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AN IDEAL THEORY FOR EXTERIOR DIFFERENTIAL EQUATIONS

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Introduction

The main purpose of this paper is to express the arithmetic notion of genus which appears in E. Cartan's Theory of exterior differential equations [2] in terms of ideal theory. Accordingly Sections 1, 2, and 3 of this paper are devoted to a discussion of ideal theory in Grassmann algebras. Let V be an n dimensional vector space and let V^* be its dual space. Let $\Lambda(V)$ and $\Lambda(V^*)$ denote the Grassmann algebra over V and V^* respectively. Now $\Lambda(V)$ and $\Lambda(V^*)$ are dually paired to the reals. Let W be a subspace of V and let $\Lambda(W)$ be the Grassmann algebra over W. Then $\Lambda(W) \subset \Lambda(V)$. We wish to study the annihilator space in $\Lambda(V^*)$ of $\Lambda(W)$. In Theorem 2.3 we show that this is exactly the homogeneous ideal generated in $\Lambda(V^*)$ by the annihilator of W. An ideal generated by a subspace of V^* will be called a one generated ideal. It should be remarked that the integral elements of E. Cartan are precisely Grassmann algebras generated by subspaces of V. We also define a concept of minimal ideal, see Definition 3.1, for a given ideal $\mathfrak{A}$ and prove that under certain conditions on $\mathfrak{A}$ the dimension of the covector space which generates a minimal ideal is an invariant of the ideal $\mathfrak{A}$. We call this number the dimension of $\mathfrak{A}$ when it exists.

Sections 4, 5, and 6 are devoted to applications of the above techniques to the study of the characteristic vector space, the genus of a differential ideal and the problem of prolongation, respectively. We prove that the characteristic vector space is contained in the intersection of the annihilator spaces of the minimal ideals for the given ideal $\mathfrak{A}$. We also find a condition for these two spaces to coincide. Theorem 5.1 shows that for one and two generated ideals $\mathfrak{A}$, the genus at a point is either one or the dimension of $\mathfrak{A}$. In Theorem 5.2 we give a necessary and sufficient condition for the genus to be one. In Section 6 we give an invariant formulation of the process of prolongation and in Theorem 6.2 we obtain a characterization of certain integral ideals in the prolonged system.

We will assume in presenting the material of this paper that the reader is familiar with [1] and [2].

The author would like to acknowledge the fact that his interest in this subject stems from a course given by Professor S. S. Chern at the University of Chicago in 1952–1953 and that much of the exposition in paragraphs 5 and 6 follow the ideas in [3].

1. Interior products

Let V be an n dimensional vector space over the reals, R, and let V^* be its dual vector space. Let $\Lambda(V)$ denote the Grassmann algebra over V. Then $\Lambda(V)$ is a graded algebra, $\Lambda(V) = \Lambda^0(V) + \Lambda^1(V) + \cdots + \Lambda^n(V)$, where $\Lambda^0(V) = R$ and $\Lambda^1(V) = V$. Let $\Lambda(V^*)$ be the Grassmann algebra over V^*, then $\Lambda(V^*)$ is also a graded algebra and $\Lambda^r(V)$ and $\Lambda^r(V^*)$, $0 \leqq r \leqq n$ are dual vector spaces. An element will be called homogeneous of degree r if it is contained in $\Lambda^r(V)$ or $\Lambda^r(V^*)$. An ideal $\mathfrak{A}$ in $\Lambda(V^*)$ will be called homogeneous if it is generated by homogeneous elements. We will henceforth use the term ideal to denote a homogeneous ideal.

Let us denote the pairing of $\Lambda^r(V)$ and $\Lambda^r(V^*)$ to the reals by $\langle \xi, z \rangle$, where $\xi \in \Lambda^r(V)$ and $z \in \Lambda^r(V^*)$. Let z be a homogeneous element of $\Lambda(V^*)$ and $\xi \in \Lambda(V)$. Then we define $\xi \llcorner z \in \Lambda(V)$ by the equation

$$\langle \xi \llcorner z, z' \rangle = \langle \xi, z \wedge z' \rangle \tag{1.1}$$

for all $z' \in \Lambda(V^*)$. If $\xi \in \Lambda^s(V)$ and $z \in \Lambda^r(V^*)$ then $\xi \llcorner z \in \Lambda^{s-r}(V)$. Similarly, we define $\xi \lrcorner z \in \Lambda(V^*)$ by the equation

$$\langle \xi', \xi \lrcorner z \rangle = \langle \xi' \wedge \xi, z \rangle \tag{1.2}$$

for all $\xi' \in \Lambda(V)$.

We shall list, without proofs, the following properties of these pairings (for a fuller discussion see [1]):

$$\xi' \lrcorner (\xi \lrcorner z) = (\xi' \wedge \xi) \lrcorner z \tag{1.3}$$

$$(\xi \llcorner z) \llcorner z' = \xi \llcorner (z \wedge z'). \tag{1.4}$$

For $z \in V^*$ and $\xi' \in \Lambda^k(V)$

$$(\xi \wedge \xi') \llcorner z = \xi \wedge (\xi' \llcorner z) + (-1)^k (\xi \llcorner z) \wedge \xi'. \tag{1.5}$$

For $\xi \in V$ and $z \in \Lambda^k(V^*)$

$$\xi \lrcorner (z \wedge z') = (\xi \lrcorner z) \wedge z' + (-1)^k z \wedge (\xi \lrcorner z'). \tag{1.6}$$

Now it is well-known that once we have chosen a base for V and a dual base for V^* we have an isomorphism between $\Lambda^r(V)$ and $\Lambda^{n-r}(V^*)$. Let us denote this isomorphism by π. Then we may consider the following diagram:

$$\begin{array}{ccc} \Lambda^r(V) & \xrightarrow{\pi} & \Lambda^{n-r}(V^*) \\ & & \downarrow \chi \\ \Lambda^{r-1}(V) & \xrightarrow{\pi} & \Lambda^{n-r+1}(V^*) \end{array}$$

where χ is the pairing of $\Lambda^{n-r}(V^*)$ and V^* to $\Lambda^{n-r+1}(V^*)$ defined by the wedge product operating to the left.

THEOREM 1.1. *$\xi \llcorner z = \delta\pi^{-1}(z \wedge \pi(\xi))$, where $z \in V^*$, $\xi \in \Lambda^r(V)$, and $\delta = 1$ if r is odd and $\delta = -1$ if r is even.*

PROOF. Let $X_1, \cdots, X_n$ be a basis for V and let $x_1, \cdots, x_n$ be a dual basis for V^*. Then

$$X_{i_1} \wedge \cdots \wedge X_{i_r}, \qquad i_1 < \cdots < i_r, \qquad 1 \leqq i \leqq n,$$

form a basis for $\Lambda^r(V)$ and

$$x_{j_1} \wedge \cdots \wedge x_{j_{n-r}}, \qquad j_1 < \cdots < j_{n-r}, \qquad 1 \leqq j \leqq n,$$

form a basis for $\Lambda^{n-r}(V^*)$. Now

$$\pi(X_{i_1} \wedge \cdots \wedge X_{i_r}) = \eta(x_{j_1} \wedge \cdots \wedge x_{j_{n-r}})$$

where $i_1, \cdots, i_r, j_1, \cdots, j_{n-r}$ form a permutation of the numbers from 1 to n and η is the sign of this permutation. Then

$$x_\alpha \wedge \pi(\xi) = \eta x_\alpha \wedge x_{j_1} \wedge \cdots \wedge x_{j_{n-r}},$$

where $\alpha = 1, \cdots, n$. Therefore $x_\alpha \wedge \pi(\xi) = 0$ if and only if α is not one of the indices $i_1, \cdots, i_r$. Hence

$$\pi^{-1}(\eta x_\alpha \wedge x_{j_1} \wedge \cdots \wedge x_{j_{n-r}}) = \eta^2 \rho X_{i_1} \wedge \cdots \wedge \hat{X}_\alpha \wedge \cdots \wedge X_{i_r}$$

where $\hat{X}_\alpha$ is omitted, $\rho = 0$ if α is not one of the indices $i_1, \cdots, i_r$ and equals the sign of the permutation which carries X_α into the last entree if α is one of the indices $i_1, \cdots, i_r$.

Now if $x = x_{j_i} \cdots x_{j_{n-r}}$, we have

$$\langle x_\alpha \wedge x, \xi\rangle = \langle x_\alpha, X_\alpha\rangle\langle x, \rho' X_{i_1} \wedge \cdots \wedge \hat{X}_\alpha \wedge \cdots \wedge X_{i_r}\rangle,$$

where $\rho' = 0$ if α is not one of the indices $i_1, \cdots, i_r$ and equals the sign of the permutation that carries X_α into the first entree. Hence

$$\xi \llcorner x_\alpha = \rho\rho'\pi^{-1}(x_\alpha \wedge \pi(\xi)).$$

This proves the theorem, since we may extend this result to any element by linearity.

THEOREM 1.2. *Let W be a subspace of V. Then $\xi \in \Lambda^r(W)$ if and only if $\xi \llcorner z \in \Lambda^{r-1}(W)$ for all $z \in V^*$.*

PROOF. Assume $\xi \in \Lambda^r(W)$. Let $X_1, \cdots, X_n$ be a basis for V such that $X_1, \cdots, X_s$ is a basis for W. Let $x_1, \cdots, x_n$ be the dual basis for V^*. Then

$$\pi(\xi) = \sum A_k z_k' \wedge x_{s+1} \wedge \cdots \wedge x_n,$$

where $z' \in \Lambda^{s-r}(V^*)$. Hence if $z = \sum_{i=s+1}^n a_i x_i$ we have from Theorem 1.1 that $\xi \llcorner z = 0$. If $z \in W^*$ then clearly $\xi \llcorner z \in \Lambda^{r-1}(W)$.

Assume $\xi \llcorner z \in \Lambda^{r-1}(W)$ for all $z \in V^*$. Then $\pi^{-1}(z \wedge \pi(\xi)) \in \Lambda^{r-1}(W)$ for all z. Therefore $z \wedge \pi(\xi)$ is of the form $\sum A_k z_k' \wedge x_{s+1} \wedge \cdots \wedge x_n$ for all z. Hence $\pi(\xi)$ itself must be of this form. This implies that $\xi \in \Lambda^r(W)$.

2. r generated ideals

We will adopt the following notational conventions: Let $\mathfrak{A}$ be a homogeneous ideal. We will denote $\mathfrak{A} \cap \Lambda^r(V^*)$ by A_r and the annihilator of A_r, contained in $\Lambda^r(V)$, will be denoted by ann. (A_r).

DEFINITION 2.1. An ideal $\mathfrak{A}$ generated, as an ideal, by its homogeneous elements of degree r will be called an r generated ideal.

LEMMA 2.1. ann. $(A_r) \llcorner z \subset$ ann. (A_{r-1}) *for all* $z \in V^*$.

PROOF. Let $\xi \in$ ann. (A_r) and let $x \in A_{r-1}$. Then $x \wedge z \in A_r$. Since $\langle \xi, z \wedge x\rangle = \langle \xi \llcorner z, x\rangle$, we have that $\langle \xi \llcorner z, x\rangle = 0$ for all $x \in A_{r-1}$. Hence $\xi \llcorner z \in$ ann. (A_{r-1}).

THEOREM 2.1. ann. $(A_r) \llcorner z \subset$ ann. (A_{r-s}) *for all* $z \in \Lambda^s(V^*)$.

This follows from Lemma 2.1 and Formula 1.4.

THEOREM 2.2. *Given a set of subvector spaces* $Q_i \subset \Lambda^i(V)$ *such that* $Q_i \llcorner z \subset Q_{i-1}$, *for all* $z \in V^*$, *there exists an ideal* $\mathfrak{A}$ *such that* ann. $(A_i) = Q_i$.

PROOF. Let $\mathfrak{A}$ be the subset of $\Lambda(V^*)$, closed under addition, which has its r homogeneous elements equal to ann. (Q_r). We assert that $\mathfrak{A}$ is an ideal. All that remains to show is that $\mathfrak{A}$ is closed under multiplication with any element of $\Lambda(V^*)$. This follows from the fact that for $x \in$ ann. (Q_{i-1}) and $z \in V^*$ we have $z \wedge x \in$ ann. (Q_i). This is because $\langle \xi, z \wedge x\rangle = \langle \xi \llcorner z, x\rangle$ and $\xi \llcorner z \in Q_{i-1}$ for all $\xi \in Q_i$. We may extend this result to any element of $\Lambda(V^*)$ in the obvious way. This proves the theorem.

DEFINITION 2.2. Let Q_i be a set of subvector spaces such that $Q_i \llcorner z \subset Q_{i-1}$ for all $z \in V^*$. We will call an ideal annihilator the module over the reals generated by this set under addition in $\Lambda(V)$.

LEMMA 2.2. *Let* $\mathfrak{G}$ *be an ideal annihilator and let* Q *denote its homogeneous elements of degree one. Then* $\Lambda(Q) \supset \mathfrak{G}$.

PROOF. Let Q_r denote the homogeneous elements of $\mathfrak{G}$ of degree r. We will prove the lemma by an induction on r. Clearly $\Lambda^1(Q) \supset Q_1$. Assume $\Lambda^r(Q) \supset Q_r$. Let $\xi \in Q_{r+1}$. Then $\xi \llcorner z \in Q_r$ or $\xi \llcorner z \in \Lambda^r(Q)$. Hence by Theorem 1.2 $\xi \in \Lambda^{r+1}(Q)$. This proves the lemma.

THEOREM 2.3. *Let* $\mathfrak{G}$ *be the ideal annihilator of the ideal* $\mathfrak{A}$. *Then a necessary and sufficient condition for* $\mathfrak{A}$ *to be one generated is that* $\mathfrak{G}$ *be a Grassmann algebra over* ann. (A_1).

PROOF. Let $\mathfrak{G}$ be a Grassmann algebra over $W \subset V$. Then, by Theorems 1.2 and 2.2, $\mathfrak{G}$ is an ideal annihilator. Let $\mathfrak{A}$ be the ideal of which $\mathfrak{G}$ is the ideal annihilator. Now let $\mathfrak{A}'$ be the one generated ideal generated by ann. (W). Then if $\mathfrak{G}'$ is the ideal annihilator for $\mathfrak{A}'$, we have $\mathfrak{G}' \subset \mathfrak{G}$. Hence $\mathfrak{A} \subset \mathfrak{A}'$. But, since $\mathfrak{A}'$ is one generated by ann. (W) and $\mathfrak{A}$ contains ann. (W), $\mathfrak{A}' \subset \mathfrak{A}$. Hence $\mathfrak{A} = \mathfrak{A}'$.

Let $\mathfrak{A}$ be one generated by A_1. Let $\mathfrak{G}'$ be the Grassmann algebra over ann. (A_1) with associated ideal $\mathfrak{A}'$. Then $\mathfrak{G}' \supset \mathfrak{G}$, where $\mathfrak{G}$ is the ideal annihilator of $\mathfrak{A}$. Hence $\mathfrak{A}' \subset \mathfrak{A}$. But we know that $\mathfrak{A}'$ is one generated by A_1 by the previous argument. Hence $\mathfrak{A}' = \mathfrak{A}$ and $\mathfrak{G}' = \mathfrak{G}$.

DEFINITION 2.3. If $\mathfrak{A}$ has the Grassmann algebra $\mathfrak{G}$ as ideal annihilator then $\mathfrak{A}$ is called the associated ideal of $\mathfrak{G}$.

3. Minimal ideals

DEFINITION 3.1. A one generated ideal $\mathfrak{M}$ will be called a minimal ideal with respect to the ideal $\mathfrak{A}$ if:

(1) $\mathfrak{M} \supset \mathfrak{A}$.

(2) There exists no one generated ideal properly contained in $\mathfrak{M}$ which satisfies Property 1.

DEFINITION 3.2. If an ideal is one generated we will call the dimension of the ideal the dimension of the subspace which generated it.

THEOREM 3.1. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two minimal ideals for $\mathfrak{A}$. Then* dim. $\mathfrak{M}_1 =$ dim. $\mathfrak{M}_2$ *if the ideal generated by $\mathfrak{M}_i$ and W is a minimal ideal for the ideal generated by $\mathfrak{A}$ and W, where W is a subvector space of V^* and $i = 1, 2$.*

PROOF. If the dimension of $\mathfrak{M}_1$ is n, then the theorem is trivially true. Assume the theorem is true for dim. $\mathfrak{M}_1 \geqq 1 + p$, $p < n$. Let dim. $\mathfrak{M}_1 = p$ and let M_1 generate $\mathfrak{M}_1$ and M_2 generate $\mathfrak{M}_2$. Then if $M_1 = A_1$ the theorem is again true. Assume therefore that $M_1 \supset A_1$. Then M_2 must contain A_1. Hence there exists $x_1 \in (M_1 - M_1 \cap M_2)$ and $x_2 \in (M_2 - M_1 \cap M_2)$. Consider the ideal $\mathfrak{A}'$ generated by $\mathfrak{A}$ and x_1 and x_2. Then clearly neither $\mathfrak{M}_1$ nor $\mathfrak{M}_2$ can be minimal ideals for $\mathfrak{A}'$. But clearly the ideals $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ generated by $\mathfrak{M}_1$ and x_2 and $\mathfrak{M}_2$ and x_1 respectively, are minimal ideals for $\mathfrak{A}'$. Hence we may apply the induction hypothesis to $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$. This and the fact that dim. $\mathfrak{M}_i + 1 =$ dim. $\mathfrak{M}_i'$, $i = 1, 2$, prove the theorem.

DEFINITION 3.3. Let $\mathfrak{M}$ be a minimal ideal for $\mathfrak{A}$, then dim. $\mathfrak{M}$ is called the dimension of $\mathfrak{A}$ when this number is an invariant.

4. Characteristic vector space

DEFINITION 4.1. The characteristic vector space S relative to $\mathfrak{A}$ is the subset of V such that $X \in S$ if and only if $X \lrcorner z \in \mathfrak{A}$ for all $z \in \mathfrak{A}$.

THEOREM 4.1. $S =$ ann. (A_1) *if and only if $\mathfrak{A}$ is one generated.*

PROOF. Let $z \in A_r$ and $\xi \in$ ann. (A_{r-1}). Then $X \lrcorner z \in A_{r-1}$ if and only if $\langle \xi, X \lrcorner z \rangle = 0$. But $\langle \xi, X \lrcorner z \rangle = \langle \xi \wedge X, z \rangle$. Hence $X \lrcorner z \in A_{r-1}$ if and only if $\xi \wedge X \in$ ann. (A_r) for all $\xi \in$ ann. (A_{r-1}) and all r. Hence $\Lambda(S)$ is contained in the ideal annihilator of $\mathfrak{A}$. But by Theorem 2.3 $\Lambda($ann. $(A_1))$ is contained in the annihilator of $\mathfrak{A}$ if and only if $\mathfrak{A}$ is one generated.

THEOREM 4.2. *S is contained in the intersection of the annihilators of all the minimal ideals of $\mathfrak{A}$.*

PROOF. Let $X \in S$. Let $\mathfrak{M}$ be any one generated ideal containing A and let $W =$ ann. (M_1). Then, if we denote by $X + W$ the vector space spanned by X and W, $X + W$ has the property that $\Lambda(X + W)$ is contained in the ideal annihilator of $\mathfrak{A}$. Hence if $\mathfrak{M}$ is a minimal ideal for $\mathfrak{A}$, X is an element of the ann. (M_1). This proves the theorem.

THEOREM 4.3. *Let $\mathfrak{A}$ equal the intersection of its minimal ideals. Then the characteristic vector space S of $\mathfrak{A}$ equals the intersection of the annihilators of the minimal ideals.*

PROOF. Let X be an element of the intersection of the annihilators of the minimal ideals. Then X has the property that $X \wedge \xi \in$ ann. (A_{i+1}) for all $\xi \in$ ann. (A_i). This follows from the fact that ann. (A_i) is exactly spanned by $\Lambda^i(W)$ for all i, where $W =$ ann. (M_1) and M is any minimal ideal for A. Hence $X \in$ ann. (M_1).

5. Genus

Let $\mathfrak{X}$ be a real analytic manifold. At each point $x \in \mathfrak{X}$ denote by $V(x)$ the space of tangent vectors at x and by $V^*(x)$ the space of covectors. Let $\Lambda(V(x))$ and $\Lambda(V^*(x))$ denote the Grassmann algebra over $V(x)$ and $V^*(x)$ respectively. By an exterior differential system on $\mathfrak{X}$, we mean that we are given a subvariety $\mathfrak{Y} \subset \mathfrak{X}$ and the association of an ideal $\mathfrak{A}_x \subset \Lambda(V^*_{(x)})$ to each point $x \in \mathfrak{Y}$ satisfying the following conditions: For every homogeneous element $\alpha \in \mathfrak{A}_x$ there exists a neighborhood U of x in $\mathfrak{Y}$ and an analytic differential form ω, on U, which belongs to $\mathfrak{A}_y$ for each $y \in U$ and which reduces to α at x. This structure will be denoted by Σ. We will say that ω belongs to Σ.

Let $\mathfrak{X}$ and Σ be given. Let $x_0 \in \mathfrak{X}$ such that $x_0 \in \mathfrak{Y}$. Consider $\mathfrak{A}_{x_0}$. Since for the rest of this discussion the point x_0 will be fixed we will drop the subscript x_0. Let $\mathfrak{A} = A_1 + A_2 + \cdots + A_n$ and let $\mathfrak{A}^p$ be the ideal generated by $\{A_1, A_2, \cdots, A_p\}$. Let I be a p dimension subvector space of $V(x_0)$. Then I is said to be a p dimensional integral element if $\Lambda(I)$ is contained in the ideal annihilator of $\mathfrak{A}^p$. Let $\mathfrak{M}$ be the associated ideal to an integral element, then $\mathfrak{M}$ is called an integral ideal and $\mathfrak{M} \supset \mathfrak{A}^p$. An integral element I_1 is called an extension of I if $I_1 \supset I$. Correspondingly $\mathfrak{M}'$ is called a contraction of $\mathfrak{M}$ if $\mathfrak{M}' \subset \mathfrak{M}$. We say that Σ has genus p at x_0 if every q dimensional integral element, $q < p$, has an extension to a p dimensional integral element, but some p dimensional integral element cannot be extended to a $p + 1$ dimensional integral element.

THEOREM 5.1. *Let $\mathfrak{A}$ be generated as an ideal by $\{A_1, \cdots, A_s\}$ and let* dim. $\mathfrak{A} = k$. *Then if $k \geqq s$ and if $\mathfrak{A}$ has genus at least s it has genus k.*

This follows in a straightforward manner from Theorem 3.1, and the definitions of the previous paragraph.

Cartan in [2] has shown that all exterior differential systems which arise from attempts to solve differential equations may always be chosen to be one and two generated.

SPECIAL CASE OF THEOREM 5.1. *Let $\mathfrak{A}$ be one and two generated. Then either the genus of $\mathfrak{A}$ is one or the genus of $\mathfrak{A}$ equals the dimension of $\mathfrak{A}$ when this is defined.*

THEOREM 5.2. *A necessary and sufficient condition for a one and two generated ideal $\mathfrak{A}$ to have genus one is that the intersection of the generators of all minimal ideals for $\mathfrak{A}$ properly contain A_1.*

PROOF. Assume the genus of $\mathfrak{A}$ is one, but that the dimension of $\mathfrak{A}$ is greater than one. Then there exists $X \in$ ann. (A_1) which cannot be extended. Hence $X \notin$ ann. (M_1) where $\mathfrak{M}$ is any minimal ideal of $\mathfrak{A}$. Hence $X \notin \bigcup$ ann. (M_1), where the union is taken over all minimal ideals for $\mathfrak{A}$. But $X \in$ ann. (A_1) and $\bigcap M_1 \supset A_1$. Hence $\bigcap M_1$ contains A_1 properly.

Assume $\bigcap M_1$ contains A_1 properly. Let $X \in$ ann. (A_1), but $X \notin \bigcup$ ann. (M_1). We assert that X cannot be extended. Assume X can be extended or there exists an $X_1 \in$ ann. (A_1) such that $X \wedge X_1 \in$ ann. (A_2). Let $\mathfrak{M}$ be the associated integral ideal. Then since $\mathfrak{A}$ is one and two generated $\mathfrak{M} \supset \mathfrak{A}$. Hence $\mathfrak{M}$ contains a minimal ideal $\mathfrak{M}'$. But since $X \in$ ann. (M_1), $X \in$ ann. (M_1'). Hence $X \in \bigcup$ ann. (M_1). This contradiction proves the theorem.

THEOREM 5.3. *Let $\mathfrak{A}$ be one and two generated and assume the genus of $\mathfrak{A}$ is greater than one. Then the characteristic vector space equals the intersection of the annihilators of the minimal ideals.*

The proof of this theorem is an immediate consequence of Theorems 5.2 and 5.3.

6. Prolongation

Assume we are given over $\mathfrak{X}$ a decomposable analytic form $\omega \in \Lambda^p(V^*(x))$ for all x. Then let $\mathfrak{M}_x$ be a p dimensional integral ideal of Σ. We will say that $\mathfrak{M}$ (dropping the subscript, since we will confine the discussion to this fixed point) is involutive with respect to ω if and only if $\omega \wedge \Lambda^{n-p}(M_1) \neq 0$. An integral element is called involutive with respect to ω if its associated integral ideal is involutive with respect to ω.

For the rest of this paper involutive integral elements or ideals will be taken with respect to a fixed ω.

Let $\mathfrak{Y}'$ denote the set of all involutive integral elements. Then $\mathfrak{Y}'$ is a subset of $\mathfrak{X}'$, where $\mathfrak{X}'$ is the associated bundle to the tangent bundle over $\mathfrak{X}$ with fiber the Grassmann manifold of p planes in n dimensional vector space V. At each point of $y' = (x, I)$, where I is a p dimensional involutive element, consider the ideal generated by $p^*(\mathfrak{M})$, where p^* is the dual mapping induced by the bundle projection and $\mathfrak{M}$ is the integral ideal associated with I. Let Σ^* denote this collection of ideals.

THEOREM 6.1. *Σ^*, $\mathfrak{X}'$, $\mathfrak{Y}'$ constitute a system of exterior differential equations.*

The proof of this theorem is straightforward and will be omitted.

Consider $p^*(V^*(x))$ at x', where $p(x') = x$. Let us fix x and x' for the rest of this discussion and omit them in the formulas. The dimension of $p^*(V^*)$ equals n. Let dimension of the tangent space to x' be n'. We will denote generically by W any $n' - n$ dimension subvector space of $V^*(x')$ such that $W + p^*(V^*)$ spans the tangent space to x'.

THEOREM 6.2. *Given a p dimensional involutive ideal $\mathfrak{M}'$ for Σ' at x' there exists a p dimensional involutive ideal $\mathfrak{M}$ at x for Σ, where $p(x') = x$, and a W such that $\mathfrak{M}'$ is the ideal generated by $p^*(\mathfrak{M})$ and W.*

PROOF. Let $x' = (x, I)$. Then $\mathfrak{M}_1' \supset p^*(\mathfrak{M}_1)$ where $\mathfrak{M}$ is the associated ideal of I. Hence

$$\mathfrak{M}_1' = p^*(\mathfrak{M}_1) + S.$$

Now since $\mathfrak{M}_1' \wedge p^*(\omega) \neq 0$ and $\mathfrak{M}_1 \wedge \omega \,\mathfrak{M}\, 0$ we have that $p^*(V^*) + S$ equals the tangent space to x'. This proves the theorem.

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