Dynamical Systems

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The following people have maintained these notes.

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Contents

Introduction

These notes are based on the course "Dynamical Systems" given by Dr. C. Baesens in Cambridge in the Lent Term 1998. These typeset notes are totally unconnected with Dr. Baesens. The recommended books for this course are discussed in the bibliography.

Other sets of notes are available for different courses. At the time of typing these courses were:

Probability	Discrete Mathematics
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Methods	Quantum Mechanics
Fluid Dynamics 1	Quadratic Mathematics
Geometry	Dynamics of D.E.'s
Foundations of QM	Electrodynamics
Methods of Math. Phys	Fluid Dynamics 2
Waves (etc.)	Statistical Physics
General Relativity	Dynamical Systems

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Chapter 1

Basic concepts

1.1 What is a dynamical system?

A dynamical system is a system whose evolution in time is uniquely determined by its current state. Time can be discrete or continuous, and in this course we concentrate almost entirely on discrete time, in which the interesting ideas are reached more easily. Also, some continuous time dynamical systems can be reduced to discrete time dynamical systems.

Discrete time dynamical systems are generated by the iteration of maps. Let X be a topological space and $f\colon X \mapsto X$ be continuous. Then $x_{n+1} = f(x_n)$ is a dynamical system, where n is the time and x_n is the state at time n. X is called state space or (for historical reasons) phase space.

For $n > 0$ define the n^{th} iterate of f, f^n as

$$f^n = \overbrace{f \circ \cdots \circ f}^{n \text{ times}}.$$

We also define $f^0 \equiv \text{id}$, and so $x_n = f^n(x_0)$. If f is invertible then we can also define $f^{-n} = \left(f^{-1}\right)^n$.

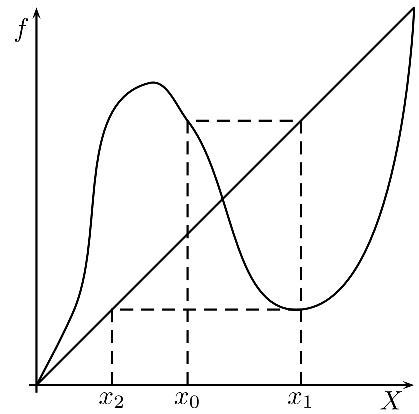
A more sophisticated view is that a dynamical system is an action of a semigroup or a group on a topological space. We have

$$\begin{aligned} \phi\colon G \times X &\mapsto X \\ \phi(g, x) &\mapsto \phi_g(x) \qquad \text{such that} \\ \phi_g(\phi_h(x)) &= \phi_{gh}(x), \end{aligned}$$

where $G = (\mathbb{R}, +)$, $(\mathbb{R}_+, +)$, $(\mathbb{Z}, +)$ or $(\mathbb{Z}_+, +)$ X is a topological space and ϕ is continuous. Then the discrete time dynamical system is the map $\phi\colon \begin{matrix}\mathbb{Z}\\ \mathbb{Z}_+\end{matrix} \times X \mapsto X$ such that $\phi(n, x) = f^n(x)$.

Definition 1.1. *The forward orbit of $x \in X$, which is denoted $O^+(x)$ is the sequence $x, f(x), f^2(x), \ldots$, that is $\left(f^n(x)\right)_{n \in \mathbb{Z}_+}$. If f is invertible we can define the (full) orbit of x as $O(x) = \left(f^n(x)\right)_{n \in \mathbb{Z}}$. We can also define the backwards orbit of x in the obvious way.*

In one dimension there is a graphical representation of iteration.



The apparent generalisation $x_{n+k} = F(x_{n+k-1}, \ldots, x_n)$ can also be viewed as a dynamical system by putting $X = \mathbb{R}^k$ and

$$f \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(k)} \end{pmatrix} = \begin{pmatrix} x^{(2)} \\ \vdots \\ x^{(k)} \\ F(x^{(k-1)}, \ldots, x^{(1)}) \end{pmatrix}.$$

1.2 Dynamical systems viewpoint

We do *not* aim to find explicit formulae but instead to understand the "qualitative features" of the dynamical system, e.g. fixed points, periodic orbits.

We say that x is a *fixed point* of f if $f(x) = x$ and x is a *periodic point of (least) period* q if $f^q(x) = x$ and $f^n(x) \neq x$ for $0 < n < q$.

More thoroughly, by "qualitative features" we mean properties which are preserved under change of co-ordinates by homeomorphism.[1] We say that $f\colon X \mapsto X$ and $g\colon Y \mapsto Y$ are *topologically conjugate* if there exists a homeomorphism $h\colon X \mapsto Y$ such that $g \circ h = h \circ f$ (h is called the conjugacy). Then the qualitative features of f and g are the same, for instance if f has a fixed point $\bar{x}$ then g has a fixed point $h(\bar{x})$.

1.3 Asymptotic behaviour

The most interesting qualitative features are those to do with the behaviour of orbits when $t \to \pm\infty$.

Definition 1.2. *The ω-limit set of $x \in X$, $\omega(x)$ is the set*

$$\omega(x) = \{y \in X : \exists (n_i) \to \infty \textit{ such that } f^{n_i}(x) \to y\}.$$

If f is invertible then we can define the α-limit set by replacing ∞ with $-\infty$. Note that $\omega(f(x)) = \omega(x)$.

For instance, if $\bar{x}$ is a fixed point or periodic point of f then $\omega(\bar{x}) = O^+(\bar{x})$.[2]

[1] homeomorphism: continuous map with continuous inverse

[2] Abuse of notation: $O^+(x)$ is now $\{f^n(x) : n \in \mathbb{Z}_+\}$.

1.4 Homeomorphisms of the interval

The simplest class of dynamical system is that of homeomorphisms of a closed interval $I \subset \mathbb{R}$. There are two possible cases

- f is orientation preserving. Then the only possible ω (resp. α) -limit sets are fixed points.

  Proof. Take $x \in I$. If $f(x) = x$ there is nothing to prove so assume $f(x) > x$, so by orientation preservation $(f^n(x))_n$ is an increasing sequence, bounded above and so tends to a limit $\bar{x}$, which by the continuity of f must be a fixed point. □

  We have obtained a complete description of the dynamics. There is a closed set of fixed points and in each complementary interval the orbits move either to the right ($f(x) > x$) or to the left ($f(x) < x$).

- f is orientation reversing. Then the only possible ω (resp. α) -limit sets are fixed points or period 2 points. To prove this, simply note that f^2 is orientation preserving.

To get more exciting ω-limit sets we can consider maps of the circle or add non-invertibility in one dimension, or we can go to higher dimensions.

Chapter 2

Maps of the circle

2.1 Generalities

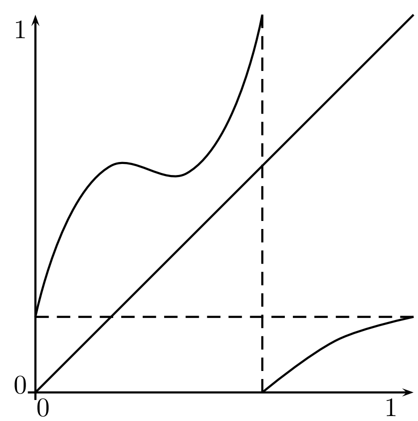
We consider continuous maps of the circle S^1 into itself. There are two ways of representing S^1,

1. $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, or equivalently $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. This is the *multiplicative notation*.

2. $S^1 = \mathbb{R}/\mathbb{Z}$, the quotient of the reals by integer translation. This is *additive notation*.

Additive notation will be more useful for our purposes. $\log$ establishes the isomorphism between the two representations.

We can represent circle maps graphically by cutting the circle at one point and regarding it as an interval.



The simplest example of circle map is rotation by angle β[1], r_β. In multiplicative and additive notation respectively we have

1. $r_\beta z = e^{2\pi\imath\beta} z$

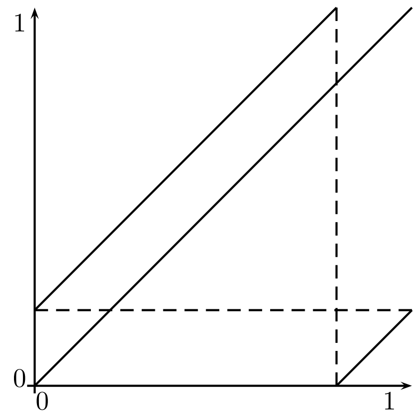
2. $r_\beta x = x + \beta \pmod 1$.

[1] $\beta \in \mathbb{R}$...

We can find the iterates of r_β in both additive and multiplicative notation:

1. $r_\beta^n z = e^{2\pi \imath n\beta} z$
2. $r_\beta^n x = x + n\beta \pmod 1$.

Represented graphically this is



There is a crucial distinction between $\beta \in \mathbb{Q}$ and $\beta \in \mathbb{R} \setminus \mathbb{Q}$. If β is rational with $\beta = \frac{p}{q}$, $q > 0$ and p and q coprime then $r_\beta^q x = x$ for all x and all points are periodic with least period q. When β is irrational things are a little more complicated.

Definition 2.1. *A subset $S \subset X$ is* invariant *under f if $fS = S$. It is* positively invariant *if $fS \subset S$ and* negatively invariant *if $f^{-1}S \subset S$.*

Definition 2.2. *A minimal set $S \subset X$ for f is a closed f-invariant subset of X with no proper closed invariant subsets. Equivalently, a minimal set is a closed invariant subset of X in which $O^+(x)$ is dense in S for all $x \in S$.*

For instance, a periodic orbit is a minimal subset.

Proposition 2.3. *If $\beta \in \mathbb{R} \setminus \mathbb{Q}$ then the orbit of every point $x \in S^1$ under rotation r_β is dense in S^1.*

Proof. Given $\beta \in \mathbb{R} \setminus \mathbb{Q}$, $x \in S^1$ and $\epsilon > 0$, the points $r_\beta^n x$, $n \in \mathbb{Z}_+$ (or $\mathbb{Z}$) are distinct, else $r_\beta^m x = r_\beta^n x$ for some m, n and $(m-n)\beta \in \mathbb{Z}$ (contradiction).

As S^1 is compact, $\exists n \neq m$ such that $0 < d(r_\beta^m x, r_\beta^n x) < \epsilon$. Let $N = |n - m|$ and $\beta_N = d(r_\beta^m x, r_\beta^n x)$. Now r_β preserves orientation and length on S^1, so that r_β^N is just a rotation by β_N. Thus the points

$$\left\{ r_\beta^{jN} x : j = 0, 1, \ldots, \left[\beta_N^{-1}\right] \right\}$$

are equally spaced and come within ϵ of every point of the circle. $\square$

Do we have similar properties for more general orientation preserving homeomorphisms of S^1?

2.2 Lift and degree

Define $\pi\colon \mathbb{R} \mapsto S^1$ by $\pi(x) = x \pmod 1$. Then given a continuous function $f\colon S^1 \mapsto S^1$ there exists a (nonunique) continuous function $F\colon \mathbb{R} \mapsto \mathbb{R}$ such that $f\pi x = \pi F x$. π is *not* a topological conjugacy — it is not invertible.

F is called a *lift* of f.

Lemma 2.4 (Properties of lifts).

1. *If $F_i\colon \mathbb{R} \mapsto \mathbb{R}$, $i = 1, 2$ are lifts of the same continuous map f then $\exists k \in \mathbb{Z}$ such that $F_1 x - F_2 x = k\ \forall x$.*

2. *Given a continuous function $f\colon S^1 \mapsto S^1$ there exists d such that for all lifts F and for all $x \in \mathbb{R}$ such that $F(x+1) = F(x) + d$. d is called the degree of f and written* $\deg f$.

3. *If F is a lift of f then F^n is a lift of f^n for all $n \in \mathbb{Z}$.*

4. $\deg f^n = (\deg f)^n$.

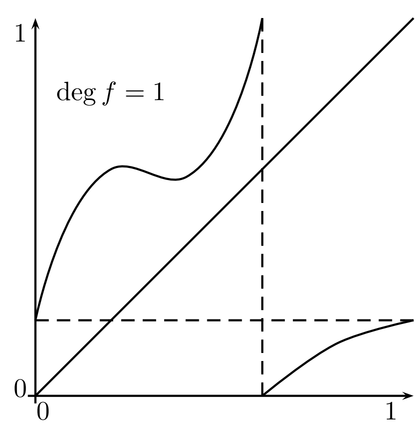
We will prove 1 and 2 leaving 3 and 4 as exercises.

Proof.

1. Take $x \in \mathbb{R}$. Then $f\pi x = \pi F_1 x = \pi F_2 x$ and so $F_1 x - F_2 x \in \mathbb{Z}$. Thus $F_1 x - F_2 x$ is constant (by connectedness of $\mathbb{R}$ and continuity of $F_1 - F_2$).

2. $\pi(x+1) = \pi x$ and so $\pi F(x+1) = f\pi(x+1) = f\pi x = \pi F x$. Thus $F(x+1) - Fx = d \in \mathbb{Z}$ for all x (argue as before). Let $\bar{F}$ be another lift. Thus $\bar{F} = F + k$ and hence $\bar{F}(x+1) - \bar{F}x = d$.

□



Intuitively, $|\deg f|$ measures how many times the circle is mapped around itself by f.

If f is a homeomorphism then $\deg f = \pm 1$. If f is orientation preserving then $\deg f = 1$ and if f is orientation reversing $\deg f = -1$.

The rotation map r_β has $\deg r_\beta = 1$. If we take lifts $R_{\beta,k}\colon \mathbb{R} \mapsto \mathbb{R}$, $x \mapsto x + \beta + k$ with $k \in \mathbb{Z}$ then we see that

$$\lim_{n\to\infty} \frac{R^n_{\beta,k}x - x}{n} == \lim_{n\to\infty} \frac{n(\beta+k)}{n} = \beta + k.$$

We want to generalise this concept to degree 1 maps of the circle. We restrict to degree 1 maps from now on.

2.2.1 Properties of degree one continuous circle maps

Lemma 2.4 gives that if f is a degree 1 circle map with lift F,

$$\begin{aligned} F(x+1) &= F(x)+1 \\ F^k(x+1) &= F^k(x)+1 \qquad k\in\mathbb{N} \\ F^k(x+m) &= F^k(x)+m \qquad m\in\mathbb{Z} \\ F(x)-x &\text{ is periodic with period 1,} \\ F^k(x)-x &\text{ is periodic with period 1.} \end{aligned} \tag{2.1}$$

2.3 Rotation number

Definition 2.5. *Let $f\colon S^1 \mapsto S^1$ be a degree 1 continuous map and F be a lift of f. Then the rotation number of $x \in S^1$ under F is*

$$\bar{\rho}(F,x) = \lim_{n\to\infty} \frac{F^n(x)-x}{n} \qquad \text{if this limit exists.}$$

Theorem 2.6.
Let $f\colon S^1 \mapsto S^1$ be an orientation preserving homeomorphism. Then:

1. *For $x\in\mathbb{R}$, $\bar{\rho}(F,x)$ exists and is independent of x. (Denoted $\bar{\rho}(F)$.)*
2. *$\rho(f) := \bar{\rho}(F) \pmod 1$ does not depend on the lift used.*
3. *$\rho(f)$ depends continuously on f.*

$\rho(f)$ is called the rotation number of f. Before we prove 2.6 we need two lemmas.

Lemma 2.7. *Given x, $y\in\mathbb{R}$, $n\in\mathbb{N}$ and F a lift of an orientation preserving homeomorphism of S^1,*

$$F^n(x)-x-1 < F^n(y)-y < F^n(x)-x+1.$$

Proof. $\exists m\in\mathbb{Z}$ such that $x \le y+m < x+1$. Then F^n is monotone (as F is), and

$$F^n(x) \le F^n(y+m) < F^n(x+1).$$

Using (2.1) freely,

$$\begin{aligned} F^n(x)-x-1 &\le F^n(y+m)-(x+1) \\ &< F^n(y+m)-(y+m) \quad < F^n(x+1)-x = F^n(x)-x+1. \end{aligned}$$

Now note that $F^n(y+m)-(y+m) = F^n(y)-y$. ∎

Lemma 2.8. *Let F be a lift of an orientation preserving homeomorphism $f\colon S^1 \mapsto S^1$ and $n \in \mathbb{N}$. Then $\exists k(n) \in \mathbb{Z}$ such that*

$$k - 1 < F^n(x) - x < k + 1 \qquad \forall x \in \mathbb{R}.$$

The proof of lemma 2.8 is left as an exercise.

Proof of theorem 2.6. We first prove that $\bar{\rho}(F, 0)$ exists. For $n, k \in \mathbb{N}$,

$$\begin{aligned} F^{nk}(0) = &\Big(F^{nk}(0) - F^{n(k-1)}(0)\Big) \\ &+ \Big(F^{n(k-1)}(0) - F^{n(k-2)}(0)\Big) \\ &\vdots \\ &+ \left(F^{2n}(0) - F^n(0)\right) \\ &+ \left(F^n(0) - 0\right). \end{aligned}$$

Using lemma 2.7 with $x = 0$ and $y = F^{n(m-1)}(0)$ for $m = 1, \dots, k$ gives the inequality

$$k\left(F^n(0) - 1\right) < F^{nk}(0) < k\left(F^n(0) + 1\right).$$

Thus

$$\left|\frac{F^{nk}(0)}{nk} - \frac{F^n(0)}{n}\right| < \frac{1}{n}.$$

However, we can exchange the rôles of n and k and using the triangle inequality,

$$\left|\frac{F^{k}(0)}{k} - \frac{F^n(0)}{n}\right| < \frac{1}{k} + \frac{1}{n}.$$

Hence the sequence $\left(\frac{F^n(0)}{n}\right)_{n \in \mathbb{N}}$ is Cauchy and so converges to a limit $\bar{\rho}(F, 0)$.

Now by lemma 2.7 we have

$$\frac{F^n(0) - 1}{n} < \frac{F^n(x) - x}{n} < \frac{F^n(0) + 1}{n}$$

for all $x \in \mathbb{R}$ and so $\bar{\rho}(F, x)$ exists and equals $\bar{\rho}(F, 0)$ for all $x \in \mathbb{R}$.

Now assume that F_1 and F_2 are two lifts of f. Then $\exists k \in \mathbb{Z}$ such that $F_2(x) = F_1(x) + k$ and so $F_2^n(x) = F_1^n(x) + nk$. Therefore

$$\lim_{n\to\infty} \frac{F_2^n(x) - x}{n} = k + \lim_{n\to\infty} \frac{F_1^n(x) - x}{n} \qquad \text{as required.}$$

We finally need to prove continuous dependence on f. Let F be a lift of f. Lemma 2.8 implies that given n, $\exists k$ such that $k - 1 < F^n(x) - x < k + 1$ for all $x \in \mathbb{R}$. Given $\epsilon > 0$ choose $n \in \mathbb{N}$ such that $\frac{2}{n} < \epsilon$.

For g close enough to f in the C^0 topology[2] we can choose a lift G of g such that

$$k - 1 < G^n(x) - x < k + 1 \qquad \forall x \text{ (same } k, n \text{ as for } F).$$

[2] $d(f, g) = \sup_{x \in S^1} |f(x) - g(x)|$

Now

$$F^{nl}(0) - 0 = \sum_{j=0}^{l-1} \Big(F^{n(j+1)}(0) - F^{nj}(0) \Big)$$

and so

$$l(k-1) < F^{nl}(0) < l(k+1).$$

We can do the same thing for G, and we find that

$$\frac{k-1}{n} \leq \bar{\rho}(F) \leq \frac{k+1}{n}$$
$$\frac{k-1}{n} \leq \bar{\rho}(G) \leq \frac{k+1}{n}$$

and thus $|\bar{\rho}(F) - \bar{\rho}(G)| \leq \frac{2}{n} < \epsilon$. □

The rotation number of an orientation preserving homeomorphism is a topological invariant.

Proposition 2.9. *Suppose $f\colon S^1 \mapsto S^1$ and $g\colon S^1 \mapsto S^1$ are orientation preserving homeomorphisms and there exists an orientation preserving homeomorphism h such that $h \circ g = f \circ h$. Then $\rho(f) = \rho(g)$.*

The proof is left as an exercise.

2.4 Orientation preserving homeomorphisms with rational rotation number

Proposition 2.10. *Let f be an orientation preserving homeomorphism of S^1. Then $\rho(f) \in \mathbb{Q}$ iff f has a periodic point. In fact, $\rho(f) = \frac{p}{q}$ with p, $q \in \mathbb{Z}$ coprime and $q > 0$ iff f has a point of least period q.*

Proof. $\Leftarrow$ Suppose f has a periodic point x_0 with least period q and let F be a lift of f. Then $\exists k \in \mathbb{Z}$ such that $F^q(x_0) = x_0 + k$. Then $F^{nq}(x_0) - x_0 = nk$, so

$$\rho(F) = \lim_{n\to\infty} \frac{F^{nq}(x_0) - x_0}{nq} = \frac{k}{q}.$$

Thus $\rho(f) = \frac{k \pmod q}{q}$.

$\Rightarrow$ Assume $\rho(f) = \frac{p}{q}$ (in lowest terms). Let $\bar{F}$ be a lift of f, so $\exists k \in \mathbb{Z}$ such that $\rho(\bar{F}) = k + \frac{p}{q}$. Then $F(x) = \bar{F} - k$ is another lift of f with $\rho(F) = \frac{p}{q}$. Also,

$$\rho(F^q - p) = \rho(F^q) - p = q\rho(F) - p = 0.$$

Let $G(x) = F^q(x) - p$ — it is enough to prove that G has a fixed point in $\mathbb{R}$. We consider $G(0)$, and there are three cases.

1. $G(0) = 0$ — trivial.

2. $G(0) > 0$. G is increasing, so $0 < G(0) < \cdots < G^n(0) < \ldots$. This has two subcases:

   (a) $0 < G^n(0) < 1$ for all n. We have an increasing sequence, bounded above. Thus $G^n(0)$ converges to a limit point which by the continuity of G is a fixed point.

   (b) $\exists k > 0$ such that $G^k(0) > 1$. Then

   $$G^{2k}(0) = G^k(G^k(0)) > G^k(1) = G^k(0) + 1 > 2.$$

   By induction, $G^{jk}(0) > j$ and $\frac{G^{jk}(0)}{jk} > \frac{1}{k}$. Thus the rotation number is bounded away from zero and we have a contradiction.

3. $G(0) < 0$. Similar reasoning applies.

□

We can now describe the dynamics of an orientation preserving homeomorphism of S^1 with the following theorem.

Theorem 2.11. *Let $f\colon S^1 \mapsto S^1$ be an orientation preserving homeomorphism with rational rotation number $\frac{p}{q}$ in lowest terms. Then every orbit is either periodic of period q or forward asymptotic to a period q orbit and backward asymptotic to a period q orbit.*

Proof. f^q can be identified with an orientation preserving homeomorphism of the closed interval by cutting S^1 at a fixed point of f^q. Then section 1.4 applies. □

The periodic orbits of an orientation preserving homeomorphism of S^1 are ordered on S^1 like those of a rigid rotation with the same rotation number. That is, if y is a periodic point and $\rho(f) = \frac{p}{q}$ that the ordering of $(y, f(y), \ldots, f^n(y), \ldots)$ is the same as $(0, \frac{p}{q}, \ldots, \frac{np}{q}, \ldots)$.

2.5 Orientation preserving homeomorphisms with irrational rotation number

Theorem 2.12. *Assume $f\colon S^1 \mapsto S^1$ is an orientation preserving homeomorphism and $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. Then*

1. *$\omega(x)$ is independent of x.*
2. *$E = \omega(x)$ is the unique minimal set of f.*
3. *E is either S^1 or a Cantor subset of S^1.*

Proof.

1. Take any two points $x, y \in S^1$. We wish to show that $\omega(x) \subseteq \omega(y)$. Since S^1 is compact, $\omega(x)$ is non-empty and since $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$, all points in $O^+(x)$ are distinct. Take $z \in \omega(x)$, so $\exists n_i \to \infty$ such that $|f^{n_i}(x) - z| \to 0$. Given $\epsilon > 0$ we can find $n_k > n_j > 0$ such that

   $$|f^{n_i}(x) - z| < \epsilon \qquad \text{for } i = k, j, \text{ and}$$
   $$|f^{n_k}(x) - f^{n_j}(x)| < \epsilon.$$

Let I be the closed interval of length $< \epsilon$ with endpoints $f^{n_k}(x)$ and $f^{n_j}(x)$ and $N = n_k - n_j$. Now $f^N(f^{n_j}(x)) = f^{n_k}(x)$ and $f^{-N}(f^{n_k}(x)) = f^{n_j}(x)$ and so $f^{-N}(I) \cap I = \{f^{n_j}(x)\}$. Arguing similarly, $\left(f^{-mN}(I)\right)_{m\in\mathbb{Z}_+}$ is a sequence of closed intervals joined end to end. Either $f^{-mN}(I)$ accumulates to some point p, which must by continuity satisfy $f^{-N}(p) = p$ — a fixed point, giving a contradiction (by proposition 2.10) or they cover S^1. Thus for all $y \in S^1$, $\exists l \in \mathbb{Z}_+$ such that $y \in f^{-lN}(I)$ and so $f^{lN}(y) \in I$ and $\left|f^{lN}(y) - z\right| < \epsilon$. Thus $\omega(x) \subseteq w(y)$ and by symmetry $\omega(x) = \omega(y)$.

2. E is closed and invariant (by construction). Let $A \subseteq S^1$ be a non-empty, closed invariant set and $x \in A$. Then $O^+(x) \subseteq A$ and so $E = \omega(x) \subseteq A$ as A is closed. Thus any non-empty closed invariant subset of S^1 contains E and E is thus the unique minimal set.

3. A *Cantor subset* of $\mathbb{R}^n$ is a compact, totally disconnected set with no isolated points. On S^1 we can replace “totally disconnected” with “empty interior”.

   We know that $\emptyset$ and E are the only closed invariant subsets of E and as the boundary ∂E is a closed invariant subset of E (exercise), $\partial E = \emptyset$ (and $E = S^1$) or $\partial E = E$.

   If $\partial E = E$ then E has an empty interior. It remains to show that E has no isolated point. Take $x \in E$. Since $E = \omega(x)$, $\exists k_n \to \infty$ such that $\lim_{n\to\infty} f^{k_n}(x) = x$. As f has no periodic point $f^{k_n}(x) \neq x$ for all n, and so x is an accumulation point of E as $f^{k_n}(x) \in E$ by invariance.

□

We have seen examples of maps with $E = S^1$ (r_β, $\beta \in \mathbb{R} \setminus \mathbb{Q}$), but do maps exist with $E \neq S^1$?

Theorem 2.13. *Assume $f\colon S^1 \mapsto S^1$ is a C^2 diffeomorphism and $\beta = \rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. Then f is topologically conjugate to the rotation r_β.*

It is actually sufficient to have f' of bounded variation. In any case the proof is technical and omitted. In this case, $E = S^1$ and all orbits are dense in S^1.

Proposition 2.14. *Let $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists a C^1 orientation preserving diffeomorphism f of S^1 such that $\rho(f) = \beta$ and $E \neq S^1$.*

Sketch proof. We want to find a map with an orbit that is not dense in S^1, so that $E \neq S^1$. The idea is to start from the rigid rotation r_β, $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and to choose an orbit $(x_n)_{n\in\mathbb{Z}}$ of r_β and “blow it up” to an orbit of closed intervals $(I_n)_{n\in\mathbb{Z}}$ with lengths l_n such that $\sum_{n\in\mathbb{Z}} l_n < \infty$ to obtain a map on a new circle $S^{1\prime}$.

We extend r_β to a map $f\colon S^{1\prime} \mapsto S^{1\prime}$ by choosing for each $n \in \mathbb{Z}$ an orientation preserving homeomorphism mapping I_n onto I_{n+1}.

If we choose $f\colon I_n \mapsto I_{n+1}$ to be affine then we create a C^0 map, but to make f C^1 we need $f' = 1$ at the endpoints of each I_n and $\max_{x\in I_n} |f'(x) - 1| \to 0$ as $n \to \infty$.

We see that f has the same rotation number as r_β and that no point in I_n ever returns to I_n under iteration. So if $p \in \operatorname{Int} I_n$ then $f^m(p) \notin \operatorname{Int} I_n$ for $m \neq 0$ and so $O(p)$ is not dense in $S^{1\prime}$.

Thus E is not the whole of $S^{1\prime}$ and so by theorem 2.12 is a Cantor subset of $S^{1\prime}$. □

In fact

$$E = S^{1\prime} \setminus \bigcup_{n \in \mathbb{Z}} \operatorname{Int} I_n$$

and the open sets $\operatorname{Int} I_n$ are the gaps in the Cantor set.

E is nowhere dense since $\bigcup_{n \in \mathbb{Z}} \operatorname{Int} I_n$ is dense in $S^{1\prime}$.

Definition 2.15. *An orbit $O(x)$ is said to be homoclinic to an invariant set $S \subset S^1 \setminus O(x)$ if $\alpha(x) = \omega(x) = S$.*

Theorem 2.16. *Let $f\colon S^1 \mapsto S^1$ be an orientation preserving homeomorphism with $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. Then every orbit is either:*

1. *dense in S^1,*
2. *dense in a Cantor set or*
3. *homoclinic to a Cantor set.*

2.6 Families of circle maps

Proposition 2.17. *Let $f\colon S^1 \mapsto S^1$ be an orientation preserving homeomorphism with lift F such that $\rho(F) = \rho(f) = \frac{p}{q} \in \mathbb{Q}$ and suppose that the graph of $F^q - p$ has points on either side of the diagonal. Then all small enough perturbations of f have rotation number $\frac{p}{q}$.*

This phenomenon is called *frequency locking*.

Proof. Now $\exists x_0$ such that $F^q(x_0) - x_0 - p > 0$ and x_1 such that $F^q(x_1) - x_1 - p < 0$. Then for all small enough perturbations $\bar{f}$ of f with corresponding lift $\bar{F}$, $\bar{F}^q(x_0) - x_0 - p > 0$ and $\bar{F}^q(x_1) - x_1 - p < 0$. These inequalities give $\frac{p}{q} \leq \rho(\bar{F}) \leq \frac{p}{q}$ and the result is thus true. □

2.6.1 Monotonicity of rotation number

Let F_1 and F_2 be lifts of f_1 and f_2. If $F_1(x) < F_2(x)$ for all x then $\rho(F_1) \leq \rho(F_2)$. (Immediate.)

At irrational values the rotation number strictly increases.

Proposition 2.18. *Let F_1 and F_2 be lifts of the orientation preserving homeomorphisms of S^1, f_1 and f_2. If $F_1(x) < F_2(x)$ for all $x \in \mathbb{R}$ then $\rho(F_1) < \rho(F_2)$.*

Proof. By continuity and periodicity, $F_2(x) - F_1(x) > \delta > 0$ for all $x \in \mathbb{R}$. Take $\frac{p}{q} \in \mathbb{Q}$ such that

$$\frac{p}{q} - \frac{\delta}{q} < \rho(F_1) < \frac{p}{q}.$$

Then $\exists x_0$ such that $F_1^q(x_0) - x_0 > p - \delta$, because otherwise

$$\rho(F_1) = \lim_{n\to\infty} \frac{F_1^{nq}(x) - x}{nq} \leq \lim_{n\to\infty} \frac{n(p-\delta)}{nq} = \frac{p}{q} - \frac{\delta}{q}.$$

Now

$$F_2^q(x_0) = F_2(F_2^{q-1}(x_0)) > F_1(F_2^{q-1}(x_0)) + \delta > F_1^q(x_0) + \delta > x_0 + p$$

and so $\rho(F_2) \geq \frac{p}{q} > \rho(F_1)$. □

2.6.2 The Arnold family

This is a 2 parameter family of circle maps $f_{k,\omega}\colon S^1 \mapsto S^1$ with lifts

$$F_{k,\omega}\colon x \mapsto x + \omega + \frac{k}{2\pi}\sin 2\pi x.$$

- $k = 0$: rigid rotation
- $0 \leq k < 1$: diffeomorphism
- $k = 1$: homeomorphism
- $k > 1$: not invertible.

We consider $k, \omega \in [0,1]$. First, fix k and vary ω. If $\omega_1 < \omega_2$ then $F_{k,\omega_1}(x) < F_{k,\omega_2}(x)$ and so $\rho(F_{k,\omega_1}) \leq \rho(F_{k,\omega_2})$. Hence ρ is a non-decreasing function of ω for fixed k. It is also continuous (by theorem 2.6).

Definition 2.19. *A monotone continuous function $\phi\colon [0,1] \mapsto \mathbb{R}$ is called a* devil's staircase *if there exists a collection $\{I_\alpha\}_{\alpha\in A}$ of disjoint closed intervals $[0,1]$ with dense union such that ϕ takes distinct constant values on these intervals.*

Proposition 2.20. *For $k \in [0,1]$, $\phi\colon \omega \mapsto \rho(f_{k,\omega})$ is a devil's staircase. $\phi^{-1}\left(\frac{p}{q}\right)$ is one of the intervals I_α for each rational $\frac{p}{q} \in [0,1]$.*

The proof is left as an exercise.

In (k,ω) parameter space, $\rho(F_{k,\omega}) = 0$ iff $\exists x$ such that $F_{k,\omega}(x) = x$. Thus $\sin 2\pi x = \frac{2\pi\omega}{k}$ and there exist fixed points if $\frac{2\pi\omega}{k} \leq 1$.

Similarly, $\rho(F_{k,\omega}) = 1$ iff $\frac{2\pi(k-\omega)}{k} \leq 1$. Regions in parameter space where ρ is rational are called *Arnold tongues*.

We want to know what happens on the boundary of Arnold tongues.

2.7 Stability, persistence and bifurcations

Definition 2.21. *In one dimension, a fixed point x^* of a differentiable map f is* hyperbolic *if $|f'(x^*)| \neq 1$. The fixed point is* stable/attracting/a source *if $|f'(x^*)| < 1$. It is* unstable/repelling/a sink *if $|f'(x^*)| > 1$.*

A period q orbit (cycle) $\{x_0, x_1, \ldots, x_{q-1}\}$ with $x_i = f^i(x_0)$ is stable if y is stable as a fixed point of f^q for y in the cycle. It is unstable if y is unstable as a fixed point of f^q for y in the cycle.

Note that the y in the cycle used does not matter, as

$$(f^q)'(y) = \prod_{i=0}^{q-1} f'\left(f^i(y)\right) = \prod_{i=0}^{q-1} f'(x_i) \qquad \text{by the chain rule.}$$

2.7.1 Persistence and bifurcation

Consider a 1 parameter family of maps $f\colon \mathbb{R}\times\mathbb{R}$ (or $S^1\times\mathbb{R}$) $\mapsto \mathbb{R}$, $(x,\lambda) \mapsto f(x,\lambda) := f_\lambda(x)$.

We need the implicit function theorem, stated here in a weakened form (and not proved).

Theorem 2.22. *Let $G\colon R \mapsto \mathbb{R}$ be C^r, $r \geq 1$, where*

$$R := \{(x,y) : a < x < b, c < y < d\},$$

with $(x_0, y_0) \in R$. If $G(x_0,y_0) = 0$ and $\left.\frac{\partial G}{\partial y}\right|_{(x_0,y_0)} \neq 0$ then there exist open intervals $I \ni x_0$ and $J \ni y_0$ with $I \times J \subset R$ and a C^r function $p\colon I \mapsto J$ such that $G(x,y) = 0$ if $I \times J$ iff $y = p(x)$.

We can now state conditions for the persistence of fixed points.

Theorem 2.23. *Assume f_λ is C^r, $r \geq 1$ and $f_{\lambda^*}(x^*) = x^*$ and $f'_{\lambda^*}(x^*) \neq 1$. Then there exist open $I \ni \lambda^*$ and $J \ni x^*$ and a C^r function $p\colon I \mapsto J$ such that $p(\lambda^*) = x^*$ and $f_\lambda(p(\lambda)) = p(\lambda)$. Moreover, f_λ has no other fixed points in I.*

Proof. Apply the implicit function theorem to $G(x, \lambda) = f_\lambda(x) - x$. Our hypotheses give $G(x^*, \lambda^*) = 0$ and $\frac{\partial G}{\partial x}\big|_{(x^*,\lambda^*)} \neq 0$.

Thus by the IFT $\exists I \ni \lambda^*$ and $J \ni x^*$ and a C^r function $p\colon I \mapsto J$ such that $p(\lambda^*) = x^*$, $G(p(\lambda), \lambda) = 0$ and $G \neq 0$ in $J \times I$ unless $x = p(\lambda)$. □

We have a curve (or "branch") of fixed points.

We can find $\frac{\mathrm{d}p}{\mathrm{d}\lambda}$ by implicit differentiation:

$$\frac{\mathrm{d}p}{\mathrm{d}\lambda} = -\left.\frac{\frac{\partial G}{\partial \lambda}}{\frac{\partial G}{\partial x}}\right|_{(p(\lambda),\lambda)} = -\frac{\frac{\partial f_\lambda}{\partial \lambda}(p(\lambda))}{f'_\lambda(p(\lambda)) - 1}.$$

Intuitively, a bifurcation takes place at (x_0, λ) when the topological nature of the dynamics near x_0 changes when λ passes through λ_0.

Fixed points (dis)appear in a saddle-node/tangent/fold bifurcation.

Theorem 2.24. *Assume f_λ is C^r with $r \geq 2$, $f_\lambda(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$, $f''_{\lambda_0}(x_0) \neq 0$ and $\left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{(x_0,\lambda_0)} \neq 0$. Then $\exists J \ni x_0$, $I \ni \lambda_0$ and a C^r function $x \mapsto g(x)$ such that $g(x_0) = \lambda_0$ and such that $f_\lambda(x) = x$ in $J \times I$ iff $\lambda = g(x)$. Moreover $g'(x_0) = 0$ and $g''(x_0) \neq 0$. The fixed points created are attracting on one side of x_0 and repelling on the other.*

Proof. Let $G(x, \lambda) = f_\lambda(x) - x$. Now $G(x, \lambda) = 0$ iff x is a fixed point of f_λ. $\frac{\partial G}{\partial x}\big|_{(x_0,\lambda_0)} = 0$ and $\frac{\partial G}{\partial \lambda}\big|_{(x_0,\lambda_0)} \neq 0$, and so, by the IFT, there exists a C^r function $g\colon J \mapsto I$ satisfying $G(x, g(x)) = 0$.

Now $0 = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial \lambda}\frac{\partial g}{\partial x}$ and so $g' = -\frac{\partial G}{\partial x}\left(\frac{\partial G}{\partial \lambda}\right)^{-1}$. In particular, $g'(x_0) = 0$. Differentiating again we get that

$$g''(x_0) = -\left[\frac{\partial^2 G}{\partial x^2}\left(\frac{\partial G}{\partial \lambda}\right)^{-1}\right]_{(x_0,\lambda_0)} \neq 0.$$

The sign of g'' determines the direction of the bifurcation. As for stability,

$$\begin{aligned}\frac{\partial f_\lambda}{\partial x} &= 1 + \frac{\partial^2 f_\lambda}{\partial x^2}(x - x_0) + \frac{\partial^2 f}{\partial x \partial \lambda}(\lambda - \lambda_0) + \text{higher order}\\ &= 1 + \frac{\partial^2 f_\lambda}{\partial x^2}(x - x_0)\end{aligned}$$

as $g(x) - \lambda_0 = \mathcal{O}(x - x_0)^2$. Thus $f'_{g(x)}(x) - 1$ takes opposite signs on either side of x_0. □

Bifurcation diagrams

As an example, consider the Arnold family of maps, $f_{k,\omega}(x) = x + \omega + \frac{k}{2\pi}\sin 2\pi x$.

We can generalise these results to periodic orbits of period $q > 1$ by applying theorems 2.23 and 2.24 to f_λ^q.

An an example consider the Arnold tongue about $\omega = \frac{p}{q}$.

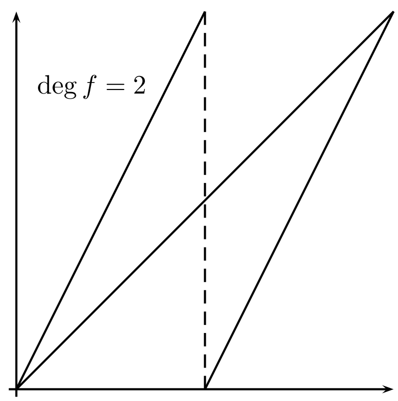
The boundaries of the Arnold tongues are lines of saddle-node bifurcation.

Chapter 3

Chaos and non-invertible one dimensional maps

3.1 Chaos

We start with an example. Consider $f\colon S^1 \mapsto S^1$ given by $f\colon z \mapsto z^2$ in multiplicative notation or $f\colon x \mapsto 2x \mod 1$ in additive notation.



We will write x as a binary expansion,

$$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i},$$

with $a_i \in \{0, 1\}$.

Note that dyadic rationals $\frac{m}{2^n}$ have 2 expansions (although this will not bother us). For instance,

$$\tfrac{1}{2} = 0.1000\ldots = 0.0111\ldots.$$

It is easy to see what $f(x)$ is. Using the binary expansion,

$$f(x) = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}.$$

We can write down a list of properties of f.

1. If $x \in \mathbb{Q}$, x is either a periodic point or eventually periodic.

   Definition 3.1. *A point x is eventually periodic of period n if x is not periodic, but $\exists k > 0$ such that $f^k(x)$ is periodic of period n.*

2. If x is irrational, x is neither periodic nor eventually periodic (as the binary expansion never repeats itself).

3. Let $P_n(f)$ be the number of periodic points of f with (not necessarily least) period n. We can show that $P_n(f) = 2^n - 1$. ($P_n(f)$ is also the number of fixed points of f^n. We require $z^{2^n} = z$, or $z^{2^n - 1} = 1$. There are $2^n - 1$ such.)

4. If $x \neq y$ then there exists $n \geq 0$ such that $|f^n(x) - f^n(y)| \geq \frac{1}{4}$.

   Proof. Take $x > y$. If $x - y \geq \frac{1}{4}$ then we are done, else $\exists n \geq 1$ such that
   $$\frac{1}{2^{n+2}} < x - y < \frac{1}{2^{n+1}}.$$
   Now $|f(x) - f(y)| = 2\,|x - y|$ and so
   $$\frac{1}{4} \leq |f^n(x) - f^n(y)| \leq \frac{1}{2}.$$
   □

5. For every open interval $J \subset S^1$, $\exists n \geq 0$ such that $f^n(J) = S^1$. (This follows from property 4.)

6. Periodic points and eventually periodic points are dense in S^1.

Some of these properties are specific to this example, but it has two properties which are of more general interest.

Definition 3.2. *A map $f \colon X \mapsto X$ is said to be* topologically transitive *on an invariant set $\Lambda \subset X$ if the forward orbit of some point $x \in \Lambda$ is dense in Λ.*

An equivalent (for most "reasonable" topological spaces) definition is:

Definition 3.3. *A map $f \colon X \mapsto X$ (X a topological space) is said to be* topologically transitive *on an invariant set $\Lambda \subset X$ if for any pair of open sets U, $V \subset \Lambda$ $\exists k > 0$ such that $f^k(U) \subset V \neq \emptyset$.*

The other interesting property is sensitive dependence on initial conditions (SDIC).

Definition 3.4. *A map $f \colon X \mapsto X$ (X a metric space) has SDIC on an invariant set Λ if $\exists \delta > 0$ such that for all $x \in \Lambda$ and any neighbourhood U of x, $\exists y \in U$ and $n > 0$ such that $d(f^n(x), f^n(y)) > \delta$.*

Definition 3.5. *A dynamical system (f, X) is* chaotic *if it has a compact invariant subset Λ on which f is both topologically transitive and has SDIC.*

For another example take $X = [-1, 1]$ and $g \colon X \mapsto X$ such that $g(x) = 2x^2 - 1$. Then g is chaotic.

Proof. Consider $h\colon S^1 \mapsto X$, $h(\theta) = \cos 2\pi\theta$. h is continuous and onto. Now

$$h(f(\theta)) = h(\cos 2\theta) = \cos 4\pi\theta = 2\cos^2 2\pi\theta - 1 = g(h(\theta)).$$

h is *not* one to one, but we don't need that. We need to prove both topological transitivity and SDIC. Given two open sets I, $J \subset X$. Now $h^{-1}(I)$ and $h^{-1}(J)$ are open in S^1 since h is continuous. Then $\exists n > 0$ such that

$$f^n(h^{-1}(I)) \cap h^{-1}(J) \neq \emptyset,$$

and so $g^n(I) \cap J \neq \emptyset$. To prove SDIC, given $x \in X$ and open $U \ni x$, $\exists n > 0$ such that $g^n(U) = X$ (as the same is true for f). Now let $y = 1$ if $g^n(x) \leq 0$ and -1 if $g^n(x) > 0$. $\exists z \in U$ such that $g^n(x) = y$ and so

$$|g^n(x) - g^n(y)| \geq 1.$$

□

This proof has introduced an important notion.

Definition 3.6. *Let $f\colon X \mapsto X$ and $g\colon Y \mapsto Y$. Then $h\colon X \mapsto Y$ is called a* topological semi-conjugacy *from f to g if*

1. *h is continuous,*
2. *h is onto,*
3. *$h \circ f = g \circ h$.*

We say that f is topologically semi-conjugate to g by h.

Semi-conjugacy means that the dynamics of f are at least as complicated as the dynamics of g.

3.2 Sequence spaces

Let

$$\Sigma_N = \{\mathbf{a} = (a_0, a_1, \dots) : a_i \in \{0, \dots, N-1\}, i \in \mathbb{Z}_+\},$$

a *sequence space on n symbols*. We make Σ_N a metric space by defining a distance

$$d(\mathbf{a}, \mathbf{b}) = \sum_{n=0}^{\infty} \frac{\gamma(a_n, b_n)}{3^n} \quad \text{where } \gamma(i,j) = \begin{cases} 0 & i = j \\ 1 & i \neq j. \end{cases}$$

Two points in Σ_N are close if they agree on a long initial segment, as follows. Suppose $\mathbf{a}, \mathbf{b} \in \Sigma_N$ with $a_i = b_i$ for $i < m$ and $a_m \neq b_m$. Then

$$\frac{1}{3^m} \leq d(\mathbf{a}, \mathbf{b}) \leq \sum_{n=m}^{\infty} \frac{1}{3^n} = \frac{2}{3^{m-1}}.$$

Σ_N is a Cantor set.[1]

[1] See example sheet.

3.3 Shift map

We define $\sigma \colon \Sigma_N \mapsto \Sigma_N$ by

$$\sigma(a_0, a_1, \dots) = (a_1, a_2, \dots).$$

Proposition 3.7.

1. *σ is continuous*
2. $P_k(\sigma) = N^k$
3. $\mathrm{Per}(\sigma)$ *(the set of periodic points of σ) is dense in Σ_N.*
4. *There exists a dense (forward) orbit in Σ_N.*
5. *σ has SDIC.*

Proof.

1. $d(\mathbf{a}, \mathbf{b}) = \gamma(a_0, b_0) + \frac{1}{3} d(\sigma(\mathbf{a}), \sigma(\mathbf{b}))$ and so $d(\sigma(\mathbf{a}), \sigma(\mathbf{b})) \leq 3d(\mathbf{a}, \mathbf{b})$. Given the usual ϵ, pick N such that $3^{-N} < \epsilon$, and then $\delta = 3^{-(N+1)}$.
2. $\sigma^k(\mathbf{a}) = \mathbf{a}$ iff $a_{k+j} = a_j$ for all $j \geq 0$. Given k there are N^k blocks of length k.
3. Given $\mathbf{a} \in \Sigma_N$ and $\epsilon > 0$ take n such that $\frac{1}{2 \cdot 3^{n-1}} < \epsilon$ and let $\mathbf{b}$ be a periodic sequence of the form $(a_0, a_1, \dots, a_n, a_0, a_1, \dots)$. Then $d(\mathbf{a}, \mathbf{b}) \leq \epsilon$.
4. Let $\mathbf{b}$ be a sequence which lists all blocks of length n for each successive $n \in \mathbb{N}$. For instance, for $N = 2$,

   $$\mathbf{b} = (0\ 1\ 00\ 01\ 10\ 11\ 000\ \dots).$$

   Then given $\mathbf{a} \in \Sigma_N$ and $k \in \mathbb{N}$, $\exists n \in \mathbb{N}$ such that $\sigma^n(\mathbf{b})$ and $\mathbf{a}$ agree on the first k places, and so $d(\sigma^n(\mathbf{b}), \mathbf{a}) \leq \frac{1}{2 \cdot 3^{k-1}}$.
5. Given $\mathbf{a} \in \Sigma$, choose $\mathbf{b} \in \Sigma$ such that $a_i = b_i$ for $i = 0, \dots, q$ but $a_i \neq b_i$ for $i > q$. Then

   $$d(\mathbf{a}, \mathbf{b}) < \frac{1}{3^q} \quad \text{but} \quad d(\sigma^q(\mathbf{a}), \sigma^q(\mathbf{b})) = \frac{1}{2}.$$

□

Properties 4 and 5 mean that σ is chaotic on Σ_N.
As an example consider the map shown

Let

$$\Lambda = \{x : f^n(x) \in I_0 \cup I_2 \ \forall n \geq 0\}.$$

Then Λ is the middle third Cantor set,

$$\Lambda = \left\{ x \in [0,1] : x = \sum_{n=0}^{\infty} \frac{a_i}{3^{n+1}}, a_i \in \{0,2\} \right\}.$$

We claim that $f|_\Lambda$ is topologically conjugate to σ on Σ_2, which we prove by exhibiting the conjugacy,

$$h\colon \sum_{n=0}^{\infty} \frac{a_n}{3^{n+1}} \mapsto \left(\tfrac{a_0}{2}, \tfrac{a_1}{2}, \dots\right).$$

Thus f is chaotic on Λ.

3.4 Subshifts of finite type

The general setting is with $f\colon I \mapsto I$ continuous, $I \subset \mathbb{R}$ or $I \subset S^1$.

Suppose that $\{I_0, I_1, \dots, I_{N-1}\}$ are disjoint closed intervals in I. We say that I_i f-covers I_j (and write $I_i \to I_j$ or $i \to j$) if $I_j \subset f(I_i)$.

Let Γ be the directed graph with N vertices indicating the f-covering relations.

We see that $I_1 \to I_2$, $I_2 \to I_1$ and $I_2 \to I_2$. The graph for this is $1 \rightleftarrows 2 \circlearrowleft$

Let $\mathcal{A}$ be the $N \times N$ matrix defined by

$$\mathcal{A}_{ij} = \begin{cases} 1 & \text{if } i \to j \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{A}$ is called the *transition matrix* associated to $\{I_0, \dots, I_{N-1}\}$. For the example above

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $\Sigma_{N,A} = \{\mathbf{a} \in \Sigma_N : \mathcal{A}_{a_n a_{n+1}} = 1 \ \forall n \geq 0\}$. Note that $\Sigma_{N,A}$ is closed and invariant under σ.

Definition 3.8. *The restriction $\sigma_A = \sigma|_A$ of σ to $\Sigma_{N,A}$ is called a* subshift of finite type *or* topological Markov chain.

Theorem 3.9. *Let $J = \bigcup_{i=0}^{n-1} I_{a_i}$. Then there exists a closed f-invariant set $\Lambda \subset J$ such that $f|_\Lambda$ is topologically semi-conjugate to σ_A.*

Before proving this theorem we need two lemmas.

Lemma 3.10. *If L and M are closed intervals and $L \to M$ then there exists a closed interval $K \subset L$ such that $f(K) = M$.*

Proof. Let $M = [a, b]$. Then $f^{-1}(a)$ and $f^{-1}(b)$ are closed and non-empty so we can choose $u \in f^{-1}(a)$ and $v \in f^{-1}(b)$ such that

$$(u, v) \cap \left(f^{-1}(a) \cup f^{-1}(b)\right) = \emptyset.$$

WLOG $u < v$. Set $K = [u, v]$ and use the IVT. $\square$

Lemma 3.11. *If $I_{a_0} \to I_{a_1} \to \cdots \to I_{a_n}$ then $\bigcup_{i=0}^{n} f^{-i}(I_{a_i})$ contains an interval $I_{a_0 a_1 \dots a_n}$ such that $f^n(I_{a_0 a_1 \dots a_n}) = I_{a_n}$ and $x \in I_{a_0 a_1 \dots a_n}$ implies that $f^i(x) \in I_{a_i}$ for all $0 \le i \le n$.*

Proof. By lemma 3.10 there exists an interval $I_{a_0 a_1} \subset I_{a_0}$ such that $f(I_{a_0 a_1}) = I_{a_0}$. Now $\exists I_{a_1 a_2} \subset I_{a_2}$ such that $f(I_{a_1 a_2}) = I_{a_2}$. Therefore $\exists I_{a_0 a_1 a_2} \subset I_{a_0 a_1}$ such that $f(I_{a_0 a_1 a_2}) = I_{a_1 a_2}$ and $f^2(I_{a_0 a_1 a_2}) = I_{a_2}$. We continue inductively to get $I_{a_0 a_1 \dots a_n}$. $\square$

Proof of Theorem 3.9. We first obtain

$$\Lambda_1 = \{I_{a_0 a_1} \subset I_{a_0} : a_0 \to a_1 \text{ and } f(I_{a_0 a_1}) = I_{a_1}\}.$$

We define Λ_n inductively, assuming that

$$\Lambda_{n-1} = \left\{I_{a_0 a_1 \dots a_{n-1}} : a_0 \to a_1 \to \cdots \to a_{n-1} \text{ and } f^n(I_{a_0 \dots a_{n-1}}) = I_{a_n}\right\}.$$

For all allowed transitions $a_{n-1} \to a_n$ define $I_{a_0 \dots a_n} \subset I_{a_0 \dots a_{n-1}}$ such that $f^{n-1}(I_{a_0 \dots a_n}) = I_{a_{n-1} a_n} \subset \Lambda_1$. Thus

$$\Lambda_n = \{I_{a_0 a_1 \dots a_n} : a_0 \to a_1 \to \cdots \to a_n \text{ and } f^n(I_{a_0 \dots a_n}) = I_{a_n}\}.$$

Now define $\Lambda = \bigcup_{n \ge 1} \Lambda_n$. Λ is non-empty (as it is the intersection of a nested sequence of closed sets). The connected components of Λ are closed intervals or possibly isolated points.

We can now define $h \colon \Lambda \to \Sigma_{N,A}$ (the *itinerary map*) as

$$x \to \mathbf{a} \text{ where } f^i(x) \in I_{a_i} \ \forall i \in \mathbb{Z}_+.$$

h is continuous, as given M, $\exists \delta > 0$ such that $x, y \in \Lambda$ with $d(x, y) < \delta$ implies that $f^i(x)$ and $f^i(y)$ are in the same I_{a_i} for $0 \le i \le M$.

By construction, h is surjective and $h \circ f|_\Lambda = \sigma_A \circ h$. $\square$

If f is monotone on each I_i then Λ is uniquely defined.

If f is expanding on J ($\exists \lambda > 1$ such that for all i, $x, y \in I_i$, $d(f(x), f(y)) \ge \lambda d(x, y)$) then h is a topological conjugacy.

3.4.1 Properties of σ_A

We call a finite string $\mathbf{w} = w_0 w_1 \dots w_k$ a *word*. Given a transition matrix $\mathcal{A}$ a word is said to be *allowed* if the transition $w_i \to w_{i+1}$ is allowed for all $0 \le i \le k-1$, equivalently

$$\mathcal{A}_{w_0 w_1} \mathcal{A}_{w_1 w_2} \dots \mathcal{A}_{w_{k-1} w_k} = 1.$$

Lemma 3.12. *Let $N_{ij}^{(n)}$ denote the number of allowed words of length $n+1$ starting in i and ending in j. Then*

$$N_{ij}^{(n)} = (\mathcal{A}^n)_{ij}\,.$$

Proof. The product $\mathcal{A}_{iw_1}\mathcal{A}_{w_1w_2}\ldots\mathcal{A}_{w_{n-1}j}$ equals one if $iw_1\ldots w_{n-1}j$ is allowed and equals zero otherwise. Then

$$N_{ij}^{(n)} = \sum_{w_1,\ldots,w_{n-1}} \mathcal{A}_{iw_1}\mathcal{A}_{w_1w_2}\ldots\mathcal{A}_{w_{n-1}j} = (\mathcal{A}^n)_{ij}\,.$$

□

Proposition 3.13. $P_n(\sigma_A) = \operatorname{Tr}\mathcal{A}^n$.

Proof. Fixed points of σ_A^n are in one-to-one correspondance with allowed words of length $n+1$ with the same start and finish. Now use lemma 3.12. □

We can compute $\operatorname{Tr}\mathcal{A}^n$ by using the Cayley-Hamilton theorem, that a matrix satisfies its own characteristic equation. For instance, for the $\mathcal{A}$ we considered earlier, we find $\operatorname{Tr}\mathcal{A}^{n+2} - \operatorname{Tr}\mathcal{A}^{n+1} - \operatorname{Tr}\mathcal{A}^n = 0$, and imposing the initial conditions $\operatorname{Tr}\mathcal{A} = 1$ and $\operatorname{Tr}\mathcal{A}^2 = 3$ we can find $\operatorname{Tr}\mathcal{A}^n$.

If N_q is the number of periodic cycles of least period n, then

$$P_n = \sum_{q|n} qN_q.$$

3.4.2 Chaos

Definition 3.14. *A matrix $\mathcal{A}$ is* irreducible *if $\forall i,j$, $\exists n$ such that $(\mathcal{A}^n)_{ij} \neq 0$ (that is there is an allowed path from i to j).*

Proposition 3.15. *If $\mathcal{A}$ is irreducible then σ_A is topologically transitive.*

Proof. We need to find a dense orbit. We can do this by choosing a sequence which contains every allowed word, with a proper choice of transition words between them. (This is always doable as $\mathcal{A}$ is irreducible.) □

Definition 3.16. *A matrix $\mathcal{A}$ is* non-trivial *if $\exists i, j_1 \neq j_2$ such that $i \to j_1$ and $i \to j_2$.*

A permutation matrix (for instance) is trivial.

Proposition 3.17. *If $\mathcal{A}$ is irreducible and non-trivial then σ_A is chaotic on Σ_A.*

Proof. By proposition 3.15 we just need to show SDIC. Given $\mathbf{a} = a_0a_1\ldots$ and $M \in \mathbb{Z}_+$ there exists an allowable word $a_Mw_{M+1}\ldots(w_k = i)$ where i can be followed by either j_1 or j_2 since $\mathcal{A}$ is non-trivial. If $w_{M+1}\ldots w_k = a_{M+1}\ldots a_k$ then choose $\mathbf{b} = a_0a_1\ldots b_{k+1}\ldots$ where $b_{k+1} \neq a_{k+1}$ and let $n = k$.

If $w_{M+1}\ldots w_k \neq a_{M+1}\ldots a_k$ then choose n to be the index of the first non-agreeing character and $\mathbf{b} = a_0\ldots a_M\ldots a_nb_{n+1}\ldots$ where $b_{n+1} \neq a_{n+1}$. Then $d(\mathbf{a},\mathbf{b}) \leq \frac{1}{23^n}$ and $d(\sigma_A^{n+1}(\mathbf{a}), \sigma_A^{n+1}(\mathbf{b})) \geq 1$. □

If $f|_\Lambda$ is semi-conjugate to an irreducible, non-trivial subshift $\sigma_A|_{\Sigma_A}$, can we deduce that $f|_\Lambda$ is chaotic?

This is not true. We need conjugacy to show that $f|_\Lambda$ is chaotic, although we can show that if $f|_\Lambda$ is semi-conjugate to σ_A then $f|_\Lambda$ has at least as many periodic cycles as σ_A.

Proposition 3.18. *For every closed path $a_0a_1 \dots (a_k = a_0)$ in Γ there exists a periodic orbit for f in Λ, $(x_0 \dots x_{k-1})^\infty$ such that $x_n \in I_{a_n}$ for all $n \geq 0$.*

We need a lemma before proving this.

Lemma 3.19. *If the closed interval K f-covers itself then f has a fixed point in K.*

Proof. Let $K = [a, b]$. Then $K \to K$ implies that $\exists c, d \in K$ such that $f(c) = a \leq c$ and $f(d) = b \geq d$. Now apply the IVT. $\square$

Proof of proposition 3.18. From lemma 3.11 $\exists I_{a_0 \dots a_k} \subset I_{a_0}$ such that

$$f^n(I_{a_0 \dots a_k}) \subset I_{a_n} \qquad \text{and} \qquad f^k(I_{a_0 \dots a_k}) = I_{a_0}. \tag{$*$}$$

In particular $I_{a_0 \dots a_k} \to I_{a_0 \dots a_k}$ and so (by lemma 3.19), f^k has a fixed point x_0 and by $(*)$ $f^k(x_0) \in I_{a_n}$ for all $n \geq 0$. $\square$

If the loop $a_0a_1 \dots (a_k = a_0)$ is of least period then x_0 has least period k.

If the I_n's are not disjoint but have disjoint interiors then we can construct Γ the same way. We cannot deduce semiconjugacy to σ_A, but we can still get lots of periodic orbits for $f|_\Lambda$ as proposition 3.18 does not use semiconjugacy. The only problem is that multiple paths in Γ give the same periodic orbit.

For instance, consider $z \mapsto z^2$ (as a map of S^1).

This has a graph:

$$\begin{tikzcd} 1 \arrow[loop left] \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

and the only problem is that $0^\infty = 1^\infty$.

3.5 Sharkovsky's theorem

Our first proposition is the (famous?) "period 3 implies chaos" result of Li and Yorke.

Proposition 3.20. *Suppose $f \colon I \mapsto I$ is continuous and has a periodic point of period 3. Then f has periodic points of all least periods.*

Proof. Let the period 3 cycle be $x < y < z$. Let $I_0 = [x, y]$ and $I_1 = [y, z]$. Suppose $f(y) = z$. Then $f^2(y) = x$ and so we have the graph

$$\begin{tikzcd} I_0 \arrow[r, shift left] & I_1 \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

(If $f(y) = x$ then just relabel I_0 and I_1.) Now for all $n \in \mathbb{N}$ there exists a loop

$$I_0 \to \underbrace{I_1 \to \cdots \to I_1}_{n-1} \to I_0,$$

a least period loop of period n. Proposition 3.18 gives a periodic point of exact period n. $\square$

This is a special case of a more general result which describes in a precise way the order in which periodic orbits of different periods appear. We first need to define the *Sharkovsky ordering* on the natural numbers. This is given by

$$\begin{gathered} 1 \prec 2 \prec 4 \prec \cdots \prec 2^n \prec 2^{n+1} \prec \ldots \\ \cdots \prec 2^{n+1} \cdot 9 \prec 2^{n+1} \cdot 7 \prec 2^{n+1} \cdot 5 \prec 2^{n+1} \cdot 3 \prec \ldots \\ \cdots \prec 2^n \cdot 9 \prec 2^n \cdot 7 \prec 2^n \cdot 5 \prec 2^n \cdot 3 \prec \cdots \prec 9 \prec 7 \prec 5 \prec 3. \end{gathered}$$

We can now state Sharkovsky's theorem.

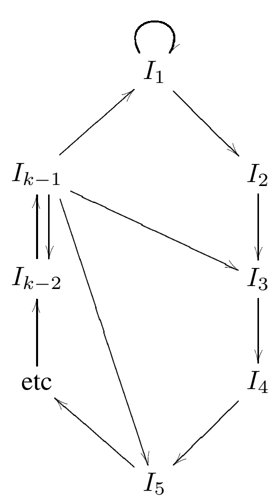
Theorem 3.21 (Sharkovsky's theorem). *Let $I \subset \mathbb{R}$ be a closed interval and $f\colon I \mapsto \mathbb{R}$ be a continuous map. If f has a periodic point of least period k then f has a periodic orbit of least period n for all $n \prec k$.*

Proof is in a number of stages. We first prove the case $k > 1$, odd.

Lemma 3.22. *If $f\colon I \mapsto \mathbb{R}$ is continuous and has a periodic point x of least period $k \geq 3$, odd, and no points of odd period n with $1 < n < k$ then f has a point of least period n for all $n > k$ and all even $n < k$ and period 1.*

Proof. Let $J = [\min \theta(x), \max \theta(x)]$. Make a partition of J by the element of $\theta(x) = \{p_1 < p_2 < \cdots < p_k\}$. We define intervals I_i of the form $[p_l, p_{l+1}]$ for $1 \leq i \leq k-1$, where l is not necessarily in the same order as i.

We aim to show that we can choose the labelling of the I_i's to obtain the following directed graph (*Stefan graph*) for the f-covering relations.



That is: a loop $I_1 \to I_2 \to \cdots \to I_{k-1} \to I_1$, a loop $I_1 \to I_1$ and directed edges from I_{k-1} to all odd vertices. Once this is established we just need to note that there are distinct closed loops of period

- $n > k : I_1^{n-k+1} \to I_2 \to \cdots \to I_{k-1} \to I_1$
- even numbers $< k : I_{k-1} \to I_{2l+1} \to I_{2l+2} \cdots \to I_{k-2} \to I_{k-1}$
- $1 : I_1 \to I_1$.

We will prove this in a series of claims.

Claim 1: $I_1 \to I_1$. Note that $f(p_1) > p_1$ and $f(p_k) < p_k$. Take $a = \max\{y \in \theta(x) : f(y) > y\}$ ($a \neq p_k$). Now let $I_1 = [a, b]$, where b is the closest point of $\theta(x)$ to the right of a. Then $f(a) \geq b$ and $f(b) \leq a$ since $b > a$. Thus $f(I_1) \supset I_1$ as required.

Claim 2: $f^{k-2}(I_1) \supset J$: i.e. there exists a path from I_1 to any other vertex. To prove this note that $f(I_1) \supset I_1$ with proper inclusion (else $k = 2$) and so $f^{j+1}(I_1) \supset f^j(I_1)$ (nested iterates). There are $k-2$ points in $\theta(x) \setminus \{a, b\}$ and so $p_k \in f^j(I_1)$ for some $0 \leq j \leq k-2$ and by the nested property $P - k \in f^{k-2}(I_1)$. Similarly $p_1 \in f^{k-2}(I_1)$ and since I_1 is connected $f^{k-2}(I_1) \supset [p_1, p_k] = J$.

Claim 3: $\exists j \neq 1$ such that $I_1 \subset f(I_j)$ (that is $I_j \to I_1$). To prove this let $B_l = \{y \in \theta(x) : y \leq a\}$ and $B_r = \{y \in \theta(x) : y \geq b\}$. Now k is odd, so that $\#B_l \neq \#B_r$. Let B be the one of B_l and B_r with more elements. Then $\exists y_1, y_2 \in B$ adjacent with $f(y_1) \in B$ and $f(y_2) \in \theta(x) \setminus B$. Take $I_j = [y_1, y_2]$, and so $I_1 \subset f(I_j)$.

Now label the intervals such that $I_1 \to I_2 \to \cdots \to I_l \to I_1$ is the shortest loop containing I_1.

Claim 4: The shortest loop with $l \geq 2$ has $l = k-1$. There are only $k-1$ distinct intervals, so the shortest loop has $l \leq k-1$. Assume that $l < k-1$. Let q be the odd number out of $\{l, l+1\}$. So $1 < q < k$. Use the loop $I_1 \to I_2 \to \cdots \to I_l \to I_1$ or $I_1 \to I_2 \to \cdots \to I_l \to I_1 \to I_1$ depending on whether $q = l$ or $q = l+1$. By proposition 3.18, $\exists y \in I_1$ such that $f^q(y) = y$. Now $y \notin \partial I_1$ since points on ∂I_1 have period $k > q$. Hence y has period $q < k$. This gives a contradiction: k is the smallest odd number such that f has a periodic orbit of least period k.

Claim 5:

1. If $f(I_i) \supset I_1$ then $i = 1$ or $k-1$.
2. For $j > i+1$, $I_i \not\to I_j$.
3. I_1 f-covers only I_1 and I_2.

Claim 4 implies 1 and the shortest loop property implies 2 and 3.

Claim 6: orderings (in terms of $\mathbb{R}$) of the I_i's and of $\theta(x)$ are either

$$\begin{cases} I_{k-1} \leq I_{k-3} \leq \cdots \leq I_2 \leq I_1 \leq I_3 \leq \cdots \leq I_{k-2} \\ f^{k-1}(a) < f^{k-3}(a) < \cdots < f^2(a) < a < f(a) < f^3(a) < \cdots < f^{k-2}(a) \end{cases}$$

or the above exactly reversed. To prove this note that $I_1 = [a, b]$ f-covers only I_1 and I_2, so by connectedness I_1 and I_2 must be adjacent. Assume $I_2 \leq I_1$ (the other case gives the reversed order). Then we must have $f(a) = b$ and $f^2(b)$ the left point of I_2.

Now $f(a) = b$ and $I_2 \not\to I_1$ (by claim 5.1) and so $f(I_2) \geq a$. Now $I_2 \to I_3$ but $I_2 \not\to I_j$ for $j > 3$ (by claim 5.2) so that I_3 is adjacent to I_1. We obtain the claimed order inductively.

Claim 7: $I_{k-1} \to I_j$ for j odd. To prove this note that $I_{k-1} = [f^{k-1}(a), f^{k-3}(a)]$ and $f(f^{k-1}(a)) = a$. Also $f^{k-3}(a) \subset I_{k-3}$ so $f(f^{k-3}(a)) \subset I_{k-2}$. Thus $f(I_{k-1}) \supset [a, f^{k-2}(a)] = I_1 \cup I_3 \cup \cdots \cup I_{k-2}$.

We have finally proved the Stefan graph and we can now complete the proof of lemma 3.22 by using proposition 3.18 on each least period loop in the Stefan graph to get a periodic point of the same least period. □

Lemma 3.23. *If f has a periodic point of least period $k = 2m$, $m \geq 1$ and no periodic points of odd period greater than or equal to 3 then f has a fixed point and f^2 has two periodic orbits of least period m. These are $\{p_1, \ldots, p_m\}$ and $\{p_{m+1}, \ldots, p_{2m}\}$.*

Proof. Define a and b and set $I_1 = [a, b]$ as before. Then $I_1 \to I_1$ and there exists a fixed point in I_1.

In lemma 3.22 we used the fact that k was odd only in claim 3 to show that $\exists I_j$, $j \neq 1$ such that $I_j \to I_1$. If we have such a I_j then we get the Stefan graph as before (but with k even) so there exists a loop of period $k-1$, odd. This is a contradiction. Hence only I_1 f-covers I_1.

Since $f(a) \geq b$ at least one point must change side with respect to I_1, so that all points in $\theta(x)$ must change sides: $f(B_l) = B_r$ and $f(B_r) = B_l$. Hence $f^2(B_l) = B_l$ and $f^2(B_r) = B_r$ — that is B_l and B_r are permuted independently by f^2. □

Proof of Sharkovsky's theorem.

- If k is odd then we are done by lemma 3.22.
- If $k = 2r$, r odd then f has a period 1 orbit and f^2 splits the orbit into two components by lemma 3.23, each of period r. Thus f^2 has a period r orbit, r odd. Hence f has a period $2m$ orbit for all $m \geq r$, a period $2p$ orbit for all even $p < r$, a period 2 orbit and a fixed point.
- If $k = 2^l r$, r odd, $l > 1$ just repeat the argument.

□

The converse of Sharkovsky's theorem is true: there are examples of maps with *exactly* the periodic orbits implied by the Sharkovsky ordering. For instance, consider the following function.

As an exercise prove that this map has no period 3 point.

For all $n \geq 3$ there exists a permutation of n elements such that there exists a periodic orbit $\{p_1 < p_2 < \cdots < p_n\}$ of least period n which realises the permutation and forces the existence of periodic points of all periods.

The Sharkovsky theorem only happens on the line in one dimension: not on $\mathbb{C}$ or S^1. Consider the map of S^1, $r_{\frac{1}{3}} : x \mapsto x + \frac{1}{3} \mod 1$, which has all orbits of period 3 and no orbits of other periods.

3.6 The quadratic family

Consider

$$F_\mu(x) = \mu x(1-x) \text{ with } \mu > 1.$$

Note that:

1. $F_\mu(0) = F_\mu(1) = 0$,
2. $F_\mu(P_\mu) = P_\mu$, where $P_\mu = \frac{\mu-1}{\mu}$.
3. If $x < 0$ or $x > 1$ then $F^n(x) \to \infty$ as $n \to \infty$.

3.6.1 $1 < \mu < 3$

Proposition 3.24. *If* $1 < \mu < 3$ *then*

1. P_μ *is attracting and the origin repelling.*
2. *if* $0 < x < 1$ *then* $\omega(x) = P_\mu$.

Proof. To prove the first part just calculate the derivative F'_μ.

When $1 < \mu \leq 2$ then if $0 < x < P_\mu$, $F_\mu(x) > x$, $F_\mu(x) < P_\mu$. If $P_\mu < x < 1$ then $0 < F_\mu(x) < P_\mu$.

When $2 < \mu < 3$, $F^2_\mu([\frac{1}{2}, P_\mu]) \subset [\frac{1}{2}, P_\mu]$ and F^2_μ is monotone on $[\frac{1}{2}, P_\mu]$.

Let $\hat{P}_\mu$ be the other preimage of P_μ. Now $F_\mu([\hat{P}_\mu, \frac{1}{2}]) = [\frac{1}{2}, P_\mu]$ and we can apply the previous result.

If $x_0 \in [0, \hat{P}_\mu]$, $F^j_\mu(x_0)$ is monotone so long as the iterates stay in $[0, \hat{P}_\mu]$. When the iterates leave $[0, \hat{P}_\mu]$ then enter $[\hat{P}_\mu, P_\mu]$. Now apply the previous results.

Finally, if $x_0 \in [P_\mu, 1]$, $F_\mu(x_0) \in [0, \hat{P}_\mu]$. □

3.6.2 $\mu = 4$

$g(x) = 2x^2 - 1$ is topologically conjugate to $F_4 = 4x(1-x)$. Now g is chaotic (see page 20), so that F_4 is chaotic.

3.6.3 $\mu > 4$

I_0 and I_1 map on to $I = [0, 1]$ so we have the graph

$$\begin{tikzcd} 1 \arrow[loop left] \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

Proposition 3.25. *If $\mu > 2 + \sqrt{5}$ then*

$$\Lambda_\mu = \{x : F_\mu^n(x) \in I \; \forall n \geq 0\}$$

is a Cantor set and $F_\mu|_{\Lambda_\mu}$ is topologically conjugate to the shift map on Σ_2.

Proof. Exercise (note that $\mu > 2 + \sqrt{5}$ implies $|F_\mu'| > 1$). □

Proposition 3.25 is in fact true for $\mu > 4$ (by negative Schwartzian derivative, not included in course.)

3.6.4 $3 \leq \mu \leq 4$

When μ increases through 3, P_μ becomes repelling and an attracting period 2 cycle appears. As μ is increased further this period 2 cycle becomes unstable and a stable period 4 cycle appears. There is a cascade of period-doubling bifurcations at $\mu_0 < \mu_1 < \ldots$, where the period 2^i cycle loses stability at μ_i and a stable 2^{i+1} cycle appears.

As $n \to \infty$, $\mu_n \to \mu_\infty = 3.569942\ldots$ and

$$\lim_{n\to\infty} \frac{\mu_n - \mu_\infty}{\mu_{n+1} - \mu_\infty} = 4.6692\ldots, \quad \text{the Feigenbaum constant.}$$

3.6.5 Period doubling bifurcation

Let $f_\lambda \colon I \mapsto \mathbb{R}$ be a C^r one-parameter family with $r \geq 3$.

Theorem 3.26. *Suppose that*

1. $f_{\lambda_0}(x_0) = x_0$,

2. $f'_{\lambda_0}(x_0) = -1$,

3. $\frac{\mathrm{d}}{\mathrm{d}\lambda} f'_\lambda(P(\lambda))\big|_{\lambda_0} \neq 0$, *or equivalently*

$$\alpha := \left(\frac{\partial f'_\lambda}{\partial \lambda} + \tfrac{1}{2}\frac{\partial f''_\lambda}{\partial \lambda}\right) \neq 0.$$

4. *the graph of* $f^2_{\lambda_0}$ *has a non-zero cubic term in its tangency with the diagonal, or equivalently*

$$f^2_{\lambda_0}(x) - x = -\beta(x-x_0)^3 + \mathcal{O}(x-x_0)^4,$$

where

$$\beta := \tfrac{1}{3} f'''_{\lambda_0}(x_0) + \tfrac{1}{2}\left(f''_{\lambda_0}(x_0)\right)^2 \neq 0.$$

Then there exists a period-doubling bifurcation at λ_0, *that is*

- *there exists a differentiable curve of fixed points* $P(\lambda)$ *of* f_λ *passing through* (x_0, λ_0), *the stability of which changes at* λ_0, *and*
- *there exists a differentiable curve* γ *which passes through* (x_0, λ_0) *such that* $\gamma \setminus (x_0, \lambda_0)$ *is the union of two hyperbolic period 2 orbits and* γ *is the graph of a function* $\lambda = h(x)$, $h'(x_0) = 0$ *and* $h''(x_0) = -\frac{\beta}{\alpha} \neq 0$.

If $\beta > 0$ *then this period two cycle is stable and if* $\beta < 0$ *this period two cycle is unstable.*

Proof. $f'_{\lambda_0}(x_0) \neq 1$, so that there exists $P(\lambda)$ such that

$$P'(\lambda_0) = -\frac{\frac{\partial f_\lambda}{\partial \lambda}(p(\lambda))}{f'_\lambda(p(\lambda)) - 1} = \tfrac{1}{2}\left.\frac{\partial f}{\partial \lambda}\right|_{(x_0,\lambda_0)}.$$

We change co-ordinates such that the origin is a fixed point for all λ near λ_0. Let

$$g(y,\lambda) = f(y + P(\lambda)) - P(\lambda),$$

so that $\frac{\partial^n g}{\partial y^n}(0,\lambda) = f^{(n)}_\lambda\Big|_{P(\lambda)}$ and

$$\left.\frac{\partial^2 g}{\partial \lambda \partial y}\right|_{(0,\lambda_0)} = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda} f'_\lambda(P(\lambda))\right|_{\lambda_0} = \alpha \neq 0.$$

Let $G(y,\lambda) = g^2(y,\lambda) - y$ so that when $G(y,\lambda) = 0$, y is either a fixed point or a period two point. Note that $\frac{\partial G}{\partial \lambda}\big|_{(0,\lambda_0)} = 0$ and so we cannot naively apply the IFT directly to G.

Now if $G(y,\lambda) = 0$ and $y \neq 0$ then y is a period two point. Define

$$H(y,\lambda) = \begin{cases} \frac{G(y,\lambda)}{y} & y \neq 0 \\ \left.\frac{\partial G}{\partial y}\right|_{(0,\lambda)} & y = 0. \end{cases}$$

(Exercise: check H is C^1) We want to apply the IFT to H, so we verify the conditions of the IFT:

- $H(0,\lambda_0) = \frac{\partial g^2}{\partial y}\Big|_{(0,\lambda_0)} - 1 = \left(f'_{\lambda_0}(x_0)\right)^2 - 1 = 0.$
- $\frac{\partial H}{\partial \lambda}\big|_{(0,\lambda_0)} = \left(\frac{\partial}{\partial \lambda}\left(g'(0,\lambda)\right)^2\right)_{\lambda=\lambda_0} = 2g'(0,\lambda_0)\,\frac{\partial}{\partial \lambda} g'(0,\lambda)\big|_{\lambda=\lambda_0} = -2\alpha \neq 0.$

Thus we can use the IFT and so there exists a differentiable function $h(y)$ with $H(y,h(y)) = 0$. Now

$$h'(0) = -\left.\frac{\frac{\partial H}{\partial y}}{\frac{\partial H}{\partial \lambda}}\right|_{(0,\lambda_0)} = 0 \quad \text{and} \quad h''(0) = -\left.\frac{\frac{\partial^2 H}{\partial y^2}}{\frac{\partial H}{\partial y}}\right|_{(0,\lambda_0)} = -\frac{\beta}{\alpha} \neq 0.$$

To prove the stability result we Taylor expand $\frac{\partial g^2(y,h(y))}{\partial y}$ about $y = 0$. This gives $\frac{\partial g^2(y,h(y))}{\partial y} = 1 - 2\beta y^2 + \mathcal{O}(y^3)$ and so this orbit is attracting if $\beta > 0$ and repelling if $\beta < 0$. □

We have the following pictures.

3.7 Accumulation of period-doublings

Suppose we have a continuous one parameter family of unimodal maps $f_\mu\colon I \mapsto I$, $\mu \in [\mu_0, \mu_1]$ with maximum at c_μ and

$$\begin{gathered} f^2(c) \leq c < f(c) \\ f^2(c) \leq f^3(c) \end{gathered}$$

for all $\mu \in [\mu_0, \mu_1]$, with $f^2_{\mu_0}(c) = c$. c is called a *superstable* period two point. We also want $f^3_{\mu_1}(c) = f^2_{\mu_1}(c)$ and so c is eventually fixed (but unstable).

Then there exists an invariant interval $J = [f^2(c), f(c)] \subset I$ and every orbit starting outside J is either asymptotic to a fixed point or eventually enters J.

Now the set of periods of f_{μ_0} is $\{1, 2\}$ and the set of periods of f_{μ_1} is $\mathbb{N}$.

As μ goes from μ_0 to μ_1 we get a subinterval $[\tilde{\mu}_0, \tilde{\mu}_1]$ for which

$$f^2(c) < c \leq f^4(c) < f^3(c) \leq f^5(c) < f(c)$$

for all $\mu \in [\tilde{\mu}_0, \tilde{\mu}_1]$, with $f^4_{\tilde{\mu}_0}(c) = c$ (c superstable period 4) and $f^5_{\tilde{\mu}_1}(c) = f^3_{\tilde{\mu}_1}(c)$ (c is eventually period 2).

Let $J_1 = [f^2(c), f^4(c)]$ and $J_0 = [f^3(c), f(c)]$. Then there exists a fixed point in $[f^4(c), f^3(c)]$ and everything in $[f^4(c), f^3(c)]$ is either asymptotic to a fixed point or eventually enters J_0. We also have $f(J_0) = J_1$ and $f(J_1) \subset J_0$ (f exchanges the intervals J_0 and J_1) so that $f^2(J_0) \subset J_0$.

Thus the set of periods of f_μ for $\mu \in [\tilde{\mu}_0, \tilde{\mu}_1]$ is $\{1\} \cup 2\mathbb{N}$. Consider $\tilde{f} = f^2\big|_{J_0}$ for $\mu \in [\tilde{\mu}_0, \tilde{\mu}_1]$. It is unimodal and satisfies our original conditions.

Repeat this process to get a nested sequence of subintervals $[\mu_0^{(i)}, \mu_1^{(i)}]$.

Denote the infinite intersection of all these subintervals by L. For $\mu \in L$ the set of periods is the set of all of the powers of 2.

Denote the interval obtained at the N^{th} stage of this process by $J_{0\ldots0}$ (N zeroes), so that

$$K_N = \bigcup_{n=0}^{2^N-1} f^n(J_{0\ldots0})$$

is the union of 2^N disjoint intervals which are cyclically permuted by f.

Now $K_\infty = \bigcap_N K_N$ is an invariant subset of f_μ and is a Cantor set if the maximum lengths of the subintervals tends to zero as $N \to \infty$.[2]

3.7.1 Dynamics on K_∞

Label the images of $J_{0\ldots0}$ at level N by

$$f^n(J_{0\ldots0}) = J_{\text{base 2 representation of } n \text{ mod } 2^N \text{ written backwards}}.$$

For all $a_0, a_1, \ldots, a_m$ we have $J_{a_0,a_1,\ldots,a_m} \subset J_{a_0,a_1,\ldots,a_{m-1}}$.
Define $h\colon K_\infty \mapsto \Sigma_2$ by

$$h\colon \bigcap_{m>0} J_{a_0,\ldots,a_m} \mapsto \mathbf{a},$$

so that $h \circ f = A \circ h$, where $A\colon \Sigma_2 \mapsto \Sigma_2$ is the *adding machine* defined by

$$A(a_0a_1a_2\ldots) = (a_0a_1a_2\ldots) + (100\ldots)$$

with carrying to the right. h is a conjugacy if K_∞ is a Cantor set, else h is a semiconjugacy.

[2]This can be proved for the quadratic family using negative Schwartzian derivative.

Chapter 4

Two dimensional invertible maps

The main difference between one dimensional maps and higher dimensional maps is the possibility of both expansion and contraction at the same points in higher dimensions.

4.1 The horseshoe map

Consider a map $f\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ with the following geometric properties.

$f(S) \cap S = V_0 \cup V_1$. S is not mapped into itself so we extend the mapping outside S by considering a stadium $D = S \cup E_1 \cup E_2$. We take f to be a contraction on $E_1 \cup E_2$ such that $f(D) \subset D$. f is $1-1$ but not onto, so that f^{-1} is not globally defined.

4.1.1 Dynamics on S

If $x \in E_1$ then $f(x) \in E_1$. Thus by the contraction mapping principle there exists a unique attracting fixed point $p \in E_1$. If $x \in E_2$ then $f(x) \in E_1$ and so $f^n(x) \to p$. For any $x \in E_1 \cup E_2$, $\omega(x) = p$.

What are the sets remaining in S for all time? The points of S which are mapped into S are in $f^{-1}(S \cap f(s)) = S \cap f^{-1}(S) = H_0 \cup H_1$ (the union of two horizontal strips).

We want $f|_{H_0 \cup H_1}$ to be affine with derivative

$$\begin{pmatrix} \pm\lambda & 0 \\ 0 & \pm\mu \end{pmatrix} \quad \text{with a } + \text{ sign on } H_0 \text{ and a } - \text{ on } H_1.$$

$f^{-2}(S \cap f(s) \cap f^2(S))$ is four horizontal strips H_{ij}, $i, j \in \{0,1\}$.

In general

$$f^{-n}(S \cap f(S) \cap \cdots \cap f^n(S)) = \bigcap_{-n \le j \le 0} f^j(S)$$

is 2^n strips of thickness μ^{-n}. Define

$$\Lambda_H = \{x \in S : f^n(x) \in S \; \forall n \ge 0\} = \bigcap_{j=-\infty}^{0} f^j(S)$$

is a Cantor set of horizontal lines. Similarly, $\bigcap_{0 \le j \le n} f^j(S)$ consists of 2^n vertical strips of thickness λ^n and

$$\Lambda_V = \bigcap_{j=0}^{\infty} f^j(S)$$

is a Cantor set of vertical lines. The set of points which remain in S for all positive and negative times is

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(S)$$

is a Cantor set of points.

Introduce $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$, the set of *doubly* infinite sequences of 0's and 1's with metric

$$d(\mathbf{a}, \mathbf{b}) = \sum_{i \in \mathbb{Z}} \frac{\gamma_i}{4^{|i|}}$$

with

$$\gamma_i = \begin{cases} 0 & a_i = b_i \\ 1 & a_i \ne b_i. \end{cases}$$

Two sequences are close if they agree on a long central block. We define the intinerary map $h\colon \Lambda \mapsto \Sigma_2$ by

$$h(x) = (a_i)_{i\in\mathbb{Z}} \text{ if } f^i(x) \in H_{a_i}\ \forall i.$$

Theorem 4.1. *$f|_\Lambda$ is topologically conjugate to the two-sided shift $\sigma|_{\Sigma_2}$ by h.*

Proof. Exercise. □

The hypotheses we took are not the strongest ones possible:

If there exist disjoint horizontal strips H_i such that $f(H_i) = V_i$ for $i = 1, \dots, N$, f contracts vertical strips and f^{-1} contracts horizontal strips uniformly then f possesses an invariant set Λ such that $f|_\Lambda$ is topologically conjugate to $\sigma|_{\Sigma_N}$.

It is possible to have geometric horseshoes that are conjugate to two-sided subshifts of finite type.

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(S_1 \cup S_2) \qquad f|_\Lambda \sim \sigma|_{\Sigma_A} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

4.2 The Hénon map

This is an example of a nonlinear map with a horseshoe.[1]

[1] A doubly invariant set on which the dynamics are conjugate to σ.

$$f_{a,b}\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}.$$

Theorem 4.2. *If $b \neq 0$ and*

$$\begin{aligned} a &\geq \left(5 + 2\sqrt{5}\right) \frac{1 + |b|^2}{4}, \\ R &= \frac{1 + |b| + \sqrt{(1 + |b|)^2 + 4a}}{2}, \\ S &= \left\{(x, y) \in \mathbb{R}^2 : |x|, |y| \leq R\right\} \text{ and} \\ \Lambda &= \bigcap_{j \in \mathbb{Z}} f^j(S), \end{aligned}$$

then $f_{a,b}|_{\Lambda} \sim \sigma|_{\Sigma_2}$.

Proof. Omitted (very technical). □

This theorem can be made plausible by considering the case when $a = 5$, $b = 0.3$ and letting $S = \{(x, y) : |x|, |y| \leq 4\}$.

4.3 (Un)stable manifolds and homoclinic points

In this section we suppose that f is at least C^1.

Definition 4.3. *A fixed point p for $f\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ is* hyperbolic *if the Jacobian matrix $Df(p)$ has no eigenvalues on the unit circle.*

A periodic point p of least period n is hyperbolic if $Df^n(p)$ has no eigenvalues on the unit circle.

There are three types of hyperbolic fixed points.

- p is a *sink* (or *attracting*) if all the eigenvalues of $Df(p)$ lie inside the unit circle.
- p is a *source* (or *repelling*) if all the eigenvalues of $Df(p)$ lie outside the unit circle.
- p is a saddle otherwise.

The following theorem is not proven (but reasonably obvious).

Theorem 4.4. *Suppose f has an attracting (repelling) fixed point p. Then there exists an open set about p in which all points tend to p under forward (backward) iteration of f.*

The largest such open set is called the *stable set / basin of attraction* (*unstable set / basin of repulsion*) and is written $W^S(p)$ ($W^U(p)$).

4.3.1 Saddle points

Theorem 4.5. *Stable/unstable manifold theorem Suppose that f has a saddle point P. Then $\exists\epsilon > 0$ and a C^1 curve $\gamma_U\colon (-\epsilon, \epsilon) \mapsto \mathbb{R}^2$ such that:*

1. $\gamma_U(0) = p$,
2. $\gamma_U' \neq 0$,
3. $\gamma_U'(0)$ *is an unstable eigenvector of* $Df(p)$,
4. γ_U *is invariant under* f^{-1},
5. $f^{-n}(\gamma_U) \to p$ *as* $n \to \infty$ *and*
6. *if* $|f^{-n}(Q) - p| < \epsilon$ *for all* $n \geq 0$ *then* $Q = \gamma_U(t)$ *for some* $t \in (-\epsilon, \epsilon)$.

γ_U *is called the local unstable manifold.*
The stable *manifold theorem is the image of the above theorem under*

$$U \mapsto S \qquad f^{-1} \mapsto f \qquad \textit{unstable} \mapsto \textit{stable}.$$

The local stable/unstable manifolds have global counterparts.

Definition 4.6. *The unstable manifold of p is*

$$W^U(p) = \bigcup_{n \geq 0} f^n(\gamma_U)$$

and the stable manifold of p is

$$W^S(p) = \bigcup_{n \geq 0} f^{-n}(\gamma_S).$$

$W^S(p)$ and $W^U(p)$ frequently cross each other.

A point $q \in W^S(p) \cap W^U(p) \setminus \{p\}$ is called a *homoclinic* point to p. q is a transverse homoclinic point if $W^S(p)$ and $W^U(p)$ intersect transversely.

Since $W^S(p)$ and $W^U(p)$ are invariant then the points

$$O(q) = \{f^n(q) : n \in \mathbb{Z}\}$$

are homoclinic. $O(q)$ is a *homoclinic orbit* with $\omega(O(q)) = \alpha(O(q)) = p$.
This leads to very complicated behaviour for the stable and unstable manifolds.

4.4 Transverse homoclinics imply chaos

The next theorem (which we do not prove) states that the existence of a transverse homoclinic point implies the existence of a horseshoe.

Theorem 4.7 (Smale-Birkhoff). *Let $f\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a diffeomorphism with a hyperbolic fixed point p and suppose that there exists a transverse homoclinic point q to p. Then $\exists n > 0$ such that f^n has an invariant set Λ on which f^n is conjugate to the two-sided shift on Σ_2.*

The idea of the proof is to find a picture which looks like Smale's horseshoe for some iterate of f.

Take a square $U \ni p$. Since $f^i(q) \to p$ as $i \to \infty$, $q \in f^l(p)$ for some $l > 0$. Similarly $q \in f^{-k}(U)$ for some $k > 0$. Let $n = l + k$ so that f^n maps $f^{-k}(U)$ to $f^l(U)$.

4.5 The Melnikov method

We have seen that a horseshoe arises from a transverse homoclinic point to a hyperbolic periodic point. We will now obtain a method to verify that certain classes of ODE's have transverse homoclinic points.

Consider a Hamiltonian system and add a periodic perturbation.

$$\dot{z} = X_H(z) + \epsilon Y(z,t), \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad X_H = \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix}, \quad Y(z,t+T) = Y(z,t), \tag{4.1}$$

where $x \in \mathbb{R}$ or S^1 and $y \in \mathbb{R}$. Assume that $\dot{z} = X_H(z)$ has a saddle equilibrium P_0 and that $\dot{z} = X_H(z)$ has a homoclinic orbit Γ for p given by $z_0(t)$.

We can express (4.1) as an autonomous system on $\mathbb{R}^2 \times S^1_T$:

$$\begin{cases} \dot{z} = X_H(z) + \epsilon Y(z,t) \\ \dot{t} = 1. \end{cases} \tag{4.2}$$

We can also consider the stroboscopic (or Poincaré) map $Q\colon \Sigma \mapsto \Sigma$ given by $z(t) \mapsto z(t+T)$, where Σ is a cross-section.

$\epsilon = 0$

Choose a normal $\hat{\mathbf{n}}$ to Γ for (4.1) and let π be the corresponding plane for (4.2). The unperturbed problem has a one parameter family of homoclinic orbits $z_\tau(t) = z_0(t+\tau)$: we choose the origin of time for z_0 at its intersection with the plane π. We want to find out which homoclinics survive perturbation — if any?

ϵ small

By the IFT a closed orbit γ_ϵ persists in $\mathbb{R}^2 \times S^1_T$ which is close to γ_0. This corresponds to a hyperbolic saddle for Q.

γ_ϵ is hyperbolic and the surfaces $W^S(\gamma_\epsilon)$ and $W^U(\gamma_\epsilon)$ vary smoothly with ϵ, so the surfaces $W^S(\gamma_\epsilon)$ and $W^U(\gamma_\epsilon)$ cross π transversely.

We want to measure the separation between W^U and W^S and see if it changes sign as τ varies. H is a good thing to use to measure this.

$$\begin{aligned}\Delta H(\tau,\epsilon) &= H(z_\epsilon^U(\tau)) - H(z_\epsilon^S(\tau))\\ &= \int_{\tau-nT}^{\tau} \frac{\mathrm{d}}{\mathrm{d}s}\left(z_\epsilon^U(s)\right)\,\mathrm{d}s + H(z_\epsilon^U(\tau-nT))\\ &\quad + \int_{\tau}^{\tau+nT} \frac{\mathrm{d}}{\mathrm{d}s}\left(z_\epsilon^S(s)\right)\,\mathrm{d}s - H(z_\epsilon^S(\tau+nT)).\end{aligned}$$

As $n \to \infty$, $z_\epsilon^U(\tau - nT)$ and $z_\epsilon^S(\tau+nT)$ both tend to $\gamma_\epsilon(\tau)$. So

$$\Delta H(\tau,\epsilon) = \lim_{n\to\infty} \int_{\tau-nT}^{\tau+nT} \frac{\mathrm{d}}{\mathrm{d}s}\left(H(\tilde{z}_\epsilon(s))\right)\,\mathrm{d}s, \qquad \tilde{z}_\epsilon = \begin{cases} z_\epsilon^U & s<\tau \\ z_\epsilon^S & s>\tau. \end{cases}$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}s} H(\tilde{z}_\epsilon(s)) = DH\cdot(X_H+\epsilon Y)(\tilde{z}_\epsilon(s),s) = \epsilon DH \cdot Y(\tilde{z}_\epsilon(s),s),$$

and $\epsilon \to 0$, $\tilde{z}_\epsilon(s) \to z_0(s+\tau)$. We write $\Delta H = \epsilon G(\tau,\epsilon)$ and define the *Melnikov* function: $M(\tau) = G(\tau,0)$. Then

$$\Delta H = \epsilon M(\tau) + \mathcal{O}(\epsilon^2).$$

Theorem 4.8. *If M has a zero at $\tau = \tau_0$ and $\frac{\partial M}{\partial \tau}\big|_{\tau_0} \neq 0$ then γ_ϵ has a transverse homoclinic orbit which is near $z_0(\tau_0)$.*

Proof. We know $\Delta H(\tau,\epsilon) = \epsilon G(\tau,\epsilon)$ and that $M(\tau_0) = G(\tau_0,0)$ and $\frac{\partial G}{\partial \tau}\big|_{\tau_0} \neq 0$. We can therefore apply the IFT to find a curve $\tau(\epsilon)$ such that $G(\tau(\epsilon),\epsilon) = 0$ (for small ϵ). □

In this case we have a horseshoe (by the Smale-Birkhoff theorem (4.7)) and so chaos.

Concluding remarks

We have found sets (horseshoes) on which the dynamics are chaotic. However the horseshoe is repelling. Proving the existence of chaotic attractors is one of the major challenges of Dynamical Systems.

References

- *Dynamics of Differential Equations*, unpublished, 1996.

  Not useful as a reference but a fairly fun related course. The course style has changed since I typed these.

- P.A. Glendenning, *Stability, Instability and Chaos*, CUP, 1994.

  An easier book than the others I've mentioned here. It is not that useful for discrete time dynamical systems, although the bit it does have is explained well.

- Guckenheimer & Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer, 1983.

  Worth the read — I like it but it's probably not that good a book for this course.

- Clark Robinson, *Dynamical Systems*, CRC Press, 1994.

  A good book for this course, but with some of the worst TEX layout I've ever seen. Still, it is an excellent book.

Contrary to most of maths this area appears to be very well served with excellent readable textbooks. That means these notes are probably futile. Oh well.