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Expanded Lie Group Transformations and Similarity Reductions of Differential Equations

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Continuous groups of transformations acting on the expanded space of variables, which includes the equation parameters in addition to independent and dependent variables, are considered. It is shown that the use of the expanded transformations enables one to enrich the concept of similarity reductions of PDEs. The expanded similarity reductions of differential equations may be used as a tool for finding changes of variables, which convert the original PDE into another (presumably simpler) PDE. A new view on the common similarity reductions as the singular expanded group transformations may be used for defining reductions of a PDE to a specific target ODE.

1 Introduction

By an *expanded* Lie group transformation of a partial differential equation (PDE) we mean a continuous group of transformations acting on the expanded space of variables which includes the equation parameters in addition to independent and dependent variables. We consider the transformations that can be found using the Lie infinitesimal criterion with the properly expanded infinitesimal group generators. In this paper, we are only concerned with groups of point transformations, leaving aside problems involving generalized (Lie–Bäcklund) symmetry groups.

An expanded group of transformations represents a particular case of the equivalence group that preserves the class of PDEs under study – roughly speaking, having the same differential structure but with arbitrary functions having different forms. The approach to finding these equivalence transformation groups with the use of the Lie infinitesimal technique was introduced by Ovsiannikov (see, e.g., [1]) who suggested using the Lie infinitesimal criterion in the properly extended space of variables including dependent and independent variables, arbitrary functions and their derivatives. The original Ovsiannikov method was further developed by Akhatov *et al.* [2]. Their ideas have also been generalized in several papers (see, e.g., [3] and references therein). The transformations in the extended space of variables obtained by adding parameters to the list of independent variables were also used in the context of the renormalization group (RG) symmetries [4, 5].

The main purpose of this paper is to show that the use of the Lie groups of transformations in the expanded space of variables including equation parameters enables one to enrich the concept of similarity reductions as applied to PDEs. In addition, we wish to draw attention to a possibility of using these groups for finding changes of variables that remove some terms from the original equation. Although such a possibility is excluded neither in the framework of the equivalence group approach nor in the context of the RG symmetries, this aspect is obscure in those theories.

The rest of this paper is organized as follows. In Section 2 we consider some illustrative examples of expanded group of transformations that can be used for removing terms from the original differential equations. A comparison with the RG symmetry approach is made. In

Sections 3 and 4 we show how the concept of similarity reductions of PDEs can be enriched in the framework of the expanded transformation groups. Finally, in Section 5 we make some remarks and suggest possible further work.

2 Examples of application of the expanded groups

2.1 Application to ODEs: a simple linear example

We will start with a simple example that applies the technique to the well known ODE of the linear damped oscillator. Of course, this example is only of illustrative value but it is interesting from the methodological point of view since this equation is frequently used to discuss some aspects of asymptotic methods. The equation of linear oscillations with linear damping is

$$u_{tt} + au_t + u = 0 \tag{1}$$

or

$$u_{tt} + u_t + bu = 0, \tag{2}$$

where the subscripts on u denote derivatives. In the context of perturbation methods, when the parameter a or b are assumed to be small, equations (1) and (2) acquire somewhat different physical meanings – an oscillator with a weak resistance for (1) and an overdamped oscillator for (2). An example of application of the expanded transformations to the ODE (1) of the linear oscillator appeared first (to the author's knowledge) in [7]. We will discuss equation (2), which was used in [5] and [6] to illustrate the approaches to the asymptotic analysis of solutions of differential equations based on the renormalization group concept.

We consider the one-parameter (ϵ) Lie group of transformations in (t, u, b):

$$\tilde{t} = g(t, u, b, \epsilon), \qquad \tilde{u} = h(t, u, b, \epsilon), \qquad \tilde{b} = \phi(b, \epsilon) \tag{3}$$

with an infinitesimal generator of the form

$$X = \tau(t, u, b)\frac{\partial}{\partial t} + \eta(t, u, b)\frac{\partial}{\partial u} + \beta(b)\frac{\partial}{\partial b}$$

which leaves (2) invariant. The invariance requirement yields the following determining equations

$$\begin{aligned} &\tau_{uu} = 0, \qquad \eta_{uu} - 2\tau_{tu} + 2\tau_u = 0, \qquad 2\eta_{tu} - \tau_{tt} + \tau_t + 3bu\tau_u = 0, \\ &\eta_{tt} - bu\eta_u + \eta_t + 2bu\tau_t + b\eta + u\beta = 0. \end{aligned} \tag{4}$$

If one is aimed at reducing a given equation to an equation with a known general solution, there is no need in defining the most general form of the group from the determining equations. It is sufficient to define a minimal subgroup arising solely due to the presence of the generator β in the equations. Such a subgroup is found from (4) as

$$\tau = \frac{2\beta t}{1-4b}, \qquad \eta = -\frac{\beta ut}{1-4b}.$$

The finite transformations (3) are defined by solving the problem

$$\begin{aligned} &\frac{dg}{d\epsilon} = \frac{2\beta g}{1-4\phi}, \qquad \frac{dh}{d\epsilon} = -\frac{\beta gh}{1-4\phi}, \qquad \frac{d\phi}{d\epsilon} = \beta, \\ &g = t, \qquad h = u, \qquad \phi = b \qquad \text{at} \quad \epsilon = 0. \end{aligned} \tag{5}$$

Using the third equation of (5) one can go over to derivatives with respect to ϕ in the first two equations and obtain solutions in the forms

$$\tilde{t} = t\left(\frac{1-4b}{1-4\tilde{b}}\right)^{1/2}, \qquad \tilde{u} = u\exp\left\{\frac{t-\tilde{t}}{2}\right\}, \qquad \tilde{b} = b+\epsilon. \tag{6}$$

We have set $\beta = 1$ in the last equation. It can be checked that applying these transformations to the equation $\tilde{u}_{\tilde{t}\tilde{t}} + \tilde{u}_{\tilde{t}} + \tilde{b}\tilde{u} = 0$ yields equation (2).

To find the transformations that remove the last term from the equation (2) by converting it into

$$\tilde{u}_{\tilde{t}\tilde{t}} + \tilde{u}_{\tilde{t}} = 0 \tag{7}$$

we specify the formulae (6) by setting $\tilde{b} = 0$, which corresponds to the specific choice of the group parameter $\epsilon = -b$. Thus, we arrive at the transformations

$$u = \tilde{u}\exp\left\{\frac{\tilde{t}-t}{2}\right\}, \qquad \tilde{t} = t\left(1-4b\right)^{1/2} \tag{8}$$

expressing the solution u of the original equation (2) through the solution $\tilde{u}$ of equation (7). Substituting the general solution of (7) $\tilde{u} = C_1 + C_2\exp\{-\tilde{t}\}$ into (8) yields a general solution of (2).

Next we will compare our approach, that is based merely on the expanded transformation group, with the approaches of [5] and [6] using the RG concept. The methods of [5] and [6] were designed to improve approximate solutions of the boundary value problems for equations depending on a small parameter. Correspondingly, the initial-value problem for equation (2) with a small parameter b is considered. In [6] the renormalization technique is developed to construct the uniformly valid asymptotics using a straightforward naive perturbation expansion as a starting point. The approach of [5] treats the RG as the Lie group of transformations for a renormgroup manifold constructed in a special way (including parameters of the equations in the list of independent variables is considered as one of the possibilities). The authors consider the initial-value problem for a system of two first order ODEs replacing (2) and find that the use of the modified RG technique permits to improve a perturbation expansion solution up to the exact solution. Thus, as a matter of fact, the expanded transformations are applied in [5] with the same result as that described above in this section. However, in the method of [5], the issue of the *exact* transformations, that would reduce the equation to the simpler one by removing some terms, is obscured by the underlying asymptotic concept, and this may conceal the possibility of finding such transformations. In addition, use of an approximate solution as a starting point and embedding the initial conditions in the framework of the method make the procedure more complicated.

2.2 Application to PDEs: two-dimensional steady-state nonlinear diffusion equation

This example applies the technique to the nonlinear equation

$$u_{xx} + u_{yy} + u_x^2 + u_y^2 + bu_x + a\left(1 + pe^{-u}\right) = 0, \tag{9}$$

where a, b and p are constants. This equation can be obtained from the steady-state nonlinear diffusion (heat conduction) equation with the source and the gradient term in the case where the coefficient of thermal conductivity and the source have exponential dependence on the concentration (temperature): $K(u) = K_0e^u$ and $Q(u) = Q_0 + Q_1e^u$. The equation including both the source and the gradient terms has been chosen to show how the technique works in the case when two equation parameters a and b are involved into transformations.

We consider again the one-parameter (ϵ) Lie group of transformations in (x, y, u, a, b) space:

$$X = \xi(x, y, u, a, b)\frac{\partial}{\partial x} + \zeta(x, y, u, a, b)\frac{\partial}{\partial y} + \eta(x, y, u, a, b)\frac{\partial}{\partial u} + \alpha(a, b)\frac{\partial}{\partial a} + \beta(a, b)\frac{\partial}{\partial b}$$

which leaves equation (9) invariant. Solving the determining equations yields

$$\xi = C_1 + \frac{2\alpha - \beta b}{b^2 - 4a}x + ky, \qquad \zeta = C_2 - kx + \frac{2\alpha - \beta b}{b^2 - 4a}y,$$
$$\eta = \left(1 + pe^{-u}\right)\left(C_3 + \frac{2\beta a - \alpha b}{b^2 - 4a}x - \frac{k}{2}by\right),$$

where C_1, C_2, C_3 and k are constants. Here considering groups more general than the minimal subgroup can be useful for constructing different solutions of the initial equation from a special solution of a simplified equation. We will show the finite transformations for the case of $C_1 = C_2 = 0$, as follows

$$\tilde{x} = S(\epsilon)\big(x\cos A(\epsilon) - y\sin A(\epsilon)\big), \qquad \tilde{y} = S(\epsilon)\big(x\sin A(\epsilon) + y\cos A(\epsilon)\big),$$
$$\tilde{u} = \ln\left\{-p + (p + e^u)\exp\left[C_3\epsilon + \frac{bx}{2} - \frac{b + \epsilon\beta}{2}S(\epsilon)\big(x\cos A(\epsilon) - y\sin A(\epsilon)\big)\right]\right\},$$
$$\tilde{a} = a + \epsilon\alpha, \qquad \tilde{b} = b + \epsilon\beta, \tag{10}$$

where

$$S = \left[\frac{b^2 - 4a}{b^2 - 4a + \epsilon(2\beta b - 4\alpha) + \epsilon^2\beta^2}\right]^{1/2},$$
$$A = -k\epsilon, \qquad \alpha = 1, \qquad b = 1 \qquad \text{for} \quad \beta = 0,$$
$$A = \frac{k}{\beta}\left\{\left[\left(b - 2\frac{\alpha}{\beta}\right)^2 + \epsilon(2\beta b - 4\alpha) + \epsilon^2\beta^2\right]^{1/2} - \left(b - 2\frac{\alpha}{\beta}\right)\right\} \qquad \text{for} \quad \beta \neq 0.$$

These transformations map equation (9) into the equation with parameters $\tilde{a}$ and $\tilde{b}$ calculated according to (10). In particular, it is possible to transform (9) into an equation without either the last source term or the term bu_x by setting respectively either $\beta = 0$ and $\epsilon = -a/\alpha$ or $\alpha = 0$ and $\epsilon = -b/\beta$ in the formulae (10).

3 Expanded similarity reductions of PDEs

In this section we will show that the symmetry reduction procedure implemented in the expanded space can lead to discovering transformations between equations that cannot be obtained by applying the technique described in the previous section. We will take as an example the Fokker–Planck equation

$$u_t = u_{xx} + xu_x + u \tag{11}$$

which is used in statistical physics to describe the evolution of probability distribution functions (see, for example, [8]). This equation does not include any parameters – two physical parameters in the original equation corresponding to the problem of a free particle in Brownian motion can be removed by the scaling of x and t. Nevertheless, to apply the technique, a parameter may be introduced into the equation "artificially", for example, as the coefficient in front of the last term of the equation

$$u_t - u_{xx} - xu_x - au = 0. \tag{12}$$

This coefficient cannot change as a result of rescaling the time and space coordinates without appearance of coefficients in front of other terms of the equation. It may seem that introducing this coefficient spoils the physics of the problem (the last two terms in the equation must have equal coefficients), but one may always set $a = 1$ in the final formulae.

We consider the one-parameter (ϵ) Lie group of infinitesimal transformations in (x, t, u, a) defined by

$$X = \xi(x,t,u,a)\frac{\partial}{\partial x} + \tau(x,t,u,a)\frac{\partial}{\partial t} + \eta(x,t,u,a)\frac{\partial}{\partial u} + \alpha(a)\frac{\partial}{\partial a}.$$

Applying the invariance criterion to equation (12) and solving the determining equations yields

$$\begin{aligned} &\xi = \left(C_1 e^{2t} - C_2 e^{-2t}\right)x + C_3 e^t + C_4 e^{-t}, \qquad \tau = C_0 + C_1 e^{2t} + C_2 e^{-2t}, \\ &\eta = \left[-C_1 e^{2t} x^2 - C_3 e^t x - C_1 e^{2t} + a\left(C_0 + C_1 e^{2t} + C_2 e^{-2t}\right) + \alpha t + C_5\right] u + \Phi(x,t), \end{aligned} \tag{13}$$

where C_0, C_1, C_2, C_3, C_4 and C_5 are constants and $\Phi(x,t)$ satisfies the equation $\Phi_{xx} + x\Phi_x - \Phi_t + a\Phi = 0$.

To illustrate the approach we will take the C_1-subgroup of the full group (13): $C_0 = C_2 = C_3 = C_4 = C_5 = \Phi = 0$. We will start with defining classical (not expanded) similarity reductions corresponding to this subgroup – one should set $\alpha = 0$ and $a = 1$ to obtain a classical group from (13). With such group generators, from the invariant surface condition

$$\xi u_x + \tau u_t - \eta = 0$$

we derive the functional form of the similarity reduction, which being substituted into equation (11) eventually produces the following

$$u(x,t) = e^{-x^2/2} w(z), \quad z = xe^{-t}, \qquad w'' = 0. \tag{14}$$

Now we will apply the same procedure, but for the expanded similarity reductions in (x, t, u, a) (instead of (x, t, u)) space. The generators of the same subgroup are now given by

$$\xi = C_1 x e^{2t}, \qquad \tau = C_1 e^{2t}, \qquad \eta = \left[-C_1 x^2 e^{2t} + (a-1)C_1 e^{2t} + \alpha t\right] u.$$

Integrating the characteristic system for the invariant surface condition of the form

$$\xi u_x + \tau u_t + \alpha u_a - \eta = 0$$

yields three similarity variables

$$z = xe^{-t}, \qquad \varphi = -\frac{C_1}{\alpha} a - \frac{1}{2} e^{-2t}, \qquad w = ue^{(1-a)t + x^2/2}.$$

We choose w as a new dependent variable and take z and φ as new independent variables to arrive at the following similarity reduction

$$u(x,t,a) = w(z,\varphi) e^{(a-1)t - x^2/2}, \qquad z = xe^{-t}, \qquad \varphi = -\frac{C_1}{\alpha} a - \frac{1}{2} e^{-2t}. \tag{15}$$

Although in terms of the expanded transformations such a similarity reduction reduces the number of variables, it seems that we gain nothing for solving the original PDE since, upon substituting the reduction into the equation, it will be reduced again to a PDE for a function $w(z, \varphi)$. However, this new equation will have a different form, which may allow new possibilities for solution. In particular, substituting (15) into (12) yields

$$w_\varphi - w_{zz} = 0. \tag{16}$$

Thus, the expanded similarity reduction (15) taken for $a = 1$ (we may also set $\alpha = 1$ without loss of generality) provides a transformation of the original equation (11) to the linear heat equation.

Such a transformation cannot arise as the result of application of the classical Lie group method – it is seen that the similarity solution (14) provided by the classical method corresponds to taking only a particular solution of (16), namely, $w = w_0 + w_1 z$ where w_0 and w_1 are arbitrary constants. The technique described in the previous section can produce it neither. It enables one to define only the transformations eliminating the last term of (12).

4 Similarity reductions as the singular expanded group transformations

In this section, we will show that including the equation parameters into the transformations allows one to treat the classical similarity reductions of a PDE as the expanded group transformations which are singular in some variables. This can also enable one to define reductions of the PDE to specific target ODEs.

We will take as an example the Generalized Boussinesq (GBQ) equation

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + u_{tt} = 0 \tag{17}$$

which has a number of equations, arising in different physical applications, as special cases. Similarity reductions for equation (17) obtained from the classical Lie group method and from the Clarkson–Kruskal direct method have been considered in [9].

To apply the method we will introduce an artificial coefficient a in front of the last term, as follows

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + au_{tt} = 0 \tag{18}$$

and consider the one-parameter Lie group of infinitesimal transformations in the expanded (x, t, u, a, p, q, r) space defined by

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + A\frac{\partial}{\partial a} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + R\frac{\partial}{\partial r}.$$

Several different families of solutions of the determining equations obtained from the invariance requirement may arise depending on the relations between the coefficients of (18). We will restrict ourselves to the case

$$p = q, \qquad r = \frac{q^2}{2a} \qquad \Longrightarrow \qquad P = Q, \qquad \frac{R}{r} = 2\frac{Q}{q} - \frac{A}{a}. \tag{19}$$

Then solving the determining equations yields

$$\xi = C_1 xt + C_2 x + C_3 t + C_4, \qquad \tau = C_1 t^2 + \left(2C_2 + \frac{A}{2a}\right)t + C_0,$$
$$\eta = \left(\frac{A}{2a} - \frac{Q}{q}\right)u + \frac{a}{q}C_1 x^2 + \frac{2a}{q}C_3 x + C_5. \tag{20}$$

Here not only the generators A and Q may depend on a and q but also the $C_0, C_1, \ldots, C_5$ are allowed to be functions of a and q due to the fact that the determining equations do not include the corresponding derivatives.

To show the idea it is sufficient to consider the subgroup of (20) defined by the conditions $C_0 = C_2 = C_3 = C_4 = C_5 = 0$ and the following

$$\frac{A}{2a} = \frac{Q}{q} \qquad \Longrightarrow \qquad R = 0 \qquad (r = \text{const}), \qquad q = \sqrt{2r}a^{1/2}, \tag{21}$$

where (19) has been used. In this special case, the finite group transformations are found by solving the problem

$$\frac{d\tilde{x}}{d\epsilon} = C_1(\tilde{a})\tilde{x}\tilde{t}, \qquad \frac{d\tilde{t}}{d\epsilon} = C_1(\tilde{a})\tilde{t}^2 + \frac{A(\tilde{a})}{2\tilde{a}}\tilde{t}, \qquad \frac{d\tilde{u}}{d\epsilon} = \frac{1}{\sqrt{2r}}\tilde{a}^{1/2}C_1(\tilde{a})\tilde{x}^2, \qquad \frac{d\tilde{a}}{d\epsilon} = A(\tilde{a}),$$
$$\tilde{x} = x, \qquad \tilde{t} = t, \qquad \tilde{u} = u, \qquad \tilde{a} = a \qquad \text{at} \quad \epsilon = 0. \tag{22}$$

The solutions of the problem (22) may be represented as

$$\tilde{t} = \frac{t\sqrt{\tilde{a}}}{\sqrt{a}[1 + tN(\tilde{a}, a)]}, \qquad \tilde{x} = \frac{x}{1 + tN(\tilde{a}, a)}, \qquad \tilde{u} = u - \frac{\sqrt{a}}{\sqrt{2r}}\left[\frac{x^2 N(\tilde{a}, a)}{1 + tN(\tilde{a}, a)}\right],$$
$$N(\tilde{a}, a) = -\frac{1}{\sqrt{a}}\int_a^{\tilde{a}} \frac{C_1(\varphi)\sqrt{\varphi}}{A(\varphi)}d\varphi. \tag{23}$$

In the formulae (23), $\tilde{a}(a, \epsilon)$ is a solution of the last equation of (22) with the corresponding initial condition.

We will look for a transformation removing the terms with derivatives with respect to t from (18) which requires $\tilde{a} = 0$, $\tilde{p} = 0$ and $\tilde{q} = 0$ – in view of the assumptions (19) and (21), it is sufficient to set $\tilde{a} = 0$. It is immediately seen that the transformation obtained in such a way is singular in the variable t: setting $\tilde{a} = 0$ in (23) yields $\tilde{t} = 0$. However, since the equation resulting from this transformation does not include derivatives with respect to t, one can treat it as an equation for the function $\tilde{u}(\tilde{x})$ of one variable, and then it does not matter what happens with the discarded variable t. The transformation obtained by setting $\tilde{a} = 0$ in (23) (having in mind the original equation (17) we may also set $a = 1$ now) is

$$u = \tilde{u}(\tilde{x}, 0) + \frac{\lambda}{\sqrt{2r}}\left(\frac{x^2}{1 + t\lambda}\right), \qquad \tilde{x} = \frac{x}{1 + t\lambda}, \qquad \lambda = N(0, 1). \tag{24}$$

This transformation reduces (17) to the following equation (it is readily checked by the direct substitution):

$$\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + r\tilde{u}_{\tilde{x}}^2\tilde{u}_{\tilde{x}\tilde{x}} = 0 \tag{25}$$

in which a number of independent variables is reduced as compared with the original equation. Thus, the expanded group transformations given by (23) with $\tilde{a} = 0$, which are singular in t, provide the following similarity reduction of the GBQ equation:

$$u = w(z) + \frac{\lambda}{\sqrt{2r}}\left(\frac{x^2}{1 + t\lambda}\right), \qquad z = \frac{x}{1 + t\lambda}, \tag{26}$$

where $w(z)$ satisfies

$$w'''' + rw'^2w'' = 0. \tag{27}$$

The forms of the functions $C_1(a)$ and $A(a)$ have not been specified in the process of derivation of the above formulae. It is evident that any function $C_1(a)$, which being substituted into (23) provides $\lambda = N(0,1) \neq 0$ (for example, $C_1 = \text{const}$), and any function $A(a)$, which permits a transformation to $\tilde{a} = 0$ (for example, $A = 1$, $\tilde{a} = a + \epsilon$), are suitable.

To obtain a similarity reduction, which is more general than (26) but reduces (17) to the same ODE (27), one should consider a more general subgroup of the group (20).

The singular expanded transformations considered above, which remove terms with derivatives with respect to t from the original equation (17), can produce only the reductions to the ODE (27). To define a reduction of the original equation to another ODE (or, at least, to check whether such a reduction is possible) we will look for the expanded transformations that remove some terms from and simultaneously add other terms (desired in the target ODE) to the original PDE. For example, if we wish to define a reduction from (18) (we consider again the particular case defined by (19) and (21)) to the equation

$$w'''' + rw'^2 w'' + kw'' = 0, \tag{28}$$

we have to consider the expanded transformations of the equation

$$u_{xxxx} + qu_t u_{xx} + qu_x u_{xt} + (q^2/2a)u_x^2 u_{xx} + au_{tt} + bu_{xx} = 0 \tag{29}$$

in the (x, t, u, a, q, b) space with a requirement for $\tilde{a} = \phi(a, q, b)$, and $\tilde{q} = \kappa(a, q, b)$ to transform respectively from a and q to zero values, and for $\tilde{b} = \mu(a, q, b)$ to transform from zero to k. We will take the subgroup which for $b = \tilde{b} = 0$ would coincide with that defined by (22) and (23). Omitting the details of calculations we will show only the resulting finite transformations

$$\begin{aligned}
&\tilde{t} = \frac{t\sqrt{\tilde{a}}}{\sqrt{a}(1+tN)}, \qquad \tilde{x} = \frac{x}{1+tN}, \\
&\tilde{u} = u - \frac{\sqrt{a}}{\sqrt{2r}}\left(\frac{x^2 N}{1+tN}\right) + \frac{b}{\sqrt{2ra}}t - \frac{\tilde{b}}{\sqrt{2ra}}\left(\frac{t}{1+tN}\right), \\
&\tilde{a} = a + \epsilon, \qquad \tilde{q} = \tilde{a}^2 \frac{q}{a^2}, \qquad \tilde{b} = \mu(a, r, b, \tilde{a}).
\end{aligned} \tag{30}$$

The reduction from (17) to (28) is obtained from (30) by setting $a = 1$, $\tilde{a} = 0$, $b = 0$, $\tilde{b} = k$, as follows

$$u = \tilde{u}(\tilde{x}, 0) + \frac{\lambda}{\sqrt{2r}}\left(\frac{x^2}{1+t\lambda}\right) + \frac{k}{\sqrt{2ra}}\left(\frac{t}{1+t\lambda}\right), \qquad \tilde{x} = \frac{x}{1+t\lambda}, \qquad \lambda = N(0,1)$$

with a subsequent change of notation $\tilde{x} \to z$, $\tilde{u}(\tilde{x}, 0) \to w(z)$.

Other similarity reductions of equation (17) can be derived in a similar way. It is worth noting that, within this framework, the reduction to equation (27) (or (25)), which includes only the terms from the original equation, is singled out from the variety of possible reductions. The fact that it possesses some special properties was marked in [9] without addressing its special nature.

5 Concluding remarks

In this paper we have demonstrated that some new applications of the Lie group method to differential equations may arise due to the use of expanded transformation groups.

Simplifying the original equation by means of eliminating some terms from the equation via expanded symmetry groups, discussed in Section 2, cannot be considered as a quite new method – as a matter of fact, it represents a particular case of the equivalence group approach. Here our purpose was to stimulate an interest in the fact that, in this pure form, the approach offers considerable promise for applications, so that it would be helpful to introduce the corresponding options into existing computer algebra packages (it does not require significant modifications). Even if this approach does not guarantee simplifying the equation it provides a way of checking whether the simplifications are possible.

The expanded similarity reductions of differential equations considered as a tool for finding changes of variables, which convert the original PDE into another (presumably simpler) PDE, represent a new method that may be applied in a promising way. Also a new view on the common similarity reductions as the singular expanded group transformations may be used for developing a technique for defining reductions to specific target ODEs.

We will also mention a possible use of the expanded transformations (not discussed here) in the context of perturbation methods. It may enable one to introduce a small parameter into a problem in the case when this cannot be done by a rescaling procedure.

The extensions of the formalism to contact and Lie–Bäclund groups are straightforward. Other generalizations of the described approaches – for example, in the spirit of the non-classical method – are also possible.

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