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APPROXIMATE SYMMETRIES

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ABSTRACT. A theory, based on the new concept of an approximate group, is developed for approximate group analysis of differential equations with a small parameter. An approximate Lie theorem is proved that enables one to construct approximate symmetries that are stable under small perturbations of the differential equations. The use of the algorithm is illustrated in detail by examples: approximate symmetries of nonlinear wave equations are considered along with a broad class of evolution equations that includes the Korteweg-de Vries and Burgers-Korteweg-de Vries equations.

Tables: 2. Bibliography: 4 titles.

Introduction

The methods of classical group analysis enable one to distinguish among all equations of mathematical physics the equations that are remarkable with respect to their symmetry properties (see, for example, [1], [2], and [3]). Unfortunately, any small perturbation of an equation disturbs the group admitted, and this reduces the practical value of these "refined" equations and of group-theoretic methods in general. Therefore, it became necessary to work out group analysis methods that are stable under small perturbations of the differential equations. In this article we develop such a method that is based on the concepts of an approximate group of transformations and approximate symmetries.

The following notation is used: $z = (z^1, \dots, z^N)$ is the independent variable; ε is a small parameter; all functions are assumed to be jointly analytic in their arguments; the vector expression $\xi(\partial/\partial z)$ is used, along with $\xi^k(\partial/\partial z^k)$ for expressions of the type $\sum_1^N \xi^k(\partial/\partial z^k)$. Everywhere below, $\theta_p(z,\varepsilon)$ denotes an infinitesimally small function of order ε^{p+1}, $p \geq 0$, i.e., $\theta_p(z,\varepsilon) = o(\varepsilon^p)$, where this equality (in the case of functions analytic in a neighborhood of $\varepsilon = 0$) is equivalent to any of the following conditions:

$$\lim_{\varepsilon\to 0} \frac{\theta_p(z,\varepsilon)}{\varepsilon^p} = 0;$$

or there exists a constant $C > 0$ such that

$$|\theta_p(z,\varepsilon)| \leq C|\varepsilon|^{p+1};$$

or there exists a function $\varphi(z,\varepsilon)$ analytic in a neighborhood of $\varepsilon = 0$ such that

$$\theta_p(z,\varepsilon) = \varepsilon^{p+1}\varphi(z,\varepsilon). \tag{0.1}$$

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Everywhere below, the approximate equality $f \approx g$ means the equality $f(z, \varepsilon) = g(z, \varepsilon) + o(\varepsilon^p)$ for some fixed value of $p \geq 0$.

The following notation is also used in §5: t and x are independent variables; u is a differentiable variable with successive derivatives (with respect to x) $u_1, u_2, \ldots$: $u_{\alpha+1} = D(u_\alpha)$, $u_0 = u$, and $D = \partial/\partial x + \sum_{\alpha \geq 0} u_{\alpha+1} \partial/\partial u_\alpha$; $\mathscr{A}$ is the space of differentiable functions, i.e., analytic functions of any finite number of variables $t, x, u, u_1, \ldots$; also $f_t = \partial/\partial t$, $f_x = \partial/\partial x$, $f_\alpha = \partial f/\partial u_\alpha$, and $f_* = \sum_{\alpha \geq 0} f_\alpha D^\alpha$.

Below we use the following variant of the theorem on continuous dependence of the solution of the Cauchy problem on the parameters.

THEOREM 1. *Suppose that the functions $f(z, \varepsilon)$ and $g(z, \varepsilon)$, which are analytic in a neighborhood of the point $(z_0, 0)$, satisfy the condition*

$$g(z, \varepsilon) = f(z, \varepsilon) + o(\varepsilon^p)$$

and let $z = z(t, \varepsilon)$ and $\tilde{z} = \tilde{z}(t, \varepsilon)$ be the solutions of the respective problems

$$dz/dt = f(z, \varepsilon), \qquad z|_{t=0} = \alpha(\varepsilon)$$

and

$$d\tilde{z}/dz = g(\tilde{z}, \varepsilon), \qquad \tilde{z}|_{t=0} = \beta(\varepsilon),$$

where $\alpha(0) = \beta(0) = z_0$ and $\beta(\varepsilon) = \alpha(\varepsilon) + o(\varepsilon^p)$. Then

$$\tilde{z}(t, \varepsilon) = z(t, \varepsilon) + o(\varepsilon^p).$$

We consider the approximate Cauchy problem

$$dz/dt \approx f(z, \varepsilon), \tag{0.2}$$

$$z|_{t=0} \approx \alpha(\varepsilon), \tag{0.3}$$

which is defined as follows. The approximate differential equation (0.2) is understood as a family of differential equations

$$dz/dt = g(z, \varepsilon) \quad \text{with } g(z, \varepsilon) \approx f(z, \varepsilon); \tag{0.2$'$}$$

the approximate initial condition (0.3) is understood similarly, namely,

$$z|_{t=0} = \beta(\varepsilon) \quad \text{with } \beta(\varepsilon) \approx \alpha(\varepsilon). \tag{0.3$'$}$$

The approximate equality in (0.2′) and (0.3′) has the same degree of accuracy p as in (0.2) and (0.3). According to Theorem 1, the solutions of all the problems of the form (0.2′), (0.3′) coincide to within $o(\varepsilon^p)$. Therefore, the *solution of the approximate Cauchy problem* (0.2), (0.3) is defined to be the solution of any of the problems (0.2′), (0.3′), considered to within $o(\varepsilon^p)$. Theorem 1 gives us the uniqueness (with the indicated accuracy) of this solution.

§1. One-parameter approximate groups

Let $z' = g(z, \varepsilon, a)$ be given (local) transformations forming a one-parameter group with respect to a, so that

$$g(z, \varepsilon, 0) = z, \qquad g(g(z, \varepsilon, a), \varepsilon, b) = g(z, \varepsilon, a + b), \tag{1.1}$$

and depending on the small parameter ε. Suppose that $f \approx g$, i.e.,

$$f(z, \varepsilon, a) = g(z, \varepsilon, a) + o(\varepsilon^p). \tag{1.2}$$

Together with the points z' we introduce the "close" points $\tilde{z}$ defined by

$$\tilde{z} = f(z, \varepsilon, a). \tag{1.3}$$

It is easy to show by substituting (1.2) in (1.1) that (1.3) gives an approximate group in the sense of the following definition.

DEFINITION 1. The transformations (1.3), or

$$z' \approx f(z, \varepsilon, a), \tag{1.3'}$$

form an *approximate one-parameter group with respect to the parameter a* if

$$f(z, \varepsilon, 0) \approx z, \tag{1.4}$$

$$f(f(z, \varepsilon, a), \varepsilon, b) \approx f(z, \varepsilon, a+b), \tag{1.5}$$

and the condition $f(z, \varepsilon, a) \approx z$ for all z implies that $a = 0$.

The main assertions about the infinitesimal description of local Lie groups remain true upon passing to approximate groups, with the exact equalities replaced by approximate equalities.

THEOREM 2 (an approximate Lie theorem). *Suppose that the transformations* (1.3′) *form an approximate group with the tangent vector field*

$$\xi(z, \varepsilon) \approx \left. \frac{\partial f(z, \varepsilon, a)}{\partial a} \right|_{a=0}. \tag{1.6}$$

Then the function $f(z, \varepsilon, a)$ *satisfies*

$$\frac{\partial f(z, \varepsilon, a)}{\partial a} \approx \xi(f(z, \varepsilon, a), \varepsilon). \tag{1.7}$$

Conversely, for any (*smooth*) *function* $\xi(z, \varepsilon)$ *the solution* (1.3′) *of the approximate Cauchy problem*

$$dz'/da \approx \xi(z', \varepsilon), \tag{1.8}$$

$$z'|_{a=0} \approx z \tag{1.9}$$

determines an approximate one-parameter group with group parameter a.

REMARK 1. Equation (1.8) will be called the *approximate Lie equation.*

PROOF. Suppose that $f(z, \varepsilon, a)$ gives an approximate group of transformations (1.3′). The (1.5) takes the form

$$f(f(z, \varepsilon, a), \varepsilon, 0) + \left. \frac{\partial f(f(z, \varepsilon, a), \varepsilon, b)}{\partial b} \right|_{b=0} \cdot b + o(b)$$
$$\approx f(z, \varepsilon, a) + \frac{\partial f(z, \varepsilon, a)}{\partial a} \cdot b + o(b)$$

after the principal terms with respect to b are singled out. The approximate equation (1.7) is obtained from this by transforming the left-hand side with the help of (1.4) and (1.6), dividing by b, and passing to the limit as $b \to 0$.

Conversely, suppose that the function (1.3′) is a solution of the approximate problem (1.8), (1.9). To prove that $f(z, \varepsilon, a)$ gives an approximate group it suffices to verify the approximate equality (1.5),

$$f(f(z, \varepsilon, a), \varepsilon, b) \approx f(z, \varepsilon, a+b).$$

Denote by $x(b, \varepsilon)$ and $y(b, \varepsilon)$ the left-hand and right-hand side of (1.5), regarded (for fixed z and a) as functions of (b, ε). By (1.8), they satisfy the same approximate Cauchy problem:

$$\partial x/\partial b \approx \xi(x, \varepsilon), \qquad x|_{b=0} \approx g(z, \varepsilon, a),$$
$$\partial y/\partial b \approx \xi(y, \varepsilon), \qquad y|_{b=0} \approx g(z, \varepsilon, a).$$

Therefore, by Theorem 1, we have the approximate equality $x(b,\varepsilon) \approx y(b,\varepsilon)$, i.e., the group property (1.5).

§2. An algorithm for constructing an approximate group

The construction of an approximate group from a given infinitesimal operator is implemented on the basis of the approximate Lie theorem. To show how to solve the approximate Lie equation (1.8) we consider first the case $p = 1$.

We seek the approximate group of transformations

$$z' \approx f_0(z,a) + \varepsilon f_1(z,a), \tag{2.1}$$

determined by the infinitesimal operator

$$X = (\xi_0(z) + \varepsilon\xi_1(z))(\partial/\partial z). \tag{2.2}$$

The corresponding approximate Lie equation

$$\frac{d(f_0 + \varepsilon f_1)}{da} \approx \xi_0(f_0 + \varepsilon f_1) + \varepsilon\xi_1(f_0 + \varepsilon f_1)$$

can be rewritten as the system

$$df_0/da \approx \xi_0(f_0), \qquad df_1/da \approx \xi_0'(f_0)f_1 + \xi_1(f_0)$$

after singling out the principal terms with respect to ε, where ξ_0' is the derivative of ξ_0. The initial condition $z'|_{a=0} \approx z$ gives us that $f_0|_{a=0} \approx z$ and $f_1|_{a=0} \approx 0$.

Thus, according to the definition of a solution of the approximate Cauchy problem (§1), to construct the approximate (to within $o(\varepsilon)$) group (2.1) from the given infinitesimal operator (2.2) it suffices to solve the following (exact) Cauchy problem:

$$\frac{df_0}{da} = \xi_0(f_0), \qquad \frac{df_1}{da} = \xi_0'(f_0)f_1 + \xi_1(f_0), \qquad f_0|_{a=0} = z, \qquad f_1|_{a=0} = 0. \tag{2.3}$$

EXAMPLE 1. Suppose that $N = 1$ and $X = (1 + \varepsilon x)(\partial/\partial x)$. The corresponding problem (2.3)

$$df_0/da = 1, \qquad df_1/da = f_0, \qquad f_0|_{a=0} = z, \qquad f_1|_{a=0} = 0$$

is easily solved, and gives us $f_0 = x + a$ and $f_1 = xa + a^2/2$. Consequently, the approximate group is determined by

$$x' \approx x + a + (xa + a^2/2)\varepsilon.$$

This formula is clearly the principal term in the Taylor series expansion with respect to ε of the exact group

$$x' = xe^{a\varepsilon} + \frac{e^{a\varepsilon-1}}{\varepsilon} = (x+a) + a\left(x + \frac{a}{2}\right)\varepsilon + \frac{a^2}{2}\left(x + \frac{a}{3}\right)\varepsilon^2 + \cdots,$$

generated by the operator $X = (1 + \varepsilon x)(\partial/\partial x)$.

EXAMPLE 2. We construct the approximate group of transformations

$$x' \approx f_0^1(x,y,a) + \varepsilon f_1^1(x,y,a), \qquad y' \approx f_0^2(x,y,a) + \varepsilon f_1^2(x,y,a)$$

determined by the operator

$$X = (1 + \varepsilon x^2)(\partial/\partial x) + \varepsilon xy(\partial/\partial y)$$

in the (x,y)-plane. After solving problem (2.3)

$$\frac{df_0^1}{da} = 1, \qquad \frac{df_0^2}{da} = 0, \qquad \frac{df_1^1}{da} = (f_0^1)^2, \qquad \frac{df_1^2}{da} = f_0^1 f_0^2,$$
$$f_0^1|_{a=0} = x, \qquad f_0^2|_{a=0} = y, \qquad f_1^1|_{a=0} = 0, \qquad f_1^2|_{a=0} = 0,$$

we get that

$$x' \approx x + a + (x^2 a + x a^2 + a^3/3)\varepsilon, \qquad y' \approx y + (xya + ya^2/2)\varepsilon.$$

To construct an approximate (to within $o(\varepsilon^p)$) group for arbitrary p we need a formula for the principal (with respect to ε) part of a function of the form $F(y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p)$. By Taylor's formula,

$$F(y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p) = F(y_0) + \sum_{|\sigma|=1}^{p} \frac{1}{\sigma!} F^{(\sigma)}(y_0)(\varepsilon y_1 + \cdots + \varepsilon^p y_p)^\sigma + o(\varepsilon^p), \quad (2.4)$$

where

$$F^{(\sigma)} = \frac{\partial^{|\sigma|} F}{(\partial z^1)^{\sigma_1} \ldots (\partial z^N)^{\sigma_N}}, \qquad (\varepsilon y_1 + \cdots + \varepsilon^p y_p)^\sigma = \prod_{k=1}^{N} (\varepsilon y_1^k + \cdots + \varepsilon^p y_p^k)^k, \quad (2.5)$$

$\sigma = (\sigma_1, \ldots, \sigma_N)$ is a multi-index, $|\sigma| = \sigma_1 + \cdots + \sigma_N$, $\sigma! = \sigma_1! \cdots \sigma_N!$, and the indices $\sigma_1, \ldots, \sigma_N$ run from 0 to p. In the last expression we single out the terms up to order ε^p:

$$\begin{aligned}
\prod_{k=1}^{N} (\varepsilon y_1^k + \cdots + \varepsilon^p y_p^k)^{\sigma_k} &= \prod_{k=1}^{N} \left(\sum_{i_1, \ldots, i_{\sigma_k}=1}^{p} y_{i_1}^k \ldots y_{i_{\sigma_k}}^k \varepsilon^{i_1 + \cdots + i_{\sigma_k}} \right) \\
&\approx \prod_{k=1}^{N} \left(\sum_{\nu_k = \sigma_k}^{p} \varepsilon^{\nu_k} \sum_{i_1 + \cdots + i_{\sigma_k} = \nu_k} y_{i_1}^k \ldots y_{i_{\sigma_k}}^k \right) \equiv \prod_{k=1}^{N} \sum_{\nu_k = \sigma_k}^{p} \varepsilon^{\nu_k} y_{(\nu_k)}^k \\
&\approx \sum_{j=|\sigma|} \varepsilon^j \left(\sum_{\nu_1 + \cdots + \nu_N = j} y_{(\nu_1)}^1 \cdots y_{(\nu_N)}^N \right) \equiv \sum_{j=|\sigma|}^{p} \varepsilon^j \sum_{|\nu|=j} y_{(\nu)}.
\end{aligned} \quad (2.6)$$

Here the notation is

$$y_{(\nu_k)}^k \equiv \sum_{i_1 + \cdots + i_{\sigma_k} = \nu_k} y_{i_1}^k \cdots y_{i_{\sigma_k}}^k, \qquad y_{(\nu)} = y_{(\nu_1)}^1 \ldots y_{(\nu_N)}^N, \quad (2.7)$$

where the indices $i_1, \ldots, i_{\sigma_k}$ run from 0 to p, and $\nu = \nu(\sigma) = (\nu_1, \ldots, \nu_N)$ is a multi-index associated with the multi-index σ in such a way that if the index σ_s in σ is equal to zero, then the corresponding index ν_s is absent in ν, and each of the remaining indices ν_k takes values from σ_k to p; for example, for $\sigma = (0, \sigma_2, \sigma_2, 0, \ldots, 0)$ with $\sigma_2, \sigma_3 \neq 0$ we have that $\nu = (\nu_2, \nu_3)$, so that $y_{(\nu)} = y_{(\nu_2)}^2 y_{(\nu_3)}^3$.

Substituting (2.6) into (2.4) and interchanging the summations over σ and j, we get the following formula for the principal part:

$$F(y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p) = F(y_0) + \sum_{j=1}^{p} \varepsilon^j \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!} F^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)} + o(\varepsilon^p), \quad (2.8)$$

where the notation in (2.5) and (2.7) has been used. For example,

$$
\begin{aligned}
&F(y_0+\varepsilon y_1+\varepsilon^2 y_2+\varepsilon^3 y_3)\\
&= F(y_0)+\varepsilon\sum_{k=1}^{N}\frac{\partial F(y_0)}{\partial z^k}y_1^k\\
&+\varepsilon^2\left(\sum_{k=1}^{N}\frac{\partial F(y_0)}{\partial z^k}y_2^k+\frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\partial^2 F(y_0)}{\partial z^k\partial z^l}y_1^k y_1^l\right)\\
&+\varepsilon^3\left(\sum_{k=1}^{N}\frac{\partial F(y_0)}{\partial z^k}y_3^k+\frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\partial^2 F(y_0)}{\partial z^k\partial z^l}\times(y_1^k y_2^l+y_1^l y_2^k)\right.\\
&\qquad\left.+\frac{1}{3!}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\frac{\partial^2 F(y_0)}{\partial z^k\partial z^l\partial z^m}y_1^k y_1^l y_1^m\right)+o(\varepsilon^3).
\end{aligned}
$$

We also need a generalization of (2.8) for the expression

$$\sum_{i=0}^{p}\varepsilon^i F_i(y_0+\varepsilon y_1+\cdots+\varepsilon^p y_p).$$

Applying (2.8) to each function F_i and introducing for brevity the notation

$$\tau_{j,i}=\sum_{|\sigma|=1}^{j}\frac{1}{\sigma!}F_i^{(\sigma)}(y_0)\sum_{|\nu|=j}y_{(\nu)},$$

we have that

$$
\begin{aligned}
\sum_{i=0}^{p}\varepsilon^i F_i(y_0+\varepsilon y_1+\cdots+\varepsilon^p y_p)&\approx\sum_{i=0}^{p}\varepsilon^i\left[F_i(y_0)+\sum_{j=1}^{p}\varepsilon^j\tau_{j,i}\right]\\
&\approx\sum_{i=0}^{p}\varepsilon^i F_i(y_0)+\sum_{i=0}^{p-1}\sum_{j=1}^{p-i}\varepsilon^{i+j}\tau_{j,i}
\end{aligned}
$$

to within $o(\varepsilon^p)$. The standard transformations are used to order the last term with respect to powers of ε:

$$\sum_{i=0}^{p-1}\sum_{j=1}^{p-i}\varepsilon^{i+j}\tau_{j,i}=\sum_{i=0}^{p-1}\sum_{l=i+1}^{p}\varepsilon^l\tau_{l-i,i}=\sum_{l=1}^{p}\varepsilon^l\sum_{i=0}^{l-1}\tau_{l-i,i}=\sum_{l=1}^{p}\varepsilon^l\sum_{j=1}^{l}\tau_{j,l-j}.$$

As a result, we arrive at the following generalization of (2.8):

$$
\begin{aligned}
&\sum_{i=0}^{p}\varepsilon^i F_i(y_0+\varepsilon y_1+\cdots+\varepsilon^p y_p)\\
&\approx F_0(y_0)+\sum_{i=1}^{p}\varepsilon^i\left[F_i(y_0)+\sum_{j=1}^{i}\sum_{|\sigma|=1}^{j}\frac{1}{\sigma!}F_{i-j}^{(\sigma)}(y_0)\sum_{|\nu|=j}y_{(\nu)}\right]
\end{aligned}
\tag{2.9}
$$

with the same notation as in (2.5) and (2.7).

We now return to the construction of an approximate group to within $o(\varepsilon^p)$ with an arbitrary p. For the infinitesimal operator

$$X=[\xi_0(z)+\varepsilon\xi_1(z)+\cdots+\varepsilon^p\xi_p(z)]\frac{\partial}{\partial z}$$

the approximate group of transformations

$$z' \approx f_0(z,a) + \varepsilon f_1(z,a) + \cdots + \varepsilon^p f_p(z,a) \tag{2.10}$$

is determined by the approximate Lie equation

$$\frac{d}{da}(f_0 + \varepsilon f_1 + \cdots + \varepsilon^p f_p) \approx \sum_{i=0}^{p} \varepsilon^i \xi_i(f_0 + \varepsilon f_1 + \cdots + \varepsilon^p f_p). \tag{2.11}$$

Transforming the right-hand side of this equation according to (2.9) and equating the coefficients of like powers of ε, we get the system of equations (in the notation of (2.5) and (2.7))

$$df_0/da = \xi_0(f_0), \tag{2.12}$$

$$\frac{df_i}{da} = \xi_i(f_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!} \xi_{i-j}^{(\sigma)}(f_0) \sum_{|\nu|=j} f_{(\nu)}, \qquad i = 1, \ldots, p, \tag{2.13}$$

which is equivalent to the approximate equation (2.11).

Accordingly, the problem of constructing the approximate group (2.10) reduces to the solution of system (2.12), (2.13) under the initial conditions

$$f_0|_{a=0} = z, \qquad f_i|_{a=0} = 0, \quad i = 1, \ldots, p. \tag{2.14}$$

For clarity we write out the first few equations of system (2.12), (2.13):

$$\begin{gathered} df_0/da = \xi_0(f_0), \\ \frac{df_1}{da} = \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_1^k + \xi_1(f_0), \\ \frac{df_2}{da} = \sum_{k=1}^{N} \frac{\partial \xi_0(f_0)}{\partial z^k} f_2^k + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^2 \xi_0(f_0)}{\partial z^k \partial z^l} f_1^k f_1^l + \sum_{k=1}^{N} \frac{\partial \xi_1(f_0)}{\partial z^k} f_1^k + \xi_2(f_0). \end{gathered} \tag{2.15}$$

EXAMPLE 3. Let us write out system (2.12), (2.13) for the operator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy \frac{\partial}{\partial y}$$

in Example 2. In this case $N = 2$, $z = (x, y)$, $f_k = (f_k^1, f_k^2)$, $k = 0, 1, \ldots, p$, $\xi_0 = (1, 0)$, $\xi_1 = (x^2, xy)$, and $\xi_l = 0$ for $l \geq 2$. From (2.15),

$$\begin{aligned} df_0^1/da &= 1, & df_0^2/da &= 0; \\ df_1^1/da &= (f_0^1)^2, & df_1^2/da &= f_0^1 f_0^2; \\ df_2^1/da &= 2 f_0^1 f_1^1, & df_2^2/da &= f_0^2 f_1^1 + f_0^1 f_1^2. \end{aligned}$$

For $i \geq 3$ equation (2.13) simplifies because of the special form of the vector ξ. Namely, since $\xi_0 = \text{const}$ and $\xi_l = 0$ for $l \geq 2$, only terms with $j = i - 1$ are present on the right-hand side of (2.13), and the latter can be written in the form

$$\frac{df_i}{da} = \sum_{|\sigma|=1}^{i-1} \frac{1}{\sigma!} \xi_1^{(\sigma)}(f_0) \sum_{|\nu|=i-1} f_{(\nu)}.$$

A further simplification of these equations has to do with the form of the vector ξ_1: since $\xi_1^1 = x^2$ and $\xi_1^2 = xy$, only $\sigma = (1, 0)$ and $\sigma = (2, 0)$ are used in the expression for the first component of the equations under consideration, and only σ equal to

$(1,0)$, $(0,1)$, and $(1,1)$ in the expression for the second component. As a result we have the following recurrence system:

$$\frac{df_i^1}{da} = 2f_0^1 f_{i-1}^1 + \sum_{i_1+i_2=i-1} f_{i_1}^1 f_{i_2}^1,$$

$$\frac{df_i^2}{da} = f_0^2 f_{i-1}^1 + f_0^1 f_{i-1}^2 + \sum_{i_1+i_2=i-1} f_{i_1}^1 f_{i_2}^2.$$

EXAMPLE 4. We compute the approximate group of transformations of order ε^p generated by the operator $X = (1+\varepsilon x)(\partial/\partial x)$ in Example 1. In this case system (2.12), (2.13) takes the form

$$\frac{df_0}{da} = 1, \quad \frac{df_i}{da} = f_{i-1}, \quad i = 1, \dots, p,$$

and, under the initial conditions (2.14), gives us $f_i = xa^i/i! + a^{i+1}/(i+1)!$, $i = 0, \dots, p$. The corresponding approximate group of transformations is determined by

$$x' \approx \sum_{i=0}^{p} \frac{a^i}{l!}\left(x + \frac{a}{i+1}\right)\varepsilon^i.$$

§3. A criterion for approximate invariance

DEFINITION 2. The approximate equation

$$F(z,\varepsilon) \approx 0 \tag{3.1}$$

is said to be *invariant with respect to the approximate group of transformations* $z' \approx f(z,\varepsilon,a)$ if

$$F(f(z,\varepsilon,a),\varepsilon) \approx 0 \tag{3.2}$$

for all $z = (z^1, \dots, z^N)$ satisfying (3.1).

THEOREM 3. *Suppose that the function* $F(z,\varepsilon) = (F^1(z,\varepsilon), \dots, F^n(z,\varepsilon))$, $n < N$, *which is jointly analytic in the variables* z *and* ε, *satisfies the condition*

$$\operatorname{rank} F'(z,0)|_{F(z,0)=0} = n, \tag{3.3}$$

where $F'(z,\varepsilon) = \|\partial F^\nu(z,\varepsilon)/\partial z^i\|$ *for* $\nu = 1, \dots, n$ *and* $i = 1, \dots, N$. *For the approximate equation*

$$F(z,\varepsilon) = o(\varepsilon^p) \tag{3.1}$$

to be invariant under the approximate group of transformations

$$z' = f(z,\varepsilon,a) + o(\varepsilon^p)$$

with infinitesimal operator

$$X = \xi(z,\varepsilon)\frac{\partial}{\partial z}, \qquad \xi = \left.\frac{\partial f}{\partial a}\right|_{a=0} + o(\varepsilon^p), \tag{3.4}$$

it is necessary and sufficient that

$$XF(z,\varepsilon)|_{(3.1)} = o(\varepsilon^p). \tag{3.5}$$

PROOF. *Necessity.* Suppose that condition (3.2) for invariance of the approximate equation (3.1) holds:

$$F(f(z,\varepsilon,a),\varepsilon)|_{(3.1)} = o(\varepsilon^p).$$

By differentiation with respect to a at $a = 0$, this yields (3.5).

Sufficiency. Suppose now that (3.5) holds for a function $F(z, \varepsilon)$ satisfying (3.3). Let us prove the invariance of the approximate equation (3.1). To do this we introduce the new variables $y^1 = F^1(z,\varepsilon), \dots, y^n = F^n(z,\varepsilon)$, $y^{n+1} = H^1(z,\varepsilon), \dots, y^N = H^{N-n}(z,\varepsilon)$ instead of $z^1, \dots, z^N$, choosing $H^1(z,\varepsilon), \dots, H^{N-n}(z,\varepsilon)$ so that the functions $F^1, \dots, F^n$, $H^1, \dots, H^{N-n}$ are functionally independent (for sufficiently small ε this is possible in view of condition (3.3)). In the new variables the original approximate equation (3.1), the operator (3.4), and condition (3.5) take the respective forms

$$y^\nu = \theta_p^\nu(y, \varepsilon), \qquad \nu = 1, \dots, n, \tag{3.1$'$}$$

$$X = \eta^i(y,\varepsilon)\frac{\partial}{\partial y^i}, \quad \text{where } \eta^i \approx \xi^j(x,\varepsilon)\frac{\partial y^i(x,\varepsilon)}{\partial x^j}, \tag{3.4$'$}$$

$$\eta^\nu(\theta_p^1, \dots, \theta_p^n, y^{n+1}, \dots, y^N) = o(\varepsilon^p), \qquad \nu = 1, \dots, n, \tag{3.5$'$}$$

where $\theta_p^\nu = o(\varepsilon^p)$ (see (0.1)). By Theorem 2, the transformations of the variables y are determined from the approximate Cauchy problem

$$dy'^\nu/da \approx \eta^\nu(y'^1, \dots, y'^n, y'^{n+1}, \dots, y'^N, \varepsilon), \qquad y'^\nu|_{a=0} = \theta_p^\nu(y,\varepsilon),$$
$$dy'^k/da \approx \eta^k(y'^1, \dots, y'^n, y'^{n+1}, \dots, y'^N, \varepsilon), \qquad y'^k|_{a=0} = y^k, \quad k = n+1, \dots, N,$$

where the initial conditions for the first subsystem are written with (3.1) taken into account. According to Theorem 1, the solution of this problem is unique (with the accuracy under consideration) and has the form $y' = (\theta_p^1, \dots, \theta_p^n, y^{n+1}, \dots, y^N)$ in view of (3.5$'$). Returning to the old variables, we get that $F^\nu(z', \varepsilon) = o(\varepsilon^p)$, $\nu = 1, \dots, n$, i.e., the approximate equation (3.2). The theorem is proved.

EXAMPLE 5. Let $N = 2$, $z = (x, y)$, and $p = 1$. We consider the approximate group of transformations (see Example 2 in §2)

$$x' \approx x + a + (x^2 a + x a^2 + a^3/3)\varepsilon, \qquad y' \approx y + (xya + ya^2/2)\varepsilon \tag{3.6}$$

with the infinitesimal operator

$$X = (1 + \varepsilon x^2)(\partial/\partial x) + \varepsilon xy(\partial/\partial y). \tag{3.7}$$

Let us show that the approximate equation

$$F(x,y,\varepsilon) \equiv y^{2+\varepsilon} - \varepsilon x^2 - 1 = o(\varepsilon) \tag{3.8}$$

is invariant with respect to the transformations (3.7).

We first verify the invariance of (3.8), following Definition 2. For this it is convenient to rewrite (3.8), while preserving the necessary accuracy, in the form

$$\tilde F(x,y,\varepsilon) \equiv y^2 - \varepsilon(x^2 - y^2 \ln y) - 1 \approx 0. \tag{3.8$'$}$$

After the transformation (3.6) we have that

$$\begin{aligned}\tilde F(x', y', \varepsilon) &= y'^2 + \varepsilon(x'^2 - y'^2 \ln y') - 1 \approx y^2 - \varepsilon(x^2 - y^2 \ln y) - 1 \\ &\quad + \varepsilon(2xa + a^2)(y^2 - 1) \\ &= \tilde F(x,y,\varepsilon) + \varepsilon(2xa + a^2)[\tilde F(x,y,\varepsilon) + \varepsilon(x^2 - y^2 \ln y)] \\ &= [1 + \varepsilon(2ax + a^2)]\tilde F(x,y,\varepsilon) + o(\varepsilon),\end{aligned}$$

which implies the necessary equality (3.2): $\tilde F(x', y', \varepsilon)|_{(3.8')} \approx 0$.

The function $F(x, y, \varepsilon)$ satisfies condition (3.3) of Theorem 3; therefore, the invariance can be established also with the help of the infinitesimal criterion (3.5). For the operator (3.7) we have that

$$XF = (2+\varepsilon)\varepsilon x y^{2+\varepsilon} - 2\varepsilon x(1+\varepsilon x^2) = 2\varepsilon x(y^{2+\varepsilon} - 1) + o(\varepsilon) = 2\varepsilon x F + o(\varepsilon),$$

so that the satisfaction of the invariance criterion (3.5) is obvious.

According to Theorem 3, the construction of the approximate group leaving the equation $F(z, \varepsilon) \approx 0$ invariant reduces to the solution of the defining equation

$$XF(x, \varepsilon)|_{F\approx 0} \approx 0 \tag{3.9}$$

for the coordinates $\xi^k(z, \varepsilon)$ of the infinitesimal operator $X = \xi(\partial/\partial z)$. To solve the defining equation (3.9) to within $o(\varepsilon^p)$ it is necessary to represent z, F, and ξ^k in the form

$$z \approx y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p, \quad F(z, \varepsilon) \approx \sum_{i=0}^{p} \varepsilon^i F_i(z), \quad \xi^k(z, \varepsilon) \approx \sum_{i=0}^{p} \varepsilon^i \xi_i^k(z), \tag{3.10}$$

substitute them in XF, and single out their principal terms. We have

$$\begin{aligned} XF &= \xi^k \frac{\partial F}{\partial z^k} \\ &= \left[\sum_{i=0}^{p} \varepsilon^i \xi_i^k(y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p)\right] \cdot \left[\sum_{j=0}^{p} \varepsilon^j \frac{\partial}{\partial z^k} F_j(y_0 + \varepsilon y_1 + \cdots + \varepsilon^p y_p)\right]. \end{aligned}$$

Using (2.9) and the notation

$$A_i^k = \xi_i^k(y_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!} (\xi_{i-j}^k)^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)}, \tag{3.11}$$

$$B_{j,k} = \frac{\partial F_j(y_0)}{\partial z^k} + \sum_{i=1}^{j} \sum_{|\omega|=1}^{i} \frac{1}{\omega!} \left(\frac{\partial F_{j-i}}{\partial z^k}\right)^{(\omega)} (y_0) \sum_{|\mu|=i} y_{(\mu)}, \tag{3.12}$$

we get that

$$XF = \left[\xi_0^k(y_0) + \sum_{i=1}^{p} \varepsilon^i A_i^k\right] \cdot \left[\frac{\partial F_0(y_0)}{\partial z^k} + \sum_{j=1}^{p} \varepsilon^j B_{j,k}\right],$$

which implies

$$\begin{aligned} XF = {} & \xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} + \varepsilon \left[\xi_0^k(y_0) B_{1,k} + A_1^k \frac{\partial F_0(y_0)}{\partial z^k}\right] \\ & + \sum_{s=2}^{p} \varepsilon^s \left[\xi_0^k(y_0) B_{s,k} + A_s^k \frac{\partial F_0(y_0)}{\partial z^k} + \sum_{i+j=s} A_i^k B_{j,k}\right]. \end{aligned} \tag{3.13}$$

Combining (3.9)–(3.13) and (2.9), we arrive at the following form of the defining equation:

$$\begin{gathered} \xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} = 0, \qquad \xi_0^k(y_0) B_{1,k} + A_1^k \frac{\partial F_0(y_0)}{\partial z^k} = 0, \\ \xi_0^k(y_0) B_{l,k} + A_l^k \frac{\partial F_0(y_0)}{\partial z^k} + \sum_{i+j=l} A_i^k B_{j,k} = 0, \qquad l = 2, \ldots, p; \end{gathered} \tag{3.14}$$

these equations hold on the set of all $y_0, \dots, y_p$ satisfying the system

$$F_0(y_0) = 0, \qquad F_i(y_0) + \sum_{j=1}^{i} \sum_{|\sigma|=1}^{j} \frac{1}{\sigma!} F_{i-j}^{(\sigma)}(y_0) \sum_{|\nu|=j} y_{(\nu)}, \qquad i = 1, \dots, p, \tag{3.15}$$

which is equivalent to the approximate equation (3.1). Thus, the problem of solving the approximate defining equation (3.9) has been reduced to the solution of the system of exact equations (3.14), (3.15).

We write the defining equations for $p = 1$. Equations (3.14) and (3.15) give us([1])

$$\xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} = 0, \tag{3.16}$$

$$\xi_1^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} + \xi_0^k(y_0) \frac{\partial F_1(y_0)}{\partial z^k} + y_1^l \frac{\partial}{\partial z^l} \left(\xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} \right) = 0 \tag{3.17}$$

under the conditions

$$F_0(y_0) = 0, \qquad F_1(y_0) + y_1^l \frac{\partial F_0(y_0)}{\partial z^l} = 0. \tag{3.18}$$

EXAMPLE 6. Let us again consider the approximate equation (3.8)

$$F(x, y, \varepsilon) \equiv y^{2+\varepsilon} - \varepsilon x^2 - 1 = o(\varepsilon)$$

from Example 5. In the notation of (3.10) (see also (3.8′)) we have

$$F_0(x, y) = y^2 - 1, \qquad F_1(x, y) = y^2 \ln y - x^2.$$

Since $y > 0$, the equations (3.18) imply that $y_0 = 1$ and $y_1 = x_0^2/2$, and the defining equations (3.16) and (3.17) can be written in the form

$$\begin{gathered} \xi_0^2(x_0, y_0) = 0, \qquad \frac{\partial \xi_0^2(x_0, y_0)}{\partial x} = 0, \\ y_0 \xi_1^2(x_0, y_0) - x_0 \xi_0^1(x_0, y_0) + \frac{x_0^2}{2} \frac{\partial \xi_0^2(x_0, y_0)}{2} = 0 \end{gathered} \tag{3.19}$$

after splitting with respect to the free variable x_1 and substituting $y_1 = x_0^2/2$. Any operator

$$X = [\xi_0^1(x, y) + \varepsilon \xi_1^1(x, y)] \frac{\partial}{\partial x} + [\xi_0^2(x, y) + \varepsilon \xi_1^2(x, y)] \frac{\partial}{\partial y}$$

with coordinates satisfying (3.19) with $y_0 = 1$ and arbitrary values of x_0 generates an approximate group leaving (3.8) invariant (to within $o(\varepsilon)$). For example,

$$X_1 = x \frac{\partial}{\partial x} + 2(y - 1) \frac{\partial}{\partial y}, \qquad X_2 = xy \frac{\partial}{\partial x} + (y^2 - 1) \frac{\partial}{\partial y}$$

are such operators, along with (3.7).

REMARK 2. If some variables z^k do not enter in the equation $F(z, \varepsilon) \approx 0$, then it is unnecessary to represent z^k in the form $\sum_{i \geq 0} y_i^k \varepsilon^i$ in the defining equation (3.14).

§4. Approximate symmetries of the equation $u_{tt} + \varepsilon u_t = (\varphi(u) u_x)_x$

The approximate symmetries (understood either as admissible approximate groups or as their infinitesimal operators) of differential equations can be computed according to the algorithm in §3 with the use of the usual technique for prolongation of

([1])Here, as everywhere in this section, the following notation is used for brevity:

$$y_1^l \frac{\partial}{\partial z^l} \left(\xi_0^k(y_0) \frac{\partial F_0(y_0)}{\partial z^k} \right) \equiv \sum_{l=1}^{N} \sum_{k=1}^{N} y_1^l \frac{\partial}{\partial z^l} \left(\xi_0^k(z) \frac{\partial F_0(z)}{\partial z^k} \right) \Bigg|_{z=y_0}.$$

the infinitesimal operators by the necessary derivatives. Below, we consider approximate symmetries of first order ($p = 1$) and classify according to such symmetries the second-order equations

$$u_{tt} + \varepsilon u_t = (\varphi(u)u_x)_x, \qquad \varphi \neq \text{const}, \tag{4.1}$$

with a small parameter, which arise in various applied problems (see, for example, [4]). The infinitesimal operator of an approximate symmetry is sought in the form

$$X = (\xi_0^1 + \varepsilon\xi_1^1)\frac{\partial}{\partial t} + (\xi_0^2 + \varepsilon\xi_1^2)\frac{\partial}{\partial x} + (\eta_0 + \varepsilon\eta_1)\frac{\partial}{\partial u}. \tag{4.2}$$

The coordinates ξ and η of the operator (4.2) depend on t, x, and u and occur in the defining equations (3.16) and (3.17), in which

$$z = (t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}), \qquad F_0 = u_{tt} - (\varphi(u)u_x)_x, \qquad F_1 = u_t;$$

according to Remark 2 (see §3), it suffices to carry out the decomposition $z = y_0 + \varepsilon y_1$ only for the differentiable variable (since t and x do not appear explicitly in (4.1)): $u = u_0 + \varepsilon u_1$, $u_x = (u_0)_x + \varepsilon(u_1)_x$, and so on.

Equation (3.16) is the defining equation for the operator

$$X^0 = \xi_0^1\frac{\partial}{\partial t} + \xi_0^2\frac{\partial}{\partial x} + \eta_0\frac{\partial}{\partial u}, \tag{4.3}$$

admitted by the zero approximate of equation (4.1), i.e., by the equation

$$u_{tt} = (\varphi(u)u_x)_x, \qquad \varphi \neq \text{const}. \tag{4.4}$$

Consequently, the first step in the classification of the equations (4.1) according to approximate symmetries is the classification of the equations (4.4) according to exact symmetries. The second step is to solve the defining equation (3.17) with known F_0 and with values ξ_0^1, ξ_0^2, η_0 of the coordinates of the operator (4.3).

A group classification of the equations (4.4) (according to exact point symmetries) was obtained in [4], and its result can be written in the form of Table 1 with the use of dilations and translations.

TABLE 1

Group classification of the equations (4.4)

	$\varphi(u)$	ξ_0^1	ξ_0^2	η_0
	Arbitrary function	$C_1t + C_2$	$C_1x + C_3$	0
1	ku^σ	$C_1t + C_2$	$C_3x + C_4$	$\frac{2}{\sigma}(C_3 - C_1)u$
2	$ku^{-\frac{4}{3}}$	$C_1t + C_2$	$C_3x^2 + C_4x + C_5$	$-\frac{3}{2}(2C_3x + C_4 - C_1)u$
3	ku^{-4}	$C_1t^2 + C_2t + C_3$	$C_4x + C_5$	$\left(C_1t + \frac{C_2 - C_4}{2}\right)u$
4	ke^u	$C_1t + C_2$	$C_3x + C_4$	$2(C_3 - C_1)$
$k = \pm 1$, σ is an arbitrary parameter, and $C_1, \dots, C_5 = \text{const}$				

We now pass to the second step in constructing approximate symmetries. Let us first consider the case of an arbitrary function $\varphi(u)$. Substituting in (3.17) the values $\xi_0^1 = C_1t + C_2$, $\xi_0^2 = C_1x + C_3$, and $\eta_0 = 0$, we get that $C_1 = 0$, $\xi_1^1 = K_1t + K_2$, $\xi_1^2 = K_1x + K_3$, and $\eta_1 = 0$, where $K_i = \text{const}$. We now observe that equation

(4.1) admits together with any admissible (exactly or approximately) operator X also the operator εX; such operators will be assumed to be inessential and omitted. In particular, the operators $\varepsilon(\partial/\partial t)$ and $\varepsilon(\partial/\partial x)$ are inessential, so that the constants K_2 and K_3 in the solution of the defining equation (3.17) can be set equal to zero. Thus, for an arbitrary function $\varphi(u)$ equation (4.1) admits three essential approximate symmetry operators, corresponding to the constants C_2, C_3, and K_1. The remaining cases in Table 1 are analyzed similarly. The result is summarized in Table 2, where for convenience in comparing approximate symmetries with exact ones we have given the operators admitted by equations (4.4) exactly, and those admitted by (4.1) exactly and approximately.

TABLE 2
Comparative table of exact and approximate symmetries

	$\varphi(u)$	Operators for (4.4)	Operators for (4.1)	
			Exact symmetries	Approximate symmetries
	Arbitrary function	$X_1^0 = \frac{\partial}{\partial t}$, $X_2^0 = \frac{\partial}{\partial x}$ $X_3^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$	$Y_1 = X_1^0$ $Y_2 = X_2^0$	$X_1 = X_1^0$, $X_2 = X_2^0$ $X_3 = \varepsilon X_3^0$
1	ku^σ	$X_4^0 = \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$	$Y_3 = X_4^0$	$\tilde{X}_3 = X_3^0 + \frac{\varepsilon}{\sigma+4}\left(\frac{\sigma}{2}t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}\right)$ $X_4 = X_4^0$
2	$ku^{-4/3}$	$X_4^0 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}$ $X_5^0 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$	$Y_3 = X_4^0$ $Y_4 = X_5^0$	$\tilde{X}_3 = X_3^0 - \frac{\varepsilon}{4}\left(t^2\frac{\partial}{\partial t} + 3tu\frac{\partial}{\partial u}\right)$ $X_4 = X_4^0$, $X_5 = X_5^0$
3	ku^{-4}	$X_4^0 = 2x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$ $X_5^0 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}$	$Y_3 = X_4^0$ $Y_4 = X_5^0$	$X_4 = X_4^0$ $X_5 = \varepsilon X_5^0$
4	ke^u	$X_4^0 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}$	$Y_3 = X_4^0$	$\tilde{X}_3 = X_3^0 + \varepsilon\left(\frac{t^2}{2}\frac{\partial}{\partial t} - t\frac{\partial}{\partial u}\right)$ $X_4 = X_4^0$

NOTE. In Table 2 bases of the admitted algebras are given for the exact symmetries, and generators for them are given for the approximate symmetries: a basis for the corresponding algebra is obtained by multiplying the generators by ε and discarding the terms of order ε^2. For example, for $\varphi(u) = ku^{-4/3}$ equation (4.4) admits a 5-dimensional algebra, and (4.1) admits a 4-dimensional algebra of exact symmetries and a 10-dimensional algebra of approximate symmetries with basis

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad \tilde{X}_3 = \left(t - \frac{1}{4}\varepsilon t^2\right)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{3}{4}\varepsilon t u\frac{\partial}{\partial u},$$

$$X_4 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}, \quad X_5 = x^2\frac{\partial}{\partial x}3xu\frac{\partial}{\partial u}, \quad X_6 = \varepsilon X_1, \quad X_7 = \varepsilon X_2,$$

$$X_8 = \varepsilon\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right), \quad X_9 = \varepsilon X_4, \quad X_{10} = \varepsilon X_5.$$

§5. Approximate symmetries of the equation $u_t = h(u) = u_1 + \varepsilon H$

We consider the class of evolution equations of the form

$$u_t = h(u)u_1 + \varepsilon H, \qquad H \in \mathscr{A}, \tag{5.1}$$

which contains, in particular, the Korteweg-de Vries equation, the Burgers-Korteweg-de Vries equation, etc.

THEOREM 4. *Equation* (5.1) *approximately* (*with any degree of accuracy*) *inherits all the symmetries of the Hopf equation*

$$u_t = h(u)u_1. \tag{5.2}$$

Namely, any canonical Lie-Bäcklund operator [3] $X^0 = f^0(\partial/\partial u) + \cdots$, *admitted by* (5.2) *gives rise to an approximate* (*of arbitrary order* p) *symmetry for* (5.1) *determined by the coordinate*

$$f = \sum_{i=0}^{p} \varepsilon^i f^i, \qquad f^i \in \mathscr{A}, \tag{5.3}$$

of the canonical operator $X = f(\partial/\partial u) + \cdots$.

PROOF. The approximate symmetries (5.3) of equation (5.1) are found from the defining equation (3.14), which in this case takes the form

$$f_t^0 - h(u)f_x^0 + \sum_{\alpha\geq 0}[D^\alpha(hu_1) - hu_{1+\alpha}]f_\alpha^0 - h'(u)u_1 f^0 = 0, \tag{5.4}$$

$$f_t^i - h(u)f_x^i + \sum_{\alpha\geq 0}[D^\alpha(hu_1) - hu_{1+\alpha}]f_\alpha^i - h'(u)u_1 f^i = \sum_{\alpha\geq 0}[D^\alpha(f^{i-1})H_\alpha - f_\alpha^{i-1}D^\alpha(H)], \qquad i = 1, \ldots, p. \tag{5.5}$$

Equation (5.4) in f^0 is a defining equation for finding the exact group of transformations admitted by (5.2). Let f^0 be an arbitrary solution of (5.4) that is a differentiable function of order $k_0 \geq 0$, and let H be a differentiable function of order $n \geq 1$, i.e.,

$$f_0 = f^0(t, x, u, \ldots, u_{k_0}), \qquad H = H(t, x, u, \ldots, u_n).$$

We look for a solution f^1 of (5.5) in the form of a differentiable function of order $k_1 = n + k_0 - 1$. Then (5.5) is a linear first-order partial differential equation in the function f^1 of the $k_1 + 3$ arguments $t, x, u, u_1, \ldots, u_{k_1}$, and is hence solvable. Substitution of any solution $f^1(t, x, u, u_1, \ldots, u_{k_1})$ in the right-hand side of (5.5) with $i = 2$ shows that f^2 can be found in the form of a differentiable function of order $k_2 = n + k_1 - 1$, and the corresponding equation for f^2 is solvable. The rest of the coefficients f^i, $i = 3, \ldots, p$, in (5.3) are determined recursively from (5.5). The theorem is proved.

It follows from Theorem 4 that, in particular, any point symmetry of (5.2) determined by the infinitesimal operator

$$Y = \theta(t,x,u)\frac{\partial}{\partial t} + [\varphi(x+tu,u) - t\psi(x+tu,u) - u\theta(t,x,u)]\frac{\partial}{\partial x} + \psi(x+tu,u)\frac{\partial}{\partial u}$$

with arbitrary functions φ, ψ, and θ, or with the corresponding canonical Lie-Bäcklund operator with coordinate

$$f^0 = [\varphi(x+tu,u) - t\psi(x+tu,u)]u_1 - \psi(x+tu,u),$$

is approximately inherited by equation (5.1). For example, the Burgers-Korteweg-de Vries equation

$$u_t = uu_1 + \varepsilon(au_3 + bu_2) \tag{5.6}$$

to within $o(\varepsilon^2)$ admits the operator

$$\begin{aligned} f_u = \varphi(u)u_1 &+ \varepsilon(a\varphi' u_3 + 2a\varphi'' u_1u_2 + \tfrac{1}{2}a\varphi''' u_1^3 + b\varphi' u_2 + b\varphi'' u_1^2) \\ &+ \varepsilon^2(\tfrac{3}{5}a^2\varphi'' u_5 + \tfrac{5}{4}ab\varphi'' u_4 + \tfrac{1}{10}ab\varphi''\tfrac{u_2u_3}{u_1} - \tfrac{1}{20}ab\varphi''\tfrac{u_2^3}{u_1^2} + \tfrac{2}{3}b^2\varphi'' u_3 \\ &\quad + \tfrac{9}{5}a^2\varphi''' u_1u_4 + 3a^2\varphi''' u_2u_3 + \tfrac{7}{2}ab\varphi''' u_1u_3 + \tfrac{23}{10}ab\varphi''' u_2^2 \\ &\quad + \tfrac{5}{3}b^2\varphi''' u_1u_2 + \tfrac{23}{10}a^2\alpha^{\mathrm{IV}} u_1^2u_3 + \tfrac{31}{10}a^2\varphi^{\mathrm{IV}} u_1u_2^2 + \tfrac{15}{4}ab\varphi^{\mathrm{IV}} u_1^2u_2 \\ &\quad + \tfrac{1}{2}b^2\varphi^{\mathrm{IV}} u_1^3 + \tfrac{8}{5}a^2\varphi^{\mathrm{V}} u_1^3u_2 + \tfrac{1}{2}ab\varphi^{\mathrm{V}} u_1^4 + \tfrac{1}{8}a^2\varphi^{\mathrm{VI}} u_1^5) + o(\varepsilon^2). \end{aligned} \tag{5.7}$$

Setting $a = 1$ and $b = 0$ in (5.7), we get an approximate symmetry of second order for the Korteweg-de Vries equation

$$u_t = uu_1 + \varepsilon u_3. \tag{5.8}$$

We remark that in this case the coefficient f^k of the approximate symmetry (5.3) is a differentiable function of order $2k+1$ containing derivatives of φ of order $\geq k$. This implies that if $\varphi(u)$ is a polynomial, then the approximate symmetry becomes an exact Lie-Bäcklund symmetry; then we can set $\varepsilon = 1$ and get exact symmetries of the equation

$$u_t = u_3 + uu_1. \tag{5.9}$$

For example, for $p = 2$ and $\varphi = u^2$ we get that $f_u = u^2u_1 + 4u_1u_2 + 2uu_3 + \frac{6}{5}u_5$ by (5.7) (cf. [3], §18.2).

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