}_1 + \mathfrak{G}\, t \;\; (\mathfrak{C}_1 = \text{constant vector}\,) \tag{27b_z}
$$

in complete analogy to (13), i.e.

The common center of mass of the electromagnetic field and the massive matter moves uniformly along a straight line.

Five remaining equations in (27) do not have the form of pure conservation laws ($\frac{\partial}{\partial t}$ of space integral = surface integral) due to presence of K if the integration area contains moving fields. Therefore the integration with respect to time cannot be performed explicitly. Nevertheless, these equations carry a certain physical meaning. In order to simplify the interpretation, let us consider the case of the free aether without heavy masses are moved through the field. Then $K = 0$, and we have again pure conservation equations which can be written as follows:

$$
\begin{aligned}
&c) && \int\int\limits_B\int (\mathfrak{r}\,\mathfrak{g}_e)\,d\tau = C_1 + E_e\,t\,,\\
&d_r) && \int\int\limits_B\int \left\{\mathfrak{r}(\mathfrak{r}\,\mathfrak{g}_e) + \left[\mathfrak{r},\,[\mathfrak{r},\,\mathfrak{g}_e]\right]\right\} d\tau = \mathfrak{E}_2 + 2\,c^2\,\mathfrak{E}_1\,t + c^2\,\mathfrak{G}_e\,t^2\,, \qquad (27)\\
&d_z) && \int\int\limits_B\int \mathfrak{r}\,\frac{W_e}{c^2}\,d\tau = C_2 + 2\,C_1\,t + E_e\,t^2\\
& && C_1,\,C_2 = \text{const.},\quad \mathfrak{E}_2 = \text{constant vector}\,.
\end{aligned}
$$

Here, the easiest equation for understanding is $d_z)$. Since $\frac{W_c}{c^2}$ = “mass density” of the electromagnetic field, the left side of $d_z)$ means the half sum of the principal moments of inertia of the “electromagnetic mass” of the field with respect to the origin of coordinate system, and we can say:

The sum of the electromagnetic principal moments of inertia of the field with respect to an arbitrary fixed point is a quadratic function of time, where the coefficient of the square of time is the double total energy of the field.

It seems to me that the equations (27 c) and $d_r)$, unlike $d_z)$, don’t have an direct analogy in mechanics. Therefore their left side integrals should be introduced as new quantities in physics. The dimension of $(\mathfrak{r}\;\mathfrak{g}_e)$ is that of the density of an action quantity, and the dimension of $\left\{\mathfrak{r}(\mathfrak{r}\,\mathfrak{g}_e) + \left[\mathfrak{r}\,[\mathfrak{r}\,\mathfrak{g}_e]\right]\right\}$ is that of the momentum of an action density.

I don’t want to finish without expressing my gratitude to Miss Emmy Noether and Mr. Prof. Paul Hertz for their encouraging interest to my work.

(Received 3 March 1921.)

Archives of ALGA
Volume 3, 2006

The answer to the question put to me by L.V. Ovsyannikov 33 years ago

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Abstract. In 1973, during a discussion of contemporary works in soliton theory at "Theoretical seminar of the Institute of Hydrodynamics" in Novosibirsk, Professor Ovsyannikov asked me if the infinite number of conservation laws for the Korteweg-de Vries equation can be obtained from its symmetries. The answer was by no means evident because the KdV equation did not have the usual Lagrangian, and hence the Noether theorem was not applicable. In the present paper I give the affirmative answer to Ovsyannikov's question by using my recent new general theorem on conservation laws applicable to arbitrary differential equations. The new theorem, which I will call here *Theorem on nonlocal conservation laws*, does not require existence of Lagrangians and is based on a concept of adjoint equations for non-linear equations. For derivation of the infinite series of conservation laws for the KdV equation, I modify the notion of self-adjoint equations and extend it to non-linear equations.

I dedicate this paper to L.V. Ovsyannikov
on the special occasion of his 88th birthday.

1 Introduction

Recall the formulation of the well-known conservation theorem proved by Noether [1] in 1918 by using the calculus of variations. Let us begin with variational integrals

$$\int_V \mathcal{L}(x, u, u_{(1)})dx, \tag{1.1}$$

where $\mathcal{L}(x, u, u_{(1)})$ is a first-order Lagrangian, i.e. it involves, along with the independent variables $x = (x^1, \ldots, x^n)$ and the dependent variables $u = (u, \ldots, u^m)$, the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ of u with respect to x, i.e. $u_i^\alpha = D_i(u^\alpha)$.

I will formulate Noether's theorem in the case of Lagrangians up to third order.

Theorem 1.1. Let the variational integral (1.1) be invariant under a group G with a generator

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u^\alpha}\,\cdot \tag{1.2}$$

Then the vector field $C = (C^1, \ldots, C^n)$ defined by

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha)\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\,, \quad i = 1, \ldots, n, \tag{1.3}$$

provides a conservation law for the Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\Big(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\Big) = 0, \quad \alpha = 1, \ldots, m. \tag{1.4}$$

In other words, the vector field (1.3) obeys the equation $\mathrm{div} C \equiv D_i(C^i) = 0$ for all solutions of Eqs. (1.4), i.e.

$$D_i(C^i)\Big|_{(1.4)} = 0. \tag{1.5}$$

Any vector field C^i satisfying (1.5) is called a *conserved vector* for Eqs. (1.4).

In the case of second-order Lagrangians $L(x, u, u_{(1)}, u_{(2)})$, the Euler-Lagrange equations (1.4) and the conserved vector (1.3) are replaced by

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\Big(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\Big) + D_i D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}\Big) = 0 \tag{1.6}$$

and

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}\Big)\right] + D_k\big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}\,, \tag{1.7}$$

respectively.

In the case of third-order Lagrangians $L(x, u, u_{(1)}, u_{(2)}, u_{(3)})$, the Euler-Lagrange equations are written:

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\Big(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\Big) + D_i D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}\Big) - D_i D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big) = 0 \tag{1.8}$$

and the conserved vector (1.3) is replaced by

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\Big) + D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big)\right] \tag{1.9}$$

$$+ D_j\big(\eta^\alpha - \xi^j u_j^\alpha\big)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big)\right] + D_j D_k\big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\,\cdot$$

Remark 1.1. It is manifest from Eq. (1.5) that any linear combination of conserved vectors is a conserved vector. Furthermore, any vector vanishing on the solutions of Eqs. (1.4) is a conserved vector, a *trivial conserved vector,* for Eqs. (1.4). In what follows, conserved vectors will be considered up to addition of trivial conserved vectors.

The invariance of the integral (1.1) implies that the Euler-Lagrange equations (1.4) admit the group G. Therefore, in order to apply Noether's theorem, one has first of all to find the symmetries of Eqs. (1.4). Then one should single out the symmetries leaving invariant the variational integral (1.4). This can be done by means of the following infinitesimal test for the invariance of the variational integral (see [2] or [3]):

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = 0, \tag{1.10}$$

where the generator X is prolonged to the first derivatives $u_{(1)}$ by the formula

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \Big[D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j)\Big] \frac{\partial}{\partial u_i^\alpha} \cdot \tag{1.11}$$

If Eq. (1.10) is satisfied, then the vector (1.3) provides a conservation law.

The invariance of the variational integral is sufficient, as said above, for the invariance of the Euler-Lagrange equations, but not necessary. Indeed, the following lemma shows that if one adds to a Lagrangian the divergence of any vector field, the Euler-Lagrange equations remain invariant.

Lemma 1.1. A function $f(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ with several independent variables $x = (x^1, \ldots, x^n)$ and several dependent variables $u = (u^1, \ldots, u^m)$ is the divergence of a vector field $H = (h^1, \ldots, h^n),\ h^i \in \mathcal{A}$, i.e.

$$f = \mathrm{div} H \equiv D_i(h^i), \tag{1.12}$$

if and only if the following equations hold identically in $x, u, u_{(1)}, \ldots$:

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{1.13}$$

Therefore, one can add to the Lagrangian $\mathcal{L}$ the divergence of an arbitrary vector field depending on the group parameter and replace the invariance condition (1.10) by the divergence condition

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(B^i). \tag{1.14}$$

Then Eqs. (1.4) are again invariant and have a conservation law $D_i\big(C^i\big) = 0$, where (1.3) is replaced by (see also Bessel-Hagen's paper in this volume)

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha) \frac{\partial \mathcal{L}}{\partial u_i^\alpha} - B^i. \tag{1.15}$$

The restriction to Euler-Lagrange equations reduces applications of Noether's theorem significantly. For example, Noether's theorem is not applicable to *all evolution type equations*, to differential equations of an *odd order,* etc. Moreover, a symmetry of Euler-Lagrange equations should satisfy an additional property to leave invariant the variational integral. In spite of the fact that certain attempts have been made to overcome these restrictions and various generalizations of Noether's theorem have been discussed, I do not know in the literature a general result associating a conservation law with *every infinitesimal symmetry of an arbitrary differential equation.*

In 1973, during a discussion of the pioneering works on soliton theory at the "Theoretical seminar of the Institute of Hydrodynamics" in Novosibirsk, Professor Lev V. Ovsyannikov asked me if the infinite number of conservation laws for the Korteweg-de Vries (KdV) equation can be obtained from its symmetries. The answer was by no means evident because the KdV equation did not have the usual Lagrangian, and hence the Noether theorem was not applicable. I give here the affirmative answer to Ovsyannikov's question by proving a new conservation theorem (Theorem 3.4) applicable, unlike Noether's theorem, to arbitrary differential equations (see also [4]).

The new theorem is based on a concept of adjoint equations for non-linear equations and does not require existence of Lagrangians. The crucial fact is that all Lie point, Lie-Bäcklund and nonlocal symmetries of any equation are inherited by the adjoint equation (Section 3.1, Theorem 3.1). I give in Section 3.2 an explicit formula for the conserved quantities associated with these symmetries. Accordingly, one can find for any differential equation with known Lie, Lie-Bäcklund or nonlocal symmetries the associated conservation laws independently on existence of classical Lagrangians. The theorem is valid also for any system of differential equations where the number of equations is equal to the number of dependent variables (Theorem 3.2).

The new conservation theorem is, in fact, a *theorem on nonlocal conservation laws,* since the conserved quantities provided by this theorem are essentially "nonlocal". Namely, they involve, along with the variables of the equations under consideration, also *adjoint variables* which can be treated as *nonlocal variables* as defined, e.g. in [5]. In order to single out *local conservation laws,* I modify the notion of "self-adjoint" equations and show that the adjoint variables can be eliminated from the conserved quantities for the self-adjoint equations. However, this elimination may reduce some of nonlocal conservation laws to trivial local conservation laws. For example, the KdV and modified KdV equations are self-adjoint. It is shown in Section 4 that the known infinite series of *local conservation laws* of the KdV equation are associated with its *nonlocal symmetries.* On the other hand, the *local symmetries* of the KdV equation lead to essentially *nonlocal conservation laws* which become trivial if one eliminates the adjoint variable.

Finally, it is shown in Section 5.1 that even for equations having Lagrangians my theorem leads to conservation laws different from those given by Noether's theorem, e.g. in the case of nonlinear equations that are not self-adjoint.

2 Preliminaries

I begin with a brief discussion of the space $\mathcal{A}$ of differential functions, the basic operators $X, \delta/\delta u^\alpha, \mathcal{N}^i$ acting in $\mathcal{A}$ and the *fundamental identity* connecting them (see [2], Chap. 4 and 5; also [3], Sections 8.4 and 9.7). Then I recall the definition of adjoint equations to arbitrary equations and a new concept of self-adjoint equations [6].

2.1 Notation

Let $x = (x^1, \ldots, x^n)$ be n independent variables, and $u = (u^1, \ldots, u^m)$ be m dependent variables. We will use the notation $u_{(1)} = \{u_i^\alpha\},\ u_{(2)} = \{u_{ij}^\alpha\},\ \ldots$, where

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_i D_j(u^\alpha), \ldots$$

with D_i denoting the total differentiation with respect to x^i :

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{2.1}$$

The variables u^α are also called *differential variables.* A function $f(x, u, u_{(1)}, \ldots)$ of a finite number of variables $x, u, u_{(1)}, u_{(2)}, \ldots$ is called a *differential function* if it is locally analytic, i.e., locally expandable in a Taylor series with respect to all arguments. The highest order of derivatives appearing in the differential function is called the order of this function. The set of all differential functions of all finite orders is denoted by $\mathcal{A}$. This set is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is defined by the usual multiplication of functions. The space $\mathcal{A}$ is closed under the total differentiations: if $f \in \mathcal{A}$ then $D_i(f) \in \mathcal{A}$.

2.2 Basic operators and the fundamental identity

The variational derivatives (the *Euler-Lagrange operator*) in $\mathcal{A}$ are defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}\,, \quad \alpha = 1, \ldots, m, \tag{2.2}$$

where the summation over the repeated indices $i_1 \ldots i_s$ runs from 1 to n.

A first-order linear differential operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots\,, \tag{2.3}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ are arbitrary differential variables, and $\zeta_i^\alpha, \zeta_{i_1 i_2}^\alpha, \ldots$ are given by

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \ldots \tag{2.4}$$

with

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, \ldots, m, \tag{2.5}$$

is called a *Lie-Bäcklund operator.* It is often written in the abbreviated form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \tag{2.6}$$

where the prolongation given by (2.3) - (2.4) is understood. The operator (2.3) is formally an infinite sum, but it truncates when acting on any differential function. Hence, *the action of Lie-Bäcklund operators is well defined on the space* $\mathcal{A}$.

The commutator $[X_1, X_2] = X_1X_2 - X_2X_1$ of any two Lie-Bäcklund operators,

$$X_\nu = \xi_\nu^i \frac{\partial}{\partial x^i} + \eta_\nu^\alpha \frac{\partial}{\partial u^\alpha} + \cdots \quad (\nu = 1, 2),$$

is identical with the Lie-Bäcklund operator given by

$$[X_1, X_2] = \big(X_1(\xi_2^i) - X_2(\xi_1^i)\big)\frac{\partial}{\partial x^i} + \big(X_1(\eta_2^\alpha) - X_2(\eta_1^\alpha)\big)\frac{\partial}{\partial u^\alpha} + \cdots. \tag{2.7}$$

The set of all Lie-Bäcklund operators is an infinite dimensional Lie algebra with respect to the commutator (2.7). It is called the *Lie-Bäcklund algebra* and denoted by $L_\mathcal{B}$. The algebra $L_\mathcal{B}$ is endowed with the following properties.

I. $D_i \in L_\mathcal{B}$. In other words, the total differentiation (2.1) is a Lie-Bäcklund operator. Furthermore,

$$X_* = \xi_*^i D_i \in L_\mathcal{B} \tag{2.8}$$

for any $\xi_*^i \in \mathcal{A}$.

II. Let L_* be the set of all Lie-Bäcklund operators of the form (2.8). Then L_* is an ideal of $L_\mathcal{B}$, i.e., $[X, X_*] \in L_*$ for any $X \in L_\mathcal{B}$. Indeed,

$$[X, X_*] = \big(X(\xi_*^i) - X_*(\xi^i)\big)D_i \in L_*.$$

III. In accordance with property II, two operators $X_1, X_2 \in L_\mathcal{B}$ are said to be *equivalent* (i.e. $X_1 \sim X_2$) if $X_1 - X_2 \in L_*$. In particular, every operator $X \in L_\mathcal{B}$ is equivalent to an operator (2.3) with $\xi^i = 0, i = 1, ..., n$. Namely, $X \sim \widetilde{X}$ where

$$\widetilde{X} = X - \xi^i D_i = (\eta^\alpha - \xi^i u_i^\alpha)\frac{\partial}{\partial u^\alpha} + \cdots. \tag{2.9}$$

The operators of the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \quad \eta^\alpha \in \mathcal{A}, \tag{2.10}$$

are called *canonical Lie-Bäcklund operators*. Hence, the property III means that any operator $X \in L_\mathcal{B}$ is equivalent to a canonical Lie-Bäcklund operator.

IV. Generators of Lie point transformation groups are operators (2.6) with the coefficients ξ^i and η^α depending only on x, u :

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\cdot \tag{2.11}$$

The Lie-Bäcklund operator (2.3) is equivalent to a generator (2.11) of a point transformation group if and only if its coordinates have the form

$$\xi^i = \xi^i_1(x,u) + \xi^i_*\,, \quad \eta^\alpha = \eta^\alpha_1(x,u) + \big(\xi^i_2(x,u) + \xi^i_*\big)u^\alpha_i\,,$$

where $\xi^i_* \in \mathcal{A}$ are any differential functions and $\xi^i_1, \xi^i_2, \eta^\alpha_1$ depend only on $x, u.$

Example 2.1. Let t, x be the independent variables. The generator of the Galilean transformation and its canonical Lie-Bäcklund form (2.9) are written:

$$X = \frac{\partial}{\partial u} - t\frac{\partial}{\partial x} \sim \widetilde{X} = (1 + tu_x)\frac{\partial}{\partial u} + \cdots\,.$$

Example 2.2. The generator of non-homogeneous dilations (see operator X_2 in Section 4) and its canonical Lie-Bäcklund representation (2.9) are written:

$$X = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} \sim \widetilde{X} = (2u + 3tu_t + xu_x)\frac{\partial}{\partial u} + \cdots\,.$$

I associate with Lie-Bäcklund operators $X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \cdots$ of the form (2.3) the infinite-order operators $\mathcal{N}^i$ $(i = 1, \ldots, n)$ defined by the following formal sums:

$$\mathcal{N}^i = \xi^i + W^\alpha\,\frac{\delta}{\delta u^\alpha_i} + \sum_{s=1}^{\infty} D_{i_1}\cdots D_{i_s}(W^\alpha)\,\frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}}\,, \tag{2.12}$$

where W^α are given by (2.5) and the variational derivatives with respect to variables $u^\alpha_i, \ldots$ are obtained from (2.2) by replacing u^α by the corresponding derivatives $u^\alpha_i, \ldots,$ e.g.

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s=1}^{\infty}(-1)^s D_{j_1}\cdots D_{j_s}\,\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}\cdot \tag{2.13}$$

The following statement holds (N.H. Ibragimov, 1979, see [3], Section 8.4.4).

Theorem 2.1. The operators (2.2), (2.3) and (2.12) are connected by the following *fundamental identity:*

$$X + D_i(\xi^i) = W^\alpha\frac{\delta}{\delta u^\alpha} + D_i\mathcal{N}^i\,, \tag{2.14}$$

where W^α are given by (2.5).

2.3 Adjoint equations

Recall that the *adjoint operator* to a linear differential operator L is defined in the classical literature as a linear operator L^* such that the equation

$$vL[u] - uL^*[v] = \mathrm{div}P(x) \tag{2.15}$$

holds for all functions u and v, where $P(x) = \big(p^1(x), \ldots, p^n(x)\big)$ is any vector and $\mathrm{div}P = D_i(p^i)$. The equation $L^*[v] = 0$ is called the adjoint equation to $L[u] = 0$. The operator L and the equation $L[u] = 0$ are said to be *self-adjoint* if $L[u] = L^*[u]$ for any function $u(x)$. For example, if L is a linear second-order differential operator,

$$L[u] = a^{ij}(x)D_iD_j(u) + b^i(x)D_i(u) + c(x)u, \tag{2.16}$$

Eq. (2.15) yields the adjoint operator L^* given by

$$L^*[v] = D_iD_j(a^{ij}(x)v) - D_i(b^i(x)v) + c(x)v. \tag{2.17}$$

The operator (2.16) is self-adjoint provided that

$$b^i(x) = D_j(a^{ij})\,, \quad i = 1, \ldots, n. \tag{2.18}$$

The definitions of the adjoint operator and the adjoint equation are the same for systems of differential equations. For example, in the case of systems of second-order equations the adjoint operator is obtained by assuming that u is an m-dimensional vector-function and that the coefficients $a^{ij}(x), b^i(x)$ and $c(x)$ of the operator (2.16) are $m \times m$-matrices. The following two second-order equations provide an example of a self-adjoint system:

$$x^2u_{xx} + u_{yy} + 2xu_x + w = 0,$$

$$w_{xx} + y^2w_{yy} + 2yw_y + u = 0.$$

Linearity of equations is crucial for defining adjoint equations by means of Eq. (2.15). The following definition of an adjoint equation [6] is applicable to any system of linear and non-linear differential equations.

Definition 2.1. Consider a system of sth-order partial differential equations

$$F_\alpha\big(x, u, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{2.19}$$

where $F_\alpha(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ are differential functions with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m),\ u = u(x)$. We introduce the differential functions

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha}\,, \quad \alpha = 1, \ldots, m, \tag{2.20}$$

where $v = (v^1, \ldots, v^m)$ are new dependent variables, $v = v(x)$, and define the system of *adjoint equations* to Eqs. (2.19) by

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m. \tag{2.21}$$

In the case of linear equations, Definition 2.1 is equivalent to the classical definition of the adjoint equation. Namely, taking for the sake of simplicity scalar equations, we can formulate the statement as follows.

Theorem 2.2. The operator L^* to a linear operator L defined by Eq. (2.15) is identical with the operator L^* given by

$$L^*[v] = \frac{\delta(vL[u])}{\delta u} \cdot \tag{2.22}$$

One can easily verify that if $L[u]$ is the second-order operator given by (2.16), then the operator $L^*[v]$ defined by Eq. (2.22) coincides with the operator $L^*[v]$ defined by Eq. (2.17). For practical use, the definition by (2.22) is simpler than by (2.17).

Remark 2.1. The adjoint equations (2.21) to linear equations $F(x, u, \ldots, u_{(s)}) = 0$ for $u(x)$ are linear equations $F^*(x, v, \ldots, v_{(s)}) = 0$ for $v(x)$. If Equations (2.19) are nonlinear, the adjoint equations are linear with respect to $v(x)$, but nonlinear in the coupled variables u and v.

The following definition [6] extends the classical concept of self-adjointness of linear operators to nonlinear equations.

Definition 2.2. A system of equations (2.19) is said to be *self-adjoint* if the system obtained from the adjoint equations (2.21) by the substitution $v = u$:

$$F^*_\alpha\big(x, u, u, \ldots, u_{(s)}, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{2.23}$$

is identical with the original system (2.19). In other words, the *self-adjoint* equations obey the condition

$$F^*_\alpha(x, u, u, \ldots, u_{(s)}, u_{(s)}) = \phi^\beta_\alpha\, F_\alpha(x, u, \ldots, u_{(s)}), \quad \alpha = 1, \ldots, m, \tag{2.24}$$

with regular coefficients $\phi^\beta_\alpha \in \mathcal{A}$.

Example 2.3. For the heat equation $u_t - u_{xx} = 0$, Eq. (2.20) yields

$$F^* = \frac{\delta}{\delta u}\big[v\big(v_t - u_{xx}\big)\big] = \Big(-D_t\,\frac{\partial}{\partial u_t} + D_x^2\,\frac{\partial}{\partial u_{xx}}\Big)\big[v\big(u_t - u_{xx}\big)\big] = -D_t(v) - D_x^2(v).$$

Hence, the adjoint equation (2.21) to the heat equation is $v_t + v_{xx} = 0$. It is manifest that the heat equation is not self-adjoint.

2.4 Formal Lagrangians

Consider the extension of the variational derivatives (2.2) to differential functions with $2m$ differential variables $(u, v) = (u^1, \ldots, u^m; v^1, \ldots, v^m)$ defined by the coupled formal sums:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}\,,$$

$$\frac{\delta}{\delta v^\alpha} = \frac{\partial}{\partial v^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_s}}\,, \tag{2.25}$$

where $\alpha = 1, \ldots, m$ and the summation over the indices $i_1 \ldots i_s$ runs from 1 to n.

Consider the differential function

$$\mathcal{L} = v^\beta F_\beta\big(x, u, \ldots, u_{(s)}\big) \tag{2.26}$$

with $2m$ differential variables (u, v), where $u = (u^1, \ldots, u^m), v = (v^1, \ldots, v^m)$. It is manifest from Eqs. (2.20) that the variational derivatives (2.25) of the function (2.26) provide the differential equations (2.19) and their adjoint equations (2.21), namely:

$$\frac{\delta \mathcal{L}}{\delta v^\alpha} = F_\alpha\big(x, u, \ldots, u_{(s)}\big), \tag{2.27}$$

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big). \tag{2.28}$$

This circumstance justifies the following definition.

Definition 2.3. The differential function (2.26) is called a *formal Lagrangian* for the differential equations (2.19).

Example 2.4. The heat equation $u_t - u_{xx} = 0$ has the *second-order* formal Lagrangian

$$\mathcal{L} = v\big(u_t - u_{xx}\big).$$

Using Lemma 1.1 and the identity $-vu_{xx} = (-vu_x)_x + u_x v_x$, one can replace it by the equivalent *first-order* formal Lagrangian

$$\mathcal{L} = vu_t + u_x v_x.$$

The variational derivatives (2.25) of both $\mathcal{L}$ provide the heat equation together with its adjoint equation $v_t + v_{xx} = 0$:

$$\frac{\delta \mathcal{L}}{\delta v} = u_t - u_{xx}, \quad \frac{\delta \mathcal{L}}{\delta u} = -(v_t + v_{xx}).$$

Let us extend Example 2.4 to any linear second-order differential equation

$$L[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = f(x). \tag{2.29}$$

The formal Lagrangian (2.26) is written $\mathcal{L} = \big(a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u - f(x)\big)v$. We rewrite it in the form

$$\mathcal{L} = D_j\big(va^{ij}u_i\big) - vu_iD_j\big(a^{ij}\big) - a^{ij}u_iv_j + vb^iu_i + cuv - f(x)v.$$

The first term at the right-hand side can be dropped by Lemma 1.1, and hence

$$\mathcal{L} = cuv + vb^i(x)u_i - vu_iD_j\big(a^{ij}\big) - a^{ij}u_iv_j - f(x)v. \tag{2.30}$$

The variational differentiation of the function (2.30) leads to Eq. (2.29) and the adjoint equation $D_iD_j\big(a^{ij}v\big) - D_i\big(b^iv\big) + cv = 0$:

$$\frac{\delta\mathcal{L}}{\delta v} = cu + b^i(x)u_i - u_iD_j\big(a^{ij}\big) + D_j\big(a^{ij}u_i\big) - f = a^{ij}u_{ij} + b^iu_i + cu - f$$

$$\frac{\delta\mathcal{L}}{\delta u} = cv - D_i(b^iv) + D_i\big(vD_j(a^{ij}v)\big) + D_i(a^{ij}v_j) = D_iD_j(a^{ij}v) - D_i(b^iv) + cv.$$

If the operator $L[u]$ is self-adjoint, then (2.30) leads to the following well known Lagrangian for Eq. (2.29):

$$\mathcal{L} = \frac{1}{2}\big[c(x)u^2 - a^{ij}(x)u_iu_j\big]. \tag{2.31}$$

Indeed, the second and the third terms in the right-hand side of Eq. (2.30) annihilate each other by the condition (2.18). Now we set $v = u$, divide by two and arrive at the Lagrangian (2.31).

Example 2.5. Consider the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + uu_x. \tag{2.32}$$

Eq. (2.20) yields $F^*(t, x, u, v, \ldots, u_{(3)}, v_{(3)}) = -(v_t - v_{xxx} - uv_x)$. It follows that the adjoint equation to the KdV equation is

$$v_t = v_{xxx} + uv_x. \tag{2.33}$$

Moreover, we see that $F^*(t, x, u, u, \ldots, u_{(3)}, u_{(3)}) = -F(t, x, u, \ldots, u_{(3)})$. Hence the *KdV equation is self-adjoint* since the condition (2.24) is satisfied with $\phi = -1$. Using Eq. (2.26), we obtain the third-order formal Lagrangian for the KdV equation:

$$\mathcal{L} = v[u_t - uu_x - u_{xxx}]. \tag{2.34}$$

It can be replaced by the second-order formal Lagrangian

$$\mathcal{L} = vu_t - vuu_x + v_x u_{xx} \tag{2.35}$$

due to Lemma 1.1 and the equation $-vu_{xxx} = (-vu_{xx})_x + v_x u_{xx}$. Furthermore, one can easily replace (2.35) by

$$\mathcal{L} = v_x u_{xx} - uv_t + \frac{1}{2}u^2 v_x. \tag{2.36}$$

Each of the functions (2.34), (2.35) and (2.36) yield the KdV equation and its adjoint:

$$\frac{\delta\mathcal{L}}{\delta v} = u_t - uu_x - u_{xxx}, \quad \frac{\delta\mathcal{L}}{\delta u} = -v_t + v_{xxx} + uv_x.$$

3 Main theorems

3.1 Symmetry of adjoint equations

Let us show that the adjoint equations (2.21) inherit all Lie and Lie-Bäcklund symmetries of Eqs. (2.19). We will begin with scalar equations.

Theorem 3.1. Consider an equation

$$F\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0 \tag{3.1}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u. The adjoint equation

$$F^*\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) \equiv \frac{\delta(vF)}{\delta u} = 0 \tag{3.2}$$

to Equation (3.1) inherits the symmetries of Equation (3.1). Namely, if Equation (3.1) admits an operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}\,, \tag{3.3}$$

where X is either a generator of a point transformation group, i.e. $\xi^i = \xi^i(x, u)$, $\eta = \eta(x, u)$, or a Lie-Bäcklund operator, i.e. $\xi^i = \xi^i(x, u, u_{(1)}, \ldots, u_{(p)})$ and $\eta = \eta(x, u, u_{(1)}, \ldots, u_{(q)})$ are any differential functions, then Eq. (3.2) admits the operator (3.3) extended to the variable v by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \eta_* \frac{\partial}{\partial v} \tag{3.4}$$

with a certain function $\eta_* = \eta_*(x, u, v, u_{(1)}, \ldots)$.

Proof. Let the operator (3.3) be a Lie point symmetry of Eq. (3.1). Then

$$X(F) = \lambda F \tag{3.5}$$

where $\lambda = \lambda(x, u, \ldots)$. In Eq. (3.5), the prolongation of X to all derivatives involved in Eq. (3.1) is understood. Furthermore, the simultaneous system (3.1), (3.2) can be obtained as the variational derivatives of the formal Lagrangian (2.26)

$$\mathcal{L} = vF \tag{3.6}$$

of Eq. (3.1). We take an extension of the operator (3.3) in the form (3.4) with an unknown coefficient η_* and require that the invariance condition (1.10) be satisfied:

$$Y(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = 0. \tag{3.7}$$

We have:

$$Y(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = Y(v)F + vX(F) + vFD_i(\xi^i)$$
$$= \eta_* F + v\lambda F + vFD_i(\xi^i) = [\eta_* + v\lambda + vD_i(\xi^i)]F.$$

Hence, the requirement (3.7) leads to the equation

$$\eta_* = -\big[\lambda + D_i(\xi^i)\big]v. \tag{3.8}$$

with λ defined by Eq. (3.5). Since Eq. (3.7) guarantees the invariance of the system (3.1), (3.2) we conclude that the adjoint equation (3.2) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} - \big[\lambda + D_i(\xi^i)\big] v \frac{\partial}{\partial v} \tag{3.9}$$

thus proving the theorem for Lie point symmetries.

Let us assume now that the symmetry (3.3) is a Lie-Bäcklund operator. Then Eq. (3.5) is replaced by (see [2])

$$X(F) = \lambda_0 F + \lambda_1^i D_i(F) + \lambda_2^{ij} D_i D_j(F) + \lambda_3^{ij} D_i D_j D_k(F) + \cdots, \tag{3.10}$$

where $\lambda_2^{ij} = \lambda_2^{ji}, \ldots$. Therefore, using the operator (3.4), we have:

$$Y(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = Y(v)F + vX(F) + vFD_i(\xi^i)$$

$$= [\eta_* + v\lambda_0 + vD_i(\xi^i)]F + v\lambda_1^i D_i(F) + v\lambda_2^{ij} D_i D_j(F) + v\lambda_3^{ijk} D_i D_j D_k(F) + \cdots .$$

Now we use the identities

$$v\lambda_1^i D_i(F) = D_i\big(v\lambda_1^i F\big) - FD_i\big(v\lambda_1^i\big),$$

$$v\lambda_2^{ij} D_i D_j(F) = D_i\big[v\lambda_2^{ij} D_j(F) - FD_j(v\lambda_2^{ij})\big] + FD_i D_j\big(v\lambda_2^{ij}\big),$$

$$v\lambda_3^{ijk} D_i D_j D_k(F) = D_i\big[\cdots\big] - FD_i D_j D_k\big(v\lambda_3^{ijk}\big),$$

etc., and obtain:

$$Y(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i\big[v\lambda_1^i\,F + v\lambda_2^{ij}D_j(F) - FD_j(v\lambda_2^{ij}) + \cdots\big]$$
$$+ \left[\eta_* + v\lambda_0 + vD_i(\xi^i) - D_i\big(v\lambda_1^i\big) + D_iD_j\big(v\lambda_2^{ij}\big) - D_iD_jD_k\big(v\lambda_3^{ijk}\big) + \cdots\right] F.$$

Finally, we complete the proof of the theorem by setting

$$\eta_* = -\big[\lambda_0 + D_i(\xi^i)\big]v + D_i\big(v\lambda_1^i\big) - D_iD_j\big(v\lambda_2^{ij}\big) + D_iD_jD_k\big(v\lambda_3^{ijk}\big) - \cdots \tag{3.11}$$

and arriving at Eq. (1.14),

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i(B^i), \tag{1.14}$$

with

$$B^i = -v\lambda_1^i\,F - v\lambda_2^{ij}D_j(F) + FD_j(v\lambda_2^{ij}) - \cdots \tag{3.12}$$

Let us prove a similar statement on symmetries of adjoint equations for systems of m equations with m dependent variables. For the sake of simplicity we will prove the theorem only for Lie point symmetries. The proof can be extended to Lie-Bäcklund symmetries as it has been done in Theorem 3.1.

Theorem 3.2. Consider a system of m equations

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{3.13}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$. The adjoint system

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{3.14}$$

inherits the symmetries of the system (3.13). Namely, if the system (3.13) admits a point transformation group with a generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}\,, \tag{3.15}$$

then the adjoint system (3.14) admits the operator (3.15) extended to the variables v^α by the formula

$$Y = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \eta^\alpha_*\frac{\partial}{\partial v^\alpha} \tag{3.16}$$

with appropriately chosen coefficients $\eta^\alpha_* = \eta^\alpha_*(x, u, v, \ldots)$.

Proof. Now the invariance condition (3.5) is replaced by

$$X(F_\alpha) = \lambda_\alpha^\beta F_\beta, \quad \alpha = 1, \ldots, m, \tag{3.17}$$

where the prolongation of X to all derivatives involved in Eqs. (3.13) is understood. We know that the simultaneous system (3.13), (3.14) can be obtained as the variational derivatives of the formal Lagrangian

$$\mathcal{L} = v^\alpha F_\alpha \tag{3.18}$$

of Eqs. (3.13). We take an extension of the operator (3.15) in the form (3.16) with undetermined coefficients η_*^α and require that the invariance condition (1.10) be satisfied:

$$Y(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = 0. \tag{3.19}$$

We have:

$$Y(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = Y(v^\alpha)F_\alpha + v^\alpha X(F_\alpha) + v^\alpha F_\alpha D_i(\xi^i)$$
$$= \eta_*^\alpha F_\alpha + \lambda_\alpha^\beta v^\alpha F_\beta + v^\alpha F_\alpha D_i(\xi^i) = [\eta_*^\alpha + \lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)]F_\alpha.$$

Therefore, the requirement (3.19) leads to the equations

$$\eta_*^\alpha = -\big[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)\big], \quad \alpha = 1, \ldots, m, \tag{3.20}$$

with λ_β^α defined by Eqs. (3.17). Since Eqs. (3.19) guarantee the invariance of the system (3.13), (3.14), the adjoint system (3.14) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} - \big[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)\big] \frac{\partial}{\partial v^\alpha} \cdot \tag{3.21}$$

This proves the theorem.

Theorem 3.3. Theorems 3.1 and 3.2 are valid for nonlocal symmetries defined in [5].

Proof. The statement is proved by adding to Eqs. (2.19), (2.21) the differential equations defining nonlocal variables involved in nonlocal symmetries and repeating the proofs of Theorems 3.1 and 3.2.

3.2 Theorem on nonlocal conservation laws

Theorem 3.4. Every Lie point and Lie-Bäcklund symmetry

$$X = \xi^i(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial u^\alpha}\,, \tag{3.22}$$

as well as nonlocal symmetry, of differential equations

$$F_\alpha\big(x, u, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{3.23}$$

provides a *nonlocal conservation law* for Eqs. (3.23). The corresponding conserved quantity involves the adjoint (i.e. nonlocal) variables v given by the adjoint equations (3.14), and hence the resulting conservation laws is, in general, *nonlocal.*

Proof. We take the extended action (3.16) of the operator (3.22),

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha} \cdot \tag{3.24}$$

In the case of one variable $u,$ the coefficient η_* in (3.24) is given by (3.8) if (3.3) is a Lie point symmetry and by (3.11) if (3.3) is Lie-Bäcklund symmetry. In the case of several variables u^α and Lie point symmetries (3.15), the η_*^α are given by (3.20). Now we extend the action of the operators (2.12) to differential functions of $2m$ variables u^α, v^α and obtain the following prolongation of the operators $\mathcal{N}^i$:

$$\begin{aligned} \widetilde{\boldsymbol{\mathcal{N}}}^i &= \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + W_*^\alpha \frac{\delta}{\delta v_i^\alpha} \\ &+ \sum_{s=1}^{\infty} \Big[D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{i i_1 \cdots i_s}^\alpha} + D_{i_1} \cdots D_{i_s}(W_*^\alpha) \frac{\delta}{\delta v_{i i_1 \cdots i_s}^\alpha} \Big], \end{aligned} \tag{3.25}$$

where (see (2.5))

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad W_*^\alpha = \eta_*^\alpha - \xi^j v_j^\alpha, \tag{3.26}$$

and (see (2.13))

$$\begin{aligned} \frac{\delta}{\delta u_i^\alpha} &= \frac{\partial}{\partial u_i^\alpha} - D_{j_1} \frac{\partial}{\partial u_{ij_1}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial u_{ij_1j_2}^\alpha} - \cdots , \\ \frac{\delta}{\delta v_i^\alpha} &= \frac{\partial}{\partial v_i^\alpha} - D_{j_1} \frac{\partial}{\partial v_{ij}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial v_{ij_1j_2}^\alpha} - \cdots , \\ \frac{\delta}{\delta u_{ii_1}^\alpha} &= \frac{\partial}{\partial u_{ii_1}^\alpha} - D_{j_1} \frac{\partial}{\partial u_{ii_1j_1}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial u_{ii_1j_1j_2}^\alpha} - \cdots , \\ \frac{\delta}{\delta v_{ii_1}^\alpha} &= \frac{\partial}{\partial v_{ii_1}^\alpha} - D_{j_1} \frac{\partial}{\partial v_{ii_1j_1}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial v_{ii_1j_1j_2}^\alpha} - \cdots , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3.27}$$

The fundamental identity (2.14) is extended likewise and has the form:

$$Y + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + W_*^\alpha \frac{\delta}{\delta v^\alpha} + D_i \widetilde{\boldsymbol{\mathcal{N}}}^i . \tag{3.28}$$

Now we act by the both sides of the operator identity (3.28) on the formal Lagrangian (3.18), invoke the equations (3.19) and (2.27)-(2.28) obtain the conservation law

$$D_i(C^i)\Big|_{(3.23),(3.14)} = 0, \tag{3.29}$$

where

$$C^i = \widetilde{\mathcal{N}}^i(\mathcal{L}). \tag{3.30}$$

Since the differential function $\mathcal{L} = v^\beta F_\beta$ given by (3.18) does not contain derivatives of the variables v^α, it follows from Eqs. (3.27) that (3.30) reduces to the form

$$C^i = \mathcal{N}^i(\mathcal{L}). \tag{3.31}$$

Substituting in (3.31) the expansion (2.12) of $\mathcal{N}^i$ up to $s = 2$, we obtain the following coordinates of the nonlocal conserved vector:

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\Big) + D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big) - \ldots\right] \tag{3.32}$$

$$+ D_j\big(W^\alpha\big)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big) + \ldots\right] + D_j D_k\big(W^\alpha\big)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\right].$$

Note that if (3.3) is a Lie-Bäcklund symmetry, we have Eq. (1.14) instead of Eq. (3.19), and therefore we should add to (3.31) the term $-B^i$ (cf. (1.15)). However, one can ignore this term since the vector B^i defined by (3.12) vanishes on the solutions of Eq. (3.23) (see Remark 1.1). Finally, we complete the proof by invoking Theorem 3.3.

Remark 3.1. Eq. (3.32) shows that for computing nonlocal conserved vectors by using formal Lagrangians in the form (2.12), $\mathcal{L} = v^\beta F_\beta$, we do not need the expressions W_*^α, and hence the coefficient η_* of the extended operator (3.24).

Remark 3.2. If one changes a formal Lagrangian $\mathcal{L} = v^\beta F_\beta$ to an equivalent form, e.g. as in Examples 2.4 and 2.5, one arrives at a formal Lagrangian containing derivatives of v^α. Then one should use Eq. (3.30) instead of (3.31), and hence one should calculate the coefficient η_* of the operator (3.24).

3.3 An example on Theorems 3.1 and 3.4

Let us consider the heat equation $u_t - u_{xx} = 0$ together with its formal Lagrangian $\mathcal{L} = v(u_t - u_{xx})$ (see Example 2.4) and apply Theorems 3.1 and 3.4 to a Lie point symmetry and a Lie-Bäcklund symmetry. Since we have one dependent variable ($m = 1$) and deal with a second-order formal Lagrangian, Eqs. (3.32) for computing the conserved vectors are written:

$$C^i = \xi^i \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}}\Big)\right] + D_j\big(W\big)\,\frac{\partial \mathcal{L}}{\partial u_{ij}}. \tag{3.33}$$

As an example of a Lie point symmetry I will take the generator

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \tag{3.34}$$

of the Galilean transformation admitted by the heat equation. Let us extend the operator (3.34) to the variable v by means of Theorem 3.1so that the extended generator will be admitted by the adjoint equation $v_t + v_{xx}$ to the heat equation. The prolongation of X to the derivatives involved in the heat equation has the form

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} - \big(xu_t + 2u_x\big)\frac{\partial}{\partial u_t} - \big(2u_x + xu_{xx}\big)\frac{\partial}{\partial u_{xx}}\,\cdot$$

The reckoning shows that Eq. (3.5) is written $X(u_t - u_{xx}) = -x(u_t - u_{xx})$, hence $\lambda = -x$. Noting that in our case $D_i(\xi^i) = 0$ and using (3.8) we obtain $\eta_* = xv$. Hence, the extension (3.9) of the operator (3.34) to v has the form

$$Y = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} + xv\frac{\partial}{\partial v}\,\cdot \tag{3.35}$$

One can readily verify that it is admitted by the system $u_t - u_{xx} = 0,\ v_t + v_{xx} = 0.$

Let us find the conservation law provided by the symmetry (3.34). Denoting $t = x^1, x = x^2$, we have for the extended operator (3.35):

$$\xi^1 = 0, \quad \xi^2 = 2t, \quad \eta = -xu, \quad \eta_* = xv, \quad W = -(xu + 2tu_x). \tag{3.36}$$

Substituting in (3.33) $\mathcal{L} = v(u_t - u_{xx})$ and invoking Eqs. (3.36) we obtain the conservation equation $D_t(C^1) + D_x(C^2) = 0$ for the vector $C = (C^1, C^2)$ with

$$C^1 = W\frac{\partial \mathcal{L}}{\partial u_t} = vW, \quad C^2 = 2t\mathcal{L} + WD_x(v) - vD_x(W).$$

Substituting here the expressions for $\mathcal{L}$ and W, we get:

$$C^1 = -v(xu + 2tu_x), \quad C^2 = v(2tu_t + u + xu_x) - (xu + 2tu_x)v_x. \tag{3.37}$$

This vector involves an arbitrary solution v of the adjoint equation $v_t + v_{xx} = 0$. Since the adjoint equation does not involve u, we can substitute in (3.37) any solution of the adjoint equation and obtain an infinite number of conservation laws for the heat equation. Let us take, e.g. the solutions $v = -1,\ v = -x$ and $v = -\mathrm{e}^t \sin x$. In the first case, we have:

$$C^1 = xu + 2tu_x, \quad C^2 = -(2tu_t + u + xu_x).$$

Noting that $D_t(2tu_x) = D_tD_x(2tu) = D_xD_t(2tu) = D_x(2u + 2tu_t)$ we can transfer the term $2tu_x$ from C^1 to C^2 in the form $2u + 2tu_t$. Then the components of the conserved vector are written simply

$$C^1 = xu,\ C^2 = u - xu_x.$$

In the second case, $v = -x$, we obtain the vector

$$C^1 = x^2u + 2txu_x, \quad C^2 = (2t - x^2)u_x - 2txu_t,$$

and simplifying it as before arrive at

$$C^1 = (x^2 - 2t)u,\ C^2 = (2t - x^2)u_x + 2xu.$$

In the case $v = -\mathrm{e}^t \sin x$, we note that

$$2tu_x\mathrm{e}^t \sin x = D_x(2tu\mathrm{e}^t \sin x) - 2tu\mathrm{e}^t \cos x$$

and simplifying as before obtain:

$$\begin{aligned} C^1 &= \mathrm{e}^t(x\sin x - 2t\cos x)u,\\ C^2 &= (u + 2tu - xu_x)\mathrm{e}^t \sin x + (xu + 2tu_x)\mathrm{e}^t \cos x. \end{aligned}$$

The heat equation has also Lie-Bäcklund symmetries. One of them is

$$X = \big(xu_{xx} + 2tu_{xxx}\big)\frac{\partial}{\partial u} \tag{3.38}$$

We prolong (3.38) to u_t and u_{xx}, denote the prolonged operator again by X and obtain

$$X(u_t - u_{xx}) = xD_x^2(u_t - u_{xx}) + 2tD_x^3(u_t - u_{xx}).$$

It follows that Eq. (3.10) is satisfied and that the only non-vanishing coefficients in (3.10) are $\lambda_2^{22} = x$ and $\lambda_3^{222} = 2t$. Accordingly, Eq. (3.11) yields:

$$\eta_* = -D_x^2(xv) + D_x^3(2tv) = -2v_x - xv_{xx} + 2tv_{xxx},$$

and hence the extension (3.4) of the operator (3.38) to the variable v is

$$Y = \eta\frac{\partial}{\partial u} + \eta_*\frac{\partial}{\partial v} \equiv \big(xu_{xx} + 2tu_{xxx}\big)\frac{\partial}{\partial u} + \big(2tv_{xxx} - 2v_x - xv_{xx}\big)\frac{\partial}{\partial v}\cdot \tag{3.39}$$

For the operator (3.39), we have

$$\xi^1 = \xi^2 = 0, \quad W = \eta = xu_{xx} + 2tu_{xxx}.$$

Therefore, (3.33) provides the conserved vector with the following components:

$$\begin{aligned} C^1 &= W\frac{\partial\mathcal{L}}{\partial u_t} = \eta v = [xu_{xx} + 2tu_{xxx}]v,\\ C^2 &= 2t\mathcal{L} - WD_x\Big(\frac{\partial\mathcal{L}}{\partial u_{xx}}\Big) + D_x(W)\frac{\partial\mathcal{L}}{\partial u_{xx}} = [2t(u_t - u_{xx}) - D_x(\eta)]v + \eta D_x(v)\\ &= [2t(u_t - u_{xx} - u_{xxxx}) - u_{xx} - xu_{xxx}]v + [u_{xx} + 2tu_{xxx}]v_x. \end{aligned}$$

4 Application to the Korteweg-de Vries equation

4.1 Generalities

The KdV equation (2.32),

$$u_t = u_{xxx} + uu_x, \tag{2.32}$$

has the formal Lagrangian (2.34),

$$\mathcal{L} = v[u_t - uu_x - u_{xxx}]. \tag{2.34}$$

Since we have one dependent variable ($m = 1$) and deal with a third-order formal Lagrangian, Eqs. (3.32) for computing the conserved vectors are written:

$$\begin{aligned} C^i = \xi^i \mathcal{L} + W\Big[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}}\Big) + D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\Big)\Big] \\ + D_j(W)\,\Big[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\Big)\Big] + D_j D_k(W)\,\frac{\partial \mathcal{L}}{\partial u_{ijk}}\,\cdot \end{aligned} \tag{4.1}$$

As before, I will set $t = x^1, x = x^2$ and write the conservation equation in the form

$$D_t(C^1) + D_x(C^2) = 0.$$

Let us begin by applying Theorem 3.4 to the generators X_1 and X_2 of the Galilean and scaling transformations, respectively:

$$X_1 = \frac{\partial}{\partial u} - t\frac{\partial}{\partial x}\,, \quad X_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\,,$$

which are obviously admitted by of the KdV equation.

The operator X_1 yields the conservation law $D_t(C^1) + D_x(C^2) = 0$, where the conserved vector $C = (C^1, C^2)$ is given by (4.1) and has the components

$$C^1 = (1 + tu_x)v, \quad C^2 = t(v_x u_{xx} - u_x v_{xx} - vu_t) - uv - v_{xx}.$$

Since the KdV equation is self-adjoint (see Example 2.5), we let $v = u$, transfer the term $tuu_x = D_x(\frac{1}{2}tu^2)$ from C^1 to C^2 in the form $tuu_t + \frac{1}{2}u^2$ and obtain

$$C^1 = u, \quad C^2 = -\frac{1}{2}u^2 - u_{xx}. \tag{4.2}$$

Let us make more detailed calculations for the operator X_2. For this operator, we have $W = (2u + 3tu_t + xu_x)$ and the vector (4.1) is written:

$$\begin{aligned} C^1 &= -3t\mathcal{L} + Wv = (3tu_{xxx} + 3tuu_x + xu_x + 2u)v, \\ C^2 &= -x\mathcal{L} - (uv + v_{xx})W + v_x D_x(W) - vD_x^2(W) = -(2u^2 + xu_t + 3tuu_t \\ &\quad + 4u_{xx} + 3tu_{txx})v + (3u_x + 3tu_{tx} + xu_{xx})v_x - (2u + 3tu_t + xu_x)v_{xx}. \end{aligned}$$

As before, we let $v = u$, simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2 and obtain

$$C^1 = u^2, \quad C^2 = u_x^2 - 2uu_{xx} - \frac{2}{3}\,u^3. \tag{4.3}$$

Remark 4.1. One can use also the second-order formal Lagrangians, (2.35) or (2.36). Since they involve the derivatives of v, one should use Eq. (3.30) for computing the conserved vectors (see Remark 3.2). Since the formal Lagrangians (2.35) and (2.36) contain only the first-order derivatives of v, one can use the operator (3.25)in the truncated form:

$$\widetilde{\boldsymbol{\mathcal{N}}}^{\,i} = \xi^i + W\,\left[\frac{\partial}{\partial u_i} - D_j\,\frac{\partial}{\partial u_{ij}}\right] + D_j(W)\,\frac{\partial}{\partial u_{ij}} + W_*\,\frac{\partial}{\partial v_i}\,\cdot \tag{4.4}$$

The reckoning shows that the extension (3.4) of X_1 to v coincides with X_1. To find the extension of X_2, we prolong it to the derivatives involved in the KdV equation:

$$X_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 5u_t\frac{\partial}{\partial u_t} + 3u_x\frac{\partial}{\partial u_x} + 4u_{xx}\frac{\partial}{\partial u_{xx}} + 5u_{xxx}\frac{\partial}{\partial u_{xxx}}\,,$$

and get $X_2(u_t - uu_x - u_{xxx}) = 5(u_t - uu_x - u_{xxx})$. Whence $\lambda = 5$. Since $D_i(\xi^i) = -4$, Eq. (3.8) yields $\eta_* = -v$. Thus, the extension (3.4) of X_2 is

$$Y_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\,\cdot$$

Eqs. (3.26) yield

$$W = 1 + tu_x, \quad W_* = tv_x$$

and

$$W = 2u + 3tu_t + xu_x, \quad W_* = -v + 3tv_t + xv_x$$

for $Y_1 = X_1$ and Y_2, respectively. Substituting these expressions for W, W_* in (4.4) and applying Eq. (3.30), e.g. to the formal Lagrangian (2.35), we arrive again the conserved vectors (4.2) and (4.3). It is manifest from these calculations that the use of the second-order formal Lagrangians does not simplify the computation of conserved vectors. Therefore, I will will use further the third-order formal Lagrangian (2.34).

4.2 Nonlocal conservation laws furnished by local symmetries

Let us find conservation laws associated with the known infinite algebra of local (Lie-Bäcklund) and nonlocal symmetries of the KdV equation (see, e.g. [7]; see also [2], Ch. 4 and [8], Ch. 5, and the references therein). I will write here only the first component of the corresponding conserved vector (4.1). Let us begin with local symmetries.

The Lie-Bäcklund symmetry of the lowest (fifth) order is

$$X_3 = f_5 \frac{\partial}{\partial u} \quad \text{with} \quad f_5 = u_5 + \frac{5}{3} u u_3 + \frac{10}{3} u_2 u_2 + \frac{5}{6} u^2 u_1, \tag{4.5}$$

where $u_1 = u_x,\ u_2 = u_{xx}, \ldots$. The reckoning shows that the invariance condition (3.10) for $F = u_t - u u_x - u_{xxx}$ is satisfied in the following form:

$$X_3(F) = \Big[\frac{5}{3}(u_3 + u u_1) + \frac{5}{6}(4u_2 + u^2) D_x + \frac{10}{3} u_1 D_x^2 + \frac{5}{3} u D_x^3 + D_x^5\Big](F).$$

The first component of the nonlocal conserved vector (4.1) is $C^1 = v f_5$, i.e.

$$C^1 = \Big(u_5 + \frac{5}{3} u u_3 + \frac{10}{3} u_2 u_2 + \frac{5}{6} u^2 u_1\Big) v. \tag{4.6}$$

Upon setting $v = u$, we have

$$\Big(u_5 + \frac{5}{3} u u_3 + \frac{10}{3} u_2 u_2 + \frac{5}{6} u^2 u_1\Big) u = D_x\Big(u u_4 - u_1 u_3 + \frac{1}{2} u_2^2 + \frac{5}{3} u^2 u_2 + \frac{5}{24} u^4\Big).$$

Hence, the Lie-Bäcklund symmetry (4.5) provides only a trivial local conserved vector, i.e. with $C^1 = 0$. The reckoning shows that all local higher-order (Lie-Bäcklund) symmetries lead to nonlocal conserved vectors, the first component of which are similar to (4.6), but contain higher-order derivatives, and vanish upon setting $v = u$. Thus, the *local higher-order symmetries lead to essentially nonlocal conservation laws.*

4.3 Local conservation laws furnished by nonlocal symmetries

Let us apply our technique to nonlocal symmetries (see Theorem 3.3). The KdV equation has an infinite set of nonlocal symmetries, namely:

$$\mathcal{X}_{n+2} = g_{n+2} \frac{\partial}{\partial u}\,, \tag{4.7}$$

where g_{n+2} are given recurrently by ([7], see also [2], Eq. (18.36))

$$g_1 = 1 + t u_1, \quad g_{n+2} = \Big(D_x^2 + \frac{2}{3}\, u + \frac{2}{3}\, D_x^{-1}\Big) g_n, \quad n = 1, 3, \ldots. \tag{4.8}$$

The operator $\mathcal{X}_1 = (1 + t u_1) \frac{\partial}{\partial u}$ corresponding to g_1 is the canonical Lie-Bäcklund representation of the generator X_1 of the Galilean transformation (Example 2.1). Eq. (4.8) yields $g_3 = (1/3)[2u + 3t(u_3 + u u_1) + x u_1] \equiv (1/3)(2u + 3t u_t + x u_x)$, hence $\mathcal{X}_3$ coincides, up to the constant factor $1/3$, with the canonical Lie-Bäcklund representation of the scaling generator X_2 (cf. Example 2.2).

Continuing the recursion (4.8), we arrive at the following nonlocal symmetry of the KdV equation:

$$\mathcal{X}_5 = g_5 \frac{\partial}{\partial u} \quad \text{with} \quad g_5 = tf_5 + \frac{x}{3}(u_3 + uu_1) + \frac{4}{3}u_2 + \frac{4}{9}u^2 + \frac{1}{9}u_1\varphi,$$

where f_5 is the coordinate of the Lie-Bäcklund operator X_3 used above and φ is a *nonlocal variable* defined by the following integrable system of equations:

$$\varphi_x = u, \quad \varphi_t = u_{xx} + \frac{1}{2}u^2 \cdot$$

The first component of the conserved vector (4.1) is given by $C^1 = vg_5$. Setting $v = u$, transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2, eliminating an immaterial constant factor and returning to the original notation $u_1 = u_x$, we arrive at a non-trivial conservation law with

$$C^1 = u^3 - 3u_x^2. \tag{4.9}$$

The nonlocal variable φ is involved in the component C^2 of the conserved vector. We can also take in $C^1 = vg_5$ the solution $v = 1$ of the adjoint equation (2.33), $v_t = v_{xxx} + uv_x$. Then we will arrive again to the conserved vector (4.3).

Dealing likewise with all nonlocal symmetries (4.7), we obtain the renown infinite set of non-trivial conservation laws of the KdV equation. For example, $\mathcal{X}_7$ yields

$$C^1 = 29\,u^4 + 852\,uu_1^2 - 252\,u_2^2. \tag{4.10}$$

5 Further discussion

5.1 Derivation of local and nonlocal conservation laws for nonlinear equations that are not self-adjoint but have a Lagrangian

Noether's theorem on local conservation laws (Theorem 1.1) and my theorem on nonlocal conservation laws (Theorem 3.4) are distinctly different even for equations having usual Lagrangians. To illustrate the difference, consider the following examples.

Example 5.1. Consider the equation

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0, \tag{5.1}$$

describing the non-steady-state potential gas flow with transonic speeds. It has the Lagrangian

$$\mathcal{L} = -u_t u_x - \frac{1}{6}u_x^3 + \frac{1}{2}u_y^2. \tag{5.2}$$

Application of Theorem 1.1, e.g. to the generator

$$X = \frac{\partial}{\partial t} \tag{5.3}$$

of the time translation group leaving invariant the variational integral with the Lagrangian (5.2) provides the *local conservation law*

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0$$

with the following components[1]:

$$C^1 = \frac{1}{2}\, u_y^2 - \frac{1}{6}\, u_x^3, \quad C^2 = u_t^2 + \frac{1}{2}\, u_t u_x^2, \quad C^3 = -u_t u_y. \tag{5.4}$$

On the other hand, Eq. (5.1) has the second-order formal Lagrangian

$$\mathcal{L}_* = (2u_{tx} + u_x u_{xx} - u_{yy})v \tag{5.5}$$

which can be replaced by the first-order formal Lagrangian (cf. (5.2))

$$\mathcal{L}_* = -2v_x u_t - \frac{1}{2} v_x u_x^2 + v_y u_y. \tag{5.6}$$

Accordingly, the adjoint equation to (5.1) is

$$2v_{tx} + u_x v_{xx} + v_x u_{xx} - v_{yy} = 0. \tag{5.7}$$

It is manifest from Eqs. (5.1) and (5.7) that Eq. (5.1) is not self-adjoint.

Application of Theorem 3.4 to any of the equivalent formal Lagrangians nonself.eq4 and nonself.eq5 furnishes the following *nonlocal conserved vector* associated with the symmetry (5.3):

$$C^1 = v_y u_y - \frac{1}{2}\, v_x u_x^2, \quad C^2 = 2v_t u_t + u_t u_x v_x + v_t u_x^2, \quad C^3 = -u_t v_y - v_t u_y. \tag{5.8}$$

Thus, one symmetry (5.3) generates two different conserved vectors, (5.4) and (5.8).

Example 5.2. The nonlinear wave equation

$$u_{tt} - \Delta u + a u^3 = 0, \quad a = \text{const.}, \tag{5.9}$$

where Δu is the three-dimensional Laplacian, has the Lagrangian

$$L = |\nabla u|^2 - u_t^2 + \frac{a}{2} u^4. \tag{5.10}$$

Let us write conservation laws in the form

$$D_t(\tau) + (\nabla \cdot \boldsymbol{\chi}) = 0,$$

where τ is the density of the conservation law and $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$.

[1] For the calculations see [2], Section 23.3, p. 329. Note that the second component is misprinted there as $C^2 = u_t^2 + \frac{1}{2}\, u_x^2$.

Application of Theorem 1.1 to the Lagrangian (5.10) and to the operator (5.3) admitted by Eq. (5.9) yields the conservation law with the density (see [2], Eq. (24.10))

$$\tau = u_t^2 + |\nabla u|^2 + \frac{a}{2}u^4. \tag{5.11}$$

On the other hand, Eq. (5.9) has the second-order formal Lagrangian

$$\mathcal{L}_* = (u_{tt} - \Delta u + au^3)v \tag{5.12}$$

which can be replaced by the first-order formal Lagrangian (cf. (5.10))

$$\mathcal{L}_* = -u_t v_t + \nabla u \cdot \nabla v + 3avu^3. \tag{5.13}$$

Accordingly, the adjoint equation to (5.9) is

$$v_{tt} - \Delta v + 3avu^2 = 0. \tag{5.14}$$

It is manifest from Eqs. (5.9) and (5.14) that Eq. (5.9) is not self-adjoint.

Application of Theorem 3.4 to any of the equivalent formal Lagrangians nonself.eq10 or nonself.eq11 and to the symmetry (5.3) yields the *nonlocal conservation law* with the density (cf. (5.11))

$$\tau = u_t v_t + |\nabla u| \cdot |\nabla v| + avu^3. \tag{5.15}$$

5.2 Determination of self-adjoint equations

Example 5.3. We have used the remarkable property of the KdV equation to be self-adjoint for deriving an infinite series of local conservation laws. Let us consider a more general set of equations containing the KdV equation as a particular case, namely:

$$u_t - u_{xxx} - f(x, u, u_x) = 0 \tag{5.16}$$

and single out all self-adjoint equations. We have:

$$\frac{\delta}{\delta u}\Big[(u_t - u_{xxx} - f)v\Big] = -v_t + v_{xxx} - vf_u + D_x\big(vf_{u_x}\big).$$

Hence, the adjoint equation to (5.16) has the form

$$-v_t + v_{xxx} + v_x f_{u_x} + \big(-f_u + f_{xu_x} + u_x f_{uu_x} + u_{xx} f_{u_x u_x}\big)v = 0.$$

Letting $v = u$ we obtain

$$-u_t + u_{xxx} + u_x f_{u_x} + \big(-f_u + f_{xu_x} + u_x f_{uu_x} + u_{xx} f_{u_x u_x}\big)u = 0. \tag{5.17}$$

Comparison with (5.16) yields $f_{u_x u_x} = 0$, whence

$$f(x, u, u_x) = \varphi(x, u)u_x + \psi(x, u). \tag{5.18}$$

Now Eqs. (5.16) and (5.17) take the form

$$u_t - u_{xxx} - \varphi u_x - \psi = 0 \tag{5.19}$$

and

$$u_t - u_{xxx} - \varphi u_x - (\varphi_x - \psi_u)u = 0, \tag{5.20}$$

respectively. Whence, $(\varphi_x - \psi_u)u = \psi$, or

$$u\frac{\partial \psi}{\partial u} + \psi = u\varphi_x. \tag{5.21}$$

Given an arbitrary function $\varphi(x, u)$, we integrate the linear first-order ordinary differential equation (5.21) for ψ with respect to the variable u and obtain

$$\psi(x, u) = \frac{1}{u}\Big[\int u\varphi_x du + \alpha(x)\Big],$$

thus arriving at the following result.

Proposition 5.1. The general self-adjoint equation of the form (5.21) is

$$u_t - u_{xxx} - \varphi(x, u)u_x - \frac{1}{u}\Big[\int u\varphi_x(x, u)du + \alpha(x)\Big] = 0, \tag{5.22}$$

where $\varphi(x, u)$ and $\alpha(x)$ are arbitrary functions. In particular, the equation

$$u_t - u_{xxx} - f(u, u_x) = 0$$

is self-adjoint if and only if it has the form

$$u_t - u_{xxx} - \varphi(u)u_x - \frac{a}{u} = 0, \quad a = \text{const}. \tag{5.23}$$

According to Proposition 5.1, the KdV equation (2.32) and the modified KdV equation

$$u_t = u_{xxx} + u^2 u_x \tag{5.24}$$

are self-adjoint. Using this property of the modified KdV equation and the known recursion operator (see, e.g. [2], Eq. (19.50))

$$\mathcal{R} = D_x^2 + \frac{2}{3}u^2 + \frac{2}{3}u_x D_x^{-1} u \tag{5.25}$$

for the modified KdV equation (cf. (5.25) and (4.8)), one can apply Theorem 3.4 to Eq. (5.24) and, proceeding as in Section 4, compute local and nonlocal conservation laws for the modified KdV equation.

Example 5.4. Let us single out the self-adjoint equations from the set of the equations

$$u_t = f(u)u_{xxx}. \tag{5.26}$$

Writing the adjoint equation:

$$\frac{\delta}{\delta u}\Big[(u_t - f(u)u_{xxx})v\Big] = -v_t - f'(u)vu_{xxx} + D_x^3\big(vf(u)\big) = 0$$

and letting $v = u$, we obtain:

$$-u_t + fu_{xxx} + 3(2f' + uf'')u_x u_{xx} + (3f'' + uf''')u_x^3 = 0. \tag{5.27}$$

Comparison of Eq. (5.27) with (5.26) yields $2f' + uf'' = 0,\ 3f'' + uf''' = 0$. Since the second equation is obtained from the first one by differentiation, we integrate the equation $2f' + uf'' = 0$ and obtain:

$$f(u) = \frac{a}{u} + b, \quad a, b = \text{const}.$$

Proposition 5.2. The general self-adjoint equation of the form (5.26) is

$$u_t = \Big(\frac{a}{u} + b\Big)u_{xxx}. \tag{5.28}$$

Example 5.5. Consider the second-order equations of the form

$$u_t = f(u)u_{xx}. \tag{5.29}$$

Proceeding as in the previous example, one can prove the following statement.

Proposition 5.3. The general self-adjoint equation of the form (5.29) is

$$u_t = \frac{a}{u}\,u_{xx}. \tag{5.30}$$

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Archives of ALGA
Volume 3, 2006

Adjoint system and conservation laws for symmetrized electromagnetic equations with a dual Ohm's law

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Introduction

In 1921, Bessel-Hagen applied Noether's theorem to the Maxwell equations in vacuum written in terms of the vector potential and the electromagnetic tensor since in this form the Maxwell equations have a Lagrangian. Bessel-Hagen employed the conformal invariance of the equations in question and found five new conservation laws along with the well-known energy-momentum tensor corresponding to the Lorentz invariance of the Maxwell equations (see [1][1] and the references therein).

We apply a new method for deriving conservation laws from symmetries, developed recently by Ibragimov [2][2], to a symmetrized version of the Maxwell-Lorentz microscopic equations, allowing magnetic charges and magnetic currents, where the latter, just as electric currents, are assumed to be described by a linear relationship between the field and the current, i.e. an Ohm's law. The conservation laws obtained in the paper contain two new adjoint vector fields which fulfil Maxwell-like equations. In particular, using the two-solution representation, we obtain the conservation laws for the electromagnetic field which are nonlocal in time.

[1] For its English translation, see the second paper in this volume.
[2] See also the third paper in this volume.

1 Formulation of the main conservation theorem

Let $x = (x^1, \ldots, x^n)$ be independent variables and $u = (u^1, \ldots, u^m)$ be dependent variables. The set of the first-order partial derivatives

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i} \qquad (\alpha = 1, \ldots, m;\ i = 1, \ldots, n),$$

will be denoted by $u_{(1)} = \{u_i^\alpha\}$. The similar notation is used for higher-order derivatives, $u_{(2)} = \{u_{ij}^\alpha\}$, etc. The symbol D_i denotes the total differentiation with respect to x^i :

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + u_{ijk}^\alpha \frac{\partial}{\partial u_{jk}^\alpha} + \cdots$$

We employ the usual convention of summation in repeated indices.

Consider an arbitrary system of sth-order partial differential equations

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{1.1}$$

where the functions $F_\alpha(x, u, u_{(1)}, \ldots, u_{(s)})$ involve the independent variables x and the dependent variables u together with their derivatives up to an arbitrary order s.

An *infinitesimal symmetry* of Eqs. (1.1) is a generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{1.2}$$

of a continuous transformation group leaving invariant Eqs. (1.1). Here $\xi^i(x, u)$ and $\eta^\alpha(x, u)$ are real-valued functions of x and u.

A vector field $C = (C^1, \ldots, C^n)$, where

$$C^i = C^i(x, u, u_{(1)}, \ldots), \quad i = 1, \ldots, n,$$

is said to be a *conserved vector* for the differential equations (1.1) if the equation

$$D_i\big(C^i\big) = 0 \tag{1.3}$$

holds for any solution of Eqs. (1.1). If one of the independent variables is time, e.g. $x^n = t$, then the conservation law (1.3) is often written, using the divergence theorem, in the form

$$\frac{dE}{dt} = 0,$$

where

$$E = \int\limits_{\mathbb{R}^{n-1}} C^n\big(x, u(x), u_{(1)}(x), \ldots\big) dx^1 \cdots dx^{n-1}. \tag{1.4}$$

Accordingly, C^n is termed the *density* of the conservation law.

We will use the recent general theorem on a connection between symmetries and conservation laws (see [2], Theorem 3.5). In the case of systems of first-order differential equations this theorem is formulated as follows.

Theorem 1.1. Let an operator (1.2) be a symmetry of a system of first-order partial differential equations

$$F_\alpha\big(x, u, u_{(1)}\big) = 0, \quad \alpha = 1, \ldots, m. \tag{1.5}$$

where $u = (u^1, \ldots, u^m)$. Then the quantities

$$C^i = v^\beta \left[\xi^i F_\beta + \big(\eta^\alpha - \xi^j u_j^\alpha\big) \frac{\partial F_\beta}{\partial u_i^\alpha}\right], \quad i = 1, \ldots, n, \tag{1.6}$$

furnish a conserved vector $C = (C^1, \ldots, C^n)$ for the equations (1.5) considered together with the adjoint system

$$F_\alpha^*\big(x, u, v, u_{(1)}, v_{(1)}\big) \equiv \frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{1.7}$$

where $v = (v^1, \ldots, v^m)$ are new dependent variables, i.e. $v = v(x)$,

$$\mathcal{L} = v^\beta F_\beta\big(x, u, u_{(1)}, \ldots, u_{(s)}\big), \tag{1.8}$$

and

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha}, \quad \alpha = 1, \ldots, m.$$

Remark 1.1. Using Eq. (1.8), one can write the conserved vector (1.6) as follows:

$$C^i = \mathcal{L}\xi + \big(\eta_\mu^\alpha - \xi u_j^\alpha\big) \frac{\partial \mathcal{L}}{\partial u_i^\alpha}, \quad i = 1, \ldots, n. \tag{1.9}$$

Remark 1.2. If Eqs. (1.5) have r symmetries $X_1, \ldots, X_r$ of the form (1.2),

$$X_\mu = \xi_\mu^i(x, u) \frac{\partial}{\partial x^i} + \eta_\mu^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad \mu = 1, \ldots, r,$$

then Eqs. (1.9) provide r conserved vectors $C_1, \ldots, C_r$ with the components

$$C_\mu^i = \mathcal{L}\xi_\mu^i + \big(\eta_\mu^\alpha - \xi_\mu^j u_j^\alpha\big) \frac{\partial \mathcal{L}}{\partial u_i^\alpha}, \quad \mu = 1, \ldots, r; \;\; i = 1, \ldots, n.$$

2 Electromagnetic equations and their symmetries

2.1 Basic and adjoint equations

Adopting Dirac's ideas on the existence of magnetic monopoles [3], one can formulate a symmetrized version of Maxwell's electromagnetic equations (see, e.g. [4], [5]). We shall use the electromagnetic equations in the following form (see [6], Eqs. (1.50)):

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} + \mu_0 \boldsymbol{j}_{\,m} = 0, \tag{2.1}$$

$$\nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \mu_0 \boldsymbol{j}_{\,e} = 0, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{E} - \mu_0 c^2 \rho_e = 0, \tag{2.3}$$

$$\nabla \cdot \boldsymbol{B} - \mu_0 \rho_m = 0, \tag{2.4}$$

together with the dual Ohm's law

$$\boldsymbol{j}_{\,e} = \sigma_e \boldsymbol{E}, \quad \boldsymbol{j}_{\,m} = \sigma_m \boldsymbol{B}. \tag{2.5}$$

The first equation in (2.5) is Ohm's law for electric currents. The second equation is a dual Ohm's law for magnetic currents, that has been introduced for symmetry reasons (see [4], Eq. (1-5), and [6], Eq. (2.20); see also Eq.(38) in [7], and its generalization Eq.(8) in [8]). For homogeneous media, considered in the most part of the present paper, the coefficients σ_e, σ_m and μ_0 are given constants.

Now we substitute Eqs. (2.5) in Eqs. (2.1)-(2.2). The resulting equations involve, along with the light velocity c, three other constants, σ_e, σ_m and μ_0. We eliminate two constants by setting

$$\begin{aligned} &\tilde{t} = ct, \quad \widetilde{\boldsymbol{B}} = c\boldsymbol{B}, \quad \tilde{\sigma}_e = c\mu_0\sigma_e, \quad \tilde{\sigma}_m = \frac{\mu_0}{c}\,\sigma_m, \\ &\tilde{\rho}_e = c^2\mu_0\rho_e, \quad \tilde{\rho}_m = c\mu_0\rho_m \end{aligned} \tag{2.6}$$

and rewrite our basic equations, discarding tilde, as follows:

$$\begin{aligned} &\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} + \sigma_m \boldsymbol{B} = 0, \\ &\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} - \sigma_e \boldsymbol{E} = 0, \\ &\nabla \cdot \boldsymbol{E} - \rho_e = 0, \\ &\nabla \cdot \boldsymbol{B} - \rho_m = 0. \end{aligned} \tag{2.7}$$

The system (2.7) has eight equations for eight dependent variables: six coordinates of the electric and magnetic vector fields $\boldsymbol{E} = (E^1, E^2, E^3)$ and $\boldsymbol{B} = (B^1, B^2, B^3)$, respectively, and two scalar quantities, namely, the electric and magnetic monopole charge densities ρ_e and ρ_m.

The expression (1.8) for Eqs. (2.7) has the following form:

$$
\begin{aligned}
\mathcal{L} =& \boldsymbol{V}\cdot\Big(\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}+\sigma_m\boldsymbol{B}\Big)+R_e\Big(\nabla\cdot\boldsymbol{E}-\rho_e\Big)\\
&+\boldsymbol{W}\cdot\Big(\nabla\times\boldsymbol{B}-\frac{\partial\boldsymbol{E}}{\partial t}-\sigma_e\boldsymbol{E}\Big)+R_m\Big(\nabla\cdot\boldsymbol{B}-\rho_m\Big),
\end{aligned}
\tag{2.8}
$$

where $\boldsymbol{V},\ \boldsymbol{W},\ R_e,\ R_m$ are the adjoint variables. We have:

$$
\begin{aligned}
&\frac{\delta\mathcal{L}}{\delta\boldsymbol{E}}=\nabla\times\boldsymbol{V}+\frac{\partial\boldsymbol{W}}{\partial t}-\sigma_e\boldsymbol{W}-\nabla R_e,\quad \frac{\delta\mathcal{L}}{\delta\rho_e}=-R_e,\\
&\frac{\delta\mathcal{L}}{\delta\boldsymbol{B}}=\nabla\times\boldsymbol{W}-\frac{\partial\boldsymbol{V}}{\partial t}+\sigma_m\boldsymbol{V}-\nabla R_m,\quad \frac{\delta\mathcal{L}}{\delta\rho_m}=-R_m,
\end{aligned}
\tag{2.9}
$$

and hence the adjoint equations to Eqs. (2.7) are written:

$$
\begin{aligned}
&\nabla\times\boldsymbol{V}+\frac{\partial\boldsymbol{W}}{\partial t}-\sigma_e\boldsymbol{W}=0,\\
&\nabla\times\boldsymbol{W}-\frac{\partial\boldsymbol{V}}{\partial t}+\sigma_m\boldsymbol{V}=0,\\
&R_e=0,\qquad R_m=0.
\end{aligned}
\tag{2.10}
$$

Let us denote the spatial coordinates x^1,x^2,x^3 by x,y,z. It is convenient for further computations to use the following coordinate representation of $\mathcal{L}$ given by Eq. (2.8):

$$
\begin{aligned}
\mathcal{L}=&\,V^1\,(E_y^3-E_z^2+B_t^1+\sigma_m B^1)+V^2\,(E_z^1-E_x^3+B_t^2+\sigma_m B^2)\\
&+V^3\,(E_x^2-E_y^1+B_t^3+\sigma_m B^3)+R_e(E_x^1+E_y^2+E_z^3-\rho_e)\\
&+W^1\,(B_y^3-B_z^2-E_t^1-\sigma_e E^1)+W^2\,(B_z^1-B_x^3-E_t^2-\sigma_e E^2)\\
&+W^3\,(B_x^2-B_y^1-E_t^3-\sigma_e E^3)+R_m(B_x^1+B_y^2+B_z^3-\rho_m).
\end{aligned}
\tag{2.11}
$$

2.2 Symmetries

The system of equations (2.7) is invariant under the translations of time t and the position vector $\boldsymbol{x}=(x,y,z)$ as well as the simultaneous rotations of the vectors $\boldsymbol{x},\ \boldsymbol{E}$ and $\boldsymbol{B}$ due to the vector formulation of Eqs. (2.7). These geometric transformations provide the following seven infinitesimal symmetries:

$$
\begin{aligned}
& X_0 = \frac{\partial}{\partial t}\,, \quad X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = \frac{\partial}{\partial z}\,, \\
& X_{12} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + E^2\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^2} + B^2\frac{\partial}{\partial B^1} - B^1\frac{\partial}{\partial B^2}\,, \\
& X_{13} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^3} + B^3\frac{\partial}{\partial B^1} - B^1\frac{\partial}{\partial B^3}\,, \\
& X_{23} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^2} - E^2\frac{\partial}{\partial E^3} + B^3\frac{\partial}{\partial B^2} - B^2\frac{\partial}{\partial B^3}\,\cdot
\end{aligned}
\tag{2.12}
$$

The infinitesimal symmetries for the adjoint system (2.10) are obtained from (2.12) by replacing the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ by $\boldsymbol{V}$ and $\boldsymbol{W}$, respectively.

Furthermore, it follows from the homogeneity of Eqs. (2.7) that they admit the simultaneous dilations of all dependent variables with the generator

$$
T = \boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{B}\cdot\frac{\partial}{\partial \boldsymbol{B}} + \rho_e\frac{\partial}{\partial \rho_e} + \rho_m\frac{\partial}{\partial \rho_m}\,, \tag{2.13}
$$

where

$$
\boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{E}} = \sum_{i=1}^{3} E^i\frac{\partial}{\partial E^i}\,, \quad \boldsymbol{B}\cdot\frac{\partial}{\partial \boldsymbol{B}} = \sum_{i=1}^{3} B^i\frac{\partial}{\partial B^i}\,\cdot
$$

Moreover, since Eqs. (2.7) are linear, they admit the usual superposition principle, i.e. they are invariant with respect to addition of any given solution. In other words, Eqs. (2.7) admit the operator

$$
S = \mathcal{E}(\boldsymbol{x},t)\cdot\frac{\partial}{\partial \boldsymbol{E}} + \mathcal{B}(\boldsymbol{x},t)\cdot\frac{\partial}{\partial \boldsymbol{B}} + \phi_e(\boldsymbol{x},t)\frac{\partial}{\partial \rho_e} + \phi_m(\boldsymbol{x},t)\frac{\partial}{\partial \rho_m}\,, \tag{2.14}
$$

where

$$
\boldsymbol{E} = \mathcal{E}(\boldsymbol{x},t), \quad \boldsymbol{B} = \mathcal{B}(\boldsymbol{x},t), \quad \rho_e = \phi_e(\boldsymbol{x},t), \quad \rho_m = \phi_m(\boldsymbol{x},t)
$$

is any particular solution of Eqs. (2.7).

Recall that the Maxwell equations in vacuum also admit the one-parameter group of *dual transformations*

$$
\overline{\boldsymbol{E}} = \boldsymbol{E}\cos\alpha - \boldsymbol{B}\sin\alpha, \quad \overline{\boldsymbol{B}} = \boldsymbol{E}\sin\alpha + \boldsymbol{B}\cos\alpha
$$

with the generator

$$
X = \boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{B}} - \boldsymbol{B}\cdot\frac{\partial}{\partial \boldsymbol{E}} \equiv \sum_{i=1}^{3}\left(E^i\frac{\partial}{\partial B^i} - B^i\frac{\partial}{\partial E^i}\right)\cdot
$$

It is shown in [2] that this group provides the conservation of energy for the Maxwell equations. Let us clarify if Eqs. (2.7) admit a similar group. Let

$$X = \boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{B}} - \boldsymbol{B}\cdot\frac{\partial}{\partial \boldsymbol{E}} + \rho_e\frac{\partial}{\partial \rho_m} - \rho_m\frac{\partial}{\partial \rho_e}\cdot \tag{2.15}$$

The prolongation of the operator (2.15) is written

$$X = \boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{B}} - \boldsymbol{B}\cdot\frac{\partial}{\partial \boldsymbol{E}} + \rho_e\frac{\partial}{\partial \rho_m} - \rho_m\frac{\partial}{\partial \rho_e} + \boldsymbol{E}_t\cdot\frac{\partial}{\partial \boldsymbol{B}_t} - \boldsymbol{B}_t\cdot\frac{\partial}{\partial \boldsymbol{E}_t} \tag{2.16}$$
$$+ \boldsymbol{E}_x\cdot\frac{\partial}{\partial \boldsymbol{B}_x} - \boldsymbol{B}_x\cdot\frac{\partial}{\partial \boldsymbol{E}_x} + \boldsymbol{E}_y\cdot\frac{\partial}{\partial \boldsymbol{B}_y} - \boldsymbol{B}_y\cdot\frac{\partial}{\partial \boldsymbol{E}_y} + \boldsymbol{E}_z\cdot\frac{\partial}{\partial \boldsymbol{B}_z} - \boldsymbol{B}_z\cdot\frac{\partial}{\partial \boldsymbol{E}_z}\cdot$$

Reckoning shows that the operator (2.16) acts on the left-hand sides of Eqs. (2.7) as follows:

$$\begin{aligned}
&X\big(\nabla\times\boldsymbol{E} + \boldsymbol{B}_t + \sigma_m\boldsymbol{B}\big) = -\big(\nabla\times\boldsymbol{B} - \boldsymbol{E}_t - \sigma_m\boldsymbol{E}\big),\\
&X\big(\nabla\times\boldsymbol{B} - \boldsymbol{E}_t - \sigma_e\boldsymbol{E}\big) = \quad \nabla\times\boldsymbol{E} + \boldsymbol{B}_t + \sigma_e\boldsymbol{B},\\
&X\big(\nabla\cdot\boldsymbol{E} - \rho_e\big) = -\big(\nabla\cdot\boldsymbol{B} - \rho_m\big),\\
&X\big(\nabla\cdot\boldsymbol{B} - \rho_m\big) = \quad \nabla\cdot\boldsymbol{E} - \rho_e.
\end{aligned}$$

It follows that the operator (2.15) is admitted by Eqs. (2.7) only in the case

$$\sigma_m = \sigma_e. \tag{2.17}$$

Note that in the case (2.17) Eqs. (2.7) admit also the operator

$$X = t\frac{\partial}{\partial t} + \boldsymbol{x}\frac{\partial}{\partial \boldsymbol{x}} - \sigma_e t\Big(\boldsymbol{E}\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{B}\frac{\partial}{\partial \boldsymbol{B}}\Big). \tag{2.18}$$

3 Conservation laws in homogeneous media

Let us derive the conservation laws associated with the symmetries of the electromagnetic equations in homogeneous media, i.e. when $\sigma_m, \sigma_e = \text{const}$.

We will write the conservation law (1.3) in the form

$$D_t(\tau) + \operatorname{div}\boldsymbol{\chi} = 0, \tag{3.1}$$

where τ is the density of the conservation law (3.1), $\boldsymbol{\chi}$ is a three-dimensional vector:

$$\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3),$$

and

$$\operatorname{div}\boldsymbol{\chi} \equiv \nabla\cdot\boldsymbol{\chi} = D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3).$$

3.1 The case $\sigma_m = \sigma_e$

Let us find the conservation law furnished by the symmetry (2.15) when the condition (2.17) is satisfied, $\sigma_m = \sigma_e$. Applying the formula (1.6) to the symmetry (2.15) we obtain the following density of the conservation law (3.1):

$$\tau = \boldsymbol{E} \cdot \boldsymbol{V} + \boldsymbol{B} \cdot \boldsymbol{W}. \tag{3.2}$$

The vector $\boldsymbol{\chi}$ is obtained likewise by applying the equations (1.6) and (2.11). For example,

$$\chi^1 = E^2W^3 - E^3W^2 - B^2V^3 + B^3V^2.$$

The other coordinates of $\boldsymbol{\chi}$ are computed likewise, and the final result is:

$$\boldsymbol{\chi} = (\boldsymbol{E} \times \boldsymbol{W}) - (\boldsymbol{B} \times \boldsymbol{V}). \tag{3.3}$$

One can readily verify that (3.2) and (3.3) provide a conservation law for Eqs. (2.7) considered together with the adjoint equations (2.10). Indeed, using the well-known formula $\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\nabla \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b})$ and Eqs. (2.7), (2.10), we obtain:

$$\begin{aligned} D_t(\tau) &= \boldsymbol{E}_t \cdot \boldsymbol{V} + \boldsymbol{E} \cdot \boldsymbol{V}_t + \boldsymbol{B}_t \cdot \boldsymbol{W} + \boldsymbol{B} \cdot \boldsymbol{W}_t \\ &= \boldsymbol{V} \cdot (\nabla \times \boldsymbol{B} - \sigma_e \boldsymbol{E}) + \boldsymbol{E} \cdot (\nabla \times \boldsymbol{W} + \sigma_m \boldsymbol{V}) \\ &- \boldsymbol{W} \cdot (\nabla \times \boldsymbol{E} + \sigma_m \boldsymbol{B}) - \boldsymbol{B} \cdot (\nabla \times \boldsymbol{V} - \sigma_e \boldsymbol{W}), \end{aligned}$$

$$\begin{aligned} \nabla \cdot \boldsymbol{\chi} &= \nabla \cdot (\boldsymbol{E} \times \boldsymbol{W}) - \nabla \cdot (\boldsymbol{B} \times \boldsymbol{V}) \\ &= \boldsymbol{W} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{W}) - \boldsymbol{V} \cdot (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \cdot (\nabla \times \boldsymbol{V}). \end{aligned}$$

Whence,

$$D_t(\tau) + \operatorname{div} \boldsymbol{\chi} = (\sigma_m - \sigma_e)(\boldsymbol{E} \cdot \boldsymbol{V} - \boldsymbol{B} \cdot \boldsymbol{W}).$$

It follows again that the conservation law is valid only if $\sigma_m = \sigma_e$.

Likewise, one can compute the conservation law furnished by the symmetry (2.18) in this case. We will give here only the density of this conservation law. It has the form

$$\tau = \boldsymbol{W} \cdot \big[t(\nabla \times \boldsymbol{B}) + (\boldsymbol{x} \cdot \nabla)\boldsymbol{E}\big] - \boldsymbol{V} \cdot \big[t(\nabla \times \boldsymbol{E}) + (\boldsymbol{x} \cdot \nabla)\boldsymbol{B}\big]. \tag{3.4}$$

Remark 3.1. The conservation law given by (3.2)-(3.3) depends on solutions $(\boldsymbol{V}, \boldsymbol{W})$ of the adjoint system (2.10). However, substituting in Eqs. (3.2), (3.3) any particular solution $(\boldsymbol{V}, \boldsymbol{W})$ of the adjoint system (2.10) with $\sigma_m = \sigma_e$ one obtains the conservation law for Eqs. (2.7) not involving $\boldsymbol{V}$ and $\boldsymbol{W}$. Let us denote $\sigma_m = \sigma_e = \sigma$ and take, e.g. the following simple solution of the adjoint system (2.10):

$$V^1 = \mathrm{e}^{\sigma t},\ V^2 = V^3 = 0; \quad W^1 = \mathrm{e}^{\sigma t},\ W^2 = W^3 = 0.$$

Then Eqs. (3.2), (3.3) yield:

$$\tau = (E^1 + B^1)\mathrm{e}^{\sigma t}; \quad \chi^1 = 0,\ \chi^2 = (E^3 - B^3)\mathrm{e}^{\sigma t},\ \chi^3 = (B^2 - E^2)\mathrm{e}^{\sigma t}.$$

Remark 3.2. The operator (2.15) generates the one-parameter group

$$\overline{\boldsymbol{E}} = \boldsymbol{E}\cos\alpha - \boldsymbol{B}\sin\alpha, \quad \overline{\boldsymbol{B}} = \boldsymbol{E}\sin\alpha + \boldsymbol{B}\cos\alpha,$$

$$\overline{\rho}^{\,e} = \rho_e\cos\alpha - \rho_m\sin\alpha, \quad \overline{\rho}^{\,m} = \rho_e\sin\alpha + \rho_m\cos\alpha.$$

Remark 3.3. In the original variables used in Eqs. (2.1)-(2.2) and (2.5), the operator (2.15) is written:

$$X = \frac{1}{c}\,\boldsymbol{E}\cdot\frac{\partial}{\partial\boldsymbol{B}} - c\,\boldsymbol{B}\cdot\frac{\partial}{\partial\boldsymbol{E}} + c\,\rho_e\frac{\partial}{\partial\rho_m} - \frac{1}{c}\,\rho_m\frac{\partial}{\partial\rho_e}\cdot$$

3.2 The case of arbitrary σ_m and σ_e

Let us turn now the case of arbitrary σ_m and σ_e. Applying the above calculations to the generator (2.13) of the dilation group provides the conservation law with

$$\tau = \boldsymbol{B}\cdot\boldsymbol{V} - \boldsymbol{E}\cdot\boldsymbol{W}, \quad \boldsymbol{\chi} = (\boldsymbol{E}\times\boldsymbol{V}) + (\boldsymbol{B}\times\boldsymbol{W}). \tag{3.5}$$

This conservation law is valid for arbitrary σ_m and σ_e. Indeed,

$$\begin{aligned} D_t(\tau) &= \boldsymbol{B}_t\cdot\boldsymbol{V} + \boldsymbol{B}\cdot\boldsymbol{V}_t - \boldsymbol{E}_t\cdot\boldsymbol{W} - \boldsymbol{E}\cdot\boldsymbol{W}_t \\ &= -\boldsymbol{V}\cdot(\nabla\times\boldsymbol{E} + \sigma_m\boldsymbol{B}) + \boldsymbol{B}\cdot(\nabla\times\boldsymbol{W} + \sigma_m\boldsymbol{V}) \\ &- \boldsymbol{W}\cdot(\nabla\times\boldsymbol{B} - \sigma_e\boldsymbol{E}) + \boldsymbol{E}\cdot(\nabla\times\boldsymbol{V} - \sigma_e\boldsymbol{W}), \\ &= -\boldsymbol{V}\cdot(\nabla\times\boldsymbol{E}) + \boldsymbol{B}\cdot(\nabla\times\boldsymbol{W}) - \boldsymbol{W}\cdot(\nabla\times\boldsymbol{B}) + \boldsymbol{E}\cdot(\nabla\times\boldsymbol{V}), \end{aligned}$$

$$\begin{aligned} \nabla\cdot\boldsymbol{\chi} &= \nabla\cdot(\boldsymbol{E}\times\boldsymbol{V}) + \nabla\cdot(\boldsymbol{B}\times\boldsymbol{W}) \\ &= \boldsymbol{V}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{V}) + \boldsymbol{W}\cdot(\nabla\times\boldsymbol{B}) - \boldsymbol{B}\cdot(\nabla\times\boldsymbol{W}). \end{aligned}$$

Hence, $D_t(\tau) + \operatorname{div}\boldsymbol{\chi} = 0$.

Let us find the conservation law provided by the symmetry $X_0 = \frac{\partial}{\partial t}$ from (2.12). The formula (1.6) yields:

$$\tau = \mathcal{L} + \boldsymbol{E}_t\cdot\boldsymbol{W} - \boldsymbol{B}_t\cdot\boldsymbol{V}.$$

Since $\mathcal{L}$ given by (2.8) vanishes on the solutions of Eqs. (2.7), we can take

$$\tau = \boldsymbol{E}_t\cdot\boldsymbol{W} - \boldsymbol{B}_t\cdot\boldsymbol{V} \tag{3.6}$$

or

$$\tau = \boldsymbol{W}\cdot[(\nabla\times\boldsymbol{B}) - \sigma_e\boldsymbol{E}] + \boldsymbol{V}\cdot[(\nabla\times\boldsymbol{E}) + \sigma_m\boldsymbol{B}]. \tag{3.7}$$

Let us calculate the vector $\boldsymbol{\chi}$. The equations (1.6) and (2.11) yield, e.g.

$$\chi^1 = -E_t^2V^3 + E_t^3V^2 - B_t^2W^3 + B_t^3W^2.$$

The other coordinates of $\boldsymbol{\chi}$ are computed likewise, and the final result is

$$\boldsymbol{\chi} = (\boldsymbol{V} \times \boldsymbol{E}_t) + (\boldsymbol{W} \times \boldsymbol{B}_t). \tag{3.8}$$

Thus, the time translational invariance of Eqs. (2.7) leads to the conservation law (3.1) with τ and $\boldsymbol{\chi}$ given by (3.6) and (3.8), respectively.

Remark 3.4. Let us substitute in Eqs. (3.7), (3.8) the following simple solution of the adjoint system (cf. Remark 3.1):

$$V^1 = \mathrm{e}^{\sigma_m t},\ V^2 = V^3 = 0; \quad W^1 = \mathrm{e}^{\sigma_e t},\ W^2 = W^3 = 0.$$

Then Eqs. (3.7), (3.8) yield:

$$\tau = (B_y^3 - B_z^2 - \sigma_e E^1)\mathrm{e}^{\sigma_e t} + (E_y^3 - E_z^2 + \sigma_m B^1)\mathrm{e}^{\sigma_m t},$$

$$\chi^1 = 0, \quad \chi^2 = -E_t^3\mathrm{e}^{\sigma_m t} - B_t^3\mathrm{e}^{\sigma_e t}, \quad \chi^3 = E_t^2\mathrm{e}^{\sigma_m t} + B_t^2\mathrm{e}^{\sigma_e t}.$$

The conservation law provided by the symmetry $X_1 = \frac{\partial}{\partial x}$ from (2.12) has the following density:

$$\tau = \boldsymbol{E}_x \cdot \boldsymbol{W} - \boldsymbol{B}_x \cdot \boldsymbol{V}.$$

For the vector $\boldsymbol{\chi}$ the formula (1.6) yields, e.g.

$$\chi^1 = \mathcal{L} - E_x^2 V^3 + E_x^3 V^2 - B_x^2 W^3 + B_x^3 W^2.$$

The other coordinates of $\boldsymbol{\chi}$ are calculated similarly:

$$\chi^2 = E_x^1 V^3 - E_x^3 V^1 + B_x^1 W^3 - B_x^3 W^1,$$

$$\chi^3 = -E_x^1 V^2 + E_x^2 V^1 - B_x^1 W^2 + B_x^2 W^1.$$

We can ignore $\mathcal{L}$ in χ^1 since $D_x\mathcal{L} = 0$ on solutions of Eqs. (2.7), (2.10) and the final result is

$$\boldsymbol{\chi} = (\boldsymbol{V} \times \boldsymbol{E}_x) + (\boldsymbol{W} \times \boldsymbol{B}_x). \tag{3.9}$$

Replacing x by y and z we obtain the following conservation laws corresponding to $X_2 = \frac{\partial}{\partial y}$ and $X_3 = \frac{\partial}{\partial z}$, respectively:

$$\tau = \boldsymbol{E}_y \cdot \boldsymbol{W} - \boldsymbol{B}_y \cdot \boldsymbol{V}, \quad \boldsymbol{\chi} = (\boldsymbol{V} \times \boldsymbol{E}_y) + (\boldsymbol{W} \times \boldsymbol{B}_y)$$

and

$$\tau = \boldsymbol{E}_z \cdot \boldsymbol{W} - \boldsymbol{B}_z \cdot \boldsymbol{V}, \quad \boldsymbol{\chi} = (\boldsymbol{V} \times \boldsymbol{E}_z) + (\boldsymbol{W} \times \boldsymbol{B}_z).$$

Applying the formula (1.6) to the symmetry X_{12} we obtain the following density of the conservation law:

$$\tau = E^2\frac{\partial\mathcal{L}}{\partial E_t^1} - E^1\frac{\partial\mathcal{L}}{\partial E_t^2} + (x\boldsymbol{E}_y - y\boldsymbol{E}_x)\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{E}_t} + B^2\frac{\partial\mathcal{L}}{\partial B_t^1} - B^1\frac{\partial\mathcal{L}}{\partial B_t^2} + (x\boldsymbol{B}_y - y\boldsymbol{B}_x)\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{B}_t}$$

$$= W^2E^1 - W^1E^2 + \boldsymbol{W}\cdot(y\boldsymbol{E}_x - x\boldsymbol{E}_y) - (V^2B^1 - V^1B^2) - \boldsymbol{V}\cdot(y\boldsymbol{B}_x - x\boldsymbol{B}_y).$$

The densities of the conservation laws for X_{13} and X_{23} are

$$\tau = W^3E^1 - W^1E^3 + \boldsymbol{W}\cdot(z\boldsymbol{E}_x - x\boldsymbol{E}_z) - (V^3B^1 - V^1B^3) - \boldsymbol{V}\cdot(z\boldsymbol{B}_x - x\boldsymbol{B}_z)$$

and

$$\tau = W^3E^2 - W^2E^3 + \boldsymbol{W}\cdot(z\boldsymbol{E}_y - y\boldsymbol{E}_z) - (V^3B^2 - V^2B^3) - \boldsymbol{V}\cdot(z\boldsymbol{B}_y - y\boldsymbol{B}_z),$$

respectively. Finally, the densities of conservation laws corresponding to the rotation generators X_{ij} can be written as one vector:

$$\boldsymbol{\tau} = \boldsymbol{W}\times\boldsymbol{E} + \boldsymbol{W}\cdot(\boldsymbol{x}\times\nabla)\boldsymbol{E} - \boldsymbol{V}\times\boldsymbol{B} - \boldsymbol{V}\cdot(\boldsymbol{x}\times\nabla)\boldsymbol{B}, \tag{3.10}$$

where $\boldsymbol{x} = (x, y, z)$.

The operator X_{12} provides the following vector $\boldsymbol{\chi}$:

$$\chi^1 = -V^3E^1 - y(E^2_xV^3 - E^3_xV^2) + x(E^2_yV^3 - E^3_yV^2)$$
$$-W^3B^1 - y(B^2_xW^3 - B^3_xW^2) + x(B^2_yW^3 - B^3_yW^2),$$

$$\chi^2 = -V^3E^2 + y(E^1_xV^3 - E^3_xV^1) - x(E^1_yV^3 - E^3_yV^1)$$
$$-W^3B^2 + y(B^1_xW^3 - B^3_xW^1) - x(B^1_yW^3 - B^3_yW^1),$$

$$\chi^3 = V^1E^1 + V^2E^2 - y(E^1_xV^2 - E^2_xV^1) + x(E^1_yV^2 - E^2_yV^1)$$
$$+W^1B^1 + W^2B^2 - y(B^1_xW^2 - B^2_xW^1) + x(B^1_yW^2 - B^2_yW^1).$$

The vector $\boldsymbol{\chi}$ for the operator X_{13} has the following form:

$$\chi^1 = V^2E^1 - z(E^2_xV^3 - E^3_xV^2) + x(E^2_zV^3 - E^3_zV^2)$$
$$+W^2B^1 - z(B^2_xW^3 - B^3_xW^2) + x(B^2_zW^3 - B^3_zW^2),$$

$$\chi^2 = -V^1E^1 - V^3E^3 + z(E^1_xV^3 - E^3_xV^1) - x(E^1_zV^3 - E^3_zV^1)$$
$$-W^1B^1 - W^3B^3 + z(B^1_xW^3 - B^3_xW^1) - x(B^1_zW^3 - B^3_zW^1),$$

$$\chi^3 = V^2E^3 - z(E^1_xV^2 - E^2_xV^1) + x(E^1_zV^2 - E^2_zV^1)$$
$$+W^2B^3 - z(B^1_xW^2 - B^2_xW^1) + x(B^1_zW^2 - B^2_zW^1).$$

The operator X_{23} provides the following vector $\boldsymbol{\chi}$:

$$\chi^1 = V^2E^2 + V^3E^3 - z(E_y^2V^3 - E_y^3V^2) + y(E_z^2V^3 - E_z^3V^2)$$
$$+W^2B^2 + W^3B^3 - z(B_y^2W^3 - B_y^3W^2) + y(B_z^2W^3 - B_z^3W^2),$$

$$\chi^2 = -V^1E^2 + z(E_y^1V^3 - E_y^3V^1) - y(E_z^1V^3 - E_z^3V^1)$$
$$-W^1B^2 + z(B_y^1W^3 - B_y^3W^1) - y(B_z^1W^3 - B_z^3W^1),$$

$$\chi^3 = -V^1E^3 - z(E_y^1V^2 - E_y^2V^1) + y(E_z^1V^2 - E_z^2V^1)$$
$$-W^1B^3 - z(B_y^1W^2 - B_y^2W^1) + y(B_z^1W^2 - B_z^2W^1).$$

Finally, applying our procedure to the infinitesimal symmetry S, (2.14) we arrive at the conservation law (3.1) with (cf. (3.5))

$$\tau = \mathcal{B}(\boldsymbol{x},t)\cdot\boldsymbol{V} - \mathcal{E}(\boldsymbol{x},t)\cdot\boldsymbol{W}, \quad \boldsymbol{\chi} = [\mathcal{E}(\boldsymbol{x},t)\times\boldsymbol{V}] + [\mathcal{B}(\boldsymbol{x},t)\times\boldsymbol{W}]. \tag{3.11}$$

3.3 Two-solution representation of conservation laws

The conserved quantities obtained above involve solutions $\boldsymbol{V}, \boldsymbol{W}$ of the adjoint equations (2.10). It may be useful for applications to give an alternative representation of the conserved quantities in terms of the electric and magnetic vector fields $\boldsymbol{E}, \boldsymbol{B}$ only.

We suggest here one of possibilities based on the observation that one can satisfy the adjoint system (2.10) by letting

$$\begin{aligned}
&\boldsymbol{V}(\boldsymbol{x},t) = \boldsymbol{B}(\boldsymbol{x},-t),\\
&\boldsymbol{W}(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{x},-t),\\
&R_e(\boldsymbol{x},t) = \nabla\cdot\boldsymbol{E}(\boldsymbol{x},-t) - \rho_e(\boldsymbol{x},-t),\\
&R_m(\boldsymbol{x},t) = \nabla\cdot\boldsymbol{B}(\boldsymbol{x},-t) - \rho_m(\boldsymbol{x},-t),
\end{aligned} \tag{3.12}$$

where $\boldsymbol{E}(\boldsymbol{x},s), \boldsymbol{B}(\boldsymbol{x},s)$ solve Eqs. (2.7) with $s=-t$. Indeed, employing the substitution (3.12) and the notation $s=-t$ we have

$$\nabla\times\boldsymbol{V} + \frac{\partial\boldsymbol{W}}{\partial t} - \sigma_e\boldsymbol{W} = \nabla\times\boldsymbol{B}(\boldsymbol{x},s) + \frac{\partial\boldsymbol{E}(\boldsymbol{x},s)}{\partial s}\frac{\partial s}{\partial t} - \sigma_e\,\boldsymbol{E}(\boldsymbol{x},s),$$
$$\nabla\times\boldsymbol{W} - \frac{\partial\boldsymbol{V}}{\partial t} + \sigma_m\boldsymbol{V} = \nabla\times\boldsymbol{E}(\boldsymbol{x},s) - \frac{\partial\boldsymbol{B}(\boldsymbol{x},s)}{\partial s}\frac{\partial s}{\partial t} + \sigma_m\boldsymbol{B}(\boldsymbol{x},s),$$
$$R_e = \nabla\cdot\boldsymbol{E}(\boldsymbol{x},s) - \rho_e(\boldsymbol{x},s), \qquad R_m = \nabla\cdot\boldsymbol{B}(\boldsymbol{x},s) - \rho_m(\boldsymbol{x},s).$$

Hence, the adjoint equations (2.10) reduce to (2.7):

$$
\begin{aligned}
&\nabla \times \boldsymbol{E}(\boldsymbol{x}, s) + \frac{\partial \boldsymbol{B}(\boldsymbol{x}, s)}{\partial s} + \sigma_m \boldsymbol{B}(\boldsymbol{x}, s) = 0,\\
&\nabla \times \boldsymbol{B}(\boldsymbol{x}, s) - \frac{\partial \boldsymbol{E}(\boldsymbol{x}, s)}{\partial s} - \sigma_e \boldsymbol{E}(\boldsymbol{x}, s) = 0,\\
&\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, s) - \rho_e(\boldsymbol{x}, s) = 0,\\
&\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, s) - \rho_m(\boldsymbol{x}, s) = 0.
\end{aligned} \tag{3.13}
$$

Let $\big(\boldsymbol{E}(\boldsymbol{x}, t), \boldsymbol{B}(\boldsymbol{x}, t)\big)$ and $\big(\boldsymbol{E}'(\boldsymbol{x}, t), \boldsymbol{B}'(\boldsymbol{x}, t)\big)$ be any two solutions of the electromagnetic equations (2.7). Substituting in (3.12) the solution $\big(\boldsymbol{E}', \boldsymbol{B}'\big)$, we obtain the *two-solution representations* of the conservation laws. For example, the conservation law given by (3.2)-(3.3) has in this representation the following coordinates:

$$
\begin{aligned}
&\tau = \boldsymbol{E}(\boldsymbol{x}, t) \cdot \boldsymbol{B}'(\boldsymbol{x}, -t) + \boldsymbol{B}(\boldsymbol{x}, t) \cdot \boldsymbol{E}'(\boldsymbol{x}, -t),\\
&\boldsymbol{\chi} = \big(\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{E}'(\boldsymbol{x}, -t)\big) - \big(\boldsymbol{B}(\boldsymbol{x}, t) \times \boldsymbol{B}'(\boldsymbol{x}, -t)\big).
\end{aligned} \tag{3.14}
$$

In particular, if the solutions $\big(\boldsymbol{E}(\boldsymbol{x}, t), \boldsymbol{B}(\boldsymbol{x}, t)\big)$ and $\big(\boldsymbol{E}'(\boldsymbol{x}, t), \boldsymbol{B}'(\boldsymbol{x}, t)\big)$ are identical, (3.14) provides the *one-solution representation:*

$$
\begin{aligned}
&\tau = \boldsymbol{E}(\boldsymbol{x}, t) \cdot \boldsymbol{B}(\boldsymbol{x}, -t) + \boldsymbol{B}(\boldsymbol{x}, t) \cdot \boldsymbol{E}(\boldsymbol{x}, -t),\\
&\boldsymbol{\chi} = \big(\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{E}(\boldsymbol{x}, -t)\big) - \big(\boldsymbol{B}(\boldsymbol{x}, t) \times \boldsymbol{B}(\boldsymbol{x}, -t)\big).
\end{aligned} \tag{3.15}
$$

All other conservation laws can be treated likewise, e.g. the conservation law given by (3.6), (3.8) has the following two-solution representation:

$$
\begin{aligned}
&\tau = \boldsymbol{E}_t(\boldsymbol{x}, t) \cdot \boldsymbol{E}'(\boldsymbol{x}, -t) - \boldsymbol{B}_t(\boldsymbol{x}, t) \cdot \boldsymbol{B}'(\boldsymbol{x}, -t),\\
&\boldsymbol{\chi} = \big[\boldsymbol{B}'(\boldsymbol{x}, -t) \times \boldsymbol{E}_t(\boldsymbol{x}, t)\big] + \big[\boldsymbol{E}'(\boldsymbol{x}, -t) \times \boldsymbol{B}_t(\boldsymbol{x}, t)\big]
\end{aligned} \tag{3.16}
$$

and one-solution representation:

$$
\begin{aligned}
&\tau = \boldsymbol{E}_t(\boldsymbol{x}, t) \cdot \boldsymbol{E}(\boldsymbol{x}, -t) - \boldsymbol{B}_t(\boldsymbol{x}, t) \cdot \boldsymbol{B}(\boldsymbol{x}, -t),\\
&\boldsymbol{\chi} = \big[\boldsymbol{B}(\boldsymbol{x}, -t) \times \boldsymbol{E}_t(\boldsymbol{x}, t)\big] + \big[\boldsymbol{E}(\boldsymbol{x}, -t) \times \boldsymbol{B}_t(\boldsymbol{x}, t)\big].
\end{aligned} \tag{3.17}
$$

In conclusion we give the one-solution representations of the conservation laws (3.5) and (3.11) corresponding to the homogeneity and linearity of the electromagnetic field equations. Namely, the one-solution representation of (3.5) is written

$$
\begin{aligned}
&\tau = \boldsymbol{B}(\boldsymbol{x}, t) \cdot \boldsymbol{B}(\boldsymbol{x}, -t) - \boldsymbol{E}(\boldsymbol{x}, t) \cdot \boldsymbol{E}(\boldsymbol{x}, -t),\\
&\boldsymbol{\chi} = [\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{B}(\boldsymbol{x}, -t)] + [\boldsymbol{B}(\boldsymbol{x}, t) \times \boldsymbol{E}(\boldsymbol{x}, -t)]
\end{aligned} \tag{3.18}
$$

and can be called the *nonlocal energy conservation law.* The similar representation of the conservation law (3.11) has the form

$$\tau = \mathcal{B}(\boldsymbol{x},t)\cdot\boldsymbol{B}(\boldsymbol{x},-t) - \mathcal{E}(\boldsymbol{x},t)\cdot\boldsymbol{E}(\boldsymbol{x},-t), \qquad (3.19)$$
$$\boldsymbol{\chi} = [\mathcal{E}(\boldsymbol{x},t)\times\boldsymbol{B}(\boldsymbol{x},-t)] + [\mathcal{B}(\boldsymbol{x},t)\times\boldsymbol{E}].$$

However, the conservation law (3.19) is not new and can be regarded as the two-solution representation of the conservation law (3.5).

4 Conservation laws for inhomogeneous media

If σ_e and σ_m are not constant but depend on time and space coordinates,

$$\sigma_e = \sigma_e(\boldsymbol{x},t), \quad \sigma_m = \sigma_m(\boldsymbol{x},t), \qquad (4.1)$$

then the operators (2.12) and (2.18) are not admitted by Eqs. (2.7) provided that $\sigma_e(\boldsymbol{x},t)$ and $\sigma_m(\boldsymbol{x},t)$ are arbitrary functions. However, the operators (2.13) and (2.14) are still admitted due to homogeneity and linearity of the equations in question. The symmetries (2.13) and (2.14) of Eqs. (2.7) with arbitrary variable coefficients (4.1) provide the conservation laws given by (3.5),

$$\tau = \boldsymbol{B}\cdot\boldsymbol{V} - \boldsymbol{E}\cdot\boldsymbol{W}, \quad \boldsymbol{\chi} = (\boldsymbol{E}\times\boldsymbol{V}) + (\boldsymbol{B}\times\boldsymbol{W}),$$

and (3.11),

$$\tau = \mathcal{B}(\boldsymbol{x},t)\cdot\boldsymbol{V} - \mathcal{E}(\boldsymbol{x},t)\cdot\boldsymbol{W}, \quad \boldsymbol{\chi} = [\mathcal{E}(\boldsymbol{x},t)\times\boldsymbol{V}] + [\mathcal{B}(\boldsymbol{x},t)\times\boldsymbol{W}].$$

respectively.

Furthermore, if $\sigma_e(\boldsymbol{x},t) = \sigma_m(\boldsymbol{x},t)$, Eqs. (2.7) admit the operator (2.15), and hence have the additional conservation law given by (3.2)-(3.3):

$$\tau = \boldsymbol{E}\cdot\boldsymbol{V} + \boldsymbol{B}\cdot\boldsymbol{W}, \quad \boldsymbol{\chi} = (\boldsymbol{E}\times\boldsymbol{W}) - (\boldsymbol{B}\times\boldsymbol{V}).$$

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