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Group classification of the Sachs equations for a radiating axisymmetric, non-rotating, vacuum space-time.

Nail H. Ibragimov[1], Ewald J. H. Wessels[2] and George F. R. Ellis[2]

[1] Research Centre ALGA: Advances in Lie Group Analysis
Blekinge Institute of Technology
SE-371 79 Karlskrona, Sweden
[2] Department of Applied Mathematics
University of Cape Town
Cape Town, South Africa

Abstract. We carry out a Lie group analysis of the Sachs equations for a time-dependent axisymmetric non-rotating space-time in which the Ricci tensor vanishes. These equations, which are the first two members of the set of Newman-Penrose equations, define the null initial-value problem for the space-time. We find a particular form for the initial data such that these equations admit a Lie symmetry, and so defines a geometrically special class of such spacetimes. These should additionally be of particular physical interest because of this special geometric feature.

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Key words: Initial value problem, Sachs equations, symmetries, equivalence transformations, group classification.

1. Introduction

The advent of gravitational wave detectors has increased the need for realistic mathematical models of radiating space-times. The first step in such modeling is the initial value problem for Einstein's field equations. Christodoulou [1] has called this mathematical problem the problem "par excellence" in general relativity.

The usual approach is to choose initial data on a space-like Cauchy slice. The initial data must then satisfy Einstein's equations on that slice (see e.g. [2] for a review).

An alternative is the "characteristic initial value" approach, which uses null geodesics, or "characteristics" as coordinate curves (see e.g. [3] for a review). In this case the initial value function is defined on a null hypersurface. One of the many advantages is the fact that analytic initial data, in the form of the first element Ψ_0 of the Weyl spinor, are then free of any constraints, other than conditions that may be imposed on the geometry of the space-time, such as the conditions that are imposed in this paper. However, in the words of Winicour, "This flexibility and control in prescribing initial data has the trade-off of limited experience with prescribing physically realistic characteristic initial data." This comment reflects the fact that initial value functions are usually constructed "by hand" to give suitable limiting values.

In the case of axisymmetric, non-rotating vacuum space-times it has been shown [4] that the initial value problem is defined by the Sachs equations when these are written out in appropriately chosen null coordinates. This pair of equations, originally discovered by Sachs [5], is the first pair of the set of Newman-Penrose equations [6] for the space-time. With a coordinate system based on a set of null geodesics emanating from a point on the axis of symmetry, they are defined on an outgoing null geodesic and relate two of the metric functions, which are labeled $Q(u,r,\theta)$ and $q(u,r,\theta)$ to the initial-value function $\psi_0(u,r,\theta)$. Here u is a time-like coordinate that labels successive null hypersurfaces, r is a space-like affine parameter that measures distance along the null geodesics, and θ is an angular elevation coordinate. The fourth coordinate, ϕ, is an azimuth and does not appear in these functions because of the rotational symmetry that is assumed to exist about the axis. Along a single null geodesic u, θ and ϕ are all constant and only r varies. With the symmetries under discussion these equations take the following form when the Ricci tensor vanishes:

$$\frac{\partial^2 q}{\partial r^2} + \Psi_0\, q = 0 \qquad \text{and} \qquad \frac{\partial^2 Q}{\partial r^2} - \Psi_0\, Q = 0\,. \tag{1.1}$$

The unknowns q and Q are the $d\theta^2$ metric function and the $d\phi^2$ metric function respectively (see the appendix). The function Ψ_0 is the first element of the Weyl spinor.

Penrose [7] has shown that Ψ_0 is not determined by the field equations but can be specified independently on an initial null hypersurface without any constraints (other than constraints that may derive from symmetry conditions imposed on the geometry.) The equations can therefore be treated as a closed set of ordinary differential equations that determine the r dependence of q and Q once the character of the space-time has been prescribed by the choice of Ψ_0 on the initial null hypersurface.

The Sachs equations (1.1) are underdetermined with respect to a solution, since the two equations contain three unknown variables. However, it has been shown [4] that group theory can provide a third constraint on the problem, in the form of a particular non-trivial Lie symmetry that can be imposed on the equations. If this symmetry (which is the only non-trivial Lie symmetry that the equations can admit for any possible form of the function Ψ_0) is imposed they have the following unique general solution:

$$\Psi_0\big|_{u=0} = \frac{a_0}{\left(b_2 r^2 + b_1 r + b_0\right)^2} \tag{1.2}$$

where a_0 and the b_i are unknown functions of θ.

The four undetermined functions of integration appear to allow sufficient latitude to cover the variety of matter distributions that belong to the class of non-rotating axisymmetric space-times (note, as remarked before, that the equations under consideration cover only the vacuum portion of such space-times where the Ricci tensor vanishes). In particular, they cover the spherically symmetric (Schwarzschild) case, in which case a_0, and therefore Ψ_0, vanish identically.

One particular choice of values for the functions of integration (when $b_1 = 0$) appears to describe the vacuum portion of a space-time corresponding to a pair of identical non-rotating massive bodies moving along an axis of symmetry towards a head-on collision with the origin at the centre of mass The full set of Newman-Penrose equations can be solved analytically for the elements of the metric tensor, in this case, in the limit when $\theta \to 0$, i.e. on the axis of symmetry [4]. The solution is regular on the axis unlike previous axisymmetric solutions, such as the Weyl solution, which contain conical singularities on the axis. It passes other reasonable tests, which suggest that the Lie symmetry solution approximates to a single Schwarzschild solution and a pair of Schwarzschild solutions respectively when a parameter, which can be identified approximately with the separation of the two massive bodies, is small or large.

Also, the "tidal force" term of the Weyl curvature on the axis, in the weak field limit, approximates to the curvature that can be calculated from Newton's laws for such a case.

These features strongly suggest that the mathematical property of the equations that produces this result is a reflection of an underlying interesting physical property of non-rotating axisymmetric relativistic space-time manifolds and that it is not just a mathematical curiosity [4].

The details of the Lie group analysis that produces the result (1.2) have not previously been published. It is the purpose of this paper to give these details and also to extend the analysis by finding the group of equivalence transformations of the Sachs equations (1.1).

2. Symmetries and the Classifying relation

Consider the system (1.1) of the Sachs equations, which can be treated as a pair of ordinary differential equations. We denote differentiation with respect to r by primes, drop the subscript from Ψ_0, and write the equations in the following form:

$$q'' + \Psi(r)\,q = 0, \qquad Q'' - \Psi(r)\,Q = 0 \tag{2.1}$$

Note that each of the two equations, considered separately, admits an 8-dimensional Lie algebra. It would be erroneous, however, to conclude that the simultaneous system of the two equations should admit a 16-dimensional Lie Algebra (cf. [8], Section 9.3.3).

The Sachs equations (2.1) are linear and homogeneous. Therefore they admit the six-parameter group composed of the dilations in q and Q and the usual superposition principle consisting of addition to q and Q of the arbitrary solutions k(r) and l(r) of the first and second equation (2.1) respectively. In other words, the Sachs equations (2.1) with an arbitrary potential $\Psi(r)$ admit:

$$X_1 = q\frac{\partial}{\partial q}, \quad X_2 = Q\frac{\partial}{\partial Q}, \quad X_3 = k(r)\frac{\partial}{\partial q}, \quad X_4 = l(r)\frac{\partial}{\partial Q} \tag{2.2}$$

where k(r) and l(r) solve the Sachs equations

$$k''(r) + \Psi(r)k(r) = 0 \qquad l''(r) - \Psi(r)\,l(r) = 0$$

Since the general solution of the latter equations depends on 4 arbitrary constants the operators (2.2) span a six-dimensional Lie algebra.

We look for the general admissible infinitesimal generators

$$X = \tau(r,q,Q)\frac{\partial}{\partial r} + \xi(r,q,Q)\frac{\partial}{\partial q} + \eta(r,q,Q)\frac{\partial}{\partial Q} \tag{2.3}$$

The determining equations are written

$$X[q'' + \Psi(r)\,q]_{(2.1)} \equiv \zeta_2^1\Big|_{(2.1)} + \xi\,\Psi(r) + q\,\tau\,\Psi'(r) = 0 \tag{2.4}$$

$$X[Q'' - \Psi(r)Q]_{(2.1)} \equiv \varsigma_2^2\Big|_{(2.1)} - \eta\Psi(r) - Q\tau\Psi'(r) = 0 \tag{2.5}$$

where $\big|_{(2.1)}$ means that q'' and Q'' should be replaced by - $\Psi(r)$ q and $\Psi(r)$ Q respectively according to the system of equations (2.1).

The quantities ς_2^1 and ς_2^2 are obtained by the usual prolongation procedure (see any book on Lie group analysis, e.g. [9], [10], [8], [11]), namely by the first prolongation

$$\begin{aligned}
\varsigma_1^1 &= D_r(\xi) - q' D_r(\tau) = \xi_r + q'\xi_q + Q'\xi_Q - q'(\tau_r + q'\tau_q + Q'\tau_Q) \\
\varsigma_1^2 &= D_r(\eta) - Q' D_r(\tau) = \eta_r + q'\eta_q + Q'\eta_Q - Q'(\tau_r + q'\tau_q + Q'\tau_Q)
\end{aligned} \tag{2.6}$$

and the second prolongation

$$\varsigma_2^1 = D_r(\varsigma_1^1) - q'' D_r(\tau) \ , \qquad \varsigma_2^2 = D_r(\varsigma_1^2) - Q'' D_r(\tau) \tag{2.7}$$

respectively.

Upon substituting the expression (2.7) for ς_2^1, (2.4) takes the following form:

$$\begin{aligned}
&\xi_{rr} + (2\xi_{rq} - \tau_{rr})q' + 2\xi_{rQ}Q' + (\xi_{qq} - 2\tau_{rq})q'^2 + 2(\xi_{qQ} - \tau_{rQ})q'Q' + \xi_{QQ}Q'^2 - \tau_{qq}q'^3 \\
&- 2\tau_{qQ}q'^2Q' - \tau_{QQ}q'Q'^2 + Q(\xi_Q - \tau_Q q')\Psi(r) - q(\xi_q - 2\tau_r - 3\tau_q q' - 2\tau_Q Q')\Psi(r) \\
&+ \xi\Psi(r) + q\tau\Psi'(r) = 0
\end{aligned} \tag{2.8}$$

We collect here the like terms and annul the coefficients of different powers of q' and Q'. The coefficients for the cubic terms q'^3, q'^2Q' and $q'Q'^2$ yield:

$$\tau_{qq} = 0, \quad \tau_{qQ} = 0, \quad \tau_{QQ} = 0,$$

whence

$$\tau = a(r)q + b(r)Q + c(r) \tag{2.9}$$

Furthermore, the coefficients for the quadratic terms $Q'^2, q'Q'$, and q'^2 yield:

$$\xi_{QQ} = 0, \quad \xi_{qQ} - \tau_{rQ} = 0, \quad \xi_{qq} - 2\tau_{rq} = 0$$

whence, invoking the expression (2.9) for τ and using (2.2) one obtains:

$$\xi = a'(r)q^2 + b'(r)qQ + A(r)q + m(r)Q + k(r)$$

Annulling the coefficients for Q' in (2.8) one obtains:

$$\xi_{rQ} + q\Psi(r)\tau_Q \equiv [b''(r) + \Psi(r)b(r)]q + m'(r) = 0$$

whence $m'(r) = 0$ and

$$b''(r) + \Psi(r)\, b(r) = 0 \tag{2.10}$$

Thus ξ has the form

$$\xi = a'(r)\, q^2 + b'(r)\, q\, Q + A(r)\, q + m\, Q + k(r)\,, \quad m = \text{const.} \tag{2.11}$$

Likewise, inspecting (2.5), by using the quadratic terms in q', Q' and the term in q' we arrive at the following equations:

$$a''(r) - \Psi(r)\, a(r) = 0 \tag{2.12}$$

and

$$\eta = a'(r)\, q\, Q + b'(r)\, Q^2 + B(r)\, Q + s\, q + l(r)\,, \quad s = \text{const.} \tag{2.13}$$

We return now to (2.8) and annul there the coefficient for q' and the term free of the derivatives q' and Q' to obtain:

$$2\,\xi_{rq} - \tau_{rr} + [3\, q\, \tau_q - Q\, \tau_Q\,]\Psi(r) = 0 \tag{2.14}$$

and

$$\xi_{rr} + [\xi + Q\,\xi_Q - q\xi_q + 2\, q\tau_r\,]\Psi(r) + q\tau\Psi'(r) = 0 \tag{2.15}$$

respectively. Upon substituting the expression (2.9) for τ and the expression (2.11) for ξ, (2.14) becomes:

$$3q[a''(r) + a(r)\Psi(r)] + Q[b''(r) - b(r)\Psi(r)] + 2A'(r) - c''(r) = 0$$

whence, taking into account (2.10) and (2.12) and assuming $\Psi(r) \neq 0$, we have $a(r) = b(r) = 0$ and

$$2A'(r) = c''(r) \tag{2.16}$$

Equation (2.15) now takes the form

$$[(A''(r) + 2c'(r)\,\Psi(r) + c(r)\Psi'(r)]q + 2mQ\Psi(r) + k''(r) + k(r)\Psi(r) = 0$$

If $\Psi(r) \neq 0$, it follows from this equation that $m = 0$,

$$A''(r) + 2c'(r)\,\Psi(r) + c(r)\Psi'(r) = 0 \tag{2.17}$$

and

$$k''(r) + k(r)\Psi(r) = 0 \tag{2.18}$$

Similar calculations with (2.5) provide the equation $s = 0$ together with the following counterparts of (2.16), (2.17), and (2.18):

$$2B'(r) = c''(r), \quad B''(r) - 2c'(r)\Psi(r) - c(r)\Psi'(r) = 0, \quad l''(r) - l(r)\Psi(r) = 0 \tag{2.19}$$

In consequence, expressions (2.9), (2.11) and (2.13) reduce to the following:

$$\tau = c\,(r), \quad \xi = A(r)\,q + k(r), \quad \eta = B(r)Q + l(r) \tag{2.20}$$

Equations (2.16) and (2.17), together with the first and second equations (2.19), yield the following two simultaneous equations:

$$\tfrac{1}{2}c'''(r) + 2\,c'(r)\,\Psi(r) + c(r)\,\Psi'(r) = 0, \quad \tfrac{1}{2}c'''(r) - 2c'(r)\Psi(r) - c(r)\Psi'(r) = 0$$

whence $c'''(r) = 0$, and

$$2\,c'(r)\,\Psi(r) + c(r)\,\Psi'(r) = 0 \tag{2.21}$$

Equation (2.21) involves the potential $\Psi(r)$ and is called a *classifying relation.* Thus, the function c(r) is at most quadratic in r,

$$c\,(r) = C_1 r^2 + C_2 r + C_3, \qquad C_i = \text{const.} \tag{2.22}$$

Furthermore, the classifying relation (2.21) written in the form

$$\frac{\Psi'(r)}{\Psi(r)} = -2\frac{c'(r)}{c(r)}$$

yields $\Psi(r) = C_4 / c^2(r)$, provided that $c(r) \neq 0$. On the other hand, if c(r) = 0, it follows from (2.16) and the first equation (2.19) that A = const., B = const., and hence the Sachs equations (2.1) have only the self-evident infinitesimal symmetries (2.2). Thus, invoking (2.16), (2.22) and the first equation (2.19) we arrive at the following statement.

Theorem 1: The Sachs equations (2.1) admit, along with the trivial infinitesimal symmetries (2.2), a nontrivial Lie algebra if, and only if, the potential $\Psi(r)$ has the following form:

$$\Psi(r) = \frac{\delta}{(\alpha r^2 + \beta r + \gamma)^2}, \tag{2.23}$$

where α, β, γ and δ are arbitrary constants, $\delta \neq 0$.

This is the result first reported by Wessels [4] in the form (1.2) where, in the context of the Newman-Penrose equations, the "constants" are functions of θ defined on the initial null hypersurface.

The infinitesimal symmetries (2.3) for the equations (2.1) with the potential (2.23) have the coordinates

$$\begin{aligned} &\tau = C_1 r^2 + C_2 r + C_3 \\ &\xi = (C_1 r + C_4)\,q + k(r), \quad k'' + k\,\Psi = 0, \\ &\eta = (C_1 r + C_5)\,Q + l(r), \quad l'' - l\Psi = 0, \end{aligned} \tag{2.24}$$

Where C_i = const. The constants C_1, C_2, and C_3 are connected with the potential (2.23) by (2.21):

$$2\left(2C_1 r + C_2\right)\Psi + \left(C_1 r^2 + C_2 r + C_3\right)\Psi' = 0 \tag{2.25}$$

If $\alpha \neq 0$, (2.25) yields the result $C_1 \neq 0$, otherwise $C_2 = C_3 = 0$ and hence one has only the trivial symmetries (2.2). Therefore, we let $C_1 = \alpha$ and obtain from (2.25) the result that $C_2 = \beta$ and $C_3 = \gamma$. Thus equations (2.1) with the potential (2.23), where $\alpha \neq 0$, admit the following non-trivial symmetry [4]:

$$X = \left(\alpha r^2 + \beta r + \gamma\right)\frac{\partial}{\partial r} + \alpha r q \frac{\partial}{\partial q} + \alpha r Q \frac{\partial}{\partial Q}. \tag{2.26}$$

In order to complete the group classification, we need to single out all equivalent potentials (2.23). First, we have to find the equivalence transformations of the system of Sachs equations (2.1).

3. Equivalence transformations

An equivalence transformation is a smooth, invertible transformation of the dependent and independent variables that leaves the form of the equations unchanged, except possibly for the form of an unspecified function appearing in the equations, which does not have to be the same before and after the transformation. Thus an equivalence transformation of the system (1.1) is an invertible transformation of the variables r, q, Q:

$$\bar{r} = f(r,q,Q), \quad \bar{q} = g(r,q,Q) \quad \overline{Q} = h(r,q,Q)$$

mapping the system (2.1) into a system of the same form,

$$\frac{d^2\bar{q}}{d\bar{r}^2} + \overline{\Psi}(\bar{r})\,\bar{q} = 0, \qquad \frac{d^2\overline{Q}}{d\bar{r}^2} - \overline{\Psi}(\bar{r})\,\overline{Q} = 0$$

where the form of the transformed function $\overline{\Psi}(\bar{r})$ can, in general, be different from the form of the original function $\Psi(r)$. An equivalence transformation can also be regarded as a mapping of the variables (r, q, Q, Ψ) into variables $\left(\bar{r}, \bar{q}, \overline{Q}, \overline{\Psi}\right)$ leaving invariant the system (2.1) written in the "extended form" :

$$q'' + q\Psi = 0, \quad Q'' - Q\Psi = 0, \quad \Psi_q = 0, \quad \Psi_Q = 0 \tag{3.1}$$

Using the latter definition and assuming that the transformed variables $\bar{r}, \bar{q}$, and $\overline{Q}$ involve only r, q, and Q, but not Ψ, one can calculate in the usual way the generators of the continuous group of equivalence transformations in the form

$$Y = \tau(r,q,Q)\frac{\partial}{\partial r} + \xi(r,q,Q)\frac{\partial}{\partial q} + \eta(r,q,Q)\frac{\partial}{\partial Q} + \mu(r,q,Q,\Psi)\frac{\partial}{\partial \Psi} \tag{3.2}$$

Then one can apply Lie's infinitesimal technique by using the prolongation of the operator Y to the derivatives involved in the extended system (3.1) as follows (see [9] and e.g. [12]):

$$\overline{Y} = \tau\frac{\partial}{\partial r} + \xi\frac{\partial}{\partial q} + \eta\frac{\partial}{\partial Q} + \mu\frac{\partial}{\partial \Psi} + \varsigma_2^1\frac{\partial}{\partial q''} + \varsigma_2^2\frac{\partial}{\partial Q''} + \omega_1\frac{\partial}{\partial \Psi_q} + \omega_2\frac{\partial}{\partial \Psi_Q} \tag{3.3}$$

The invariance test for equations (3.1) requires that the following system of *determining equations* holds:

$$\varsigma_2^1\Big|_{(3.1)} + \xi\Psi + q\mu = 0 \qquad \varsigma_2^2\Big|_{(3.1)} - \eta\Psi - Q\mu = 0 \tag{3.4}$$

$$\omega_1\Big|_{(3.1)} = 0 \qquad \omega_2\Big|_{(3.1)} = 0 \tag{3.5}$$

Here ς_2^1 and ς_2^2 are given by the previous prolongation formulae (2.6) and (2.7), whereas ω_1 and ω_2 are determined by

$$\begin{aligned}\omega_1 &= \widetilde{D}_q(\mu) - \Psi_q\widetilde{D}_q(\xi) - \Psi_Q\widetilde{D}_q(\eta) - \Psi_r\,\widetilde{D}_q(\tau),\\ \omega_2 &= \widetilde{D}_Q(\mu) - \Psi_q\widetilde{D}_Q(\xi) - \Psi_Q\widetilde{D}_Q(\eta) - \Psi_r\,\widetilde{D}_Q(\tau)\end{aligned} \tag{3.6}$$

where

$$\widetilde{D}_q = \frac{\partial}{\partial q} + \Psi_q\frac{\partial}{\partial \Psi}, \qquad \widetilde{D}_Q = \frac{\partial}{\partial Q} + \Psi_Q\frac{\partial}{\partial \Psi} \tag{3.7}$$

are the "new" total differentiations for the extended system (3.1).

The restriction to the equations (3.1) means, in particular, that we set $\Psi_q = \Psi_Q = 0$. The expressions (3.7) and (3.6) take the form:

$$\widetilde{D}_q = \frac{\partial}{\partial q}, \qquad \omega_1 = \mu_q - \Psi_r\tau_q$$

$$\widetilde{D}_Q = \frac{\partial}{\partial Q}, \qquad \omega_2 = \mu_Q - \Psi_r\tau_Q$$

Let us begin with the equations (3.5):

$$\omega_1 \equiv \mu_q - \Psi_r\tau_q = 0, \qquad \omega_2 \equiv \mu_Q - \psi_r\tau_Q = 0$$

Since Ψ and hence Ψ_r are arbitrary functions, the above equations yield:

$$\tau_q = 0, \quad \mu_q = 0, \quad \tau_Q = 0, \quad \mu_Q = 0.$$

Thus the operator (3.2) reduces to the form:

$$Y = \tau(r)\frac{\partial}{\partial r} + \xi(r,q,Q)\frac{\partial}{\partial q} + \eta(r,q,Q)\frac{\partial}{\partial Q} + \mu(r,\Psi)\frac{\partial}{\partial \Psi} \tag{3.8}$$

Comparing equations (3.4) with the determining equations for the symmetries we notice that equations (3.4) are obtained from (2.4) and (2.5) merely by replacing $\tau\Psi'(r)$ by μ. Hence we can apply the calculations of the previous section and arrive at the equations (2.24). We note that since Ψ is regarded now as an arbitrary variable, the equations $k''(r)+k(r)\Psi(r)=0$ and $l''(r)-l(r)\Psi(r)=0$ yield k(r) = 0 and l(r) = 0, respectively. Furthermore we make the above mentioned replacement $\tau\Psi'(r)\mapsto\mu$ in (2.25).

Summing up, we obtain the following coordinates of the equivalence generator (3.8):

$$\begin{aligned}
&\tau = C_1 r^2 + C_2 r + C_3,\\
&\xi = (C_1 r + C_4) q\\
&\eta = (C_1 r + C_5) Q\\
&\mu = -(4C_1 r + 2C_2)\Psi
\end{aligned} \tag{3.9}$$

Thus, the system (2.1) has the five-dimensional equivalence Lie algebra spanned by the following equivalence generators:

$$\begin{aligned}
&Y_1 = q\frac{\partial}{\partial q}, \quad Y_2 = Q\frac{\partial}{\partial Q}, \quad Y_3 = \frac{\partial}{\partial r}, \quad Y_4 = r\frac{\partial}{\partial r} - 2\Psi\frac{\partial}{\partial \Psi},\\
&Y_5 = r^2\frac{\partial}{\partial r} + rq\frac{\partial}{\partial q} + rQ\frac{\partial}{\partial Q} - 4r\Psi\frac{\partial}{\partial \Psi}
\end{aligned} \tag{3.10}$$

The operators Y_1, Y_2, and Y_3 generate the groups of dilations in q and Q and the translations in r respectively. The operator Y_4 generates the group of dilations in r together with a coupled dilation of Ψ, i.e. $\bar{r} = ar$, $\overline{\Psi} = \frac{\Psi}{a^2}$. The operator Y_5 generates the group

$$\bar{r} = \frac{r}{1-ar}, \quad \bar{q} = \frac{q}{1-ar}, \quad \overline{Q} = \frac{Q}{1-ar}, \quad \overline{\Psi} = (1-ar)^4\Psi \tag{3.11}$$

4. Classification

Let us find all the potentials (2.23) that are equivalent to the constant potential

$$\Psi = C, \qquad C \neq 0 \tag{4.1}$$

In order to find them, we substitute $\overline{\Psi} = \delta$, where δ is a constant, in the equivalence transformation (3.11), $(1-ar)^4\Psi = \delta$, and obtain

$$\Psi(r) = \frac{\delta}{(1-ar)^4} \equiv \frac{\delta}{(a^2r^2 - 2ar + 1)^2} \tag{4.2}$$

This is a potential of the form (2.23) with $\alpha = a^2, \beta = -2a,$ and $\gamma = 1$. For this potential, the expression $\alpha r^2 + \beta r + \gamma$ has the vanishing discriminant

$$\Delta = \beta^2 - 4\alpha\gamma, \tag{4.3}$$

i.e. the equation $\alpha r^2 + \beta r + \gamma = 0$ has two equal roots. One can readily verify that the remaining equivalence transformations, i.e. the dilations and the translation of r leave invariant the equation $\Delta = 0$. Therefore, we can obtain from $\overline{\Psi}(\bar{r}) = C$ all potentials (2.23) with the vanishing discriminant (4.3). Vice versa, all potentials (2.23) with the vanishing discriminant (4.3) are equivalent to the constant potential, since the equivalence transformations are invertible. For the constant potential (4.1), (2.25) yields $C_1 = C_2 = 0$. Hence, the Sachs equations (2.1) with the constant potential (4.1) have, along with the six trivial symmetries (2.2), the additional symmetry

$$X_7 = \frac{\partial}{\partial r} \tag{4.4}$$

Likewise, one readily finds the potentials that are equivalent to

$$\Psi(r) = \frac{C}{r^2}, \qquad C \neq 0 \tag{4.5}$$

Namely, substituting $\overline{\Psi} = \frac{\delta}{\bar{r}^2}$ in the transformation (3.11),

$$(1 - a\,r)^4 \Psi = \frac{(1 - a\,r)^2}{r^2}\delta$$

one obtains:

$$\Psi(r) = \frac{\delta}{r^2(1 - a\,r)^2} \equiv \frac{\delta}{(a\,r^2 - r)^2}$$

This is a potential of the form (2.23) with $\alpha = a$, $\beta = -1$, and $\gamma = 0$. It has the positive discriminant $\Delta > 0$. In other words, the equation $\alpha r^2 + \beta r + \gamma = 0$ has two real roots. As above, we conclude that all potentials (2.23) with the positive discriminant (4.3) are equivalent to the potential (4.5), since the equivalence transformations are invertible and do not change the sign of the discriminant (4.3). For the potential (4.5), (2.25) yields $C_1 = C_3 = 0$. Hence, the equations (2.1) with the potential (4.5) have the following additional symmetry (written upon subtracting the operators X_1 and X_2):

$$X_7 = r\frac{\partial}{\partial r} \tag{4.6}$$

Furthermore, one can readily verify that the potentials (2.23) with the negative discriminant (4.3), $\Delta < 0$, are equivalent to the potential

$$\Psi(r) = \frac{C}{(r^2 + 1)^2}, \qquad C \neq 0 \tag{4.7}$$

The equations (2.1) with the potential (4.7) have the additional symmetry (2.26) with $\alpha = 1, \beta = 0$ and $\gamma = 1$:

$$X_7 = (1 + r^2)\frac{\partial}{\partial r} + r q \frac{\partial}{\partial q} + r Q \frac{\partial}{\partial Q} \tag{4.8}$$

Thus we have proved the following theorem.

Theorem 2. The Sachs equations (2.1) with the potential (2.23),

$$\Psi(r) = \frac{\delta}{\left(\alpha r^2 + \beta r + \gamma\right)^2},$$

can be reduced by a proper equivalence transformation to the equation with the standard potential (4.1), (4.5) or (4.7). Specifically,, we can set

$$\begin{aligned} \Psi &= C, && \text{if} \quad \Delta = 0 \\ \Psi &= \frac{C}{r^2}, && \text{if} \quad \Delta > 0 \\ \Psi &= \frac{C}{(r^2+1)^2} && \text{if} \quad \Delta < 0 \end{aligned} \tag{4.9}$$

where $\Delta = \beta^2 - 4\alpha\gamma$.

Therefore it suffices to solve the Sachs equations (2.1) with the standard potentials (4.9).

Example. Let us obtain the solution of equations (2.1) with the potential (4.2) where we let $C = \omega^2$:

$$q'' + \frac{\omega^2 q}{(a^2 r^2 - 2 a r + 1)^2} = 0, \qquad Q'' - \frac{\omega^2 Q}{(a^2 r^2 - 2 a r + 1)^2} = 0 \tag{4.10}$$

In the new variables $\bar{q}, \bar{Q}$, and $\bar{r}$ obtained by the equivalence transformations (3.11), the system (4.10) takes the form (cf. (4.1))

$$\bar{q}'' + \omega^2 \bar{q} = 0, \qquad \bar{Q}'' - \omega^2 \bar{Q} = 0$$

Whence

$$\bar{q} = A_1 \cos(\omega \bar{r}) + B_1 \sin(\omega \bar{r}), \qquad \bar{Q} = A_2 e^{\omega \bar{r}} + B_2 e^{-\omega \bar{r}}.$$

According to (3.11) we substitute here

$$\bar{r} = \frac{r}{1 - a r},$$

insert the expression obtained for $\overline{q}$ and $\overline{Q}$ into

$$q = (1 - a\,r)\,\overline{q}, \qquad Q = (1 - a\,r)\,\overline{Q}$$

and arrive at the following general solution of the system (4.10):

$$\begin{aligned} q &= (1 - a\,r)\left[A_1 \cos\left(\frac{\omega r}{1 - a\,r}\right) + B_1 \sin\left(\frac{\omega r}{1 - a\,r}\right)\right] \\ Q &= (1 - a\,r)\left[A_2 \exp\left(\frac{\omega r}{1 - a\,r}\right) + B_2 \exp\left(-\frac{\omega r}{1 - a\,r}\right)\right] \end{aligned} \tag{4.11}$$

We recall that the above results were obtained under the assumption that $\Psi \neq 0$. Therefore, to complete the group classification of the systems admitting more general groups than those generated by (2.1), it remains to provide the symmetries of equations (2.1) with the vanishing potential Ψ:

$$q'' = 0, \qquad Q'' = 0\,. \tag{4.12}$$

This system admits the 15-dimensional Lie algebra spanned by the following generators (cf. [8] Section 9.3.3):

$$\begin{aligned} & X_1 = \frac{\partial}{\partial r}, \quad X_2 = \frac{\partial}{\partial q}, \quad X_3 = \frac{\partial}{\partial Q}, \quad X_4 = r\frac{\partial}{\partial r}, \quad X_5 = q\frac{\partial}{\partial r}, \\ & X_6 = Q\frac{\partial}{\partial r}, \quad X_7 = r\frac{\partial}{\partial q}, \quad X_8 = q\frac{\partial}{\partial q}, \quad X_9 = Q\frac{\partial}{\partial q}, \quad X_{10} = r\frac{\partial}{\partial Q}, \\ & X_{11} = q\frac{\partial}{\partial Q}, \quad X_{12} = Q\frac{\partial}{\partial Q}, \quad X_{13} = r^2\frac{\partial}{\partial r} + r\,q\frac{\partial}{\partial q} + r\,Q\frac{\partial}{\partial Q}, \\ & X_{14} = r\,q\frac{\partial}{\partial r} + q^2\frac{\partial}{\partial q} + q\,Q\frac{\partial}{\partial Q}, \quad X_{15} = r\,Q\frac{\partial}{\partial r} + q\,Q\frac{\partial}{\partial q} + Q^2\frac{\partial}{\partial Q} \end{aligned} \tag{4.13}$$

5. Discussion and Conclusions

We start with a discussion of the equivalence transformations derived in section 3.

Since the fundamental postulate of the theory of special relativity is that space-time is a differentiable manifold endowed with a Lorentz group structure, and the Newman-Penrose equations assume that this structure is preserved in the fibre bundle of tangent spaces over the points of a curved space-time, it should not be a surprise that the equivalence transformations of equations (1.1) include the elements of the group SL (2R), the special linear group over the real numbers of order 2: This group is just the group of Lorentz transformations corresponding to a boost in the r - θ plane.

In the coordinates employed in the derivation of equations (1.1) the imaginary axis lies in the direction $\frac{\partial}{\partial \phi}$. It follows that, while the general Lorentz group SL (2C) includes the sub-group of boosts across the direction of the axis of symmetry, the group SL (2R) excludes this sub-group. (For a discussion of the relation between the group SL (2C) and the Lorentz transformations see [13] chapter 1 or [14] chapter 13.) Boosts in the direction of the axis of symmetry, which preserve the axisymmetric character of the space-time and of the coordinates, and therefore must preserve the form of the Sachs equations, belong to the group SL (2R).

The translations in the parameter r that displace the origin of the coordinate system along the axis of symmetry must similarly preserve the form of the Sachs equations.

For the remaining equivalence transformations, the group of translations in the parameter r, in the general case when this displaces the origin away from the axis of symmetry, and boosts in the r - θ plane that are not oriented in the direction of the axis of symmetry, the situation is more complicated. Both these groups of transformations destroy the axisymmetric nature of the coordinate system (though not of the manifold, provided the cross-axis component of the boost vanishes when r = 0). In general, therefore, such transformations can be expected to change the form of the field equations.

However, it should be borne in mind that the Sachs equations, considered in isolation, are defined on a single null geodesic {u,θ,ϕ = const} emanating from a point on the axis of symmetry. Neither a displacement of the origin along such a coordinate curve, nor a boost in the r - θ plane changes the direction $\frac{\partial}{\partial\phi}$ on the curve itself, where this direction remains aligned to the Killing field. The symmetry of the metric tensor under the reflection $d\phi \rightarrow -d\phi$ is therefore preserved in the transformed coordinates and the off-diagonal elements of the fourth row and the fourth column of the metric tensor must still vanish in the new coordinates. As a result the metric tensor on the single null geodesic in question retains the general form that leads to the form of the Sachs equations (see the appendix) and these equations must therefore retain their form after the transformation, with some initial value function Ψ_0 (not necessarily the same function). This result, of course, does not apply to the remainder of the Newman-Penrose equations.

We have therefore arrived at the main conclusion of this paper: Any initial value function Ψ_0 that is an exact representation of an axisymmetric, non-rotating space-time manifold in which the Newman-Penrose equations are valid, and in which the Ricci tensor vanishes, determines the metric functions q and Q in accordance with the Sachs equations in the form (1.1). Such an initial value function must remain a representation of the same space-time manifold after any of the equivalence transformations corresponding to the infinitesimal generators (3.10) and it must remain related to the transformed metric functions by equations of the form (1.1). Some particular functions Ψ_0 may be invariant in form under the action on the Sachs equations of a particular sub-group of equivalence transformations, in which case this sub-group will constitute a symmetry group and the system of equations will admit a corresponding Lie symmetry. As demonstrated in section 2 and section 4, the function (2.23) is the only function that meets these invariance criteria. (The infinitesimal generator of the corresponding Lie symmetry is given by (2.26).) All forms of this function, corresponding to different values of the functions of integration, are transformed by the equivalence transformations into forms of the same general function, and therefore remain consistent with the Sachs equations. The combination of equivalence transformations corresponding to the generator (2.26) leaves both the form of the equations and the form $\overline{\Psi}_0$, of the function that appears in the transformed system of equations, unaltered.[1] A space-time corresponding to

[1] Note that we are considering only transformations of Ψ_0 within the context of the Sachs equations, in which all the variables are transformed simultaneously. Considered in isolation, the function is not invariant under the group of symmetry transformations that are applicable to the system of equations as a whole.

this function will therefore have special properties in that its system of Sachs equations possesses the associated non-trivial Lie symmetry.

The geometric interpretation of the Lie symmetry (2.26) is that for any displacement of the origin along the coordinate curve {u, θ, ϕ = const}, both the Sachs equations and the function Ψ_0, defined at a point on the same coordinate curve, are form-invariant provided the transformation of r corresponding to a particular displacement (a transformation $\bar{r} = r - a$ where a is some constant) is coupled with a specific dilation (corresponding to a transformation $\bar{r} = a\,r$) and/or a specific rotation of the direction $\frac{\partial}{\partial\theta}$ (which entails the inversion $\bar{r} = \frac{r}{1 - a\,r}$), where the relative magnitudes of these coupled elements of the real sub-group of the Poincare group (also called the inhomogeneous Lorentz group iSL (2R)) are given by the "constants" α,β and γ (more generally, these are functions of θ defined on a given null hypersurface).

This is consistent with the fact that a general curved axisymmetric space-time, in which the Ricci tensor vanishes and which is also symmetric with respect to the reflection $d\,\phi \rightarrow -d\,\phi$, is completely specified within a coordinate patch once the direction of the time vector and the dilation of r have been specified at every point along the null geodesics that constitute the coordinate curves. This follows from the fact that in an orthonormal tetrad, constructed at any point along the geodesic, only the direction $\frac{\partial}{\partial\theta}$ and the direction of the time vector can vary under the action of the null rotations that constitute the gauge freedom of the Newman-Penrose equations: the directions $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial\phi}$ are fixed in an invariant manner by the tangent to the geodesic {u, θ, ϕ = const} and the Killing vector respectively. Once the direction of the time vector has been fixed, after a null rotation, the direction $\frac{\partial}{\partial\theta}$ follows from the fact that it has to be orthogonal to the directions of the other three tetrad vectors.

It remains an open question whether there are any other functions that have the property that they remain a representation of the same space-time and remain consistent with the Sachs equations in the form (1.1) under the action of the group of equivalence transformations corresponding to the infinitesimal generators (3.10). If not, then the initial value function for all space-times with the geometric symmetries under discussion, and in which the Ricci tensor vanishes, must take the form (1.2). The space-times corresponding to this form of Ψ_0, which are special in the sense noted above, presumably also have special physical significance. As remarked in the introduction it has been shown [4] that (1.2) reproduces a number of the features that would be expected, in the appropriate limits, of the vacuum portion of a space-time representing a pair of non-rotating masses moving towards a head-on collision. .

Appendix: Derivation of the coordinate form of the Sachs equations

Co-ordinate system

We follow Bondi [15] in introducing a co-ordinate system based on a set of null cones generated from the time-like curve followed by a point O that remains on the axis of symmetry. In the words of Bondi, van der Burg and Metzner [16]: "From the axial symmetry the azimuth angle ϕ is readily defined in an invariant manner. Suppose now we put a source of light at a point O on the axis of symmetry and surround it by a small sphere on which we can produce the azimuth co-ordinate ϕ together with a co-latitude θ and a time co-ordinate u. We then define the u, θ, ϕ co-ordinates of an arbitrary event E to be the u, θ, ϕ co-ordinates of the event at which the light ray OE intersects the small sphere. In other words, along an outward radial light ray the three co-ordinates u, θ, ϕ are constant." An affine parameter r, defined up to a linear transformation, can be associated with the null geodesics. We take the affine parameter as the second co-ordinate and we assign numerical references 1,2,3,4 to u, r, θ and ϕ in that order.

We now follow Newman and Unti [17] in constructing a null tetrad within the co-ordinate neighbourhood of the origin. For the NP equations to be valid, this co-ordinate neighbourhood has to be a spinor space. This means that it must be possible to construct a null tetrad with a unique orientation at each point. The domain of validity U of the analysis therefore is a simply connected neighbourhood that extends outwards from the origin and terminates if the null geodesics, that constitute the co-ordinate curves, intersect one another.

On each null hypersurface, u = const, a vector $\ell_\alpha = u_{,\alpha}$ can be chosen orthogonal to the hypersurface. Taking u as the co-ordinate x^1, it follows that

$$\ell_\alpha = \delta^1_\alpha$$

where δ is the Kronecker delta.

Since the hypersurfaces are null, the vectors ℓ_α will also be tangent to the null geodesics lying within the surfaces. The tangents to the geodesics are given by:

$$\ell^\mu = \frac{dx^\mu}{dr} = g^{\mu\nu}\ell_\nu = g^{\mu 1} = \delta^\mu_2$$

It follows that $g^{21} = g^{12} = 1$ while all the remaining elements of the first column and of the first row of the metric tensor $g^{\mu\nu}$ are zero. Similarly,

$$\ell_\mu = g_{\mu\nu}\ell^\nu = g_{\mu 2} = \delta^1_\mu$$

Hence $g_{12} = g_{21} = 1$ while all the other elements of the second column and the second row of the metric tensor $g_{\mu\nu}$ are zero.

From the reflection symmetry under the transformation $d\phi \to -d\phi$ it follows that all the off-diagonal elements of the metric tensor $g_{\mu\nu}$ vanish in the last row and the last column.

It follows from this analysis that the metric tensor has to take the form:

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & 1 & g_{13} & 0 \\ 1 & 0 & 0 & 0 \\ g_{13} & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{22} & g^{23} & 0 \\ 0 & g^{23} & g^{33} & 0 \\ 0 & 0 & 0 & g^{44} \end{pmatrix}$$

Tetrad and Metric Tensor
A tetrad of four vectors that are normal to one another is constructed at each point of U, using the tangent vectors ℓ^{μ} as reference. A second null vector n^{μ} is chosen so that $\ell^{\mu} n_{\mu} = 1$. The tetrad is then completed with a pair of complex null vectors, m^{μ} and $\overline{m}^{\mu}$, that are normal to each other and to the pair of real null vectors. These complex null vectors are defined from an orthonormal pair of real unit space-like vectors, a^{μ} and b^{μ} such that

$$m^{\mu} = \tfrac{1}{\sqrt{2}}(a^{\mu} + i\,b^{\mu}) \quad \text{and} \quad \overline{m}^{\mu} = \tfrac{1}{\sqrt{2}}(a^{\mu} - i\,b^{\mu})$$

The pair-wise inner products of the tetrad vectors are determined by their normality conditions:

$$\ell_{\mu}\ell^{\mu} = m_{\mu}m^{\mu} = \overline{m}_{\mu}\overline{m}^{\mu} = n_{\mu}n^{\mu} = 0$$
$$\ell_{\mu}n^{\mu} = -m_{\mu}\overline{m}^{\mu} = 1$$
$$\ell_{\mu}m^{\mu} = \ell_{\mu}\overline{m}^{\mu} = n_{\mu}m^{\mu} = n_{\mu}\overline{m}^{\mu} = 0$$

The tetrad vectors are related to the elements of the metric tensor by the relation

$$g^{\mu\nu} = \ell^{\mu}\, n^{\nu} + n^{\mu}\,\ell^{\nu} - m^{\mu}\,\overline{m}^{\nu} - \overline{m}^{\mu}\, m^{\nu}$$

A similar relation holds for the covariant form of the metric tensor.

To satisfy the normality conditions the vectors n^{μ} and m^{μ} must take the form

$$n^{\mu} = \delta_1^{\mu} + U\,\delta_2^{\mu} + X^{i}\delta_i^{\mu} \qquad (i = 3, 4)$$

$$m^{\mu} = \omega\delta_2^{\mu} + \xi^{i}\delta_i^{\mu}$$

where, using the notation of Newman, Penrose , and their co-workers, U, X^i , ω and ξ^i are unknown, complex-valued functions of u, r, and θ. Using these relations gives the elements of the metric tensor $g^{\mu\nu}$ in terms of the functions that appear in the tetrad vectors.

It is possible to choose the vector n^{μ} such that its spatial projection lies in the $r\!-\!\theta$ plane at the origin. This implies $X^4 = 0$. As a result of the axial symmetry, the spatial projection of n^{μ} must then remain in the $r\!-\!\theta$ plane as the tetrad propagates outwards, if it is parallel propagated, so that $X^4 = 0$ for all r. The index is then redundant and it is possible to put $X^3 \equiv X$.

If the imaginary axis is chosen to lie along the co-ordinate direction x^4 and one puts $\xi^3 = \dfrac{1}{q}$ and $\xi^4 = \dfrac{i}{Q}$ where both q and Q are real-valued functions of the co-ordinates the result is that all the metric functions become real-valued as do all the Newman-Penrose equations and all the functions that appear in them. The metric tensor resulting from this analysis is:

$$ds^2 = (-2U - \tfrac{1}{2}X^2q^2 + 2X\omega q)du^2 + 2dudr - (2\omega q - q^2X)dud\theta - \tfrac{1}{2}q^2d\theta^2 - \tfrac{1}{2}Q^2d\phi^2$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2(U-\omega^2) & X - \dfrac{2\omega}{q} & 0 \\ 0 & X - \dfrac{2\omega}{q} & \dfrac{-2}{q^2} & 0 \\ 0 & 0 & 0 & \dfrac{-2}{Q^2} \end{pmatrix}$$

There is an apparent redundancy since five functions are used to specify four elements of the metric tensor. However, when parallel propagation is imposed as a condition, a relation between ω and X results, which removes the redundancy (if the tetrad is parallel propagated then the spin coefficient π must vanish).

Transformations of the co-ordinates
Two sets of transformations can be applied to the tetrad without changing the metric tensor, violating the normality conditions, or changing the direction of the tangent vector ℓ^μ:

- Rotation of the pair of space-like vectors in their plane without changing their orientation relative to each other, i.e.

$$\ell^{\mu'} = \ell^\mu, \qquad n^{\mu'} = n^\mu, \qquad m^{\mu'} = m^\mu e^{iC} \qquad \text{(C real)}$$

  Null rotations that leave ℓ^μ fixed, i.e.

$$\ell^{\mu'} = \ell^\mu, \qquad n^{\mu'} = n^\mu + \overline{B} m^\mu + B\overline{m}^\mu + B\overline{B}\ell^\mu, \qquad m^{\mu'} = m^\mu + B\ell^\mu$$

In general the parameter B is complex. However, if the condition is imposed that the imaginary space-like direction must remain aligned with the Killing field $\frac{\partial}{\partial\phi}$, then B becomes real. Under this condition the parameter C must also vanish, leaving only the real null rotations as a gauge freedom.

The Sachs equations
In the notation of Newman and Penrose, the two Sachs equations take the following form in the current case where the Ricci tensor vanishes and all the variables are real-valued:

$$\frac{\partial\rho}{\partial r} = \rho^2 + \sigma^2 \quad \text{and} \quad \frac{\partial\sigma}{\partial r} = 2\rho\sigma + \Psi_0$$

where the function Ψ_0 is the first term of the Weyl spinor and ρ and σ, the so-called optical scalars, are the expansion and shear, respectively, of the null geodesic congruence. They are defined by:

$$\rho = \ell_{\mu;\nu} m^\mu \overline{m}^\nu \ \text{and} \ \sigma = \ell_{\mu;\nu} m^\mu m^\nu$$

Expanding the expressions for ρ and σ in terms of the above metric tensor yields the equations in the form (1.1).

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A discussion of conservation laws for over-determined systems with application to the Maxwell equations in vacuum

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

Abstract. The evolution equations of Maxwell's equations has a Lagrangian written in terms of the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields, but admit neither Lorentz nor conformal transformations. The additional equations $\nabla\cdot\boldsymbol{E} = 0,\ \nabla\cdot\boldsymbol{H} = 0$ guarantee the Lorentz and conformal invariance, but the resulting system is overdetermined, and hence does not have a Lagrangian. The aim of the present paper is to attain a harmony between these two contradictory properties.

Introduction

It was discovered by Lorentz [1] that that the Maxwell equations in vacuum are invariant under the 10-parameter group of isometric motions in the four-dimensional flat space-time known as the Minkowsky space. Accordingly, this group is known as the Lorentz group. Then it was shown by Bateman [2], [3] and Cunningham [4] that the Maxwell equations in vacuum admit a wider group, namely the 15-parameter group of the conformal transformations in the Minkowsky space. It was proved later [5] that the conformal transformations, together with the simultaneous dilations and *duality rotations* of the electric and magnetic fields, as well as the obvious infinite group expressing the linear superposition principle, furnish the maximal local group of Lie point transformations admitted by the Maxwell equations.

Thus, the Maxwell equations in vacuum admit a 17-dimensional Lie algebra, L_{17}; along with the infinite-dimensional Lie algebra existing for all linear equations due to the linear superposition principle.

Bessel-Hagen [6] applied Noether's theorem to the 15-dimensional Lie algebra L_{15} of the conformal group (L_{15} is a subalgebra of L_{17}) and, using the variational formulation of Maxwell's equations in terms of the 4-potential for the electromagnetic field, derived 15 conservation laws. In this way, he obtained, along with the well-known theorems on conservation of energy, momentum, angular momentum and the relativistic center-of-mass theorem, five new conservation laws. He wrote about the latter: "The future will show if they have any physical significance". To the best of my knowledge, a physical interpretation and utilization of Bessel-Hagen's new conservation laws is still an open question. Meanwhile, Lipkin [7] discovered ten new conservation laws involving first derivatives of the electric and magnetic vectors. It was shown later [8] that Lipkin's conservation laws were associated with translations and Lorentz transformations. For a review of further investigations in this direction, see [9], Section 8.6, and the references therein. See also [10].

The present paper is a continuation of my recent work [11] with an emphasis on applicability of Noether's theorem to overdetermined systems hinted in [12]. Namely, the evolution equations of the system of Maxwell equations has a Lagrangian written in terms of the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ vectors, but admit neither Lorentz nor conformal transformations. The additional equations $\nabla \cdot \boldsymbol{E} = 0, \ \nabla \cdot \boldsymbol{H} = 0$ guarantee the Lorentz and conformal invariance, but destroy the Lagrangian. The aim of the present paper is to attain a harmony between these two contradictory properties.

1 Generalities

1.1 The Maxwell equations

Consider the Maxwell equations in vacuum:

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0. \tag{1.2}$$

We will also use the coordinate notation

$$\boldsymbol{x} = (x, y, z), \quad \boldsymbol{E} = (E^1, E^2, E^3), \quad \boldsymbol{H} = (H^1, H^2, H^3)$$

and write Eqs. (1.1)–(1.2) in the coordinate form as well:

$$\begin{array}{ll} E^3_y - E^2_z + H^1_t = 0, & H^3_y - H^2_z - E^1_t = 0, \\ E^1_z - E^3_x + H^2_t = 0, & H^1_z - H^3_x - E^2_t = 0, \\ E^2_x - E^1_y + H^3_t = 0, & H^2_x - H^1_y - E^3_t = 0, \end{array} \tag{1.1$'$}$$

$$E^1_x + E^2_y + E^3_z = 0, \qquad H^1_x + H^2_y + H^3_z = 0. \tag{1.2$'$}$$

1.2 Symmetries

Eqs. (1.1)-(1.2) are invariant under the translations of time t and the position vector $\boldsymbol{x}$ as well as the simultaneous rotations of the vectors $\boldsymbol{x}, \boldsymbol{E}$ and $\boldsymbol{H}$ due to the vector formulation of Maxwell's equations. The corresponding generators provide the following seven infinitesimal symmetries:

$$X_0 = \frac{\partial}{\partial t}; \quad X_1 = \frac{\partial}{\partial x}; \quad X_2 = \frac{\partial}{\partial y}; \quad X_3 = \frac{\partial}{\partial z};$$

$$X_{12} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + E^2\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^2} + H^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^2};$$

$$X_{13} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^3} + H^3\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^3}; \qquad (1.3)$$

$$X_{23} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^2} - E^2\frac{\partial}{\partial E^3} + H^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial H^3} \cdot$$

Besides, the Maxwell equations admit the Lorentz transformations (hyperbolic rotations) with the generators

$$X_{01} = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} + E^2\frac{\partial}{\partial H^3} + H^3\frac{\partial}{\partial E^2} - E^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial E^3};$$

$$X_{02} = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} + E^3\frac{\partial}{\partial H^1} + H^1\frac{\partial}{\partial E^3} - E^1\frac{\partial}{\partial H^3} - H^3\frac{\partial}{\partial E^1}; \qquad (1.4)$$

$$X_{03} = t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} + E^1\frac{\partial}{\partial H^2} + H^2\frac{\partial}{\partial E^1} - E^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial E^2};$$

and the *duality rotations*

$$\overline{E} = E\cos\alpha - H\sin\alpha; \quad \overline{H} = H\cos\alpha + E\sin\alpha$$

with the generator

$$Z_0 = E\cdot\frac{\partial}{\partial H} - H\cdot\frac{\partial}{\partial E} \equiv \sum_{k=1}^{3}\left(E^k\frac{\partial}{\partial H^k} - H^k\frac{\partial}{\partial E^k}\right)\cdot \qquad (1.5)$$

Furthermore, Eqs. (1.1)-(1.2) admit, due to their homogeneity and linearity, the dilation generators

$$Z_1 = E\cdot\frac{\partial}{\partial E} + H\cdot\frac{\partial}{\partial H} \equiv \sum_{k=1}^{3}E^k\frac{\partial}{\partial E^k} + \sum_{k=1}^{3}H^k\frac{\partial}{\partial H^k} \qquad (1.6)$$

and

$$Z_2 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}; \qquad (1.7)$$

and the superposition generator

$$S = E_*(x;t)\cdot\frac{\partial}{\partial E} + H_*(x;t)\cdot\frac{\partial}{\partial H}; \qquad (1.8)$$

where the vectors $\boldsymbol{E} = \boldsymbol{E}_*(\boldsymbol{x}, t), \ \boldsymbol{H} = \boldsymbol{H}_*(\boldsymbol{x}, t)$ solve Eqs. (1.1)-(1.2).

Finally, Eqs. (1.1)-(1.2) admit the conformal transformations with the generators

$$
\begin{aligned}
Y_1 = {} & \left(x^2 - y^2 - z^2 + t^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + 2xt\frac{\partial}{\partial t} \\
& - \left(4xE^1 + 2yE^2 + 2zE^3\right)\frac{\partial}{\partial E^1} - \left(4xH^1 + 2yH^2 + 2zH^3\right)\frac{\partial}{\partial H^1} \\
& - \left(4xE^2 - 2yE^1 - 2tH^3\right)\frac{\partial}{\partial E^2} - \left(4xH^2 - 2yH^1 + 2tE^3\right)\frac{\partial}{\partial H^2} \\
& - \left(4xE^3 - 2zE^1 + 2tH^2\right)\frac{\partial}{\partial E^3} - \left(4xH^3 - 2zH^1 - 2tE^2\right)\frac{\partial}{\partial H^3}, \\
Y_2 = {} & 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2 - z^2 + t^2\right)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} + 2yt\frac{\partial}{\partial t} \\
& - \left(4yE^1 - 2xE^2 - 2tH^3\right)\frac{\partial}{\partial E^1} - \left(4yH^1 - 2xH^2 + 2tE^3\right)\frac{\partial}{\partial H^1} \\
& - \left(4yE^2 + 2xE^1 + 2zE^3\right)\frac{\partial}{\partial E^2} - \left(4yH^2 + 2xH^1 + 2zH^3\right)\frac{\partial}{\partial H^2} \\
& - \left(4yE^3 - 2zE^2 + 2tH^1\right)\frac{\partial}{\partial E^3} - \left(4yH^3 - 2zH^2 - 2tE^1\right)\frac{\partial}{\partial H^3}, \\
Y_3 = {} & 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + \left(z^2 - x^2 - y^2 + t^2\right)\frac{\partial}{\partial z} + 2zt\frac{\partial}{\partial t} \\
& - \left(4zE^1 - 2xE^3 + 2tH^2\right)\frac{\partial}{\partial E^1} - \left(4zH^1 - 2xH^3 - 2tE^2\right)\frac{\partial}{\partial H^1} \\
& - \left(4zE^2 - 2yE^3 - 2tH^1\right)\frac{\partial}{\partial E^2} - \left(4zH^2 - 2yH^3 + 2tE^1\right)\frac{\partial}{\partial H^2} \\
& - \left(4zE^3 + 2yE^2 + 2xE^1\right)\frac{\partial}{\partial E^3} - \left(4zH^3 + 2yH^2 + 2xH^1\right)\frac{\partial}{\partial H^3}, \\
Y_4 = {} & 2tx\frac{\partial}{\partial x} + 2ty\frac{\partial}{\partial y} + 2tz\frac{\partial}{\partial z} + \left(x^2 + y^2 + z^2 + t^2\right)\frac{\partial}{\partial t} \\
& - \left(4tE^1 + 2yH^3 - 2zH^2\right)\frac{\partial}{\partial E^1} - \left(4tH^1 - 2yE^3 + 2zE^2\right)\frac{\partial}{\partial H^1} \\
& - \left(4tE^2 + 2zH^1 - 2xH^3\right)\frac{\partial}{\partial E^2} - \left(4tH^2 - 2zE^1 + 2xE^3\right)\frac{\partial}{\partial H^2} \\
& - \left(4tE^3 - 2yH^1 + 2xH^2\right)\frac{\partial}{\partial E^3} - \left(4tH^3 + 2yE^1 - 2xE^2\right)\frac{\partial}{\partial H^3}.
\end{aligned}
\tag{1.9}
$$

It has been proved in [5] that the operators (1.3)–(1.9) span the Lie algebra of the maximal continuous point transformation group admitted by Eqs. (1.1)–(1.2). This algebra comprises the 17-dimensional subalgebra spanned by the operators (1.3)-(1.7) and (1.9), and the infinite-dimensional ideal consisting of the operators (1.8).

1.3 Conservation equations

Conservation laws for the Maxwell equations are written in the differential form as

$$\left(D_t(\tau) + \operatorname{div}\chi\right)\Big|_{(1.1)-(1.2)} = 0, \tag{1.10}$$

where τ and $\chi = (\chi^1, \chi^2, \chi^3)$ are termed the *density* of the conservation law and the *flux*, respectively. The divergence $\operatorname{div}\chi$ of the flux vector is given by

$$\operatorname{div}\chi \equiv \nabla\cdot\chi = D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3). \tag{1.11}$$

The symbol $\big|_{(1.1)-(1.1)}$ means that Eq. (1.10) is satisfied on the solutions of the Maxwell equations (1.1)-(1.2). The density τ and flux χ may depend, in general, on the variables t, x as well as on the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{H}$ and their derivatives.

It is assumed in what follows that the time derivatives of $\boldsymbol{E}, \boldsymbol{H}$ have been eliminated by using Eqs. (1.1), and hence τ and χ^1, χ^2, χ^3 depend, in general, on $t, x, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z, \boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z, \ldots$. Then Eq. (1.10) is written:

$$\left(D_t(\tau)\big|_{(1.1)} + \operatorname{div}\chi\right)\Big|_{(1.2)} = 0, \tag{1.12}$$

where $D_t(\tau)\big|_{(1.1)}$ is obtained from $D_t(\tau)$ by substituting the values of $\boldsymbol{E}_t, \boldsymbol{H}_t$ given by Eqs. (1.1), whereas the sign $\big|_{(1.2)}$ indicates that the terms proportional to $\operatorname{div}\boldsymbol{E}$ and $\operatorname{div}\boldsymbol{H}$ have been eliminated by using Eqs. (1.2).

Often the differential conservation law (1.10) is written, using the divergence theorem, in the integral form:

$$\frac{d}{dt}\int \tau \, dxdydz = 0. \tag{1.13}$$

The integral form of conservation laws is preferable from physical point of view because is has a clear physical interpretation. Namely, Eq. (1.13) means that the integral

$$\frac{d}{dt}\int \tau \, dxdydz \tag{1.14}$$

taken for any solution of Maxwell's equations is constant in time.

1.4 Test for conservation densities of Eqs. (1.1)

Sometimes conservation laws are formulated by stating that the integral (1.14) with a certain τ is constant in time, but the corresponding flux χ is not given. In this case, one can verify that the conservation equation (1.13) is satisfied by using the *tests for conservation densities* given in this and the next sections. We will need the following property of divergencies.

Lemma 1.1. Let $\psi^l = \psi^l(t; x; E; H; E_x; E_y; E_z; H_x; H_y; H_z; \ldots); \ (l = 1; 2; 3)$ be any smooth functions. Then the following equations are satisfied:

$$\frac{\delta}{\delta E}(\nabla \cdot \chi) = 0; \quad \frac{\delta}{\delta H}(\nabla \cdot \chi) = 0: \tag{1.15}$$

Here the vector valued Euler-Lagrange operators (variational derivatives) are given by

$$\frac{\delta}{\delta E} = \frac{\partial}{\partial E} - D_i \frac{\partial}{\partial E_i} + D_i D_j \frac{\partial}{\partial E_{ij}} - \cdots; \tag{1.16}$$

$$\frac{\delta}{\delta H} = \frac{\partial}{\partial H} - D_i \frac{\partial}{\partial H_i} + D_i D_j \frac{\partial}{\partial H_{ij}} - \cdots; \tag{1.17}$$

where the indices $i; j$ assume the values 1, 2, 3, and the following notation is used:

$$\begin{array}{lll} D_1 = D_x; & D_2 = D_y; & D_3 = D_z; \\ E_1 = E_x; & E_2 = E_y; & E_3 = E_z; \\ H_1 = H_x; & H_2 = H_y; & H_3 = H_z; \\ E_{11} = E_{xx}; & E_{12} = E_{xy}; \ \ldots; & E_{33} = E_{zz}; \\ H_{11} = H_{xx}; & H_{12} = H_{xy}; \ \ldots; & H_{33} = H_{zz}: \end{array} \tag{1.18}$$

Proof. The Lemma is a particular case of the general result (see [13], Section 8.4.1, Exercise 3; see also [11], p. 747, Lemma 2.3) stating that a function $f(x; u; u_{(1)}); \ldots; u_{(s)})$ is a divergence, i.e.

$$f = \operatorname{div} \phi \equiv \sum_{i=1}^{n} D_i(\phi^i);$$

where $\phi = (\phi^1; \ldots; \phi^n)$ is an n-dimensional vector with components

$$\phi^1 = \phi^1(x; u; u_{(1)}); \ldots; u_{(s-1)}); \ldots; \phi^n = \phi^n(x; u; u_{(1)}); \ldots; u_{(s-1)});$$

if and only if

$$\frac{\delta f}{\delta u^\alpha} = 0; \quad \alpha = 1; \ldots; m:$$

Here $x = (x^1; \ldots; x^n)$ are independent variables, $u = (u^1; \ldots; u^m)$ are dependent variables, $u_{(1)}$ is the set of the first-order derivatives u_i^α of the variables u^α with respect to x^i; and $u_{(2)}$ is the set of the second-order derivatives, etc.

I give below an independent proof of Eqs. (1.15). Let us write these equations in the expanded form used in (1.11):

$$\frac{\delta}{\delta E}\left[D_x(\psi^1) + D_y(\psi^2) + D_z(\psi^3)\right] = 0;$$

$$\frac{\delta}{\delta H}\left[D_x(\psi^1) + D_y(\psi^2) + D_z(\psi^3)\right] = 0: \tag{1.15$'$}$$

Due to linearity of the variational derivatives (1.16) -(1.17) and the symmetry of Eqs. (1.15′) with respect to the variables $x, y, z,$ it suffices to prove that

$$\frac{\delta}{\delta E} D_x(\chi^l) = 0, \quad \frac{\delta}{\delta H} D_x(\chi^l) = 0, \quad l = 1, 2, 3. \tag{1.19}$$

Let us begin with the case when χ^l do not depend on derivatives of E and H, i.e.

$$\chi^l = \chi^l(t, x, E, H), \quad l = 1, 2, 3. \tag{1.20}$$

In this case the variational derivatives (1.16)-(1.17) reduce to

$$\frac{\delta}{\delta E} = \frac{\partial}{\partial E} - D_x \frac{\partial}{\partial E_x} - D_y \frac{\partial}{\partial E_y} - D_z \frac{\partial}{\partial E_z}, \tag{1.21}$$

$$\frac{\delta}{\delta H} = \frac{\partial}{\partial H} - D_x \frac{\partial}{\partial H_x} - D_y \frac{\partial}{\partial H_y} - D_z \frac{\partial}{\partial H_z} \tag{1.22}$$

and have the components

$$\frac{\delta}{\delta E^k} = \frac{\partial}{\partial E^k} - D_x \frac{\partial}{\partial E^k_x} - D_y \frac{\partial}{\partial E^k_y} - D_z \frac{\partial}{\partial E^k_z},$$

$$\frac{\delta}{\delta H^k} = \frac{\partial}{\partial H^k} - D_x \frac{\partial}{\partial H^k_x} - D_y \frac{\partial}{\partial H^k_y} - D_z \frac{\partial}{\partial H^k_z}, \quad k = 1, 2, 3.$$

It is manifest that the total differentiation D_x with respect to $x,$

$$D_x = \frac{\partial}{\partial x} + E_x \cdot \frac{\partial}{\partial E} + H_x \cdot \frac{\partial}{\partial H},$$

commutes with the partial differentiations with respect to E and H :

$$\frac{\partial}{\partial E} D_x - D_x \frac{\partial}{\partial E} = 0, \quad \frac{\partial}{\partial H} D_x - D_x \frac{\partial}{\partial H} = 0. \tag{1.23}$$

The similar equations hold for the total differentiations D_y and D_z with respect to y and $z,$ respectively. The equation

$$D_x(\chi^l) = \frac{\partial \chi^l}{\partial x} + E_x \cdot \frac{\partial \chi^l}{\partial E} + H_x \cdot \frac{\partial \chi^l}{\partial H},$$

yields:

$$\frac{\partial}{\partial E_x} D_x(\chi^l) = \frac{\partial \chi^l}{\partial E}, \quad \frac{\partial}{\partial E_y} D_x(\chi^l) = \frac{\partial}{\partial E_z} D_x(\chi^l) = 0. \tag{1.24}$$

Using the variational derivative (1.21) and invoking Eqs. (1.24), (1.23) we have:

$$\frac{\delta}{\delta \boldsymbol{E}} D_x(\Phi^l) = \frac{\partial}{\partial \boldsymbol{E}} D_x(\Phi^l) - D_x \frac{\partial \Phi^l}{\partial \boldsymbol{E}} = \left(\frac{\partial}{\partial \boldsymbol{E}} D_x - D_x \frac{\partial}{\partial \boldsymbol{E}}\right) \Phi^l = 0.$$

The proof of the second equation (1.19) is similar.

One can use an alternative proof of Eqs. (1.19) in coordinates. Then one has, e.g.

$$\frac{\delta}{\delta E^1} D_x(\Phi^l) = \frac{\partial^2 \Phi^l}{\partial x \partial E^1} + E^i_x \frac{\partial^2 \Phi^l}{\partial E^i \partial E^1} + H^i_x \frac{\partial^2 \Phi^l}{\partial H^i \partial E^1} - D_x\left(\frac{\partial \Phi^l}{\partial E^1}\right).$$

It follows that

$$\frac{\delta}{\delta E^1} D_x(\Phi^l) = 0$$

since

$$D_x\left(\frac{\partial \Phi^l}{\partial E^1}\right) = \frac{\partial^2 \Phi^l}{\partial x \partial E^1} + E^i_x \frac{\partial^2 \Phi^l}{\partial E^i \partial E^1} + H^i_x \frac{\partial^2 \Phi^l}{\partial H^i \partial E^1} \cdot$$

Consider now the case when τ and Φ^l involve first derivatives of $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\tau = \tau(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z, \boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z), \tag{1.25}$$
$$\Phi^l = \Phi^l(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z, \boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z), \quad l = 1, 2, 3.$$

The equation

$$D_x(\Phi^l) = \frac{\partial \Phi^l}{\partial x} + \boldsymbol{E}_x \cdot \frac{\partial \Phi^l}{\partial \boldsymbol{E}} + \boldsymbol{H}_x \cdot \frac{\partial \Phi^l}{\partial \boldsymbol{H}} + \boldsymbol{E}_{xi} \cdot \frac{\partial \Phi^l}{\partial \boldsymbol{E}_i} + \boldsymbol{H}_{xi} \cdot \frac{\partial \Phi^l}{\partial \boldsymbol{H}_i},$$

yields:

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{E}_x} D_x(\Phi^l) &= \frac{\partial \Phi^l}{\partial \boldsymbol{E}} + D_x\left(\frac{\partial \Phi^l}{\partial \boldsymbol{E}_x}\right), \\ \frac{\partial}{\partial \boldsymbol{E}_y} D_x(\Phi^l) &= D_x\left(\frac{\partial \Phi^l}{\partial \boldsymbol{E}_y}\right), \\ \frac{\partial}{\partial \boldsymbol{E}_z} D_x(\Phi^l) &= D_x\left(\frac{\partial \Phi^l}{\partial \boldsymbol{E}_z}\right). \end{aligned} \tag{1.26}$$

Using the notation (1.18) and the Kronecker symbol δ^i_j, one can write Eqs. (1.26) as

$$\frac{\partial}{\partial \boldsymbol{E}_i} D_x(\Phi^l) = \delta^i_1 \frac{\partial \Phi^l}{\partial \boldsymbol{E}} + D_x\left(\frac{\partial \Phi^l}{\partial \boldsymbol{E}_i}\right). \tag{1.27}$$

Eqs. (1.16), (1.27) yield:

$$\frac{\delta}{\delta E} D_x(\chi^l) = \frac{\partial}{\partial E} D_x(\chi^l) - D_i\left[\delta_1^i \frac{\partial \chi^l}{\partial E} + D_x\left(\frac{\partial \chi^l}{\partial E_i}\right)\right] + D_x D_i\left(\frac{\partial \chi^l}{\partial E_i}\right)$$

$$= \frac{\partial}{\partial E} D_x(\chi^l) - D_x\left(\frac{\partial \chi^l}{\partial E}\right) - D_i D_x\left(\frac{\partial \chi^l}{\partial E_i}\right) + D_x D_i\left(\frac{\partial \chi^l}{\partial E_i}\right).$$

Using the commutation relations $D_i D_x = D_x D_i$ and $\frac{\partial}{\partial E} D_x = D_x \frac{\partial}{\partial E}$ (see the first equation (1.23)) we obtain the first equation (1.19):

$$\frac{\delta}{\delta E} D_x(\chi^l) = 0.$$

The proof of the second equation (1.19) is similar. The general case when χ^l depend on higher derivatives is treated likewise.

We turn now to a characterization of conservation densities. Let us first prove the necessary and sufficient conditions for conservation densities for the equations (1.1) taken alone, without Eqs. (1.2).

Theorem 1.1. A function $\tau(t, x, E, H, E_x, H_x, \ldots)$ is a conservation density for Eqs. (1.1) if and only if it satisfies the equations

$$\frac{\delta}{\delta E}\left[D_t(\tau)\big|_{(1.1)}\right] = 0, \quad \frac{\delta}{\delta H}\left[D_t(\tau)\big|_{(1.1)}\right] = 0. \tag{1.28}$$

Proof. The differential equation defining conservation laws for Eqs. (1.1), considered without Eqs. (1.2), is written (cf. Eqs. (1.12))

$$D_t(\tau)\big|_{(1.1)} + \operatorname{div} \chi = 0,$$

Let us write it, using the notation (1.11), as follows:

$$D_t(\tau)\big|_{(1.1)} + D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3) = 0. \tag{1.29}$$

Taking the variational derivatives and using Eqs. (1.15) we arrive at Eqs. (1.28).

Example 1.1. Let us test $\tau = \frac{1}{2}\left(|E|^2 + |H|^2\right)$ for conservation density. We have:

$$\begin{aligned} D_t(\tau)\big|_{(1.1)} &= \left(E \cdot E_t + H \cdot H_t\right)\big|_{(1.1)} \\ &= E \cdot (\nabla \times H) - H \cdot (\nabla \times E) \\ &= E^1(H_y^3 - H_z^2) + E^2(H_z^1 - H_x^3) + E^3(H_x^2 - H_y^1) \\ &\quad - H^1(E_y^3 - E_z^2) - H^2(E_z^1 - E_x^3) - H^3(E_x^2 - E_y^1). \end{aligned}$$

Therefore

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^1} = H^3_y - H^2_z + D_z(H^2) - D_y(H^3) = 0,$$

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^2} = H^1_z - H^3_x - D_z(H^1) + D_x(H^3) = 0,$$

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^3} = H^2_x - H^1_y + D_y(H^1) - D_x(H^2) = 0.$$

Thus, the first equation (1.28) is satisfied. Verification of the second equation (1.28) is similar. Hence,

$$\tau = \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big) \tag{1.30}$$

is a conservation density and therefore the integral

$$\int \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big)dxdydz \tag{1.31}$$

commonly accepted as the electromagnetic *energy* is constant in time for Eqs. (1.1). The differential equation for conservation of energy has the form (see also Section 3.5)

$$D_t\left(\frac{|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2}{2}\right)\bigg|_{(1.1)} + \operatorname{div}\big(\boldsymbol{E} \times \boldsymbol{H}\big) = 0. \tag{1.32}$$

The reckoning shows that Eq. (1.32) is satisfied identically, without using Eqs. (1.2).

Example 1.2. Let us check if the Poynting vector $\boldsymbol{\sigma} = \boldsymbol{E} \times \boldsymbol{H}$ is a conservation density for Eqs. (1.1). Consider, e.g. its first component,

$$\sigma^1 = E^2H^3 - E^3H^2.$$

Its time derivative is (see further Eq. (1.38))

$$D_t(\sigma^1)\big|_{(1.1)} = E^2(E^1_y - E^2_x) + H^3(H^1_z - H^3_x) + E^3(E^1_z - E^3_x) + H^2(H^1_y - H^2_x).$$

Therefore, e.g.

$$\frac{\delta D_t(\sigma^1)\big|_{(1.1)}}{\delta E^1} = -E^2_y - E^3_z \neq 0.$$

Eqs. (1.28) are not satisfied, and hence σ^1 is not a conservation density for Eqs. (1.1) considered without Eqs. (1.2). The same is true for all components of $\boldsymbol{\sigma} = \boldsymbol{E} \times \boldsymbol{H}.$ See also further Example 1.3.

1.5 Test for conservation densities of Eqs. (1.1)-(1.2)

The following theorem provides convenient necessary conditions for conservation densities for the Maxwell equations (1.1)-(1.2).

Theorem 1.2. Let $\tau(t, x, E, H, E_t, H_t, E_x, H_x, \ldots)$ be a conservation density for the Maxwell equations (1.1)-(1.2). Then the following equations are satisfied:

$$\frac{\delta}{\delta E}\left[\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E}\right|_{(1.2)}\right] = 0, \quad \frac{\delta}{\delta H}\left[\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E}\right|_{(1.2)}\right] = 0,$$
$$\frac{\delta}{\delta E}\left[\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H}\right|_{(1.2)}\right] = 0, \quad \frac{\delta}{\delta H}\left[\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H}\right|_{(1.2)}\right] = 0. \tag{1.33}$$

Proof. The conservation equation (1.12) can be equivalently written in the form

$$D_t(\tau)\big|_{(1.1)} + \nabla \cdot \chi = \alpha\, \nabla \cdot E + \beta\, \nabla \cdot H \tag{1.34}$$

with certain coefficients $\alpha = \alpha(t, x, E, H, \ldots),\ \beta = \beta(t, x, E, H, \ldots).$ Therefore, using Eqs. (1.15) we obtain:

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E} = \frac{\delta}{\delta E}\big(\alpha\, \nabla \cdot E + \beta\, \nabla \cdot H\big),$$
$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H} = \frac{\delta}{\delta H}\big(\alpha\, \nabla \cdot E + \beta\, \nabla \cdot H\big). \tag{1.35}$$

Let us first assume that α and β do not depend upon derivatives of E and H:

$$\alpha = \alpha(t, x, E, H), \quad \beta = \beta(t, x, E, H),$$

and calculate, e.g.

$$\frac{\delta}{\delta E^1}\big(\alpha\, \nabla \cdot E + \beta\, \nabla \cdot H\big) = \Big(\frac{\partial}{\partial E^1} - D_j \frac{\partial}{\partial E^1_j}\Big)\big(\alpha\, \nabla \cdot E + \beta\, \nabla \cdot H\big).$$

We have:

$$\Big(\frac{\partial}{\partial E^1} - D_j \frac{\partial}{\partial E^1_j}\Big)\big(\alpha\, \nabla \cdot E + \beta\, \nabla \cdot H\big) = \frac{\partial \alpha}{\partial E^1}\, \nabla \cdot E + \frac{\partial \beta}{\partial E^1}\, \nabla \cdot H - D_j(\alpha \delta^j_1),$$

where δ^j_1 is the Kronecker symbol (cf. Eq. (1.27)). Therefore

$$\frac{\delta}{\delta E^1}\Big(\alpha\, \nabla \cdot E + \beta\, \nabla \cdot H\Big)\Big|_{(1.2)} = -D_x(\alpha),$$

and hence, invoking the first equation (1.35):

$$\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^1}\right|_{(1.2)} = -D_x(\alpha).$$

It is manifest now that

$$\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^k}\right|_{(1.2)} = -D_k(\alpha), \tag{1.36}$$

$$\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H^k}\right|_{(1.2)} = -D_k(\beta), \quad k = 1,2,3.$$

Therefore Eqs. (1.19) provide the coordinate form of Eqs. (1.33),

$$\frac{\delta}{\delta E^i}\left\{\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^k}\right|_{(1.2)}\right\} = 0, \quad \frac{\delta}{\delta H^i}\left\{\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^k}\right|_{(1.2)}\right\} = 0,$$

$$\frac{\delta}{\delta E^i}\left\{\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H^k}\right|_{(1.2)}\right\} = 0, \quad \frac{\delta}{\delta H^i}\left\{\left.\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H^k}\right|_{(1.2)}\right\} = 0, \quad i,k = 1,2,3,$$

thus proving the theorem.

Example 1.3. Let us test for conservation density the Poynting vector $\boldsymbol{\sigma} = \boldsymbol{E} \times \boldsymbol{H}$ (cf. Example 1.2). Consider, e.g. its first component,

$$\sigma_1 = E^2H^3 - E^3H^2. \tag{1.37}$$

We have:

$$\begin{aligned} D_t(\sigma_1)\big|_{(1.1)} &= \left(E^2H^3_t + H^3E^2_t - E^3H^2_t - H^2E^3_t\right)\big|_{(1.1)} \\ &= E^2(E^1_y - E^2_x) + H^3(H^1_z - H^3_x) \\ &\quad + E^3(E^1_z - E^3_x) + H^2(H^1_y - H^2_x). \end{aligned} \tag{1.38}$$

Hence

$$\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^1} = -E^2_y - E^3_z, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^2} = E^1_y, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^3} = E^1_z,$$

$$\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^1} = -H^2_y - H^3_z, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^2} = H^1_y, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^3} = H^1_z.$$

Therefore Eqs. (1.33) are satisfied, and we have in accordance with Eqs. (1.36):

$$\left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^1}\right|_{(1.2)} = E^1_x, \quad \left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^2}\right|_{(1.2)} = E^1_y, \quad \left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^3}\right|_{(1.2)} = E^1_z,$$

$$\left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^1}\right|_{(1.2)} = H^1_x, \quad \left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^2}\right|_{(1.2)} = H^1_y, \quad \left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^3}\right|_{(1.2)} = H^1_z.$$

Accordingly, σ_1 is the density of the conservation equation of the form (1.34):

$$D_t(\sigma_1)\big|_{(1.1)} + D_x(T^1_1) + D_y(T^2_1) + D_z(T^3_1) = -E^1\,\nabla\cdot\boldsymbol{E} - H^1\,\nabla\cdot\boldsymbol{H}, \tag{1.39}$$

where

$$T^1_1 = \frac{1}{2}\left[(E^2)^2 + (E^3)^2 - (E^1)^2 + (H^2)^2 + (H^3)^2 - (H^1)^2\right],$$
$$T^2_1 = -\left(E^1E^2 + H^1H^2\right), \quad T^3_1 = -\left(E^1E^3 + H^1H^3\right),$$

or

$$T^i_1 = -E^1E^i - H^1H^i + \frac{1}{2}\delta^i_1\left(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\right), \quad i = 1,2,3. \tag{1.40}$$

The obvious generalization of (1.40) is written:

$$T^i_k = -E^kE^i - H^kH^i + \frac{1}{2}\delta^i_k\left(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\right), \quad k,i = 1,2,3. \tag{1.41}$$

It is called *Maxwell's tension tensor.*

Eq. (1.39) together with the similar conservation equations associated with the components

$$\sigma_2 = E^3H^1 - E^1H^3, \quad \sigma_3 = E^1H^2 - E^2H^1 \tag{1.42}$$

of the vector $\boldsymbol{\sigma} = \boldsymbol{E} \times \boldsymbol{H}$ provide the following three conservation equations for the Maxwell equations (1.1)-(1.2):

$$D_t(\sigma_k)\big|_{(1.1)} + D_x(T^1_k) + D_y(T^2_k) + D_z(T^3_k) = -E^k\,\nabla\cdot\boldsymbol{E} - H^k\,\nabla\cdot\boldsymbol{H}, \tag{1.43}$$

where $k = 1,2,3.$ The quantities σ_k are determined by Eqs. (1.37), (1.42), and T^i_k is Maxwell's tension tensor (1.41).

Thus, the vector $\boldsymbol{\sigma} = \boldsymbol{E} \times \boldsymbol{H}$ is a conservation density for the system of equations (1.1)-(1.2), and hence the integral (known in physics as the *linear momentum*)

$$\int \left(\boldsymbol{E} \times \boldsymbol{H}\right) dxdydz \tag{1.44}$$

is constant in time for the Maxwell equations.

Example 1.4. Let us test for conservation density the vector $\mu = x \times (E \times H).$ Consider, e.g. its first component,

$$\mu^1 = y(E^1H^2 - E^2H^1) - z(E^3H^1 - E^1H^3). \tag{1.45}$$

We have:

$$D_t(\mu^1)\big|_{(1.1)} = y\big[H^2(H^3_y - H^2_z) + E^1(E^3_x - E^1_z) + H^1(H^3_x - H^1_z) + E^2(E^3_y - E^2_z)\big] + z\big[H^1(H^1_y - H^2_x) + E^3(E^3_y - E^2_z) + H^3(H^3_y - H^2_z) + E^1(E^1_y - E^2_x)\big];$$

whence

$$\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^1} = yE^3_x - zE^2_x;$$
$$\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^2} = yE^3_y + E^3 + zE^3_z + zE^1_x;$$
$$\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^3} = -yE^1_x - yE^2_y - E^2 - zE^2_z.$$

Hence, Eqs. (1.28) are not satisfied. However Eqs. (1.33) are satisfied because, e.g.

$$\left.\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^1}\right|_{(1.2)} = D_x(yE^3 - zE^2);$$
$$\left.\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^2}\right|_{(1.2)} = yE^3_y + E^3 - zE^2_y = D_y(yE^3 - zE^2);$$
$$\left.\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^3}\right|_{(1.2)} = yE^3_z - E^2 - zE^2_z = D_z(yE^3 - zE^2).$$

Therefore, the integral (known in physics as the *angular momentum*)

$$\int \big[x \times (E \times H)\big] dxdydz \tag{1.46}$$

is constant in time for the Maxwell equations (1.1)-(1.2).

2 Lagrangian and symmetries for the evolution equations of Maxwell's system

2.1 The Lagrangian

The system (1.1)-(1.2) contains six dependent variables, namely, the components of the electric field $E = (E^1, E^2, E^3)$ and the magnetic field $H = (H^1, H^2, H^3)$, and eight equations, i.e. it is *over-determined*. Therefore the system (1.1)-(1.2) does not have a Lagrangian.

I used in [11] the fact that the system of equations (1.1), without Eqs (1.2), is a self adjoint system and therefore has a Lagrangian. Namely,

$$\mathcal{L} = \boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{H}}{\partial t}\right)+\boldsymbol{H}\cdot\left(\nabla\times\boldsymbol{H}-\frac{\partial\boldsymbol{E}}{\partial t}\right) \tag{2.1}$$

is a Lagrangian for Eqs. (1.1).

For Eqs. (1.1), written in the coordinate form (1.1′):

$$\begin{array}{ll} E^3_y - E^2_z + H^1_t = 0; & H^3_y - H^2_z - E^1_t = 0;\\ E^1_z - E^3_x + H^2_t = 0; & H^1_z - H^3_x - E^2_t = 0;\\ E^2_x - E^1_y + H^3_t = 0; & H^2_x - H^1_y - E^3_t = 0; \end{array} \tag{1.1′}$$

the Lagrangian (2.1) is written:

$$\begin{aligned} \mathcal{L} = {} & E^1(E^3_y - E^2_z + H^1_t) + E^2(E^1_z - E^3_x + H^2_t) + E^3(E^2_x - E^1_y + H^3_t)\\ & + H^1(H^3_y - H^2_z - E^1_t) + H^2(H^1_z - H^3_x - E^2_t) + H^3(H^2_x - H^1_y - E^3_t). \end{aligned} \tag{2.2}$$

However, by separating Eqs. (1.1) from Eqs. (1.2) we loose certain symmetries of the Maxwell equations. This is discussed in the next section.

Remark 2.1. In the case of the Maxwell equations with electric charges and currents,

$$\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{H}}{\partial t} = 0, \quad \nabla\times\boldsymbol{H}-\frac{\partial\boldsymbol{E}}{\partial t} - \boldsymbol{j} = 0, \tag{2.3}$$

$$\nabla\cdot\boldsymbol{E} = \rho, \quad \nabla\cdot\boldsymbol{H} = 0, \tag{2.4}$$

the Lagrangian (2.1) is replaced by[1]

$$\mathcal{L} = \boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{H}}{\partial t}\right)+\boldsymbol{H}\cdot\left(\nabla\times\boldsymbol{H}-\frac{\partial\boldsymbol{E}}{\partial t}-2\boldsymbol{j}\right). \tag{2.5}$$

2.2 The symmetries

We know that the Lie algebra L admitted by Maxwell's system of equations (1.1)–(1.2) is spanned by the operators (1.3)–(1.9). Let us prove the following statement.

Theorem 2.1. The maximal subalgebra $K \subset L$ admitted by the evolutionary equations (1.1) of Maxwell's system comprises the 10-dimensional algebra spanned by the operators (1.3), (1.5), (1.6), (1.7) and the infinite dimensional ideal (1.8).

[1] Recently I learned that the Lagrangian (2.5) was noted in [14] and in [15]. It was also used in [8]. I thank Professor Bo Thidé for drawing my attention to these papers.

Proof. It is geometrically evident that Eqs. (1.1) admit the operators (1.3), (1.6), (1.7) and (1.8). In order to verify that they admit also the duality rotations, let us write the duality generator (1.5) in the prolonged form:

$$Z_0 = E\frac{\partial}{\partial H} - H\frac{\partial}{\partial E} + E_t\frac{\partial}{\partial H_t} - H_t\frac{\partial}{\partial E_t} + E_x\frac{\partial}{\partial H_x} - H_x\frac{\partial}{\partial E_x} + \cdots . \tag{2.6}$$

Acting by the operator (2.6) on the Lagrangian (2.1) we have:

$$\begin{aligned} Z_0\,(\nabla\times E + H_t) &= E_t - \nabla\times H,\\ Z_0\big(\nabla\times H - E_t\big) &= \nabla\times E + H_t. \end{aligned} \tag{2.7}$$

Hence, Z_0 is admitted by Eqs. (1.1).

It remains to show that the generators (1.4) of the Lorentz transformations and the generators (1.9) of the conformal transformations are not admitted by Eqs. (1.1) if one takes Eqs. (1.1) alone, without Eqs. (1.2).

Let us begin with the Lorentz transformations. Consider, e.g. the first operator X_{01} from (1.4) and verify that it is not admitted by Eqs. (1.1′). Computing the first prolongation of X_{01} and denoting the prolonged operator again by X_{01} we have:

$$\begin{aligned} X_{01} = {} & t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} + E^2\frac{\partial}{\partial H^3} + H^3\frac{\partial}{\partial E^2} - E^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial E^3}\\ & - E^1_t\frac{\partial}{\partial E^1_x} - E^1_x\frac{\partial}{\partial E^1_t} + \big(H^3_x - E^2_t\big)\frac{\partial}{\partial E^2_x} + H^3_y\frac{\partial}{\partial E^2_y} + H^3_z\frac{\partial}{\partial E^2_z}\\ & + \big(H^3_t - E^2_x\big)\frac{\partial}{\partial E^2_t} - \big(H^2_x + E^3_t\big)\frac{\partial}{\partial E^3_x} - H^2_y\frac{\partial}{\partial E^3_y} - H^2_z\frac{\partial}{\partial E^3_z}\\ & - \big(H^2_t + E^3_x\big)\frac{\partial}{\partial E^3_t} - H^1_t\frac{\partial}{\partial H^1_x} - H^1_x\frac{\partial}{\partial H^1_t} - \big(E^3_x + H^2_t\big)\frac{\partial}{\partial H^2_x}\\ & - E^3_y\frac{\partial}{\partial H^2_y} - E^3_z\frac{\partial}{\partial H^2_z} - \big(E^3_t + H^2_x\big)\frac{\partial}{\partial H^2_t} + \big(E^2_x - H^3_t\big)\frac{\partial}{\partial H^3_x}\\ & + E^2_y\frac{\partial}{\partial H^3_y} + E^2_z\frac{\partial}{\partial H^3_z} + \big(E^2_t - H^3_x\big)\frac{\partial}{\partial H^3_t}\,\cdot \end{aligned} \tag{2.8}$$

Acting by the operator (2.8) on the left-hand sides of Eqs. (1.1′) we get:

$$\begin{aligned} & X_{01}\big(E^3_y - E^2_z + H^1_t\big) = -\big(H^1_x + H^2_y + H^3_z\big);\\ & X_{01}\big(E^1_z - E^3_x + H^2_t\big) = 0;\\ & X_{01}\big(E^2_x - E^1_y + H^3_t\big) = 0;\\ & X_{01}\big(H^3_y - H^2_z - E^1_t\big) = E^1_x + E^2_y + E^3_z;\\ & X_{01}\big(H^1_z - H^3_x - E^2_t\big) = 0;\\ & X_{01}\big(H^2_x - H^1_y - E^3_t\big) = 0. \end{aligned} \tag{2.9}$$

Therefore, Eqs. (1.2′) are required for the invariance of Eqs. (1.1′). Hence, X_{01} is not admitted by Eqs. (1.1). The same is true for all operators (1.4).

Let us apply the similar procedure to the operator Y_1 from (1.9). In order to calculate the action of the prolonged operator Y_1 on the first equation of the system (1.1′), we compute the prolongation to the variables E^3_y, E^2_z, H^1_t and obtain:

$$\begin{aligned} Y_1 = {} & \left(x^2 - y^2 - z^2 + t^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + 2xt\frac{\partial}{\partial t} \\ & - \left(4xE^1 + 2yE^2 + 2zE^3\right)\frac{\partial}{\partial E^1} - \left(4xH^1 + 2yH^2 + 2zH^3\right)\frac{\partial}{\partial H^1} \\ & - \left(4xE^2 - 2yE^1 - 2tH^3\right)\frac{\partial}{\partial E^2} - \left(4xH^2 - 2yH^1 + 2tE^3\right)\frac{\partial}{\partial H^2} \\ & - \left(4xE^3 - 2zE^1 + 2tH^2\right)\frac{\partial}{\partial E^3} - \left(4xH^3 - 2zH^1 - 2tE^2\right)\frac{\partial}{\partial H^3} \\ & - \left(6xH^1_t + 2yH^2_t + 2zH^3_t + 2tH^1_x\right)\frac{\partial}{\partial H^1_t} - \left(6xE^3_y - 2yE^3_x\right. \\ & \left. - 2zE^1_y + 2tH^2_y\right)\frac{\partial}{\partial E^3_y} - \left(6xE^2_z - 2yE^1_z - 2zE^2_x - 2tH^3_z\right)\frac{\partial}{\partial E^2_z} \cdot \end{aligned} \tag{2.10}$$

Therefore

$$\begin{aligned} Y_1\left(E^3_y - E^2_z + H^1_t\right) = {} & -6x\left(E^3_y - E^2_z + H^1_t\right) - 2y\left(E^1_z - E^3_x + H^2_t\right) \\ & - 2z\left(E^2_x - E^1_y + H^3_t\right) - 2t\left(H^1_x + H^2_y + H^3_z\right). \end{aligned}$$

Applying the procedure to all equations (1.1′), we obtain the following result:

$$\begin{aligned} Y_1\left(E^3_y - E^2_z + H^1_t\right) = {} & -6x\left(E^3_y - E^2_z + H^1_t\right) - 2y\left(E^1_z - E^3_x + H^2_t\right) \\ & - 2z\left(E^2_x - E^1_y + H^3_t\right) - 2t\left(H^1_x + H^2_y + H^3_z\right); \\ Y_1\left(E^1_z - E^3_x + H^2_t\right) = {} & -6x\left(E^1_z - E^3_x + H^2_t\right) + 2y\left(E^3_y - E^2_z + H^1_t\right); \\ Y_1\left(E^2_x - E^1_y + H^3_t\right) = {} & -6x\left(E^2_x - E^1_y + H^3_t\right) + 2z\left(E^3_y - E^2_z + H^1_t\right); \\ Y_1\left(H^3_y - H^2_z - E^1_t\right) = {} & -6x\left(H^3_y - H^2_z - E^1_t\right) - 2y\left(H^1_z - H^3_x - E^2_t\right) \\ & - 2z\left(H^2_x - H^1_y - E^3_t\right) + 2t\left(E^1_x + E^2_y + E^3_z\right); \\ Y_1\left(H^1_z - H^3_x - E^2_t\right) = {} & -6x\left(H^1_z - H^3_x - E^2_t\right) + 2y\left(H^3_y - H^2_z - E^1_t\right); \\ Y_1\left(H^2_x - H^1_y - E^3_t\right) = {} & -6x\left(H^2_x - H^1_y - E^3_t\right) + 2z\left(H^3_y - H^2_z - E^1_t\right). \end{aligned} \tag{2.11}$$

The first and fourth equations (2.11) show that the infinitesimal tes for the invariance of Eqs. (1.1) under Y_1 requires Eqs. (1.2). Making similar calculations for all operators (1.9), one can verify that the conformal group is not admitted by Eqs. (1.1) taken alone. This completes the proof of the theorem.

Remark 2.2. The complete form of the prolonged terms of $\mathbf{Y_1}$ is

$$
\begin{aligned}
Y_1 = {} & \left(x^2 - y^2 - z^2 + t^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + 2xt\frac{\partial}{\partial t} + \ldots \\
& - \left(6xE^1_t + 2yE^2_t + 2zE^3_t + 2tE^1_x\right)\frac{\partial}{\partial E^1_t} - \left(6xH^1_t + 2yH^2_t + 2zH^3_t\right. \\
& \left. + 2tH^1_x\right)\frac{\partial}{\partial H^1_t} - \left(6xE^1_x + 4E^1 + 2yE^2_x + 2zE^3_x + 2yE^1_y + 2zE^1_z + 2tE^1_t\right)\frac{\partial}{\partial E^1_x} \\
& - \left(6xH^1_x + 4H^1 + 2yH^2_x + 2zH^3_x + 2yH^1_y + 2zH^1_z + 2tH^1_t\right)\frac{\partial}{\partial H^1_x} \\
& - \left(6xE^1_y + 2E^2 + 2yE^2_y + 2zE^3_y - 2yE^1_x\right)\frac{\partial}{\partial E^1_y} - \left(6xH^1_y + 2H^2 + 2yH^2_y + 2zH^3_y\right. \\
& \left. - 2yH^1_x\right)\frac{\partial}{\partial H^1_y} - \left(6xE^1_z + 2E^3 + 2yE^2_z + 2zE^3_z - 2zE^1_x\right)\frac{\partial}{\partial E^1_z} - \left(6xH^1_z + 2H^3\right. \\
& \left. + 2yH^2_z + 2zH^3_z - 2zH^1_x\right)\frac{\partial}{\partial H^1_z} - \left(6xE^2_t - 2H^3 - 2yE^1_t - 2tH^3_t + 2tE^2_x\right)\frac{\partial}{\partial E^2_t} \\
& - \left(6xH^2_t + 2E^3 - 2yH^1_t + 2tE^3_t + 2tH^2_x\right)\frac{\partial}{\partial H^2_t} - \left(6xE^2_x + 4E^2 - 2yE^1_x - 2tH^3_x\right. \\
& \left. + 2yE^2_y + 2zE^2_z + 2tE^2_t\right)\frac{\partial}{\partial E^2_x} - \left(6xH^2_x + 4H^2 - 2yH^1_x + 2tE^3_x + 2yH^2_y + 2zH^2_z\right. \\
& \left. + 2tH^2_t\right)\frac{\partial}{\partial H^2_x} - \left(6xE^2_y - 2E^1 - 2yE^1_y - 2tH^3_y - 2yE^2_x\right)\frac{\partial}{\partial E^2_y} - \left(6xH^2_y - 2yH^1_y\right. \\
& \left. - 2H^1 + 2tE^3_y - 2yH^2_x\right)\frac{\partial}{\partial H^2_y} - \left(6xE^2_z - 2yE^1_z - 2zE^2_x - 2tH^3_z\right)\frac{\partial}{\partial E^2_z} - \left(6xH^2_z\right. \\
& \left. - 2yH^1_z - 2zH^2_x + 2tE^3_z\right)\frac{\partial}{\partial H^2_z} - \left(6xE^3_t + 2H^2 - 2zE^1_t + 2tH^2_t + 2tE^3_x\right)\frac{\partial}{\partial E^3_t} \\
& - \left(6xH^3_t - 2E^2 - 2zH^1_t - 2tE^2_t + 2tH^3_x\right)\frac{\partial}{\partial H^3_t} - \left(6xE^3_x + 4E^3 - 2zE^1_x + 2tH^2_x\right. \\
& \left. + 2yE^3_y + 2zE^3_z + 2tE^3_t\right)\frac{\partial}{\partial E^3_x} - \left(6xH^3_x + 4H^3 - 2zH^1_x - 2tE^2_x + 2yH^3_y + 2zH^3_z\right. \\
& \left. + 2tH^3_t\right)\frac{\partial}{\partial H^3_x} - \left(6xE^3_y - 2yE^3_x - 2zE^1_y + 2tH^2_y\right)\frac{\partial}{\partial E^3_y} - \left(6xH^3_y - 2yH^3_x - 2zH^1_y\right. \\
& \left. - 2tE^2_y\right)\frac{\partial}{\partial H^3_y} - \left(6xE^3_z - 2E^1 - 2zE^1_z - 2zE^3_x + 2tH^2_x\right)\frac{\partial}{\partial E^3_z} \\
& - \left(6xH^3_z - 2H^1 - 2zH^1_z - 2zH^3_x - 2tE^2_x\right)\frac{\partial}{\partial H^3_z}.
\end{aligned}
\tag{2.12}
$$

Remark 2.3. Acting by the operator (2.8) on the left-hand sides of Eqs. (1.2′) we get:

$$X_{01}\left(E^1_x + E^2_y + E^3_z\right) = H^3_y - H^2_z - E^1_t; \qquad (2.13)$$

$$X_{01}\left(H^1_x + H^2_y + H^3_z\right) = -\left(E^3_y - E^2_z + H^1_t\right).$$

Eqs. (2.9) and (2.13) show that X_{01} is admitted by the simultaneous system (1.1)–(1.2).

Remark 2.4. Acting by the operator (2.12) on Eqs. (1.2′) we get:

$$\begin{aligned} Y_1\left(E^1_x + E^2_y + E^3_x\right) &= -6x\left(E^1_x + E^2_y + E^3_x\right) - 2t\left(H^3_y - H^2_z - E^1_t\right); \\ Y_1\left(H^1_x + H^2_y + H^3_x\right) &= -6x\left(H^1_x + H^2_y + H^3_x\right) + 2t\left(E^3_y - E^2_z + H^1_t\right). \end{aligned} \qquad (2.14)$$

Eqs. (2.11) and (2.14) show that Y_1 is admitted by the simultaneous system (1.1)–(1.2).

The counterparts of Eqs. (2.11) for the operators Y_2 and Y_3 are obtained from (2.11) by the cyclic permutations of (x, y, z) and the corresponding coordinates of $\boldsymbol{E}$ and $\boldsymbol{H}$.

Remark 2.5. For the operator Y_4 Eqs. (2.11) are replaced by the following equations:

$$\begin{aligned} Y_4\left(E^3_y - E^2_z + H^1_t\right) &= -6t\left(E^3_y - E^2_z + H^1_t\right) - 2x\left(H^1_x + H^2_y + H^3_z\right); \\ Y_4\left(E^1_z - E^3_x + H^2_t\right) &= -6t\left(E^1_z - E^3_x + H^2_t\right) - 2y\left(H^1_x + H^2_y + H^3_z\right); \\ Y_4\left(E^2_x - E^1_y + H^3_t\right) &= -6t\left(E^2_x - E^1_y + H^3_t\right) - 2z\left(H^1_x + H^2_y + H^3_z\right); \\ Y_4\left(H^3_y - H^2_z - E^1_t\right) &= -6t\left(H^3_y - H^2_z - E^1_t\right) + 2x\left(E^1_x + E^2_y + E^3_z\right); \\ Y_4\left(H^1_z - H^3_x - E^2_t\right) &= -6t\left(H^1_z - H^3_x - E^2_t\right) + 2y\left(E^1_x + E^2_y + E^3_z\right); \\ Y_4\left(H^2_x - H^1_y - E^3_t\right) &= -6t\left(H^2_x - H^1_y - E^3_t\right) + 2z\left(E^1_x + E^2_y + E^3_z\right). \end{aligned} \qquad (2.15)$$

3 Calculation of conservation laws for Eqs. (1.1)

3.1 Notation

I will denote by x^i $(i = 0, 1, 2, 3)$ the independent variables:

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \qquad (3.1)$$

by u^α $(\alpha = 1, \ldots, 6)$ the dependent variables:

$$u^1 = E^1, \quad u^2 = E^2, \quad u^3 = E^3, \quad u^4 = H^1, \quad u^5 = H^2, \quad u^6 = H^3, \qquad (3.2)$$

and write the symmetry generators (1.3), (1.4), (1.6), (1.5) and (1.8) in the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\cdot \tag{3.3}$$

For the symmetries leaving invariant the variational integral with the Lagrangian $L,$ the conserved vectors $C = (C^0,\ldots,C^3)$ are calculated by

$$C^i = \xi^i L + \left(\eta^\alpha - \xi^j u^\alpha_j\right)\frac{\partial L}{\partial u^\alpha_i}\cdot \tag{3.4}$$

Note that the infinitesimal test for the invariance of the variational integral is written

$$X(L) + L D_i(\xi^i) = 0, \tag{3.5}$$

where the first prolongation of X is understood. If the invariance condition (3.5) is replaced by the *divergence condition*

$$X(L) + L D_i(\xi^i) = D_i(B^i), \tag{3.6}$$

then Eq. (3.4) for the conserved vector is replaced by

$$C^i = \xi^i L + \left(\eta^\alpha - \xi^j u^\alpha_j\right)\frac{\partial L}{\partial u^\alpha_i} - B^i. \tag{3.7}$$

Since it suffices to require validity of Eq. (3.5) or Eq. (3.6) on the solutions of Maxwell's equations and since the Lagrangian (2.1) vanishes on the solutions of Eqs. (1.1), one can use Eqs. (3.5), (3.6), (3.4) and (3.7) in the reduced forms

$$X(L) = 0, \tag{3.8}$$

$$X(L) = D_i(B^i), \tag{3.9}$$

$$C^i = \left(\eta^\alpha - \xi^j u^\alpha_j\right)\frac{\partial L}{\partial u^\alpha_i}, \quad i = 0,\ldots,3, \tag{3.10}$$

and

$$C^i = \left(\eta^\alpha - \xi^j u^\alpha_j\right)\frac{\partial L}{\partial u^\alpha_i} - B^i, \quad i = 0,\ldots,3, \tag{3.11}$$

respectively. I will denote

$$C^0 = \tau, \quad C^1 = \chi^1, \quad C^2 = \chi^2, \quad C^3 = \chi^3 \tag{3.12}$$

and write the conservation laws in the form (1.12):

$$D_t(\tau) + \operatorname{div}\chi = 0.$$

3.2 Time translation

Let us apply the formula (3.10) to the geometric symmetries (1.3) of Eqs. (1.1). Taking the generator

$$X_0 = \frac{\partial}{\partial t}$$

of time translations we have:

$$\xi^0 = 1; \xi^1 = \xi^2 = \xi^3 = 0; \quad \eta^\alpha = 0; \quad \eta^\alpha - \xi^j u^\alpha_j = -u^\alpha_t.$$

Denoting the density τ of the conservation law provided by X_0 by π_0, we obtain from Equations (2.2) and (3.10):

$$\pi_0 = -u^\alpha_t \frac{\partial L}{\partial u^\alpha_t} = -E^k_t \frac{\partial L}{\partial E^k_t} - H^k_t \frac{\partial L}{\partial H^k_t} = \sum_{k=1}^{3} \left(H^k E^k_t - E^k H^k_t\right);$$

or

$$\pi_0 = \boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t. \tag{3.13}$$

The flux components are calculated likewise. Eqs. (3.10), (3.12) and (2.2) yield, e.g.

$$\chi^1_0 = -u^\alpha_t \frac{\partial L}{\partial u^\alpha_x} = E^2 E^3_t - E^3 E^2_t + H^2 H^3_t - H^3 H^2_t$$

or

$$\chi^1_0 = \left(\boldsymbol{E} \times \boldsymbol{E}_t\right)^1 + \left(\boldsymbol{H} \times \boldsymbol{H}_t\right)^1.$$

Computing the other components of χ we obtain the flux

$$\chi_0 = \left(\boldsymbol{E} \times \boldsymbol{E}_t\right) + \left(\boldsymbol{H} \times \boldsymbol{H}_t\right). \tag{3.14}$$

Summarizing Equations (1.12), (3.13) and (3.14), we arrive at the following equation:

$$D_t\left(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\right) + \nabla \cdot \left(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\right) = 0. \tag{3.15}$$

Remark 3.1. Verification of Eq. (3.15) is straightforward. Using Eqs. (1.1) we have:

$$\begin{aligned} D_t\left(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\right) &= \boldsymbol{H} \cdot \boldsymbol{E}_{tt} - \boldsymbol{E} \cdot \boldsymbol{H}_{tt} \\ &= \boldsymbol{H} \cdot \left(\nabla \times \boldsymbol{H}_t\right) + \boldsymbol{E} \cdot \left(\nabla \times \boldsymbol{E}_t\right) \end{aligned}$$

and

$$\begin{aligned} &\nabla \cdot \left(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\right) = \nabla \cdot \left(\boldsymbol{E} \times \boldsymbol{E}_t\right) + \nabla \cdot \left(\boldsymbol{H} \times \boldsymbol{H}_t\right) \\ &= \boldsymbol{E}_t \cdot \left(\nabla \times \boldsymbol{E}\right) - \boldsymbol{E} \cdot \left(\nabla \times \boldsymbol{E}_t\right) + \boldsymbol{H}_t \cdot \left(\nabla \times \boldsymbol{H}\right) - \boldsymbol{H} \cdot \left(\nabla \times \boldsymbol{H}_t\right). \end{aligned}$$

Therefore

$$\begin{aligned} &D_t\left(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\right) + \nabla \cdot \left(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\right) \\ &= \boldsymbol{E}_t \cdot \left(\nabla \times \boldsymbol{E}\right) + \boldsymbol{H}_t \cdot \left(\nabla \times \boldsymbol{H}\right) = -\boldsymbol{E}_t \cdot \boldsymbol{H}_t + \boldsymbol{H}_t \cdot \boldsymbol{E}_t = 0. \end{aligned}$$

Representing the differential conservation equation (3.15) in the integral form (1.13), we can formulate the result as follows.

Lemma 3.1. Time translational invariance of Eqs. (1.1) leads to the conservation law

$$\frac{d}{dt}\int \big(\boldsymbol{H}\cdot \boldsymbol{E}_t - \boldsymbol{E}\cdot \boldsymbol{H}_t\big)\, dxdydz = 0. \tag{3.16}$$

Remark 3.2. Eliminating in (3.13) the time-derivatives by using Eqs. (1.1), one has:

$$\pi_0 = \boldsymbol{E}\cdot(\nabla\times \boldsymbol{E}) + \boldsymbol{H}\cdot(\nabla\times \boldsymbol{H}). \tag{3.17}$$

Then the differential conservation law (3.15) is written[2]

$$D_t\big[\boldsymbol{E}\cdot(\nabla\times \boldsymbol{E}) + \boldsymbol{H}\cdot(\nabla\times \boldsymbol{H})\big] + \operatorname{div}\big[\boldsymbol{E}\times \boldsymbol{E}_t + \boldsymbol{H}\times \boldsymbol{H}_t\big] = 0. \tag{3.18}$$

3.3 Spatial translations

For the operator

$$X_1 = \frac{\partial}{\partial x}$$

from (1.3) we have:

$$\xi^0 = 0, \quad \xi^1 = 1, \quad \xi^2 = \xi^3 = 0, \quad \eta^\alpha = 0, \quad \eta^\alpha - \xi^j u^\alpha_j = -u^\alpha_x.$$

Denoting the density τ of the conservation law provided by X_1 by π_1, we obtain from Eqs. (2.2), (3.10):

$$\pi_1 = -u^\alpha_x \frac{\partial L}{\partial u^\alpha_t} = -E^k_x \frac{\partial L}{\partial E^k_t} - H^k_x \frac{\partial L}{\partial H^k_t} = \sum_{k=1}^{3}\big(H^k E^k_x - E^k H^k_x\big).$$

Hence,

$$\pi_1 = \boldsymbol{H}\cdot \boldsymbol{E}_x - \boldsymbol{E}\cdot \boldsymbol{H}_x. \tag{3.19}$$

Let us find the corresponding flux χ_1. Proceeding as above, we have:

$$\chi^1_1 = -u^\alpha_x \frac{\partial L}{\partial u^\alpha_x} = -E^k_x \frac{\partial L}{\partial E^k_x} - H^k_x \frac{\partial L}{\partial H^k_x}$$
$$= E^2 E^3_x - E^3 E^2_x + H^2 H^3_x - H^3 H^2_x.$$

Thus,

$$\chi^1_1 = \big(\boldsymbol{E}\times \boldsymbol{E}_x\big)^1 + \big(\boldsymbol{H}\times \boldsymbol{H}_x\big)^1.$$

[2] Eq. (3.18) was discovered by Lipkin [7] (see his Eq. (1)). He expressed his new conservation law in a tensor notation and found nine additional new conservation laws akin to Eq. (3.18). See also [8].

Computing the other components of χ_1 we have:

$$\chi_1 = \left(E \times E_x\right) + \left(H \times H_x\right). \tag{3.20}$$

Let us verify that (3.19) and (3.20) satisfy the conservation law (1.12). We have:

$$\begin{aligned} D_t\left(H \cdot E_x - E \cdot H_x\right) &= H \cdot E_{tx} + H_t \cdot E_x - E \cdot H_{tx} - E_t \cdot H_x \\ &= H \cdot \left(\nabla \times H_x\right) - E_x \cdot \left(\nabla \times E\right) + E \cdot \left(\nabla \times E_x\right) - H_x \cdot \left(\nabla \times H\right) \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\nabla \cdot \left(E \times E_x\right) + \nabla \cdot \left(H \times H_x\right) \\ &= E_x \cdot \left(\nabla \times E\right) - E \cdot \left(\nabla \times E_x\right) + H_x \cdot \left(\nabla \times H\right) - H \cdot \left(\nabla \times H_x\right). \end{aligned} \tag{3.22}$$

It is manifest from Eqs. (3.21), (3.22) that the conservation equation (1.12) is satisfied.

Thus, the invariance under the x-translation group with the generator X_1 provides the following conservation equation (1.12):

$$D_t\left(H \cdot E_x - E \cdot H_x\right) + \nabla \cdot \left(E \times E_x + H \times H_x\right) = 0.$$

Applying the above procedure to X_2 and X_3 from (1.3), we arrive at the densities

$$\begin{aligned} \pi_1 &= H \cdot E_x - E \cdot H_x, \\ \pi_2 &= H \cdot E_y - E \cdot H_y, \\ \pi_3 &= H \cdot E_z - E \cdot H_z \end{aligned} \tag{3.23}$$

and the corresponding fluxes

$$\begin{aligned} \chi_1 &= \left(E \times E_x\right) + \left(H \times H_x\right), \\ \chi_2 &= \left(E \times E_y\right) + \left(H \times H_y\right), \\ \chi_3 &= \left(E \times E_z\right) + \left(H \times H_z\right) \end{aligned} \tag{3.24}$$

of the following differential conservation laws:

$$D_t\left(\pi_k\right) + \nabla \cdot \left(E \times E_k + H \times H_k\right) = 0, \quad k = 1,2,3, \tag{3.25}$$

where

$$E_k = \frac{\partial E}{\partial x^k}, \quad H_k = \frac{\partial H}{\partial x^k} \cdot$$

Using the integral form (1.13) of conservation laws, I formulate the result as follows.

Lemma 3.2. The invariance of Eqs. (1.1) under the translations with respect to the spatial variables leads to the vector valued conservation law

$$\frac{d}{dt}\int \pi \, dxdydz = 0, \tag{3.26}$$

where $\pi = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (3.23).

Remark 3.3. According to Eqs. (3.21), (3.23), we have:

$$D_t(\pi_1) = H\cdot(\nabla\times H_x) - E_x\cdot(\nabla\times E) + E\cdot(\nabla\times E_x) - H_x\cdot(\nabla\times H),$$
$$D_t(\pi_2) = H\cdot(\nabla\times H_y) - E_y\cdot(\nabla\times E) + E\cdot(\nabla\times E_y) - H_y\cdot(\nabla\times H),$$
$$D_t(\pi_3) = H\cdot(\nabla\times H_z) - E_z\cdot(\nabla\times E) + E\cdot(\nabla\times E_z) - H_z\cdot(\nabla\times H).$$

3.4 Rotations

Applying the formula (3.10) with $i = 0$ to the rotation generator

$$X_{12} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + E^2\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^2} + H^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^2}$$

from (1.3) we obtain the conservation density

$$\tau_{12} = (\eta^\alpha - \xi^j u^\alpha_j)\frac{\partial L}{\partial u^\alpha_t},$$

which upon substitution of the coordinates of X_{12} becomes:

$$\tau_{12} = (E^2 + xE^1_y - yE^1_x)\frac{\partial L}{\partial E^1_t} - (E^1 - xE^2_y + yE^2_x)\frac{\partial L}{\partial E^2_t} + (xE^3_y - yE^3_x)\frac{\partial L}{\partial E^3_t}$$
$$+ (H^2 + xH^1_y - yH^1_x)\frac{\partial L}{\partial H^1_t} - (H^1 - xH^2_y + yH^2_x)\frac{\partial L}{\partial H^2_t} + (xH^3_y - yH^3_x)\frac{\partial L}{\partial H^3_t}\cdot$$

Invoking the expression (2.2) for the Lagrangian, we obtain

$$\tau_{12} = (E^1 - xE^2_y + yE^2_x)H^2 - (E^2 + xE^1_y - yE^1_x)H^1 - (xE^3_y - yE^3_x)H^3$$
$$+ (H^2 + xH^1_y - yH^1_x)E^1 - (H^1 - xH^2_y + yH^2_x)E^2 + (xH^3_y - yH^3_x)E^3,$$

or

$$\tau_{12} = 2(E^1H^2 - E^2H^1) + y(H\cdot E_x - E\cdot H_x) - x(H\cdot E_y - E\cdot H_y).$$

Using the notation (3.23), we write it in the form

$$\tau_{12} = 2(E^1H^2 - E^2H^1) + y\pi_1 - x\pi_2, \tag{3.27}$$

or in terms of the vector $\pi = (\pi_{41}, \pi_{42}, \pi_{43})$:

$$\tau_{12} = 2\big(E \times H\big)^3 + \big(x \times \pi\big)^3. \qquad (3.28)$$

The components of the flux $\chi_{12} = (\chi^1_{12}, \chi^2_{12}, \chi^3_{12})$ are computed by the formulae

$$\chi^1_{12} = \big(\xi^\alpha - \eta^j u^\alpha_j\big)\frac{\partial L}{\partial u^\alpha_x} = \big(xE^2_y - yE^2_x - E^1\big)\frac{\partial L}{\partial E^2_x} + \big(xE^3_y - yE^3_x\big)\frac{\partial L}{\partial E^3_x}$$
$$+ \big(xH^2_y - yH^2_x - H^1\big)\frac{\partial L}{\partial H^2_x} + \big(xH^3_y - yH^3_x\big)\frac{\partial L}{\partial H^3_x};$$
$$\chi^2_{12} = \big(\xi^\alpha - \eta^j u^\alpha_j\big)\frac{\partial L}{\partial u^\alpha_y} = \big(xE^1_y - yE^1_x + E^2\big)\frac{\partial L}{\partial E^1_y} + \big(xE^3_y - yE^3_x\big)\frac{\partial L}{\partial E^3_y}$$
$$+ \big(xH^1_y - yH^1_x + H^2\big)\frac{\partial L}{\partial H^1_y} + \big(xH^3_y - yH^3_x\big)\frac{\partial L}{\partial H^3_y};$$
$$\chi^3_{12} = \big(\xi^\alpha - \eta^j u^\alpha_j\big)\frac{\partial L}{\partial u^\alpha_z} = \big(xE^1_y - yE^1_x + E^2\big)\frac{\partial L}{\partial E^1_z} + \big(xE^2_y - yE^2_x - E^1\big)\frac{\partial L}{\partial E^2_z}$$
$$+ \big(xH^1_y - yH^1_x + H^2\big)\frac{\partial L}{\partial H^1_z} + \big(xH^2_y - yH^2_x - H^1\big)\frac{\partial L}{\partial H^2_z}.$$

Substituting here the expression (2.2) for the Lagrangian L, we obtain the following flux components:

$$\chi^1_{12} = \big(xE^2_y - yE^2_x - E^1\big)E^3 - \big(xE^3_y - yE^3_x\big)E^2$$
$$+ \big(xH^2_y - yH^2_x - H^1\big)H^3 - \big(xH^3_y - yH^3_x\big)H^2;$$
$$\chi^2_{12} = -\big(xE^1_y - yE^1_x + E^2\big)E^3 + \big(xE^3_y - yE^3_x\big)E^1$$
$$- \big(xH^1_y - yH^1_x + H^2\big)H^3 + \big(xH^3_y - yH^3_x\big)H^1;$$
$$\chi^3_{12} = \big(xE^1_y - yE^1_x + E^2\big)E^2 - \big(xE^2_y - yE^2_x - E^1\big)E^1$$
$$+ \big(xH^1_y - yH^1_x + H^2\big)H^2 - \big(xH^2_y - yH^2_x - H^1\big)H^1.$$

It is useful to rewrite them by collecting the terms with x and y :

$$
\begin{aligned}
\chi^1_{12} &= -\left(E^1E^3 + H^1H^3\right) + x\left(E^3E^2_y - E^2E^3_y + H^3H^2_y - H^2H^3_y\right)\\
&\qquad - y\left(E^3E^2_x - E^2E^3_x + H^3H^2_x - H^2H^3_x\right);\\
\chi^2_{12} &= -\left(E^2E^3 + H^2H^3\right) + x\left(E^1E^3_y - E^3E^1_y + H^1H^3_y - H^3H^1_y\right)\\
&\qquad - y\left(E^1E^3_x - E^3E^1_x + H^1H^3_x - H^3H^1_x\right); \qquad (3.29)\\
\chi^3_{12} &= \left(E^1\right)^2 + \left(E^2\right)^2 + \left(H^1\right)^2 + \left(H^2\right)^2\\
&\qquad + x\left(E^2E^1_y - E^1E^2_y + H^2H^1_y - H^1H^2_y\right)\\
&\qquad - y\left(E^2E^1_x - E^1E^2_x + H^2H^1_x - H^1H^2_x\right).
\end{aligned}
$$

In order to verify that τ_{12} given by (3.27) is a conservation density, we differentiate (3.27) with respect to t, use Eqs. (1.1′) and Remark 3.3, and obtain:

$$
\begin{aligned}
D_t(\tau_{12}) &= 2\left[E^1\left(E^3_x - E^1_z\right) + H^2\left(H^3_y - H^2_z\right) - E^2\left(E^2_x - E^3_y\right) - H^1\left(H^1_z - H^3_x\right)\right]\\
&+ y\left[\boldsymbol{H}\cdot\left(\nabla\times\boldsymbol{H}_x\right) - \boldsymbol{E}_x\cdot\left(\nabla\times\boldsymbol{E}\right) + \boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{E}_x\right) - \boldsymbol{H}_x\cdot\left(\nabla\times\boldsymbol{H}\right)\right]\\
&- x\left[\boldsymbol{H}\cdot\left(\nabla\times\boldsymbol{H}_y\right) - \boldsymbol{E}_y\cdot\left(\nabla\times\boldsymbol{E}\right) + \boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{E}_y\right) - \boldsymbol{H}_y\cdot\left(\nabla\times\boldsymbol{H}\right)\right]. \qquad (3.30)
\end{aligned}
$$

Calculating of the divergence of the vector $\chi_{12} = (\chi^1_{12}, \chi^2_{12}, \chi^3_{12})$ with the components (3.29) one can check that $\nabla\cdot\chi_{12}$ is equal to the right-hand side of (3.30) taken with the opposite sign. Hence, (3.27), (3.29) satisfy the conservation equation (1.12).

Proceeding likewise with other rotation generators and using the integral form (1.13) of conservation laws, we arrive at the following result.

Lemma 3.3. The invariance of Eqs. (1.1) under the rotations with the generators X_{12}, X_{13}, X_{23} from (1.3) provides *two* vector valued conservation law

$$
\frac{d}{dt}\int\left[2\left(\boldsymbol{E}\times\boldsymbol{H}\right) + \left(\boldsymbol{x}\times\boldsymbol{\pi}\right)\right]dxdydz = 0, \qquad (3.31)
$$

where $\pi = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (3.23).

3.5 Duality rotations

Consider the duality rotations with the generator (1.5). Acting by the prolonged operator Z_0 given by (2.6) on the Lagrangian (2.1) and using Eqs. (2.7) we have:

$$
\begin{aligned}
Z_0(L) &= -\boldsymbol{H}\cdot\left(\nabla\times\boldsymbol{E} + \boldsymbol{H}_t\right) + \boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{H} - \boldsymbol{E}_t\right)\\
&\quad + \boldsymbol{E}\cdot\left(-\nabla\times\boldsymbol{H} + \boldsymbol{E}_t\right) + \boldsymbol{H}\cdot\left(\nabla\times\boldsymbol{E} + \boldsymbol{H}_t\right) = 0.
\end{aligned}
$$

Hence, Eq. (3.8) is satisfied, and Eq. (3.10) yields:

$$\tau = E\cdot\frac{\partial L}{\partial H_t} - H\cdot\frac{\partial L}{\partial E_t} = E\cdot E - H\cdot(-H) = E\cdot E + H\cdot H.$$

Therefore, τ is identical (up to the inessential factor 1/2) with the energy density (1.30),

$$\tau = \frac{1}{2}\left(|E|^2 + |H|^2\right).$$

Furthermore, the reckoning shows that the flux χ is the Poynting vector

$$\sigma = (E\times H).$$

Thus, the invariance of Eqs. (1.1) with respect to the duality rotations provides the differential equation (1.32) for the conservation of energy,

$$D_t\left(\frac{|E|^2 + |H|^2}{2}\right)\Bigg|_{(1.1)} + \nabla\cdot\left(E\times H\right) = 0,$$

or Eq. (1.31) in the integral form:

$$\frac{d}{dt}\int\frac{1}{2}\left(|E|^2 + |H|^2\right)dxdydz = 0. \tag{1.31}$$

3.6 The dilations

The invariance test (3.8) is satisfied for the dilation generators Z_1 and Z_2 given by (1.6) and (1.7), respectively. The reckoning shows that for Z_1 Eq. (3.10) yields $\tau = 0, \chi = 0.$ Hence, the invariance with respect to the dilations of the dependent variables with the generator Z_1 does not provide a nontrivial conservation law.

Let us consider the dilations of the independent variables with the generator $Z_2.$ For this operator, Eq. (3.10) for $\tau = C^0$ is written

$$\tau = -\left(tE_t + xE_x + yE_y + zE_z\right)\frac{\partial L}{\partial E_t} - \left(tH_t + xH_x + yH_y + zH_z\right)\frac{\partial L}{\partial H_t}\cdot$$

It follows:

$$\begin{aligned}\tau &= \left(tE_t + xE_x + yE_y + zE_z\right)\cdot H - \left(tH_t + xH_x + yH_y + zH_z\right)\cdot E\\ &= t\left(H\cdot E_t - E\cdot H_t\right) + x\left(H\cdot E_x - E\cdot H_x\right) + y\left(H\cdot E_y - E\cdot H_y\right)\\ &\quad + z\left(H\cdot E_z - E\cdot H_z\right) = t\pi_0 + x\pi_1 + y\pi_2 + \pi_3,\end{aligned}$$

or

$$\tau = t\pi_0 + x\cdot\pi, \tag{3.32}$$

where π^0 is given by (3.13) or equivalently by (3.17), and $\boldsymbol{\pi} = (\pi^1, \pi^2, \pi^3)$ is the vector with the components (3.23). The flux components are computed likewise, e.g.

$$
\begin{aligned}
\chi^1 &= -\big(tE_t + xE_x + yE_y + zE_z\big)\frac{\partial L}{\partial E_x} - \big(tH_t + xH_x + yH_y + zH_z\big)\frac{\partial L}{\partial H_x} \\
&= \big(tE^3_t + xE^3_x + yE^3_y + zE^3_z\big)E^2 - \big(tE^2_t + xE^2_x + yE^2_y + zE^2_z\big)E^3 \\
&= \big(tH^3_t + xH^3_x + yH^3_y + zH^3_z\big)H^2 - \big(tH^2_t + xH^2_x + yH^2_y + zH^2_z\big)H^3.
\end{aligned}
$$

Collecting the like terms, we have:

$$
\begin{aligned}
\chi^1 = {} & t\big(E^2E^3_t - E^3E^2_t + H^2H^3_t - H^3H^2_t\big) + x\big(E^2E^3_x - E^3E^2_x + H^2H^3_x - H^3H^2_x\big) \\
& + y\big(E^2E^3_y - E^3E^2_y + H^2H^3_y - H^3H^2_y\big) + z\big(E^2E^3_z - E^3E^2_z + H^2H^3_z - H^3H^2_z\big),
\end{aligned}
$$

or, using the vectors $\boldsymbol{\chi}_0 = \big(\chi^1_0, \chi^2_0, \chi^3_0\big)$ and $\boldsymbol{\chi}_i = \big(\chi^1_i, \chi^2_i, \chi^3_i\big),\ i = 1, 2, 3,$ defined by Eqs. (3.14) and (3.24), respectively:

$$
\chi^1 = t\chi^1_0 + x\chi^1_1 + y\chi^1_2 + z\chi^1_3.
$$

Computing the other flux components, we obtain the following flux:

$$
\boldsymbol{\chi} = t\boldsymbol{\chi}_0 + x\boldsymbol{\chi}_1 + y\boldsymbol{\chi}_2 + z\boldsymbol{\chi}_3. \tag{3.33}
$$

Thus, we have arrived at the following differential conservation equation:

$$
D_t\big(t\pi^0 + \boldsymbol{x}\cdot\boldsymbol{\pi}\big) + \nabla\cdot\big(t\boldsymbol{\chi}_0 + x\boldsymbol{\chi}_1 + y\boldsymbol{\chi}_2 + z\boldsymbol{\chi}_3\big) = 0. \tag{3.34}
$$

Representing it in the integral form, we can formulate the result as follows.

Lemma 3.4. The invariance of Eqs. (1.1) with respect to the group of dilations of the independent variables with the generator Z_1 leads to the conservation law

$$
\frac{d}{dt}\int \big(t\pi^0 + \boldsymbol{x}\cdot\boldsymbol{\pi}\big)\, dxdydz = 0. \tag{3.35}
$$

3.7 Conservation laws due to superposition

For the superposition generator (1.8),

$$
S = \boldsymbol{E}_*(\boldsymbol{x}, t)\cdot\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{H}_*(\boldsymbol{x}, t)\cdot\frac{\partial}{\partial \boldsymbol{H}}, \tag{1.8}
$$

Eq. (3.10) for calculating the conservation density is written

$$
\tau = \boldsymbol{E}_*\frac{\partial L}{\partial \boldsymbol{E}_t} + \boldsymbol{H}_*\frac{\partial L}{\partial \boldsymbol{H}_t}
$$

and yields:

$$\tau = \boldsymbol{H}_* \cdot \boldsymbol{E} - \boldsymbol{E}_* \cdot \boldsymbol{H}. \tag{3.36}$$

The flux components are computed likewise, e.g.

$$\begin{aligned}\psi^1 &= E_*^k \frac{\partial L}{\partial E_x^k} + H_*^k \frac{\partial L}{\partial H_x^k} = E_*^2 E^3 - E_*^3 E^2 + H_*^2 H^3 - H_*^3 H^2 \\ &= \left(\boldsymbol{E}_* \times \boldsymbol{E}\right)^1 + \left(\boldsymbol{H}_* \times \boldsymbol{H}\right)^1.\end{aligned}$$

Hence, computing the other flux components, we obtain the following flux:

$$\chi = \left(\boldsymbol{E}_* \times \boldsymbol{E}\right) + \left(\boldsymbol{H}_* \times \boldsymbol{H}\right). \tag{3.37}$$

Thus, we have arrived at the following differential conservation equation:

$$D_t\left(\boldsymbol{H}_* \cdot \boldsymbol{E} - \boldsymbol{E}_* \cdot \boldsymbol{H}\right) + \nabla \cdot \left(\boldsymbol{E}_* \times \boldsymbol{E} + \boldsymbol{H}_* \times \boldsymbol{H}\right) = 0. \tag{3.38}$$

Representing it in the integral form, we can formulate the result as follows.

Lemma 3.5. The linear superposition principle provides the infinite set of conservation laws

$$\frac{d}{dt} \int \left(\boldsymbol{H}_* \cdot \boldsymbol{E} - \boldsymbol{E}_* \cdot \boldsymbol{H}\right) dxdydz = 0 \tag{3.39}$$

involving any two solutions, $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{E}_*, \boldsymbol{H}_*)$, of Eqs. (1.1)–(1.2).

Example 3.1. Taking the trivial solution $\boldsymbol{E}_* = \text{const.}, \boldsymbol{H}_* = \text{const.}$ we obtain from (3.39) the conservation equations

$$\frac{d}{dt} \int \boldsymbol{E}\, dxdydz = 0, \quad \frac{d}{dt} \int \boldsymbol{H}\, dxdydz = 0 \tag{3.40}$$

corresponding to the particular cases of the superposition symmetry (1.8), namely,

$$S_1 = \frac{\partial}{\partial \boldsymbol{E}}, \quad S_2 = \frac{\partial}{\partial \boldsymbol{H}} \cdot$$

Example 3.2. Substituting in (3.39) the travelling wave solution

$$\boldsymbol{E}_* = \begin{pmatrix} 0 \\ f(x-t) \\ g(x-t) \end{pmatrix}, \quad \boldsymbol{H}_* = \begin{pmatrix} 0 \\ -g(x-t) \\ f(x-t) \end{pmatrix}$$

of Eqs. (1.1)-(1.2) we obtain the conservation law

$$\frac{d}{dt} \int \left\{ f(x-t)\left[E^3 - H^2\right] - g(x-t)\left[E^2 + H^3\right] \right\} dxdydz = 0 \tag{3.41}$$

with two arbitrary functions, $f(x-t)$ and $g(x-t)$.

4 Calculation of conservation laws for Eqs. (1.1)–(1.2)

4.1 Splitting of the conservation law (3.31) by Eqs. (1.2)

Comparing τ_{12} given by Eq. (3.27) with σ_3 given in (1.42), and the flux components (3.29) with the quantities

$$\chi_3^1 = -\left(E^1E^3 + H^1H^3\right), \quad \chi_3^2 = -\left(E^2E^3 + H^2H^3\right),$$
$$\chi_3^3 = \frac{1}{2}\left[(E^1)^2 + (E^2)^2 - (E^3)^2 + (H^1)^2 + (H^2)^2 - (H^3)^2\right]$$

given by Eqs. (1.41), we conclude that the conservation equation with the density (3.27) and the flux (3.29) satisfied for for Eqs. (1.1) splits into two conservation equations for Eqs. (1.1)-(1.2). One of them is the conservation equation (1.43) for the linear momentum with the density $\sigma = \boldsymbol{E} \times \boldsymbol{H}.$ The other has the density

$$\tau_* = y\pi_1 - x\pi_2 \tag{4.1}$$

and the flux χ_* with the components

$$\begin{aligned}
\chi_*^1 = E^1E^3 + H^1H^3 &+ x\left(E^3E_y^2 - E^2E_y^3 + H^3H_y^2 - H^2H_y^3\right)\\
&- y\left(E^3E_x^2 - E^2E_x^3 + H^3H_x^2 - H^2H_x^3\right),\\
\chi_*^2 = E^2E^3 + H^2H^3 &+ x\left(E^1E_y^3 - E^3E_y^1 + H^1H_y^3 - H^3H_y^1\right)\\
&- y\left(E^1E_x^3 - E^3E_x^1 + H^1H_x^3 - H^3H_x^1\right),\\
\chi_*^3 = \left(E^3\right)^2 + \left(H^3\right)^2 &+ x\left(E^2E_y^1 - E^1E_y^2 + H^2H_y^1 - H^1H_y^2\right)\\
&- y\left(E^2E_x^1 - E^1E_x^2 + H^2H_x^1 - H^1H_x^2\right).
\end{aligned} \tag{4.2}$$

Proceeding likewise with all components of the vector valued conservation equation (3.31) we arrive at the following result.

Lemma 4.1. The equations (1.2) of Maxwell's system split the conservation equation (3.31) into two equations. In consequence, the group of rotations generated by X_{12}, X_{13}, X_{23} from (1.3) provide *two* vector valued conservation laws for the Maxwell equations (1.1)-(1.2), namely (cf. the linear momentum (1.44)):

$$\frac{d}{dt}\int \left(\boldsymbol{E} \times \boldsymbol{H}\right) dxdydz = 0 \tag{4.3}$$

and

$$\frac{d}{dt}\int \left(\boldsymbol{x} \times \boldsymbol{\pi}\right) dxdydz = 0, \tag{4.4}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (3.23).

4.2 Conservation laws provided by the Lorentz transformations

Acting by the operator (2.8) on the Lagrangian (2.2) and using Eqs. (2.9) we obtain:

$$X_{01}(L)\Big|_{(1:1)} = ¡\, E^1(r ¢ H) + H^1(r ¢ E):$$

Therefore the operator X_{01} satisfies the invariance test (3.8) for the solutions of the Maxwell equations (1.1)–(1.2):

$$X_{01}(L)\Big|_{(1:1)-(1:2)} = 0:$$

This is true for all generators $X_{0i};\ i = 1;2;3;$ (see (1.4)) of the Lorentz transformations. Consequently, the conservation laws associated with the Lorentz transformations can be computed by means of Eqs. (3.10). I will calculate here only the densities $¿_{0i}$ of these conservation laws.

Let us consider X_{01}: Eqs. (3.10) yield:

$$¿_{01} = (tE^1_x + xE^1_t)H^1 \ ¡\ (H^3 \ ¡\ tE^2_x \ ¡\ xE^2_t)H^2 + (H^2 + tE^3_x + xE^3_t)H^3$$
$$¡\ (tH^1_x + xH^1_t)E^1 \ ¡\ (E^3 + tH^2_x + xH^2_t)E^2 + (E^2 \ ¡\ tH^3_x \ ¡\ xH^3_t)E^3:$$

We rewrite it in the form

$$¿_{01} = x\left(H^1E^1_t + H^2E^2_t + H^3E^3_t \ ¡\ E^1H^1_t \ ¡\ E^2H^2_t \ ¡\ E^2H^3_t\right)$$
$$+\, t\left(H^1E^1_x + H^2E^2_x + H^3E^3_x \ ¡\ E^1H^1_x \ ¡\ E^2H^2_x \ ¡\ E^2H^3_x\right)$$
$$= x(H ¢ E_t \ ¡\ E ¢ H_t) + t(H ¢ E_x \ ¡\ E ¢ H_x):$$

Finally, using the conservation densities $¼_0$ and $¼_1$ given by Eqs. (3.13) and (3.23), respectively, we obtain:

$$¿_{01} = x¼_0 \ ¡\ t¼_1:$$

Proceeding likewise with all generators X_{0i} from (1.4), we arrive at the following.

Lemma 4.2. The Lorentz invariance of Maxwell's equations (1.1)–(1.2) provides the vector valued conservation law

$$\frac{d}{dt}\int (¼_0 x \ ¡\ t\pi)\, dxdydz = 0; \tag{4.5}$$

where

$$\pi = (¼_1; ¼_2; ¼_3) \tag{4.6}$$

is the vector with the components (3.23).

4.3 Conservation laws provided by the conformal transformations

Acting by the operator Y_1 from (1.9) on the Lagrangian (2.2) and using Eqs. (2.11) we see that the invariance test (3.8) is satisfied:

$$Y_1(L)\Big|_{(1.1)-(1.2)} = 0.$$

Therefore the conservation laws associated with the conformal transformations can be computed by means of Eqs. (3.10). Let us calculate the density τ of the conservation law provided by the operator Y_1. Eqs. (3.10) yield:

$$\tau = (\eta^\alpha - \xi^j u^\alpha_j)\frac{\partial L}{\partial u^\alpha_t} = -\Big[4xE^1 + 2yE^2 + 2zE^3 + (x^2 - y^2 - z^2 + t^2)E^1_x + 2xyE^1_y + 2xzE^1_z + 2xtE^1_t\Big]\frac{\partial L}{\partial E^1_t} - \cdots - \Big[4xH^3 - 2zH^1 - 2tE^2 + (x^2 - y^2 - z^2 + t^2)H^3_x + 2xyH^3_y + 2xzH^3_z + 2xtH^3_t\Big]\frac{\partial L}{\partial H^3_t}\cdot$$

Substituting

$$\frac{\partial L}{\partial E^1_t} = -H^1, \ldots, \frac{\partial L}{\partial H^3_t} = E^3$$

and arranging in terms of different powers of $x, y, z, t,$ we have:

$$\begin{aligned}\tau = {} & 4y\big(E^2H^1 - E^1H^2\big) + 4z\big(E^3H^1 - E^1H^3\big)\\ & + (x^2 - y^2 - z^2 + t^2)\big(H^1E^1_x + H^2E^2_x + H^3E^3_x - E^1H^1_x - E^2H^2_x - E^2H^3_x\big)\\ & + 2xy\big(H^1E^1_y + H^2E^2_y + H^3E^3_y - E^1H^1_y - E^2H^2_y - E^2H^3_y\big)\\ & + 2xz\big(H^1E^1_z + H^2E^2_z + H^3E^3_z - E^1H^1_z - E^2H^2_z - E^2H^3_z\big)\\ & + 2xt\big(H^1E^1_t + H^2E^2_t + H^3E^3_t - E^1H^1_t - E^2H^2_t - E^2H^3_t\big);\end{aligned}$$

or

$$\begin{aligned}\tau = {} & -4y(E \times H)^3 + 4z(E \times H)^2\\ & + (x^2 - y^2 - z^2 + t^2)(H \cdot E_x - E \cdot H_x) + 2xy(H \cdot E_y - E \cdot H_y)\\ & + 2xz(H \cdot E_z - E \cdot H_z) + 2xt(H \cdot E_t - E \cdot H_t).\end{aligned}$$

Finally, using the quantities π_0 and π_1, π_2, π_3 given by Eq. (3.13) and Eqs. (3.23), respectively, we write the resulting conservation density in the form

$$\tau = -4\big[x \times (E \times H)\big]^1 + (t^2 - x^2 - y^2 - z^2)\pi_1 + 2x(x\pi_1 + y\pi_2 + z\pi_3 + t\pi_0). \quad (4.7)$$

Using the test for conservation densities (Section 1.5) one can verify that the terms of (4.7) that are linear in the independent variables and those quadratic in these variables provide two independent conservation densities (cf. Example 1.4). Treating likewise the operators Y_2 and Y_3 we arrive at the following result.

Lemma 4.3. The invariance of Maxwell's equations (1.1)–(1.2) with respect to the conformal transformations generated by Y_1, Y_2, Y_3 provides two vector valued conservation laws. Namely, the conservation of the angular momentum (1.46):

$$\frac{d}{dt}\int \left[\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{H})\right] dxdydz = 0, \tag{4.8}$$

and the following new vector valued conservation law:

$$\frac{d}{dt}\int \left[2(\boldsymbol{x} \cdot \boldsymbol{\pi} + t\pi_0)\boldsymbol{x} + (t^2 - x^2 - y^2 - z^2)\boldsymbol{\pi}\right] dxdydz = 0, \tag{4.9}$$

where $\boldsymbol{x} = (x, y, z),\ \boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3).$

The reckoning shows that the operator Y_4 from (1.9) leads to the following conservation density instead of (4.7):

$$\tau = 2t(x\pi_1 + y\pi_2 + z\pi_3) + (t^2 + x^2 + y^2 + z^2)\pi_0. \tag{4.10}$$

Hence, we have the following result.

Lemma 4.4. The invariance of Maxwell's equations (1.1)–(1.2) with respect to the conformal transformations generated by Y_4 provides the conservation law

$$\frac{d}{dt}\int \left[2t(\boldsymbol{x} \cdot \boldsymbol{\pi}) + (t^2 + x^2 + y^2 + z^2)\pi_0\right] dxdydz = 0. \tag{4.11}$$

5 Summary

It is exhibited that the Noether theorem can be applied to overdetermined systems of differential equations, provided that the system in question contains a sub-system admitting a variational formulation. In the case of the Maxwell equations the appropriate sub-system is provided by the evolution equations (1.1) of Maxwell's system. Using the Lagrangian (2.1) of this sub-system, one obtains an infinite set of conservation laws containing the conservation of the classical electromagnetic energy, linear and angular momenta, as well as Lipkin's conservation laws. This set of conservation laws does not contain, however, the relativistic center-of-mass theorem and the five conservation laws associated by Bessel-Hagen with the conformal transformations.

Tests for conservation densities are proved in Sections 1.4 and 1.5.

The results on conservation laws obtained in the previous sections are summarized in the following theorem. For the sake of brevity, the conservation equations are written in the integral form.

Theorem 5.1. The symmetries (1.3)–(1.9) applied to the Lagrangian (2.1) of the evolutionary part (1.1) of Maxwell's vacuum equations provide the following conservation laws for the Maxwell equations (1.1)–(1.2):

Classical conservation laws:

$$\frac{d}{dt}\int \left(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\right) dxdydz = 0 \quad \text{(energy conservation)}; \tag{5.1}$$

$$\frac{d}{dt}\int \left(\boldsymbol{E}\times\boldsymbol{H}\right) dxdydz = 0 \quad \text{(linear momentum)}; \tag{5.2}$$

$$\frac{d}{dt}\int \left[\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{H})\right] dxdydz = 0 \quad \text{(angular momentum)}; \tag{5.3}$$

Non-classical conservation laws:

$$\frac{d}{dt}\int \sigma^0\, dxdydz = 0; \tag{5.4}$$

$$\frac{d}{dt}\int \pi\, dxdydz; \tag{5.5}$$

$$\frac{d}{dt}\int \left(\boldsymbol{x}\times\pi\right) dxdydz = 0; \tag{5.6}$$

$$\frac{d}{dt}\int \left(t\sigma^0 + \boldsymbol{x}\cdot\pi\right) dxdydz = 0; \tag{5.7}$$

$$\frac{d}{dt}\int (\sigma^0\boldsymbol{x} - t\pi)\, dxdydz = 0; \tag{5.8}$$

$$\frac{d}{dt}\int \left[2(\boldsymbol{x}\cdot\pi + t\sigma^0)\boldsymbol{x} + (t^2 - x^2 - y^2 - z^2)\pi\right] dxdydz = 0; \tag{5.9}$$

$$\frac{d}{dt}\int \left[2t(\boldsymbol{x}\cdot\pi) + (t^2 + x^2 + y^2 + z^2)\sigma^0\right] dxdydz = 0; \tag{5.10}$$

$$\frac{d}{dt}\int \left(\boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}\right) dxdydz = 0. \tag{5.11}$$

In Eqs. (5.4)–(5.10), σ^0 is given by Eq. (3.13) and $\pi = (\sigma^1, \sigma^2, \sigma^3)$ is the vector with the components (3.23). The infinite set of conservation equations (5.11) involves two arbitrary solutions, $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{E}_*, \boldsymbol{H}_*)$, of the Maxwell equations (1.1)–(1.2).

Remark 5.1. If the solution $(\boldsymbol{E}, \boldsymbol{H})$ is identical with the solution $(\boldsymbol{E}_*, \boldsymbol{H}_*)$, then the conservation law (5.11) is trivial, i.e. its density $\boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}$ vanishes. This is in accordance with the fact that in the case $\boldsymbol{E}_* = \boldsymbol{E}, \boldsymbol{H}_* = \boldsymbol{H}$ the superposition generator S coincides with the dilation generator Z_1 which does not provide a nontrivial conservation law (see the preamble to Section 3.6).

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Quasi-self-adjoint differential equations

Nail H. Ibragimov
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

1 Self-adjoint equations

Recall that for any (linear or non-linear) system of differential equations

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{1.1}$$

where $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots, u^m)$ denote the independent and dependent variables, respectively, the *adjoint equations* are defined by

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{1.2}$$

where

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \ldots, m. \tag{1.3}$$

Here $v = (v^1, \ldots, v^m)$ are new dependent variables, $u_{(1)}$ and $v_{(1)}$ denote the first-order derivatives u^α_i and v^α_i of u and v with respect to $x,$ etc.

I called (see [1]) the system (1.1) *self-adjoint* if the system of adjoint equations (1.2) becomes equivalent to the original system (1.1) upon the substitution $v = u.$ It means that the following equations are satisfied

$$F^*_\alpha(x, u, u, \ldots, u_{(s)}, u_{(s)}) = \lambda^\beta_\alpha F_\beta(x, u, \ldots, u_{(s)}), \quad \alpha = 1, \ldots, m, \tag{1.4}$$

with undetermined coefficients $\lambda^\beta_\alpha.$ In general, the coefficients λ^β_α may be variable. This definition of self-adjoint equations was motivated by the fact that if a *linear* equation $L(u) = 0$ is self-adjoint, then the adjoint equation $L_*(v) = 0$ coincides with the original equation after substituting $v = u;$ namely, $L_*(u) = L(u).$

Remark 1.1. Self-adjoint equations have a remarkable property that the "nonlocal" variables v can be eliminated, e.g. from conservation laws. This important fact was applied in [2] to the Korteweg-de Vries equation.

The following examples illustrate the approach to determining self-adjoint equations. They are taken from [2].

Example 1.1. Consider the second-order evolution equations of the form

$$u_t = f(u)u_{xx}. \tag{1.5}$$

According to Eq. (1.3), we have:

$$\begin{aligned}\frac{\delta}{\delta u}\Big[(u_t - f(u)u_{xx})v\Big] = & -v_t - f'(u)vu_{xx} - D_x^2\big(vf(u)\big)\\ & -v_t - f'(u)vu_{xx} - D_x\big(f(u)v_x + vf'(u)u_x\big)\\ & -v_t - 2f'(u)vu_{xx} - f(u)v_{xx} - 2f'(u)u_xv_x - vf''(u)u_x^2.\end{aligned}$$

Therefore, the adjoint equation (1.2) to Eq. (1.5) is

$$v_t = -2f'(u)vu_{xx} - f(u)v_{xx} - 2f'(u)u_xv_x - vf''(u)u_x^2. \tag{1.6}$$

Upon setting $v = u$ it becomes:

$$u_t = -\big[2uf'(u) + f(u)\big]u_{xx} - \big[2f'(u) + uf''(u)\big]u_x^2. \tag{1.7}$$

In our case Eq. (1.4) is written:

$$u_t + \big[2uf'(u) + f(u)\big]u_{xx} + \big[2f'(u) + uf''(u)\big]u_x^2 = \lambda u_t - \lambda f(u)u_{xx}, \tag{1.8}$$

whence $\lambda = 1$ and

$$2uf'(u) + f(u) = -f(u), \quad 2f'(u) + uf''(u) = 0. \tag{1.9}$$

The second equation in (1.9) is the differential consequence of the first one. Therefore, we solve the first equation in (1.9),

$$uf'(u) + f(u) = 0,$$

and obtain

$$f(u) = \frac{a}{u}, \quad a = \text{cons}.$$

Hence, Eq. (1.5) is self-adjoint if and only if it has the form

$$u_t = \frac{a}{u}u_{xx}, \quad a = \text{cons}. \tag{1.10}$$

Example 1.2. Let us find all self-adjoint equations among the equations

$$u_t = f(u)u_{xxx}. \tag{1.11}$$

We have:

$$\frac{\delta}{\delta u}\Big[(u_t - f(u)u_{xxx})v\Big] = -v_t - f'(u)vu_{xxx} + D_x^3\big[vf(u)\big]$$

$$= -v_t + fv_{xxx} + 3\big[f'v_x + f''vu_x\big]u_{xx} + 3f'u_xv_{xx} + 3f''v_xu_x^2 + f'''vu_x^3.$$

Hence, the adjoint equation to Eq. (1.11) is

$$v_t = fv_{xxx} + 3\big[f'v_x + f''vu_x\big]u_{xx} + 3f'u_xv_{xx} + 3f''v_xu_x^2 + f'''vu_x^3. \tag{1.12}$$

Letting in (1.12) $v = u,$ we have:

$$u_t = fu_{xxx} + 3(2f' + uf'')u_xu_{xx} + (3f'' + uf''')u_x^3 = 0. \tag{1.13}$$

Comparison of Eqs. (1.13) and (1.11) yields the system

$$2f' + uf'' = 0, \quad 3f'' + uf''' = 0.$$

Since the second equation of this system is obtained from the first one by differentiation, we integrate the equation

$$2f' + uf'' = 0$$

and obtain:

$$f(u) = \frac{a}{u} + b, \quad a, b = \text{const}.$$

Hence, the general self-adjoint equation of the form (1.11) is

$$u_t = \Big(\frac{a}{u} + b\Big)u_{xxx}, \quad a, b = \text{const}. \tag{1.14}$$

These examples show that, e.g. the nonlinear equations

$$u_t = u^2u_{xx} \tag{1.15}$$

and

$$u_t = u^3u_{xxx} \tag{1.16}$$

having remarkable symmetry properties and physical significance (see, e.g. [3] and [4], Section 20) are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from conservation laws of these equations by setting $v = u.$

I will generalize now the concept of self-adjoint equations by introducing the definition of *quasi-self-adjoint equations.* Then Eqs. (1.15) and (1.16) will be self-adjoint in the generalized meaning.

2 Quasi-self-adjoint equations

Definition 2.1. The system (1.1) is said to be quasi-self-adjoint if the the adjoint system (1.2) is equivalent to the original system (1.1) upon the substitution

$$v = \varphi(u) \tag{2.1}$$

with a certain function $\varphi(u)$ such that $\varphi'(u) \neq 0.$

Example 2.1. Consider again the second-order equation (1.5) from Example 1.1:

$$u_t = f(u)u_{xx}. \tag{1.5}$$

We us substitute

$$v = \varphi(u);\ v_t = \varphi' u_t;\ v_x = \varphi' u_x;\ v_{xx} = \varphi' u_{xx} + \varphi'' u_x^2$$

in the adjoint equation (1.6),

$$v_t + 2f'(u)vu_{xx} + f(u)v_{xx} + 2f'(u)u_x v_x + vf''(u)u_x^2 = 0, \tag{1.6}$$

and obtain:

$$\varphi' u_t + 2f'\varphi u_{xx} + \left[\varphi' u_{xx} + \varphi'' u_x^2\right] f + 2f'\varphi' u_x^2 + \varphi f'' u_x^2 = 0. \tag{2.2}$$

Therefore Eq. (1.4) is written:

$$\varphi' u_t + 2f'\varphi u_{xx} + \left[\varphi' u_{xx} + \varphi'' u_x^2\right] f + 2f'\varphi' u_x^2 + \varphi f'' u_x^2 = \lambda u_t - \lambda f u_{xx},$$

whence $\lambda = \varphi'$ and

$$2f'\varphi + f\varphi' = -f\varphi'; \quad f\varphi'' + 2f'\varphi' + \varphi f'' = 0. \tag{2.3}$$

The second equation in (2.3) is the differential consequence of the first one. The first equation in (2.3) is written

$$f'\varphi + f\varphi' \equiv (f\varphi)' = 0$$

and yields

$$\varphi(u) = \frac{a}{f(u)}, \quad a = \text{const}.$$

We can take $a = 1.$ Hence, Eq. (1.5) is quasi-self-adjoint for any function $f(u).$ Namely, the adjoint equation (1.6) becomes equivalent to the original equation (1.5) upon the substitution

$$v = \frac{1}{f(u)}. \tag{2.4}$$

In particular, the adjoint equation

$$v_t + 4uvu_{xx} + u^2v_{xx} + 4uu_xv_x + 2vu_x^2 = 0 \tag{2.5}$$

to Eq. (1.15),

$$u_t = u^2u_{xx}; \tag{1.15}$$

becomes equivalent with Eq. (1.15) after the substitution

$$v = \frac{1}{u^2}\text{¢} \tag{2.6}$$

Example 2.2. Proceeding as above, one can verify that the third-order equation (1.11) from Example 1.2:

$$u_t = f(u)u_{xxx} \tag{1.11}$$

is also quasi-self-adjoint for any function $f(u)$: The adjoint equation (1.12) becomes equivalent to the original equation (1.11) upon the substitution

$$v = \frac{1}{f(u)}\text{¢} \tag{2.7}$$

In particular, the adjoint equation

$$v_t = u^3v_{xxx} + 9\left[u^2v_x + 2uvu_x\right]u_{xx} + 9u^2u_xv_{xx} + 18uv_xu_x^2 + 6vu_x^3 \tag{2.8}$$

to Eq. (1.16),

$$u_t = u^3u_{xxx}$$

becomes equivalent with Eq. (1.16) after the substitution

$$v = \frac{1}{u^3}\text{¢} \tag{2.9}$$

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Existence of integrating factors for higher-order ordinary differential equations

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

SALAVAT V. KHABIROV
Institute of Mechanics, Ufa Centre of Russian Academy of Sciences,
October Avenue, 71, Ufa 450054, Russia

Abstract. We investigate the concept of integrating factors for higher-order ordinary differential equations (ODEs) introduced in [1]. It is shown in [1] that the integrating factors for ODEs of order higher than one are determined by overdetermined systems. Therefore one can expect that not all higher-order equations have integrating factors. We prove in this paper that in fact they have. Moreover, we demonstrate that every ODE of order n has precisely n functionally independent integrating factors.

1 Introduction

Recall the following definition and theorems from [1].

Definition 1.1. A function $\mu\big(x, y, y', \ldots, y^{(n-1)}\big)$ is called an integrating factor for an ordinary differential equation of order n,

$$a\big(x, y, y', \ldots, y^{(n-1)}\big)\, y^{(n)} + b\big(x, y, y', \ldots, y^{(n-1)}\big) = 0, \tag{1.1}$$

if multiplication by μ converts the left-hand side of Eq. (1.1) into a total derivative of a function $\psi\big(x, y, y', \ldots, y^{(n-1)}\big)$:

$$\mu a y^{(n)} + \mu b = D_x(\psi), \tag{1.2}$$

where

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots + y^{(s+1)} \frac{\partial}{\partial y^{(s)}} + \cdots$$

is the operator of total differentiation.

The definition of integrating factors can be naturally extended to systems of ordinary differential equations of any order.

Theorem 1.1. The integrating factors for Eq. (1.1) are determined by the equation

$$\frac{\delta}{\delta y}\left(\mu a y^{(n)} + \mu b\right) = 0, \tag{1.3}$$

where $\delta/\delta y$ is the variational derivative

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'} + D_x^2 \frac{\partial}{\partial y''} - D_x^3 \frac{\partial}{\partial y'''} + \cdots .$$

Eq. (1.3) should be satisfied identically in the variables $x, y, y', \ldots, y^{(2n-2)}.$

Let us consider the second-order equations

$$a(x, y, y')y'' + b(x, y, y') = 0, \quad a \neq 0. \tag{1.4}$$

An analysis of Eq. (1.3) leads to the following result.

Theorem 1.2. The integrating factors $\mu(x, y, y')$ for the second-order equations (1.4) are determined by the following system of two equations (see [1], Theorem 2.2):

$$y'(\mu a)_{yy'} + (\mu a)_{xy'} + 2(\mu a)_y - (\mu b)_{y'y'} = 0, \tag{1.5}$$

$$y'^2(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx} - y'(\mu b)_{yy'} - (\mu b)_{xy'} + (\mu b)_y = 0, \tag{1.6}$$

where the subscripts denote the respective partial differentiations.

One can find likewise that the integrating factors for any higher-order equation should satisfy an overdetermined system.. Therefore one can expect that not all second-order or higher-order equations have integrating factors. However, we prove in the next section that the system of equations (1.5)-(1.6) is compatible for any equation (1.4). Hence, any second-order ODE has an integrating factor. The result can be extended to higher-order equations as well.

2 Existence theorem for second-order equations

We will write Eq. (1.4), dividing it by $a,$ in the form

$$y'' + b(x, y, y') = 0. \tag{2.1}$$

Furthermore, we denote $y' = p$ and write Eqs. (1.5)-(1.6) for the integrating factors $\mu(x, y, p)$ in the form

$$p\mu_{yp} + \mu_{xp} + 2\mu_y - (\mu b)_{pp} = 0, \tag{2.2}$$

$$p^2\mu_{yy} + 2p\mu_{xy} + \mu_{xx} - p(\mu b)_{yp} - (\mu b)_{xp} + (\mu b)_y = 0. \tag{2.3}$$

Lemma 2.1. The system of equations (2.2)-(2.3) is compatible for any smooth function $b(x,y,p).$

Proof. Let us rewrite Eq. (2.2) in the form

$$\mu_y = (b\mu)_{pp} - p\mu_{yp} - \mu_y - \mu_{xp} = ((b\mu)_p - p\mu_y - \mu_x)_p.$$

Then, introducing the *potential* $\psi(x,y,p)$ of the integrating factor by setting $\mu = \psi_p,$ we write the above equation in the form

$$\psi_{py} = [(b\psi_p)_p - p\psi_{py} - \psi_{px}]_p,$$

or

$$(b\psi_p - p\psi_y - \psi_x)_{pp} = 0.$$

Whence, integrating twice in $p,$ we get:

$$b\psi_p = p\psi_y + \psi_x + pA(x,y) + B(x,y), \tag{2.4}$$

where $A(x,y)$ and $B(x,y)$ are arbitrary functions.

One can readily verify that the substitution of the representation $\mu = \psi_p$ and of the expression for $b\psi_p$ given by (2.4) in the left-hand side of (2.3) reduces Eq. (2.3) to

$$B_y - A_x = 0,$$

whence $B = \varphi_x,\ A = \varphi_y$ with an arbitrary function $\varphi(x,y).$

Since the potential ψ is determined up to addition of an arbitrary function of x and $y,$ we replace ψ by $\psi + \varphi$ and get $A = B = 0.$ Then Eq. (2.4) yields the following linear homogeneous first-order partial differential equation:

$$\psi_x + p\psi_y - b\psi_p = 0. \tag{2.5}$$

The characteristic system for Eq. (2.5)

$$dx = \frac{dy}{p} = -\frac{dp}{b} \tag{2.6}$$

has two first integrals

$$\alpha(x,y,p) = C_1, \quad \beta(x,y,p) = C_2 \tag{2.7}$$

with functionally independent α and $\beta.$ The general solution of Eq. (2.5) is given by

$$\psi(x,y,p) = F\big(\alpha(x,y,p), \beta(x,y,p)\big). \tag{2.8}$$

Since $\alpha(x,y,p)$ and $\beta(x,y,p)$ are functionally independent, ψ_p cannot vanish for all functions F in (2.8). Taking a solution (2.8) such that $\psi_p \neq 0$ one obtains a solution $\mu = \psi_p$ to the system (2.2)-(2.3). This completes the proof.

Lemma 2.2. The potential ψ of an integrating factor obtained in Lemma 2.1 is a first integral of Eq. (2.1), i.e. it is constant on the solutions of Eq. (2.1).

Proof. Indeed, Eq. (2.5) can be written

$$D_x(\psi)\big|_{y''+b=0} = 0.$$

Hence

$$\psi(x, y, y') = C \tag{2.9}$$

on the solutions of Eq. (2.1).

Combining Lemma 2.1 and Lemma 2.2, we arrive at the following general statement on integrating factors and first integrals of second-order equations.

Theorem 2.1. Any second-order equation (2.1) has an integrating factor. The potential $\psi(x, y, p)$ of an integrating factor defined in Lemma 2.1 provides a first integral (2.9) of Eq. (2.1). Moreover, Eq. (2.1) has two distinctly different integrating factors whose potentials are defined by two functionally independent solutions $\psi_1 = \alpha(x, y, p)$ and $\psi_2 = \beta(x, y, p)$ of Eq. (2.6). Then Eqs. (2.7) provide two independent first integrals

$$\alpha(x, y, y') = C_1, \quad \beta(x, y, y') = C_2 \tag{2.10}$$

of Eq. (2.1), and hence its general solution. Conversely, any first integral (2.9) of Eq. (2.1) defines an integrating factor.

Remark 2.1. If one knows an integrating factor $\mu(x, y, y')$, one can find a first integral $\phi(x, y, y') = C$ by Definition 1.1, namely by using Eq. (1.2),

$$(ay'' + b)\mu = D_x(\phi), \tag{2.11}$$

and solving it for ϕ. Theorem 2.1 furnishes an alternative way and often simplifies the calculations. See examples in Section 4.

Remark 2.2. Functional independence of two integrating factors μ_1 and μ_2 does not, in general, imply that their potentials ψ_1 and ψ_2 are also functionally independent. See Examples 4.1 and 4.3 in Section 4.

Remark 2.3. The characteristic equations (2.6):

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = -b$$

are equivalent to Eq. (2.6) written as a system of first-order equations.

Theorem 2.2. Let $b(x;y;y')$ be an analytical function with respect to the variable y' and let $b(x;y;0) \neq 0.$ Then the equation

$$y'' + b(x;y;y') = 0$$

has an integrating factor whose potential is an analytical function $\tilde{A}(x;y;p)$ with respect to the variable $p = y'.$

Proof. Let

$$b = \sum_{k\geq 0} b_k(x;y)p^k; \quad b_0(x;y) \neq 0.$$

We look for a potential analytic in p in the form

$$\tilde{A} = \sum_{j\geq 0} \tilde{A}_j p^j.$$

Substitution in Eq. (2.5) yields

$$\sum_{i\geq 0} \tilde{A}_{ix} p^i + \sum_{i\geq 0} \tilde{A}_{iy} p^{i+1} = \sum_{k\geq 0} b_k p^k \sum_{j\geq 1} j\tilde{A}_j p^{j-1}$$

or

$$\sum_{i\geq 0} \tilde{A}_{ix} p^i + \sum_{i\geq 0} \tilde{A}_{iy} p^{i+1} = \sum_{i\geq 0} p^i \sum_{j\geq 1} j\tilde{A}_j b_{i-j+1}.$$

Comparing the coefficients of $p^i;$ we obtain

$$\begin{aligned} i = 0: &\quad \tilde{A}_{0x} = \tilde{A}_1 b_0; \\ i = 1: &\quad \tilde{A}_{1x} + \tilde{A}_{0y} = \tilde{A}_1 b_1 + 2\tilde{A}_2 b_0; \\ i > 1: &\quad \tilde{A}_{ix} + \tilde{A}_{i-1y} = \sum_{j=1}^{i+1} j\tilde{A}_j b_{i+1-j}. \end{aligned}$$

It follows that starting with an arbitrary smooth function $\tilde{A}_0(x;y) \neq$ const., we can consecutively determine all coefficients $\tilde{A}_j(x;y);\ j \geq 1;$ e.g.

$$\tilde{A}_1 = \frac{1}{b_0}\tilde{A}_{0x}; \quad \tilde{A}_2 = \frac{1}{2b_0}(\tilde{A}_{0y} + \tilde{A}_{1x} - b_1\tilde{A}_1);$$

thus proving the theorem.

Remark 2.4. If $b_0 = 0;$ one can set $\tilde{A}_0 = 0$ since the potential is determined up to addition of an arbitrary function of x and $y.$ Then $\tilde{A}_j$ with $j \geq 1$ are obtained by integrating simple equations.

3 Arbitrary case

Let us apply our approach to the first-order equations

$$y' + b(x, y) = 0. \tag{3.1}$$

Taking integrating factors $\mu(x, y)$ in the form $\mu = \psi_y$ and substituting in the equation

$$(b\mu)_y - \mu_x = 0$$

for the integrating factors, we get

$$(b\psi_y)_y - \psi_{xy} = 0,$$

whence

$$b\psi_y - \psi_x = A(x).$$

Replacing ψ by $\psi + \int A(x)dx,$ we obtain:

$$b\,\psi_y - \psi_x = 0. \tag{3.2}$$

Since $\psi_x - b\,\psi_y = D_x(\psi)\big|_{y'+b=0},$ Eq. (3.2) shows that

$$\psi(x, y) = C \tag{3.3}$$

is a first integral for Eq. (3.1).

Consider now the higher-order equations and extend Theorem 2.1 to equations of any order. Let us rewrite Eq. (1.1), dividing it by $a\big(x, y, y', \ldots, y^{(n-1)}\big),$ in the form

$$y^{(n)} + b\big(x, y, y', \ldots, y^{(n-1)}\big) = 0. \tag{3.4}$$

Theorem 3.1. An arbitrary nth-order equation (3.4) has n integrating factors

$$\mu_i = \frac{\partial \psi_i}{\partial y^{(n-1)}}, \quad i = 1, \ldots, n,$$

with functionally independent *potentials* $\psi_i = \psi_i\big(x, y, y', \ldots, y^{(n-1)}\big), \ i = 1, \ldots, n.$

Proof. Let us take the general solution

$$y = h(x, C_1, \ldots, C_n) \tag{3.5}$$

of Eq. (3.4) and eliminate $n - 1$ constants, e.g. $C_2, \ldots, C_n$ by differentiating (3.5) $n - 1$ times. Solving the resulting equations with respect to $C_1,$ we obtain a first integral

$$\psi_1\big(x, y, y', \ldots, y^{(n-1)}\big) = C_1. \tag{3.6}$$

It follows that

$$D_x\Big(\psi_1\big(x, y, y', \ldots, y^{(n-1)}\big)\Big) = 0,$$

and hence

$$\mu_1 = \frac{\partial \psi_1\big(x, y, y', \ldots, y^{(n-1)}\big)}{\partial y^{(n-1)}} \tag{3.7}$$

is an integrating factor for Eq. (3.4). Applying the procedure to all constants $C_i,$ we obtain n integrating factors with functionally independent *potentials* $\psi_1, \ldots, \psi_n.$

4 Examples

Example 4.1. Consider the equation

$$(y^2 - 3x^2)dy + 2xydx = 0 \tag{4.1}$$

with a known integrating factor $\mu = 1/y^4$ (see, e.g. [2], Example 3.2.2). Dividing the equation in question by $y^2 - 3x^2$, we obtain an equation of the form (3.1),

$$y' + \frac{2xy}{y^2 - 3x^2} = 0, \tag{4.2}$$

with the integrating factor

$$\mu = \frac{y^2 - 3x^2}{y^4} \cdot \tag{4.3}$$

Integrating, we obtain the potential:

$$\psi = \int \mu dy = \int \Big(-\frac{3x^2}{y^4} + \frac{1}{y^2} \Big) dy = \frac{x^2 - y^2}{y^3} + h(x). \tag{4.4}$$

Substituting the expression (4.4) in Eq. (3.2) with $b = 2xy/(y^2 - 3x^2)$ we obtain $h'(x) = 0$. Hence

$$\psi = \frac{x^2 - y^2}{y^3} + h, \quad h = \text{const}. \tag{4.5}$$

Eq. (3.3) with the potential (4.5) provides the following first integral for Eq. (4.2):

$$\frac{x^2 - y^2}{y^3} = C. \tag{4.6}$$

It is interesting to compare the above calculations with the derivation of Eq. (4.6) given in [2] (p. 107) by using the traditional method. Note also that one can find in [2], p. 107, the second integrating factor for Eq. (4.1), namely $\nu = 1/(y^3 - x^2 y)$. Hence, we know, along with (4.3), the following integrating factor for Eq. (4.2):

$$\nu = \frac{y^2 - 3x^2}{y(y^2 - x^2)} \cdot \tag{4.7}$$

The functions μ and ν given by (4.3) and (4.7), respectively, are functionally independent. Proceeding as in calculating (4.5), we obtain the following potential for ν:

$$\tilde{\psi} = \ln \left| k \frac{y^3}{y^2 - x^2} \right|, \quad k = \text{const}. \tag{4.8}$$

The potentials (4.5) and (4.8) are functionally dependent (see Remark 2.2).

Example 4.2. Consider the second-order equation

$$y'' + \frac{y'^2}{y} + 3\frac{y'}{x} = 0. \tag{4.9}$$

Its two integrating factors

$$\mu_1 = xy, \quad \mu_2 = x^3 y \tag{4.10}$$

are calculated in [1], Example 2.1 (see also [2], p. 240, Example 6.6.4). They are found by looking for the integrating factors of the form $\mu(x, y)$ and solving Eqs. (2.2)-(2.3). Integrating μ_1 and μ_2 with respect to p we obtain:

$$\Psi_1 = xyp + \varphi(x, y), \quad \Psi_2 = x^3 yp + \psi(x, y). \tag{4.11}$$

The functions $\varphi(x, y)$ and $\psi(x, y)$ are determined by substituting the above expressions for Ψ_1 and Ψ_2 in Eq. (2.5) with the function $b(x, y, p)$ taken from Eq. (4.9), e.g. for Ψ_1 we have:

$$yp + \varphi_x + p(xp + \varphi_y) - \left(\frac{p^2}{y} + 3\frac{p}{x}\right)xy = 0$$

or

$$\varphi_x + (\varphi_y - 2y)p = 0.$$

It follows that $\varphi_x = 0, \varphi_y = 2y,$ and hence $\varphi = y^2 + k.$ Thus, the potential for the integrating factor μ_1 is

$$\Psi_1 = xyp + y^2 + k, \quad k = \text{const}. \tag{4.12}$$

The similar calculation for Ψ_2 from (4.11) show that $\psi = \text{const}.$ Hence:

$$\Psi_2 = x^3 yp + l, \quad l = \text{const}. \tag{4.13}$$

The potentials (4.12) and (4.13) are functionally independent. Consequently, they provide two independent first integrals (hence, the general solution to Eq. (4.9))

$$xyy' + y^2 = C_1, \quad x^3 yy = C_2$$

obtained in [1] in a different way.

Example 4.3. Consider the second-order equation

$$yy'' - y'^2 - (x + x^2)y' + (2x + 1)y = 0. \tag{4.14}$$

Looking for the integrating factors as functions of two variables, e.g. $\mu = \mu(x, y),$ and solving Eqs. (1.5)-(1.6) one can find the integrating factor

$$\mu_1 = \frac{1}{y^2}, \tag{4.15}$$

Likewise, letting $\mu = \mu(x; y');$ one can easily find the second integrating factor

$$\mu_2 = \frac{1}{(y' + x + x^2)^2} \cdot \tag{4.16}$$

If we rewrite Eq. (4.14) in the form (2.1):

$$y'' - \frac{1}{y}y'^2 - \frac{x + x^2}{y}y' + 2x + 1 = 0; \tag{4.17}$$

then (4.15) and (4.16) give rise to the integrating factors

$$\mu_1 = \frac{1}{y}; \quad \mu_2 = \frac{y}{(y' + x + x^2)^2} \tag{4.18}$$

for Eq. (4.17); see also [2], Example 6.6.5.

The integrating factors (4.18) are functionally independent, but they do not lead to different first integrals. To understand the reason, we calculate the potentials. Applying our procedure, we find the following potentials for(4.18):

$$\psi_1(x; y; p) = \frac{p + x + x^2}{y} + k; \quad \psi_2(x; y; p) = -\frac{y}{p + x + x^2} + l; \tag{4.19}$$

where $k; l = \text{const}:$ The potentials (4.19) are functionally dependent and therefore do not provide independent first integrals.

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Integration of second-order linear equations via linearization of Riccati's equations

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

Abstract. The test for linearization of the Riccati equations by a change of the dependent variable [1] is utilized for integration of second-order linear equations by quadratures.

1 Second-order equations integrable via linearization of the associated Riccati equations

The homogeneous linear equation

$$y'' + a(x)y' + b(x)y = 0 \tag{1.1}$$

is invariant under the group of dilations of the dependent variable y with the generator

$$X = y\frac{\partial}{\partial y}\cdot \tag{1.2}$$

The standard substitution

$$u = \ln|y| \tag{1.3}$$

reduces the generator (1.2) to the form

$$X = \frac{\partial}{\partial u};$$

and Equation (1.1) becomes

$$u'' + u'^2 + a(x)u' + b(x) = 0.$$

Setting here $z = u'$ we obtain the Riccati equation

$$z' + z^2 + a(x)z + b(x) = 0. \tag{1.4}$$

Provided that the integral $z = \phi(x; C_1)$ to Equation (1.4) is known, the general solution of the original equation (1.1) is obtained by quadrature:

$$u = \int \phi(x; C_1)dx + K. \tag{1.5}$$

Using Equation (1.3) and setting $K = \ln|C_2|$, we obtain from Equation (1.5) the following solution to Equation (1.1):

$$y = C_2\, e^{\int \phi(x;\, C_1)dx}. \tag{1.6}$$

According to the linearization test (see [1], Section 11.2.5, and the references therein), the Riccati equation (1.4) is linearizable by a change of the dependent variable z if and only if

$$b(x) = k[a(x) - k], \quad k = \text{const}. \tag{1.7}$$

Since in this case Equation (1.4) can be integrated by quadrature using the two-dimensional Vessiot-Guldberg-Lie algebra, we arrive at the following statement.

Theorem 1.1. Any equation (1.1) of the form

$$y'' + a(x)y' + k[a(x) - k]y = 0 \quad (k = \text{const.}) \tag{1.8}$$

with an arbitrary function $a(x)$ can be integrated by two quadratures via linearization of the auxiliary Riccati equation (1.4).

2 Examples

Example 2.1. Consider the following Riccati equation:

$$z' + z^2 + \left(1 - \frac{2}{x^2}\right) z - \frac{2}{x^2} = 0. \tag{2.1}$$

Its Vessiot-Guldberg-Lie algebra is the two-dimensional algebra spanned by

$$X_1 = z(1 + z)\frac{\partial}{\partial z}, \quad X_2 = (1 + z)\frac{\partial}{\partial z} \cdot \tag{2.2}$$

We have

$$[X_1, X_2] = -(X_1 + X_2).$$

Therefore we take the new basis

$$Z_1 = X_1 + X_2 = (1 + z)^2\frac{\partial}{\partial z}, \quad Z_2 = -X_2 = -(1 + z)\frac{\partial}{\partial z} \tag{2.3}$$

and obtain $[Z_1, Z_2] = Z_1.$ Hence a linearizing transformation $v = v(z)$ is determined by the equations

$$Z_1(v) \equiv (1+z)^2 \frac{dv}{dz} = 1, \quad Z_2(v) \equiv -(1+z)\frac{dv}{dz} = v. \tag{2.4}$$

Integrating the second equation one obtains

$$v = \frac{C}{1+z},$$

Substitution of the result into the first equation yields $C = -1.$ Thus, the linearizing transformation is

$$v = -\frac{1}{1+z}, \tag{2.5}$$

We have

$$z = -\frac{1+v}{v} \tag{2.6}$$

and Equation (2.1) becomes

$$\frac{dv}{dx} + \left(1 + \frac{2}{x^2}\right)v + 1 = 0. \tag{2.7}$$

This linear equation is readily solved and yields:

$$v = -\left[C_1 + \int e^{x-\frac{2}{x}} dx\right] e^{\frac{2}{x}-x}. \tag{2.8}$$

Substituting in (2.6), we obtain the general solution to Equation (2.1):

$$z = -1 + \frac{e^{x-\frac{2}{x}}}{C_1 + \int e^{x-\frac{2}{x}} dx}, \tag{2.9}$$

Example 2.2. Consider the equation

$$y'' + \left(1 - \frac{2}{x^2}\right) y' - \frac{2}{x^2} y = 0. \tag{2.10}$$

It has the form (1.8) with

$$k = 1, \quad a(x) = 1 - \frac{2}{x^2},$$

The corresponding Riccati equation (1.4) has the form (2.1). Substituting its solution (2.9) in Equation (1.5) we obtain:

$$u = \int \left(-1 + \frac{e^{x-\frac{2}{x}}}{C_1 + \int e^{x-\frac{2}{x}} dx} \right) dx + \ln|C_2|. \tag{2.11}$$

Denoting

$$\int e^{x-\frac{2}{x}}dx = \hat{A}(x),$$

we have

$$e^{x-\frac{2}{x}}dx = d\hat{A}$$

and write Equation (2.11) in the form

$$\ln|y| = -x + \int \frac{d\hat{A}}{C_1 + \hat{A}} + \ln|C_2| = \ln|C_2| - x + \ln|C_1 + \hat{A}|.$$

Hence, the solution to Equation (2.10) is given by

$$y = C_2\, e^{-x}\Big[C_1 + \int e^{x-\frac{2}{x}}dx\Big]. \tag{2.12}$$

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Linearization of second-order ordinary differential equations by changing the order

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden
SERGEY V. MELESHKO
School of Matematics,
Suranaree University of Technology,
Nakhon Ratchasima, 3000, Thailand

Abstract Criteria for second-order ordinary differential equations be linearizable after differentiating them or after the Ricatti substitution is given in the paper.

1 Introduction

The problem of linearization of a nonlinear ordinary differential equation has attracted a lot of attention of scientists. The first linearization problem for ordinary differential equations was solved by S.Lie [1]. He found the general form of all ordinary differential equations of second-order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients.

A different approach for tackling the equivalence problem of second-order ordinary differential equations was developed by E.Cartan [2]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form.

These approaches were also applied to third-order ordinary differential equations. Linearization problem of a third-order ordinary differential equation with respect to point transformations were studied in [3]. Complete criteria for linearization were obtained in [4, 5].

S.Lie also noted that all second-order ordinary differential equations can be transformed to each other by means of contact transformations, and that this is not so for third-order ordinary differential equations. Hence, the linearization problem by a contact transformation also becomes interesting for a third-order ordinary differential equation. This problem were studied in [6, 7, 8, 9, 10]. Solutions of the linearization problem were given in [11] and [5][1]. In explicit form the criteria for linearization is presented in [5].

The linearization problem for a third-order ordinary differential equation were also investigated with respect to a generalized Sundman transformation[2] [15, 16]. Criteria for a third-order ordinary differential equation to be equivalent to the linear equation $u''' = 0$ with respect to a generalized Sundman transformation were presented in [16].

It is known that all discussed above methods (point, contact or generalized Sundman transformations) are complementary: there are equations that can only be linearized by one of these methods.

There are attempts to tackle the linearization problem by increasing order of the equation [17, 18], reducing the order [19] or their combination [20].

This manuscript is devoted to give criteria for a second-order ordinary differential equation to be linearizable by increasing on one time the order of the equation using either differentiation of the equation or the Ricatti substitution.

In the next subsections we will recall some notations for second- and third-order ordinary differential equations which are used in the manuscript.

1.1 Second-order equations

The criteria for linearization of the equation

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 = 0 \tag{1.1}$$

is solved through the relative invariants

$$L_1 = b_{4yy} + 3b_4 b_{2y} - 2b_4 b_{1x} - 6b_3 b_{3y} + 3b_3 b_{2x} + 3b_2 b_{4y} - b_1 b_{4x} - 2b_{3xy} + b_{2xx}$$

$$L_2 = b_{3yy} + b_4 b_{1y} - 3b_3 b_{1x} - 3b_2 b_{3y} + 6b_2 b_{2x} + 2b_1 b_{4y} - 3b_1 b_{3x} - 2b_{2xy} + b_{1xx}$$

S.Lie showed that equation (1.1) is linearized if and only if

$$L_1 = 0; \qquad L_2 = 0. \tag{1.2}$$

We also use the following relative invariants of higher order given in [21, 22, 23]:

$$\begin{aligned} v_5 = {} & L_2(L_1 L_{2x} - L_2 L_{1x}) + L_1(L_2 L_{1y} - L_1 L_{2y}) \\ & - b_1 L_1^3 + 3b_2 L_1^2 L_2 - 3L_1 L_2^2 + b_4 L_2^3 \end{aligned} \tag{1.3}$$

[1]The authors of [5] discovered [11] after obtaining the results of [5]. Because the criteria for linearization of a third-order ordinary differential equation by a contact transformation given in [11] were in implicit form, it was decided to include the obtained results in [5].

[2]Generalized Sundman transformation was also applied to second-order ordinary differential equations in [12, 13, 14].

$$\begin{aligned} w_1 = L_1^{-4}\big[& -L_1^3(\Pi_{12}L_1 - \Pi_{11}L_2) + R_1(L_1^2)_x \\ & - L_1^2 R_{1x} + L_1R_1(b_3L_1 - b_4L_2)\big], \end{aligned} \tag{1.4}$$

and

$$I_2 = 3R_1L_1^{-1} + L_{2x} - L_{1y}, \tag{1.5}$$

where

$$\begin{aligned} \Pi_{11} &= 2(b_3^2 - b_2b_4) + b_{3t} - b_{4u}, \\ \Pi_{22} &= 2(b_2^2 - 3b_1b_3) + b_{1t} - b_{2u}, \\ \Pi_{12} &= b_2b_3 - b_1b_4 + b_{2t} - b_{3u}. \end{aligned} \tag{1.6}$$

$$R_1 = L_1L_{2x} - L_2L_{1x} + b_2L_1^2 - 2b_3L_1L_2 + b_4L_2^2,$$

$$I_4 = L_1I_{2y} - L_2I_{2x} + 2I_2(L_{2x} - L_{1y})$$

If $I_2 \neq 0$, then $J_4 = I_4/I_2^2$ is invariant.

1.2 Third-order equations

The following theorems are proven in [5].

Theorem 1. A linearizable with respect to a point transformation third-order ordinary differential equation has to have one of the forms: either

$$y''' + (A_1y' + A_0)y'' + B_3y'^3 + B_2y'^2 + B_1y' + B_0 = 0 \tag{1.7}$$

or

$$\begin{aligned} y''' + \frac{1}{y' + r}\Big[& -3(y'')^2 + (C_2y'^2 + C_1y' + C_0)y'' \\ & + D_5y'^5 + D_4y'^4 + D_3y'^3 + D_2y'^2 + D_1y' + D_0\Big] = 0, \end{aligned} \tag{1.8}$$

where the coefficients $r(x,y)$, $A_i(x,y)$, $B_j(x,y)$, $C_k(x, y)$, $D_l(x,y)$, $(i = 0,1;\ j = 0,1,2,3;\ k = 0,1,2;\ l = 0,1,2,3,4,5)$.

The sufficient conditions for linearizability are formulated in the next theorems.

Theorem 2. Equation (1.7) is linearizable if and only if its coefficients obey the following five equations:

$$A_{0y} - A_{1x} = 0, \quad (3B_1 - A_0^2 - 3A_{0x})_y = 0, \tag{1.9}$$

$$3A_{1x} + A_0A_1 - 3B_2 = 0, \quad 3A_{1y} + A_1^2 - 9B_3 = 0, \tag{1.10}$$

$$(9B_1 - 6A_{0x} - 2A_0^2)A_{1x} + 9(B_{1x} - A_1B_0)_y + 3B_{1y}A_0 - 27B_{0yy} = 0. \tag{1.11}$$

Theorem 3. Equation (1.8) is linearizable if and only if its coefficients obey the following equations:

$$C_0 = 6r\frac{\partial r}{\partial y} - 6\frac{\partial r}{\partial x} + rC_1 - r^2C_2, \tag{1.12}$$

$$6\frac{\partial^2 r}{\partial y^2} = \frac{\partial C_2}{\partial x} - \frac{\partial C_1}{\partial y} + r\frac{\partial C_2}{\partial y} + C_2\frac{\partial r}{\partial y}, \tag{1.13}$$

$$\begin{aligned}18D_0 = {} & 3r^2\left[r\tfrac{\partial C_1}{\partial y} - 2\tfrac{\partial C_1}{\partial x} - r\tfrac{\partial C_2}{\partial x} + 3r^2\tfrac{\partial C_2}{\partial y} - 12\tfrac{\partial^2 r}{\partial x\partial y}\right] - 54\left(\tfrac{\partial r}{\partial x}\right)^2 \\ & +6r\left[3\tfrac{\partial^2 r}{\partial x^2} + 15\tfrac{\partial r}{\partial x}\tfrac{\partial r}{\partial y} - 6r\left(\tfrac{\partial r}{\partial y}\right)^2 + (3C_1 - rC_2)\tfrac{\partial r}{\partial x}\right] \\ & +r^2\left[9(rC_2 - 2C_1)\tfrac{\partial r}{\partial y} - 2C_1^2 + 2rC_1C_2 + 4r^2C_2^2 + 18r^2D_4 - 72r^3D_5\right];\end{aligned} \tag{1.14}$$

$$\begin{aligned}18D_1 = {} & 9r^2\frac{\partial C_1}{\partial y} - 12r\frac{\partial C_1}{\partial x} - 27r^2\frac{\partial C_2}{\partial x} + 33r^3\frac{\partial C_2}{\partial y} - 36r\frac{\partial^2 r}{\partial x\partial y} \\ & +18\frac{\partial^2 r}{\partial x^2} + 6(3C_1 + 4rC_2)\frac{\partial r}{\partial x} - 3r(6C_1 + 7rC_2)\frac{\partial r}{\partial y} + 18r\left(\frac{\partial r}{\partial y}\right)^2 \\ & -18\frac{\partial r}{\partial x}\frac{\partial r}{\partial y} - 4rC_1^2 - 2r^2C_1C_2 + 20r^3C_2^2 + 72r^3D_4 - 270r^4D_5;\end{aligned} \tag{1.15}$$

$$\begin{aligned}9D_2 = {} & 3r\frac{\partial C_1}{\partial y} - 3\frac{\partial C_1}{\partial x} - 21r\frac{\partial C_2}{\partial x} + 21r^2\frac{\partial C_2}{\partial y} + 15C_2\frac{\partial r}{\partial x} \\ & -15rC_2\frac{\partial r}{\partial y} - C_1^2 - 5rC_1C_2 + 14r^2C_2^2 + 54r^2D_4 - 180r^3D_5;\end{aligned} \tag{1.16}$$

$$3D_3 = 3r\frac{\partial C_2}{\partial y} - 3\frac{\partial C_2}{\partial x} - C_1C_2 + 2rC_2^2 + 12rD_4 - 30r^2D_5; \tag{1.17}$$

$$\begin{aligned}54\frac{\partial D_4}{\partial x} = {} & 18\frac{\partial^2 C_1}{\partial y^2} + 3C_2\frac{\partial C_1}{\partial y} - 72\frac{\partial^2 C_2}{\partial x\partial y} - 39C_2\frac{\partial C_2}{\partial x} \\ & +18r\frac{\partial^2 C_2}{\partial y^2} - 3rC_2\frac{\partial C_2}{\partial y} + \left(72\frac{\partial C_2}{\partial y} + 33C_2^2\right)\frac{\partial r}{\partial y} + 108D_4\frac{\partial r}{\partial y} \\ & +270D_5\frac{\partial r}{\partial x} + 378r\frac{\partial D_5}{\partial x} - 108r^2\frac{\partial D_5}{\partial y} - 540rD_5\frac{\partial r}{\partial y} \\ & +36rC_1D_5 - 8rC_2^3 - 36rC_2D_4 + 108r^2C_2D_5 + 54rH;\end{aligned} \tag{1.18}$$

and

$$\frac{\partial H}{\partial x} = 3H\frac{\partial r}{\partial y} + r\frac{\partial H}{\partial y}, \tag{1.19}$$

where

$$\begin{aligned}H = {} & \frac{\partial D_4}{\partial y} - 2\frac{\partial D_5}{\partial x} - 3r\frac{\partial D_5}{\partial y} - 5D_5\frac{\partial r}{\partial y} - 2rC_2D_5 \\ & +\frac{1}{3}\left[\frac{\partial^2 C_2}{\partial y^2} + 2C_2\frac{\partial C_2}{\partial y} - 2C_1D_5 + 2C_2D_4\right] + \frac{4}{27}C_2^3.\end{aligned} \tag{1.20}$$

2 Linearization on Differentiation

A second-order ordinary differential equation

$$y'' + f(x, y, p) = 0 \tag{2.21}$$

after differentiating becomes

$$y''' + f_x + pf_y + y''f_y = 0, \tag{2.22}$$

where $p = y'$.

By virtue of necessary conditions for linearization of third-order differential equation by point transformations (1.7) (1.8), equation (2.22) has to be a second degree polynomial with respect to the second-order derivative y''. Hence, one has to study the problem of linearizing by point transformation the equation

$$y''' + f_x + pf_y + y''f_p + \alpha(y'' + f)^2 + (\beta + y''\gamma)(y'' + f) = 0, \tag{2.23}$$

where $\alpha(x, y, p)$, $\beta(x, y, p)$ and $\gamma(x, y, p)$ are some functions. Equation (2.23) becomes

$$y''' + y''^2(\alpha + \gamma) + y''(f_p + f(2\alpha + \gamma) + \beta) + f_x + pf_y + f(\alpha f + \beta) = 0. \tag{2.24}$$

The problem is to find r, A_i, B_j, C_k, D_l, $(i = 0, 1;\ j = 0, 1, 2, 3;\ k = 0, 1, 2;\ l = 0, 1, 2, 3, 4, 5)$ through the function f, and then satisfy the conditions for linearization of equation (2.24). These conditions depend on the value of $\alpha + \gamma$ [4, 5]: $\alpha + \gamma = 0$ or $\alpha + \gamma = -3(p + r)^{-1}$.

2.1 First case of linearization ($\alpha + \gamma = 0$)

Let $\alpha + \gamma = 0$. In this case the necessary conditions for linearization give

$$f_p + \alpha f + \beta = pA_1 + A_0, \tag{2.25}$$

$$f_x + pf_y + f(\alpha f + \beta) = p^3B_3 + p^2B_2 + pB_1 + B_0, \tag{2.26}$$

where A_i, B_j $(i = 1, 2;\ j = 1, 2, 3)$ are some functions of x and y. Hence, from (2.25) one finds $\beta = -\alpha f + pA_1 + A_0 - f_p$. Equation (2.26) becomes

$$S = p^3B_3 + p^2B_2 + pB_1 + B_0, \tag{2.27}$$

where $S = g + f(pA_1 + A_0)$ and $g = f_x + pf_y - ff_p$. Differentiating equation (2.27) four and five times with respect to p, one obtains

$$S_{pppp} = A_0 f_{pppp} + A_1(f_{pppp}p + 4f_{ppp}) + g_{pppp} = 0, \tag{2.28}$$

$$S_{ppppp} = A_0 f_{ppppp} + A_1(f_{ppppp}p + 5f_{pppp}) + g_{ppppp} = 0. \tag{2.29}$$

Equations (2.28), (2.29) compose a system of linear algebraic equations with respect to A_0 and A_1 with the determinant

$$Q = 4f_{ppppp}f_{ppp} - 5f_{pppp}^2.$$

2.1.1 Case $Q = 0$

The general solution of the equation $Q = 0$ is

$$f = \frac{h}{p+\varphi} + q_1p^3 + 3q_2p^2 + 3q_3p + q_4,$$

where $h(x,y)$, $\varphi(x,y)$ and $q_i(x,y)$, $(i = 1,2,3)$ are some functions, and $hq_1 = 0$. Then equation (2.28) leads to a polynomial of eighth degree with respect to p. The coefficient with p^8 is equal to q_1^2. Hence, $q_1 = 0$ and this equation becomes a polynomial of second degree with respect to p. Splitting this equation with respect to p, one only obtains one condition $h = 0$. Hence, the function f is a polynomial of second degree with respect to p:

$$f = 3p^2q_2 + 3pq_3 + q_4. \tag{2.30}$$

In this case

$$B_3 = 3A_1q_2 + 3(q_{2y} - 6q_2^2),$$
$$B_2 = 3A_0q_2 + 3A_1q_3 + 3(q_{2x} + q_{3y} - 9q_2q_3),$$
$$B_1 = 3A_0q_3 + A_1q_4 + 3q_{3x} + q_{4y} - 6q_2q_4 - 9q_3^2,$$
$$B_0 = A_0q_4 + q_{4x} - 3q_3q_4,$$

and conditions (1.9)-(1.11) for linearization of equation (2.24) are

$$(A_1 - 9q_2)(A_1 - 18q_2) + 3(A_1 - 9q_2)_y = 0, \tag{2.31}$$

$$(A_1 - 9q_2)(9q_3 - A_0) + 3(-A_{1x} + 3q_{2x} + 3q_{3y}) = 0, \tag{2.32}$$

$$A_{0y} - A_{1x} = 0, \tag{2.33}$$

$$\begin{array}{c} A_0^2(2A_{1x} - 9q_{3y}) + 6A_0A_1q_{4y} + 3A_0(-3A_{0y}q_3 - 9A_{1x}q_3 + 2A_{1y}q_4 \\ +6q_{2y}q_4 - 12q_{3xy} + 18q_{3y}q_3 + 8q_{4yy} + 6q_{4y}q_2) + 9A_1(A_{0y}q_4 - A_{1x}q_4 \\ -3q_{3y}q_4 - 3q_{4y}q_3) + 3(-9A_{0xy}q_3 + 2A_{0x}A_{1x} - 9A_{0x}q_{3y} + 9A_{0yy}q_4 \\ -9A_{0y}q_{3x} + 18A_{0y}q_{4y} - 3A_{1xy}q_4 - 9A_{1x}q_{3x} - 6A_{1x}q_{4y} + 18A_{1x}q_2q_4 \\ +27A_{1x}q_3^2 - 9A_{1y}q_3q_4 + 18q_{2xy}q_4 + 18q_{2x}q_{4y} + 18q_{2y}q_{4x} \\ +54q_{3xy}q_3 - 9q_{3xxy} + 54q_{3x}q_{3y} - 27q_{3yy}q_4 - 54q_{3y}q_{4y} + 6q_{4xyy} \\ +18q_{4xy}q_2 - 27q_{4yy}q_3) = 0, \end{array} \tag{2.34}$$

$$\begin{array}{c} A_0(2A_{1x} - 9q_{3y}) - 3A_1q_{4y} + 3(-3A_{0y}q_3 + A_{1xx} - A_{1y}q_4 + 6q_{2y}q_4 \\ -3q_{3xy} + 18q_{3y}q_3 - q_{4yy} + 6q_{4y}q_2) = 0. \end{array} \tag{2.35}$$

If the function $f(x,y,y')$ is given (2.30), then the coefficients $q_j(x,y)$, $(j = 0,1,2)$ are defined and the relative invariants are

$$L_1 = q_{4yy} + 3q_4q_{2y} - 6q_3q_{3y} + 3q_3q_{2x} + 3q_2q_{4y} - 2q_{3xy} + q_{2xx},$$
$$L_2 = q_{3yy} - 3q_2q_{3y} + 6q_2q_{2x} - 2q_{2xy}.$$

Equations (2.31)-(2.35) except the functions $q_j(x,y)$, $(j = 0,1,2)$ include the unknown coefficients $A_0(x,y)$ and $A_1(x,y)$. In the process of obtaining involutive conditions for the overdetermined system of equations (2.31)-(2.35) these coefficients A_0 and A_1 are defined through the function f, and necessary and sufficient conditions for linearization of equation (2.21) are derived. These conditions can be written through the relative invariants L_1, L_2, v_5 and w_1 of the second-order equation (2.21) with (2.30).

Theorem 4. Nonlinearizable equation (2.21) with (2.30) is linearizable on differentiation if and only if

$$L_2 = 0, \quad q_{3y} = 2q_{2x}, \tag{2.36}$$

$$5L_1L_{1yy} = 3(2L_{1y}^2 + 3L_{1y}L_1q_2 + 5q_{2y}L_1^2 - 12L_1^2q_2^2), \tag{2.37}$$

$$5L_1L_{1xy} = 3(2L_{1x}L_{1y} - L_{1x}L_1q_2 + 2L_{1y}L_1q_3 + 5q_{2x}L_1^2 - 6L_1^2q_2q_3), \tag{2.38}$$

$$\begin{array}{c} 3(L_{1y} - 3L_1q_2)(5L_1L_{1xx} - 6L_{1x}^2 + 3L_1q_3L_{1x} - 5L_1q_4L_{1y} \\ +30q_{3x}L_1^2 - 25q_{4y}L_1^2 - 60L_1^2q_2q_4 + 54L_1^2q_3^2) + 125L_1^4 = 0. \end{array} \tag{2.39}$$

Remark. Since it is assumed that equation (2.21) is nonlinearizable and $L_2 = 0$, then the relative invariant $L_1 \neq 0$. Notice also that conditions (2.36)-(2.39) lead to $I_2 = -(L_{1y} - 3L_1q_2) \neq 0$, the invariant $J_4 = 4/5$, the relative invariants $w_1 = 0, \; v_5 = 0$. The coefficients

$$A_0 = 3(-L_{1x} + 9L_1q_3)/(5L_1), \quad A_1 = 3(-L_{1y} + 18L_1q_2)/(5L_1).$$

Remark. If conditions (2.36)-(2.39) are satisfied then equation (2.24) can be transformed by the mapping $t = \varphi(x), \quad u = \psi(x,y)$ into the equation is $u''' = 0$. These conditions also guarantee that equation (2.24) can be linearizable by contact and generalized Sundman transformations.

Proof

Assume that $A_1 = 9q_2$, then equation (2.32) gives $q_{3y} = 2q_{2x}$. By virtue of (2.33) and (2.35) this leads to $L_1 = 0$ and $L_2 = 0$. Hence, one concludes that $\mu_2 = A_1 - 9q_2 \neq 0$.

From equations(2.31), (2.33) and (2.32) one finds

$$A_{1y} = (-A_1^2 + 27A_1q_2 + 27(q_{2y} - 6q_2^2))/3,$$

$$A_{0y} = (-A_0A_1 + 9A_0q_2 + 9A_1q_3 + 3(5q_{2x} + 3\mu - 27q_2q_3))/3,$$

$$A_{1x} = A_{0y}, \tag{2.40}$$

where $\mu = q_{3y} - 2q_{2x}/3$. Notice that

$$\begin{array}{c} L_1 = (-6\mu_x - q_{2xx} - 3q_{2x}q_3 + 9q_{2y}q_4 + 3q_{4yy} + 9q_{4y}q_2 - 18\mu q_3)/3, \\ L_2 = (3\mu_y - 4q_{2xy} + 12q_{2x}q_2 - 9\mu q_2)/3 \end{array}$$

or

$$\mu_x = (-q_{2xx} - 3q_{2x}q_3 + 9q_{2y}q_4 + 3q_{4yy} + 9q_{4y}q_2 - 3L_1 - 18\mu q_3)/6, \tag{2.41}$$

$$\mu_y = (4q_{2xy} - 12q_{2x}q_2 + 3L_2 + 9\mu q_2)/3. \tag{2.42}$$

Taking mixed derivatives $(A_{1x})_y - (A_{1y})_x = 0$, one obtains

$$4(A_1 - 9q_2)(3\mu - 4q_{2x}) + L_2 = 0. \tag{2.43}$$

Since $\mu_2 \neq 0$, equation (2.43) gives

$$\mu = (16q_{2x}\mu_2 - 9L_2)/(12\mu_2).$$

Substituting μ into (2.41) and (2.42), one finds from these equations

$$q_{4yy} = (-4A_0L_2\mu_2^2 - 12L_{2x}\mu_2^2 + 24q_{2xx}\mu_2^3 + 72q_{2x}\mu_2^3q_3 - 24q_{2y}\mu_2^3q_4 - 24q_{4y}\mu_2^3q_2 + 8L_1\mu_2^3 - 27L_2^2)/(8\mu_2^3);$$

$$L_{2y} = L_2(-5\mu_2 + 18q_2)/3.$$

Equation (2.35) gives

$$A_{0x} = (-8A_0^2\mu_2^4 + 72A_0\mu_2^2(2L_2 + \mu_2^2q_3) + 3(36L_{2x}\mu_2^2 + 72q_{3x}\mu_2^4 - 24q_{4y}\mu_2^4 - 24L_1\mu_2^3 + 81L_2^2 - 324L_2\mu_2^2q_3 + 8\mu_2^5q_4 - 72\mu_2^4q_2q_4))/(24\mu_2^4). \tag{2.44}$$

Composing the equation $(A_{0x})_y - (A_{0y})_x = 0$, one has

$$L_{2x} = (-40A_0L_2\mu_2^2 - 24L_{1y}\mu_2^2 - 40L_1\mu_2^3 + 72L_1\mu_2^2q_2 + 135L_2^2 + 288L_2\mu_2^2q_3)/(24\mu_2^2). \tag{2.45}$$

Taking the mixed derivatives $(L_{2x})_y - (L_{2y})_x = 0$, one finds

$$L_{1yy} = (-240L_{1y}\mu_2^2 + 648L_{1y}\mu_2q_2 + 216q_{2x}L_2\mu_2 + 216q_{2y}L_1\mu_2 - 160L_1\mu_2^3 + 720L_1\mu_2^2q_2 - 1296L_1\mu_2q_2^2 - 1053L_2^2)/(72\mu_2).$$

Equation (2.34) defines

$$L_{1xy} = (10A_0\mu_2^2(-8L_{1y}\mu_2^2 - 32L_1\mu_2^3 + 24L_1\mu_2^2q_2 - 21L_2^2) - 208L_{1x}\mu_2^5 + 144L_{1x}\mu_2^4q_2 - 360L_{1y}L_2\mu_2^2 + 432L_{1y}\mu_2^4q_3 + 144q_{2x}L_1\mu_2^4 + 48q_{4y}L_2\mu_2^4 - 84L_1L_2\mu_2^3 + 1080L_1L_2\mu_2^2q_2 + 1632L_1\mu_2^5q_3 - 1296L_1\mu_2^4q_2q_3 + 2835L_2^3 + 1890L_2^2\mu_2^2q_3 + 128L_2\mu_2^5q_4 + 144L_2\mu_2^4q_2q_4)/(48\mu_2^4). \tag{2.46}$$

The equation $(L_{1xy})_y - (L_{1yy})_x = 0$ gives

$$L_{1x} = (-40A_0L_1\mu_2^2 + 3(45L_1L_2 + 72L_1\mu_2^2q_3 + 8L_2\mu_2^2q_4))/(24\mu_2^2). \tag{2.47}$$

Substituting L_{1x} into (2.47), it becomes

$$L_2(-70A_0L_2\mu_2^2 + 35(-6L_{1y}\mu_2^2 - 8L_1\mu_2^3 + 18L_1\mu_2^2q_2 + 27L_2^2 + 18L_2\mu_2^2q_3)) = 0.$$

Assume that $L_2 \neq 0$. Then from the last equation one can find A_0

$$A_0 = (-6L_{1y}\mu_2^2 - 8L_1\mu_2^3 + 18L_1\mu_2^2 q_2 + 27L_2^2 + 18L_2\mu_2^2 q_3)/(2L_2\mu_2^2).$$

Substituting A_0 into (2.44), one obtains the contradiction $L_2 = 0$. Hence, $L_2 = 0$ and $L_1 \neq 0$.

From equation (2.45) one finds

$$\mu_2 = 3(-L_{1y} + 3L_1 q_2)/(5L_1).$$

From equation (2.47) one finds

$$A_0 = 3(-L_{1x} + 9L_1 q_3)/(5L_1).$$

Substitution A_0 into equation (2.44) gives (2.39).

2.1.2 Case $Q \neq 0$

If $Q \neq 0$, then solving the linear algebraic system of equations (2.28), (2.29), one gets

$$A_0 = Q^{-1}(g_{ppppp}(-f_{pppp}p - 4f_{ppp}) + g_{pppp}(f_{ppppp}p + 5f_{pppp}));$$
$$A_1 = Q^{-1}(g_{ppppp}f_{pppp} - g_{pppp}f_{ppppp}). \tag{2.48}$$

Since $A_{0p} = 0$ and $A_{1p} = 0$, then $A_{0p} + pA_{1p} = 0$. From this equation one finds

$$g_{pppppp} = Q^{-1}(2g_{ppppp}(2f_{pppppp}f_{ppp} - 3f_{ppppp}f_{pppp}) + g_{pppp}(-5f_{pppppp}f_{pppp} + 6f_{ppppp}^2)) \tag{2.49}$$

Condition (2.49) provides that

$$A_{0p} = 0, \quad A_{1p} = 0.$$

The function S becomes

$$S = Q^{-1}(-4g_{ppppp}f_{ppp}f + 5g_{pppp}f_{pppp}f + g(4f_{ppppp}f_{ppp} - 5f_{pppp}^2)).$$

Condition (2.49) also guarantees that $S_{pppp} = 0$, which means the function $S(x, y, p)$ is a polynomial of third degree with respect to p. Hence, the other coefficients B_j, $(j = 0, 1, 2, 3)$ are found by the relations

$$B_3 = S_{ppp}/6, \ B_2 = (S - B_3p^3)_{pp}/2, \ B_1 = (S - B_3p^3 - B_2p^2)_p,$$
$$B_0 = S - B_3p^3 - B_2p^2 - B_1p. \tag{2.50}$$

For linearizable equation (2.23) the coefficients $A_i, B_j, (i = 0, 1; \ j = 0, 1, 2, 3)$ have to satisfy conditions (1.9)-(1.11).

Theorem 5. Equation (2.21) with $Q \neq 0$ and satisfying (2.49) is linearizable on differentiation if and only if the coefficients $A_i, B_j, (i = 0, 1; \ j = 0, 1, 2, 3)$ defined by (2.48), (2.50) satisfy conditions (1.9)-(1.11).

Remark. Since $Q = 4f_{ppppp}f_{ppp} - 5f_{pppp}^2 \neq 0$, then $f_{pppp} \neq 0$. Hence, the second-order ordinary differential equation $y'' + f(x, y, y') = 0$ is not linearizable by a point transformation.

3 Second case of linearization on differentiation

Let $\alpha + \gamma = -\frac{3}{p+r}$. In this case the necessary conditions for linearization give

$$(p + r)(\alpha f + \beta) = 3f - f_p(p + r) + p^2C_2 + pC_1 + C_0, \tag{3.51}$$

$$\begin{gathered}(p + r)(f_x + pf_y - ff_p) + f(p^2C_2 + pC_1 + C_0) + 3f^2 \\ = p^5D_5 + p^4D_4 + p^3D_3 + p^2D_2 + pD_1 + D_0,\end{gathered} \tag{3.52}$$

where r, C_i, D_j, $(i = 1,2,3;\ j = 1,2,3,4,5)$ are some functions of x and y.

The problem is for a given function $f(x,y,p)$ to find the coefficients r, C_i, D_j, $(i = 1,2,3;\ j = 1,2,3,4,5)$ from equation (3.51), (3.52). If the coefficients are found, then one has to substitute them into the linearization conditions (1.12)-(1.19).

Theorem 6. If the coefficients r, C_i, D_j, $(i = 1,2,3;\ j = 1,2,3,4,5)$ cannot be found from equation (3.52), then the function $f(x,y,p)$ has one of the representations:

$$f = \frac{\sum_{j=0}^{5} p^j h_j}{\sum_{i=0}^{2} p^i q_i}, \tag{3.53}$$

$$f = \sum_{j=0}^{3} p^i q_i + \sqrt{\sum_{j=0}^{6} p^j h_j} \tag{3.54}$$

where $a_k(x,y),\ q_i(x,y),\ h_j(x,y)$, $(k = 1,2,3,4;\ i = 0,1,2,3;\ j = 0,1,2,\dots,6)$.

Proof

Let us study the functions $f(x,y,p)$ such that the coefficients r, C_i, D_j, $(i = 1,2,3;\ j = 1,2,3,4,5)$ cannot be found from equation (3.52). As before let us define $g = f_x + pf_y - ff_p$. Equation (3.51) becomes

$$C_0 + pC_1 + p^2C_2 = \alpha_6 r + \sum_{i=0}^{5} \alpha_i D_i - (3f^2 + pg)/f, \tag{3.55}$$

where

$$\alpha_6 = -\frac{g}{f},\ \alpha_i = \frac{p^i}{f},\ (i = 0,1,\dots,5).$$

Since r, C_i, D_j, $(i = 1,2,3;\ j = 1,2,3,4,5)$ do not depend on p, differentiating (3.55) with respect to p, one obtains a linear system of algebraic equations with respect to these coefficients. For example,

$$C_0 = -pC_1 - p^2C_2 + \alpha_6 r + \sum_{i=0}^{5} \alpha_i D_i - (3f^2 + pg)/f.$$

Differentiating C_0 with respect to p, one finds

$$C_1 = -2pC_2 + \alpha_6' r + \sum_{i=0}^{5} \alpha_i' D_i - \left((3f^2 + pg)/f\right)',$$

where prime means a derivative with respect to p: $\alpha_i' = \partial\alpha_i/\partial p$, $(i = 1, 2, \ldots, 6)$. Similar, one gets

$$2C_2 = \alpha_6'' r + \sum_{i=0}^{5} \alpha_i'' D_i - \left((3f^2 + pg)/f\right)''$$

and

$$\alpha_6''' r + \sum_{i=0}^{5} \alpha_i''' D_i = \left((3f^2 + pg)/f\right)'''. \tag{3.56}$$

If at least one of $\alpha_i''' = 0$, then the corresponding coefficient cannot be defined. If all $\alpha_i''' \neq 0$, then one can proceed the process of obtaining the coefficients. For example,

$$r = -\sum_{i=0}^{5} (\alpha_6''')^{-1} \alpha_i''' D_i + (\alpha_6''')^{-1} \left((3f^2 + pg)/f\right)''';$$

and

$$\sum_{i=0}^{5} \left((\alpha_6''')^{-1} \alpha_i'''\right)' D_i - \left((\alpha_6''')^{-1} \left((3f^2 + pg)/f\right)'''\right)' = 0.$$

Again, if one of the coefficients $((\alpha_6''')^{-1}\alpha_i''')' = 0$, then the coefficients D_i cannot be defined. The equation $((\alpha_6''')^{-1}\alpha_i''')' = 0$ means that there is a function $h(x, y)$ such that $\alpha_i''' + h\alpha_6''' = 0$ or

$$\alpha_i + h\alpha_6 = q_0 + pq_1 + p^2 q_2,$$

where $q_j(x, y)$, $(j = 0, 1, 2)$ some functions. Proceeding one obtains that the coefficients r, D_j, $(j = 1, 2, 3, 4, 5)$ cannot be defined only if there is a set of functions $h_j(x, y)$, $(j = 0, 1, \ldots, 5)$ and $h(x, y)$ such that

$$\sum_{j=0}^{5} \alpha_j h_j + h\alpha_6 = q_0 + pq_1 + p^2 q_2. \tag{3.57}$$

If $h = 0$, then after substituting α_j one obtains the representation (3.53) for the function f:

$$f = \frac{\sum_{j=0}^{5} p^j h_j}{q_0 + pq_1 + p^2 q_2}.$$

If $h \neq 0$, then without loss of generality one can assume that $h = 1$. In this case after substituting α_j one obtains

$$g = f_x + pf_y - ff_p = \sum_{j=0}^{5} p^j h_j - f(q_0 + pq_1 + p^2 q_2). \qquad (3.58)$$

Equation (3.52) becomes

$$3f^2 + fH_3 + H_6 = 0, \qquad (3.59)$$

where H_3 and H_6 are polynomial with respect to p degree three and six, correspondingly:

$$H_3 = C_0 + pC_1 + p^2 C_2 - (p + r)(q_0 + pq_1 + p^2 q_2),$$
$$H_6 = \sum_{j=0}^{5} p^j \left((p + r)h_j - D_j\right).$$

$$3f^2 + f\left(C_0 + pC_1 + p^2 C_2 - (p + r)(q_0 + pq_1 + p^2 q_2)\right)$$
$$+ \sum_{j=0}^{5} p^j \left((p + r)h_j - D_j\right) = 0.$$

In this case the function $f(x, y, p)$ has the representation (3.54).

3.1 Case (3.53)

Assume that the function $f(x, y, p)$ has the representation (3.53). Substituting the function $f(x, y, p)$ into (3.52) and splitting this equations with respect to p, one obtains 12 equations. One has to also satisfy the linearizability conditions (1.12)-(1.19). Compatibility analysis[3] of this overdetermined system of partial differential equations gives that these equations can be compatible only if the function f is a third degree polynomial with respect to p

$$f = a_1 p^3 + 3a_2 p^2 + 3a_3 p + a_4. \qquad (3.60)$$

For this case

$$D_5 = a_{1y} - 3a_1^2 r + 3a_1 a_2 + a_1 C_2,$$

$$D_4 = a_{1x} + a_{1y} r + 3a_{2y} - 15a_1 a_2 r + 6a_1 a_3 + a_1 C_1 + 9a_2^2 + 3a_2 C_2,$$

$$D_3 = a_{1x} r + 3a_{2x} + 3a_{2y} r + 3a_{3y} - 12a_1 a_3 r + 3a_1 a_4 + a_1 C_0$$
$$- 18a_2^2 r + 27a_2 a_3 + 3a_2 C_1 + 3a_3 C_2,$$

$$D_2 = 3a_{2x} r + 3a_{3x} + 3a_{3y} r + a_{4y} - 3a_1 a_4 r - 27a_2 a_3 r + 12a_2 a_4$$
$$+ 3a_2 C_0 + 18a_3^2 + 3a_3 C_1 + a_4 C_2,$$

$$D_1 = 3a_{3x} r + a_{4x} + a_{4y} r - 6a_2 a_4 r - 9a_3^2 r + 15a_3 a_4 + 3a_3 C_0 + a_4 C_1,$$

$$D_0 = a_{4x} r - 3a_3 a_4 r + 3a_4^2 + a_4 C_0.$$

[3] For computations we used Reduce [24]. These computations are cumbersome and omitted here.

From (1.12) and (1.13) one finds

$$
\begin{aligned}
r_x &= (6r_y r - C_0 + C_1 r - C_2 r^2)/6;\\
r_{yy} &= (-C_{1y} + C_{2x} + C_{2y} r + r_y C_2)/6.
\end{aligned}
\tag{3.61}
$$

The equation $(r_x)_{yy} - (r_{yy})_x = 0$ becomes

$$
\begin{gathered}
6C_{0yy} - C_{0y}C_2 - 6C_{1xy} + 6C_{1y}r_y + C_{1y}C_1 - C_{1y}C_2 r + 6C_{2xx}\\
-12C_{2x}r_y - C_{2x}C_1 + 2C_{2x}C_2 r - C_{2y}C_0 = 0.
\end{gathered}
\tag{3.62}
$$

From equations (1.15)-(1.17) one finds

$$
\begin{gathered}
C_{0x} = (-180a_{1x}r^3 + 360a_{1y}r^4 + 162a_{2x}r^2 - 540a_{2y}r^3 + 54a_{3x}r\\
+162a_{3y}r^2 - 36a_{4x} + 18a_{4y}r + 6C_{0y}r + 60C_{2x}r^2 - 60C_{2y}r^3\\
-1620a_1^2r^5 + 4320a_1a_2r^4 - 1080a_1a_3r^3 - 162a_1a_4r^2 - 180a_1C_1r^3\\
+540a_1C_2r^4 - 1620a_2^2r^3 - 1458a_2a_3r^2 + 864a_2a_4r + 162a_2C_0r\\
-540a_2C_2r^3 + 1296a_3^2r - 540a_3a_4 - 108a_3C_0 + 162a_3C_1r - 36a_4C_1\\
+54a_4C_2r - 7C_0C_1 + 9C_0C_2r + 5C_1^2r + 10C_1C_2r^2 - 35C_2^2r^3)/6;
\end{gathered}
\tag{3.63}
$$

$$
\begin{gathered}
C_{1x} = (108a_{1x}r^2 - 252a_{1y}r^3 - 54a_{2x}r + 324a_{2y}r^2 - 54a_{3x} - 54a_{3y}r\\
-18a_{4y} + 6C_{1y}r - 42C_{2x}r + 42C_{2y}r^2 + 1080a_1^2r^4 - 2700a_1a_2r^3\\
+648a_1a_3r^2 + 54a_1a_4r + 108a_1C_1r^2 - 360a_1C_2r^3 + 972a_2^2r^2\\
+486a_2a_3r - 216a_2a_4 - 54a_2C_0 + 324a_2C_2r^2 - 324a_3^2 - 54a_3C_1\\
-18a_4C_2 - 5C_0C_2 - 2C_1^2 - 5C_1C_2r + 23C_2^2r^2)/6;
\end{gathered}
$$

$$
\begin{gathered}
C_{2x} = (9a_{1x}r - 18a_{1y}r^2 - 9a_{2x} + 27a_{2y}r - 9a_{3y} + 3C_{2y}r + 90a_1^2r^3\\
-270a_1a_2r^2 + 108a_1a_3r - 9a_1a_4 - 3a_1C_0 + 12a_1C_1r - 30a_1C_2r^2\\
+162a_2^2r - 81a_2a_3 - 9a_2C_1 + 36a_2C_2r - 9a_3C_2 - C_1C_2 + 2C_2^2r)/3.
\end{gathered}
$$

From equation (1.14) one finds

$$
C_0 = 6a_1r^3 - 18a_2r^2 + 18a_3r - 6a_4 + C_1r - C_2r^2.
$$

One can check that equation (3.63) is satisfied.

Let us introduce

$$
Q = 3r_y + 9a_1r^2 - 18a_2r + 9a_3 + C_1 - 2C_2r
$$

or

$$
r_y = (-9a_1r^2 + 18a_2r - 9a_3 - C_1 + 2C_2r + Q)/3 \tag{3.64}
$$

Then equation (3.61) becomes

$$
\begin{gathered}
6Q_y = 9a_{1x}r + 36a_{1y}r^2 - 9a_{2x} - 81a_{2y}r + 45a_{3y} + 3C_{1y}\\
-6C_{2y}r - 252a_1^2r^3 + 756a_1a_2r^2 - 270a_1a_3r + 9a_1a_4 - 27a_1C_1r\\
+72a_1C_2r^2 + 36a_1Qr - 486a_2^2r + 243a_2a_3 + 27a_2C_1 - 90a_2C_2r\\
-36a_2Q + 18a_3C_2 + 2C_1C_2 - 4C_2^2r - 3C_2Q.
\end{gathered}
\tag{3.65}
$$

If $Q = 0$, then equation (3.65) gives

$$
\begin{gathered}
C_{1y} = (-9a_{1x}r - 36a_{1y}r^2 + 9a_{2x} + 81a_{2y}r - 45a_{3y} + 6C_{2y}r + 252a_1^2r^3 \\
- 756a_1a_2r^2 + 270a_1a_3r - 9a_1a_4 + 27a_1C_1r - 72a_1C_2r^2 + 486a_2^2r \\
- 243a_2a_3 - 27a_2C_1 + 90a_2C_2r - 18a_3C_2 - 2C_1C_2 + 4C_2^2r)/3.
\end{gathered}
$$

Equations (3.62) and (1.18) lead to the contradiction $L_1 = 0$, $L_2 = 0$. Hence, one obtains that $Q \neq 0$.

Assuming that $Q \neq 0$, one finds from equation (3.62)

$$
\begin{gathered}
C_{1y} = (-18a_{1x}Qr - 72a_{1y}Qr^2 + 18a_{2x}Q + 162a_{2y}Qr - 90a_{3y}Q + 12C_{2y}Qr \\
+504a_1^2Qr^3 - 1512a_1a_2Qr^2 + 540a_1a_3Qr - 18a_1a_4Q + 54a_1C_1Qr \\
-144a_1C_2Qr^2 - 36a_1Q^2r + 972a_2^2Qr - 486a_2a_3Q - 54a_2C_1Q + 180a_2C_2Qr \\
+36a_2Q^2 - 36a_3C_2Q - 4C_1C_2Q + 8C_2^2Qr + 2C_2Q^2 + 27L_1 - 27L_2r)/(6Q).
\end{gathered}
$$

The equation $(r_x)_y - (r_y)_x = 0$ gives

$$
\begin{gathered}
12QQ_x = Q^2(-72a_1r^2 + 180a_2r - 108a_3 \\
-8C_1 + 12C_2r + 4Q) + 27r(L_1 - L_2r).
\end{gathered}
$$

From the equation $(Q_x)_y - (Q_y)_x = 0$ one obtains

$$
\begin{gathered}
3L_{1x} = 3L_{1y}r + 3L_{2x}r - 3L_{2y}r^2 - 63a_1L_1r^2 + 60a_1L_2r^3 + 126a_2L_1r \\
-117a_2L_2r^2 - 63a_3L_1 + 54a_3L_2r + 3a_4L_2 - 5(L_1 - L_2r)(c_1 - 2c_2r).
\end{gathered}
$$

The equation $(c_{1y})_x - (c_{1x})_y = 0$ is satisfied. From equation (1.18) one finds C_{2y}. Composing $(c_{2y})_x - (c_{2x})_y = 0$, one obtains

$$
\begin{gathered}
L_{1y} = (-24L_{2x}Q^2 + 48L_{2y}Q^2r + 288a_1L_1Q^2r - 720a_1L_2Q^2r^2 \\
-288a_2L_1Q^2 + 1152a_2L_2Q^2r - 432a_3L_2Q^2 - 40C_1L_2Q^2 - 40C_2L_1Q^2 \\
+120C_2L_2Q^2r - 135L_1^2 + 270L_1L_2r - 135L_2^2r^2)/(24Q^2).
\end{gathered}
$$

From the equations $(L_{1y})_x - (L_{1x})_y = 0$ and (1.19) one can find L_{2xx} and L_{2xy}, respectively. Then the equation $(L_{2xx})_y - (L_{2xy})_x = 0$ gives

$$
\begin{gathered}
L_{2y} = (8Q^2(-3a_1L_1 + 30a_1L_2r - 27a_2L_2 - 5C_2L_2) \\
-135L_2(L_1 - L_2r))/(24Q^2).
\end{gathered}
$$

The equation $L_{2xy} - (L_{2y})_x = 0$ becomes

$$
\begin{gathered}
(L_1 - L_2r)(8Q^2(3L_{2x} - 6a_1L_1r + 24a_1L_2r^2 + 9a_2L_1 \\
-72a_2L_2r + 45a_3L_2 + 4C_1L_2 + C_2L_1 - 4C_2L_2r) \\
+27(4L_1^2 - 3L_1L_2r - L_2^2r^2)) = 0
\end{gathered}
\tag{3.66}
$$

The assumption

$$L_1 - L_2r \neq 0$$

leads to a contradiction. In fact, from equation (3.66) one can find

$$\begin{gathered}3L_{2x} = -(24a_1L_2r^2 - 6a_1L_1r + 9a_2L_1 - 72a_2L_2r + 45a_3L_2\\ +4C_1L_2 + C_2L_1 - 4C_2L_2r) - 27(4L_1^2 - 3L_1L_2r - L_2^2r^2)/8Q^2\end{gathered}$$

After that the equations $L_{2xx} - (L_{2x})_x = 0$ and $L_{2xy} - (L_{2y})_x = 0$ give $L_1 = 0$ and $L_2 = 0$. Hence,

$$L_1 - L_2r = 0.$$

Notice also that $L_2 \neq 0$. Thus,

$$r = L_1/L_2$$

Substituting the value of r into (3.64) one obtains

$$L_{2x} = (-18a_1L_1^2 + 63a_2L_1L_2 - 45a_3L_2^2 - 4C_1L_2^2 + 3C_2L_1L_2 - L_2^2Q)/(3L_2).$$

Thus, one obtains

$$\begin{gathered}3L_2^4QC_{1x} = 45a_{2x}L_1L_2^3Q - 27a_{3x}L_2^4Q - 9a_{3y}L_1L_2^3Q - 9a_{4y}L_2^4Q\\ +72a_1^2L_1^4Q - 324a_1a_2L_1^3L_2Q - 45a_1a_4L_1L_2^3Q - 24a_1C_2L_1^3L_2Q\\ -6a_1L_1^2L_2^2Q^2 + 324a_2^2L_1^2L_2^2Q + 81a_2a_3L_1L_2^3Q + 54a_2a_4L_2^4Q\\ +9a_2C_1L_1L_2^3Q + 54a_2C_2L_1^2L_2^2Q + 18a_2L_1L_2^3Q^2 - 162a_3^2L_2^4Q\\ -27a_3C_1L_2^4Q + 6a_4C_2L_2^4Q - C_1^2L_2^4Q\\ +2C_2^2L_1^2L_2^2Q + C_2L_1L_2^3Q^2 + 18L_1^2L_2^3,\end{gathered} \tag{3.67}$$

$$\begin{gathered}3L_2^3QC_{1y} = 9a_{1x}L_1L_2^2Q + 9a_{2x}L_2^3Q + 27a_{2y}L_1L_2^2Q - 45a_{3y}L_2^3Q\\ +36a_1^2L_1^3Q - 324a_1a_2L_1^2L_2Q + 54a_1a_3L_1L_2^2Q - 9a_1a_4L_2^3Q\\ +9a_1C_1L_1L_2^2Q - 18a_1C_2L_1^2L_2Q - 6a_1L_1L_2^2Q^2 + 486a_2^2L_1L_2^2Q\\ -243a_2a_3L_2^3Q - 27a_2C_1L_2^3Q + 72a_2C_2L_1L_2^2Q + 18a_2L_2^3Q^2\\ -18a_3C_2L_2^3Q - 2C_1C_2L_2^3Q + 2C_2^2L_1L_2^2Q + C_2L_2^3Q^2 + 18L_1L_2^3,\end{gathered} \tag{3.68}$$

$$\begin{gathered}3L_2^3QC_{2x} = 18a_{1x}L_1L_2^2Q - 9a_{2x}L_2^3Q - 9a_{3y}L_2^3Q - 36a_1^2L_1^3Q\\ -54a_1a_3L_1L_2^2Q + 9a_1a_4L_2^3Q + 6a_1L_1L_2^2Q^2 + 162a_2^2L_1L_2^2Q\\ -81a_2a_3L_2^3Q - 9a_2C_1L_2^3Q + 27a_2C_2L_1L_2^2Q - 9a_3C_2L_2^3Q\\ -C_1C_2L_2^3Q + C_2^2L_1L_2^2Q + 9L_1L_2^3,\end{gathered} \tag{3.69}$$

$$\begin{gathered}3L_2^2QC_{2y} = 9a_{1x}L_2^2Q + 18a_{1y}L_1L_2Q - 27a_{2y}L_2^2Q - 108a_1^2L_1^2Q\\ +216a_1a_2L_1L_2Q - 108a_1a_3L_2^2Q - 9a_1C_1L_2^2Q + 27a_1C_2L_1L_2Q\\ +6a_1L_2^2Q^2 - 9a_2C_2L_2^2Q - C_2^2L_2^2Q + 9L_2^3,\end{gathered} \tag{3.70}$$

$$\begin{gathered}3L_2^2Q_x = Q(L_2^2Q - 18a_1L_1^2 + 45a_2L_1L_2\\ -27a_3L_2^2 - 2C_1L_2^2 + 3C_2L_1L_2),\end{gathered} \tag{3.71}$$

$$3L_2Q_y = Q(9a_1L_1 - 9a_2L_2 - C_2L_2), \tag{3.72}$$

$$\begin{gathered}3L_2^2L_{1x} = -30a_1L_1^3 + 90a_2L_1^2L_2 - 63a_3L_1L_2^2\\ +3a_4L_2^3 - 5C_1L_1L_2^2 + 5C_2L_1^2L_2,\end{gathered} \tag{3.73}$$

$$3L_2L_{1y} = 18a_1L_1^2 - 9a_2L_1L_2 - 9a_3L_2^2 - C_1L_2^2 - 3C_2L_1L_2 + L_2^2Q; \tag{3.74}$$

$$3L_2L_{2x} = -18a_1L_1^2 + 63a_2L_1L_2 - 45a_3L_2^2 - 4C_1L_2^2 + 3C_2L_1L_2 - L_2^2Q; \tag{3.75}$$

$$3L_{2y} = 27a_1L_1 - 27a_2L_2 - 5C_2L_2; \tag{3.76}$$

and

$$r = L_1/L_2; \quad C_0 = (6a_1L_1^3 - 18a_2L_1^2L_2 + 18a_3L_1L_2^2 - 6a_4L_2^3 + C_1L_1L_2^2 - C_2L_1^2L_2)/L_2^3 \tag{3.77}$$

All mixed derivatives are satisfied. Here equations (3.74), (3.75) and (3.76) compose a linear system of equations with respect to $C_1; C_2$ and Q. Solving these equations for $C_1; C_2$ and Q:

$$5C_1L_2 = 54a_2L_1 - 54a_3L_2 - 3(L_{1y} + L_{2x}) \tag{3.78}$$

$$5C_2L_2 = 27a_1L_1 - 27a_2L_2 - 3L_{2y} \tag{3.79}$$

$$5QL_2^2 = 3(4L_2L_{1y} - L_2L_{2x} - 3L_1L_{2y} - 3a_1L_1^2 + 6a_2L_1L_2 - 3a_3L_2^2) \tag{3.80}$$

and substituting them into equations (3.67)-(3.73) one obtains sufficient conditions for linearization of equation (3.60).

Remark. All coefficients r, C_i, D_j, $(i = 1; 2; 3;\ j = 1; 2; 3; 4; 5)$ are found through the function $f(x; y; p)$.

Remark. Obtained conditions guarantee that the relative invariants $v_5 = 0$, $w_1 = 0$, and the invariant $J_4 = 4/5$. Notice that the third-order equation is linearized to $u''' = 0$. It is also linearized by a contact transformation. It cannot be linearized by the generalized Sundman transformation.

Theorem 7. A second-order equation $y'' + f(x; y; y') = 0$ with

$$f(x; y; p) = p^3a_1(x; y) + 3p^2a_2(x; y) + 3pa_3(x; y) + a_4(x; y)$$

and $L_2 \neq 0$ is linearizable by differentiating if the coefficients $a_i(x; y);\ (i = 1; 2; 3; 4)$ satisfy equations (3.67)-(3.73), where the functions $C_i(x; y);\ (i = 0; 1; 2);\ r(x; y)$ and $Q(x; y)$ are defined by formulae (3.77)-(3.80).

Remark. Under a point transformation the quantities L_1 and L_2 are transformed as follows [21]:

$$\tilde{L}_1 = \Delta(L_1\varphi_x + L_2\psi_x); \quad \tilde{L}_2 = \Delta(L_1\varphi_y + L_2\psi_y); \tag{3.81}$$

where Δ is the Jacobian of the change. If $L_1 = 0$ and $L_2 \neq 0$, then the change

$$t = y; \ u = x$$

leads to the case $L_1 \neq 0$ and $L_2 = 0$. Applying a change of the variables satisfying the condition

$$L_1\varphi_y + L_2\psi_y = 0 \tag{3.82}$$

any nonlinearizable equation (3.60) is transformed to an equation of the same form with (3.60) with $L_2 = 0$.

3.2 Case of f is (3.54)

Assume that the function $f(x,y,p)$ has the representation (3.54):

$$f = \sum_{j=0}^{3} p^i q_i + \sqrt{\sum_{j=0}^{6} p^j h_j}$$

where $q_i(x,y)$ and $h_j(x,y)$ $(i = 0,1,2,3;\ j = 0,1,2,\ldots,6)$ are some functions. Substituting the function f into (3.52) and splitting this equations with respect to p, one obtains an overdetermined system of equations. From these equations one defines $C_i,\ (i = 0,1,2)$ and $D_i,\ (i = 0,1,\ldots,5)$. Compatibility analysis[4] of the overdetermined system of partial differential equations obtained from the conditions that one cannot define r and linearization conditions (1.12)-(1.19) gives that

$$q_3 = 0,\quad \sum_{j=0}^{6} p^j h_j = (p + r)^2 \left(\sum_{j=0}^{4} p^j h_j\right).$$

This means that the function $r(x,y)$ is also defined by the function $f(x,y,p)$.

Theorem 8. Let a second-order equation $y'' + f(x,y,y') = 0$ with $f(x,y,p)$ (3.54) which has linearized equation (2.23). If the function $r(x,y)$ cannot be defined from equation (3.52), then

$$q_3 = 0,\quad \sum_{j=0}^{6} p^j h_j = (p + r)^2 \left(\sum_{j=0}^{4} p^j h_j\right). \tag{3.83}$$

Remark. The theorem does not state the existence of functions (3.54) which has linearized equation (2.23). The nonexistence can be proven by the following way. Let us change the variables

$$t = \varphi(x,y),\quad u = \psi(x,y) \tag{3.84}$$

in a second-order equation

$$u'' + b_{33}q^3 + b_{32}q^2 + b_{31}q + b_{30}$$
$$+ (aq + r)\sqrt{b_{64}q^4 + b_{63}q^3 + b_{62}q^2 + b_{61}q + b_{60}} = 0$$

where $b_{i,j} = b_{i,j}(t,u),\ q = u'$. The transformed equation has the same form

$$y'' + b_{33}p^3 + b_{32}p^2 + b_{31}p + b_{30}$$
$$+ (cp + d)\sqrt{b_{64}p^4 + b_{63}p^3 + b_{62}p^2 + b_{61}p + b_{60}} = 0,$$

where

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x,\ c = \Delta^{-1}(\varphi_y r + \psi_y a),\ d = \Delta^{-1}(\varphi_x r + \psi_x a),$$

[4]For computations we used Reduce [24]. These computations are cumbersome and omitted here.

$$a_{33} = \Delta^{-1}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + \varphi_y^3 b_{30} + \varphi_y^2\psi_y b_{31} + \varphi_y\psi_y^2 b_{32} + \psi_y^3 b_{33});$$
$$a_{32} = \Delta^{-1}(-2\varphi_{xy}\psi_y + 3\varphi_x\varphi_y^2 b_{30} + 2\varphi_x\varphi_y\psi_y b_{31} + \varphi_x\psi_{yy} + \varphi_x\psi_y^2 b_{32} - \varphi_{yy}\psi_x + \varphi_y^2\psi_x b_{31} + 2\varphi_y\psi_{xy} + 2\varphi_y\psi_x\psi_y b_{32} + 3\psi_x\psi_y^2 b_{33});$$
$$a_{31} = \Delta^{-1}(-2\varphi_{xy}\psi_x - \varphi_{xx}\psi_y + 3\varphi_x^2\varphi_y b_{30} + \varphi_x^2\psi_y b_{31} + 2\varphi_x\varphi_y\psi_x b_{31} + 2\varphi_x\psi_{xy} + 2\varphi_x\psi_x\psi_y b_{32} + \varphi_y\psi_{xx} + \varphi_y\psi_x^2 b_{32} + 3\psi_x^2\psi_y b_{33});$$
$$a_{30} = \Delta^{-1}(-\varphi_{xx}\psi_x + \varphi_x^3 b_{30} + \varphi_x^2\psi_x b_{31} + \varphi_x\psi_{xx} + \varphi_x\psi_x^2 b_{32} + \psi_x^3 b_{33});$$
$$a_{44} = \varphi_y^4 b_{40} + \varphi_y^3\psi_y b_{41} + \varphi_y^2\psi_y^2 b_{42} + \varphi_y\psi_y^3 b_{43} + \psi_y^4 b_{44};$$
$$a_{43} = 4\varphi_x\varphi_y^3 b_{40} + 3\varphi_x\varphi_y^2\psi_y b_{41} + 2\varphi_x\varphi_y\psi_y^2 b_{42} + \varphi_x\psi_y^3 b_{43} + \varphi_y^3\psi_x b_{41} + 2\varphi_y^2\psi_x\psi_y b_{42} + 3\varphi_y\psi_x\psi_y^2 b_{43} + 4\psi_x\psi_y^3 b_{44};$$
$$a_{42} = 6\varphi_x^2\varphi_y^2 b_{40} + 3\varphi_x^2\varphi_y\psi_y b_{41} + \varphi_x^2\psi_y^2 b_{42} + 3\varphi_x\varphi_y^2\psi_x b_{41} + 4\varphi_x\varphi_y\psi_x\psi_y b_{42} + 3\varphi_x\psi_x\psi_y^2 b_{43} + \varphi_y^2\psi_x^2 b_{42} + 3\varphi_y\psi_x^2\psi_y b_{43} + 6\psi_x^2\psi_y^2 b_{44};$$
$$a_{41} = 4\varphi_x^3\varphi_y b_{40} + \varphi_x^3\psi_y b_{41} + 3\varphi_x^2\varphi_y\psi_x b_{41} + 2\varphi_x^2\psi_x\psi_y b_{42} + 2\varphi_x\varphi_y\psi_x^2 b_{42} + 3\varphi_x\psi_x^2\psi_y b_{43} + \varphi_y\psi_x^3 b_{43} + 4\psi_x^3\psi_y b_{44};$$
$$a_{40} = \varphi_x^4 b_{40} + \varphi_x^3\psi_x b_{41} + \varphi_x^2\psi_x^2 b_{42} + \varphi_x\psi_x^3 b_{43} + \psi_x^4 b_{44}.$$

For example, one can find a change of the variables (3.84) such that $a_{33} \neq 0$. This can be done by choosing the functions (3.84) satisfying the equation

$$\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + \varphi_y^3 b_{30} + \varphi_y^2\psi_y b_{31} + \varphi_y\psi_y^2 b_{32} + \psi_y^3 b_{33} = 1.$$

The transformed equation does not satisfy the first relation (3.83).

4 Ricatti substitution

Here we consider increasing order of a second-order ordinary differential equation

$$y'' + f(x, y, y') = 0 \tag{4.85}$$

by using the Ricatti substitution $y = v'/v$. After Ricatti substitution equation (4.85) becomes

$$v'''/v - 3v'v''/v^2 + 2(v'/v)^3 + f\left(x, v'/v, v''/v - (v'/v)^2\right) = 0 \tag{4.86}$$

Since the necessary conditions for equation (4.86) to be linearizable is that this equation is at most second degree polynomial with respect to v'', one obtains that

$$f(x, y, y') = q_2(x, y)y'^{\,2} + q_1(x, y)y' + q_0(x, y), \tag{4.87}$$

where $q_i(x, y)$, $(i = 1, 2, 3)$ are some functions. Equation (4.86) becomes

$$vv''' - 3v'v'' + v''^{\,2}q_2 + v''v(q_1 - 2q_2y^2) + v^2(q_0 - q_1y^2 + q_2y^4 + 2y^3)$$

Comparing coefficient with $v''^{\,2}$, one obtains that there are two possibilities: either $q_2 = 0$ or $q_2 = -3/y$.

4.1 Case $q_2 = 0$

If $q_2 = 0$, then comparing terms with v'' in equation (4.86) and in the representation of a linearizable equation, one obtains

$$q_1 = A_0 + A_1 v y + 3y,$$

$$q_0 = A_0 y^2 + A_1 v y^3 + B_0 v^{-1} + B_1 y + B_2 v y^2 + B_3 v^2 y^3 + y^3$$

with some coefficients $A_i(x, v)$ and $B_j(x, v)$, $(i = 0, 1,\ j = 0, 1, 2, 3)$. Since q_0 and q_1 do not depend on v, one finds

$$B_0 = k_0 v,\ \ B_1 = k_1,\ \ B_2 = k_2 v^{-1},\ \ B_3 = k_3 v^{-2},\ \ A_0 = \lambda_0,\ \ A_1 = \lambda_1 v^{-1}$$

where $k_i(x)$, $(i = 0, 1, 2, 3)$, $\lambda_0(x)$ and $\lambda_1(x)$ are only functions of x. Hence,

$$q_0 = y^3(k_3 + \lambda_1 + 1) + y^2(k_2 + \lambda_0) + y k_1 + k_0,$$

$$q_1 = y(\lambda_1 + 3 + \lambda_0.$$

Thus, necessary conditions for third-order equation (4.86) to be linearizable give that this equation has the form

$$v'^{\,3} k_3 + v'^{\,2} k_2 v + v' v'' \lambda_1 v + v' k_1 v^2 + v'' \lambda_0 v^2 + v''' v^2 + k_0 v^3 = 0.$$

This equation is linearizable with respect to point transformations if and only if

$$\lambda_1' = 0,\ \ k_2 = \lambda_0 \lambda_1 / 3,\ \ k_3 = \lambda_1(\lambda_1/3 - 1)/3.$$

Notice that in this case equation ... is also linearizable by point transformation.

4.2 Case $q_2 = -3/y$

Assuming that $q_2 = -3/y$, one obtains $r = 0$. Comparing terms with v'' in equation (4.86) and in the representation of a linearizable equation, one obtains

$$\begin{aligned} q_1 &= C_0 v^{-1} y^{-1} + C_1 + C_2 v y - 3y, \\ q_0 &= C_0 v^{-1} y + C_1 y^2 + C_2 v y^3 + D_0 v^{-2} y^{-1} + D_1 v^{-1} \\ &\quad + D_2 y + D_3 v y^2 + D_4 v^2 y^3 + D_5 v^3 y^4 - 2y^3, \end{aligned} \tag{4.88}$$

with some coefficients $C_i(x, v)$ and $D_j(x, v)$, $(i = 0, 1, 2,\ j = 0, 1, \ldots, 5)$. As in the previous case, independence of q_0 and q_1 of v gives

$$\begin{aligned} &D_0 = k_0 v^2,\ \ D_1 = k_1 v,\ \ D_2 = k_2,\ \ D_3 = k_3 v^{-1}, \\ &D_4 = k_4 v^{-2},\ \ D_5 = k_5 v^{-3},\ \ C_0 = \mu_0 v,\ \ C_1 = \mu_1, \end{aligned}$$

with some functions $\mu_i(x)$ and $k_j(x)$, $(i = 0,1,2;\ j = 0,1,\ldots,5)$. In this case

$$q_0 = k_0 y^{-1} + k_1 + (\mu_0 + k_2)y + (\mu_1 + k_3)y^2 + (\mu_2 + k_4 - 2)y^3 + k_5 y^4,$$
$$q_1 = \mu_0 y^{-1} + \mu_1 + (\mu_2 - 3)y, \quad q_2 = -3y^{-1}.$$

and equation (4.86) becomes

$$\begin{gathered} v^3 v' v''' - 3v^3 (v'')^2 + v''(\mu_0 v^4 + \mu_1 v^3 v' + \mu_2 v^2 v'^2) + k_0 v^5 \\ + k_1 v^4 v' + k_2 v^3 v'^2 + k_3 v^2 v'^3 + k_4 v v'^4 + k_5 v'^5 = 0 \end{gathered} \tag{4.89}$$

This equation is linearizable by point transformation if and only if

$$\begin{gathered} \mu_0 = 0, \ \mu_2' = 0, \ k_0 = 0, \ k_1 = 0, \ k_2 = -(\mu_1' + \mu_1^2/3)/3, \\ k_3 = -\mu_1 \mu_2/3, \ k_4' = 0, \ k_5' = -k_5 \mu_1/3 + \lambda, \end{gathered} \tag{4.90}$$

where λ is an arbitrary constant.

Notice that second order equation (4.85) with (4.87) and (4.88) is linearizable if and only if

$$\mu_0 = 0, \ \mu_2 = 3, \ k_3 = -\mu_1, \ k_1 = 0, \ k_0 = 0, \ k_5 = 0, \tag{4.91}$$

or if it has the form

$$y'' - 3y'^2/y + \mu_1(x) y' + k_2(x) y + (1 + k_4(x)) y^3 = 0, \tag{4.92}$$

where $\mu_1(x)$, $k_2(x)$ and $k_4(x)$ are arbitrary functions.

4.3 Examples

4.3.1 Example 1

The second-order ordinary differential equation

$$y'' - 3\frac{y'^2}{y} + x y^4 = 0 \tag{4.93}$$

is not of the form (4.92). Hence it is not linearizable. Since equation (4.90) satisfies conditions (4.90), this equation after the Ricatti substitution becomes linearizable by point transformations.

4.3.2 Example 2

Another example of a second-order ordinary differential equation which is linearizable after the Ricatti substitution is the equation[5]

$$y'' - 3\frac{y'^2}{y} + k y^4 = 0. \tag{4.94}$$

[5]Later it was noted that the substitution $y' = h(y)$ reduces this equation to a linear equation $(h^2)' - 6(h^2)/y + 2ky^4 = 0$.

This equation after the Ricatti substitution becomes

$$v^3v'v''' - 3v^3v''^{\,2} + 3v^2v'^2v'' + kv'^5 - vv'^4 = 0. \tag{4.95}$$

There is a change of the dependent and independent variables in equation (4.95) which transforms this equation into the simple equation $u''' = 0$. In contrast to equation (4.93) this equation is linearizable by contact transformations. Notice also that equation (4.95) belongs to the class of equations linearizable by the method [19]: reducing the order of equation (4.95) by the substitution $v' = z(v)$, one obtains the second-order ordinary differential equation

$$v^3(z''z - 2z'^2) + 3v^2zz' + z^2(kz - v) = 0, \tag{4.96}$$

which is linearizable. Equation (4.94) shows that combinations of the Ricatti substitution and reducing the order can be applied for linearization of second-order ordinary differential equations. This method was applied in [20] for a special class of second-order ordinary differential equations.

4.3.3 Example 3

The second-order ordinary differential equations

$$\begin{aligned} y'' - 3\frac{y'^2}{y} + y^3(x+1) = 0,\\ y'' - 3\frac{y'^2}{y} + y^2(y^2+1) = 0 \end{aligned} \tag{4.97}$$

are linearizable by a point transformation, but the third-order equations obtained from them after the Ricatti substitution are not linearizable equations. Even the second equation of (4.97) is reduced by the substitution $y' = h(y)$ to the linear equation $(h^2)' - 6(h^2)/y + 2y^2(y^2+1) = 0$.

5 Linear equations of third-order

Recently [25] it was shown that the linear third order equations any linear third-order ordinary differential equation

$$\begin{gathered} y''' + f(x)y'' + y' + f(x)y = 0,\\ y''' - y = 0, \end{gathered}$$

and

$$y''' = 0 \tag{5.98}$$

are equivalent under nonlocal transformations. These equations are representative equations of linear third-order equations with four, five and seven (the maximum) point symmetries, respectively.

Here it is proven that any linear third-order ordinary differential equation

$$y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0; \tag{5.99}$$

after reducing the order by the Ricatti substitution $y' = wy$ becomes equivalent to the simple second-order ordinary differential equation $w'' = 0$.

In fact, applying the Ricatti substitution $y' = wy$ to equation (5.99), this equation is reduced to the equation

$$y(w'' + 3ww' + w^3 + a_1(w' + w^2) + a_2w + a_3) = 0:$$

Using the Lie test, one obtains that the second-order ordinary differential equation

$$w'' + 3ww' + w^3 + a_1(w' + w^2) + a_2w + a_3 = 0 \tag{5.100}$$

can be mapped by a point transformation to the free particle equation

$$u'' = 0:$$

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Second-order ordinary differential equations equivalent to $y'' = H(y)$

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden
SERGEY V. MELESHKO
School of Matematics,
Suranaree University of Technology,
Nakhon Ratchasima, 3000, Thailand

Abstract The main feature of equations $y'' = H(y)$ is that their solutions can be represented in quadratures. The paper gives criteria for a second-order ordinary differential equation to be equivalent to an equation of the form $y'' = H(y)$.

1 Introduction

One of classes of second-order ordinary differential equations

$$y'' = f(x;y;y') \tag{1.1}$$

which can be solved in quadratures is the class of equations:

$$y'' = H(y): \tag{1.2}$$

In fact, assuming that $y' = q(y)$, Equation (1.2) is reduced to the first-order ordinary differential equation

$$qq' = H(y):$$

Hence, a solution of (1.2) is obtained from the representation

$$x = \int \frac{dy}{\sqrt{2\int H(y)\,dy}}:$$

Thus, there is the interest in describing all second-order ordinary differential Equations (1.1) which can be mapped by a change of the dependent and independent variables into an equation of the form (1.2).

2 Candidates for equations $y'' = H(y)$

It is known that all equations obtained from the equation

$$y'' = H(y), \tag{2.3}$$

by the change of variables

$$t = \varphi(x, y), \quad u = \psi(x, y), \tag{2.4}$$

are contained in the family of the equations of the form

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 = 0. \tag{2.5}$$

It is also known that any Equation (2.5) is transformed by a change of variables (2.4) into an equation of the same form:

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 = 0, \tag{2.6}$$

and that the coefficients of Equations (2.5) and (2.6) are related by the following equations:

$$\begin{aligned}
a_1 = {} & \Delta^{-1}\big[\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3\big];\\
a_2 = {} & \Delta^{-1}\big[b_4\varphi_x\varphi_y^2 + b_3\varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x) + b_2\psi_y(\varphi_x\psi_y + 2\varphi_y\psi_x)\\
& + b_1\psi_x\psi_y^2 + (\varphi_x\psi_{yy} - \varphi_{yy}\psi_x - 2\varphi_{xy}\psi_y + 2\varphi_y\psi_{xy})/3\big];\\
a_3 = {} & \Delta^{-1}\big[b_4\varphi_x^2\varphi_y + b_3\varphi_x(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_2\psi_x(2\varphi_x\psi_y + \varphi_y\psi_x)\\
& + b_1\psi_x^2\psi_y + (\varphi_y\psi_{xx} - \varphi_{xx}\psi_y - 2\varphi_{xy}\psi_x + 2\varphi_x\psi_{xy})/3\big];\\
a_4 = {} & \Delta^{-1}\big[b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}\big].
\end{aligned} \tag{2.7}$$

3 Some knowledge

We will use the following information about invariants of Equations (2.5). S. Lie [1] showed that any second-order ordinary differential equation obtained from the linear equation $y'' = 0$ by the change of variables (2.4) belongs to the family of Equations (2.5) and obtained the necessary and sufficient conditions for Equations (2.5) to be equivalent to the linear equation. Lie's linearization test can be expressed by means of the equations $L_1 = 0,\ L_2 = 0$, where

$$\begin{aligned}
L_1 = {} & a_{2xx} - 2a_{3xy} + a_{4yy} - 2a_{1x}a_4 + 3a_{2x}a_3 + 3a_{2y}a_4\\
& - 6a_{3y}a_3 - a_{4x}a_1 + 3a_{4y}a_2;\\
L_2 = {} & a_{1xx} + a_{3yy} - 2a_{2xy} - 3a_{1x}a_3 + a_{1y}a_4 + 6a_{2x}a_2\\
& - 3a_{3x}a_1 - 3a_{3y}a_2 + 2a_{4y}a_1.
\end{aligned} \tag{3.8}$$

The change of variables (2.4) converts the quantities (3.8) into the following relative invariants for Equation (2.5):

$$\widetilde{L}_1 = \Delta(L_1\varphi_x + L_2\psi_x); \quad \widetilde{L}_2 = \Delta(L_1\varphi_y + L_2\psi_y); \tag{3.9}$$

where $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is the Jacobian of the change. For Equation (2.3), the relative invariants (3.9) are

$$\widetilde{L}_1 = -H_{yy}; \quad \widetilde{L}_2 = 0. \tag{3.10}$$

Hence, for nonlinearizable equation $H_{yy} \neq 0$ and the following statement is valid.

Lemma. For all Equations (2.5) obtained from Equation (2.3) by a change of variables, at least one of the relative invariants L_1, L_2 does not vanish, and the corresponding change of the variables (2.4) obeys the equation

$$L_1\varphi_y + L_2\psi_y = 0. \tag{3.11}$$

We will also use the following relative invariants of higher order given in [2, 3, 4]:

$$\begin{aligned} v_5 = {} & L_2(L_1L_{2t} - L_2L_{1t}) + L_1(L_2L_{1u} - L_1L_{2u}) - b_1L_1^3 \\ & + 3b_2L_1^2L_2 - 3b_3L_1L_2^2 + b_4L_2^3, \end{aligned} \tag{3.12}$$

$$\begin{aligned} w_1 = {} & L_1^{-4}\big[-L_1^3(\beta_{12}L_1 - \beta_{11}L_2) + R_1(L_1^2)_t \\ & - L_1^2R_{1t} + L_1R_1(b_3L_1 - b_4L_2)\big], \end{aligned} \tag{3.13}$$

and

$$I_2 = 3R_1L_1^{-1} + L_{2t} - L_{1u}, \tag{3.14}$$

where

$$\begin{aligned} \beta_{11} &= 2(b_3^2 - b_2b_4) + b_{3t} - b_{4u}, \\ \beta_{12} &= b_2b_3 - b_1b_4 + b_{2t} - b_{3u}, \\ R_1 &= L_1L_{2t} - L_2L_{1t} + b_2L_1^2 - 2b_3L_1L_2 + b_4L_2^2. \end{aligned}$$

If the relative invariant $I_2 \neq 0$, there is the set of absolute invariants

$$J_{2m} = I_{2m}I_2^{-m}, \quad (m \geq 1),$$

where

$$I_{2m+2} = L_1\frac{\partial I_{2m}}{\partial u} - L_2\frac{\partial I_{2m}}{\partial t} + 2mI_{2m}(L_{2t} - L_{1u}).$$

Similar relative invariants for Equation (2.6) are denoted by $\widetilde{v}_5, \widetilde{w}_1, \widetilde{I}_2$ and $\widetilde{J}_4$. For Equation (2.3), invoking Equations (3.10), we obtain:

$$\widetilde{v}_5 = 0, \quad \widetilde{w}_1 = 0, \quad \widetilde{I}_2 = H_{yyy}. \tag{3.15}$$

Hence, we have the following necessary conditions for Equations (2.5) obtained from Equation (2.3) by a change of variables[1]:

$$v_5 = 0, \quad w_1 = 0. \tag{3.16}$$

Hence, these conditions are also necessary for equations equivalent to $y'' = H(y)$.

In this paper we obtain the necessary and sufficient conditions for a nonlinearizable Equation (2.5) to be transformable by a change of variables (2.4) into equation of the form (2.3).

4 Statement of the problem

One has to find the conditions for the coefficients $b_1(t,u)$, $b_2(t,u)$, $b_3(t,u)$ and $b_4(t,u)$ that guarantee the existence of the functions $\varphi(x,y)$ and $\psi(x,y)$ such that the change of variables (2.4) transforms the coefficients of Equation (2.5) into

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = -H(y), \quad (H_{yy} \neq 0),$$

where a_1, a_2, a_3, a_4 are defined by formulae (2.7). Thus, we have to investigate the consistency of the following over-determined system:

For finding criteria for a second-order ordinary differential equation

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 = 0 \tag{4.17}$$

to be transformed by a point transformation

$$t = \varphi(x,y), \quad u = \psi(x,y)$$

to an equation

$$y'' = H(y)$$

one needs to solve with respect to the functions $\varphi(x,y)$ and $\psi(x,y)$ the equations

$$b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3 - \varphi_{yy}\psi_y + \varphi_y\psi_{yy} = 0 \tag{4.18}$$

$$\begin{gathered} 3b_4\varphi_x\varphi_y^2 + 6b_3\varphi_x\varphi_y\psi_y + 3b_3\varphi_y^2\psi_x + 3b_2\varphi_x\psi_y^2 + 6b_2\varphi_y\psi_x\psi_y \\ +3b_1\psi_x\psi_y^2 - 2\varphi_{xy}\psi_y + \varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\varphi_y\psi_{xy} = 0 \end{gathered} \tag{4.19}$$

$$\begin{gathered} 3b_4\varphi_x^2\varphi_y + 3b_3\varphi_x^2\psi_y + 6b_3\varphi_x\varphi_y\psi_x + 6b_2\varphi_x\psi_x\psi_y + 3b_2\varphi_y\psi_x^2 \\ +3b_1\psi_x^2\psi_y - 2\varphi_{xy}\psi_x - \varphi_{xx}\psi_y + 2\varphi_x\psi_{xy} + \varphi_y\psi_{xx} = 0 \end{gathered} \tag{4.20}$$

$$\begin{gathered} b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx} \\ +(\varphi_x\psi_y - \varphi_y\psi_x)H = 0 \end{gathered} \tag{4.21}$$

[1]These conditions are necessary and sufficient [4] for a second-order ordinary differential equation to be equivalent to an equation of the form $y'' = h(x,y)$.

Without loss of the generality one can assume that $L_1 \neq 0$. In this case, because of (3.11) and the Jacobian $\Delta \neq 0$, one has

$$\psi_y(\varphi_x L_1 + \psi_x L_2) \neq 0.$$

In [4] it is shown that the necessary and sufficient conditions for any Equation (4.17) to be transformed to the form $y'' = f(x, y)$ are

$$v_5 = 0, \quad w_1 = 0.$$

Other necessary and sufficient conditions depend on vanishing the quantity L_2.

5 Case $L_2 \neq 0$

If $L_2 \neq 0$, then by virtue of (3.11) and $\Delta \neq 0$, one has that $\varphi_y \neq 0$. From Equations (4.18)-(4.21) one finds φ_{xx} and[2]

$$\psi_{yy} = (\psi_y^2(-3b_4L_2^2 + 6b_3L_1L_2 - 3b_2L_1^2 + L_{1t}L_2 - L_{1u}L_1 - I_2L_1))/(2L_1^2); \quad (5.22)$$

$$\begin{aligned}\psi_{xx} = (&-2b_4\varphi_x^2\psi_yL_1^2 - 4b_4\varphi_x\psi_x\psi_yL_1L_2 + b_4\psi_x^2\psi_yL_2^2 - 6b_3\psi_x^2\psi_yL_1L_2 \\ &+3b_2\psi_x^2\psi_yL_1^2 - L_{1t}\psi_x^2\psi_yL_2 + L_{1u}\psi_x^2\psi_yL_1 + 4\psi_{xy}\psi_xL_1^2 + \psi_x^2\psi_yI_2L_1 \\ &-2\psi_y^2HL_1^2)/(2\psi_yL_1^2).\end{aligned}$$

Composing the equation $(\psi_{xx})_{yy} - (\psi_{yy})_{xx} = 0$, one finds

$$H_{yy} = -(\varphi_xL_1 + \psi_xL_2)^2\psi_y/L_1. \quad (5.23)$$

Since the function H do not depend on x, then the equation $(H_{yy})_x = 0$ gives

$$\begin{aligned}\psi_{xy} = (&2L_1\varphi_x(6b_4L_2 - L_{1t} - 6b_3L_1) + \psi_x(-3b_4L_2^2 + 18b_3L_1L_2 \\ &-15b_2L_1^2 + 3L_{1t}L_2 - 5L_{1u}L_1 - 3I_2L_1))\psi_y)/(10L_1^2).\end{aligned} \quad (5.24)$$

From the equations $(\psi_{xy})_x - (\psi_{xx})_y = 0$ and $(\psi_{xy})_y - (\psi_{yy})_x = 0$ one obtains, respectively

$$\begin{aligned}H_y = -(&\psi_x^2(15b_4I_2L_1L_2^2 - 30b_3I_2L_1^2L_2 + 15b_2I_2L_1^3 + 10I_{2t}L_1^2L_2 \\ &-10I_{2u}L_1^3 - 15L_{1t}I_2L_1L_2 + 15L_{1u}I_2L_1^2 + 3I_2^2L_1^2 - Q_6L_2^2) \\ &-\varphi_x^2Q_6L_1^2 - 2\varphi_x\psi_xQ_6L_1L_2 - 10\psi_yHI_2L_1^3)/(50L_1^4);\end{aligned} \quad (5.25)$$

$$\varphi_xL_1Q_2 + \psi_xQ_1 = 0; \quad (5.26)$$

[2] Equation for φ_{xx} is cumbersome and it is omitted. For symbolic calculations we used the system REDUCE [5].

where

$$Q_1 = -(3((5L_{1u} + I_2)L_1 - L_{1t}L_2)I_2 - 10I_{2u}L_1^2 - 3((6b_3L_2 - 5b_2L_1)L_1 \\ - b_4L_2^2)I_2),$$

$$Q_2 = 2(6b_4I_2L_2 - 6b_3I_2L_1 + 5I_{2t}L_1 - 6L_{1t}I_2),$$

$$Q_6 = -(2(5((5b_{4u} - 6b_{3t})L_1 + b_{4t}L_2) - 54b_3^2L_1)L_1 \\ - 3b_4^2L_2^2 + (5(3L_{1u} + I_2)L_1 - 9L_{1t}L_2 + 3(2b_3L_2 + 35b_2L_1)L_1)b_4 \\ - 2(5L_{1tt}L_1 - 6L_{1t}^2 + 3b_3L_{1t}L_1)).$$

The derivatives ψ_{xx} and φ_{xx} become

$$10L_1^2\psi_{xx} = -10b_4\varphi_x^2L_1^2 - 4\varphi_x\psi_xL_1(6b_3L_1 - b_4L_2 + L_{1t}) \\ - \psi_x^2(b_4L_2^2 - 6b_3L_1L_2 + 15b_2L_1^2 - L_{1t}L_2 + 5L_{1u}L_1 + I_2L_1) - 10\psi_yHL_1^2, \tag{5.27}$$

$$10L_1^2\varphi_{xx} = 2L_1\varphi_x^2(2b_4L_2 + 3b_3L_1 - 2L_{1t}) \\ \varphi_x\psi_x(-b_4L_2^2 + 6b_3L_1L_2 + 15b_2L_1^2 + L_{1t}L_2 - 5L_{1u}L_1 - I_2L_1) \\ +10b_1\psi_x^2L_1^2 + 10\psi_yHL_1L_2. \tag{5.28}$$

5.1 Case $Q_1 \neq 0$

Assume that $Q_1 \neq 0$, then one finds ψ_x from Equation (5.26):

$$\psi_x = -\varphi_xL_1Q_2/Q_1. \tag{5.29}$$

Substituting (5.25) into (5.23), one obtains

$$10H\psi_yQ_1^2(L_2Q_2 - Q_1) - \varphi_x^2Q_3 = 0, \tag{5.30}$$

where

$$Q_3 = 7b_4L_2^3Q_2^3 + 24b_4L_2^2Q_1Q_2^2 - 36b_4L_2Q_1^2Q_2 + 10b_4Q_1^3 \\ - 42b_3L_1L_2^2Q_2^3 + 36b_3L_1L_2Q_1Q_2^2 + 6b_3L_1Q_1^2Q_2 + 15b_2L_1^2L_2Q_2^3 \\ - 30b_2L_1^2Q_1Q_2^2 + 10b_1L_1^3Q_2^3 - 7L_{1t}L_2^2Q_2^3 - 44L_{1t}L_2Q_1Q_2^2 + 26L_{1t}Q_1^2Q_2 \\ +55L_{1u}L_1L_2Q_2^3 - 40L_{1u}L_1Q_1Q_2^2 + 10Q_{1u}L_1^2Q_2^2 + 20Q_{2t}L_1L_2Q_1Q_2 \\ - 10Q_{2t}L_1Q_1^2 - 20Q_{2u}L_1^2L_2Q_2^2 + 7I_2L_1L_2Q_2^3 - 4I_2L_1Q_1Q_2^2.$$

Since the assumption $Q_1 - L_2Q_2 = 0$ leads to $\Delta = 0$, one has $Q_1 - L_2Q_2 \neq 0$, and then Equation (5.30) gives

$$H = (\varphi_x^2Q_3)/(10\psi_yQ_1^2(L_2Q_2 - Q_1)). \tag{5.31}$$

The equations $\psi_{xx} - (\psi_x)_x = 0$, $\psi_{xy} - (\psi_x)_y = 0$, $H_y - (H)_y = 0$ $H_x = 0$ produce only condition

$$
\begin{aligned}
10L_1^2Q_2^2(L_2Q_2 - Q_1)Q_{1u} = (&10Q_1^2(b_4Q_1^2 - Q_{2t}L_1Q_1 + Q_{2t}L_1L_2Q_2) \\
&+2Q_1^3Q_2(3b_3L_1 - 13b_4L_2 + 13L_{1t}) \\
&+2Q_1^2Q_2^2(6b_4L_2^2 + 9b_3L_1L_2 - 15b_2L_1^2 - 11L_{1t}L_2 - 20L_{1u}L_1 - 2I_2L_1) \\
&+2Q_1Q_2^3(2b_4L_2^3 - 12b_3L_1L_2^2 + 15b_2L_1^2L_2 + 5b_1L_1^3 - 2L_{1t}L_2^2 + 20L_{1u}L_1L_2 \\
&+2I_2L_1L_2) - (Q_1 + L_2Q_2)Q_3 - 10b_1L_1^3L_2Q_2^4).
\end{aligned}
\tag{5.32}
$$

Hence, one obtains that the necessary and sufficient conditions for Equation (2.5) to be equivalent to equation of the form (2.3) are $v_5 = 0, \ w_1 = 0$, and (5.32). The involutive system of equations for the functions $\varphi(x,y)$, $\psi(x,y)$ and $H(x,y)$ consists of the Equations (5.28), (5.22), (5.29), and (5.31):

$$
\begin{aligned}
&L_1\varphi_y + \psi_yL_2 = 0, \quad \psi_xQ_1 + \varphi_xL_1Q_2 = 0, \quad \varphi_{xx} = \varphi_x^2\widetilde{Q}_4, \\
&2L_1^2\psi_{yy} - \psi_y^2Q_4 = 0, \quad 10\psi_yQ_1^2(Q_1 - L_2Q_2)H + \varphi_x^2Q_3 = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{Q}_4 = (&-14b_4L_2^3Q_1Q_2^2 + 18b_4L_2^2Q_1^2Q_2 - 4b_4L_2Q_1^3 + 24b_3L_1L_2^2Q_1Q_2^2 \\
&-18b_3L_1L_2Q_1^2Q_2 - 6b_3L_1Q_1^3 - 30b_2L_1^2L_2Q_1Q_2^2 + 30b_2L_1^2Q_1^2Q_2 \\
&+10b_1L_1^3L_2Q_2^3 - 10b_1L_1^3Q_1Q_2^2 + 4L_{1t}L_2^2Q_1Q_2^2 - 8L_{1t}L_2Q_1^2Q_2 + 4L_{1t}Q_1^3 \\
&-4I_2L_1L_2Q_1Q_2^2 + 4I_2L_1Q_1^2Q_2 - 5L_2Q_1Q_2^2Q_4 + L_2Q_3 \\
&+5Q_1^2Q_2Q_4)/(10L_1Q_1^2(L_2Q_2 - Q_1))
\end{aligned}
$$

5.2 Case $Q_1 = 0$

If $Q_1 = 0$, then (5.26) becomes $\varphi_xQ_2 = 0$.

Let $Q_2 \neq 0$. Then $\varphi_x = 0$ and Equation (5.28) gives

$$
H = -b_1\psi_x^2L_1/(\psi_yL_2). \tag{5.33}
$$

Differentiating $\varphi_y = \psi_yL_2/L_1$, found from (3.11), and H with respect to x, one obtains

$$
5L_1L_2L_{1u} = -b_4L_2^3 + 6b_3L_1L_2^2 + 15b_2L_1^2L_2 - 20b_1L_1^3 + L_{1t}L_2^2 - I_2L_1L_2, \tag{5.34}
$$

$$
6b_2b_1L_2 - 6b_1^2L_1 - b_{1u}L_2 = 0. \tag{5.35}
$$

Substitution of H into (5.25) do not give any new relation. Notice that (5.32) is also satisfied.

Hence, in the case $Q_1 = 0$, and $Q_2 \neq 0$ one obtains that the necessary and sufficient conditions are $v_5 = 0, \ w_1 = 0$, and (5.34), (5.35). The involutive system of equations for the functions $\varphi(y)$, $\psi(x,y)$ and $H(y)$ consists of the equations (5.22), (5.27), (5.24), and (5.33).

Let $Q_2 = 0$. The relation $(H_y)_x = 0$ gives

$$\psi_x Q_5 L_2 - \varphi_x L_1(3I_2 Q_6 L_1 - Q_5 - 250L_1^5) = 0, \tag{5.36}$$

where

$$Q_5 = 5Q_{6t}L_1L_2 + Q_6(6\lambda_3L_1L_2 - 6\lambda_4L_2^2 - 14L_{1t}L_2 + 3I_2L_1) - 250L_1^5.$$

If $Q_5 = 0$, then

$$3I_2Q_6L_1 - 250L_1^5 = 0 \tag{5.37}$$

and the system (5.22), (5.24), (5.25) , (5.27), and (5.28) is involutive.

If $Q_5 \neq 0$, then by virtue of $\Delta \neq 0$ one also has $(3I_2Q_6 - 250L_1^4) \neq 0$. Solving Equation (5.36) with respect to ψ_x, and substituting it into Equations (5.24), (5.27), one obtains

$$H = \frac{Q_7L_1}{20Q_5^2L_2^2}\frac{\varphi_x^2}{\psi_y}, \tag{5.38}$$

where

$$\begin{aligned} Q_7 = &-(((99Q_5 - 250L_1^5 + 3I_2Q_6L_1)L_{1t} - 20Q_{5t}L_1)(3I_2Q_6 - 250L_1^4)L_2 \\ &+((3I_2Q_6L_1 + 16Q_5 - 250L_1^5)I_2 \\ &+5(3I_2Q_6 - 250L_1^4)L_{1u}L_1)(Q_5 + 250L_1^5 - 3I_2Q_6L_1) \\ &+((21Q_5 + 250L_1^5 - 3I_2Q_6L_1)\lambda_4L_2^2 \\ &-3(Q_5 + 250L_1^5 - 3I_2Q_6L_1)(2\lambda_3L_2 + 5\lambda_2L_1)L_1)(3I_2Q_6 - 250L_1^4)). \end{aligned}$$

In this case Equation (5.25) is satisfied, and the relation $H_x = 0$ gives

$$\begin{aligned} &30Q_{7u}Q_6I_2L_1^2L_2 - 2500Q_{7u}L_1^6L_2 - 36Q_6^3I_2^2Q_5L_2^3 \\ &+6000Q_6^2I_2Q_5L_1^4L_2^3 + 3Q_6Q_7I_2(\lambda_4L_2^3 - 6\lambda_3L_1L_2^2 - 15\lambda_2L_1^2L_2 \\ &+20\lambda_1L_1^3 - L_{1t}L_2^2 - 115L_{1u}L_1L_2 + I_2L_1L_2) - 250000Q_6Q_5L_1^8L_2^3 \\ &+10Q_7(-25\lambda_4L_1^4L_2^3 + 150\lambda_3L_1^5L_2^2 + 375\lambda_2L_1^6L_2 - 500\lambda_1L_1^7 + 25L_{1t}L_1^4L_2^2 \\ &+2875L_{1u}L_1^5L_2 - 2I_2Q_5L_2 - 25I_2L_1^5L_2) = 0. \end{aligned} \tag{5.39}$$

Hence, the necessary and sufficient conditions are $v_5 = 0, \ w_1 = 0$, (5.35), and (5.39). The involutive system of equations for the functions $\varphi(x,y)$, $\psi(x,y)$ and $H(y)$ consists of the equations (5.22), (5.28), (5.36), and (5.38).

Thus, in the case $L_2 \neq 0$ the result can be summarized as the following theorem.

Theorem 1. The necessary and sufficient conditions for a nonlinearizable Equation (2.5) to be equivalent with respect to a change of variables (2.4) to equation of the form (2.3) are $v_5 = 0, \ w_1 = 0$, and:

(a) if $Q_1 \neq 0$, then (5.32);

(b) if $Q_1 = 0$, and $Q_2 \neq 0$, then (5.34) and (5.35);

(c) if $Q_1 = 0$, $Q_2 = 0$, and $Q_5 = 0$, then (5.37);

(d) if $Q_1 = 0$, $Q_2 = 0$, and $Q_5 \neq 0$, then (5.35), and (5.39).

6 Case $L_2 = 0$

If $L_2 = 0$, then $\varphi_y = 0$ and $\varphi_x \neq 0$. The equations $v_5 = 0$ and $w_1 = 0$ are reduced to $b_1 = 0$ and $b_{3u} = 2b_{2t}$. From Equations (4.19)-(4.21) one finds

$$\psi_{yy} = -3b_2\psi_y^2, \quad (6.40)$$

$$2\psi_{xy} = \psi_y(\varphi_{xx}/\varphi_x - 3(b_3\varphi_x + 2b_2\psi_x)), \quad (6.41)$$

$$\psi_{xx} = \varphi_{xx}\psi_x/\varphi_x - b_4\varphi_x^2 - 3b_3\varphi_x\psi_x - 3b_2\psi_x^2 - \psi_y H. \quad (6.42)$$

The equations $(\psi_{xy})_x - (\psi_{xx})_y = 0$ and $(\psi_{xy})_y - (\psi_{yy})_x = 0$ give

$$2\varphi_{xxx} = 3\varphi_{xx}^2/\varphi_x + \varphi_x^3(9b_3^2 - 12b_4b_2 - 4b_{4u} + 6b_{3t}) - 4\varphi_x H_y. \quad (6.43)$$

Differentiating (6.43) with respect to y, one finds

$$H_{yy} = -\varphi_x^2\psi_y L_1. \quad (6.44)$$

Differentiating (6.44) with respect to x, one obtains

$$\varphi_{xx} = \varphi_x(\varphi_x(3b_3L_1 - 2L_{1t}) + 2\psi_x I_2)/(5L_1). \quad (6.45)$$

Since $\varphi_y = 0$, after differentiating (6.45) with respect to y, one gets

$$\varphi_x Q_2 + \psi_x Q_1 = 0, \quad (6.46)$$

where[3]

$$Q_1 = 30b_2I_2L_1 - 5I_{2u}L_1 - 6I_2^2, \quad Q_2 = 6L_{1t}I_2 + 6b_3I_2L_1 - 5I_{2t}L_1.$$

Substituting φ_{xx} from (6.45) into (6.43), one has

$$\begin{aligned} 25L_1^2H_y = {} & \varphi_x^2(5L_{1tt}L_1 + 3b_3L_{1t}L_1 - 6L_{1t}^2 - 75b_4b_2L_1^2 + 5b_4I_2L_1 + 54b_3^2L_1^2 \\ & - 25b_{4u}L_1^2 + 30b_{3t}L_1^2) + 2\varphi_x\psi_x(6b_3I_2L_1 - 5I_{2t}L_1 + 6L_{1t}I_2) \\ & + \psi_x^2(30b_2I_2L_1 - 5I_{2u}L_1 - 6I_2^2) + 5\psi_y H I_2L_1. \end{aligned} \quad (6.47)$$

Let $Q_1 \neq 0$, then

$$\psi_x = -\varphi_x Q_2/Q_1. \quad (6.48)$$

Substituting H_y into (6.44), one obtains

$$H = \frac{\varphi_x^2}{\psi_y}\frac{Q_9}{5L_1Q_1^3}, \quad (6.49)$$

[3] Notations of Q_i used in this section do not coincide with the notations used in the previous section where $L_2 \neq 0$.

where

$$Q_9 = -(10Q_{2u}L_1Q_1Q_2 + 11L_{1t}Q_1^2Q_2 - 5Q_{2t}L_1Q_1^2 - 5Q_{1u}L_1Q_2^2 + 11I_2Q_1Q_2^2 - 30b_2L_1Q_1Q_2^2 - 9b_3L_1Q_1^2Q_2 + 5b_4L_1Q_1^3).$$

Substituting H and ψ_x into the equations $H_x = 0$, and (6.47), (6.42), (6.41), one obtains

$$6b_3Q_9L_1Q_1^2 - 135b_2Q_9L_1Q_1Q_2 - 5Q_{9t}L_1Q_1^2 + 5Q_{9u}L_1Q_1Q_2 + 41L_{1t}Q_9Q_1^2 - 15Q_{1u}Q_9L_1Q_2 + 15Q_{2u}Q_9L_1Q_1 + 41I_2Q_9Q_1Q_2 = 0, \quad (6.50)$$

$$(L_{1t}Q_1 + I_2Q_2 - 15b_2L_1Q_2 + 6b_3L_1Q_1)Q_1 + 5(Q_{1u}Q_2 - Q_{2u}Q_1)L_1 = 0 \quad (6.51)$$

Notice that in this case

$$5L_1Q_1\varphi_{xx} = (3b_3L_1Q_1 - 2(L_{1t}Q_1 + I_2Q_2))\varphi_x^2, \quad \psi_{yy} = -3b_2\psi_y^2. \quad (6.52)$$

Hence, if $Q_1 \neq 0$, then the necessary and sufficient conditions are $v_5 = 0, \ w_1 = 0$, (6.50) and (6.51). The involutive system of equations for the functions $\varphi(x)$, $\psi(x,y)$ and $H(y)$ consists of the equations (6.48), (6.49) and (6.52).

If $Q_1 = 0$, then by virtue of $\varphi_x \neq 0$, Equation (6.46) gives $Q_2 = 0$. Since $(H_y)_x = 0$, one obtains

$$(6b_3Q_8L_1 + 5Q_{8t}L_1 - 14L_{1t}Q_8)\varphi_x + (3I_2Q_8 - 125L_1^4)\psi_x = 0, \quad (6.53)$$

where

$$Q_8 = 5L_{1tt}L_1 - 6L_{1t}^2 + 3b_3L_1(18b_3L_1 + L_{1t}) - 5L_1^2(5b_{4u} - 6b_{3t}) - 5b_4L_1(15b_2L_1 - I_2).$$

Assume that $3I_2Q_8 - 125L_1^4 = 0$, then

$$6b_3Q_8L_1 + 5Q_{8t}L_1 - 14L_{1t}Q_8 = 0, \quad (6.54)$$

and there is no other restrictions. In this case the system consisting of equations (6.42), (6.41), (6.45) and (6.47) is involutive.

Assume that $3I_2Q_8 - 125L_1^4 \neq 0$. Then

$$\psi_x = -(6b_3Q_8L_1 + 5Q_{8t}L_1 - 14L_{1t}Q_8)\varphi_x/(3I_2Q_8 - 125L_1^4)$$

Equations (6.42), (6.41) give

$$H = (\varphi_x^2Q_7)/(\psi_yL_1(9I_2^2Q_8^2 - 750I_2Q_8L_1^4 + 15625L_1^8)), \quad (6.55)$$

where

$$Q_7 = (125L_1^4 - 3I_2Q_8)(Q_{3t}L_1 + 27b_3L_1Q_3 - b_4L_1(125L_1^4 - 3I_2Q_8)) - 21Q_{3u}L_1Q_3 + 189b_2L_1Q_3^2 - 68I_2Q_3^2,$$

$$Q_3 = -(5Q_{8t}L_1 - 14L_{1t}Q_8 + 6b_3Q_8L_1).$$

Composing the equation $H_x = 0$, one obtains

$$\begin{aligned} &25L_1(Q_{7t} + 60b_3Q_7)(3I_2Q_8 - 125L_1^4) - 10800b_2Q_7L_1Q_3 \\ &+1200Q_{3u}Q_7L_1 + Q_3Q_8(3I_2Q_8 - 125L_1^4)^2 + 3890I_2Q_7Q_3 = 0. \end{aligned} \tag{6.56}$$

Hence, in the case $Q_1 = 0$ the necessary and sufficient conditions are $v_5 = 0, \; w_1 = 0$, $Q_2 = 0$ and (6.56). The involutive system of equations for the functions $\varphi(x)$, $\psi(x, y)$ and $H(y)$ consists of the Equations (6.53), (6.40), (6.55), and

$$\begin{aligned} 5\varphi_{xx}L_1(3I_2Q_8 - 125L_1^4) = \varphi_x^2(&15b_3L_1(3I_2Q_8 - 125L_1^4) \\ &- 90b_2L_1Q_3 + 34I_2Q_3 + 10Q_{3u}L_1). \end{aligned}$$

Thus, in the case $L_2 = 0$ the result can be summarized as the following theorem.

Theorem 2. The necessary and sufficient conditions for a nonlinearizable Equation (2.5) to be equivalent with respect to a change of variables (2.4) to equation of the form (2.3) are $v_5 = 0, \; w_1 = 0$, and:

(a) if $Q_1 \neq 0$, then (6.51);

(b) if $Q_1 = 0$, and $3I_2Q_8 - 125L_1^4 = 0$, then $Q_2 = 0$ and (6.54);

(c) if $Q_1 = 0$, and $3I_2Q_8 - 125L_1^4 \neq 0$, then $Q_2 = 0$ and (6.56).

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Linearization of fourth-order ordinary differential equations by point transformations

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

SERGEY V. MELESHKO AND SUPAPORN SUKSERN
School of Matematics,
Suranaree University of Technology,
Nakhon Ratchasima, 30000, Thailand

Abstract. We present here the solution of the problem on linearization of fourth-order equations by means of point transformations. We show that all fourth-order equations that are linearizable by point transformations are contained in the class of equations which is linear in the third-order derivative. We provide the linearization test and describe the procedure for obtaining the linearizing transformations as well as the linearized equation.

1 Introduction

The problem on linearization of second-order ordinary differential equations by means of point transformations was solved by Sophus Lie [1] in 1883. More specifically, he showed that the linearizable equations are at most cubic in the first-order derivative and gave the linearization test in terms of the coefficients of these equations.

In 1997, G. Grebot [2] studied the linearization of third-order equations by means of a restricted class of point transformations, namely $t = \varphi(x), u = \psi(x, y).$ However, the problem was not completely solved.

In 2004, N.H. Ibragimov and S.V. Meleshko [3] solved the problem of linearization of third-order equations by means of point transformations. They showed that all third-order equations that are linearizable by point transformations are contained either in the class of equations which is linear in the second-order derivative, or in the class of equations which is quadratic in the second-order derivative. They provided the linearization test for each of these classes and describe the procedure for obtaining the linearizing transformations as well as the linearized equation.

The present paper is devoted to obtain criteria for a fourth-order equation to be linearizable by change of the dependent and independent variables. In our calculations we used computer algebra packages. The final results were checked by comparing with theoretical results on invariants as well as by applying to numerous known and new examples of linearization. The paper is organized as follows.

2 Point transformations of fourth-order equations

We consider the fourth-order ordinary differential equation

$$y^{(4)} = f(x, y, y', y'', y''') . \tag{2.1}$$

We apply a point transformation

$$t = \varphi(x, y), \quad u = \psi(x, y) \tag{2.2}$$

to equation (2.1).

We begin with investigating the necessary conditions for linearization. The general form of (2.1) that can be obtained from linear equations by any point transformations (2.2) is found on this step. In consequence, we identify two candidates for linearization.

A linear fourth-order ordinary differential equation we use in the Laguerre form. In 1879, E. Laguerre showed that in linear ordinary differential equation of order $n \geq 3$ the two terms of orders next below the highest can be simultaneously removed by equivalence transformation (see [4], Section 10.2.1 and the references therein). Therefore, we write the general linear fourth-order equation in Laguerre's form

$$u^{(4)} + \alpha(t)\, u' + \beta(t)\, u = 0, \tag{2.3}$$

where t and u are the independent and dependent variables, respectively.

2.1 The candidates for linearization

Considering t and u as the new independent and dependent variables, respectively, one obtains the following transformation of the first-order derivative

$$u' = \frac{D_x(\psi)}{D_x(\varphi)} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}, \tag{2.4}$$

where $\varphi_x = \partial\varphi/\partial x, \ \varphi_y = \partial\varphi/\partial y,$ etc., and

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + y^{(4)}\frac{\partial}{\partial y'''} + \cdots$$

is the total derivative. Likewise, one obtains the transformation of derivatives of the second and higher order. Namely, denoting by $P(x;y;y')$ the right-hand side of (2.4),

$$P(x;y;y') = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}$$

one has

$$u'' = \frac{D_x(P)}{D_x(\varphi)} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x + y'\varphi_y} = \frac{\Delta}{(\varphi_x + y'\varphi_y)^3} y'' + \cdots. \tag{2.5}$$

Denoting by $Q(x;y;y';y'')$ the right-hand side of (2.5),

$$Q = \frac{\Delta}{(\varphi_x + y'\varphi_y)^3} y'' + \cdots$$

one has

$$\begin{aligned} u''' &= \frac{D_x(Q)}{D_x(\varphi)} = \frac{Q_x + y'Q_y + y''Q_{y'} + y'''Q_{y''}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}\left[(\varphi_x + y'\varphi_y)y''' - 3\varphi_y(y'')^2\right] + \cdots. \end{aligned} \tag{2.6}$$

Denoting by $R(x;y;y';y'';y''')$ the right-hand side of (2.6),

$$R = \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}\left[(\varphi_x + y'\varphi_y)y''' - 3\varphi_y(y'')^2\right] + \cdots$$

hence,

$$\begin{aligned} u^{(4)} &= \frac{D_x(R)}{D_x(\varphi)} = \frac{R_x + y'R_y + y''R_{y'} + y'''R_{y''} + y^{(4)}R_{y'''}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^7}\left[(\varphi_x + y'\varphi_y)^2 y^{(4)}\right] + \cdots. \end{aligned} \tag{2.7}$$

Thus, (2.3) becomes

$$\begin{aligned} &\frac{1}{(\varphi_x + y'\varphi_y)^7}\Big[(\varphi_x + \varphi_y y')^2 \Delta y^{(4)} + [-10\Delta(\varphi_x + \varphi_y y')\varphi_y y'' \\ &- 2\big(2(5\varphi_{xy}\Delta - \varphi_y\Delta_x)\varphi_y + (5\varphi_{yy}\Delta - 4\varphi_y\Delta_y)\varphi_x\big)y'^2 \\ &- 2\big(5(2\varphi_{xy}\varphi_x + \varphi_{xx}\varphi_y)\Delta - 2(\varphi_x\Delta_y + 2\varphi_y\Delta_x)\varphi_x\big)y' + \cdots]y''' + \cdots\Big] = 0. \end{aligned} \tag{2.8}$$

Here

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$$

is the Jacobian of the change of variables (2.2). It is manifest from (2.8) that the transformations (2.2) with $\varphi_y = 0$ and $\varphi_y \neq 0$, respectively, provide two distinctly different candidates for linearization.

If $\varphi_y = 0$ we work out the missing terms in (2.8), substitute the resulting expression in (2.3) and obtain the following equation

$$\begin{aligned} y^{(4)} & +(A_1 y' + A_0)y''' + B_0 y''^2 + (C_2 y'^2 + C_1 y' + C_0)y'' \\ & + D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0 = 0, \end{aligned} \tag{2.9}$$

where

$$A_1 = 4(\psi_y)^{-1}\psi_{yy}, \tag{2.10}$$

$$A_0 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy}), \tag{2.11}$$

$$B_0 = 3(\psi_y)^{-1}\psi_{yy}, \tag{2.12}$$

$$C_2 = 6(\psi_y)^{-1}\psi_{yyy}, \tag{2.13}$$

$$C_1 = -6(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy}), \tag{2.14}$$

$$C_0 = -(\varphi_x^2\psi_y)^{-1}\Big[(4\varphi_{xxx}\varphi_x - 15\varphi_{xx}^2)\psi_y + 6(3\varphi_{xx}\psi_{xy} - \varphi_x\psi_{xxy})\varphi_x\Big], \tag{2.15}$$

$$D_4 = (\psi_y)^{-1}\psi_{yyyy}, \tag{2.16}$$

$$D_3 = -2(\varphi_x\psi_y)^{-1}(3\varphi_{xx}\psi_{yyy} - 2\varphi_x\psi_{xyyy}), \tag{2.17}$$

$$D_2 = -(\varphi_x^2\psi_y)^{-1}(4\varphi_{xxx}\varphi_x\psi_{yy} - 15\varphi_{xx}^2\psi_{yy} + 18\varphi_{xx}\varphi_x\psi_{xyy} - 6\varphi_x^2\psi_{xxyy}), \tag{2.18}$$

$$\begin{aligned} D_1 = & -(\varphi_x^3\psi_y)^{-1}\Big[3(5\varphi_{xx}^2\psi_y - 10\varphi_{xx}\varphi_x\psi_{xy} + 6\varphi_x^2\psi_{xxy})\varphi_{xx} \\ & - (\varphi_x^3\psi_y\alpha + 4\psi_{xxxy})\varphi_x^3 - 2(5\varphi_{xx}\psi_y - 4\varphi_x\psi_{xy})\varphi_{xxx}\varphi_x + \varphi_{xxxx}\varphi_x^2\psi_y\Big], \end{aligned} \tag{2.19}$$

$$\begin{aligned} D_0 = & -(\varphi_x^3\psi_y)^{-1}\Big[(15\varphi_{xx}^3 - \varphi_x^6\alpha + \varphi_{xxxx}\varphi_x^2)\psi_x - (10\varphi_{xxx}\varphi_{xx}\psi_x \\ & - 4\varphi_{xxx}\varphi_x\psi_{xx} + 15\varphi_{xx}^2\psi_{xx} - 6\varphi_{xx}\varphi_x\psi_{xxx} + \varphi_x^6\beta + \varphi_x^2\psi_{xxxx})\varphi_x\Big]. \end{aligned} \tag{2.20}$$

Definition 2.1. We call (2.9) with arbitrary coefficients $A_0 = A_0(x,y)$, $A_1 = A_1(x,y)$, $B_0 = B_0(x,y)$, $C_0 = C_0(x,y)$, $C_1 = C_1(x,y)$, $C_2 = C_2(x,y)$, and $D_i = D_i(x,y), (i = 0,\ldots,4)$, the first candidate for linearization.

If $\varphi_y \neq 0$, we proceed likewise and setting $r(x,y) = \frac{\varphi_x}{\varphi_y}$, arrive at the following equation

$$\begin{aligned} & y^{(4)} + \tfrac{1}{y'+r}(-10y'' + F_2y'^2 + F_1y' + F_0)y''' \\ & + \tfrac{1}{(y'+r)^2}\big[15y''^3 + (H_2y'^2 + H_1y' + H_0)y''^2 \\ & + (J_4y'^4 + J_3y'^3 + J_2y'^2 + J_1y' + J_0)y'' \\ & + K_7y'^7 + K_6y'^6 + K_5y'^5 + K_4y'^4 \\ & + K_3y'^3 + K_2y'^2 + K_1y' + K_0\big] = 0, \end{aligned} \tag{2.21}$$

where

$$F_2 = -2(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y); \quad (2.22)$$

$$F_1 = 4(\varphi_y\Delta)^{-1}\Big[(\Delta_x + \Delta_y r - 5r_y\Delta)\varphi_y - 5\varphi_{yy}r\Delta\Big]; \quad (2.23)$$

$$F_0 = -2(\varphi_y\Delta)^{-1}\Big[((5r_y\Delta - 2\Delta_x)r + 5r_x\Delta)\varphi_y + 5\varphi_{yy}r^2\Delta\Big]; \quad (2.24)$$

$$H_2 = 6(\varphi_y\Delta)^{-1}(5\varphi_{yy}\Delta - 2\varphi_y\Delta_y); \quad (2.25)$$

$$H_1 = -3(\varphi_y\Delta)^{-1}\Big[(5\Delta_x + 3\Delta_y r - 25r_y\Delta)\varphi_y - 20\varphi_{yy}r\Delta\Big]; \quad (2.26)$$

$$H_0 = 3(\varphi_y\Delta)^{-1}\Big[(5(3r_x + 2r_y r)\Delta - (5\Delta_x - \Delta_y r)r)\varphi_y + 10\varphi_{yy}r^2\Delta\Big]; \quad (2.27)$$

$$J_4 = -(\varphi_y^2\Delta)^{-1}(10\varphi_{yyy}\varphi_y\Delta - 45\varphi_{yy}^2\Delta + 30\varphi_{yy}\varphi_y\Delta_y - 6\varphi_y^2\Delta_{yy}); \quad (2.28)$$

$$\begin{aligned} J_3 = {} & 2(\varphi_y^2\Delta)^{-1}\Big[3((2(\Delta_{xy} + \Delta_{yy}r - 5r_y\Delta_y) - 5r_{yy}\Delta)\varphi_y^2 \\ & - 5((\Delta_x + 3\Delta_y r - 4r_y\Delta)\varphi_y - 6\varphi_{yy}r\Delta)\varphi_{yy}) - 20\varphi_{yyy}\varphi_y r\Delta\Big]; \end{aligned} \quad (2.29)$$

$$\begin{aligned} J_2 = {} & 6(\varphi_y^2\Delta)^{-1}\Big[(\Delta_{xx} + \Delta_{yy}r^2 + 4\Delta_{xy}r - 5(2\Delta_x + 3\Delta_y r - 5r_y\Delta)r_y \\ & - 10r_{yy}r\Delta - 5r_x\Delta_y - 5r_{xy}\Delta)\varphi_y^2 - 5(((3(\Delta_x + \Delta_y r) - 10r_y\Delta)r \\ & - 2r_x\Delta)\varphi_y - 9\varphi_{yy}r^2\Delta)\varphi_{yy} - 10\varphi_{yyy}\varphi_y r^2\Delta\Big]; \end{aligned} \quad (2.30)$$

$$\begin{aligned} J_1 = {} & -2(\varphi_y^2\Delta)^{-1}\Big[((5(3(3\Delta_x + \Delta_y r) - 14r_y\Delta)r_y - 6(\Delta_{xy}r + \Delta_{xx}) \\ & + 20r_{yy}r\Delta)r + 5(3(\Delta_x + \Delta_y r) - 16r_y\Delta)r_x + 5r_{xx}\Delta + 20r_{xy}r\Delta)\varphi_y^2 \\ & + 15(((3\Delta_x + \Delta_y r - 8r_y\Delta)r - 4r_x\Delta)\varphi_y - 6\varphi_{yy}r^2\Delta)\varphi_{yy}r \\ & + 20\varphi_{yyy}\varphi_y r^3\Delta\Big]; \end{aligned} \quad (2.31)$$

$$\begin{aligned} J_0 = {} & -(\varphi_y^2\Delta)^{-1}\Big[((2((5r_{yy}r\Delta - 3\Delta_{xx})r + 5r_{xx}\Delta + 5r_{xy}r\Delta) \\ & - 5(7r_y\Delta - 6\Delta_x)r_y r)r - 5(2(7r_y\Delta - 3\Delta_x)r + 9r_x\Delta)r_x)\varphi_y^2 \\ & - 5(3(2((2r_y\Delta - \Delta_x)r + 2r_x\Delta)\varphi_y + 3\varphi_{yy}r^2\Delta)\varphi_{yy} \\ & - 2\varphi_{yyy}\varphi_y r^2\Delta)r^2\Big]; \end{aligned} \quad (2.32)$$

$$
\begin{aligned}
\mathbf{K_7} = \; & ¡\; ('^{2}_{y}¢)^{-1}\Big[\,'_{yyyy}\,'^{2}_{y}Ã_y \;¡\; 10\,'_{yyy}\,'_{yy}\,'_{y}Ã_y + 4\,'_{yyy}\,'^{2}_{y}Ã_{yy} + 15\,'^{3}_{yy}Ã_y \\
& ¡\; 15\,'^{2}_{yy}\,'_{y}Ã_{yy} + 6\,'_{yy}\,'^{2}_{y}Ã_{yyy} \;¡\; '^{7-}_{y}Ã \;¡\; '^{6}_{y}Ã_y® \;¡\; '^{3}_{y}Ã_{yyyy}\Big]; \qquad (2.33)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{K_6} = \; & ('^{3}_{y}¢)^{-1}\Big[3(5((7\,'_{y}Ã_{yy}\mathbf{r} \;¡\; 6¢_y)\,'_{y} \;¡\; 7('_{y}Ã_y\mathbf{r} \;¡\; ¢)\,'_{yy})\,'_{yy} \\
& ¡\; 2(7\,'_{y}Ã_{yyy}\mathbf{r} \;¡\; 5¢_{yy})\,'^{2}_{y})\,'_{yy} + (7\,'^{5-}_{y}Ã\mathbf{r} + 7\,'^{4}_{y}Ã_y®\mathbf{r} \;¡\; '^{3}_{y}®¢ \\
& + 7\,'_{y}Ã_{yyyy}\mathbf{r} \;¡\; 4¢_{yyy})\,'^{3}_{y} + 2(35\,'_{yy}\,'_{y}Ã_y\mathbf{r} \;¡\; 30\,'_{yy}¢ \;¡\; 14\,'^{2}_{y}Ã_{yy}\mathbf{r} \\
& + 10\,'_{y}¢_y)\,'_{yyy}\,'_{y} \;¡\; (7\,'_{y}Ã_y\mathbf{r} \;¡\; 5¢)\,'_{yyyy}\,'^{2}_{y}\Big]; \qquad (2.34)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{K_5} = \; & ¡\; ('^{3}_{y}¢)^{-1}\Big[(2(3(¢_{xyy} + 3¢_{yyy}\mathbf{r} \;¡\; 5\mathbf{r}_y¢_{yy} \;¡\; 5\mathbf{r}_{yy}¢_y) \;¡\; 5\mathbf{r}_{yyy}¢) \\
& ¡\; 3(7\,'^{4-}_{y}Ã\mathbf{r} + 7\,'^{3}_{y}Ã_y®\mathbf{r} \;¡\; 2\,'^{2}_{y}®¢ + 7Ã_{yyyy}\mathbf{r})\,'_{y}\mathbf{r})\,'^{3}_{y} \\
& ¡\; 3(2(5(¢_{xy} + 5¢_{yy}\mathbf{r} \;¡\; 4\mathbf{r}_y¢_y \;¡\; 2\mathbf{r}_{yy}¢) \;¡\; 21\,'_{y}Ã_{yyy}\mathbf{r}^2)\,'^{2}_{y} \\
& ¡\; 15((¢_x + 11¢_y\mathbf{r} \;¡\; 3\mathbf{r}_y¢ \;¡\; 7\,'_{y}Ã_{yy}\mathbf{r}^2)\,'_{y} \\
& + 7('_{y}Ã_y\mathbf{r} \;¡\; 2¢)\,'_{yy}\mathbf{r})\,'_{yy})\,'_{yy} \;¡\; 2((5(¢_x + 11¢_y\mathbf{r} \;¡\; 3\mathbf{r}_y¢) \\
& ¡\; 42\,'_{y}Ã_{yy}\mathbf{r}^2)\,'_{y} + 15(7\,'_{y}Ã_y\mathbf{r} \;¡\; 12¢)\,'_{yy}\mathbf{r})\,'_{yyy}\,'_{y} \\
& + 3(7\,'_{y}Ã_y\mathbf{r} \;¡\; 10¢)\,'_{yyyy}\,'^{2}_{y}\mathbf{r}\Big]; \qquad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{K_4} = \; & ¡\; ('^{3}_{y}¢)^{-1}\Big[(2(45\mathbf{r}_{yy}\mathbf{r}_y¢ \;¡\; 10\mathbf{r}_{yy}¢_x \;¡\; 55\mathbf{r}_{yy}¢_y\mathbf{r} + 50\mathbf{r}^2_y¢_y \\
& ¡\; 20\mathbf{r}_y¢_{xy} \;¡\; 50\mathbf{r}_y¢_{yy}\mathbf{r} + 11¢_{xyy}\mathbf{r} + 2¢_{xxy} + 17¢_{yyy}\mathbf{r}^2 \\
& ¡\; 20\mathbf{r}_{yyy}\mathbf{r}¢ \;¡\; 5\mathbf{r}_x¢_{yy} \;¡\; 10\mathbf{r}_{xy}¢_y \;¡\; 5\mathbf{r}_{xyy}¢) \\
& ¡\; 5(7\,'^{4-}_{y}Ã\mathbf{r} + 7\,'^{3}_{y}Ã_y®\mathbf{r} \;¡\; 3\,'^{2}_{y}®¢ + 7Ã_{yyyy}\mathbf{r})\,'_{y}\mathbf{r}^2)\,'^{3}_{y} \\
& + 15((3((5(¢_x + 5¢_y\mathbf{r}) \;¡\; 14\mathbf{r}_y¢)\mathbf{r} \;¡\; \mathbf{r}_x¢) \;¡\; 35\,'_{y}Ã_{yy}\mathbf{r}^3)\,'_{y} \\
& + 35('_{y}Ã_y\mathbf{r} \;¡\; 3¢)\,'_{yy}\mathbf{r}^2)\,'^{2}_{yy} \;¡\; 10(¢_{xx} + 31¢_{yy}\mathbf{r}^2 + 13¢_{xy}\mathbf{r} \\
& ¡\; 8(¢_x + 6¢_y\mathbf{r} \;¡\; 2\mathbf{r}_y¢)\mathbf{r}_y \;¡\; 26\mathbf{r}_{yy}\mathbf{r}¢ \;¡\; 4\mathbf{r}_x¢_y \;¡\; 4\mathbf{r}_{xy}¢ \\
& ¡\; 21\,'_{y}Ã_{yyy}\mathbf{r}^3)\,'_{yy}\,'^{2}_{y} \;¡\; 10(((5(¢_x + 5¢_y\mathbf{r}) \;¡\; 14\mathbf{r}_y¢)\mathbf{r} \;¡\; \mathbf{r}_x¢ \\
& ¡\; 14\,'_{y}Ã_{yy}\mathbf{r}^3)\,'_{y} + 5(7\,'_{y}Ã_y\mathbf{r} \;¡\; 18¢)\,'_{yy}\mathbf{r}^2)\,'_{yyy}\,'_{y} \\
& + 5(7\,'_{y}Ã_y\mathbf{r} \;¡\; 15¢)\,'_{yyyy}\,'^{2}_{y}\mathbf{r}^2\Big]; \qquad (2.36)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{K_3} = {} & ¡\ ('^3_y¢)^{-1}\Big[((13¢_{xxy} + 35¢_{yyy}\mathbf{r}^2)\mathbf{r} + ¢_{xxx} + 31¢_{xyy}\mathbf{r}^2 \\
& ¡\ 5(3¢_{xx} + 26¢_{yy}\mathbf{r}^2 + 23¢_{xy}\mathbf{r}\ ¡\ (15¢_x + 49¢_y\mathbf{r}\ ¡\ 25\mathbf{r}_y¢)\mathbf{r}_y)\mathbf{r}_y \\
& ¡\ 5(13¢_x + 32¢_y\mathbf{r}\ ¡\ 50\mathbf{r}_y¢)\mathbf{r}_{yy}\mathbf{r}\ ¡\ 65\mathbf{r}_{yyy}\mathbf{r}^2¢\ ¡\ 5(3¢_{xy} + 5¢_{yy}\mathbf{r} \\
& ¡\ 16\mathbf{r}_y¢_y\ ¡\ 7\mathbf{r}_{yy}¢)\mathbf{r}_x\ ¡\ 5\mathbf{r}_{xx}¢_y\ ¡\ 5\mathbf{r}_{xxy}¢ \\
& ¡\ 5(3¢_x + 11¢_y\mathbf{r}\ ¡\ 15\mathbf{r}_y¢)\mathbf{r}_{xy}\ ¡\ 30\mathbf{r}_{xyy}\mathbf{r}¢\ ¡\ 5(7'^4_y¯\mathbf{Ã}\mathbf{r} + 7'^3_y\mathbf{Ã}_y®\mathbf{r} \\
& ¡\ 4'^2_y®¢ + 7\mathbf{Ã}_{yyyy}\mathbf{r})'_y\mathbf{r}^3)'^3_y\ ¡\ 5(2((2(2¢_{xx} + 17¢_{yy}\mathbf{r}^2 + 11¢_{xy}\mathbf{r}) \\
& ¡\ (29¢_x + 75¢_y\mathbf{r}\ ¡\ 51\mathbf{r}_y¢)\mathbf{r}_y\ ¡\ 45\mathbf{r}_{yy}\mathbf{r}¢)\mathbf{r}\ ¡\ (3¢_x + 13¢_y\mathbf{r}\ ¡\ 13\mathbf{r}_y¢)\mathbf{r}_x \\
& ¡\ \mathbf{r}_{xx}¢\ ¡\ 14\mathbf{r}_{xy}\mathbf{r}¢\ ¡\ 21'_y\mathbf{Ã}_{yyy}\mathbf{r}^4)'^2_y\ ¡\ 3((6((5(¢_x + 3¢_y\mathbf{r})\ ¡\ 13\mathbf{r}_y¢)\mathbf{r} \\
& ¡\ 2\mathbf{r}_x¢)\ ¡\ 35'_y\mathbf{Ã}_{yy}\mathbf{r}^3)'_y + 35('_y\mathbf{Ã}_y\mathbf{r}\ ¡\ 4¢)'_{yy}\mathbf{r}^2)'_{yy}\mathbf{r})'_{yy} \\
& ¡\ 10(2((5(¢_x + 3¢_y\mathbf{r})\ ¡\ 13\mathbf{r}_y¢)\mathbf{r}\ ¡\ 2\mathbf{r}_x¢\ ¡\ 7'_y\mathbf{Ã}_{yy}\mathbf{r}^3)'_y \\
& + 5(7'_y\mathbf{Ã}_y\mathbf{r}\ ¡\ 24¢)'_{yy}\mathbf{r}^2)'_{yyy}'_y\mathbf{r} + 5(7'_y\mathbf{Ã}_y\mathbf{r}\ ¡\ 20¢)'_{yyyy}'^2_y\mathbf{r}^3\Big]; \qquad (2.37)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{K_2} = {} & ¡\ ('^3_y¢)^{-1}\Big[((3((5¢_{xxy} + 7¢_{yyy}\mathbf{r}^2)\mathbf{r} + ¢_{xxx} + 7¢_{xyy}\mathbf{r}^2) \\
& ¡\ (3(13¢_{xx} + 28¢_{yy}\mathbf{r}^2 + 39¢_{xy}\mathbf{r}) + (204\mathbf{r}_y¢\ ¡\ 161¢_x\ ¡\ 217¢_y\mathbf{r})\mathbf{r}_y)\mathbf{r}_y \\
& ¡\ (79¢_x + 116¢_y\mathbf{r}\ ¡\ 264\mathbf{r}_y¢)\mathbf{r}_{yy}\mathbf{r}\ ¡\ 54\mathbf{r}_{yyy}\mathbf{r}^2¢)\mathbf{r} \\
& ¡\ (3(2¢_{xx} + 7¢_{yy}\mathbf{r}^2 + 11¢_{xy}\mathbf{r}) + (171\mathbf{r}_y¢\ ¡\ 64¢_x\ ¡\ 140¢_y\mathbf{r})\mathbf{r}_y \\
& ¡\ 72\mathbf{r}_{yy}\mathbf{r}¢\ ¡\ 18\mathbf{r}_x¢_y)\mathbf{r}_x\ ¡\ (4¢_x + 11¢_y\mathbf{r}\ ¡\ 21\mathbf{r}_y¢)\mathbf{r}_{xx}\ ¡\ 12\mathbf{r}_{xxy}\mathbf{r}¢ \\
& ¡\ \mathbf{r}_{xxx}¢\ ¡\ ((37¢_x + 53¢_y\mathbf{r}\ ¡\ 150\mathbf{r}_y¢)\mathbf{r}\ ¡\ 33\mathbf{r}_x¢)\mathbf{r}_{xy}\ ¡\ 33\mathbf{r}_{xyy}\mathbf{r}^2¢ \\
& ¡\ 3(7'^4_y¯\mathbf{Ã}\mathbf{r} + 7'^3_y\mathbf{Ã}_y®\mathbf{r}\ ¡\ 5'^2_y®¢ + 7\mathbf{Ã}_{yyyy}\mathbf{r})'_y\mathbf{r}^4)'^3_y \\
& ¡\ 3(2(5((2¢_{xx} + 7¢_{yy}\mathbf{r}^2 + 6¢_{xy}\mathbf{r}\ ¡\ (13¢_x + 19¢_y\mathbf{r}\ ¡\ 20\mathbf{r}_y¢)\mathbf{r}_y \\
& ¡\ 13\mathbf{r}_{yy}\mathbf{r}¢)\mathbf{r}^2\ ¡\ ((3¢_x + 5¢_y\mathbf{r}\ ¡\ 11\mathbf{r}_y¢)\mathbf{r}\ ¡\ \mathbf{r}_x¢)\mathbf{r}_x\ ¡\ \mathbf{r}_{xx}\mathbf{r}¢ \\
& ¡\ 6\mathbf{r}_{xy}\mathbf{r}^2¢)\ ¡\ 21'_y\mathbf{Ã}_{yyy}\mathbf{r}^5)'^2_y\ ¡\ 15((2((5(¢_x + 2¢_y\mathbf{r})\ ¡\ 12\mathbf{r}_y¢)\mathbf{r} \\
& ¡\ 3\mathbf{r}_x¢)\ ¡\ 7'_y\mathbf{Ã}_{yy}\mathbf{r}^3)'_y + 7('_y\mathbf{Ã}_y\mathbf{r}\ ¡\ 5¢)'_{yy}\mathbf{r}^2)'_{yy}\mathbf{r}^2)'_{yy} \\
& ¡\ 2(2(5((5(¢_x + 2¢_y\mathbf{r})\ ¡\ 12\mathbf{r}_y¢)\mathbf{r}\ ¡\ 3\mathbf{r}_x¢)\ ¡\ 21'_y\mathbf{Ã}_{yy}\mathbf{r}^3)'_y \\
& + 15(7'_y\mathbf{Ã}_y\mathbf{r}\ ¡\ 30¢)'_{yy}\mathbf{r}^2)'_{yyy}'_y\mathbf{r}^2 + 3(7'_y\mathbf{Ã}_y\mathbf{r}\ ¡\ 25¢)'_{yyyy}'^2_y\mathbf{r}^4\Big]; \\
& \qquad (2.38)
\end{aligned}
$$

$$
\begin{aligned}
K_1 = & -(\varphi_y^3\Delta)^{-1}\Big[((7(\Delta_{xxy} + \Delta_{yyy}r^2)r + 3\Delta_{xxx} + 7\Delta_{xyy}r^2 \\
& - (33\Delta_{xx} + 28\Delta_{yy}r^2 + 49\Delta_{xy}r + 2(59r_y\Delta - 56\Delta_x - 42\Delta_y r)r_y)r_y \\
& - (43\Delta_x + 42\Delta_y r - 128r_y\Delta)r_{yy}r - 23r_{yyy}r^2\Delta)r^2 \\
& - ((12\Delta_{xx} + 7\Delta_{yy}r^2 + 21\Delta_{xy}r + 2(86r_y\Delta - 49\Delta_x - 35\Delta_y r)r_y \\
& - 49r_{yy}r\Delta)r + (85r_y\Delta - 15\Delta_x - 21\Delta_y r)r_x)r_x \\
& - ((8\Delta_x + 7\Delta_y r - 32r_y\Delta)r - 10r_x\Delta)r_{xx} - 9r_{xxy}r^2\Delta - 2r_{xxx}r\Delta \\
& - ((29\Delta_x + 21\Delta_y r - 95r_y\Delta)r - 46r_x\Delta)r_{xy}r - 16r_{xyy}r^3\Delta \\
& - (7\varphi_y^{4}\beta\psi r + 7\varphi_y^3\psi_y\alpha r - 6\varphi_y^2\alpha\Delta + 7\psi_{yyyy}r)\varphi_y r^5)\varphi_y^3 \\
& - (2(5((4\Delta_{xx} + 7\Delta_{yy}r^2 + 7\Delta_{xy}r - (23\Delta_x + 21\Delta_y r - 31r_y\Delta)r_y \\
& - 17r_{yy}r\Delta)r^2 - ((9\Delta_x + 7\Delta_y r - 27r_y\Delta)r - 6r_x\Delta)r_x - 3r_{xx}r\Delta \\
& - 10r_{xy}r^2\Delta) - 21\varphi_y\psi_{yyy}r^5)\varphi_y^2 - 15((3((5\Delta_x + 7\Delta_y r - 11r_y\Delta)r \\
& - 4r_x\Delta) - 7\varphi_y\psi_{yy}r^3)\varphi_y + 7(\varphi_y\psi_y r - 6\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}r \\
& - 2((5((5\Delta_x + 7\Delta_y r - 11r_y\Delta)r - 4r_x\Delta) - 14\varphi_y\psi_{yy}r^3)\varphi_y \\
& + 5(7\varphi_y\psi_y r - 36\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^3 + (7\varphi_y\psi_y r - 30\Delta)\varphi_{yyyy}\varphi_y^2 r^5\Big]; \qquad (2.39)
\end{aligned}
$$

$$
\begin{aligned}
K_0 = & (\varphi_y^3\Delta)^{-1}\Big[((((2(r_{xxy} + 2r_{yyy}r^2)r + r_{xxx} + 3r_{xyy}r^2)\Delta \\
& + 3(3\Delta_x + 2\Delta_y r - 8r_y\Delta)r_{yy}r^2)r - ((10r_x + 11r_y r)\Delta \\
& - (4\Delta_x + \Delta_y r)r)r_{xx} - ((13r_x + 20r_y r)\Delta - (7\Delta_x + 3\Delta_y r)r)r_{xy}r \\
& + ((\varphi_y^{4}\beta\psi + \varphi_y^3\psi_y\alpha + \psi_{yyyy})r - \varphi_y^2\alpha\Delta)\varphi_y r^5 + (9\Delta_{xx} + 4\Delta_{yy}r^2 \\
& + 7\Delta_{xy}r - 2(13\Delta_x + 6\Delta_y r - 12r_y\Delta)r_y)r_y r^2 - ((\Delta_{xxy} + \Delta_{yyy}r^2)r \\
& + \Delta_{xxx} + \Delta_{xyy}r^2)r^2)r - ((2((17\Delta_x + 5\Delta_y r - 23r_y\Delta)r_y + 6r_{yy}r\Delta) \\
& - (6\Delta_{xx} + \Delta_{yy}r^2 + 3\Delta_{xy}r))r^2 - (5(3r_x + 8r_y r)\Delta \\
& - 3(5\Delta_x + \Delta_y r)r)r_x)r_x)\varphi_y^3 - ((2((5(r_{xx} + 3r_{yy}r^2 + 2r_{xy}r)\Delta \\
& + 3\varphi_y\psi_{yyy}r^4 + 5(5\Delta_x + 3\Delta_y r - 6r_y\Delta)r_y r - 5(\Delta_{xx} + \Delta_{yy}r^2 + \Delta_{xy}r)r)r \\
& - 5((3r_x + 7r_y r)\Delta - (3\Delta_x + \Delta_y r)r)r_x)\varphi_y^2 - 15((3(r_x + 2r_y r)\Delta \\
& + \varphi_y\psi_{yy}r^3 - 3(\Delta_x + \Delta_y r)r)\varphi_y - (\varphi_y\psi_y r - 7\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy} \\
& + (2((5(r_x + 2r_y r)\Delta + 2\varphi_y\psi_{yy}r^3 - 5(\Delta_x + \Delta_y r)r)\varphi_y \\
& - 5(\varphi_y\psi_y r - 6\Delta)\varphi_{yy}r^2)\varphi_{yyy} + (\varphi_y\psi_y r - 5\Delta)\varphi_{yyyy}\varphi_y r^2)\varphi_y r^2)r^2\Big]. \qquad (2.40)
\end{aligned}
$$

Definition 2.2. We call (2.21) with arbitrary coefficients $r = r(x,y)$, $F_0 = F_0(x,y)$, $F_1 = F_1(x,y)$, $F_2 = F_2(x,y)$, $H_0 = H_0(x,y)$, $H_1 = H_1(x,y)$, $H_2 = H_2(x,y)$, $J_0 = J_0(x,y)$, $J_1 = J_1(x,y)$, $J_2 = J_2(x,y)$, $J_3 = J_3(x,y)$, $J_4 = J_4(x,y)$, and $K_i = K_i(x,y)$, $(i = 0,\ldots,7)$, the second candidate for linearization.

Thus, we showed that every linearizable fourth-order equations belong either to the class of (2.9) or to the class of (2.21). In Sections 2.2 and 2.3, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing point transformations for each candidate. Proofs of the main theorems and illustrative examples are provided in the subsequent sections.

2.2 The linearization test for equation (2.9)

Consider the first canditate for linearization, i.e. equation (2.9). In this case, the linearizing transformations (2.2) have the form

$$t = \varphi(x), \quad u = \psi(x, y). \tag{2.41}$$

Theorem 2.1. Equation (2.9)

$$\begin{aligned} y^{(4)} &+ (A_1 y' + A_0) y''' + B_0 y''^2 + (C_2 y'^2 + C_1 y' + C_0) y'' \\ &+ D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0 = 0, \end{aligned}$$

is linearizable if and only if its coefficients obey the following ten equations

$$A_{0y} - A_{1x} = 0, \tag{2.42}$$

$$4B_0 - 3A_1 = 0, \tag{2.43}$$

$$12A_{1y} + 3A_1^2 - 8C_2 = 0, \tag{2.44}$$

$$12A_{1x} + 3A_0A_1 - 4C_1 = 0, \tag{2.45}$$

$$32C_{0y} + 12A_{0x}A_1 - 16C_{1x} + 3A_0^2A_1 - 4A_0C_1 = 0, \tag{2.46}$$

$$4C_{2y} + A_1C_2 - 24D_4 = 0, \tag{2.47}$$

$$4C_{1y} + A_1C_1 - 12D_3 = 0, \tag{2.48}$$

$$16C_{1x} - 12A_{0x}A_1 - 3A_0^2A_1 + 4A_0C_1 + 8A_1C_0 - 32D_2 = 0, \tag{2.49}$$

$$\begin{aligned} &192D_{2x} + 36A_{0x}A_0A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 - 288D_{1y} + 9A_0^3A_1 \\ &- 12A_0^2C_1 - 36A_0A_1C_0 + 48A_0D_2 + 32C_0C_1 = 0, \end{aligned} \tag{2.50}$$

$$
\begin{aligned}
&384D_{1xy} - \Big[3((3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1)\\
&- 16(A_1C_0 - D_2)A_0)A_0 - 32(4(C_1D_1 - 2C_2D_0 + C_0D_2)\\
&+ (3A_1D_0 - C_0^2)A_1) - 96D_{1y}A_0 + 384D_{0y}A_1 + 1536D_{0yy}\\
&- 16(3A_0A_1 - 4C_1)C_{0x} + 12((3A_0A_1 - 4C_1)A_0\\
&- 4(A_1C_0 - 4D_2))A_{0x}\Big] = 0. \qquad (2.51)
\end{aligned}
$$

Provided that the conditions (2.42)-(2.51) are satisfied, the linearizing transformation (2.41) is defined by a fourth-order ordinary differential equation for the function $\varphi(x)$, namely by the Riccati equation

$$
40\frac{d\chi}{dx} - 20\chi^2 = 8C_0 - 3A_0^2 - 12A_{0x}, \qquad (2.52)
$$

for

$$
\chi = \frac{\varphi_{xx}}{\varphi_x}, \qquad (2.53)
$$

and by the following integrable system of partial differential equations for $\psi(x, y)$

$$
4\psi_{yy} = \psi_y A_1, \qquad (2.54)
$$

$$
4\psi_{xy} = \psi_y(A_0 + 6\chi), \qquad (2.55)
$$

and

$$
\begin{aligned}
&1600\psi_{xxxx} = 9600\psi_{xxx}\chi + 160\psi_{xx}(-12A_{0x} - 3A_0^2 - 90\chi^2 + 8C_0)\\
&+ 40\psi_x(12A_{0x}A_0 + 72A_{0x}\chi - 16C_{0x} + 3A_0^3 + 18A_0^2\chi - 12A_0C_0\\
&+ 120\chi^3 - 48\chi C_0 + 24D_1 - 8\Omega) + \psi(144A_{0x}^2 + 72A_{0x}A_0^2 - 352A_{0x}C_0\\
&- 160C_{0xx} - 80C_{0x}A_0 - 1600D_{0y} + 640D_{1x} - 80\Omega_x + 9A_0^4 - 88A_0^2C_0\\
&+ 160A_0D_1 + 30A_0\Omega - 400A_1D_0 + 300\chi\Omega + 144C_0^2) + 1600\psi_y D_0, \qquad (2.56)
\end{aligned}
$$

where χ is given by (2.53) and Ω is the following expression

$$
\Omega = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx}. \qquad (2.57)
$$

Finally, the coefficients α and β of the resulting linear equation (2.3) is given by

$$
\alpha = \frac{\Omega}{8\varphi_x^3}, \qquad (2.58)
$$

and

$$
\begin{aligned}
\beta = {} &(1600\varphi_x^4)^{-1}(-144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0\\
&+ 1600D_{0y} - 640D_{1x} + 80\Omega_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\Omega\\
&+ 400A_1D_0 - 300\chi\Omega - 144C_0^2). \qquad (2.59)
\end{aligned}
$$

Remark 2.1. Since the system of equations (2.42)-(2.51) provides the necessary and sufficient conditions for linearization, it is invariant with respect to the transformations (2.41). It means that the left-hand sides of (2.42)-(2.51) are *relative invariants* (of the second-order) for the equivalence group (2.41).

2.3 The linearization test for equation (2.21)

The following theorem provides the test for linearization of the second candidate. The necessary and sufficient conditions comprise *eighteen* differential equations (2.60)-(2.77) for *twenty one* coefficients of the (2.21). The linearizing change of variables (2.2) is determined by (2.78)-(2.81) for the functions $\varphi(x, y)$ and $\psi(x, y)$:

Theorem 2.2. Equation (2.21)

$$
\begin{aligned}
& y^{(4)} + \frac{1}{y'+r}(-10y'' + F_2 y'^2 + F_1 y' + F_0)y''' \\
& + \frac{1}{(y'+r)^2}[15y''^3 + (H_2 y'^2 + H_1 y' + H_0)y''^2 \\
& + (J_4 y'^4 + J_3 y'^3 + J_2 y'^2 + J_1 y' + J_0)y'' \\
& + K_7 y'^7 + K_6 y'^6 + K_5 y'^5 + K_4 y'^4 \\
& + K_3 y'^3 + K_2 y'^2 + K_1 y' + K_0] = 0,
\end{aligned}
$$

is linearizable if and only if its coefficients obey the following equations

$$10r_{yy} = -(F_{1y} + F_{2x} + F_{2y}r + r_yF_2); \tag{2.60}$$

$$10r_x = 10r_yr - F_0 + F_1r - F_2r^2; \tag{2.61}$$

$$H_2 = -3F_2; \tag{2.62}$$

$$4H_1 = -3(5F_1 - 2F_2r); \tag{2.63}$$

$$4H_0 = -3(6F_0 - F_1r); \tag{2.64}$$

$$10F_{1yy} = -(F_{1y}F_2 - 40F_{2xy} - 16F_{2x}F_2 + 20F_{2yy}r + 40F_{2y}r_y + 14F_{2y}F_2r + 20J_{4x} - 20J_{4y}r + 14r_yF_2^2 - 40r_yJ_4); \tag{2.65}$$

$$12F_{2x} = 12F_{2y}r - 3F_1F_2 + 6F_2^2r + 4J_3 - 16J_4r; \tag{2.66}$$

$$60F_{1x} = 60F_{1y}r - 36F_0F_2 - 15F_1^2 + 66F_1F_2r - 36F_2^2r^2 + 40J_2 - 80J_3r + 80J_4r^2; \tag{2.67}$$

$$60F_{0x} = 60F_{0y}r - 51F_0F_1 + 66F_0F_2r + 36F_1^2r - 72F_1F_2r^2 + 36F_2^2r^3 + 60J_1 - 80J_2r + 80J_3r^2 - 80J_4r^3; \tag{2.68}$$

$$20J_0 = 9F_0^2 - 18F_0F_1r + 18F_0F_2r^2 + 9F_1^2r^2 - 18F_1F_2r^3 + 9F_2^2r^4 + 20J_1r - 20J_2r^2 + 20J_3r^3 - 20J_4r^4; \tag{2.69}$$

$$\begin{aligned}120J_{3yy} = {} & 216F_{1y}F_{2y} + 54F_{1y}F_2^2 - 48F_{1y}J_4 + 360F_{2yy}r_y + 90F_{2yy}F_1 \\ & - 180F_{2yy}F_2r - 432F_{2y}^2r + 324F_{2y}r_yF_2 + 189F_{2y}F_1F_2 - 486F_{2y}F_2^2r \\ & - 192F_{2y}J_3 + 864F_{2y}J_4r - 60J_{3y}F_2 + 720J_{4xy} + 180J_{4x}F_2 - 240J_{4yy}r \\ & - 1200J_{4y}r_y + 60J_{4y}F_2r + 720K_{6x} - 720K_{6y}r - 5040K_{7x}r + 5040K_{7y}r^2 \\ & + 36r_yF_2^3 - 432r_yF_2J_4 - 2160r_yK_6 + 15120r_yK_7r + 504F_0K_7 + 36F_1F_2^3 \\ & - 102F_1F_2J_4 - 504F_1K_7r - 72F_2^4r - 48F_2^2J_3 + 396F_2^2J_4r + 504F_2K_7r^2 \\ & + 136J_3J_4 - 544J_4^2r;\end{aligned} \tag{2.70}$$

$$
\begin{aligned}
240J_{4xyy} = {} & ¡(36F_{1y}F_{2yy} + 162F_{1y}F_{2y}F_2 ¡ 72F_{1y}J_{4y} + 36F_{1y}F_2^3 \\
& ¡ 168F_{1y}F_2J_4 ¡ 72F_{1y}K_6 ¡ 168F_{1y}K_7r ¡ 72F_{2yy}F_{2y}r + 144F_{2yy}r_yF_2 \\
& + 54F_{2yy}F_1F_2 ¡ 108F_{2yy}F_2^2r ¡ 72F_{2yy}J_3 + 288F_{2yy}J_4r + 432F_{2y}^2r_y \\
& + 108F_{2y}^2F_1 ¡ 540F_{2y}^2F_2r ¡ 144F_{2y}J_{3y} + 528F_{2y}J_{4x} + 192F_{2y}J_{4y}r \\
& + 324F_{2y}r_yF_2^2 ¡ 1008F_{2y}r_yJ_4 + 162F_{2y}F_1F_2^2 ¡ 132F_{2y}F_1J_4 ¡ 396F_{2y}F_2^3r \\
& ¡ 180F_{2y}F_2J_3 + 1320F_{2y}F_2J_4r + 144F_{2y}K_6r ¡ 336F_{2y}K_7r^2 ¡ 36J_{3y}F_2^2 \\
& + 176J_{3y}J_4 + 120J_{4xy}F_2 + 132J_{4x}F_2^2 ¡ 432J_{4x}J_4 ¡ 240J_{4yyy}r ¡ 960J_{4yy}r_y \\
& ¡ 120J_{4yy}F_2r ¡ 768J_{4y}r_yF_2 ¡ 138J_{4y}F_1F_2 + 288J_{4y}F_2^2r + 184J_{4y}J_3 \\
& ¡ 1008J_{4y}J_4r + 960K_{6xy} + 240K_{6x}F_2 ¡ 960K_{6yy}r ¡ 3840K_{6y}r_y \\
& ¡ 240K_{6y}F_2r ¡ 1920K_{7xy}r ¡ 2400K_{7xx} + 2880K_{7x}r_y ¡ 600K_{7x}F_1 \\
& ¡ 480K_{7x}F_2r + 4320K_{7yy}r^2 + 24000K_{7y}r_yr + 432K_{7y}F_0 + 168K_{7y}F_1r \\
& + 912K_{7y}F_2r^2 + 20160r_y^2K_7 + 1728r_yF_1K_7 + 36r_yF_2^4 ¡ 264r_yF_2^2J_4 \\
& ¡ 1248r_yF_2K_6 + 5280r_yF_2K_7r + 160r_yJ_4^2 + 408F_0F_2K_7 + 150F_1^2K_7 \\
& + 27F_1F_2^4 ¡ 120F_1F_2^2J_4 ¡ 168F_1F_2K_6 + 168F_1F_2K_7r ¡ 54F_2^5r ¡ 36F_2^3J_3 \\
& + 384F_2^3J_4r + 336F_2^2K_6r ¡ 1344F_2^2K_7r^2 + 160F_2J_3J_4 ¡ 640F_2J_4^2r \\
& ¡ 400J_2K_7 + 224J_3K_6 ¡ 368J_3K_7r ¡ 896J_4K_6r + 3872J_4K_7r^2 \\
& + 672F_{0y}K_7);
\end{aligned} \tag{2.71}
$$

$$
4J_{4x} = 4J_{4y}r ¡ F_1J_4 + 2F_2J_4r ¡ 4K_5 + 24K_6r ¡ 84K_7r^2; \tag{2.72}
$$

$$
\begin{aligned}
60F_{0yy} = {} & ¡(30F_{0y}F_2 + 36F_{1y}F_1 ¡ 36F_{1y}F_2r ¡ 60F_{2yy}r^2 + 24F_{2y}F_0 \\
& ¡ 36F_{2y}F_1r ¡ 54F_{2y}F_2r^2 ¡ 40J_{2y} + 40J_{3y}r + 80J_{4y}r^2 \\
& ¡ 36r_yF_1F_2 + 36r_yF_2^2r + 40r_yJ_3 ¡ 80r_yJ_4r + 6F_0F_2^2 ¡ 6F_0J_4 \\
& + 9F_1^2F_2 ¡ 18F_1F_2^2r ¡ 12F_1J_3 + 24F_1J_4r ¡ 6F_2^3r^2 ¡ 10F_2J_2 \\
& + 22F_2J_3r + 26F_2J_4r^2 ¡ 60K_4 + 180K_5r ¡ 180K_6r^2 ¡ 420K_7r^3);
\end{aligned} \tag{2.73}
$$

$$
\begin{aligned}
20J_{2x} = {} & 20J_{2y}r + 20J_{3x}r ¡ 20J_{3y}r^2 ¡ 14F_0J_3 + 28F_0J_4r ¡ 5F_1J_2 \\
& + 19F_1J_3r ¡ 28F_1J_4r^2 + 10F_2J_2r ¡ 24F_2J_3r^2 + 28F_2J_4r^3 \\
& ¡ 120K_3 + 360K_4r ¡ 640K_5r^2 + 840K_6r^3 ¡ 840K_7r^4;
\end{aligned} \tag{2.74}
$$

$$
\begin{aligned}
60J_{1x} = {} & 60J_{1y}r ¡ 40J_{3x}r^2 + 40J_{3y}r^3 ¡ 42F_0J_2 + 42F_0J_3r ¡ 70F_0J_4r^2 \\
& ¡ 15F_1J_1 + 42F_1J_2r ¡ 52F_1J_3r^2 + 70F_1J_4r^3 + 30F_2J_1r \\
& ¡ 42F_2J_2r^2 + 62F_2J_3r^3 ¡ 70F_2J_4r^4 ¡ 600K_2 + 1080K_3r ¡ 1380K_4r^2 \\
& + 1700K_5r^3 ¡ 2100K_6r^4 + 2100K_7r^5;
\end{aligned} \tag{2.75}
$$

$$\begin{aligned}80K_1 = {} & 3F_0^2F_1 - 6F_0^2F_2r - 6F_0F_1^2r + 18F_0F_1F_2r^2 - 12F_0F_2^2r^3 - 8F_0J_1 \\ & + 16F_0J_2r - 24F_0J_3r^2 + 32F_0J_4r^3 + 3F_1^3r^2 - 12F_1^2F_2r^3 + 15F_1F_2^2r^4 \\ & + 8F_1J_1r - 16F_1J_2r^2 + 24F_1J_3r^3 - 32F_1J_4r^4 - 6F_2^3r^5 - 8F_2J_1r^2 \\ & + 16F_2J_2r^3 - 24F_2J_3r^4 + 32F_2J_4r^5 + 160K_2r - 240K_3r^2 + 320K_4r^3 \\ & - 400K_5r^4 + 480K_6r^5 - 560K_7r^6; \end{aligned} \tag{2.76}$$

$$\begin{aligned}400K_0 = {} & -(6F_0^3 - 33F_0^2F_1r + 48F_0^2F_2r^2 + 48F_0F_1^2r^2 - 126F_0F_1F_2r^3 \\ & + 78F_0F_2^2r^4 + 40F_0J_1r - 80F_0J_2r^2 + 120F_0J_3r^3 - 160F_0J_4r^4 - 21F_1^3r^3 \\ & + 78F_1^2F_2r^4 - 93F_1F_2^2r^5 - 40F_1J_1r^2 + 80F_1J_2r^3 - 120F_1J_3r^4 + 160F_1J_4r^5 \\ & + 36F_2^3r^6 + 40F_2J_1r^3 - 80F_2J_2r^4 + 120F_2J_3r^5 - 160F_2J_4r^6 - 400K_2r^2 \\ & + 800K_3r^3 - 1200K_4r^4 + 1600K_5r^5 - 2000K_6r^6 + 2400K_7r^7). \end{aligned} \tag{2.77}$$

Provided that the conditions (2.60)-(2.77) are satisfied, the transformations (2.2) mapping equation (2.21) to a linear equation (2.3) is obtained by solving the following compatible system of equations for the functions $\varphi(x, y)$ and $\psi(x, y)$

$$\varphi_x = r\varphi_y; \tag{2.78}$$

$$\varphi_y\psi_x = r\varphi_y\psi_y - \Delta; \tag{2.79}$$

$$10\Delta\varphi_{yy} = \varphi_y(4\Delta_y - F_2\Delta); \tag{2.80}$$

and

$$\begin{aligned}500\varphi_y\psi_{yyyy}\Delta^3 = {} & 300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - F_2\Delta) + 5\psi_{yy}\varphi_y\Delta(-120F_{2y}\Delta^2 \\ & - 144\Delta_y^2 + 72\Delta_yF_2\Delta - 39F_2^2\Delta^2 + 80J_4\Delta^2) + \psi_y\varphi_y(-500\varphi_y^3\alpha\Delta^3 \\ & - 150F_{2yy}\Delta^3 + 360F_{2y}\Delta_y\Delta^2 - 165F_{2y}F_2\Delta^3 + 100J_{4y}\Delta^3 + 96\Delta_y^3 \\ & - 72\Delta_y^2F_2\Delta + 108\Delta_yF_2^2\Delta^2 - 240\Delta_yJ_4\Delta^2 - 24F_2^3\Delta^3 + 60F_2J_4\Delta^3) \\ & - 500\psi\varphi_y^5\beta\Delta^3 + 500K_7\Delta^4. \end{aligned} \tag{2.81}$$

The coefficients α and β of the resulting linear equation (2.3) is given by

$$\alpha = \frac{\Lambda}{8\varphi_y^3}; \tag{2.82}$$

and

$$\begin{aligned}\beta = {} & (1600\Delta\varphi_y^4)^{-1}\Big[\Delta(-144F_{2y}^2 - 72F_{2y}F_2^2 + 352F_{2y}J_4 + 160J_{4yy} \\ & + 80J_{4y}F_2 + 640K_{6y} - 1600K_{7x} - 2880K_{7y}r + 80\Lambda_y - 4480r_yK_7 \\ & - 400F_1K_7 - 9F_2^4 + 88F_2^2J_4 + 160F_2K_6 - 320F_2K_7r - 144J_4^2) \\ & - 120\Delta_y\Lambda\Big]; \end{aligned} \tag{2.83}$$

where Θ is the following expression

$$\Theta = (F_2^2 - 4J_4)F_2 - 8(K_6 - 7K_7 r) - 8J_{4y} + 6F_{2y}F_2 + 4F_{2yy}. \qquad (2.84)$$

Remark 2.2. The equations (2.60)-(2.77) define eighteen *relative invariants* of the third-order for the general point transformation group (2.2).

3 Proof of the linearization theorems

The proof of the linearization theorems formulated above requires investigation of integrability conditions for the equations given in Section 2.1. We will consider the problem for the candidates (2.9) and (2.21) separately. The problem is formulated as follows. Given the coefficients $A_i(x, y), B_i(x, y)$ and $C_i(x, y), D_i(x, y), F_i(x, y), H_i(x, y), J_i(x, y), K_i(x, y)$ of the equations (2.9) and (2.21), respectively, find the integrability conditions of the respective equations for the functions φ and ψ.

3.1 Proof of Theorem 2.1

Let us turn to the proof of Theorem 2.1 on linearization of (2.9). Namely, given the coefficients $A_i(x, y), B_i(x, y), C_i(x, y), D_i(x, y)$ of (2.9), we have to find the necessary and sufficient conditions for integrability of the over-determined system (2.10)-(2.20) for the unknown functions $\varphi(x)$ and $\psi(x, y)$.

We first rewrite the expressions (2.10) and (2.11) for A_1 and A_0 in the following form

$$\psi_{yy} = \frac{\psi_y A_1}{4}, \qquad (3.1)$$

$$\psi_{xy} = \frac{(6\varphi_{xx} + \varphi_x A_0)}{4\varphi_x}\psi_y. \qquad (3.2)$$

Comparing the mixed derivative $(\psi_{yy})_x = (\psi_{xy})_y$, one arrives at (2.42)

$$A_{0y} = A_{1x}.$$

Then (2.12), (2.13) and (2.14) are written in the form

$$3A_1 - 4B_0 = 0,$$

$$3A_1^2 - 8C_2 + 12A_{1y} = 0,$$

and

$$12A_{1x} + 3A_0A_1 - 4C_1 = 0,$$

respectively. So that one obtains (2.43), (2.44) and (2.45) respectively. Furthermore, (2.15) for C_0 becomes

$$'_{xxx} = ¡\frac{(12A_{0x}'^2_x ¡ 60'^2_{xx} + 3'^2_xA_0^2 ¡ 8'^2_xC_0)}{40'_x}: \qquad (3.3)$$

Differentiation of (3.3) with respect to y yields

$$12A_{0x}A_1 + 32C_{0y} ¡ 16C_{1x} + 3A_0^2A_1 ¡ 4A_0C_1 = 0:$$

Thus one gets (2.46). Therefore (2.16), (2.17) and (2.18) can be written in the form of (2.47), (2.48) and (2.49), respectively.

One can determine ® from (2.19), as the following

$$® = \frac{4A_{0xx} + 6A_{0x}A_0 ¡ 8C_{0x} + A_0^3 ¡ 4A_0C_0 + 8D_1}{8'^3_x}: \qquad (3.4)$$

Since $' = '(x)$, we have $®_y = 0$ yields (2.50)

$$D_{2x} = ¡\frac{1}{192}\Big[36A_{0x}A_0A_1 ¡ 48A_{0x}C_1 ¡ 48C_{0x}A_1 ¡ 288D_{1y} + 9A_0^3A_1$$
$$¡ 12A_0^2C_1 ¡ 36A_0A_1C_0 + 48A_0D_2 + 32C_0C_1\Big]:$$

From (2.20) one finds

$$Ã_{xxxx} = ¡\frac{1}{40'^3_x}\Big[32A_{0xx}'^3_xÃ_x ¡ 72A_{0x}'_{xx}'^2_xÃ_x + 48A_{0x}'^3_xÃ_{xx}$$
$$+ 36A_{0x}'^3_xÃ_xA_0 ¡ 48C_{0x}'^3_xÃ_x ¡ 120'^3_{xx}Ã_x + 360'^2_{xx}'_xÃ_{xx}$$
$$¡ 240'_{xx}'^2_xÃ_{xxx} ¡ 18'_{xx}'^2_xÃ_xA_0^2 + 48'_{xx}'^2_xÃ_xC_0 + 40'^7_x—Ã$$
$$+ 12'^3_xÃ_{xx}A_0^2 ¡ 32'^3_xÃ_{xx}C_0 + 5'^3_xÃ_xA_0^3 ¡ 20'^3_xÃ_xA_0C_0$$
$$+ 40'^3_xÃ_xD_1 ¡ 40'^3_xÃ_yD_0\Big]: \qquad (3.5)$$

Forming the mixed derivative $(Ã_{xxxx})_y = (Ã_{xy})_{xxx}$ one obtains

$$— = \frac{1}{1600'^5_x}\Big[320A_{0xxx}'_x ¡ 1200A_{0xx}'_{xx} + 360A_{0xx}'_xA_0 + 336A_{0x}^2'_x$$
$$¡ 1800A_{0x}'_{xx}A_0 ¡ 12A_{0x}'_xA_0^2 + 32A_{0x}'_xC_0 ¡ 480C_{0xx}'_x$$
$$+ 2400C_{0x}'_{xx} + 1600D_{0y}'_x ¡ 300'_{xx}A_0^3 + 1200'_{xx}A_0C_0$$
$$¡ 2400'_{xx}D_1 ¡ 39'_xA_0^4 + 208'_xA_0^2C_0 ¡ 400'_xA_0D_1$$
$$+ 400'_xA_1D_0 ¡ 144'_xC_0^2\Big]: \qquad (3.6)$$

Since $\varphi = \varphi(x)$, we have $\beta_y = 0$ yields (2.51)

$$\begin{aligned} D_{1xy} = \frac{3}{384}\Big[& [(3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1) - 16(A_1C_0 - D_2)A_0]A_0 \\ & - 32[4(C_1D_1 - 2C_2D_0 + C_0D_2) + (3A_1D_0 - C_0^2)A_1] \\ & - 96D_{1y}A_0 + 384D_{0y}A_1 + 1536D_{0yy} - 16(3A_0A_1 - 4C_1)C_{0x} \\ & + 12[(3A_0A_1 - 4C_1)A_0 - 4(A_1C_0 - 4D_2)]A_{0x}\Big]. \end{aligned}$$

From (3.3) one can rewrite the representation for C_0 upon denoting $\chi = \frac{\varphi_{xx}}{\varphi_x}$ leads to (2.52) and the representations for ψ_{yy} and ψ_{xy} in the equations (3.1) and (3.2) become (2.54) and (2.55). Rewriting the representation for α from (3.4) in the form

$$\alpha = \frac{\lambda}{8\varphi_x^3},$$

where

$$\lambda = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx};$$

and thus β of (3.6) becomes

$$\begin{aligned} \beta = (1600\varphi_x^4)^{-1}(& -144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0 \\ & + 1600D_{0y} - 640D_{1x} + 80\lambda_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\lambda \\ & + 400A_1D_0 - 300\chi\lambda - 144C_0^2). \end{aligned}$$

Finally, one obtains (3.5) in the form

$$\begin{aligned} 1600\psi_{xxxx} = {} & 9600\psi_{xxx}\chi + 160\psi_{xx}(-12A_{0x} - 3A_0^2 - 90\chi^2 + 8C_0) \\ & + 40\psi_x(12A_{0x}A_0 + 72A_{0x}\chi - 16C_{0x} + 3A_0^3 + 18A_0^2\chi - 12A_0C_0 \\ & + 120\chi^3 - 48\chi C_0 + 24D_1 - 8\lambda) + \psi(144A_{0x}^2 + 72A_{0x}A_0^2 - 352A_{0x}C_0 \\ & - 160C_{0xx} - 80C_{0x}A_0 - 1600D_{0y} + 640D_{1x} - 80\lambda_x + 9A_0^4 - 88A_0^2C_0 \\ & + 160A_0D_1 + 30A_0\lambda - 400A_1D_0 + 300\chi\lambda + 144C_0^2) + 1600\psi_yD_0. \end{aligned}$$

Hence we complete the proof of Theorem 2.1.

3.2 Proof of Theorem 2.2

In the case of (2.21), the problem is formulated as follows. Given the coefficients $F_i(x, y), H_i(x, y), J_i(x, y), K_i(x, y)$ of (2.21), find the necessary and sufficient conditions for integrability of the over-determined system of equations (2.22)-(2.40) for the unknown functions $\varphi(x, y)$ and $\psi(x, y)$. Recall that, according to our notation, the following equations hold

$$\varphi_x = r\varphi_y, \quad \psi_x = \frac{\psi_y\varphi_x - \Delta}{\varphi_y}, \tag{3.7}$$

and

$$\alpha_x = \frac{\varphi_x}{\varphi_y}\alpha_y; \qquad \beta_x = \frac{\varphi_x}{\varphi_y}\beta_y.$$

Let us simplify the expression (2.22) as follow

$$\varphi_{yy} = \Big[(4\Delta_y - \mathcal{F}_2\Delta)\varphi_y\Big]/(10\Delta). \tag{3.8}$$

Comparing the mixed derivative $(\varphi_x)_{yy} = (\varphi_{yy})_x$ one obtains

$$\Delta_{xy} = \Big[\mathcal{F}_{2x}\Delta^2 - \mathcal{F}_{2y}\nabla\Delta^2 + 10\nabla_{yy}\Delta^2 + 4\nabla_y\Delta_y\Delta - \nabla_y\mathcal{F}_2\Delta^2 + 4\Delta_x\Delta_y + 4\Delta_{yy}\nabla\Delta - 4\Delta_y^2\nabla\Big]/(4\Delta).$$

Rewriting (2.23) in the form

$$\Delta_x = (20\nabla_y\Delta + 4\Delta_y\nabla + \mathcal{F}_1\Delta - 2\mathcal{F}_2\nabla\Delta)/4.$$

Forming the mixed derivative $\Delta_{xy} = (\Delta_x)_y$ one arrives at (2.60)

$$\nabla_{yy} = -(\mathcal{F}_{1y} - \mathcal{F}_{2x} - \mathcal{F}_{2y}\nabla - \nabla_y\mathcal{F}_2)/10.$$

Then (2.24)-(2.27) are written in the form of (2.61)-(2.64), respectively. Furthermore, (2.28) becomes

$$\Delta_{yy} = -(20\mathcal{F}_{2y}\Delta^2 - 48\Delta_y^2 + 4\Delta_y\mathcal{F}_2\Delta + 7\mathcal{F}_2^2\Delta^2 - 20\mathcal{J}_4\Delta^2)/(40\Delta).$$

Now, consider the equation $(\Delta_{yy})_x = (\Delta_x)_{yy}$, one gets (2.65)

$$\begin{aligned}\mathcal{F}_{1yy} = \; & -(\mathcal{F}_{1y}\mathcal{F}_2 - 40\mathcal{F}_{2xy} - 16\mathcal{F}_{2x}\mathcal{F}_2 + 20\mathcal{F}_{2yy}\nabla + 40\mathcal{F}_{2y}\nabla_y \\ & +14\mathcal{F}_{2y}\mathcal{F}_2\nabla + 20\mathcal{J}_{4x} - 20\mathcal{J}_{4y}\nabla + 14\nabla_y\mathcal{F}_2^2 - 40\nabla_y\mathcal{J}_4)/10.\end{aligned}$$

Thus equations (2.29)-(2.32) yield (2.66)-(2.69), and from (2.33) one finds

$$\begin{aligned}\psi_{yyyy} = \Big[& 300\psi_{yyy}\varphi_y\Delta^2(4\Delta_y - \mathcal{F}_2\Delta) + 5\psi_{yy}\varphi_y\Delta(-120\mathcal{F}_{2y}\Delta^2 - 144\Delta_y^2 \\ & + 72\Delta_y\mathcal{F}_2\Delta - 39\mathcal{F}_2^2\Delta^2 + 80\mathcal{J}_4\Delta^2) + \psi_y\varphi_y(-500\varphi_y^3\alpha\Delta^3 \\ & - 150\mathcal{F}_{2yy}\Delta^3 + 360\mathcal{F}_{2y}\Delta_y\Delta^2 - 165\mathcal{F}_{2y}\mathcal{F}_2\Delta^3 + 100\mathcal{J}_{4y}\Delta^3 \\ & + 96\Delta_y^3 - 72\Delta_y^2\mathcal{F}_2\Delta + 108\Delta_y\mathcal{F}_2^2\Delta^2 - 240\Delta_y\mathcal{J}_4\Delta^2 - 24\mathcal{F}_2^3\Delta^3 \\ & + 60\mathcal{F}_2\mathcal{J}_4\Delta^3) - 500\psi\varphi_y^5\beta\Delta^3 + 500\mathcal{K}_7\Delta^4\Big]/(500\varphi_y\Delta^3).\end{aligned} \tag{3.9}$$

One can determine α from (2.34), as the following

$$\alpha = (4\mathcal{F}_{2yy} + 6\mathcal{F}_{2y}\mathcal{F}_2 - 8\mathcal{J}_{4y} + \mathcal{F}_2^3 - 4\mathcal{F}_2\mathcal{J}_4 - 8\mathcal{K}_6 + 56\mathcal{K}_7\nabla)/8\varphi_y^3. \tag{3.10}$$

Now the equation $\alpha_x - \nabla\alpha_y = 0$ leads to (2.70). Furthermore, one considers $(\psi_x)_{yyyy} = (\psi_{yyyy})_x$, yields

$$\begin{aligned}\beta = \; & 120\Delta_y(-4F_{2yy} - 6F_{2y}F_2 + 8J_{4y} - F_2^3 + 4F_2J_4 + 8K_6 - 56K_7\nabla) \\ & + \Delta(320F_{2yyy} + 480F_{2yy}F_2 + 336F_{2y}^2 + 168F_{2y}F_2^2 + 32F_{2y}J_4 - 480J_{4yy} \\ & - 240J_{4y}F_2 - 1600K_{7x} + 1600K_{7y}\nabla - 400F_1K_7 - 9F_2^4 + 88F_2^2J_4 \\ & + 160F_2K_6 - 320F_2K_7\nabla - 144J_4^2)/1600\Delta\varphi_y^4.\end{aligned} \tag{3.11}$$

The equation $\beta_x - \nabla\beta_y = 0$ leads to (2.71). Therefore, (2.35)-(2.40) become (2.72)-(2.77), respectively.

Let us turn now to the integrability problem. One can find all fourth-order derivatives of the functions φ and ψ by using (3.7), (3.8) and (3.9). So that one obtains at (2.78)-(2.81). Finally, the coefficients α and β of the resulting linear equations (3.10) and (3.11) are given by

$$\alpha = \frac{\Theta}{8\varphi_y^3},$$

$$\begin{aligned}\beta = \; & (-144F_{2y}^2\Delta - 72F_{2y}F_2^2\Delta + 352F_{2y}J_4\Delta + 160J_{4yy}\Delta + 80J_{4y}F_2\Delta \\ & + 640K_{6y}\Delta - 1600K_{7x}\Delta - 2880K_{7y}\nabla\Delta - 4480\nabla_yK_7\Delta + 80\Theta_y\Delta \\ & - 120\Delta_y\Theta - 400F_1K_7\Delta - 9F_2^4\Delta + 88F_2^2J_4\Delta + 160F_2K_6\Delta \\ & - 320F_2K_7\nabla\Delta - 144J_4^2\Delta)/(1600\varphi_y^4\Delta),\end{aligned}$$

where

$$\Theta = (F_2^2 - 4J_4)F_2 - 8(K_6 - 7K_7\nabla) - 8J_{4y} + 6F_{2y}F_2 + 4F_{2yy}.$$

Hence we complete the proof of Theorem 2.2.

4 Illustration of the linearization theorems

4.1 An example on Theorem 2.1

Example 1. Consider the nonlinear ordinary differential equation

$$x^2y(2y^{(4)} + y) + 8x^2y'y''' + 16xyy''' + 6x^2y''^2 + 48xy'y'' + 24yy'' + 24y'^2 = 0. \tag{4.1}$$

It is an equation of the form (2.9) with the coefficients

$$\begin{aligned}& A_1 = \frac{4}{y},\; A_0 = \frac{8}{x},\; B_0 = \frac{3}{y},\; C_2 = 0,\; C_1 = \frac{24}{xy},\; C_0 = \frac{12}{x^2}, \\ & D_4 = 0,\; D_3 = 0,\; D_2 = \frac{12}{x^2y},\; D_1 = 0,\; D_0 = \frac{y}{2}\Delta\end{aligned} \tag{4.2}$$

One can check that the coefficients (4.2) obey the conditions (2.42)-(2.51). Thus, the equation (4.1) is linearizable. We have

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \tag{4.3}$$

and the equation (2.52) is written as

$$2\frac{d\chi}{dx} - \chi^2 = 0.$$

Let us take its simplest solution $\chi = 0$. Then invoking (2.53), we let

$$\varphi = x.$$

Now the equations (2.54)-(2.55) are written

$$\frac{\psi_{yy}}{\psi_y} = \frac{1}{y}, \quad \frac{\psi_{xy}}{\psi_y} = \frac{2}{x}$$

and yield

$$\psi_y = Kx^2y, \qquad K = \text{const}.$$

Hence

$$\psi = K\frac{x^2y^2}{2} + f(x).$$

Since one can use any particular solution, we set $K = 2, \ f(x) = 0$ and take

$$\psi = x^2y^2.$$

Invoking (4.3) and noting that (2.57) yields $\gamma = 0$, one can readily verify that the function $\psi = x^2y^2$ solves equation (2.56) as well. Hence, one obtains the following transformations

$$t = x, \quad u = x^2y^2. \tag{4.4}$$

Since $\gamma = 0$, equations (2.58) and (2.59) give

$$\alpha = 0, \quad \beta = \frac{1}{\varphi_x^4} = 1$$

Hence, the equation (4.1) is mapped by the transformations (4.4) to the linear equation

$$u^{(4)} + u = 0.$$

Example 2. The third-order member of the Riccati Hierarchy is given by Euler et al. [5] as

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + 4y^4 = 0. \tag{4.5}$$

Applying [6], and [7] one checks that equation cannot be linearized by a point transformation or contact transformation or generalized Sundman transformation. Under the Riccati transformation $y = \frac{a\omega'}{\omega}$ the equation (4.5) becomes [8]

$$\begin{aligned} &\omega^3\omega^{(4)} + 4(a-1)\,\omega^2\omega'\omega''' + 3(a-1)\,\omega^2\omega''^2 \\ &+ 6(a-1)(a-2)\,\omega\,\omega'^2\omega'' + (a-1)(a-2)(a-3)\,\omega'^4 = 0. \end{aligned} \tag{4.6}$$

It is an equation of the form (2.9) with the coefficients

$$\begin{aligned} A_1 &= \frac{4(a-1)}{\omega};\ A_0 = 0;\ B_0 = \frac{3(a-1)}{\omega}; \\ C_2 &= \frac{6(a^2-3a+2)}{\omega^2};\ C_1 = 0;\ C_0 = 0; \\ D_4 &= \frac{a^3-6a^2+11a-6}{\omega^3};\ D_3 = 0;\ D_2 = 0;\ D_1 = 0;\ D_0 = 0. \end{aligned} \tag{4.7}$$

One can verify that the coefficients (4.7) obey the linearization conditions (2.42)-(2.51). Furthermore,

$$8C_0 - 3A_0^2 - 12A_{0x} = 0 \tag{4.8}$$

and the equation (2.52) is written as

$$2\frac{d\chi}{dx} - \chi^2 = 0.$$

We take its simplest solution $\chi = 0$ and obtain from (2.53) the equation $\varphi'' = 0$, whence $\varphi = x.$ Equations (2.54) and (2.55) have the form

$$\frac{\psi_{\omega\omega}}{\psi_\omega} = \frac{a-1}{\omega}; \qquad \psi_{x\omega} = 0$$

and yield

$$\psi_\omega = K\,\omega^{(a-1)}; \qquad K = \text{const}.$$

Hence

$$\psi = K\frac{\omega^a}{a} + f(x).$$

Since one can use any particular solution, we set $K = a;\ \ f(x) = 0$ and take

$$\psi = \omega^a.$$

Invoking (4.8) and noting that (2.57) yields $\lambda = 0$, one can readily verify that the function $\psi = \omega^a$ solves equation (2.56) as well. So that one obtains the following transformations

$$t = x; \quad u = \omega^a. \tag{4.9}$$

Since $\lambda = 0$, equations (2.58) and (2.59) gives

$$\alpha = 0; \quad \beta = 0.$$

Hence, the equation (4.6) is mapped by the transformations (4.9) to the linear equation

$$u^{(4)} = 0.$$

Example 3. Let us consider the Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0. \tag{4.10}$$

Of particular interest among the solutions of the Boussinesq equation are travelling wave solutions:

$$u(x, t) = H(x - Dt).$$

Substituting the representation of a solution into (4.10), one finds

$$H^{(4)} + (H + D^2)H'' + H'^2 = 0. \tag{4.11}$$

It is an equation of the form (2.9) with the coefficients

$$\begin{gathered} A_1 = 0, \ A_0 = 0, \ B_0 = 0, \ C_2 = 0, \ C_1 = 0, \ C_0 = D^2 + H, \\ D_4 = 0, \ D_3 = 0, \ D_2 = 1, \ D_1 = 0, \ D_0 = 0. \end{gathered} \tag{4.12}$$

Since the coefficients (4.12) do not satisfy the linearization conditions (2.46), (2.49) and (2.51), hence, the equation (4.11) is not linearizable.

Example 4. Consider the non-linear equation

$$y^{(4)} - \frac{10}{y'}y''y''' + \frac{1}{y'^2}\left(15y''^3 - xy'^7 - y'^6\right) = 0. \tag{4.13}$$

It has the form (2.21) with the following coefficients:

$$\begin{aligned} &r = 0, \ F_2 = 0, \ F_1 = 0, \ F_0 = 0, \ H_2 = 0, \ H_1 = 0, \ H_0 = 0, \\ &J_4 = 0, \ J_3 = 0, \ J_2 = 0, \ J_1 = 0, \ J_0 = 0, \ K_7 = -x, \\ &K_6 = -1, \ K_5 = 0, \ K_4 = 0, \ K_3 = 0, \ K_2 = 0, \ K_1 = 0, \ K_0 = 0. \end{aligned} \tag{4.14}$$

Let us test the equation (4.13) for linearization by using Theorem 2.2. It is manifest that the equations (2.60)-(2.77) are satisfied by the coefficients (4.14). Thus, the equation (4.13) is linearizable, and we can proceed further.

Let us take its simplest solution $\varphi = y$ and $\psi = x$ which satisfy the compatible system of equations (2.78)-(2.81). So that one obtains the following transformations

$$t = y, \quad u = x. \tag{4.15}$$

Since $\lambda = 8$, equations (2.82) and (2.83) give

$$\alpha = 1, \quad \beta = 1.$$

Hence, Eq. (4.13) is mapped by the transformations (4.15) to the linear equation

$$u^{(4)} + u' + u = 0.$$

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Non-local conservation laws in fluid dynamics

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden
EMRULLAH YAŞAR
Department of Mathematics, Faculty of Science and Art,
Uludag University,
16059 Bursa, Turkey

Abstract A general theorem on symmetries and conservation laws is applied to gasdynamic equations considered together with the adjoint equations. In particular, new non-local conservation laws for the polytropic gas are obtained.

1 Introduction

Finding conservation laws is important in the study of Mathematical Physics. They provide information on the basic properties of solutions of differential equations. Well known examples of conservation laws are constants of motion. The conserved density, when integrated, provides us with a constant of motion of the system.

The fundamental mathematical model in fluid dynamics is provided by the system of nonlinear partial differential equations of the first order describing motions of a compressible fluid (gas). The symmetries of gasdynamic equations had appeared in the literature by Ovsyannikov for the first time, and their classification was made. Later on, non-local symmetries of these equations have been found in [1] . As well-known, the theorem of E. Noether states that if a given system of differential equations has a variational principle then there exists a correspondence between symmetries of a system with a Lagrangian and conservation laws of associated Euler-Lagrange equations. But, because of the restrictions of this theorem (such as, the system should have a Lagrangian), it is not possible to find conservation laws of every differential equation. Consequently, an important problem is how to derive conservation laws for given differential equations. In this paper we use recent theorem [3] (see also [4]) for deriving new conservation laws for the one-dimensional gasdynamic equations.

Theorem

Let an operator (2.2) be a symmetry of a system of first-order partial differential equations

$$F_\alpha(x, u, u_{(1)}) = 0, \alpha = 1, \ldots, m \tag{2.4}$$

where $v = (v^1, \ldots v^m)$. Then the quantities

$$C^i = v^\beta \left[\xi^i F_\beta + (\eta^\alpha - \xi^j u_j^\alpha) \frac{\partial F_\beta}{\partial u_i^\alpha} \right], \ i = 1, \ldots, n \tag{2.5}$$

furnish a conserved vector $C = (C^1, \ldots, C^n)$ for the equation (2.4) considered together with the adjoint system

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}) \equiv \frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m \tag{2.6}$$

where L is the *formal Lagrangian* given by [4]

$$L = v^\beta F_\beta(x, u, u_{(1)}, \ldots, u_{(s)}), \tag{2.7}$$

$v = (v^1, \ldots v^m)$ are new dependent variables, i.e. $v = v(x),$ and

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha}, \alpha = 1, \ldots, m;$$

This is a modified version of Theorem 3.5 (p.13) in [3]. It is obvious that this new theorem can be generalized to higher orders[1].

3 Adjoint equations to gasdynamic equations

We shall use the following systems of one-dimensional gas equations

$$\begin{aligned} \rho(u_t + u u_x) + p_x &= 0 : F_1 \\ \rho_t + u\rho_x + \rho u_x &= 0 : F_2 \\ p_t + u p_x + A(p, \rho) u_x &= 0 : F_3 \end{aligned} \tag{3.8}$$

where $A(p, \rho)$ is an arbitrary function connected with the entropy $S(p, \rho)$ by

$$A(p, \rho) = -\rho \frac{\partial S / \partial \rho}{\partial S / \partial p} \tag{3.9}$$

[1] After publishing the papers [4] and [3], I learned that the adjoint equations for nonlinear equations were considered earlier in [2], and a possibility of their use for conservation laws was mentioned in [5], Exercises to Ch.5. I thank Emrullah Yaşar and Sergey Svirshchevskii for drawing my attention to these results. *N. Ibragimov.*

The dependent variables are the velocity $u,$ the pressure p and the density ρ of the fluid. The independent variables are the time t and the spatial variable $x.$ In the case of so-called *polytropic flows*, the equation (3.9) has the form $A = \gamma p,$ where γ is a constant known as an *adiabatic (polytropic) exponent.*

The formal Lagrangian (2.7) for Eq. (3.8) has the form

$$L = U[\rho(u_t + uu_x) + p_x] + R[\rho_t + u\rho_x + \rho u_x] + P[p_t + up_x + A(p,\rho)u_x] \quad (3.10)$$

where U, R, P are adjoint variables. With this Lagrangian we have

$$F_1^* = \frac{\delta L}{\delta u} = Pp_x - U_t\rho - U\rho_t - U_x\rho u - U\rho_x u - R_x\rho - PA_p p_x - PA_\rho\rho_x - A(p,\rho)P_x$$

$$F_2^* = \frac{\delta L}{\delta \rho} = U(u_t + uu_x) + PA_\rho u_x - R_t - uR_x$$

$$F_3^* = \frac{\delta L}{\delta p} = A_p P u_x - P_t - U_x - uP_x - u_x P \quad (3.11)$$

Thus, one obtains the following adjoint equation system with the new dependent variables U, R, P:

$$P(p_x - A_p p_x - A_\rho\rho_x) - U(\rho_t + \rho_x u) - U_t\rho - U_x\rho u - R_x\rho - A(p,\rho)P_x = 0;$$

$$U(u_t + uu_x) + PA_\rho u_x - R_t - uR_x = 0;$$

$$P(A_p u_x - u_x) - P_t - U_x - uP_x = 0 \quad (3.12)$$

Equation (3.8) are invariant under the translation of time t and along the x-axis as well as the scaling and Galilean transformations. These transformations provide the following four infinitesimal symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \quad (3.13)$$

If one considers polytropic flow , i.e, $A = \gamma p$ then there exist two additional scaling symmetries

$$X_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + 2\rho\frac{\partial}{\partial \rho}, \quad X_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho} \quad (3.14)$$

4 Calculation of conservation laws

We will write down conservation law (2.3) in the form

$$D_t(C^0) + D_x(C^1) = 0 \tag{4.15}$$

Let us find conservation law provided by the symmetry $X_1 = \frac{\partial}{\partial t}$ from (3.13). Therefore equation (2.5) yields

$$\begin{aligned} C_1^0 &= U\left[F_1 - u_t\frac{\partial F_1}{\partial u_t}\right] + R\left[F_2 - \rho_t\frac{\partial F_2}{\partial \rho_t}\right] + P\left[F_3 - p_t\frac{\partial F_3}{\partial p_t}\right] \\ &= U(\rho u u_x + p_x) + R(u\rho_x + \rho u_x) + P\left[u p_x + A(p,\rho)u_x\right] \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} C_1^1 &= U\left[-u_t\frac{\partial F_1}{\partial u_x} - p_t\frac{\partial F_1}{\partial p_x}\right] + R\left[-u_t\frac{\partial F_2}{\partial u_x} - \rho_t\frac{\partial F_2}{\partial \rho_x}\right] + P\left[-u_t\frac{\partial F_3}{\partial u_x} - p_t\frac{\partial F_3}{\partial p_x}\right] \\ &= -U\left[\rho u u_t + p_t\right] - R\left[\rho u_t + u\rho_t\right] - P\left[A(p,\rho)u_t + u p_t\right]. \end{aligned} \tag{4.17}$$

The conservation law provided by the symmetry $X_2 = \frac{\partial}{\partial x}$ from (3.13) has the following density and current, respectively:

$$\begin{aligned} C_2^0 &= U\left[-u_x\frac{\partial F_1}{\partial u_t}\right] + R\left[-\rho_x\frac{\partial F_2}{\partial \rho_t}\right] + P\left[-p_x\frac{\partial F_3}{\partial p_t}\right] \\ &= -\rho U u_x - R\rho_x - P p_x \end{aligned} \tag{4.18}$$

$$\begin{aligned} C_2^1 &= U\left[F_1 - u_x\frac{\partial F_1}{\partial u_x} - p_x\frac{\partial F_1}{\partial p_x}\right] + R\left[F_2 - u_x\frac{\partial F_2}{\partial u_x} - \rho_x\frac{\partial F_2}{\partial \rho_x}\right] + \\ &\quad + P\left[F_3 - u_x\frac{\partial F_3}{\partial u_x} - p_x\frac{\partial F_3}{\partial p_x}\right] \\ &= \rho U u_t + R\rho_t + P p_t. \end{aligned} \tag{4.19}$$

Applying formula (2.5) to the symmetry X_3 and X_4, we obtain the following densities of the conservation law

$$\begin{aligned} C_3^0 &= U\left[tF_1 - (tu_t + xu_x)\frac{\partial F_1}{\partial u_t}\right] + R\left[tF_2 - (t\rho_t + x\rho_x)\frac{\partial F_2}{\partial \rho_t}\right] + \\ &\quad + P\left[tF_3 - (tp_t + xp_x)\frac{\partial F_3}{\partial p_t}\right] = U\left[\rho u_x(tu - x) + tp_x\right] \\ &\quad + R\left[\rho_x(tu - x) + t\rho u_x\right] + P\left[p_x(tu - x) + tA(p,\rho)u_x\right]; \end{aligned} \tag{4.20}$$

$$C_4^0 = (1 - tu_x)U\rho - tR\rho_x - tPp_x. \tag{4.21}$$

The operator X_3 provides the following current vector

$$C_3^1 = U(x\rho u_t - t\rho uu_t - tp_t) + R(x\rho_t - t\rho u_t - tu\rho_t) + P[xp_t - tA(p,\rho)u_t - tup_t]. \tag{4.22}$$

The current vector C^1 for the operator X_4 has the following form

$$C_4^1 = U\rho(tu_t + u) + R(t\rho_t + \rho) + P[tp_t + A(p,\rho)]. \tag{4.23}$$

Now let us considers polytropic flow equations. We know that first four symmetries are the same. So, in the conservation laws above it's sufficient to write γp instead of $A(p,\rho)$. Then following conservation laws are obtained

$$\begin{aligned} C_1^0 &= U(\rho uu_x + p_x) + R(u\rho_x + \rho u_x) + P(up_x + \gamma pu_x);\\ C_1^1 &= -U(u\rho u_t + p_t) - R(\rho u_t + u\rho_t) - P(\gamma pu_t + up_t) \end{aligned}$$

$$C_2^0 = -U\rho u_x - R\rho_x - Pp_x, \; C_2^1 = U\rho u_t + R\rho_t + Pp_t$$

$$\begin{aligned} C_3^0 &= U[t(\rho uu_x + p_x) - x\rho u_x] + P[t(up_x + \gamma pu_x) - xp_x] +\\ &\quad +R[t(u\rho_x + \rho u_x) - x\rho_x];\\ C_3^1 &= U[x\rho u_t - t(\rho uu_t + p_t)] + P[xp_t - t(\gamma pu_t + up_t)] +\\ &\quad +R[x\rho_t - t(\rho u_t + u\rho_t)] \end{aligned}$$

$$C_4^0 = (1 - tu_x)U\rho - tR\rho_x - tPp_x; \; C_4^1 = U\rho(tu_t+u)+R(t\rho_t+\rho)+P(tp_t+\gamma p). \tag{4.24}$$

In a similar way aforementioned above, one can construct conservation laws corresponding X_5 and X_6:

$$\begin{aligned} C_5^0 &= U[t(\rho uu_x + p_x) - u\rho] + R[t(u\rho_x + \rho u_x) + 2\rho] + P[t(up_x + \gamma pu_x)];\\ C_5^1 &= -U[t(u\rho u_t + p_t) + \rho u^2] - R[t(\rho u_t + u\rho_t) - \rho u] - P[t(\gamma pu_t + up_t) + \gamma pu] \end{aligned}$$

$$C_6^0 = \rho R + pP; \; C_6^1 = Up + Ppu + R\rho u. \tag{4.25}$$

Remark

The conservation laws given by both cases depend on the solutions (U,R,P) of the adjoint system equation (3.12). However, substituting into equation (4.20) and equation (4.22) any particular solution (U,R,P) of the adjoint system (3.12) with $A(p,\rho) = p$, one obtains the conservation law for equation (3.8) not involving U, R and P. Let us denote $A(p,\rho) = p$ and take the following constraints:

i) Consider the following simple solution of the adjoint system (3.12):

$$U = R = 0;\ P = 1;\ A(p;\rho) = p$$

Then equation (4.20) and equation (4.22) yield

$$C_3^0 = tup_x + tpu_x - xp_x;\ C_3^1 = xp_t - tpu_t - tup_t.$$

These vectors, corresponding to adjoint solutions, yields us after some simplifications:

$$C_3^0 = p;\ C_3^1 = up.$$

If one considers polytropic flow with $\gamma = 1$ the same result is obtained. This property is also valid for X_6. In both cases for other transformation groups trivial conservation laws are found.
ii) Let $R = 1$, $U = P = 0$, $A(p;\rho) = p.$ In a similar way we get the following conservation laws:

$$C_3^0 = tu\rho_x + t\rho u_x - x\rho_x; C_3^1 = x\rho_t - t\rho u_t - tu\rho_t.$$

Accordingly, for the case under consideration, the conservation laws given by identity (2.5) yields us, after some simplifications,

$$C_3^0 = \rho;\ C_3^1 = u\rho.$$

If one considers polytropic flow with $\gamma = 1$ again the same result is obtained. This property is also valid for X_6 . Similarly in both cases for other transformation groups trivial conservation laws can be obtained.

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