Nonlinear Dynamics **30:** 287–294, 2002.

Solution of Ordinary Differential Equations with a Large Lie Symmetry Group ⋆

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(Received: 20 December 2001; accepted: 7 February 2002)

Abstract. The group method of solving ODEs without any quadrature goes back to Lie. In order to apply it, the number of symmetries admitted by a given ODE has to be greater by one than the number of arbitrary constants in the general solution of the equation. In this paper, we use the technique of canonical Lie–Bäcklund symmetries that makes the proof of the statement concerning integrals of ODEs more evident. The method is extended to the solution of system of ODEs with a small parameter of arbitrary order.

Keywords: Integrals, ordinary differential equations, Lie–Bäcklund symmetries.

1. Introduction

The symmetry properties provide different ways of finding the solution of a differential equation. If the number of admitted symmetries is sufficiently large, the solution of Ordinary Differential Equation (ODE) can be found by means of only differentiation and algebraic operations. It is well known that if the first-order ODE $y' = F(x, y)$ is invariant under Lie point symmetry

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y},$$

then $(\eta - F\xi)^{-1}$ is its integrating factor. The canonical Lie–Bäcklund operator corresponding to X is

$$X - \xi D = f(x, y)\frac{\partial}{\partial y},$$

where D is a total derivative. The quotient of two integrating factors $(\eta_1 - F\xi_1)/(\eta_2 - F\xi_2) = C$ gives the solution of the equation [1, 2]. In terms of canonical symmetries, this solution has the simple form $f_1/f_2 = C$.

Lie calculated integrals of nth-order ODE using determinants of coordinates of $n + 1$ admitted point symmetries. These formulas appear in [1] for a second-order ODE (see also this result in [3]). As we have just noted for the first-order ODE, and this is valid for an ODE of arbitrary order, consideration of canonical Lie–Bäcklund symmetries simplifies the form of the obtained integrals. The technique of Lie–Bäcklund symmetries, developed in works of Ibragimov (see, e.g., [4] and references therein), allows the evident extension of the method in order to solve systems of ODEs. A similar approach based on approximate Lie–Bäcklund symmetries (that were introduced in [5]) is applicable to equations with a small parameter.

⋆ Contributed by Professor N. H. Ibragimov.

In this paper, we consider the system of nth-order ODEs, solved with respect to higher-order derivatives

$$
\begin{aligned}
y_1^{(n)} &\approx F_1(x, y_1, \dots, y_m, \dots, y_1^{(n-1)}, \dots, y_m^{(n-1)}, \varepsilon),\\
y_2^{(n)} &\approx F_2(x, y_1, \dots, y_m, \dots, y_1^{(n-1)}, \dots, y_m^{(n-1)}, \varepsilon),\\
&\dots\\
y_m^{(n)} &\approx F_m(x, y_1, \dots, y_m, \dots, y_1^{(n-1)}, \dots, y_m^{(n-1)}, \varepsilon).
\end{aligned}
\tag{1}
$$

Here ε is a small parameter and the equality $u \approx v$ means that $u = v + o(\varepsilon^p)$ with some accuracy $o(\varepsilon^p)$, $p \geq 0$. Suppose that all functions to be considered are represented as series in nonnegative powers of ε, e.g. in (1),

$$F_j = F_{j,(0)} + \varepsilon F_{j,(1)} + \cdots + \varepsilon^p F_{j,(p)} + o(\varepsilon^p), \quad j = 1, \dots, m.$$

Approximate Lie–Bäcklund symmetry for system (1) is given by the operator [5]

$$X \approx \sum_{j=1}^{m} f^j(x, y_1, \dots, y_m^{(n-1)}, \varepsilon) \frac{\partial}{\partial y_j}. \tag{2}$$

Operator (2) is written in curtailed form (without prolongation to y_j', y_j'', ...) and the higher-order derivatives are eliminated by means of system (1). The coordinates f^j are defined by determining equations [4]

$$X(y_j^{(n)} - F_j)\Big|_{[y^{(n)} \approx F]} \approx 0, \quad j = 1, \dots, m, \tag{3}$$

where $[y^{(n)} \approx F]$ denotes system (1) with its differential consequences. Operator (2) prolonged to the derivatives, has the form

$$X \approx \sum_{j=1}^{m} \left(f^j \frac{\partial}{\partial y_j} + D_x f^j \frac{\partial}{\partial y_j'} + \cdots + D_x^n f^j \frac{\partial}{\partial y_j^{(n)}} \right),$$

where

$$D_x = \frac{\partial}{\partial x} + \sum_{j=1}^{m} \left(y_j' \frac{\partial}{\partial y_j} + \cdots + y_j^{(n)} \frac{\partial}{\partial y_j^{(n-1)}} + \cdots \right)$$

is the operator of the total derivative.

For system (1), we prove that its integral can be found by means of differentiation and algebraic operations from the coordinates of its $mn + 1$ Lie–Bäcklund symmetries. If one has mn integrals constructed in this way, the general solution is a result of elimination of lower-order derivatives.

2. Integrals of Systems of ODEs

The function $\varphi(x, y_1, \dots, y_m, \dots, y_1^{(n-1)}, \dots, y_m^{(n-1)}, \varepsilon)$ is called an essential integral of system (1), if

1. φ is expanded into power series in ε with nonzero coefficient $\varphi_{(0)}$,
2. an approximate equality $D\varphi \approx 0$ holds.

Here

$$D = \frac{\partial}{\partial x} + \sum_{j=1}^{m} \left(y'_j \frac{\partial}{\partial y_j} + \cdots + F_j \frac{\partial}{\partial y_j^{(n-1)}} \right)$$

is the operator of the total derivative by virtue of system (1). Using the operator D, determining equations (3) take the form

$$D^n f^j - \sum_{l=1}^{m} \sum_{k=0}^{n-1} F_{j y_l^{(k)}} D^k f^l \approx 0, \quad j = 1, \ldots, m. \tag{4}$$

Let system (1) be invariant under $r = mn + 1$ approximate Lie–Bäcklund operators

$$X_i \approx \sum_{j=1}^{m} f_i^j (x, y_1, \ldots, y_m^{(n-1)}, \varepsilon) \frac{\partial}{\partial y_j}, \quad i = 1, \ldots, r. \tag{5}$$

We denote Δ_i, $i = 1, \ldots, r$, the determinant of the mn-dimensional matrix on coordinates of prolonged operators (5), except for the operator X_i,

$$\Delta_i \approx \begin{vmatrix} f_1^1 & \ldots & f_1^m & Df_1^1 & \ldots & Df_1^m & \ldots & D^{n-1} f_1^1 & \ldots & D^{n-1} f_1^m \\ & \ldots & & & \ldots & & \ldots & & \ldots & \\ f_{i-1}^1 & \ldots & f_{i-1}^m & Df_{i-1}^1 & \ldots & Df_{i-1}^m & \ldots & D^{n-1} f_{i-1}^1 & \ldots & D^{n-1} f_{i-1}^m \\ f_{i+1}^1 & \ldots & f_{i+1}^m & Df_{i+1}^1 & \ldots & Df_{i+1}^m & \ldots & D^{n-1} f_{i+1}^1 & \ldots & D^{n-1} f_{i+1}^m \\ & \ldots & & & \ldots & & \ldots & & \ldots & \\ f_r^1 & \ldots & f_r^m & Df_r^1 & \ldots & Df_r^m & \ldots & D^{n-1} f_r^1 & \ldots & D^{n-1} f_r^m \end{vmatrix}.$$

We deal with the operator D that implies all elements of this matrix depend only on $x, y_1, \ldots, y_m^{(n-1)}, \varepsilon$.

PROPOSITION 1. *If for $r = mn + 1$ operators* (5) *admitted by system* (1)*, two determinants Δ_{i_1}, Δ_{i_2} are such that $\Delta_{i_1,(0)} \neq 0$, $\Delta_{i_2,(0)} \neq 0$, then the quotient $\Delta_{i_1}/\Delta_{i_2}$ is an essential integral of system* (1).

Proof. Without loss of generality, we calculate only $D\Delta_1$. The derivative of the determinant Δ_1 is the sum of all mn successive determinants that are equal to Δ_1 besides one of the columns (from first to mnth), whose elements are changed by their derivatives. The first $m(n-1)$ summands in this sum have two equal columns and are therefore equal to zero. In the remaining expression

$$\begin{aligned} D\Delta_1 \approx & \begin{vmatrix} f_2^1 & \ldots & f_2^m & \ldots & D^n f_2^1 & D^{n-1} f_2^2 & \ldots & D^{n-1} f_2^m \\ & \ldots & & \ldots & & & \ldots & \\ f_r^1 & \ldots & f_r^m & \ldots & D^n f_r^1 & D^{n-1} f_r^2 & \ldots & D^{n-1} f_r^m \end{vmatrix} \\ & + \begin{vmatrix} f_2^1 & \ldots & f_2^m & \ldots & D^{n-1} f_2^1 & D^n f_2^2 & \ldots & D^{n-1} f_2^m \\ & \ldots & & \ldots & & & \ldots & \\ f_r^1 & \ldots & f_r^m & \ldots & D^{n-1} f_r^1 & D^n f_r^2 & \ldots & D^{n-1} f_r^m \end{vmatrix} + \cdots \\ & + \begin{vmatrix} f_2^1 & \ldots & f_2^m & \ldots & D^{n-1} f_2^1 & D^{n-1} f_2^2 & \ldots & D^n f_2^m \\ & \ldots & & \ldots & & & \ldots & \\ f_r^1 & \ldots & f_r^m & \ldots & D^{n-1} f_r^1 & D^{n-1} f_r^2 & \ldots & D^n f_r^m \end{vmatrix} \end{aligned}$$

we change $D^n f_i^j$ according to (4). Since adding the linear combination of others to some column does not effect the determinant,

$$
\begin{aligned}
D\Delta_1 \;\approx\; & \begin{vmatrix} f_2^1 & \dots & f_2^m & \dots & F_{1y_1^{(n-1)}} D^{n-1} f_2^1 & D^{n-1} f_2^2 & \dots & D^{n-1} f_2^m \\ & \dots & & \dots & & & \dots & \\ f_r^1 & \dots & f_r^m & \dots & F_{1y_1^{(n-1)}} D^{n-1} f_r^1 & D^{n-1} f_r^2 & \dots & D^{n-1} f_r^m \end{vmatrix} \\
& + \begin{vmatrix} f_2^1 & \dots & f_2^m & \dots & D^{n-1} f_2^1 & F_{2y_2^{(n-1)}} D^{n-1} f_2^2 & \dots & D^{n-1} f_2^m \\ & \dots & & \dots & & & \dots & \\ f_r^1 & \dots & f_r^m & \dots & D^{n-1} f_r^1 & F_{2y_2^{(n-1)}} D^{n-1} f_r^2 & \dots & D^{n-1} f_r^m \end{vmatrix} + \cdots \\
& + \begin{vmatrix} f_2^1 & \dots & f_2^m & \dots & D^{n-1} f_2^1 & D^{n-1} f_2^2 & \dots & F_{my_m^{(n-1)}} D^{n-1} f_2^m \\ & \dots & & \dots & & & \dots & \\ f_r^1 & \dots & f_r^m & \dots & D^{n-1} f_r^1 & D^{n-1} f_r^2 & \dots & F_{my_m^{(n-1)}} D^{n-1} f_r^m \end{vmatrix} \\
\approx\; & \Delta_1 \sum_{j=1}^{m} F_{jy_j^{(n-1)}}.
\end{aligned}
$$

Similarly proceeding with other Δ_i, we have

$$D\Delta_i \approx \Delta_i \sum_{j=1}^{m} F_{jy_j^{(n-1)}}, \quad i = 1, \dots, r. \tag{6}$$

Thus, by virtue of system (1), the derivative of $\Delta_{i_1}/\Delta_{i_2}$ equals

$$D\left(\frac{\Delta_{i_1}}{\Delta_{i_2}}\right) \approx \frac{\Delta_{i_2} D\Delta_{i_1} - \Delta_{i_1} D\Delta_{i_2}}{\Delta_{i_2}^2} \approx \frac{\Delta_{i_2}\Delta_{i_1} - \Delta_{i_1}\Delta_{i_2}}{\Delta_{i_2}^2} \sum_{j=1}^{m} F_{jy_j^{(n-1)}} \approx 0,$$

i.e. $\Delta_{i_1}/\Delta_{i_2}$ is an integral of system (1). This completes the proof. □

If all $\Delta_{1,(0)}, \dots, \Delta_{r,(0)}$ are nonzero, we have mn integrals of system (1). After the elimination of the $m(n-1)$ derivatives $y_1', \dots, y_m', \dots, y_1^{(n-1)}, \dots, y_m^{(n-1)}$ from the integrals

$$\frac{\Delta_1}{\Delta_{mn+1}} \approx C_1, \quad \frac{\Delta_2}{\Delta_{mn+1}} \approx C_2, \quad \dots, \quad \frac{\Delta_{mn}}{\Delta_{mn+1}} \approx C_{mn},$$

m relations between $x, y_1, \dots, y_m, C_1, \dots, C_{mn}$ remain. In this case, if

$$\frac{\partial\left(\frac{\Delta_{1,(0)}}{\Delta_{r,(0)}}, \frac{\Delta_{2,(0)}}{\Delta_{r,(0)}}, \dots, \frac{\Delta_{mn,(0)}}{\Delta_{r,(0)}}\right)}{\partial(y_1, \dots, y_m, \dots, y_1^{(n-1)}, \dots, y_m^{(n-1)})} \neq 0,$$

the general solution of (1) is obtained without any quadrature.

Remark. In order to find integrals of the system of different order ODEs,

$$y_j^{(n_j)} \approx F_j(x, y_1, \dots, y_1^{(n_1-1)}, \dots, y_m, \dots, y_m^{(n_m-1)}, \varepsilon), \quad j = 1, \dots, m,$$

the number of admitted symmetries has to be equal to $r = n_1 + \cdots + n_m + 1$. The corresponding $(r-1)$-dimensional determinants Δ_i, $i = 1, \dots, r$, consist of f_k^j, Df_k^j,…, $D^{n_j-1} f_k^j$, $j = 1, \dots, m$, $k \neq i$.

The described method gives the solution of exact ODEs if we assume $\varepsilon = 0$ and consider all equalities as exact.

EXAMPLE 1. The exact equation

$$y'' = \frac{1}{2}(a\mathrm{e}^{-2y} - 1 - y'^2), \quad a = \text{const}, \tag{7}$$

admits a Lie algebra generated by the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \sin x \frac{\partial}{\partial x} + \cos x \frac{\partial}{\partial y}, \quad X_3 = \cos x \frac{\partial}{\partial x} - \sin x \frac{\partial}{\partial y}.$$

The corresponding canonical Lie–Bäcklund operators have the form

$$X_1 = y' \frac{\partial}{\partial y}, \quad X_2 = (y' \sin x - \cos x)\frac{\partial}{\partial y}, \quad X_3 = (y' \cos x + \sin x)\frac{\partial}{\partial y}.$$

Here $m = 1$, $n = 2$, $F = (1/2)(a\mathrm{e}^{-2y} - 1 - y'^2)$ and

$$\Delta_1 = \begin{vmatrix} y' \sin x - \cos x & (1+F)\sin x + y' \cos x \\ y' \cos x + \sin x & (1+F)\cos x - y' \sin x \end{vmatrix} = -\frac{1}{2}(a\,\mathrm{e}^{-2y} + 1 + y'^2),$$

$$\Delta_2 = \begin{vmatrix} y' & F \\ y' \cos x + \sin x & (1+F)\cos x - y' \sin x \end{vmatrix} = y' \cos x - \frac{1}{2}(a\,\mathrm{e}^{-2y} - 1 + y'^2)\sin x,$$

$$\Delta_3 = \begin{vmatrix} y' & F \\ y' \sin x - \cos x & (1+F)\sin x + y' \cos x \end{vmatrix} = y' \sin x + \frac{1}{2}(a\mathrm{e}^{-2y} - 1 + y'^2)\cos x.$$

One can express

$$y' = \frac{C_1 \cos x + C_2 \sin x}{C_1 \sin x - C_2 \cos x - 1}$$

from the integrals $\Delta_2/\Delta_1 = C_1$, $\Delta_3/\Delta_1 = C_2$ and then obtain the general solution

$$y = \ln(C_1 \sin x - C_2 \cos x - 1) - \frac{1}{2}\ln\left(\frac{1 - C_1^2 - C_2^2}{a}\right)$$

of Equation (7) from one of these integrals.

EXAMPLE 2. The system of two second-order ODEs

$$\begin{aligned} u'' &= \frac{2u'^2}{u+v} + 4\varepsilon\frac{\beta u'}{u+v} + o(\varepsilon), \\ v'' &= \frac{2v'^2}{u+v} - 4\varepsilon\frac{\beta v'}{u+v} + o(\varepsilon), \quad \beta = \text{const}, \end{aligned} \tag{8}$$

is invariant (with first-order precision) under the operators

$$X_1 \approx \frac{\partial}{\partial u} - \frac{\partial}{\partial v}, \quad X_2 \approx \frac{\partial}{\partial x}, \quad X_3 \approx (u + \varepsilon\beta x)\frac{\partial}{\partial u} + (v - \varepsilon\beta x)\frac{\partial}{\partial v},$$

$$X_4 \approx x\frac{\partial}{\partial x} - \varepsilon\beta x\frac{\partial}{\partial u} + \varepsilon\beta x\frac{\partial}{\partial v}, \quad X_5 \approx (u^2 + 2\varepsilon\beta xu)\frac{\partial}{\partial u} + (2\varepsilon\beta xv - v^2)\frac{\partial}{\partial v}.$$

The canonical form of the operators X_2, X_4 is

$$X_2 \approx u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}, \quad X_4 \approx (xu' + \varepsilon\beta x)\frac{\partial}{\partial u} + (xv' - \varepsilon\beta x)\frac{\partial}{\partial v}.$$

Since one of the corresponding determinants

$$\Delta_1 \approx 2u'v'(u^2v' - v^2u') + 2\varepsilon\beta((uv' + vu')^2 - u'v'(u^2 + v^2) + 2xu'v'(uv' + vu')),$$

$$\Delta_2 \approx -2u'v'(u + v)^2 + 2\varepsilon\beta(u' - v')(u + v)^2,$$

$$\Delta_3 \approx -4u'v'(uv' + vu') + 4\varepsilon\beta(vu'^2 - uv'^2 + 2u'v'(u - v) + xu'v'(u' - v')),$$

$$\Delta_4 \approx 0, \quad \Delta_5 \approx 2u'v'(v' - u') + 2\varepsilon\beta(u'^2 + v'^2 - 4u'v')$$

equals zero, there are only three integrals of system (8). The equations $\Delta_1/\Delta_5 \approx C_1$, $\Delta_2/\Delta_5 \approx C_2$, $\Delta_3/\Delta_5 \approx 2C_3$ provide expressions for the derivatives

$$u' \approx \frac{C_3 + u}{C_2}(u + v) + \varepsilon\beta\left(\frac{x}{C_2}(u + v) - 1\right),$$

$$v' \approx \frac{C_3 - v}{C_2}(u + v) + \varepsilon\beta\left(\frac{x}{C_2}(u + v) + 1\right)$$

and a relation between u, v,

$$uv + C_3(v - u) - C_1 + \varepsilon\beta x(v - u - 2C_3) - \varepsilon\beta C_2 \approx 0.$$

Hence the solution of system (8) is obtained by one quadrature ($\omega = ax + C_4$):

$$\left.\begin{aligned} u &\approx -C_3 - \left(C_2a + \frac{\varepsilon\beta}{2a}\right)\mathrm{ctg}\omega + \frac{\varepsilon}{2}\beta x(\mathrm{ctg}^2\omega - 1), \\ v &\approx C_3 - \left(C_2a + \frac{\varepsilon\beta}{2a}\right)\mathrm{tg}\omega + \frac{\varepsilon}{2}\beta x\left(1 - \mathrm{tg}^2\omega\right) \end{aligned}\right\} \quad \text{if } C_1 - C_3^2 = a^2C_2^2;$$

$$\left.\begin{aligned} u &\approx -C_3 - \left(C_2a - \frac{\varepsilon\beta}{2a}\right)\mathrm{cth}\omega - \frac{\varepsilon}{2}\beta x(1 + \mathrm{cth}^2\omega), \\ v &\approx C_3 + \left(C_2a - \frac{\varepsilon\beta}{2a}\right)\mathrm{th}\omega + \frac{\varepsilon}{2}\beta x(1 + \mathrm{th}^2\omega) \end{aligned}\right\} \quad \text{if } C_1 - C_3^2 = -a^2C_2^2;$$

$$u \approx -C_3 + \varepsilon\beta C_4, \quad v \approx C_3 + \frac{C_2}{x + C_4} + \frac{\varepsilon}{3}\beta(2x - C_4) \quad \text{if } C_1 - C_3^2 = 0.$$

Remark. The statement immediately follows from relation (6).

PROPOSITION 2. *If system (1) is invariant under mn operators* (5) *and its right-hand side satisfies* $\sum_{j=1}^{m} F_{jy_j^{(n-1)}} \approx 0$, *then the determinant*

$$I \approx \begin{vmatrix} f_1^1 & \dots & f_1^m & \dots & D^{n-1} f_1^1 & \dots & D^{n-1} f_1^m \\ & \dots & & \dots & & \dots & \\ f_{mn}^1 & \dots & f_{mn}^m & \dots & D^{n-1} f_{mn}^1 & \dots & D^{n-1} f_{mn}^m \end{vmatrix} \tag{9}$$

is an integral of system (1).

EXAMPLE 3. The system of two ODEs

$$\begin{aligned} u'' &= \frac{\alpha}{v-u}\left(u'^2 + 2\varepsilon\beta u'\right) + o(\varepsilon), \\ v'' &= \frac{-2\alpha}{v-u}\left(u'v' + \varepsilon\beta(u'+v')\right) + o(\varepsilon), \quad \alpha, \beta = \text{const}, \end{aligned} \tag{10}$$

satisfies the condition of Proposition 2 and admits (with an accuracy $o(\varepsilon)$) four operators

$$X_1 \approx \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad X_2 \approx (u + \varepsilon\beta x)\frac{\partial}{\partial u} + (v + \varepsilon\beta x)\frac{\partial}{\partial v},$$

$$X_3 \approx \frac{\partial}{\partial x}, \quad X_4 \approx x\frac{\partial}{\partial x} - \varepsilon\beta x\frac{\partial}{\partial u} - \varepsilon\beta x\frac{\partial}{\partial v}.$$

The canonical form of operators X_3, X_4 is

$$X_3 \approx u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}, \quad X_4 \approx x(u' + \varepsilon\beta)\frac{\partial}{\partial u} + x(v' + \varepsilon\beta)\frac{\partial}{\partial v}.$$

According to (9) an integral of (10) is equal to

$$\begin{aligned} I &\approx \begin{vmatrix} 1 & 1 & 0 & 0 \\ u + \varepsilon\beta x & v + \varepsilon\beta x & u' + \varepsilon\beta & v' + \varepsilon\beta \\ u' & v' & u'' & v'' \\ x(u' + \varepsilon\beta) & x(v' + \varepsilon\beta) & xu'' + u' + \varepsilon\beta & xv'' + v' + \varepsilon\beta \end{vmatrix} \\ &\approx (v - u)(u''(v' + \varepsilon\beta) - v''(u' + \varepsilon\beta)). \end{aligned}$$

The substitution of u'', v'' from system (10) yields the following integral of (10)

$$I \approx 3\alpha(u'^2 v' + \varepsilon\beta(u'^2 + 2u'v')).$$

3. Conclusion

In this work we show that the general solution of system (1) with $mn + 1$ Lie–Bäcklund symmetries in some cases can be found without integration. For the particular type of system (1), invariant under mn symmetries, its integral is also obtained by means of only algebraic operations. Note that the symmetries are not required to span the Lie algebra. The method is applicable, without much modification to ODEs with a small parameter, invariant under approximate Lie–Bäcklund symmetries.

Acknowledgements

The author would like to thank Professor R. K. Gazizov for his interest in this work which was supported in part by the Russian Foundation for Basic Research grant 01-01-00931.

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