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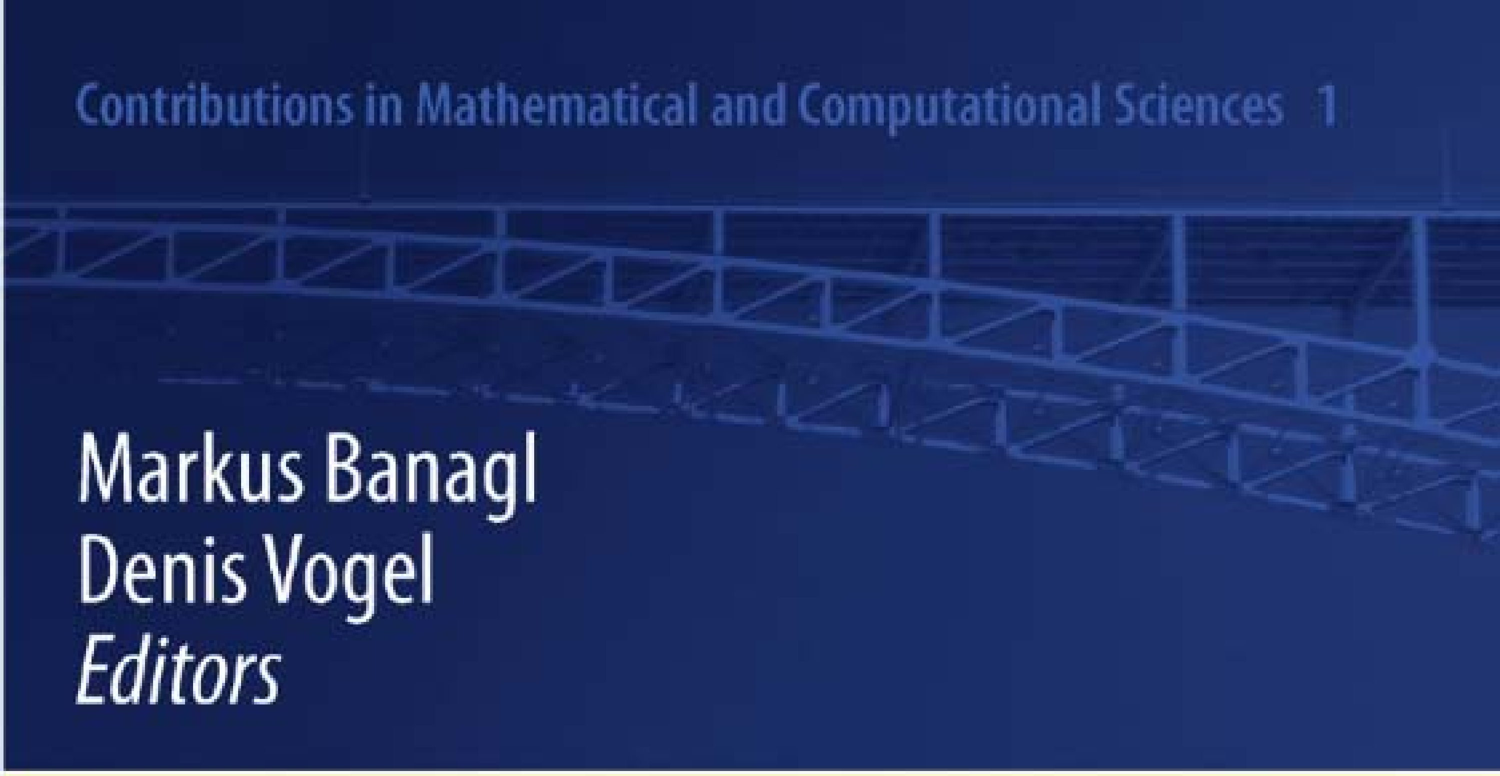

Markus Banagl
Denis Vogel
Editors

The Mathematics of Knots

Theory and Application

MATCH
IWR
Springer

Contributions in Mathematical and Computational Sciences • Volume 1

Editors
Hans Georg Bock
Willi Jäger

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Editors

The Mathematics of Knots

Theory and Application

Springer

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ISBN 978-3-642-15636-6 e-ISBN 978-3-642-15637-3
DOI 10.1007/978-3-642-15637-3
Springer Heidelberg Dordrecht London New York

Mathematics Subject Classification (2010): 57M25, 57Q45, 92C37, 81T45

Cover design: deblik

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface to the Series

Contributions in Mathematical and Computational Sciences

Mathematical theories and methods and effective computational algorithms are crucial in coping with the challenges arising in the sciences and in many areas of their application. New concepts and approaches are necessary in order to overcome the complexity barriers particularly created by nonlinearity, high-dimensionality, multiple scales and uncertainty. Combining advanced mathematical and computational methods and computer technology is an essential key to achieving progress, often even in purely theoretical research.

The term mathematical sciences refers to mathematics and its genuine sub-fields, as well as to scientific disciplines that are based on mathematical concepts and methods, including sub-fields of the natural and life sciences, the engineering and social sciences and recently also of the humanities. It is a major aim of this series to integrate the different sub-fields within mathematics and the computational sciences, and to build bridges to all academic disciplines, to industry and other fields of society, where mathematical and computational methods are necessary tools for progress. Fundamental and application-oriented research will be covered in proper balance.

The series will further offer contributions on areas at the frontier of research, providing both detailed information on topical research, as well as surveys of the state-of-the-art in a manner not usually possible in standard journal publications. Its volumes are intended to cover themes involving more than just a single "spectral line" of the rich spectrum of mathematical and computational research.

The Mathematics Center Heidelberg (MATCH) and the Interdisciplinary Center for Scientific Computing (IWR) with its Heidelberg Graduate School of Mathematical and Computational Methods for the Sciences (HGS) are in charge of providing and preparing the material for publication. A substantial part of the material will be acquired in workshops and symposia organized by these institutions in topical areas of research. The resulting volumes should be more than just proceedings collecting papers submitted in advance. The exchange of information and the discussions during the meetings should also have a substantial influence on the contributions.

Starting this series is a venture posing challenges to all partners involved. A unique style attracting a larger audience beyond the group of experts in the subject areas of specific volumes will have to be developed.

The first volume covers the mathematics of knots in theory and application, a field that appears excellently suited for the start of the series. Furthermore, due to the role that famous mathematicians in Heidelberg like Herbert Seifert (1907–1996) played in the development of topology in general and knot theory in particular, Heidelberg seemed a fitting place to host the special activities underlying this volume.

Springer Verlag deserves our special appreciation for its most efficient support in structuring and initiating this series.

Heidelberg University, Germany

Willi Jäger
Hans Georg Bock

Preface

This volume is based on the themes of, and records advances achieved as a result of, the *Heidelberg Knot Theory Semester*, held in winter 2008/09 at Heidelberg University under the sponsorship of the Mathematics Center Heidelberg (MATCH), organized by M. Banagl and D. Vogel. In the preceding summer semester an introductory seminar on knots aimed at providing non-experts and young mathematicians with some of the foundational knowledge required to participate in the events of the winter semester. These comprised expository lecture series by several leading experts, representing rather diverse aspects of knot theory and its applications, and a concluding workshop held December 15 to 19, 2008.

Knots seem to be a deep structure, whose peculiar feature it is to surface unexpectedly in many different and a priori unrelated areas of mathematics and the natural sciences, such as algebra and number theory, topology and geometry, analysis, mathematical physics (in particular statistical mechanics), and molecular biology. Its relevance in topology, apart from its intrinsic interest, is partly due to the fact that every closed, oriented 3-manifold can be obtained by surgery on a framed link in the 3-sphere. Modern topology has also obtained information on high-dimensional knots, that is, embeddings of an n-sphere in an $(n+2)$-sphere with n larger than one. In algebra, representations of quantum groups lead to a multitude of knot invariants. Based on ideas of B. Mazur in number theory, one can assign to two prime ideals of a number field a linking number in analogy with classical knot theory. This number-theoretic linking number plays a role in studying the structure of Galois groups of certain extensions of the number field. Analysis touches on knot theory by means of operator algebras and their connection to the Jones polynomial. As far as geometry is concerned, results by Fenchel on the curvature of a closed space curve date back to the 1920s. Milnor showed in 1949 that the curvature must exceed 4π if the curve is knotted. One also considers "real" knots as physical objects in 3-space and studies various natural energy functionals on them. Sums taken over all states of suitable models originating in statistical mechanics, describing large ensembles of particles, can express knot invariants such as L. Kauffman's bracket polynomial. The discovery of the Jones polynomial entailed ties with mathematical physics based on a curious congruity of five relations, namely the Artin-relation in braid groups, a fundamental relation in certain operator algebras due to Hecke, the third Reidemeister move, the classical Yang-Baxter equation, and its quantum version. This lead to the construction of topological quantum field theories by Witten and Atiyah. Cellular DNA is a long molecule, which may be closed (as e.g. the genome of certain bacteria) and knotted or linked with other DNA strands. Enzymes such as topoisomerase or recombinase operate on DNA changing the topological knot or link type.

The objective of the Heidelberg Knot Theory Semester was to do justice to this diversity by bringing together representatives of most of the above research avenues, accompanied by the hope that such a meeting might foster inspiration and synergy across the various questions and approaches. Certainly, a fairly comprehensive portrait of the current state-of-the-art in knot theory and its applications emerged as a result.

Four lecture series were given: DeWitt Sumners gave 5 lectures on scientific applications of knot theory, discussing DNA topology, a tangle model for DNA site-specific recombination, random knotting, topoisomerase, spiral waves and viral DNA packing. Kent Orr's 3 lectures explained knot concordance and surgery techniques, while Louis Kauffman's 2 lectures introduced virtual knots and detailed parallels to elementary particles. The topic of Masanori Morishita's 6 lectures were the aforementioned analogies between knot theory and number theory.

The 21 speakers of the final workshop "The Mathematics of Knots" reported on a variety of interesting current developments. Many of these accounts are mirrored in the papers of the present volume. Among the low-dimensional topics were virtual knots and associated invariants such as arrow and Jones polynomials, the HOMFLY polynomial, questions about Dehn filling, Legendrian knots, Khovanov homology, surface knots, slice knots, fibered knots and property R, colorings by metabelian groups, singular knots, Gram determinants of planar curves, as well as geometric structures such as surfaces associated with knots, and the fibering of 3-manifolds when the product of the manifold with a circle is known to be symplectic. High-dimensional topics concerned the Cohn noncommutative localization of rings and its application to knots via algebraic K- and L-theory, as well as high-dimensional non-locally flat embeddings and the role of knot theory vis-à-vis transformation groups. Scientific talks discussed random knotting, viral DNA packing, and the topology of DNA-protein interactions.

We wish to extend our sincere thanks to the contributors of this volume and to all participants of the Heidelberg Knot Theory Semester, especially to the lecturers giving mini-courses, for the energy and time they have devoted to this event and the preparation of the present collection. Paul Seyfert receives the editors' thanks for technical help in typesetting this volume. Furthermore, we are grateful to Dorothea Heukäufer for her efficient handling of numerous logistical issues. Finally, we would like to express our gratitude to Willi Jäger and MATCH, whose financial support made the Heidelberg Knot Theory Semester possible.

Heidelberg University, Germany

Markus Banagl
Denis Vogel

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Chapter 1
Knots, Singular Embeddings, and Monodromy

Markus Banagl, Sylvain E. Cappell,
and Julius L. Shaneson

Abstract The Goresky-MacPherson L-class of a PL pseudomanifold piecewise-linearly embedded in a PL manifold in a possibly nonlocally flat way, can be computed in terms of the Hirzebruch-Thom L-class of the manifold and twisted L-classes associated to the singularities of the embedding, as was shown by Cappell and Shaneson. These formulae are refined here by analyzing the twisted classes. We treat the case of Blanchfield local systems that extend into the singularities as well as cases where they do not extend. In the latter situation, we consider fibered embeddings of strata and 4-dimensional singular sets, using work of Banagl. Rho-invariants enter the picture.

1.1 Introduction

Let M^{n+2} be a closed, oriented, connected PL manifold of dimension $n+2$ and X^n a closed, oriented, connected PL pseudomanifold of dimension n. Let $i : X \hookrightarrow M$ be a

The first author was partially supported by a grant of the Deutsche Forschungsgemeinschaft. The second and third authors were partially supported by grants of the Defense Advanced Research Projects Agency.

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_1,

not necessarily locally flat PL embedding. Let $L^* = 1 + L^1(p_1) + L^2(p_1, p_2) + \cdots$ be the total Hirzebruch L-polynomial,

$$L^1(p_1) = \frac{1}{3}p_1, \qquad L^2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2), \ldots .$$

Let $P(M) \in H^*(M; \mathbb{Z})$ be the total Pontrjagin class of M and the Euler class $\chi \in H^2(M; \mathbb{Z})$ be the Poincaré dual of $i_*[X] \in H_n(M; \mathbb{Z})$, where $[X]$ is the fundamental class of X. Set

$$L_*(M, X) = [X] \cap i^* L^*(P(M) \cup (1 + \chi^2)^{-1}) \in H_*(X; \mathbb{Q}).$$

Recall that the sequence $L^1, L^2, \ldots$ of polynomials is the multiplicative sequence associated to the even power series defined by $x/\tanh(x)$. Thus

$$L^*(1 + \chi^2) = \frac{\chi}{\tanh(\chi)} = 1 + \frac{1}{3}\chi^2 - \frac{1}{45}\chi^4 \pm \cdots$$

and by the multiplicativity of $\{L^j\}$,

$$L^*((1 + \chi^2)^{-1}) = \frac{\tanh(\chi)}{\chi} = 1 - \frac{1}{3}\chi^2 + \frac{2}{15}\chi^4 \mp \cdots .$$

Hence the above defining expression for $L_*(M, X)$ may alternatively be written as

$$\begin{aligned} L_*(M, X) &= [X] \cap \left(\frac{\tanh i^*\chi}{i^*\chi} \cup i^* L^*(PM) \right) \\ &= [X] \cap \left(\left(1 - \frac{1}{3} i^*\chi^2 + \frac{2}{15} i^*\chi^4 \mp \cdots \right) \cup i^* L^*(PM) \right). \end{aligned}$$

When this formula is pushed on into M, one obtains

$$\begin{aligned} i_* L_*(M, X) &= i_* \left([X] \cap i^* \left(\frac{\tanh \chi}{\chi} \cup L^*(PM) \right) \right) \\ &= i_*[X] \cap \left(\frac{\tanh \chi}{\chi} \cup L^*(PM) \right) \\ &= ([M] \cap \chi) \cap \left(\frac{\tanh \chi}{\chi} \cup L^*(PM) \right) \\ &= [M] \cap (\tanh \chi \cup L^*(PM)). \end{aligned}$$

If the embedding is nonsingular, that is, X is a locally flat submanifold, then

$$L_*(X) = L_*(M, X),$$

where $L_*(X)$ is the Poincaré dual of the Hirzebruch L-class of X. In particular, the signature $\sigma(X) = L_0(X)$ is given by

$$\sigma(X) = L_0(M, X).$$

If the embedding is singular, the singularities of X and the singularities of the embedding induce a stratification of the pair (M, X). Under the assumption that there are no strata of odd codimension, it was shown in [CS91] that the Goresky-MacPherson L-class $L_*(X) \in H_*(X; \mathbb{Q})$ of X, defined using middle-perversity intersection homology, can be computed as

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*} L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}}), \tag{1.1}$$

where the sum ranges over all connected components V of pure strata of X that have codimension at least two, $i_V : \overline{V} \hookrightarrow X$ is the inclusion of the closure $\overline{V}$ of V into X and $L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) \in H_*(\overline{V}; \mathbb{Q})$ is the Goresky-MacPherson L-class of $\overline{V}$ twisted by a local coefficient system $\mathcal{B}_V^{\mathbb{R}}$. This local system is endowed with a nonsingular symmetric or skew-symmetric form $\mathcal{B}_V^{\mathbb{R}} \otimes \mathcal{B}_V^{\mathbb{R}} \to \mathbb{R}$ and arises as Trotter's "scalar product" [Tro73] of a certain Blanchfield local system $\mathcal{B}_V \otimes \mathcal{B}_V^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$, $\Lambda = \mathbb{Q}[t, t^{-1}]$. The systems are defined on V and do not in general extend as local systems to the closure $\overline{V}$. They do, of course, extend as intersection chain sheaves by applying Deligne's pushforward/truncation-formula to $\mathcal{B}_V^{\mathbb{R}}$, and $L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$ is defined as the L-class of this self-dual sheaf complex on $\overline{V}$. (For an introduction to the L-class of self-dual sheaves see [Ban07].)

In the present paper, we refine formula (1.1) by computing the twisted classes $L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$ further. Two cases are to be distinguished: The systems $\mathcal{B}_V^{\mathbb{R}}$ either extend as local systems from V to $\overline{V}$ or they do not. In the former situation, the results of [BCS03] apply and yield the formula (Theorem 6)

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*}(\widetilde{\mathrm{ch}}[\mathcal{B}_V^{\mathbb{R}}]_K \cap L_*(\overline{V})), \tag{1.2}$$

where the modification $\widetilde{\mathrm{ch}}$ of the Chern character is given by precomposing with the second Adams operation, $\widetilde{\mathrm{ch}} = \mathrm{ch} \circ \psi^2$ and $[\mathcal{B}_V^{\mathbb{R}}]_K$ denotes the K-theory signature of $\mathcal{B}_V^{\mathbb{R}}$, an element of $\mathrm{KO}(X)$ if the form on $\mathcal{B}_V^{\mathbb{R}}$ is symmetric, and of $\mathrm{KU}(X)$ if it is skew-symmetric. In the situation of nonextendable systems, formulae of type (1.2), even when the right hand side is defined, cease to hold as counterexamples of [Ban08] show. The main results presented here, then, are concerned with understanding the twisted signatures $\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$ when $\mathcal{B}_V^{\mathbb{R}}$ does not extend as a local system into the singularities of $\overline{V}$. Theorem 10 asserts that

$$\sigma(X) = L_0(M, X)$$

when all embeddings $\overline{V} - V \hookrightarrow \overline{V}$ are locally flat spherical fibered knots. In particular if $M = S^{n+2}$ is a sphere, we have $\sigma(X) = 0$, since $L_0(S^{n+2}, X) = 0$. The remaining results all assume that $i : X \hookrightarrow M$ has a 4-dimensional singular set such that the $\overline{V}$ are 4-manifolds and the bottom stratum consists of locally flat 2-spheres (see Examples 1 and 2). If the 2-spheres have zero self-intersection numbers and $\mathcal{B}_V^{\mathbb{R}}$ is positive ($\epsilon_V = 1$) or negative ($\epsilon_V = -1$) definite of rank r_V, then

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}),$$

with V ranging over all connected components V of the pure 4-stratum (Theorem 7). Again we obtain a corollary for the case where M is a sphere:

$$\sigma(X) + \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}) = 0.$$

Similar corollaries for embeddings in spheres can be deduced for the following results as well.

More generally, if the structure group of the form on V is $O(p_V, q_V)$, then

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} (p_V - q_V)\sigma(\overline{V}) - \sum_{V \subset X_4 - X_2} \langle 2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}}), [\overline{V}]\rangle,$$

where $2(c_1^2 - 2c_2)$ is an $H^4(\overline{V}; \mathbb{Z})$-valued characteristic class (Theorem 8). As a corollary (Corollary 4) we deduce that $\sigma(X) - L_0(M, X)$ is divisible by 8 if every $\overline{V}$ is a 4-sphere. When the 2-spheres have nonzero self-intersection numbers, then rho-invariants enter. Theorem 9 for positive, say, definite forms asserts that

$$\begin{aligned}\sigma(X) &= L_0(M, X) \\ &\quad - \sum_{V \subset X_4 - X_2} \left(r_V \sigma(V) + \sum_{i=1}^{n_V} (\text{c-rk}(\mathcal{B}_V^{\mathbb{C}}|_{L_i}) \operatorname{sign}[S_i^2]^2 - \rho_{\alpha_i}(p_i, q_i)) \right),\end{aligned}$$

where $\sigma(V)$ denotes the (Novikov-) signature of the exterior of the 2-spheres

$$\bigsqcup_{i=1}^{n_V} S_i^2 \subset \overline{V},$$

$L_i = L(p_i, q_i)$, a lens space, is the boundary of a regular neighborhood of S_i^2 in $\overline{V}$, and α_i is obtained by restricting $\mathcal{B}_V^{\mathbb{C}}$ to L_i. The function $\rho_\alpha(p, q)$ is given by an explicit formula, see Sect. 1.9, p. 26, where the constancy-rank c-rk($\mathcal{S}$) of a local system $\mathcal{S}$ is defined as well.

Organization Section 1.2 reviews Blanchfield forms and their relation to Seifert manifolds. Fundamental results of Levine, Trotter, Kearton and Kervaire are recalled. Blanchfield and Poincaré local coefficient systems are defined. In Sect. 1.3 we review the Trotter trace $T : \mathbb{Q}(t)/\Lambda \to \mathbb{Q}$, which allows us to pass from Blanchfield local systems to Poincaré local systems. An important point here is that this passage reverses symmetry properties: if the Blanchfield form is Hermitian, then the real Poincaré form is skew-symmetric and if the Blanchfield form is skew-Hermitian, then the Poincaré form is symmetric. Section 1.4 serves mainly to set up notation concerning the complexification of real sheaf complexes, forms, etc. Various characterizations of extendability of a local system from the top stratum of a stratified pseudomanifold into the singular strata are discussed in Sect. 1.5. The K-theory signature of a Poincaré local system is recalled. Section 1.6 reviews the twisted L-class formula of [BCS03]. The Cappell-Shaneson L-class formula for singular embeddings, [CS91], is discussed in Sect. 1.7, where details on the stratification induced by a singular embedding, together with an example, are also to be found. Embeddings are always assumed to induce only strata of even codimension and to be of finite local type and of finite type. The final two sections contain the results of this paper; Sect. 1.8 for local systems that extend and Sect. 1.9 for systems that do not extend.

1.2 Blanchfield and Poincaré Local Systems

Let R be a Dedekind domain, for example $R = \Lambda = \mathbb{Q}[t, t^{-1}]$, the ring of Laurent polynomials. Let F be the quotient field of R and let A and B be finitely generated torsion R-modules. A pairing

$$A \otimes_R B \longrightarrow F/R$$

is called *perfect*, if the induced map

$$A \longrightarrow \mathrm{Hom}_R(B, F/R)$$

is an isomorphism. Suppose R is equipped with an involution $r \mapsto \bar{r}$. Then B^{op} will denote the R-module obtained by composing the module structure of B with the involution. A pairing

$$\beta : B \otimes_R B^{\mathrm{op}} \longrightarrow F/R$$

is called *Hermitian* if

$$\beta(a \otimes b) = \beta(b \otimes a)^{-},$$

and *skew-Hermitian* if

$$\beta(a \otimes b) = -\beta(b \otimes a)^{-}.$$

We will be primarily concerned with the ring $R = \Lambda$ of Laurent polynomials. For this ring, the quotient field F is $F = \mathbb{Q}(t)$, the rational functions. The involution on R is given by replacing t with t^{-1}.

Definition 1 An (abstract) *Blanchfield pairing* is a perfect Hermitian or skew-Hermitian pairing

$$B \otimes_\Lambda B^{\mathrm{op}} \longrightarrow \mathbb{Q}(t)/\Lambda,$$

where B is a finitely generated torsion Λ-module.

A locally flat knot $S^{2n-1} \subset S^{2n+1}$ possesses two related kinds of abelian invariants associated to the map of the knot group to its abelianization $\mathbb{Z}$: those arising from the infinite cyclic cover of the knot exterior and those arising from choices of Seifert manifolds. For later reference, we recall here some well-known facts about the relation between the Blanchfield pairing and Seifert matrices. A *Seifert manifold* for the knot is a codimension 1 framed compact submanifold of S^{2n+1} whose boundary is the knot. Every (locally flat) knot has a Seifert manifold. Let K be the exterior of the knot. The knot is *simple*, if $\pi_i(K) \cong \pi_i(S^1)$ for $1 \leq i < n$. The $(2n-1)$-knot is simple if and only if it bounds an $(n-1)$-connected Seifert manifold, [Lev65, Theorem 2]. A choice of Seifert manifold M^{2n} together with a choice of basis $\{b_i\}$ for the torsion-free part of $H_n(M)$ determines a Seifert matrix A by defining the (i, j)-entry to be the linking number of a cycle representing b_i with a translate in the positive normal direction to M of a cycle representing b_j. Any such A has the property that $A + (-1)^n A^T$ is unimodular. In fact, $A + (-1)^n A^T$ is the matrix of the intersection form of M. Two square integral matrices are *S-equivalent* if they can be obtained from each other by a finite sequence of elementary enlargements, reductions and unimodular congruences. An *elementary enlargement* of A is any matrix of the form

$$\begin{pmatrix} A & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & \beta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where α is a row vector and β is a column vector. A matrix is an *elementary reduction* of any of its elementary enlargements.

Theorem 1 (Levine [Lev70]) *Seifert matrices of isotopic knots of any odd dimension are S-equivalent.*

Trotter [Tro73] abstractly calls a square integral matrix A with $A + A^T$ or $A - A^T$ unimodular a *Seifert matrix*. Such an A must be even dimensional. Any Seifert matrix A determines a $\mathbb{Z}[t, t^{-1}]$-module B_A presented by the matrix $tA + (-1)^n A^T$. The determinant of the latter matrix is the *Alexander polynomial* of the knot,

$$\Delta(t) = \det(tA + (-1)^n A^T),$$

defined up to multiplication with a unit of $\mathbb{Z}[t, t^{-1}]$. Moreover, A determines a non-singular $(-1)^{n+1}$-Hermitian pairing

$$\beta_A : B_A \otimes B_A^{\mathrm{op}} \longrightarrow F/\mathbb{Z}[t, t^{-1}],$$

given by the matrix $(1-t)(tA + (-1)^n A^T)^{-1}$, where F is the field of fractions of $\mathbb{Z}[t, t^{-1}]$.

Theorem 2 (Trotter [Tro73]) *If A_1 and A_2 are S-equivalent Seifert matrices, then there is an isometry $(B_{A_1}, \beta_{A_1}) \cong (B_{A_2}, \beta_{A_2})$.*

Thus (B_A, β_A) is an invariant of the knot; B_A is called the *knot module* of the knot. Let K_∞ be the infinite cyclic cover of the exterior K of the knot. The homology group $H_n(K_\infty)$ is a $\mathbb{Z}[t, t^{-1}]$-module via the action of the Deck-transformation group, generated by t. Assume that the knot is simple. The Blanchfield pairing

$$b : H_n(K_\infty) \otimes H_n(K_\infty)^{\mathrm{op}} \longrightarrow F/\mathbb{Z}[t, t^{-1}]$$

is nonsingular and $(-1)^{n+1}$-Hermitian.

Theorem 3 (Kearton [Kea73]) *If A is any Seifert matrix of a simple knot, then there is an isometry $(B_A, \beta_A) \cong (H_n(K_\infty), b)$.*

Particularly agreeable representatives of S-equivalence classes are provided by the following result.

Proposition 1 (Trotter [Tro73]) *Any Seifert matrix is S-equivalent to a nonsingular matrix.*

If A is a nonsingular Seifert matrix, then the $\mathbb{Q}$-vector space $B_A \otimes_{\mathbb{Z}} \mathbb{Q}$ has dimension

$$\dim_{\mathbb{Q}}(B_A \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{rk} A. \tag{1.3}$$

We conclude this review with a geometric realization result due to Kervaire.

Theorem 4 (Kervaire [Ker65]) *Let $n > 2$ be an integer and A a square integral matrix such that $A + (-1)^n A^T$ is unimodular. Then there exists a simple locally flat $(2n-1)$-knot with Seifert matrix A.*

Let $(X^n, \partial X)$ be a pseudomanifold with (possibly empty) boundary and filtration

$$X^n = X_n \supset X_{n-2} \supset X_{n-3} \supset \cdots \supset X_0 \supset \varnothing,$$

where the strata are indexed by dimension, the $X_i \cap \partial X$ stratify ∂X, and the $X_i - \partial X$ stratify $X - \partial X$; $\Sigma = X_{n-2}$ is the singular set. For a ring R, let R_X denote the constant sheaf with stalk R on X.

Definition 2 A *Blanchfield local system on X* is a locally constant sheaf $\mathcal{B}$ on X together with a pairing

$$\beta : \mathcal{B} \otimes \mathcal{B}^{\mathrm{op}} \longrightarrow (\mathbb{Q}(t)/\Lambda)_X,$$

such that for every $x \in X$, the stalk $\mathcal{B}_x$ is a finitely generated torsion Λ-module and the restriction

$$\beta_x : \mathcal{B}_x \otimes \mathcal{B}_x^{\mathrm{op}} \longrightarrow \mathbb{Q}(t)/\Lambda$$

is a Blanchfield pairing.

Definition 3 A *Poincaré local system on* X is a locally constant sheaf $\mathcal{P}$ on X together with a pairing

$$\phi : \mathcal{P} \otimes \mathcal{P} \to \mathbb{R}_X,$$

such that for every $x \in X$, the stalk $\mathcal{P}_x$ is a finite dimensional real vector space and the restriction

$$\phi_x : \mathcal{P}_x \otimes \mathcal{P}_x \to \mathbb{R}$$

is perfect and either symmetric or skew-symmetric.

1.3 Passage from Blanchfield Systems to Poincaré Systems

The method of partial fraction decomposition enables us to write any rational function $f \in \mathbb{Q}(t)$ uniquely in the form

$$f(t) = p(t) + \sum_{i=1}^{k} \frac{A_i}{t^i} + g(t),$$

where $p \in \mathbb{Q}[t]$, $A_i \in \mathbb{Q}$, and

$$g(t) = \sum_{j=1}^{l} \sum_{i=1}^{k_j} \frac{p_{i,j}(t)}{q_j(t)^i},$$

$p_{i,j}, q_j \in \mathbb{Q}[t]$, the q_j are distinct, irreducible, and prime to t, $\deg p_{i,j} < i \deg q_j$. Since t does not divide q_j, $q_j(0) \neq 0$ and thus $g(0) \in \mathbb{Q}$ is defined. The *Trotter trace* $T : \mathbb{Q}(t) \to \mathbb{Q}$ is the $\mathbb{Q}$-linear map

$$T(f) = g(0).$$

If $f \in \Lambda \subset \mathbb{Q}(t)$, then $g = 0$ and so $T(f) = 0$. Thus T passes from $\mathbb{Q}(t)$ to the quotient $\mathbb{Q}(t)/\Lambda$, $T : \mathbb{Q}(t)/\Lambda \to \mathbb{Q}$.

Let B be a Λ-module. By restricting the coefficients to the subring $\mathbb{Q} \subset \Lambda$, we may regard B as a $\mathbb{Q}$-vector space $B^{\mathbb{Q}}$. If B is finitely generated and torsion, then $B^{\mathbb{Q}}$ is finite dimensional. Using the standard embedding $\mathbb{Q} \subset \mathbb{R}$, we define the real vector space $B^{\mathbb{R}} = B^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. Let $\beta : B \otimes_{\Lambda} B^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$ be a Blanchfield pairing. Define

$$\beta^{\mathbb{R}} : B^{\mathbb{R}} \otimes_{\mathbb{R}} B^{\mathbb{R}} \longrightarrow \mathbb{R}$$

by

$$\beta^{\mathbb{R}}((a \otimes_{\mathbb{Q}} \lambda) \otimes_{\mathbb{R}} (b \otimes_{\mathbb{Q}} \mu)) = \lambda\mu T\beta(a \otimes_{\Lambda} b),$$

$a, b \in B$, $\lambda, \mu \in \mathbb{R}$. Then $\beta^{\mathbb{R}}$ is a perfect pairing on $B^{\mathbb{R}}$. The passage from β to $\beta^{\mathbb{R}}$ reverses symmetry properties: if β is Hermitian, then $\beta^{\mathbb{R}}$ is skew-symmetric and if

β is skew-Hermitian, then $\beta^{\mathbb{R}}$ is symmetric. We denote the signature of this pairing by $\sigma(\beta^{\mathbb{R}})$; it is zero in the skew-symmetric case.

If $B = H_n(K_\infty)$ is the knot module of a locally flat knot $S^{2n-1} \subset S^{2n+1}$ and β the Blanchfield pairing for this knot, then the signature of $\beta^{\mathbb{R}}$ can be computed as

$$\sigma(\beta^{\mathbb{R}}) = \sigma(M^{2n}), \tag{1.4}$$

where M^{2n} is any Seifert manifold for the knot and $\sigma(M^{2n})$ denotes the (Novikov-) signature of its intersection form, see [CS91]. Note also that

$$\sigma(M^{2n}) = \sigma(A + (-1)^n A^T), \tag{1.5}$$

where A is the corresponding Seifert matrix, since $A + (-1)^n A^T$ is a matrix representation of the intersection form.

If $\mathcal{B}$ is a local system of Λ-modules on a space X, then $\mathcal{B}^{\mathbb{R}} = \mathcal{B}^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, where $\mathcal{B}^{\mathbb{Q}}$ is the local system of $\mathbb{Q}$-vector spaces with stalks $(\mathcal{B}^{\mathbb{Q}})_x = (\mathcal{B}_x)^{\mathbb{Q}}$ obtained from restricting coefficients to $\mathbb{Q}$. Let $\beta : \mathcal{B} \otimes_\Lambda \mathcal{B}^{\mathrm{op}} \to (\mathbb{Q}(t)/\Lambda)_X$ be a Blanchfield local system on a pseudomanifold X. Then the pairings $(\beta_x)^{\mathbb{R}} : (\mathcal{B}_x)^{\mathbb{R}} \otimes_{\mathbb{R}} (\mathcal{B}_x)^{\mathbb{R}} \to \mathbb{R}$ define a Poincaré local system

$$\beta^{\mathbb{R}} : \mathcal{B}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{B}^{\mathbb{R}} \longrightarrow \mathbb{R}_X.$$

1.4 Passage from Poincaré Systems to Complex Hermitian Systems

Given a real vector space V of dimension n, let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote its complexification, a complex vector space of dimension n. For example, $\mathbb{R}^n_{\mathbb{C}} = \mathbb{C}^n$. Taking complex conjugation as the involution to be composed with the scalar multiplication, we get the complex vector space $V_{\mathbb{C}}^{\mathrm{op}}$. In fact, $V_{\mathbb{C}}^{\mathrm{op}} = V \otimes_{\mathbb{R}} (\mathbb{C}^{\mathrm{op}})$. If $B = \{v_1, \dots, v_n\}$ is a basis for V, then $B_{\mathbb{C}} = \{v_1 \otimes 1, \dots, v_n \otimes 1\}$ is a basis for $V_{\mathbb{C}}$. As regards pairings, let us concentrate on the symmetric case, the skew-symmetric case is treated in a similar way. To a symmetric perfect pairing $\phi : V \otimes V \to \mathbb{R}$, we can associate a Hermitian perfect pairing $\phi_{\mathbb{C}} : V_{\mathbb{C}} \otimes V_{\mathbb{C}}^{\mathrm{op}} \to \mathbb{C}$ by setting

$$\phi_{\mathbb{C}}((v \otimes \lambda) \otimes (w \otimes \mu)) = \lambda\overline{\mu}\phi(v, w),$$

$v, w \in V$, $\lambda, \mu \in \mathbb{C}$. The canonical example is $\gamma : \mathbb{R}^{p+q} \otimes \mathbb{R}^{p+q} \to \mathbb{R}$ given by

$$\begin{aligned} &\gamma((x_1, \dots, x_{p+q}) \otimes (y_1, \dots, y_{p+q})) \\ &\quad = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}. \end{aligned}$$

For this pairing, $\gamma_{\mathbb{C}} : \mathbb{C}^{p+q} \otimes (\mathbb{C}^{p+q})^{\mathrm{op}} \to \mathbb{C}$ is

$$\begin{aligned} &\gamma_{\mathbb{C}}((z_1, \dots, z_{p+q}) \otimes (u_1, \dots, u_{p+q})) \\ &\quad = z_1 \overline{u}_1 + \cdots + z_p \overline{u}_p - z_{p+1} \overline{u}_{p+1} - \cdots - z_{p+q} \overline{u}_{p+q}. \end{aligned}$$

If $\mathrm{Mat}_B(\phi) = (\phi(v_i \otimes v_j))$ denotes the matrix representation of ϕ with respect to the basis B, then the matrix representation of $\phi_{\mathbb{C}}$ with respect to $B_{\mathbb{C}}$ is simply

$$\mathrm{Mat}_{B_{\mathbb{C}}}(\phi_{\mathbb{C}}) = \mathrm{Mat}_B(\phi),$$

viewed as a complex matrix. This is a Hermitian matrix because it is real and symmetric. In fact, we can choose B so that $\mathrm{Mat}_B(\phi)$ is diagonal. Then $\mathrm{Mat}_{B_{\mathbb{C}}}(\phi_{\mathbb{C}})$ is diagonal with the same entries, which shows that the signature does not change under complexification, $\sigma(\phi) = \sigma(\phi_{\mathbb{C}}) \in \mathbb{Z}$.

The symmetric perfect pairing ϕ may alternatively be described by a self-duality isomorphism $d : V \xrightarrow{\cong} V^*$, where $V^* = \mathrm{Hom}(V, \mathbb{R})$, given by $d(v) = \phi(v \otimes -)$. The symmetry property is equivalent to asserting that

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "\mathrm{ev}"', "\cong"] & V^* \\ V^{**} \arrow[ur, "d^*"'] & \end{tikzcd}$$

commutes, where ev is the canonical evaluation isomorphism. Similarly, if W is a complex vector space and $\psi : W \otimes W^{\mathrm{op}} \to \mathbb{C}$ a perfect Hermitian pairing, then ψ can alternatively be described by a self-duality isomorphism $D : W \xrightarrow{\cong} W^\dagger$, where $W^\dagger = \mathrm{Hom}(W^{\mathrm{op}}, \mathbb{C})$, by setting $D(w) = \psi(w \otimes -)$. The Hermitian symmetry is equivalent to asserting that

$$\begin{tikzcd} W \arrow[r, "D"] \arrow[d, "\mathrm{ev}"', "\cong"] & W^\dagger \\ W^{\dagger\dagger} \arrow[ur, "D^\dagger"'] & \end{tikzcd}$$

commutes. In particular, we get $D : V_{\mathbb{C}} \xrightarrow{\cong} V_{\mathbb{C}}^\dagger$ for $(W, \psi) = (V_{\mathbb{C}}, \phi_{\mathbb{C}})$.

Let X be a path-connected space and $(\mathcal{P}, \phi)$ a Poincaré local system on X, $\phi : \mathcal{P} \otimes \mathcal{P} \to \mathbb{R}_X$. Applying complexification stalkwise, we obtain a Hermitian local system $\phi_{\mathbb{C}} : \mathcal{P}_{\mathbb{C}} \otimes \mathcal{P}_{\mathbb{C}}^{\mathrm{op}} \to \mathbb{C}_X$. A monodromy-theoretic description of this passage runs as follows: Let p and q be such that $p + q = \mathrm{rk}\,\mathcal{P}$ and $p - q = \sigma(\phi_x)$, $x \in X$. Let $O(p, q)$ be the group of all matrices in $GL(p + q, \mathbb{R})$ that preserve the form γ, that is,

$$O(p, q) = \{A \in GL(p + q, \mathbb{R}) \;:\; A^T \cdot I_{p,q} \cdot A = I_{p,q}\},$$

where

$$I_{p,q} = \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix}.$$

Then $(\mathcal{P}, \phi)$ determines, and is determined by, a representation $\pi_1(X) \to O(p,q)$. Let $U(p,q)$ be the group of all matrices in $GL(p+q,\mathbb{C})$ that preserve the form $\gamma_{\mathbb{C}}$, that is,

$$U(p,q) = \{A \in GL(p+q,\mathbb{C}) \; : \; \overline{A}^T \cdot I_{p,q} \cdot A = I_{p,q}\}.$$

Note that $O(p,q) \subset U(p,q)$ is a subgroup. The Hermitian local system $(\mathcal{P}_{\mathbb{C}}, \phi_{\mathbb{C}})$ determines, and is determined by, a representation $\pi_1(X) \to U(p,q)$, and this representation is the composition

$$\pi_1(X) \longrightarrow O(p,q) \hookrightarrow U(p,q).$$

Let X^n be a PL stratified pseudomanifold. The real dualizing complex $\mathbb{D}^\bullet_X(\mathbb{R})$ on X may be defined as the complex of sheaves of real vector spaces which has the sheafification of the presheaf

$$U \mapsto C_j(X, X-U;\mathbb{R}), \qquad U \subset X \text{ open},$$

in degree $-j$, where C_j denotes singular chains of dimension j. Similarly, $\mathbb{D}^\bullet_X(\mathbb{C})$ is the sheafification of $U \mapsto C_j(X, X-U;\mathbb{C})$. Since $C_j(X, X-U;\mathbb{C}) = C_j(X, X-U;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and $-\otimes_{\mathbb{R}} \mathbb{C}$ commutes with direct limits, it follows that

$$\mathbb{D}^\bullet_X(\mathbb{C}) = \mathbb{D}^\bullet_X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}_X.$$

Let $\mathbf{S}^\bullet \in D^b_c(X;\mathbb{R})$ be an object of the constructible bounded derived category of sheaf complexes of real vector spaces on X. Since we are working over fields, $\overset{L}{\otimes} = \otimes$. Define the complexification of $\mathbf{S}^\bullet$ by $\mathbf{S}^\bullet_{\mathbb{C}} = \mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbb{C}_X \in D^b_c(X;\mathbb{C})$. Given $\mathbf{A}^\bullet \in D^b_c(X;\mathbb{C})$, we may apply composition with complex conjugation in a stalkwise fashion to define $(\mathbf{A}^\bullet)^{\mathrm{op}} \in D^b_c(X;\mathbb{C})$. We have $(\mathbf{S}^\bullet_{\mathbb{C}})^{\mathrm{op}} = \mathbf{S}^\bullet \otimes_{\mathbb{R}} (\mathbb{C}^{\mathrm{op}}_X)$. Given $\mathbf{T}^\bullet \in D^b_c(X;\mathbb{R})$, there is a canonical isomorphism

$$\begin{aligned} \mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}} &\cong (\mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbf{T}^\bullet) \otimes_{\mathbb{R}} (\mathbb{C}_X \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{op}}_X), \\ (v \otimes \lambda) \otimes (w \otimes \mu) &\mapsto (v \otimes w) \otimes (\lambda \otimes \mu). \end{aligned}$$

Let

$$\begin{aligned} m : \mathbb{C}_X \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{op}}_X &\longrightarrow \mathbb{C}_X \\ \lambda \otimes \mu &\mapsto \lambda\overline{\mu} \end{aligned}$$

be the canonical multiplication. To a pairing

$$\phi : \mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbf{T}^\bullet \longrightarrow \mathbb{D}^\bullet_X(\mathbb{R})$$

into the dualizing complex, we can associate a pairing

$$\phi_{\mathbb{C}} : \mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbb{D}^\bullet_X(\mathbb{C})$$

by taking $\phi_{\mathbb{C}}$ to be the composition

$$\mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}} = (\mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbf{T}^\bullet) \otimes_{\mathbb{R}} (\mathbb{C}_X \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{op}}_X) \xrightarrow{\phi \otimes m} \mathbb{D}^\bullet_X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}_X = \mathbb{D}^\bullet_X(\mathbb{C}).$$

Under the canonical identifications

$$\begin{gathered} \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet \otimes \mathbf{T}^\bullet, \mathbb{D}^\bullet_X(\mathbb{R})) \cong \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet, \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{T}^\bullet, \mathbb{D}^\bullet_X(\mathbb{R}))), \\ \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}, \mathbb{D}^\bullet_X(\mathbb{C})) \cong \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet_{\mathbb{C}}, \mathrm{R}\mathbf{Hom}^\bullet((\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}, \mathbb{D}^\bullet_X(\mathbb{C}))), \end{gathered}$$

the above procedure associates to a morphism

$$d : \mathbf{S}^\bullet \longrightarrow \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{T}^\bullet, \mathbb{D}^\bullet_X(\mathbb{R})) = \mathcal{D}_{X,\mathbb{R}}(\mathbf{T}^\bullet)$$

in $D^b_c(X; \mathbb{R})$ with codomain the real Verdier-dual of $\mathbf{T}^\bullet$ a morphism

$$D : \mathbf{S}^\bullet_{\mathbb{C}} \longrightarrow \mathrm{R}\mathbf{Hom}^\bullet((\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}, \mathbb{D}^\bullet_X(\mathbb{C})) = \mathcal{D}_{X,\mathbb{C}}((\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}})$$

in $D^b_c(X; \mathbb{C})$ to the complex Verdier-dual of $(\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}$. If d is an isomorphism, then so is D. If d is symmetric, that is, $\mathcal{D}_{X,\mathbb{R}}(d) = d$, then D is Hermitian, that is, $\mathcal{D}_{X,\mathbb{C}}(D^{\mathrm{op}}) = D$. In particular, if $\mathbf{S}^\bullet$ is a symmetric self-dual real sheaf, then $\mathbf{S}^\bullet_{\mathbb{C}}$ is a Hermitian self-dual complex sheaf.

Suppose the dimension n of X is a multiple of 4 and that X is oriented, closed and has only even codimensional strata. Let $\phi : \mathcal{P} \otimes \mathcal{P} \to \mathbb{R}_X$ be a symmetric Poincaré local system on the top stratum of X. Then ϕ extends uniquely to a symmetric self-duality isomorphism

$$d : \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}) \cong \mathcal{D}_{X,\mathbb{R}} \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P})[n].$$

Similarly, the associated Hermitian local system $\phi_{\mathbb{C}} : \mathcal{P}_{\mathbb{C}} \otimes (\mathcal{P}_{\mathbb{C}})^{\mathrm{op}} \to \mathbb{C}_X$ extends uniquely to a Hermitian self-duality isomorphism

$$\delta : \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_{\mathbb{C}}) \cong \mathcal{D}_{X,\mathbb{C}} \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_{\mathbb{C}})^{\mathrm{op}}[n].$$

As, for an open inclusion i, the derived pushforward $\mathrm{R}\, i_*$ commutes with $- \otimes_{\mathbb{R}} \mathbb{C}$, and the truncation functor $\tau_{\leq k}$ commutes with $- \otimes_{\mathbb{R}} \mathbb{C}$ as well, we have

$$\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_{\mathbb{C}}) = \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P})_{\mathbb{C}}.$$

Moreover, $\delta = D$, where D is the complexification of d as described above.

Let $\mathbf{S}^\bullet \in D^b_c(X; \mathbb{R})$ be a symmetric self-dual sheaf on X, $d : \mathbf{S}^\bullet \cong \mathcal{D}_{X,\mathbb{R}} \mathbf{S}^\bullet[n]$, $\mathcal{D}_{X,\mathbb{R}} d[n] = d$. The isomorphism d induces a symmetric isomorphism on the middle-dimensional hypercohomology groups

$$\mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet) \xrightarrow{\cong} \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet)^*,$$

i.e. a symmetric perfect pairing

$$\psi : \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet) \otimes_{\mathbb{R}} \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet) \longrightarrow \mathbb{R}.$$

Let $\sigma(\mathbf{S}^\bullet) \in \mathbb{Z}$ denote the signature of this pairing. The complexification $\mathbf{S}^\bullet_\mathbb{C}$ is a Hermitian self-dual sheaf. Its self-duality isomorphism $D : \mathbf{S}^\bullet_\mathbb{C} \cong \mathcal{D}_{X,\mathbb{C}}(\mathbf{S}^\bullet_\mathbb{C})^{\mathrm{op}}[n]$ induces a perfect Hermitian pairing

$$\eta : \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet_\mathbb{C}) \otimes_\mathbb{C} \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet_\mathbb{C})^{\mathrm{op}} \longrightarrow \mathbb{C}.$$

Let $\sigma(\mathbf{S}^\bullet_\mathbb{C}) \in \mathbb{Z}$ denote the signature of η. With $V = \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet)$, we have $\mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet_\mathbb{C}) = V_\mathbb{C}$ and $\eta = \psi_\mathbb{C}$, whence $\sigma(\mathbf{S}^\bullet_\mathbb{C}) = \sigma(\psi_\mathbb{C}) = \sigma(\psi) = \sigma(\mathbf{S}^\bullet)$. Given a Poincaré local system $(\mathcal{P}, \phi)$, the twisted signature $\sigma(X; \mathcal{P})$ is by definition the signature of the self-dual sheaf $(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}), d)$. We conclude that

$$\begin{aligned}\sigma(X; \mathcal{P}) &= \sigma(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}), d)\\ &= \sigma(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P})_\mathbb{C}, D)\\ &= \sigma(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_\mathbb{C}), \delta)\\ &= \sigma(X; \mathcal{P}_\mathbb{C}),\end{aligned}$$

where the last equality is a definition.

1.5 Strongly Transverse Poincaré Local Systems

Let $\epsilon \in \{\pm 1\}$, let $(\mathcal{P}, \phi)$ be an ϵ-symmetric Poincaré local system of stalk dimension m on the space X^n and let $\Pi_1(X)$ denote the fundamental groupoid of X. By $\mathfrak{Vect}_m$ denote the category whose objects are pairs (V, ψ), with V an m-dimensional real vector space and $\psi : V \times V \to \mathbb{R}$ a perfect ϵ-symmetric bilinear pairing, and whose morphisms are isometries of the pairings. The system $(\mathcal{P}, \phi)$ induces a covariant functor

$$\mu(\mathcal{P}) : \Pi_1(X) \longrightarrow \mathfrak{Vect}_m$$

as follows: For $x \in X$, let

$$\mu(\mathcal{P})(x) = (\mathcal{P}_x, \phi_x)$$

and for a path class $[\omega] \in \pi_1(X, x_1, x_2) = \mathrm{Hom}_{\Pi_1(X)}(x_2, x_1)$, $\omega : I \to X$, $\omega(0) = x_1$, $\omega(1) = x_2$, define the linear operator

$$\mu(\mathcal{P})[\omega] : \mu(\mathcal{P})(x_2) \longrightarrow \mu(\mathcal{P})(x_1)$$

to be the composition

$$\mu(\mathcal{P})(x_2) = \mathcal{P}_{\omega(1)} \cong (\omega^*\mathcal{P})_1 \underset{\text{restr}}{\overset{\simeq}{\leftarrow}} \Gamma(I, \omega^*\mathcal{P}) \underset{\text{restr}}{\overset{\simeq}{\rightarrow}} (\omega^*\mathcal{P})_0 \cong \mathcal{P}_{\omega(0)} = \mu(\mathcal{P})(x_1).$$

If we choose a base-point $x \in X$, then restricting $\mu(\mathcal{P})$ to the fundamental group $\pi_1(X, x) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$ gives an assignment of a linear automorphism on the stalk $\mathcal{P}_x$,

$$\mu(\mathcal{P})_x(g) : \mathcal{P}_x \longrightarrow \mathcal{P}_x,$$

preserving the pairing $\phi_x : \mathcal{P}_x \times \mathcal{P}_x \to \mathbb{R}$, to each $g \in \pi_1(X, x)$. Thus one obtains the monodromy representation

$$\mu(\mathcal{P})_x : \pi_1(X, x) \longrightarrow O(p, q)$$

when $\epsilon = 1$ ($p + q = m$ is the rank of $\mathcal{P}$, $p - q$ the signature of ϕ_x), and

$$\mu(\mathcal{P})_x : \pi_1(X, x) \longrightarrow Sp(2r; \mathbb{R})$$

when $\epsilon = -1$ ($m = 2r$ is the rank of $\mathcal{P}$). Conversely, a given functor $\mu : \Pi_1(X) \to \mathfrak{Vect}_m$ determines a Poincaré local system: Let X_0 be a path component of X, and $x_0 \in X_0$. Then $\pi(X_0, x_0)$ acts on $\mu(x_0) = (V, \phi)$ by the restriction μ_{x_0} and we have the associated local system

$$\mathcal{P}|_{X_0} = \widetilde{X_0} \times_{\pi_1(X_0, x_0)} V$$

over X_0 with an induced pairing ϕ, where $\widetilde{X_0}$ denotes the universal cover of X_0.

Definition 4 Let X be a stratified pseudomanifold with singular set Σ and let $\mathcal{X}$ denote the set of components of open strata of X of codimension at least 2. Each $Z \in \mathcal{X}$ has a link $\mathrm{Lk}(Z)$. Call a Poincaré local system $\mathcal{P}$ on $X - \Sigma$ *strongly transverse to* Σ if the composite functor

$$\Pi_1(\mathrm{Lk}(Z) - \Sigma) \xrightarrow{\mathrm{incl}_*} \Pi_1(X - \Sigma) \xrightarrow{\mu(\mathcal{P})} \mathfrak{Vect}_m$$

is isomorphic to the trivial functor for all $Z \in \mathcal{X}$.

On normal spaces, strong transversality of local systems characterizes those systems that extend as local systems over the whole space:

Proposition 2 *Let X^n be normal. A Poincaré local system $\mathcal{P}$ on $X - \Sigma$ is strongly transverse to Σ if and only if it extends as a Poincaré local system over all of X. Such an extension is unique.*

The normality assumption is not necessary for the "if"-direction. The assumption cannot be omitted in the "only if"-direction and in the uniqueness statement.

Corollary 1 *Let X^n be normal. A Poincaré local system $\mathcal{P}$ on $X - \Sigma$ is strongly transverse to Σ if and only if its monodromy functor $\mu(\mathcal{P}) : \Pi_1(X - \Sigma) \to \mathfrak{Vect}_m$ factors (up to isomorphism of functors) through $\Pi_1(X)$:*

$$\begin{tikzcd} \Pi_1(X - \Sigma) \arrow[r, "\mathrm{incl}_*"] \arrow[dr, "\mu(\mathcal{P})"'] & \Pi_1(X) \arrow[d] \\ & \mathfrak{Vect}_m \end{tikzcd}$$

Let X^n be normal. A Poincaré local system $(\mathcal{P}, \phi)$ on $X^n - \Sigma$ strongly transverse to Σ has a K-theory signature

$$[\mathcal{P}]_K \in \begin{cases} \mathrm{KO}(X), & \text{if } \epsilon = 1 \\ \mathrm{KU}(X), & \text{if } \epsilon = -1, \end{cases}$$

as we shall now explain. By Proposition 2, $(\mathcal{P}, \phi)$ has a unique extension to a Poincaré local system $(\bar{\mathcal{P}}, \bar{\phi})$ on X. We now proceed as in [Mey72]. Let $\mathcal{P}^c$ denote the flat vector bundle associated to the locally constant sheaf $\bar{\mathcal{P}}$, that is

$$\mathcal{P}^c|_{X_0} = \widetilde{X_0} \times_\pi \mathbb{R}^m$$

over a path component X_0 of X, where $\mathbb{R}^m$ is given the usual topology, $\pi = \pi_1(X_0)$, and π acts on $\mathbb{R}^m$ by means of the monodromy $\mu(\bar{\mathcal{P}})$ of $\bar{\mathcal{P}}$. A suitable choice of Euclidean metric on $\mathcal{P}^c$ induces (using $\bar{\phi}$) a vector bundle automorphism

$$A : \mathcal{P}^c \longrightarrow \mathcal{P}^c$$

such that $A^2 = 1$ (if $\bar{\phi}$ is symmetric, i.e. $\epsilon = 1$) or $A^2 = -1$ (if $\bar{\phi}$ is skew-symmetric, i.e. $\epsilon = -1$). Thus in the case $\epsilon = 1$, $\mathcal{P}^c$ decomposes as a direct sum of vector bundles

$$\mathcal{P}^c = \mathcal{P}_+ \oplus \mathcal{P}_-$$

corresponding to the ± 1-eigenspaces of A. Put

$$[\mathcal{P}]_K = [\mathcal{P}_+] - [\mathcal{P}_-] \in \mathrm{KO}(X).$$

In the case $\epsilon = -1$, A defines a complex structure on $\mathcal{P}^c$ and we obtain the complex vector bundle $\mathcal{P}_\mathbb{C}$ and its conjugate bundle $\mathcal{P}_\mathbb{C}^*$; we put

$$[\mathcal{P}]_K = [\mathcal{P}_\mathbb{C}^*] - [\mathcal{P}_\mathbb{C}] \in \mathrm{KU}(X).$$

Similar remarks apply to perfect complex Hermitian local coefficient systems $\mathcal{S}$. They are determined over connected components $X_0 \subset X$ by monodromy representations $\mu(\mathcal{S}) : \pi_1(X_0) \to U(p, q)$ and their K-theory signature is defined as $[\mathcal{S}]_K = [\mathcal{S}_+] - [\mathcal{S}_-] \in \mathrm{KU}(X)$, where $\mathcal{S}^c = \mathcal{S}_+ \oplus \mathcal{S}_-$ is a nonflat splitting such that the Hermitian form is positive definite on $\mathcal{S}_+$ and negative definite on $\mathcal{S}_-$, corresponding to a reduction of the structure group from $U(p, q)$ to the maximal compact subgroup $U(p) \times U(q)$, see [Lus71].

1.6 Computing Twisted L-Classes for Strongly Transverse Coefficients

Let X be a closed Witt space with singular set Σ, and $(\mathcal{P}, \phi)$ a Poincaré local system on $X - \Sigma$ such that a self-dual extension $(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}), \bar{\phi})$ exists. The *twisted L-classes*

$$L_k(X; \mathcal{P}) \in H_k(X; \mathbb{Q})$$

of X with coefficients in $\mathcal{P}$ are the L-classes of the self-dual sheaf $\mathbf{S}^\bullet = \mathbf{IC}^\bullet_{\bar{m}}(X;\mathcal{P})$. In [BCS03], we show:

Theorem 5 *Let X^n be a closed oriented Whitney stratified normal Witt space with singular set Σ, and let $(\mathcal{P}, \phi)$ be a Poincaré local system on $X - \Sigma$, strongly transverse to Σ. Then*

$$L_*(X;\mathcal{P}) = \widetilde{\mathrm{ch}}[\mathcal{P}]_K \cap L_*(X). \tag{1.6}$$

Recall the

Definition 5 X^n is *supernormal*, if for any components Z, Z' of open strata with $\dim Z' > \dim Z \leq n-2$, the link $Lk(Z) \cap Z'$ is simply connected.

Theorem 5 implies

Corollary 2 *If X^n is supernormal, then for any Poincaré local system $(\mathcal{P}, \phi)$ on $X - \Sigma$*

$$L_*(X;\mathcal{P}) = \widetilde{\mathrm{ch}}[\mathcal{P}]_K \cap L_*(X).$$

To obtain the conclusion of the corollary, less than supernormality is actually needed. Indeed it is sufficient to require that X be normal and that the image of $\pi_1(Lk(Z) - \Sigma)$ in $\pi_1(X - \Sigma)$ vanishes for all $Z \in \mathcal{X}$.

In [Ban06], the first author has extended formula (1.6) to spaces that are not Witt, but still support self-dual perverse sheaves, given by Lagrangian structures, so that the L-class is still defined.

1.7 The Cappell-Shaneson L-Class Formula for Singular Embeddings

Let X^n be an oriented connected PL pseudomanifold of real dimension n, piecewise linearly embedded in an oriented, connected PL manifold M^m of dimension $m = n+2$. Since (M, X) is a PL pair, there exists a filtration

$$\begin{aligned} M = M_m \supset M_{m-1} = X \supset M_{m-2} = X \supset M_{m-3} = X \supset M_{m-4} \\ \supset M_{m-5} \supset \cdots \supset M_0 \supset M_{-1} = \varnothing, \end{aligned}$$

such that for each $y \in M_i - M_{i-1}$ there exists a distinguished neighborhood U of y in M, a compact Hausdorff pair (G, F), a filtration

$$G = G_{m-i-1} \supset \cdots \supset G_0 \supset G_{-1} = \varnothing,$$

and a PL homeomorphism

$$\phi : D^i \times c(G, F) \longrightarrow (U, U \cap X)$$

that maps $D^i \times c(G_{j-1}, G_{j-1} \cap F)$ onto $(M_{i+j}, M_{i+j} \cap X)$, where cY denotes the cone on a space Y. The link pair (G, F) depends up to PL homeomorphism only on the connected component V of $M_i - M_{i-1}$ that contains y. Since M is a manifold, $G = S^{m-i-1} = S^{n-i+1}$ is a sphere. As in [CS91], we will henceforth assume that embeddings are of finite local type and of finite type. (This guarantees finite dimensionality of intersection sheaf stalks and global intersection homology groups. Algebraic knots, for example, are always of finite type.) An induced PL stratification of X is given by

$$X = X_n \supset X_{n-1} = M_{m-3} = X \supset X_{n-2} = M_{m-4}$$
$$\supset X_{n-3} = M_{m-5} \supset \cdots \supset X_0 = M_0 \supset X_{-1} = \varnothing.$$

The link in X of a component V of a stratum $X_i - X_{i-1} = M_i - M_{i-1}$ at a point $y \in V$ is the above F. Let $\mathcal{X}$ be the collection of connected components of pure strata $X_i - X_{i-1}$, $i \leq n-2$. It is worthwhile to discuss the case of X a manifold. Since the embedding of X in M may not be locally flat, the pair (M, X) will in general still receive a nontrivial stratification, but the links of components in X will be spheres $F = S^{n-i-1}$. The link pairs in (M, X) will thus be knots (S^{n-i+1}, S^{n-i-1}). The closed strata $X_i \subset X$ induced by the embedding $X \subset M$ may or may not be submanifolds of X and the embeddings $X_i \subset X$ may or may not be locally flat.

Example 1 Let $S(Y)$ denote the unreduced suspension of a space Y. We shall discuss the stratification of $X^n = S^2 \times S(S^1 \times S^{n-4})$ induced by a certain nonlocally flat embedding $X^n \subset S^2 \times S^n = M^{n+2}$, where $n \geq 6$. We start out with a nontrivial locally flat PL knot $\kappa : S^{n-5} \hookrightarrow S^{n-3}$ and suspend it to obtain an embedding $S\kappa : S^{n-4} = S(S^{n-5}) \hookrightarrow S(S^{n-3}) = S^{n-2}$. Denote the two suspension points in S^{n-4} by p_+ and p_-. Think of S^{n-2} as the one-point compactification $S^{n-2} = \mathbb{R}^{n-2} \cup \{\infty\}$ of $\mathbb{R}^{n-2}$, with ∞ not in the image of $S\kappa$. Then by restricting $S\kappa$, we obtain an embedding $\sigma : S^{n-4} \hookrightarrow \mathbb{R}^{n-2}$, which is not flat at p_+ and p_- because the link pair at $p_\pm$ is the knot κ. On the complement $S^{n-4} - \{p_\pm\}$, σ is locally flat. Crossing with a circle S^1, we get an embedding $\mathrm{id}_{S^1} \times \sigma : S^1 \times S^{n-4} \hookrightarrow S^1 \times \mathbb{R}^{n-2}$ with link pair κ for the singular stratum $S^1 \times \{p_\pm\} \subset S^1 \times S^{n-4}$ where $\mathrm{id}_{S^1} \times \sigma$ is not locally flat. Embed S^1 in the plane $\mathbb{R}^2$ as the unit circle and $\mathbb{R}^2$ in $\mathbb{R}^{n-1}$ in the standard way, $(x, y) \mapsto (x, y, 0, 0, \ldots, 0)$. Then the normal bundle of $S^1 \hookrightarrow \mathbb{R}^{n-1}$ is $S^1 \times \mathbb{R}^{n-2}$ and defines an open embedding $S^1 \times \mathbb{R}^{n-2} \hookrightarrow \mathbb{R}^{n-1} \hookrightarrow S^{n-1}$. The composition of $\mathrm{id}_{S^1} \times \sigma$ with this open embedding gives an embedding $f : S^1 \times S^{n-4} \hookrightarrow S^{n-1}$. Since the open embedding does not change the link types, f still has singular stratum $S^1 \times \{p_\pm\}$. Let q_+ and q_- be the two suspension points of $S(S^1 \times S^{n-4})$. Suspending f, we obtain an embedding $Sf : S(S^1 \times S^{n-4}) \hookrightarrow S(S^{n-1}) = S^n$. The points $q_\pm$ are singularities of the pseudomanifold $S(S^1 \times S^{n-4})$ and thus must appear as a stratum of the pair $(S^n, S(S^1 \times S^{n-4}))$. The stratification of $(S^n, S(S^1 \times S^{n-4}))$ is given by

$$S^n \supset S(S^1 \times S^{n-4}) \supset S(S^1 \times \{p_\pm\}) \supset \{q_\pm\}.$$

Finally, $\mathrm{id}_{S^2} \times Sf : S^2 \times S(S^1 \times S^{n-4}) \hookrightarrow S^2 \times S^n$ defines an embedding $X^n \subset M^{n+2}$. The pair (M, X) is stratified by

$$\begin{aligned} M = S^2 \times S^n \supset X = S^2 \times S(S^1 \times S^{n-4}) \\ \supset X_4 = S^2 \times S(S^1 \times \{p_\pm\}) \supset X_2 = S^2 \times \{q_\pm\}. \end{aligned}$$

The collection $\mathcal{X}$ is given by

$$\mathcal{X} = \{S^2 \times \mathring{I} \times S^1 \times \{p_+\}, S^2 \times \mathring{I} \times S^1 \times \{p_-\}, S^2 \times \{q_+\},\ S^2 \times \{q_-\}\},$$

where $\mathring{I} = (0, 1)$ denotes the open unit interval. The closure $\overline{V}$ of the pure component $V = S^2 \times \mathring{I} \times S^1 \times \{p_+\}$ in X is

$$\overline{V} = S^2 \times S(S^1 \times \{p_+\}),$$

PL homeomorphic to the 4-manifold $S^2 \times S^2$, and the singular set $\overline{V} - V$ of $\overline{V}$ is $\overline{V} - V = S^2 \times \{q_\pm\}$, the disjoint union of two 2-spheres. The embedding $\overline{V} - V \subset \overline{V}$ is locally flat and has trivial normal bundle. The link pair of V is the knot κ that we started with.

We return to the general case of an oriented pseudomanifold $X \subset M$. Assume that all strata in $\mathcal{X}$ have even codimension in X. Let $V \in \mathcal{X}$ be a component of codimension $2c = m - i \geq 4$ in M and let $x \in V$ be a point with link pair $(G_x, F_x) = (S_x^{2c-1}, F_x)$. The fundamental class $[F_x]$ maps trivially to $H_{2c-3}(S_x^{2c-1}) = 0$. Thus $F_x \subset S_x^{2c-1}$ has a Seifert-pseudomanifold that can be used to define a linking number. For $\alpha \in \pi_1(S_x^{2c-1} - F_x)$, let $\mathrm{lk}(F_x, \alpha) \in \mathbb{Z}$ denote the linking number. The assignment $\alpha \mapsto t^{\mathrm{lk}(F_x,\alpha)}$ determines a local system $\mathcal{L}_x$ with stalks Λ on $S_x^{2c-1} - F_x$. The complex of sheaves $\mathbf{IC}^\bullet_{\bar{m}}(S_x^{2c-1}; \mathcal{L}_x)$ is defined by the Deligne extension process for the lower middle perversity $\bar{m}$ applied to $\mathcal{L}_x$. The pairing

$$\begin{aligned} \mathcal{L}_x \otimes \mathcal{L}_x^{\mathrm{op}} &\longrightarrow \Lambda, \\ f(t) \otimes g(t) &\mapsto f(t)g(t^{-1}) \end{aligned}$$

is perfect and Hermitian. Assuming that $F_x \subset S_x^{2c-1}$ is of finite local type, this pairing extends to a Verdier-superduality isomorphism

$$\mathbf{IC}^\bullet_{\bar{l}}(S_x^{2c-1}; \mathcal{L}_x)^{\mathrm{op}} \cong \mathcal{D}(\mathbf{IC}^\bullet_{\bar{m}}(S_x^{2c-1}; \mathcal{L}_x))[2c-1],$$

where $\bar{l}$ is the logarithmic perversity of [CS91], that is, $\bar{l}(s) = [(s+1)/2]$ so that $\bar{m}(s) + \bar{l}(s) = s - 1$ ($\bar{m}$ and $\bar{l}$ are "superdual"). If $F_x \subset S_x^{2c-1}$ is in addition of finite type, then this isomorphism induces upon taking hypercohomology an isomorphism

$$\begin{aligned} IH_i^{\bar{l}}(S_x^{2c-1}; \mathcal{L}_x)^{\mathrm{op}} &\cong \mathrm{Ext}(IH^{\bar{m}}_{2c-i-2}(S_x^{2c-1}; \mathcal{L}_x), \Lambda) \\ &= \mathrm{Hom}(IH^{\bar{m}}_{2c-i-2}(S_x^{2c-1}; \mathcal{L}_x), \mathbb{Q}(t)/\Lambda). \end{aligned}$$

With $(\mathcal{B}_V)_x = \mathrm{Image}(IH^{\bar{m}}_{c-1}(S^{2c-1}_x; \mathcal{L}_x) \to IH^{\bar{l}}_{c-1}(S^{2c-1}_x; \mathcal{L}_x))$, we thus get for $i = c-1$ a Blanchfield pairing

$$(\mathcal{B}_V)_x \otimes (\mathcal{B}_V)^{\mathrm{op}}_x \longrightarrow \mathbb{Q}(t)/\Lambda.$$

Remark 1 When $F_x = S^{2c-3}$ and $F_x \subset S^{2c-1}_x$ is locally flat, this is the classical Blanchfield pairing. If $F \subset S^{2c-1}$ is any locally flat submanifold, then according to [CS91, p. 339],

$$IH^{\bar{p}}_i(S^{2c-1}; \mathcal{L}) = \begin{cases} H_i(K; \mathcal{L}), & \bar{p}(2) = 0, \\ H_i(K, \partial K; \mathcal{L}), & \bar{p}(2) = 1, \end{cases}$$

where K is the exterior of F. In this situation, then, the above map $IH^{\bar{m}}_{c-1}(S^{2c-1}; \mathcal{L}) \to IH^{\bar{l}}_{c-1}(S^{2c-1}; \mathcal{L})$ becomes the map

$$H_{c-1}(K; \mathcal{L}) \longrightarrow H_{c-1}(K, \partial K; \mathcal{L})$$

induced by inclusion. For $F = S^{2c-3}$, $c \geq 3$, we have $\partial K = S^{2c-3} \times S^1$ and the map is an isomorphism.

Letting x vary over V, we obtain a Blanchfield local system

$$\mathcal{B}_V \otimes \mathcal{B}^{\mathrm{op}}_V \longrightarrow \mathbb{Q}(t)/\Lambda$$

over V. Again by the Deligne extension process, the associated Poincaré local system

$$\mathcal{B}^{\mathbb{R}}_V \otimes \mathcal{B}^{\mathbb{R}}_V \longrightarrow \mathbb{R}$$

extends to a self-duality isomorphism

$$\mathbf{IC}^{\bullet}_{\bar{m}}(\overline{V}; \mathcal{B}^{\mathbb{R}}_V) \cong \mathcal{D}\mathbf{IC}^{\bullet}_{\bar{m}}(\overline{V}; \mathcal{B}^{\mathbb{R}}_V)[m-2c].$$

This self-dual sheaf has L-classes

$$L_j(\overline{V}; \mathcal{B}^{\mathbb{R}}_V) \in H_j(\overline{V}; \mathbb{Q}),$$

assuming now that X is compact. Let $i_V : \overline{V} \hookrightarrow X$ be the inclusion. The Cappell-Shaneson L-class formula [CS91] for singular embeddings asserts that

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*} L_*(\overline{V}; \mathcal{B}^{\mathbb{R}}_V). \tag{1.7}$$

When $M = S^{n+2}$, $n > 0$, is a sphere, we have $i_*[X] = 0 \in H_n(S^{n+2})$ so that $\chi = 0$ and $L^*(P(M)) = 1$. Therefore,

$$L_*(S^{n+2}, X) = [X] \cap i^* L^*(P(M) \cup (1+\chi^2)^{-1}) = [X] \cap 1 = [X],$$

and in particular in degree 0,

$$L_0(S^{n+2}, X) = 0. \tag{1.8}$$

1.8 Embeddings and Strongly Transverse Coefficients

A synthesis of the characteristic class formula of Theorem 5 and the Cappell-Shaneson formula (1.7) yields the following result.

Theorem 6 *Let $i : X^n \hookrightarrow M^{n+2}$ be a PL embedding of an oriented compact pseudomanifold in an oriented compact manifold such that the pair (M, X) is stratifiable without odd-codimensional strata. Assume that all Poincaré local systems $\mathcal{B}_V^{\mathbb{R}}$ are strongly transverse to the singular set $\overline{V} - V$, $V \in \mathcal{X}$. Then*

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*}(\widetilde{\mathrm{ch}}[\mathcal{B}_V^{\mathbb{R}}]_K \cap L_*(\overline{V})). \tag{1.9}$$

Formula (1.9) holds automatically if X happens to be a manifold and the singular set of the embedded X has codimension at least 3. For in that case, the links in X are spheres of dimension 2 or higher which are simply connected. Thus we find ourselves in the supernormal situation of Corollary 2.

1.9 Nontransverse Coefficient Systems

We shall first consider singular embeddings $X^n \subset M^{n+2}$ which have at most 4-dimensional singularities whose pure components have definite real Blanchfield form. The 4-stratum may contain a 2-stratum which is a disjoint union of 2-spheres, embedded in the 4-stratum in a locally flat way and with zero self-intersection number. The following is an example of such a situation.

Example 2 Let A be a square integral matrix such that $A + A^T$ is unimodular. According to the realization theorem of Kervaire (Theorem 4), there exists a simple locally flat 7-knot $\kappa : S^7 \hookrightarrow S^9$ with Seifert matrix A. Applying the construction of Example 1, we obtain an embedding

$$i : X^{12} = S^2 \times S(S^1 \times S^8) \subset S^2 \times S^{12} = M^{14}.$$

The induced stratification has the form

$$M \supset X \supset X_4 = S^2 \times S(S^1 \times \{p_\pm\}) \supset X_2 = S^2 \times \{q_\pm\},$$

where $X_4 - X_2$ has 2 connected components $V_\pm = S^2 \times \mathring{I} \times S^1 \times \{p_\pm\}$ with closures $\overline{V}_\pm = S^2 \times S^2$, a 4-manifold, and $\overline{V}_+ - V_+ = S^2 \times \{q_\pm\} = \overline{V}_- - V_-$. The embeddings $S^2 \times \{q_\pm\} \subset \overline{V}_\pm = S^2 \times S^2$ are locally flat and have zero self-intersection number. The link pair of both V_- and V_+ is the 7-knot κ. Taking for instance the nonsingular matrix

$$A=\begin{pmatrix}1&1&0&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&1&1&0&0&0\\0&0&0&0&1&1&0&1\\0&0&0&0&0&1&1&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix},$$

we have $A+A^T=E_8$, which is unimodular with $\sigma(E_8)=8$. By Kervaire's theorem, there exists a simple locally flat knot $\kappa : S^7\subset S^9$ with Seifert matrix A. By formulae (1.4) and (1.5), the signature of the skew-Hermitian Blanchfield pairing β of κ is

$$\sigma(\beta^{\mathbb{R}})=\sigma(A+A^T)=8.$$

Since A is nonsingular, formula (1.3) shows that the knot $\mathbb{Z}[t,t^{-1}]$-module B_A determined algebraically by A as described in Sect. 1.2 has rational dimension

$$\dim_{\mathbb{Q}}(B_A\otimes_{\mathbb{Z}}\mathbb{Q})=\operatorname{rk}A=8.$$

By Kearton's theorem (Theorem 3), $B_A\cong H_4(K_\infty)$, where K_∞ is the infinite cyclic cover of the exterior K of κ. Thus $H_4(K_\infty;\mathbb{Q})$ has dimension 8 over $\mathbb{Q}$. We conclude that the symmetric real Blanchfield form of κ is positive definite.

Theorem 7 *Let $i : X^n\hookrightarrow M^{n+2}$, $n\equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M which induces a stratification of the form*

$$X=X_n\supset X_4\supset X_2\supset X_{-1}=\varnothing,$$

such that

(i) *for every connected component V of X_4-X_2, the closure $\overline{V}$ is a 4-manifold,*
(ii) *the link pair of every such V is a (necessarily nontrivial but locally flat) spherical knot (S^{n-3},S^{n-5}) with definite real Blanchfield form of rank r_V,*
(iii) *X_2 is a disjoint union of 2-spheres, and*
(iv) *for every such S^2 and 4-dimensional V with $S^2\subset\overline{V}$, the latter embedding is locally flat with zero self-intersection number.*

Then

$$\sigma(X)=L_0(M,X)-\sum_{V\subset X_4-X_2}\epsilon_V r_V\sigma(\overline{V}),$$

where the sum ranges over all connected components V of X_4-X_2 and $\epsilon_V=1$ if the real Blanchfield form on V is positive definite and $\epsilon_V=-1$ if it is negative definite.

Proof Write $n = 4k$. Let V be a connected component of $X_4 - X_2$ and $\beta_V : \mathcal{B}_V \otimes \mathcal{B}_V^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$ the associated Blanchfield local system with stalk

$$(\mathcal{B}_V)_x = IH_{2k-2}^{\bar{m}}(S_x^{4k-3}; \mathcal{L}_x) = H_{2k-2}(K_x; \mathcal{L}_x) \cong H_{2k-2}(K_x, \partial K_x; \mathcal{L}_x)$$

by Remark 1. At $x \in V$, $(\beta_V)_x$ is the classical Blanchfield pairing of the locally flat link pair (S_x^{4k-3}, S_x^{4k-5}). This pairing is skew-Hermitian. Its Poincaré local system $\beta_V^{\mathbb{R}} : \mathcal{B}_V^{\mathbb{R}} \otimes \mathcal{B}_V^{\mathbb{R}} \to \mathbb{R}$ is obtained using the Trotter trace as described in Sect. 1.3. This system is symmetric and by assumption definite of rank r_V. In principle, we shall use Theorem 4.1 of [Ban08] to compute the twisted signature $\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$. That theorem was proven in a slightly different context, namely for complex Hermitian local systems and for smooth embeddings. The first issue is easily resolved by passing to the complexification $\mathcal{B}_{\mathbb{C}}$ of $\mathcal{B}_V^{\mathbb{R}}$ as described in Sect. 1.4. As we have seen, the signature does not change under complexification,

$$\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) = \sigma(\overline{V}; \mathcal{B}_{\mathbb{C}}).$$

The second issue presents no problem either, since the proof of Theorem 4.1 [Ban08] essentially carries over to the PL category, with one minor addition concerning smoothability. Let us recall the argument. The stratum X_2 is comprised of pairwise disjoint two-spheres. Those two-spheres that lie in $\overline{V}$ are embedded there in a locally flat manner, whence they have a normal (block) bundle. That bundle is trivial by the assumption on the self-intersection number. We can thus do surgery on these two-spheres in $\overline{V}$ and obtain a closed PL manifold M^4. The surgery replaces each two-sphere by a circle, and M minus these circles is homeomorphic to V. Thus the local system $\mathcal{B}_{\mathbb{C}}$ on V is naturally defined on M minus the circles. The key observation is that the circles have high enough codimension (namely 3) in M in order for $\mathcal{B}_{\mathbb{C}}$ to extend (uniquely) onto all of M. (The link of a circle in M is a 2-sphere, which is simply connected. Thus $\mathcal{B}_{\mathbb{C}}$ is constant on the links of the circle and extends (uniquely) to the cone on the link.) Let us call this unique extension $\overline{\mathcal{B}}_{\mathbb{C}}$. In fact it is not hard to see that $\mathcal{B}_{\mathbb{C}}$ and $\overline{\mathcal{B}}_{\mathbb{C}}$ extend further as local systems over the trace W of the surgery, so that W together with this extension is a bordism between $(\overline{V}; \mathcal{B}_{\mathbb{C}})$ and $(M; \overline{\mathcal{B}}_{\mathbb{C}})$. By bordism invariance of the twisted signature,

$$\sigma(\overline{V}; \mathcal{B}_{\mathbb{C}}) = \sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}).$$

At this point, the proof of Theorem 4.1 [Ban08] is able to invoke W. Meyer's twisted signature formula [Mey72] because in that context the manifold M is smooth. Our present M however is piecewise linear. The Hirsch-Mazur obstructions to smoothing M lie in $H^i(M; \pi_{i-1}(PL/O))$. They all vanish because PL/O is 6-connected and M is 4-dimensional. Thus M is smoothable and we may indeed call on Meyer's formula

$$\sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}) = \langle \widetilde{\mathrm{ch}}[\overline{\mathcal{B}}_{\mathbb{C}}]_K \cup L^*(M), [M] \rangle,$$

where $L^*(M) = L^*(P(M))$. Since $\beta_V^{\mathbb{R}}$, and thus also the complexification of $\beta_V^{\mathbb{R}}$, is definite of rank r_V, the K-theory signature $[\overline{\mathcal{B}}_{\mathbb{C}}]_K$ of $\overline{\mathcal{B}}_{\mathbb{C}}$ is given by

$$[\overline{\mathcal{B}}_{\mathbb{C}}]_K = \epsilon_V [\overline{\mathcal{B}}_{\mathbb{C}}] \in \mathrm{KU}^0(M),$$

where we regard $\overline{\mathcal{B}}_{\mathbb{C}}$ as a flat complex vector bundle of rank r_V. The positive dimensional rational Chern classes of $\overline{\mathcal{B}}_{\mathbb{C}}$ vanish by flatness, so that $\widetilde{\mathrm{ch}}[\overline{\mathcal{B}}_{\mathbb{C}}]_K = \epsilon_V r_V$. Therefore,

$$\begin{aligned}\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) &= \sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}) \\ &= \epsilon_V r_V \langle L^*(M), [M]\rangle \\ &= \epsilon_V r_V \sigma(M) \\ &= \epsilon_V r_V \sigma(\overline{V}),\end{aligned}$$

using the Hirzebruch signature theorem and bordism invariance.

Let S be a connected component of X_2 and $\beta_S : \mathcal{B}_S \otimes \mathcal{B}_S^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$ the associated Blanchfield local system with stalk

$$(\mathcal{B}_S)_x = \mathrm{Image}(IH^{\bar{m}}_{2k-1}(S_x^{4k-1}; \mathcal{L}_x) \longrightarrow IH^{\bar{l}}_{2k-1}(S_x^{4k-1}; \mathcal{L}_x))$$

at $x \in V$. The link pair at x has the form (S_x^{4k-1}, F_x) with F_x a PL pseudomanifold of dimension $4k-3$. The pairing β_S is Hermitian as $2k-1$ is odd. By assumption, S is a 2-sphere, in particular simply connected. This implies that $\mathcal{B}_S$ is constant (untwisted) on S. Thus the corresponding Poincaré local system $\mathcal{B}_S^{\mathbb{R}}$ is constant and $\beta_S^{\mathbb{R}} : \mathcal{B}_S^{\mathbb{R}} \otimes \mathcal{B}_S^{\mathbb{R}} \to \mathbb{R}$ is skew-symmetric. It follows that the signature of any stalk $(\mathcal{B}_S^{\mathbb{R}})_x$ is zero, $\sigma((\mathcal{B}_S^{\mathbb{R}})_x) = 0$. Since $\mathcal{B}_S^{\mathbb{R}}$ is constant on S, the twisted signature factors as

$$\sigma(S; \mathcal{B}_S^{\mathbb{R}}) = \sigma((\mathcal{B}_S^{\mathbb{R}})_x) \cdot \sigma(S) = 0.$$

Assembling the above information using the Cappell-Shaneson L-class formula (1.7) for singular embeddings, we obtain

$$\begin{aligned}\sigma(X) &= L_0(X) \\ &= L_0(M, X) - \sum_{V \subset X_4 - X_2} \sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) - \sum_{S \subset X_2} \sigma(S; \mathcal{B}_S^{\mathbb{R}}) \\ &= L_0(M, X) - \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}).\end{aligned}$$

□

Corollary 3 *Let $i : X^n \hookrightarrow S^{n+2}$, $n \equiv 0(4)$, $n > 0$, be a PL embedding of a compact oriented PL pseudomanifold X in a sphere satisfying the hypotheses of Theorem 7. Then*

$$\sigma(X) + \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}) = 0.$$

Proof Observe that $L_0(S^{n+2}, X) = 0$ according to (1.8). □

Although not always explicitly stated, similar corollaries for embeddings in spheres can be deduced in the contexts of the subsequent results as well.

For a space Y, let $Sh_{\mathrm{lch}}(Y)$ denote the collection of isomorphism classes of locally constant perfect complex Hermitian sheaves of finite rank on Y. The following theorem extends Theorem 7 to the case of an indefinite structure group $U(p,q)$. Its conclusion reduces to the conclusion of Theorem 7 when $p=0$ or $q=0$. We do maintain the zero self-intersection assumption for now.

Theorem 8 *Let $i: X^n \hookrightarrow M^{n+2}$, $n \equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M which induces a stratification of the form*

$$X = X_n \supset X_4 \supset X_2 \supset X_{-1} = \varnothing,$$

such that

(i) *for every connected component V of $X_4 - X_2$, the closure $\overline{V}$ is a 4-manifold,*
(ii) *the complexified Blanchfield system $\mathcal{B}_V^{\mathbb{C}}$ of the link pair of every such V has structure group $U(p_V, q_V)$,*
(iii) *X_2 is a disjoint union of 2-spheres, and*
(iv) *for every such S^2 and 4-dimensional V with $S^2 \subset \overline{V}$, the latter embedding is locally flat with zero self-intersection number.*

Then there exists an integral characteristic class

$$2(c_1^2 - 2c_2): Sh_{\mathrm{lch}}(V) \longrightarrow H^4(\overline{V}; \mathbb{Z})$$

such that

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} (p_V - q_V)\sigma(\overline{V}) - \sum_{V \subset X_4 - X_2} \langle 2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}}), [\overline{V}]\rangle,$$

where the two sums range over all connected components V of $X_4 - X_2$.

Proof Let V be a connected component of $X_4 - X_2$ with associated real Blanchfield system $\mathcal{B}_V^{\mathbb{R}}$. The pairing $\mathcal{B}_V^{\mathbb{R}} \otimes \mathcal{B}_V^{\mathbb{R}} \to \mathbb{R}$ is symmetric. Thus the complexified form $\mathcal{B}_V^{\mathbb{C}} \otimes (\mathcal{B}_V^{\mathbb{C}})^{\mathrm{op}} \to \mathbb{C}_V$ is Hermitian with structure group $U(p_V, q_V)$. In order to compute the twisted signature $\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$, we modify the proof of Theorem 4.3 [Ban08] so that it applies to PL spaces. As in the proof of the previous Theorem 7, we can do surgery on $X_2 \cap \overline{V}$ to obtain a PL manifold M^4 with a perfect Hermitian local system $\overline{\mathcal{B}}_{\mathbb{C}}$ defined everywhere on M such that

$$\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) = \sigma(M; \overline{\mathcal{B}}_{\mathbb{C}})$$

via the trace of the surgery. The manifold M is smoothable because it is 4-dimensional, and therefore

$$\begin{aligned}\sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}) &= \langle \widetilde{\mathrm{ch}}[\overline{\mathcal{B}}_{\mathbb{C}}]_K \cup L^*(M), [M]\rangle \\ &= \Big\langle ((p_V - q_V) + 2c_1[\overline{\mathcal{B}}_{\mathbb{C}}]_K + 2(c_1^2 - 2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K)\end{aligned}$$

$$\cup\left(1+\frac{1}{3}p_1(M)\right),[M]\right\rangle$$
$$=(p_V-q_V)\sigma(M)+2\langle(c_1^2-2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K,[M]\rangle.$$

There is a unique isomorphism

$$\phi: H^4(M)\xrightarrow{\cong} H^4(\overline{V})$$

such that

$$\begin{tikzcd}
H_c^4(M-\bigsqcup(D^3\times S^1)) \arrow[r, equal] \arrow[d, "\cong"'] & H_c^4(V) \arrow[d, "\cong"] \\
H^4(M) \arrow[r, "\cong", "\phi"'] & H^4(\overline{V})
\end{tikzcd}$$

commutes, where $H_c^*(-)$ denotes cohomology with compact supports and the vertical maps are given by extension by zero. We set

$$2(c_1^2-2c_2)(\mathcal{B}_V^{\mathbb{C}})=\phi(2(c_1^2-2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K)\in H^4(\overline{V};\mathbb{Z}).$$

Let γ be a 4-dimensional PL cochain on M representing the cohomology class $2(c_1^2-2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K$. Since extension by zero is here an isomorphism in dimension 4, we may assume that γ has compact support in $M-\bigsqcup(D^3\times S^1)$. As

$$M-\bigsqcup(D^3\times S^1)\cong V,$$

γ is a cochain on V with compact support and thus, by extension by zero, a cochain on $\overline{V}$. This cochain represents $2(c_1^2-2c_2)(\mathcal{B}_V^{\mathbb{C}})$ and

$$2\langle(c_1^2-2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K,[M]\rangle=\langle(2(c_1^2-2c_2)(\mathcal{B}_V^{\mathbb{C}}),[\overline{V}]\rangle.$$

Since $\sigma(M)=\sigma(\overline{V})$ by bordism invariance, we have

$$\sigma(\overline{V},\mathcal{B}_V^{\mathbb{R}})=(p_V-q_V)\sigma(\overline{V})+\langle(2(c_1^2-2c_2)(\mathcal{B}_V^{\mathbb{C}}),[\overline{V}]\rangle.$$

As in the proof of Theorem 7, $\sigma(S;\mathcal{B}_S^{\mathbb{R}})=0$ for every connected component S of X_2. The result follows from substituting the above information into the Cappell-Shaneson L-class formula

$$\sigma(X)=L_0(M,X)-\sum_{V\subset X_4-X_2}\sigma(\overline{V};\mathcal{B}_V^{\mathbb{R}})-\sum_{S\subset X_2}\sigma(S;\mathcal{B}_S^{\mathbb{R}}).$$

□

Theorem 8 together with Corollary 4.4 [Ban08] and Proposition 4.5 [Ban08] imply:

Corollary 4 *Let (M^{n+2}, X^n) be stratified as in Theorem 8 and assume that $\overline{V}$ is a 4-sphere for every connected component V of $X_4 - X_2$. Then*

$$\sigma(X) - L_0(M, X)$$

is divisible by 8. If for every V, $\overline{V} - V$ is connected and $X_2 \cap \overline{V} \hookrightarrow \overline{V}$ is the Artin spin of a classical knot, then

$$\sigma(X) = L_0(M, X).$$

When a lower stratum has nonzero self-intersection inside a higher one, rho-invariants enter into signature formulae, as the next theorem illustrates. Let (p, q) be coprime integers such that $0 \leq q < p$ and write $\mathbb{Z}/_p = \{1, \xi, \xi^2, \ldots, \xi^{p-1}\}$, ξ a primitive p-th root of unity. For a representation $\alpha : \mathbb{Z}/_p \to U(k)$, let $\chi_\alpha : \mathbb{Z}/_p \to \mathbb{C}$ denote the character of α. Set

$$\rho_\alpha(p, q) = \frac{1}{p} \sum_{j=1}^{p-1} (k - \chi_\alpha(\xi^j)) \cot \frac{j\pi}{p} \cot \frac{j\pi q}{p}.$$

The *constancy rank*, c-rk$(\mathcal{S})$, of a local system $\mathcal{S}$ on a connected space with cyclic fundamental group is defined to be the rank of the 1-eigenspace of the monodromy matrix of $\mathcal{S}$.

Theorem 9 *Let $i : X^n \hookrightarrow M^{n+2}$, $n \equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M which induces a stratification of the form*

$$X = X_n \supset X_4 \supset X_2 \supset X_{-1} = \varnothing,$$

such that

(i) *for every connected component V of $X_4 - X_2$, the closure $\overline{V}$ is a 4-manifold,*
(ii) *the link pair of every such V is a (necessarily nontrivial but locally flat) spherical knot (S^{n-3}, S^{n-5}) with positive, say, definite complex Blanchfield form $\mathcal{B}_V^{\mathbb{C}}$ of rank r_V,*
(iii) *X_2 is a disjoint union of 2-spheres, and*
(iv) *for every such S^2 and 4-dimensional V with $S^2 \subset \overline{V}$, the latter embedding is locally flat with nonzero self-intersection number.*

Then

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} \left(r_V \sigma(V) + \sum_{i=1}^{n_V} (\text{c-rk}(\mathcal{B}_V^{\mathbb{C}}|_{L_i}) \,\text{sign}[S_i^2]^2 - \rho_{\alpha_i}(p_i, q_i)) \right),$$

where the sum ranges over all connected components V of $X_4 - X_2$, $\sigma(V)$ denotes the (Novikov-) signature of the exterior of the link $\overline{V} \cap X_2 = \bigsqcup_{i=1}^{n_V} S_i^2 \subset \overline{V}$, $L_i =$

$L(p_i, q_i)$, a lens space, is the boundary of a regular neighborhood of S_i^2 in $\overline{V}$, and α_i is obtained by restricting $\mathcal{B}_V^{\mathbb{C}}$ to L_i.

Proof Let V be a connected component of $X_4 - X_2$. The locally flat PL-2-link

$$\overline{V} \cap X_2 = \bigsqcup S_i^2 \hookrightarrow \overline{V}$$

is isotopic to a smooth 2-link

$$\bigsqcup S_i^2 \overset{C^\infty}{\hookrightarrow} M^4,$$

where M is a smooth 4-manifold homeomorphic to $\overline{V}$. The isotopy ensures that $\mathcal{B}_V^{\mathbb{C}}$ defines a complex Blanchfield local system on the complement of the smooth 2-link and

$$\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{C}}) = \sigma(M; \mathcal{B}_V^{\mathbb{C}}).$$

The latter signature can be computed using Theorem 4.8 of [Ban08]. Let us recall the method. Let $(E^4, \partial E)$ be the exterior of the smooth 2-link. Its boundary $\partial E = \bigsqcup L_i$ is a disjoint union of lens spaces $L_i = L(p_i, q_i)$ with finite fundamental group $\mathbb{Z}/_{p_i}$, $p_i \geq 1$, since S_i^2 has nonzero self-intersection number by (iv). Let $(N_i^4, \partial N_i = L_i)$ be the total space of the disc bundle of $S_i^2 \subset M$ so that

$$\partial E = \bigsqcup \partial N_i, \qquad M = E \cup_{\partial E} \bigcup N_i.$$

By Novikov additivity,

$$\sigma(M; \mathcal{B}_V^{\mathbb{C}}) = \sigma(E; \mathcal{B}_V^{\mathbb{C}}) + \sum_{i=1}^{n_V} \sigma(N_i; \mathcal{B}_V^{\mathbb{C}}).$$

Let us first discuss the terms $\sigma(N_i; \mathcal{B}_V^{\mathbb{C}})$, where $\mathcal{B}_V^{\mathbb{C}}$ is only given on the complement of the zero-section. This complement deformation retracts onto $\partial N_i = L_i$, whence $\mathcal{B}_V^{\mathbb{C}}$ is determined by a unitary representation $\alpha_i : \pi_1(L_i) = \mathbb{Z}/_{p_i} \to U(r_V)$, given by a monodromy matrix $A \in U(r_V)$. Diagonalizing A, we obtain a decomposition $\mathcal{B}_V^{\mathbb{C}}|_{L_i} \cong \mathcal{B}_{(1)} \oplus \mathcal{B}'$, where $\mathcal{B}_{(1)}$ is a constant sheaf of rank c-rk$(\mathcal{B}_V^{\mathbb{C}}|_{L_i})$, corresponding to the eigenvalue 1 (if it is present) of A, and the monodromy matrix of $\mathcal{B}'$ does not have 1 among its eigenvalues. Since $\sigma(N_i) = \mathrm{sign}[S_i^2]^2$, where S_i^2 is the zero-section of N_i, we have

$$\sigma(N_i; \mathcal{B}_{(1)}) = \text{c-rk}(\mathcal{B}_V^{\mathbb{C}}|_{L_i}) \cdot \mathrm{sign}[S_i^2]^2.$$

Since 1 is not an eigenvalue of $\mathcal{B}'$, the intersection chain sheaf $\mathbf{IC}^\bullet_{\bar{m}}(\mathring{N}_i; \mathcal{B}')$ is zero over $S_i^2 \subset \mathring{N}_i$ and thus for the middle hypercohomology,

$$\mathcal{H}_c^{-2}(\mathring{N}_i; \mathbf{IC}^\bullet_{\bar{m}}(\mathcal{B}')) \cong H_c^2(\mathring{N}_i - S_i^2; \mathcal{B}').$$

From a transfer-map argument involving the universal cover $S^3 \to L(p_i, q_i)$, one infers $H_c^2(\overset{\circ}{N_i} - S_i^2; \mathcal{B}') = 0$. In particular $\sigma(N_i; \mathcal{B}') = 0$, and therefore

$$\sigma(N_i; \mathcal{B}_V^{\mathbb{C}}) = \sigma(N_i; \mathcal{B}_{(1)}).$$

By [APS75], the difference between the untwisted and the twisted signature of E is a differential invariant of ∂E, the rho-invariant

$$\rho(\partial E; \mathcal{B}_V^{\mathbb{C}}) = r_V \sigma(E) - \sigma(E; \mathcal{B}_V^{\mathbb{C}}).$$

Thus

$$\begin{aligned}\sigma(E; \mathcal{B}_V^{\mathbb{C}}) &= r_V \sigma(E) - \rho\Big(\bigsqcup_i L_i; \bigsqcup_i \mathcal{B}_V^{\mathbb{C}}|_{L_i}\Big) \\ &= r_V \sigma(V) - \sum_i \rho(L_i; \mathcal{B}_V^{\mathbb{C}}|_{L_i}) \\ &= r_V \sigma(V) - \sum_i \rho_{\alpha_i}(p_i, q_i).\end{aligned}$$

□

We shall now turn our attention to fibered embeddings of strata; the dimension of the singular set is arbitrary.

Theorem 10 *Let $i : X^n \hookrightarrow M^{n+2}$, $n \equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M with stratification*

$$X = X_n \supset X_{n-2} \supset X_{n-4} \supset \cdots \supset X_{-1} = \varnothing.$$

If $\overline{V} - V \hookrightarrow \overline{V}$ is a locally flat spherical fibered knot for all $V \in \mathcal{X}$, then

$$\sigma(X) = L_0(M, X).$$

Proof This follows from the proof of Theorem 4.7 in [Ban08], by using block bundles instead of fiber bundles. Let us recall the argument briefly. Let V be a component in $\mathcal{X}$. By assumption, the embedding $\overline{V} - V \hookrightarrow \overline{V}$ is a locally flat spherical fibered knot $S^k \hookrightarrow S^{k+2}$. The complement of this knot carries the complexified Blanchfield form $\mathcal{B}_V^{\mathbb{C}} \otimes (\mathcal{B}_V^{\mathbb{C}})^{\mathrm{op}} \to \mathbb{C}_V$. By assumption, the exterior E PL-fibers over a circle with Seifert manifold fiber F, i.e. E is a block bundle over S^1: the circle is triangulated by a finite simplicial complex K, for every simplex $\Delta \in K$, there is a block $\Delta \times F$, and E is obtained from the disjoint union of all these blocks by gluing $\Delta^0 \times F$ to $\Delta^1 \times F$ for every 1-simplex $\Delta^1 \in K$ and 0-simplex $\Delta^0 \in K$ such that Δ^0 is a face of Δ^1. The gluing is effected by a PL-homeomorphism $f(\Delta^1, \Delta^0) : F \to F$, which is the identity on $\partial F = S^k$. Set $F' = F \cup_{\partial F} D^{k+1}$. Since every $f(\Delta^1, \Delta^0)$ is the identity on ∂F, we can extend it to a PL-homeomorphism

$f'(\Delta^1, \Delta^0) : F' \to F'$ by taking f' to be the identity on D^{k+1}. Using the system $\{f'(\Delta^1, \Delta^0)\}$ to glue the blocks $\Delta \times F'$, $\Delta \in K$, we obtain the total space M^{k+2} of an F'-block bundle over S^1. This manifold M is the result of surgery on the knot. Let P be the total space of the cone-block bundle associated to the blocking of M. That is, if PL-homeomorphisms $cf'(\Delta^1, \Delta^0) : cF' \to cF'$ on the cone cF' of F' are defined by coning $f'(\Delta^1, \Delta^0)$, then P is obtained by using the system $\{cf'(\Delta^1, \Delta^0)\}$ to glue the blocks $\Delta \times cF'$, $\Delta \in K$. Let $\mathcal{B}^{\mathbb{C}}_M$ be the unique extension of $\mathcal{B}^{\mathbb{C}}_V$ to M. Then, as in the proof of Theorem 7, $\sigma(\overline{V}; \mathcal{B}^{\mathbb{C}}_V) = \sigma(M; \mathcal{B}^{\mathbb{C}}_M)$. The space P is a stratified pseudomanifold-with-boundary, with stratification $P_{k+3} = P \supset P_1 = S^1$, $\partial P = M$. The singular stratum contains the cone-points of the cF'. On the interior $M \times (0, 1)$ of the top stratum, $\mathcal{B}^{\mathbb{C}}_M$ defines a perfect ± 1-Hermitian local system, which can be extended into the singular stratum P_1 by the middle-perversity Deligne step. The result is an intersection chain sheaf $\mathbf{IC}^{\bullet}_{\bar{m}}(P - \partial P; \mathcal{B}^{\mathbb{C}}_M)$ which is self-dual, as P_1 has even codimension $k + 2$ in P. Thus

$$(P; \mathbf{IC}^{\bullet}_{\bar{m}}(P - \partial P; \mathcal{B}^{\mathbb{C}}_M))$$

is a null-cobordism for $(M; \mathcal{B}^{\mathbb{C}}_M)$ and $\sigma(M; \mathcal{B}^{\mathbb{C}}_M) = 0$. Thus the contributions of all V in the Cappell-Shaneson L-class formula (1.7) vanish. □

Corollary 5 *Let $i : X^n \hookrightarrow S^{n+2}$, $n \equiv 0(4)$, $n > 0$, be a PL embedding of a compact oriented PL pseudomanifold X in a sphere with stratification*

$$X = X_n \supset X_{n-2} \supset X_{n-4} \supset \cdots \supset X_{-1} = \varnothing.$$

If $\overline{V} - V \hookrightarrow \overline{V}$ is a locally flat spherical fibered knot for all $V \in \mathcal{X}$, then

$$\sigma(X) = 0.$$

Proof We have $L_0(S^{n+2}, X) = 0$ by (1.8). □

Acknowledgements The authors thank Laurentiu Maxim for correcting an expository error in Sect. 1.7.

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Chapter 2
Lower Bounds on Virtual Crossing Number and Minimal Surface Genus

Kumud Bhandari, H.A. Dye, and Louis H. Kauffman

Abstract We compute lower bounds on the virtual crossing number and minimal surface genus of virtual knot diagrams from the arrow polynomial. In particular, we focus on several interesting examples.

2.1 The Arrow Polynomial

The arrow polynomial is an invariant of oriented link diagrams introduced in [DK]. This polynomial is an element of the ring $\mathbb{Z}[\![A, A^{-1}, K_1, K_2, \ldots]\!]$ and is invariant under both the classical and virtual Reidemeister moves. This polynomial is equivalent to the simplified extended bracket [Kau] and the Miyazawa polynomial [Kam07a, Kam07b, KM05, Miy08].

A *virtual link diagram* [Kau99] is a decorated immersion of n copies of S^1 into the plane, with two types of double points: virtual and classical crossings. Classical crossings are indicated by over under markings and virtual crossings are indicated by a solid, circled crossing. A *virtual link* is an equivalence class of virtual link diagrams. Two virtual link diagrams are equivalent if one can be transformed into the other by a sequence of classical and virtual Reidemeister moves. The classical Reidemeister moves are shown in Fig. 2.1 and the virtual Reidemeister moves are shown in Fig. 2.2.

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*, Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_2,

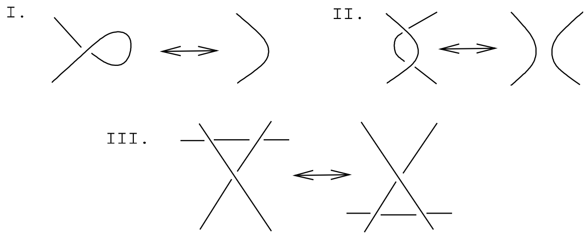


Fig. 2.1 Classical Reidemeister moves

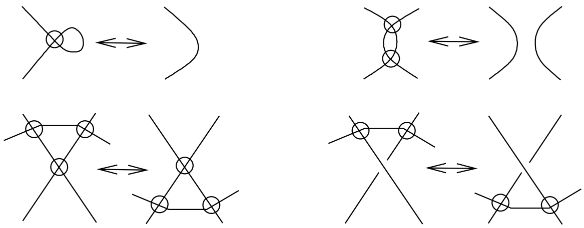


Fig. 2.2 Virtual Reidemeister moves

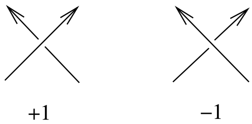


Fig. 2.3 Crossing sign

An oriented virtual link diagram is determined by assigning an orientation to each component of the diagram. For an n component link, there are 2^n possible orientations. Based on the orientation of each component, we can determine a numerical value associated with each classical crossing v as shown in Fig. 2.3. This numerical value associated with a classical crossing v is called the *crossing sign* and denoted $sgn(v)$. Based on the crossing sign, we can compute the *writhe* of a link diagram L, denoted $w(L)$. The writhe is computed by summing over all classical crossings v in the diagram. That is,

$$w(L) = \sum_{v} sgn(v) \tag{2.1}$$

The arrow polynomial, defined in [DK], is an invariant of oriented, virtual link diagrams. These polynomials are elements of the commutative ring:

$$\mathbb{Z}[\![A, A^{-1}, K_1, K_2, \ldots]\!]$$

where the K_i form an infinite set of variables. The arrow polynomial is obtained from the oriented skein relation shown in Fig. 2.4. Applying this skein relation to each classical crossing results in a weighted sum of states with coefficients in $\mathbb{Z}[\![A, A^{-1}]\!]$. We obtain a state of the diagram by choosing either a horizontal or vertical smoothing for each classical crossing in the diagram. This state consists of a collection of closed loops, possibly with virtual crossings. Each loop contains a

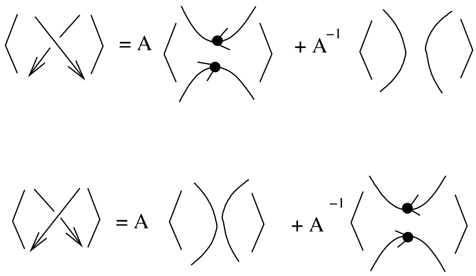


Fig. 2.4 Arrow polynomial skein relation

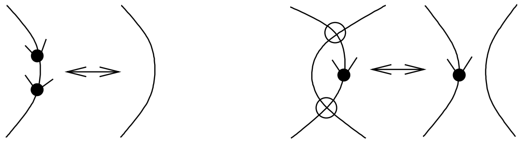


Fig. 2.5 Virtual Reidemeister move with a nodal arrow

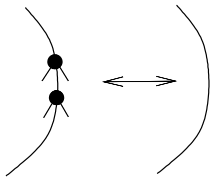


Fig. 2.6 Reducing the total number of arrows

non-negative, even number of nodal arrows. We can simplify each state into a collection of disjoint loops possibly containing nodal arrows by applying the virtual Reidemeister moves to the diagram and using the move shown in Fig. 2.5. The total number of nodal arrows in a component is reduced using the convention shown in Fig. 2.6. We evaluate a single loop C with $2n$ arrows as follows:

$$\langle C \rangle = K_n. \tag{2.2}$$

Then for a state S with

$$S = \prod_{i=1}^{n} C_i$$

we find that

$$\langle S \rangle = \prod_{i=1}^{n} \langle C_i \rangle. \tag{2.3}$$

Let $d = -A^2 - A^{-2}$ and let L denote a virtual link, then the arrow polynomial of L is:

$$\langle L \rangle_A = \sum_S A^{\alpha-\beta} d^{|S|-1} \langle S \rangle \tag{2.4}$$

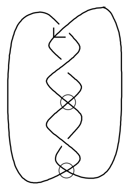


Fig. 2.7 Virtualized trefoil

where α is the number of smoothings in the state S with coefficient A and β is the number of smoothings with coefficient A^{-1}, and $|S|$ denote the number of closed loops in the state. Recall that $\langle L \rangle_A$ is invariant under the virtual Reidemeister moves and the classical Reidemeister moves II and III [DK, Kau]. The normalized arrow polynomial, denoted $\langle L \rangle_{NA}$, is invariant under all classical and virtual Reidemeister moves. The normalized arrow polynomial of a link L is:

$$(-A^3)^{-w(L)} \langle L \rangle_A. \tag{2.5}$$

For example, the arrow polynomial of the knot shown in Fig. 2.7 is:

$$-A^{-3}(-A^{-5} + K_1^2 A^{-5} - K_1^2 A^3). \tag{2.6}$$

The arrow polynomial determines a lower bound on both the genus and the virtual crossing number of a virtual link. Recall that the *virtual crossing number* of a link L (denoted $v(L)$) is the minimum number of virtual crossings in any virtual link diagram equivalent to L. Notice that individual summands of $\langle L \rangle_A$ have the form:

$$A^m K_{i_1}^{p_1} K_{i_2}^{p_2} \dots K_{i_n}^{p_n}$$

The *k-degree* of the summand is:

$$i_1 \times p_1 + i_2 \times p_2 + \dots + i_n \times p_n$$

The *maximum k-degree* of $\langle L \rangle_A$ is the maximum k-degree of the summands. In [DK], the following theorem was proved.

Theorem 1 *Let K be a virtual link diagram. Then the virtual crossing number of K, $v(K)$, is greater than or equal to the maximum k-degree of $\langle K \rangle_A$.*

Hence the maximum k-degree provides a lower bound on the virtual crossing number.

Recall that virtual links are in one to one correspondence with representations of virtual links [CKS02, DK05]. A representation of a virtual link, denoted (F, L), is an embedding of the link L in $F \times I$ where F is a closed, two dimensional, oriented surface modulo Dehn twists, isotopy of the link with in $F \times I$, and handle addition/subtraction. Representations are in one to one correspondence with virtual links. Kuperberg proved the following theorem in [Kup03]:

Theorem 2 *Every stable equivalence class of links in thickened surfaces has a unique irreducible representative.*

Hence, each virtual link corresponds to a representation with a minimum genus surface. In [DK], it was shown that the arrow polynomial can determine a lower bound on the mimimum genus.

Theorem 3 *Let S be an oriented, closed, 2-dimensional surface with genus $g \geq 1$. If $g = 1$ then S contains at most 1 nonintersecting, essential curve and if $g > 1$ then S contains at most $3g - 3$ non-intersecting, essential curves.*

That is, if $\langle L \rangle_A$ contains a K_i in any summand, the minimum genus is at least one. If a single summand contains a factor of the form $K_i K_j$ then the minimum genus is at least two.

2.2 Computations

In the Tables 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7, the arrow polynomial has been computed for all virtual knots with four or fewer classical crossings. This tabulation is based on Jeremy Green's knot tables, available at: http://www.math.toronto.edu/drorbn/Students/GreenJ/ [Gre]. Images of all the knots in this paper are available at this website. Since virtual knots have only one component, it is not necessary to specify the orientation of the links. From the arrow polynomial, we have computed both a lower bound on the virtual crossing number and the genus which is also listed in the table. There are only four knots with arrow polynomial one: (4.46, 4.72, 4.98, 4.107). In comparison, 24 four crossing knots (out of 108 knots) have Jones polynomial one.

The maximum lower bound on virtual crossing number is three, and based on the computational results we make the following conjecture:

Conjecture 1 Given a virtual knot, K, an upper bound on the number of virtual crossings is determined by the minimum number of classical crossings.

Unlike the virtual crossing number, the classical crossing number can be determined from the Gauss diagram. We focus on several specific examples in the remainder of the paper.

2.2.1 Knot 4.01

The arrow polynomial of knot 4.01 (Fig. 2.8) is:

$$A^8 K_1^2 - 3K_1^2 + 2 - 2A^4 K_1^2 + 2K_2^1 - 2A^2 K_1 + 2A^{-2} K_1 + A^{-4}. \tag{2.7}$$

Table 2.1 Bounds: Knots 1–15

Knot	Arrow polynomial	$v(K)$	$g(K)$
4.01	$A^8K_1^2 - 3K_1^2 + 2 - 2A^4K_1^2 + 2K_2^1 - 2A^2K_1 + 2A^{-2}K_1 + A^{-4}$	2	1
4.02	$-A^6K_1 - A^4K_1^2 + 2K_2 + 3 - 2K_1^2 + A^2K_1 + A^{-2}K_1 - A^{-4}K_1^2 - A^{-6}K_1$	2	1
4.03	$A^8K_1^2 - A^4 - K_1^2 + 1 - 2A^2K_1^1 - A^4K_2 + K_2 + 2A^{-2}K_1^1 + A^{-4}$	2	1
4.04	$A^2 - A^4K_1^1 - 2A^2K_1^2 - 2A^{-2}K_1^2 + A^{-2}K_2 + A^2K_2 + 2A^{-2} + 1K_1$	2	1
4.05	$-A^4K_1^1 + A^2 - 2A^2K_1^2 - 2A^{-2}K_1^2 + A^{-2}K_2 + A^2K_2 + K_1 + 2A^{-2}$	2	1
4.06	$-A^6K_1 - A^4K_1^2 + K_2 + A^{-4}K_1^2 - A^{-4}K_2 + A^2K_1 + 2 + A^{-2}K_1 - A^{-4} - A^{-6}K_1$	2	1
4.07	$A^8K_1^2 - 3K_1^2 + 2K_2^1 - 2A^4K_1^2 + 2 - 2A^2K_1^1 + 2A^{-2}K_1^1 + A^{-4}$	2	1
4.08	$-A^6K_1 - A^4K_1^2 + 3 + 2K_2 - 2K_1^2 + A^2K_1 + A^{-2}K_1 - A^{-4}K_1^2 - A^{-6}K_1$	2	1
4.09	$-A^4 - A^2K_1 + A^{-2}K_1 + A^{-4}$	1	1
4.10	$-A^6 - A^4K_1 + 2A^2 + 2K_1 + A^{-2} - A^2K_1^2 - A^{-2}K_1^2 + A^{-2}K_2 - A^{-4}K_1$	2	1
4.11	$-A^{-2}K_1^2 + A^2 - A^4K_1 - A^2K_1^2 + A^{-2}K_2 + K_1 + A^{-2}$	2	1
4.12	$-A^6K_1 - A^4K_2 + A^2K_1 + 1 + 2K_2 - A^{-4}K_2 + A^{-2}K_1 - A^{-6}K_1$	2	1
4.13	$A^4 - A^{-4}K_1^2 + 1 - A^4K_1^2 - 2K_1^2 + 2K_2 + A^{-4}$	2	1
4.14	$-A^6K_1 - 2A^{-4}K_1^2 + A^2K_1 + 2K_2 + 2 - 2K_1^2 + A^{-4}$	2	1
4.15	$K_2 - A^4K_1^2 - K_1^2 + 1 - A^2K_1 + A^{-2}K_1 + A^{-4}$	2	1

Table 2.2 Bounds: Knots 16–30

Knot	Arrow polynomial	$v(K)$	$g(K)$
4.16	$-A^6 - A^4K_1 + A^2 + 2K_1 + A^{-2} - A^{-4}K_1$	1	1
4.17	$-A^6K_1 - A^4K_2 + A^2K_1 + 2 + 2K_2 - A^{-4}K_1^2 - K_1^2 + A^{-2}K_1 - A^{-6}K_1$	2	1
4.18	$-2A^{-2}K_1^2 - A^4K_1 + A^2K_2 - 2A^2K_1^2 + 2A^{-2} + A^2 + K_1 + A^{-2}K_2$	2	1
4.19	$A^4 + K_2 + 1 - A^4K_1^2 - K_1^2$	2	1
4.20	$-A^6K_1 + K_2 + A^2K_1 + 2 - A^{-4}K_1^2 - K_1^2$	2	1
4.21	$A^2 + A^2K_2 - A^4K_1 - A^2K_1^2 - 2A^{-2}K_1^2 + 2A^{-2} + 2K_1 + A^{-2}K_2 - A^{-4}K_1 - A^{-6}K_1^2$	2	1
4.22	$-A^6K_1 + A^2K_1 + K_2 + 2 - A^2K_1K_2 - A^{-2}K_1K_2 - K_1^2 - A^{-4}K_1^2 + A^2K_1^3 + 2A^{-2}K_1^3 + A^{-6}K_1^3 - A^{-2}K_1 - A^{-6}K_1$	3	2
4.23	$A^8K_1 - 2A^4K_1 + 2A^{-2} - A^2K_1^2 - A^{-2}K_1^2 + K_1 + A^{-2}K_2$	2	1
4.24	$A^2 - 2A^4K_1 + A^4K_1^3 + 2K_1^3 + A^{-4}K_1^3 - A^{-4}K_1 + A^{-2} + K_1 + A^{-2}K_2 - 1K_1K_2 - A^{-4}K_1K_2 - A^{-2}K_1^2 - A^{-6}K_1^2$	3	2
4.25	$-A^6K_1 - A^{-2}K_1 + A^{-6}K_1 + A^2K_1 - A^{-4} + 1 + A^{-8}$	1	1
4.26	$A^2K_1 - A^{-2}K_1K_2 - A^2K_1K_2 + 1 + A^{-2}K_3$	3	2
4.27	$A^{-6} - A^{-2} + A^2 - A^{-4}K_1 + A^{-8}K_1$	1	1
4.28	$A^2 + K_3 + K_1 - A^{-4}K_1K_2 - K_1K_2$	3	2
4.29	$1 - A^4K_1^2 - K_1^2 + K_2 - A^2K_1 + A^{-2}K_1 + A^{-4}$	2	1
4.30	$-2A^{-2}K_1^2 + A^2K_2 + A^2 - 2A^2K_1^2 + A^{-2}K_2 - A^4K_1 + 2A^{-2} + K_1$	2	1

Table 2.3 Bounds: Knots 30–45

Knot	Arrow polynomial	$v(K)$	$g(K)$
4.31	$-A^6K_1^2 - 2A^2K_1^2 + 2A^2K_2 - A^{-2}K_1^2 - A^4K_1 + 2K_1 + 2A^{-2} + A^2 - A^{-4}K_1$	2	1
4.32	$-A^6K_1 + A^2K_1 + 3 - A^4K_1^2 - K_1^2 - A^{-4} + K_2 + A^{-2}K_1 - A^{-6}K_1$	2	1
4.33	$-A^4K_1 + K_1 + A^{-2}$	1	1
4.34	$-A^6K_1 + A^2K_1 + 2 - A^{-4}K_1^2 + K_2 - K_1^2$	2	1
4.35	$-K_1^2 - A^{-4}K_1^2 + 2 + A^{-4}K_2$	2	1
4.36	$A^2 + 2K_1 - A^4K_1 - A^{-4}K_1 + A^{-2}K_2 - A^{-6}K_2$	2	1
4.37	$K_2 - A^4K_2 - A^2K_1 + A^{-2}K_1 + A^{-4}$	2	1
4.38	$A^2K_2 - A^4K_1 - A^2K_1^2 - A^{-2}K_1^2 + 2A^{-2} + K_1$	2	1
4.39	$A^2K_2 - A^4K_1K_2 - K_1K_2 - A^2K_1^2 - A^{-2}K_1^2 - A^4K_1 + 2A^{-2} - A^{-4}K_1 + A^4K_1^3 + 2K_1^3 + A^{-4}K_1^3$	3	2
4.40	$-A^6K_1 - A^{-4} + 2 + A^2K_1$	1	1
4.41	$A^8K_1 - 2A^4K_1 - A^2 + K_1 + 2A^{-2}$	1	1
4.42	$-A^6K_1 + A^6K_1^3 + 2A^2K_1^3 + A^{-2}K_1^3 - A^2K_1 + 2 - A^2K_1K_2 - A^{-2}K_1K_2 - K_1^2 - A^{-4}K_1^2 + A^{-4}K_2$	2	2
4.43	$-A^6K_1 - A^{-2}K_1 + A^{-6}K_1 + A^2K_1 - A^{-4} + 1 + A^{-8}$	1	1
4.44	$-A^4K_1 + A^{-2} + K_1$	1	1
4.45	$-K_1K_2 - A^4K_1K_2 + K_1 + K_3 + A^{-2}$ 3	1	

Table 2.4 Bounds: Knots 46–60

Knot	Arrow polynomial	$v(K)$	$g(K)$
4.46	1	0	0
4.47	$A^2K_3^1 + 1 - A^{-2}K_1K_2 - A^2K_1K_2 + A^{-2}K_1$	3	2
4.48	$A^4 - 2A^4K_1^2 - 2K_1^2 + 1 + 2K_2 - A^2K_1 + A^{-2}K_1 + A^{-4}$	2	1
4.49	$A^2K_2 - A^{-2}K_1^2 - A^2K_1^2 - A^4K_1 + K_1 + 2A^{-2}$	2	1
4.50	$-A^6K_1^2 - A^2K_1^2 + A^2 + 2K_1 + A^2K_2 - A^4K_1 - A^{-4}K_1 + A^{-2}$	2	1
4.51	$-A^6K_1 + A^2K_1 + 3 - A^4K_1^2 - 2K_1^2 + 2K_2 - A^{-4}K_1^2 + A^{-2}K_1 - A^{-6}K_1$	2	1
4.52	$-A^6K_1 + A^2K_1 + 2 - A^{-4}$	1	1
4.53	$A^8 - 2A^4 - 2A^2K_1 + 1 + 2A^{-2}K_1 + A^{-4}$	1	1
4.54	$-A^6 - A^4K_1 + A^6K_1^2 + A^{-2} - A^2K_2 + A^2 + K_1 - A^{-2}K_1^2 + A^{-2}K_2$	2	1
4.55	$A^4 + 2K_2 + 1 - A^4K_1^2 - 2K_1^2 - A^{-4}K_1^2 + A^{-4}$	2	1
4.56	$A^4 + 1 - A^4K_1^2 + 2K_2 - 2K_1^2 - A^{-4}K_1^2 + A^{-4}$	2	1
4.57	$-A^6K_2 - A^4K_1 + 2A^2K_2 + 2K_1 + 2A^{-2} - A^2K_1^2 - A^{-2}K_1^2 - A^{-4}K_1$	2	1
4.58	$-A^6K_1 + A^2K_1 + 3 - A^4 + A^{-2}K_1 - A^{-4} - A^{-6}K_1$	1	1
4.59	$A^4K_2 - A^{-4}K_1^2 - A^4K_1^2 - 2K_1^2 + 3 + A^{-4}K_2$	2	1
4.60	$-A^6K_1 - 2A^{-4}K_1^2 + A^2K_1 + 3 + K_2 - 2K_1^2 + A^{-4}K_2$	2	1

Table 2.5 Bounds: Knots 61–75

Knot	Arrow polynomial	$v(K)$	$g(K)$
4.61	$1 - A^4 - A^2K_1 + A^{-2}K_1 + A^{-4}$	1	1
4.62	$A^2 - A^4K_1K_2 - K_1K_2 - A^2K_1^2 - A^{-2}K_1^2 - A^4K_1 + A^{-2}K_2 - A^{-4}K_1 + A^{-2} + A^4K_1^3 + 2K_1^3 + A^{-4}K_1^3$	2	1
4.63	$A^2 - A^4K_1 - A^2K_1^2 - A^{-2}K_1^2 + A^{-2}K_2 + K_1 + A^{-2}$	2	1
4.64	$-A^6K_1 - A^{-4}K_2 + K_2 + 1 + A^2K_1$	2	1
4.65	$A^8K_1 - 2A^4K_1 + A^{-2} - A^2K_2 + K_1 + A^{-2}K_2$	2	1
4.66	$-A^6K_1 + A^{-2}K_1^3 + 1 + A^6K_1^3 + 2A^2K_1^3 - A^{-4}K_1^2 + K_2 - A^2K_1 - A^2K_1K_2 - A^{-2}K_1K_2 - K_1^2 + A^{-4}$	3	2
4.67	$-A^{-4}K_1^2 + 1 + K_2 - K_1^2 + A^{-4}$	2	1
4.68	$A^2 - A^4K_1 + A^{-2} + 2K_1 - A^{-4}K_1 - A^{-6}$	1	1
4.69	$A^4K_2 - 2A^4K_1^2 - 2K_1^2 + K_2 + 2 - A^2K_1 + A^{-2}K_1 + A^{-4}$	2	1
4.70	$-A^6K_1^2 - A^2K_1^2 + A^2K_2 + 2K_1 - A^4K_1 + A^2 - A^{-4}K_1 + A^{-2}$	2	1
4.71	$-A^6K_1 + A^2K_1 + 2K_2 - A^4K_1^2 - 2K_1^2 + 3 - A^{-4}K_1^2 + A^{-2}K_1 - A^{-6}K_1$ 2	1	
4.72	1	0	0
4.73	$A^8K_2 - 2A^4K_2 - 2A^2K_1 + K_2 + 2A^{-2}K_1 + A^{-4}$	2	1
4.74	$-A^6K_2 - A^4K_1 + A^6K_1^2 + 2A^{-2} - A^2 + A^2K_2 + K_1 - A^{-2}K_1^2$	2	1
4.75	$-A^6K_1 - A^4 + 3 + A^2K_1 + A^{-2}K_1 - A^{-4} - A^{-6}K_1$	1	1

Table 2.6 Bounds: Knots 76–90

Knot	Arrow polynomial	$v(K)$	$g(K)$
4.76	$A^4K_2 - A^4K_1^2 + 3 - 2K_1^2 - A^{-4}K_1^2 + A^{-4}K_2$	2	1
4.77	$A^4K_2 - 2K_1^2 - A^4K_1^2 + 3 - A^{-4}K_1^2 + A^{-4}K_2$	2	1
4.78	$-A^6K_1K_2 - A^2K_1K_2 - A^4K_1^2 - K_1^2 + A^{-2}K_1^3 + K_2 + 1 + A^6K_1^3 + 2A^2K_1^3 - 2A^2K_1 + A^{-4}$	3	2
4.79	$A^8K_1^3 + 2A^4K_1^3 + K_1^3 - 3A^4K_1 + 2A^{-2} - A^4K_1K_2 - K_1K_2 - A^2K_1^2 - A^{-2}K_1^2 + K_1 + A^{-2}K_2$	3	2
4.80	$-A^6K_1K_2 - A^2K_1K_2 - A^{-2}K_1 + A^2K_1 + A^{-6}K_1 + A^2K_3 - A^{-4} + 1 + A^{-8}$	3	2
4.81	$A^4K_3 - A^{-2} + A^2 + A^{-6} - A^4K_1K_2 - K_1K_2 - A^{-4}K_1 + K_1 + A^{-8}K_1$	3	2
4.82	$-A^6K_1 + A^{-4}K_1^2 - A^4K_1^2 + 2 - 2A^{-4} + A^2K_1 + A^{-8}$	2	1
4.83	$-K_1K_2 - A^4K_1K_2 + K_1 + A^{-2} + K_3$	3	2
4.84	$2 + A^2K_1 - A^{-4} - A^{-2}K_1 - K_1^2 + A^{-8}K_1^2$	2	1
4.85	$-A^2K_1^2 + A^2 + A^{-6}K_1^2$	2	1
4.86	$A^8 - A^4 + 2 - A^{-4} - K_1^2 + A^{-8}K_1^2$	2	1
4.87	$-A^6K_1K_2 - A^2K_1K_2 - A^4K_1^2 - 2A^{-4} + A^2K_1 + 2 + A^2K_3 + A^{-4}K_1^2 + A^{-8}$	3	2
4.88	$2 + A^2K_1 + A^2K_3 - A^{-4} - A^2K_1K_2 - A^{-2}K_1K_2 - K_1^2 + A^{-8}K_1^2$	3	2
4.89	$A^4 - 2A^4K_1^2 + 2A^{-4}K_1^2 - 2A^{-4} + 1 + A^{-8}$	2	1
4.90	$A^8K_1^2 - A^4 - 2K_1^2 + 3 - A^{-4} + A^{-8}K_1^2$	2	1

Table 2.7 Bounds: Knots 90–108

Knot	Arrow Polynomial	$v(K)$	$g(K)$
4.91	$-A^{10}K_1^3 - A^6K_1^3 + A^2K_1^3 + 2A^6K_1 + A^{-2}K_1^3 - 2A^2K_1 + A^{-4}$	3	1
4.92	$-A^6K_3 - A^4 + 2 + A^2K_3 - A^{-4} + A^{-8}$	3	1
4.93	$A^4K_3 + A^{-6}K_1^2 + A^2 - A^2K_1^2 - A^4K_1K_2 - K_1K_2 + K_1$	3	2
4.94	$-A^6K_1 + 2 - A^4 + A^2K_1 - A^{-4} + A^{-8}$	1	1
4.95	$-A^4K_3 + A^{-2} + K_3$	3	1
4.96	$A^{-6}K_1^2 - A^2K_1^2 + A^2$	2	1
4.97	$-A^2K_1K_2 - A^{-2}K_1K_2 + A^2K_3 + 1 + A^{-2}K_1$	3	1
4.98	1	0	0
4.99	$A^8 - A^4 - A^{-4} + 1 + A^{-8}$	0	0
4.100	$-A^{10}K_1 + A^6K_1 - A^2K_1 + A^{-2}K_1 + A^{-4}$	1	1
4.101	$A^8K_1 + K_1^3 - A^{-4}K_1 + A^{-2} - A^8K_1^3 - A^4K_1^3 + A^{-4}K_1^3$	3	1
4.102	$-A^6K_1 - A^{-2}K_1^3 + 1 + A^6K_1^3 + A^2K_1^3 - A^{-6}K_1^3 - A^2K_1 + A^{-6}K_1 + A^{-2}K_1$	3	1
4.103	$A^4K_1 + A^2 + K_3 - A^4K_1K_2 - K_1K_2 - A^2K_1^2 + A^{-6}K_1^2$	3	2
4.104	$A^2K_3 - A^{-2}K_3 - A^{-4} + 1 + A^{-8}$	3	1
4.105	$-A^4 + 1 + A^{-8}$	0	0
4.106	$A^2 - A^2K_1^2 + A^{-6}K_1^2$	2	1
4.107	1	0	0
4.108	$A^8 - A^4 - A^{-4} + 1 + A^{-8}$	0	0

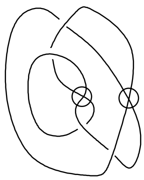


Fig. 2.8 Knot 4.01

The lower bound on the virtual crossing number is two and the lower bound on the genus is one. However, this minimal genus of this virtual knot is two.

2.2.2 Knot 4.09

The arrow polynomial of the knot 4.09 (Fig. 2.9) is:

$$-A^4 - A^2K_1 + A^{-2}K_1 + A^{-4}.$$

The lower bound on the virtual crossing number and the mimimal genus is one. Note a single detour move reduces the number of virtual crossings to three.

Fig. 2.9 Knot 4.09

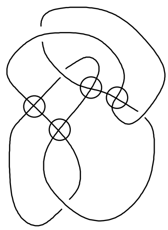


Fig. 2.10 Knot 4.22 and an equivalent knot

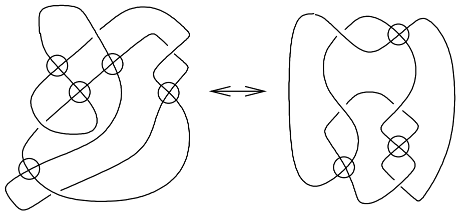
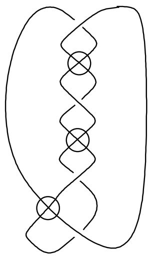


Fig. 2.11 Knot 4.47



2.2.3 Knot 4.22

The arrow polynomial of the knot 4.22 (Fig. 2.10) is:

$$-A^6K_1 + A^2K_1 + K_2 + 2 - A^2K_1K_2 - A^{-2}K_1K_2 - K_1^2$$
$$- A^{-4}K_1^2 + A^2K_1^3 + 2A^{-2}K_1^3 + A^{-6}K_1^3 - A^{-2}K_1 - A^{-6}K_1 \qquad (2.8)$$

The lower bound on the virtual crossing number is 3 and the lower bound on the minimal genus is two. A sequence of Reidemeister moves reduces the number of virtual crossings to three. The knot pictured in the right hand side of Fig. 2.10 is equivalent to knot 4.22. This demonstrates that the virtual crossing number is three.

2.2.4 Knot 4.47

The arrow polynomial of knot 4.47 (Fig. 2.11) is:

$$A^2K_3^1 + 1 - A^{-2}K_1K_2 - A^2K_1K_2 + A^{-2}K_1$$

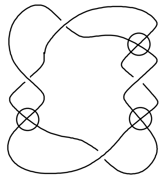


Fig. 2.12 Knot 4.91

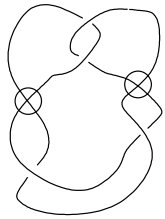


Fig. 2.13 Knot 4.99

The virtual crossing number is three and the minimal genus is at least two. The Jones polynomial does not differentiate between this knot and the unknot.

2.2.5 Knot 4.91

The arrow of polynomial of knot 4.91 (Fig. 2.12) is:

$$-A^{10}K_1^3 - A^6K_1^3 + A^2K_1^3 + 2A^6K_1 + A^{-2}K_1^3 - 2A^2K_1 + A^{-4}$$

The virtual crossing number of the knot 4.91 is three and the minimal genus is one as predicted by the arrow polynomial.

2.2.6 Knot 4.99

The arrow polynomial of the knot 4.99 (Fig. 2.13) is:

$$A^8 - A^4 - A^{-4} + 1 + A^{-8}.$$

This results in a lower bound on the virtual crossing number and the minimal genus of zero. Lower bounds on crossing number and virtual genus have been considered in [AM] and [Man03]. We observe that under virtualization, this knot is equivalent to a classical knot.

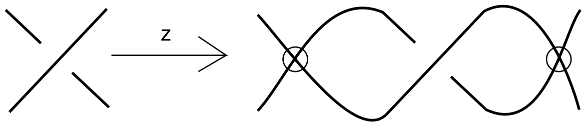


Fig. 2.14 Z-equivalence

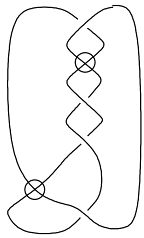


Fig. 2.15 Knot 4.46

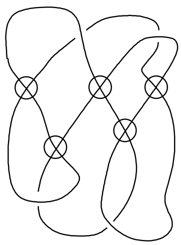


Fig. 2.16 Knot 4.72

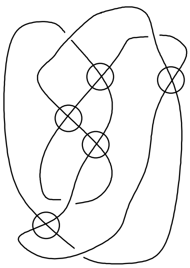


Fig. 2.17 Knot 4.98

2.2.7 Knots with Arrow Polynomial One

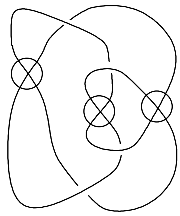
The knots 4.46, 4.72, 4.98, and 4.107 have arrow polynomial one and are equivalent to the unknot via a sequence of classical and virtual Reidemeister moves and the Z-equivalence (shown in Fig. 2.14).

The knots in Figs. 2.15, 2.16, 2.17, and 2.18 have arrow polynomial one.

In the paper [FKM05], the authors (Fenn, Kauffman, and Manturov) made the following conjecture:

Conjecture 2 Let K be a virtual knot. If the bracket polynomial of K, $\langle K\rangle = 1$ then K is Z-equivalent to the unknot.

Fig. 2.18 Knot 4.107



Note that if $\langle K \rangle_A = 1$ (arrow polynomial) then $\langle K \rangle = 1$ (bracket polynomial). This fact and our experimental evidence support the following conjecture:

Conjecture 3 Let K be a virtual knot. If $\langle K \rangle_A = 1$ then K is Z-equivalent to the unknot.

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Chapter 3
A Survey of Twisted Alexander Polynomials

Stefan Friedl and Stefano Vidussi

Abstract We give a short introduction to the theory of twisted Alexander polynomials of a 3-manifold associated to a representation of its fundamental group. We summarize their formal properties and we explain their relationship to twisted Reidemeister torsion. We then give a survey of the many applications of twisted invariants to the study of topological problems. We conclude with a short summary of the theory of higher order Alexander polynomials.

3.1 Introduction

In 1928 Alexander introduced a polynomial invariant for knots and links which quickly got referred to as the Alexander polynomial. His definition was later recast in terms of Reidemeister torsion by Milnor [Mi62] and it was extended by Turaev [Tu75, Tu86] to an invariant of 3-manifolds. More precisely, to a 3-manifold with empty or toroidal boundary N we can associate its Alexander polynomial Δ_N which lies in the group ring $\mathbb{Z}[H]$, where H is the maximal abelian quotient of $H_1(N;\mathbb{Z})$. The Alexander polynomial of knots, links and 3-manifolds in general is closely related to the topology properties of the underlying space. For example it is known to contain information on the knot genus [Se35], knot concordance [FM66], fiberedness and symmetries.

The Alexander polynomial carries only metabelian information on the fundamental group. This limitation explain why in all the above cases the Alexander

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_3, © Springer-Verlag Berlin Heidelberg 2011

polynomial carries partial, but not complete information. The idea behind twisted invariants is to associate a polynomial invariant to a 3-manifold *together* with a choice of a representation of its fundamental group. This approach makes it possible to extract more powerful topological information.

The twisted Alexander polynomial for a knot $K \subset S^3$ was first introduced by Xiao–Song Lin in 1990 (cf. [Lin01]). Whereas Lin's original definition used 'regular Seifert surfaces' of knots, later extensions to links and 3-manifolds either generalized the Reidemeister–Milnor–Turaev torsion (cf. [Wa94, Ki96, KL99a, FK06]) or generalized the homological definition of the Alexander polynomial (cf. [JW93, KL99a, Ch03, FK06, HKL10]).

In most cases the setup for twisted invariants is as follows: Let N be a 3-manifold with empty or toroidal boundary, $\psi : \pi_1(N) \to F$ an epimorphism onto a free abelian group F and $\gamma : \pi_1(N) \to GL(k, R)$ a representation with R a domain. In that case one can define the twisted Reidemeister torsion $\tau(N, \gamma \otimes \psi)$, an invariant which in general lives in the quotient field of the group ring $R[F]$. If R is furthermore a Noetherian unique factorization domain (e.g. $R = \mathbb{Z}$ or R a field), then the twisted Alexander polynomials $\Delta_{N,i}^{\gamma\otimes\psi} \in R[F]$ is defined to be the order of the twisted Alexander module $H_i(N; R[F]^k)$.

These two invariants are closely related, for example in the case that $\operatorname{rank}(F) \geq 2$ we will see that

$$\tau(N, \gamma \otimes \psi) = \Delta_{N,1}^{\gamma\otimes\psi} \in R[F].$$

In fact in many papers the twisted Reidemeister torsion $\tau(N, \gamma \otimes \psi)$ is referred to as the twisted Alexander polynomial (cf. e.g. [Wa94]).

The most important raison d'être of these invariants lies in the fact that they contain deep information on the underlying topology while at the same time being, as we will see, very computable invariants.

We now give a short outline of the paper. In Sect. 3.2 we define twisted Reidemeister torsion and twisted Alexander polynomials of 3-manifolds, and we show how to calculate these invariants. In Sect. 3.3 we discuss basic properties of twisted invariants, in particular we discuss the relationship between twisted Reidemeister torsion and twisted Alexander polynomials and we discuss the effect of Poincaré duality on twisted invariants. Section 3.4 contains applications to distinguishing knots and links using twisted invariants. In Sect. 3.5 we outline the results of Kirk and Livingston regarding the behavior of twisted invariants under knot concordance and we extend the results to the study of doubly slice knots and ribbon knots. In Sect. 3.6 we show that twisted invariants give lower bounds on the knot genus and the Thurston norm, and we show that they give obstructions to the fiberedness of 3-manifolds. Sections 3.7, 3.8 and 3.9 contains a discussion of the many generalizations and further applications of twisted invariants. In Sect. 3.10 we give an overview of the closely related theory of higher-order Alexander polynomials, this theory was initiated by Cochran and Harvey. Finally in Sect. 3.11 we provide a list of open questions and problems.

Conventions and Notation Unless we say otherwise we adopt the following conventions:

(1) rings are commutative domains with unit element,
(2) 3-manifolds are compact, connected and orientable,
(3) homology is taken with integral coefficients,
(4) groups are finitely generated.

We also use the following notation: Given a ring R we denote by $Q(R)$ its quotient field and given a link $L \subset S^3$ we denote by νL a (open) tubular neighborhood of L in S^3.

Remark For space reasons we unfortunately have to exclude from our exposition several important aspects of the subject. Among the most relevant omissions, we mention Turaev's torsion function, and the relation between torsion invariants on the one hand and Seiberg–Witten theory and Heegaard–Floer homology on the other. Turaev's torsion function is defined using Reidemeister torsion corresponding to one-dimensional abelian representations. This theory and its connection to Seiberg–Witten invariants, first unveiled by Meng and Taubes in [MT96] (cf. also [Do99]), is treated beautifully in Turaev's original papers [Tu97, Tu98] and in Turaev's books [Tu01, Tu02a]. We also refer to [OS04] for the relation of Turaev's torsion function to Heegaard–Floer homology.

Remark Almost all the results of this survey paper appeared already in previous paper. We hope that we correctly stated the results of the many authors who worked on twisted Alexander polynomials. For the definite statements we nonetheless refer to the original papers. The only new results are some theorems in Sect. 3.5.2 on knot and link concordance and Theorem 10 on Reidemeister torsion of fibered manifolds.

3.2 Definition and Basic Properties

3.2.1 Twisted Reidemeister Torsion

Let N be a 3-manifold with empty or toroidal boundary, F a torsion-free abelian group and $\alpha : \pi_1(N) \to GL(k, R[F])$ a representation. Recall that we denote by $Q(R[F])$ the quotient field of $R[F]$.

We endow N with a finite CW-structure. We denote the universal cover of N by $\tilde{N}$. Recall that there exists a canonical left $\pi_1(N)$-action on the universal cover $\tilde{N}$ given by deck transformations. We consider the cellular chain complex $C_*(\tilde{N})$ as a right $\mathbb{Z}[\pi_1(N)]$-module by defining $\sigma \cdot g := g^{-1}\sigma$ for a chain σ. The representation α induces a representation $\alpha : \pi_1(N) \to GL(k, R[F]) \to GL(k, Q(R[F]))$ which gives rise to a left action of $\pi_1(N)$ on $Q(R[F])^k$. We can therefore consider the $Q(R[F])$-complex

$$C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} Q(R[F])^k.$$

We now endow the free $\mathbb{Z}[\pi_1(N)]$-modules $C_*(\tilde{N})$ with a basis by picking lifts of the cells of N to $\tilde{N}$. Together with the canonical basis for $Q(R[F])^k$ we can now view the $Q(R[F])$-complex $C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} Q(R[F])^k$ as a complex of based $Q(R[F])$-modules.

If this complex is not acyclic, then we define $\tau(N,\alpha) = 0$. Otherwise we denote by $\tau(N,\alpha) \in Q(R[F]) \setminus \{0\}$ the Reidemeister torsion of this based $Q(R[F])$-complex. We will not recall the definition of Reidemeister torsion, referring instead to the many excellent expositions, e.g. [Mi66, Tu01] and [Nic03]. However, in the next section we will present a method for computing explicitly the twisted Reidemeister torsion of a 3-manifold.

It follows from Chapman's theorem [Chp74] and from standard arguments (cf. the above literature) that up to multiplication by an element in

$$\{\pm \det(\alpha(g)) \mid g \in \pi_1(N)\}$$

the Reidemeister torsion $\tau(N,\alpha)$ is well-defined, i.e. up to that indeterminacy $\tau(N,\alpha)$ is independent of the choice of underlying CW-structure and the choice of the lifts of the cells. In the following, given $w \in Q(R[F])$ we write

$$\tau(N,\alpha) \doteq w$$

if there exists a representative of $\tau(N,\alpha)$ which equals w.

Note that if $\gamma : \pi_1(N) \to GL(k,R)$ is a representation and $\psi : \pi_1(N) \to F$ a homomorphism to a free abelian group, then we get a tensor representation

$$\begin{aligned} \gamma \otimes \psi : \pi_1(N) &\to \mathrm{GL}(k, R[F]) \\ g &\mapsto \gamma(g) \cdot \psi(g) \end{aligned}$$

and the corresponding Reidemeister torsion $\tau(N, \gamma \otimes \psi)$. Except for parts of Sect. 3.5.2 we will always consider the twisted Reidemeister torsion corresponding to such a tensor representation. In that case, specializing the previous formula, $\tau(N, \gamma \otimes \psi)$ is well-defined up to multiplication by an element in

$$\{\pm \det(\gamma(g)) f \mid g \in \pi_1(N), f \in F\}.$$

In particular, if $\gamma : \pi_1(N) \to SL(k,R)$ is a representation to a special linear groups, then $\tau(N, \gamma \otimes \psi) \in Q(R[F])$ is well-defined up to multiplication by an element in $\pm F$. Furthermore, if k is even, then $\tau(N, \gamma \otimes \psi) \in Q(R[F])$ is in fact well-defined up to multiplication by an element in F (cf. e.g. [GKM05]).

Finally we adopt the following notation:

1. Given a homomorphism $\gamma : \pi \to G$ to a finite group G we get an induced representation $\pi \to \mathrm{Aut}(\mathbb{Z}[G]) \cong GL(|G|, \mathbb{Z})$ given by left multiplication. In our notation we will not distinguish between a homomorphism to a finite group and the corresponding representation over $\mathbb{Z}$.
2. If N is the exterior of a link $L \subset S^3$, $\psi : \pi_1(S^3 \setminus \nu L) \to F$ the abelianization and $\gamma : \pi \to GL(k,R)$ a representation, then we write $\tau(L,\gamma)$ for $\tau(S^3 \setminus \nu L, \gamma \otimes \psi)$.

3.2.2 Computation of Twisted Reidemeister Torsion

Let N be a 3-manifold with empty or toroidal boundary, $\psi : \pi_1(N) \to F$ a non-trivial homomorphism to a free abelian group F and $\gamma : \pi_1(N) \to GL(k, R)$ a representation. In this section we will give an algorithm for computing $\tau(N, \gamma \otimes \psi)$ which is based on ideas of Turaev (cf. in particular [Tu01, Theorem 2.2]).

We will first consider the case that N is closed. We write $\pi = \pi_1(N)$. We endow N with a CW-structure with one 0-cell, n 1-cells, n 2-cells and one 3-cell. It is well-known that such a CW-structure exists (cf. e.g. [McM02, Theorem 5.1] or [FK06, Proof of Theorem 6.1]). Using this CW-structure we have the cellular chain complex

$$0 \to C_3(\tilde{N}) \xrightarrow{\partial_3} C_2(\tilde{N}) \xrightarrow{\partial_2} C_1(\tilde{N}) \xrightarrow{\partial_1} C_0(\tilde{N}) \to 0$$

where $C_i(\tilde{N}) \cong \mathbb{Z}[\pi]$ for $i = 0, 3$ and $C_i(\tilde{N}) \cong \mathbb{Z}[\pi]^n$ for $i = 1, 2$. Let $A_i, i = 1, 2, 3$ be the matrices over $\mathbb{Z}[\pi]$ corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$ with respect to the bases given by the lifts of the cells of N to $\tilde{N}$. We can arrange the lifts such that

$$\begin{aligned} A_3 &= (1 - g_1, 1 - g_2, \ldots, 1 - g_n)^t, \\ A_1 &= (1 - h_1, 1 - h_2, \ldots, 1 - h_n) \end{aligned}$$

with $g_1, \ldots, g_n, h_1, \ldots, h_n \in \pi$. Note that $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ are generating sets for π since N is a closed 3-manifold. Since ψ is non-trivial there exist r, s such that $\psi(g_r) \neq 0$ and $\psi(h_s) \neq 0$. Let B_3 be the r-th row of A_3. Let B_2 be the result of deleting the r-th column and the s-th row from A_2. Let B_1 be the s-th column of A_1.

Given a $p \times q$ matrix $B = (b_{rs})$ with entries in $\mathbb{Z}[\pi]$ we write $b_{rs} = \sum b_{rs}^g g$ for $b_{rs}^g \in \mathbb{Z}$, $g \in \pi$. We then define $(\gamma \otimes \psi)(B)$ to be the $p \times q$ matrix with entries

$$\sum b_{rs}^g (\gamma \otimes \psi)(g) = \sum b_{rs}^g \gamma(g) \cdot \psi(g) \in R[F].$$

Since each such entry is a $k \times k$ matrix with entries in $R[F]$ we can think of $(\gamma \otimes \psi)(B)$ as a $pk \times qk$ matrix with entries in $R[F]$.

Now note that

$$\det((\gamma \otimes \psi)(B_3)) = \det(\mathrm{id} - \gamma(g_r) \cdot \psi(g_r)) \neq 0$$

since $\psi(g_r) \neq 0$. Similarly $\det((\gamma \otimes \psi)(B_1)) \neq 0$. The following theorem is an immediate application of [Tu01, Theorem 2.2].

Theorem 1 *We have*

$$\tau(N, \gamma \otimes \psi) \doteq \prod_{i=1}^{3} \det((\gamma \otimes \psi)(B_i))^{(-1)^i}.$$

In particular, $H_*(N; Q(R[F])^k) = 0$ *if and only if* $\det((\gamma \otimes \psi)(B_2)) \neq 0$.

We now consider the case that N has non-empty toroidal boundary. Let X be a CW-complex with the following two properties:

1. X is simple homotopy equivalent to a CW-complex of N,
2. X has one 0-cell, n 1-cells and $n-1$ 2-cells.

It is well-known that such a CW-structure exists. For example, if N is the complement of a non-split link $L \subset S^3$, then we can take X to be the 2-complex corresponding to a Wirtinger presentation of $\pi_1(S^3 \setminus \nu L)$.

We now consider

$$0 \to C_2(\tilde{X}) \xrightarrow{\partial_2} C_1(\tilde{X}) \xrightarrow{\partial_1} C_0(\tilde{X}) \to 0$$

where $C_0(\tilde{X}) \cong \mathbb{Z}[\pi]$, $C_i(\tilde{X}) \cong \mathbb{Z}[\pi]^n$ and $C_i(\tilde{X}) \cong \mathbb{Z}[\pi]^{n-1}$. Let A_i, $i = 1, 2$ over $\mathbb{Z}[\pi]$ be the matrices corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$. As above we can arrange that

$$A_1 = (1 - h_1, 1 - h_2, \ldots, 1 - h_n)$$

where $\{h_1, \ldots, h_n\}$ is a generating set for π. Since ψ is non-trivial there exists an s such that $\psi(h_s) \neq 0$. Let B_2 be the result of deleting the s-th row from A_2. Let B_1 be the s-th column of A_1. As above we have $\det((\gamma \otimes \psi)(B_1)) \neq 0$. The following theorem is again an immediate application of [Tu01, Theorem 2.2].

Theorem 2 *We have*

$$\tau(N, \gamma \otimes \psi) \doteq \prod_{i=1}^{2} \det((\gamma \otimes \psi)(B_i))^{(-1)^i}.$$

In particular, we have $H_*(N; Q(R[F])^k) = 0$ *if and only if* $\det((\gamma \otimes \psi)(B_2)) \neq 0$.

3.2.3 Torsion Invariants

Let S be a Noetherian unique factorization domain (henceforth UFD). Examples of Noetherian UFD's are given by $\mathbb{Z}$ and by fields, furthermore if R is a Noetherian UFD and F a free abelian group, then $R[F]$ is again a Noetherian UFD.

For a finitely generated S-module A we can find a presentation

$$S^r \xrightarrow{P} S^s \to A \to 0$$

since S is Noetherian. Let $i \geq 0$ and suppose $s - i \leq r$. We define $E_i(A)$, *the i-th elementary ideal* of A, to be the ideal in S generated by all $(s-i) \times (s-i)$ minors of P if $s - i > 0$ and to be S if $s - i \leq 0$. If $s - i > r$, we define $E_i(A) = 0$. It is known that $E_i(A)$ does not depend on the choice of a presentation of A (cf. [CF77]).

Since S is a UFD there exists a unique smallest principal ideal of S that contains $E_0(A)$. A generator of this principal ideal is defined to be the *order of* A and denoted by $\mathrm{ord}(A) \in S$. The order is well-defined up to multiplication by a unit in S. Note that A is S-torsion if and only if $\mathrm{ord}(A) \neq 0$. For more details, we refer to [Tu01].

3.2.4 Twisted Alexander Invariants

Let N be a 3-manifold and let $\alpha : \pi_1(N) \to GL(k, R[F])$ be a representation with R a Noetherian UFD. Similarly to Sect. 3.2.1 we define the $R[F]$-chain complex $C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} R[F]^k$. For $i \geq 0$, we define *the i-th twisted Alexander module of* (N, α) to be the $R[F]$-module

$$H_i(N; R[F]^k) := H_i(C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} R[F]^k),$$

where $\pi_1(N)$ acts on $R[F]^k$ by α. Since N is compact and $R[F]$ is Noetherian these modules are finitely presented over $R[F]$.

Definition 1 The *i-th twisted Alexander polynomial of* (N, α) is defined to be $\mathrm{ord}(H_i(N; R[F]^k)) \in R[F]$ and denoted by $\Delta^{\alpha}_{N,i}$.

Recall that by the discussion of Sect. 3.2.3 twisted Alexander polynomials are well-defined up to multiplication by a unit in $R[F]$. Note that the units of $R[F]$ are of the form rf with r a unit in R and $f \in F$. In the following, given $p \in R[F]$ we write

$$\Delta^{\alpha}_{N,i} \doteq p$$

if there exists a representative of $\Delta^{\alpha}_{N,i}$ which equals p.

We often write Δ^{α}_{N} instead of $\Delta^{\alpha}_{N,1}$, and we refer to it as the *twisted Alexander polynomial of* (N, α). We recall that given a representation $\gamma : \pi_1(N) \to GL(k, R)$ and a non-trivial homomorphism $\psi : \pi_1(N) \to F$ to a free abelian group F we get a tensor representation $\gamma \otimes \psi$ and in particular twisted Alexander polynomials $\Delta^{\gamma\otimes\psi}_{N,i}$. In almost all cases we will consider twisted Alexander polynomials corresponding to such a tensor representation.

When we consider twisted Alexander polynomials of links we adopt the following notational conventions:

1. We identify $R[\mathbb{Z}]$ with $R[t^{\pm 1}]$ and $R[\mathbb{Z}^m]$ with $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.
2. Given a link $L \subset S^3$ together with the abelianization $\psi : \pi_1(S^3 \setminus \nu L) \to F$ and a representation $\gamma : \pi_1(S^3 \setminus \nu L) \to GL(k, R)$ with R a Noetherian UFD, we write $\Delta^{\gamma}_{L,i}$ instead of $\Delta^{\gamma\otimes\psi}_{S^3\setminus\nu L,i}$.
3. If L is an ordered oriented link, then we have a canonical isomorphism $F \cong \mathbb{Z}^m$ and we can identify $R[F]$ with $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.

4. We sometimes record the fact that the twisted Alexander polynomial of a link L is a (multivariable) Laurent polynomial in the notation, i.e. given an oriented knot $K \subset S^3$ we sometimes write $\Delta^{\gamma}_{K,i}(t) = \Delta^{\gamma}_{K,i} \in R[t^{\pm 1}]$ and given an ordered oriented m-component link $L \subset S^3$ we sometimes write $\Delta^{\gamma}_{L,i}(t_1, \dots, t_m) = \Delta^{\gamma}_{L,i} \in R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.
5. Finally given a link L we also drop the representation from the notation when the representation is the trivial representation to $GL(1, \mathbb{Z})$.

With all these conventions, given a knot $K \subset S^3$, the polynomial $\Delta_K(t) = \Delta_K \in \mathbb{Z}[t^{\pm 1}]$ is just the ordinary Alexander polynomial.

3.2.5 Computation of Twisted Alexander Polynomials

Let N be a 3-manifold with empty or toroidal boundary, $\alpha : \pi_1(N) \to GL(k, R[F])$ a representation with R a Noetherian UFD and F a free abelian group. Given a finite presentation for $\pi_1(N)$ the polynomials $\Delta^{\alpha}_{N,1} \in R[F]$ and $\Delta^{\alpha}_{N,0} \in R[F]$ can be computed efficiently using Fox calculus (cf. e.g. [CF77, p. 98] and [KL99a]). We point out that because we view $C_*(\tilde{N})$ as a *right* module over $\mathbb{Z}[\pi_1(N)]$ we need a slightly different definition of Fox derivatives than the one commonly used. We refer to [Ha05, Sect. 6] for details. Finally, Proposition 5 allows us to compute $\Delta^{\alpha}_{N,2} \in R[F]$ using the algorithm for computing the zeroth twisted Alexander polynomial. In particular all the twisted Alexander polynomials $\Delta^{\alpha}_{N,i}$ can be computed from a finite presentation of the fundamental group.

3.3 Basic Properties of Twisted Invariants

In this section we summarize various basic algebraic properties of twisted Reidemeister torsion and twisted Alexander polynomials.

3.3.1 Relationship Between Twisted Invariants

The following proposition is [Tu01, Theorem 4.7].

Proposition 1 *Let N be a 3-manifold with empty or toroidal boundary and let $\alpha : \pi_1(N) \to GL(k, R[F])$ a representation where R is a Noetherian UFD and F a free abelian group. If $\Delta^{\alpha}_{N,i} \neq 0$ for $i = 0, 1, 2$, then*

$$\tau(N, \alpha) \doteq \prod_{i=0}^{2} \left(\Delta^{\alpha}_{N,i}\right)^{(-1)^{i+1}}.$$

The following is a mild extension of [FK06, Proposition 2.5], [FK08a, Lemmas 6.2 and 6.3] and [FK08a, Theorem 6.7]. Most of the ideas go back to work of Turaev (cf. e.g. [Tu86] and [Tu01]). The third statement is proved in [DFJ10].

Proposition 2 *Let N be a 3-manifold with empty or toroidal boundary, $\psi : \pi_1(N) \to F$ a non-trivial homomorphism to a free abelian group F and $\gamma : \pi_1(N) \to GL(k, R)$ a representation where R is a Noetherian UFD. Then the following hold*:

1. $\Delta_{N,0}^{\gamma\otimes\psi} \neq 0$ *and* $\Delta_{N,i}^{\gamma\otimes\psi} = 1$ *for* $i \geq 3$.
2. *If rank(Im*$\{\pi_1(N) \to F\}) > 1$, *then* $\Delta_{N,0}^{\gamma\otimes\psi} \doteq 1$.
3. *If γ is irreducible and if γ restricted to* $\ker(\psi)$ *is non-trivial, then* $\Delta_{N,0}^{\gamma\otimes\psi} \doteq 1$.
4. *If* $\Delta_{N,1}^{\gamma\otimes\psi} \neq 0$, *then* $\Delta_{N,2}^{\gamma\otimes\psi} \neq 0$.
5. *If N has non-empty boundary and if* $\Delta_{N,1}^{\gamma\otimes\psi} \neq 0$, *then* $\Delta_{N,2}^{\gamma\otimes\psi} \doteq 1$.
6. *If rank(Im*$\{\pi_1(N) \to F\}) > 1$ *and if* $\Delta_{N,1}^{\gamma\otimes\psi} \neq 0$, *then* $\Delta_{N,2}^{\gamma\otimes\psi} \doteq 1$.
7. *We have* $\Delta_{N,1}^{\gamma\otimes\psi} = 0$ *if and only if* $\tau(N, \gamma \otimes \psi) = 0$.
8. *If* $\Delta_{N,1}^{\gamma\otimes\psi} \neq 0$, *then*

$$\tau(N, \gamma \otimes \psi) \doteq \prod_{i=0}^{2} \left(\Delta_{N,i}^{\gamma\otimes\psi}\right)^{(-1)^{i+1}}.$$

A few remarks regarding the equalities of Proposition 1 and Proposition 2 (8) are in order:

1. Note that twisted Reidemeister torsion has in general a smaller indeterminacy than twisted Alexander polynomials. In particular the equality holds up to the indeterminacy of the twisted Alexander polynomials.
2. As pointed out in Sect. 3.2.5, the twisted Alexander polynomials $\Delta_{N,i}^{\gamma\otimes\psi}$ can be computed from a presentation of the fundamental group, whereas the computation of $\tau(N, \gamma \otimes \psi)$ requires in general an understanding of the CW-structure of N (cf. Sect. 3.2.2). In particular the equality of Proposition 2 (8) is often a faster method for computing $\tau(N, \gamma \otimes \psi)$ (at the price of a higher indeterminacy).
3. The twisted Alexander polynomial is only defined for representations over a Noetherian UFD, whereas the twisted Reidemeister torsion is defined for a finite dimensional representation over any commutative ring.
4. It is an immediate consequence of Proposition 2 that $\tau(N, \gamma \otimes \psi)$ lies in $R[F]$, i.e. is a polynomial, if rank(Im$\{\pi_1(N) \to F\}) > 1$.

Remark 1

1. Given a link $L \subset S^3$ and a representation $\gamma : \pi_1(S^3 \setminus L) \to GL(k, R)$ Wada [Wa94] introduced an invariant, which in this paragraph we refer to as $W(L, \gamma)$.

Wada's invariant is in many papers referred to as the twisted Alexander polynomial of a link. Kitano [Ki96] showed that $W(L, \gamma)$ agrees with the Reidemeister torsion $\tau(L, \gamma)$ (with the same indeterminacy). This can also be shown using the arguments of Sect. 3.2.2. In particular, in light of Proposition 2 we see that $W(L, \gamma) \doteq \Delta_L^\gamma$ if L has more than one component.
2. Lin's original definition [Lin01] of the twisted Alexander polynomial of a knot uses 'regular Seifert surfaces' and is rather different in character to the algebra-topological approach taken in the subsequent papers. The relation between the definitions of twisted Alexander polynomials given by Lin [Lin01], Jiang and Wang [JW93] and Sect. 3.2.4 is explained in [JW93, Proposition 3.3] and [KL99a, Sect. 4].

3.3.2 Twisted Invariants for Conjugate Representations

Given a group π we say that two representations $\gamma_1, \gamma_2 : \pi \to GL(k, R)$ are *conjugate* if there exists $P \in GL(k, R)$ such that $\gamma_1(g) = P\gamma_2(g)P^{-1}$ for all $g \in \pi$. We recall the following elementary lemma.

Lemma 1 *Let N be a 3-manifold with empty or toroidal boundary and let $\psi : \pi_1(N) \to F$ a non-trivial homomorphism to a free abelian group. If γ_1 and γ_2 are conjugate representations of $\pi_1(N)$, then*

$$\tau(N, \gamma_1 \otimes \psi) \doteq \tau(N, \gamma_2 \otimes \psi),$$

if R is furthermore a Noetherian UFD, then for any i we have

$$\Delta_{N,i}^{\gamma_1 \otimes \psi} \doteq \Delta_{N,i}^{\gamma_2 \otimes \psi}.$$

Note that non-conjugate representations do not necessarily give different Alexander polynomials (cf. [LX03, Theorem B]).

3.3.3 Change of Variables

In this section we will show how to reduce the number of variables in twisted Alexander polynomials, in particular this discussion will show how to obtain one-variable twisted Alexander polynomials from multi-variable twisted Alexander polynomials.

Throughout this section let N be a 3-manifold with empty or toroidal boundary, let $\psi : \pi_1(N) \to F$ be a non-trivial homomorphism to a free abelian group F and let $\gamma : \pi_1(N) \to GL(k, R)$ be a representation. Furthermore let $\phi : F \to H$ also be a homomorphism to a free abelian group such that $\phi \circ \psi$ is non-trivial. We denote the induced ring homomorphism $R[F] \to R[H]$ by ϕ as well. Let

$$S = \{f \in R[F] \,|\, \phi(R[F]) \neq 0 \in R[H]\}.$$

Note that ϕ induces a homomorphism $R[F]S^{-1} \to Q(R[H])$ which we also denote by ϕ.

The following is a slight generalization of [FK08a, Theorem 6.6], which in turn builds on ideas of Turaev (cf. [Tu86] and [Tu01]).

Proposition 3 *We have* $\tau(N, \gamma \otimes \psi) \in R[F]S^{-1}$, *and*

$$\tau(N, \gamma \otimes \psi \circ \phi) \doteq \phi(\tau(N, \gamma \otimes \psi)).$$

The following is now an immediate corollary of the previous proposition and Proposition 2.

Corollary 1 *Assume that* $\gamma : \pi_1(N) \to GL(k, R)$ *is a representation with* R *a Noetherian UFD. If* $\phi(\Delta_N^{\gamma\otimes\psi}) \neq 0$, *then we have the following equality*:

$$\prod_{i=0}^{2} \left(\Delta_{N,i}^{\gamma\otimes\phi\circ\psi}\right)^{(-1)^{i+1}} \doteq \phi\left(\prod_{i=0}^{2} (\Delta_{N,i}^{\gamma\otimes\psi})^{(-1)^{i+1}}\right).$$

In particular, if rank$\{Im\{\pi_1(N) \xrightarrow{\phi\circ\psi} H\}\} \geq 2$, *then*

$$\Delta_N^{\gamma\otimes\phi\circ\phi} \doteq \phi\left(\Delta_N^{\gamma\otimes\psi}\right).$$

3.3.4 Duality for Twisted Invariants

Let R be a ring with a (possibly trivial) involution $r \mapsto \overline{r}$. Let F be a free abelian group, with its natural involution. We extend the involution on R to the group ring $R[F]$ and the quotient field $Q(R[F])$ in the usual way. We equip $R[F]^k$ with the standard hermitian inner product $\langle v, w\rangle = v^t\overline{w}$ (where we view elements in $R[F]^k$ as column vectors).

Let N be a 3-manifold with empty or toroidal boundary and let $\alpha : \pi_1(N) \to GL(k, R[F])$ a representation. We denote by $\overline{\alpha} : \pi_1(N) \to GL(k, R[F])$ the representation given by

$$\langle \alpha(g^{-1})v, w\rangle = \langle v, \overline{\alpha}(g)w\rangle$$

for all $v, w \in R[F]^k$, $g \in \pi_1(N)$. Put differently, for any $g \in \pi_1(N)$ we have

$$\overline{\alpha}(g) = \overline{(\alpha(g)^{-1})^t} \in GL(k, R[F]).$$

We say that a representation is *unitary* if $\overline{\alpha} = \alpha$.

Note that if $\psi : \pi_1(N) \to F$ is a non-trivial homomorphism to a free abelian group F and $\gamma : \pi_1(N) \to GL(k, R)$ a representation, then

$$\overline{\gamma \otimes \psi} = \overline{\gamma} \otimes \psi.$$

The following duality theorem can be proved using the ideas of [Ki96] and [KL99a, Corollary 5.2].

Proposition 4 *Let N be a 3-manifold with empty or toroidal boundary and let $\alpha : \pi_1(N) \to GL(k, R[F])$ a representation. Then*

$$\overline{\tau(N, \alpha)} \doteq \tau(N, \overline{\alpha}) \in R[F].$$

In particular if $\psi : \pi_1(N) \to F$ is a non-trivial homomorphism to a free abelian group F and $\gamma : \pi_1(N) \to GL(k, R)$ is a unitary representation, then $\tau(N, \gamma \otimes \psi)$ is reciprocal, i.e.

$$\overline{\tau(N, \gamma \otimes \psi)} = \tau(N, \gamma \otimes \psi) \in R[F].$$

It is easy to see that in general the twisted Reidemeister torsion is not reciprocal if one considers representations α such that $\alpha(g) \neq \overline{\alpha(g)}$ for some $g \in \pi_1(N)$. Hillman, Silver and Williams [HSW09] give much more subtle examples which show that there also exist knots K together with special linear representations such that the corresponding twisted Reidemeister torsion is not reciprocal.

The following proposition follows from the discussion in [FK06].

Proposition 5 *Let N be a closed 3-manifold and let $\alpha : \pi_1(N) \to GL(k, R[F])$ a representation with R a Noetherian UFD. Assume that $\Delta^{\alpha}_{N,i} \neq 0$ for all i. Then the following equalities hold*:

$$\overline{\Delta^{\alpha}_{N,2}} \doteq \Delta^{\overline{\alpha}}_{N,0} \quad \text{and} \quad \overline{\Delta^{\alpha}_{N}} \doteq \Delta^{\overline{\alpha}}_{N}.$$

3.3.5 Shapiro's Lemma for Twisted Invariants

Let N be a 3-manifold with empty or toroidal boundary. Let $p : \hat{N} \to N$ be a finite cover of degree d. Let F be a free abelian group and let $\hat{\alpha} : \pi_1(\hat{N}) \to GL(k, R[F])$ a representation.

Now consider the $R[F]^k$-module

$$\mathbb{Z}[\pi_1(N)] \otimes_{\mathbb{Z}[\pi_1(\hat{N})]} R[F]^k.$$

If $g_1, \ldots, g_d$ are representatives of $\pi_1(N)/\pi_1(\hat{N})$ and if $e_1, \ldots, e_k$ is the canonical basis of $R[F]^k$, then it is straightforward to see that the above $R[F]$-module is a free $R[F]$-module with basis $g_i \otimes e_j, i = 1, \dots, d, j = 1, \dots, k$. The group $\pi_1(N)$ acts on

$$\mathbb{Z}[\pi_1(N)] \otimes_{\mathbb{Z}[\pi_1(\hat{N})]} R[F]^k = R[F]^{kd}$$

via left multiplication which defines a representation $\pi_1(N) \to GL(kd, R[F])$ which we denote by α.

Remark 2

1. Let $\hat{\gamma} : \pi_1(\hat{N}) \to GL(k, R)$ be a representation. Let $\psi : \pi_1(N) \to F$ be a non-trivial homomorphism to a free abelian group F. We write $\hat{\psi} = \psi \circ p_*$ and we denote by γ the representation given by left multiplication by $\pi_1(N)$ on

$$\mathbb{Z}[\pi_1(N)] \otimes_{\mathbb{Z}[\pi_1(\hat{N})]} R^k = R^{kd}.$$

   Given $\hat{\alpha} = \hat{\gamma} \otimes \hat{\psi}$ we have in that case $\alpha = \gamma \otimes \psi$.
2. If $\hat{\gamma}$ is the trivial one-dimensional representation, and $\hat{N}$ the cover of N corresponding to an epimorphism $\varphi : \pi_1(N) \to G$ to a finite group, then we have $\gamma = \varphi$.

The following is now a variation on Shapiro's lemma (cf. [FV08a, Lemma 3.3] and [HKL10, Sect. 3]).

Theorem 3 *We have*

$$\tau(\hat{N}, \hat{\alpha}) \doteq \tau(N, \alpha).$$

If R is furthermore a Noetherian UFD, then

$$\Delta^{\hat{\alpha}}_{\hat{N},i} \doteq \Delta^{\alpha}_{N,i}.$$

In its simplest form Theorem 3 says that given an epimorphism $\gamma : \pi_1(N) \to G$ to a finite group the corresponding twisted Alexander polynomials of N are just untwisted Alexander polynomials of the corresponding finite cover.

3.3.6 Twisted Invariants of Knots and Links

Let $L = L_1 \cup \dots \cup L_m \subset S^3$ be an ordered oriented link. Recall that given a representation $\gamma : \pi_1(S^3 \setminus \nu L) \to GL(k, R)$ with R a Noetherian UFD we can consider the twisted Alexander polynomial Δ^{γ}_L as an element in the Laurent polynomial ring $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and we write $\Delta^{\gamma}_{L,i}(t_1, \dots, t_m) = \Delta^{\gamma}_{L,i} \in R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

Given $\epsilon_j \in \{\pm 1\}$ for $j = 1, \dots, m$ we denote by L^{ϵ} the link $\epsilon_1 L_1, \dots, \epsilon_m L_m$, i.e. the oriented link obtained from L by reversing the orientation of all components with $\epsilon_j = -1$.

The following lemma is now an immediate consequence of the definitions:

Lemma 2 *Given* $\epsilon_j \in \{\pm 1\}$ *for* $j = 1, \dots, m$ *we have*

$$\Delta^{\gamma}_{L^{\epsilon},i}(t_1, \dots, t_m) = \Delta^{\gamma}_{L,i}(t_1^{\epsilon_1}, \dots, t_m^{\epsilon_m}).$$

We now turn to the study of twisted Alexander polynomials of sublinks. Given a link $L = L_1 \cup \dots L_{k-1} \cup L_k \subset S^3$ Torres [To53] showed how to relate the Alexander polynomials of L and $L' = L_1 \cup \dots L_{k-1} \subset S^3$. The following theorem of Morifuji [Mo07, Theorem 3.6] gives a generalization of the Torres condition to twisted Reidemeister torsion.

Theorem 4 *Let* $L = L_1 \cup \dots L_{k-1} \cup L_k \subset S^3$ *be a link. Write* $L' = L_1 \cup \dots L_{k-1} \subset S^3$ *and let* $\gamma' : \pi_1(S^3 \setminus \nu L') \to SL(n, \mathbb{F})$ *be a representation where* $\mathbb{F}$ *is a field. Denote by* γ *the representation* $\pi_1(S^3 \setminus \nu L) \to \pi_1(S^3 \setminus \nu L') \to SL(n, \mathbb{F})$, *then*

$$\tau(L, \gamma)(t_1, \dots, t_{k-1}, 1) \doteq \left(T^n + \sum_{k=1}^{n-1} \epsilon_i T^i + (-1)^n \right) \cdot \tau(L', \gamma')(t_1, \dots, t_{k-1})$$

where

$$T := \prod_{i=1}^{k-1} t_i^{lk(L_i, L_k)}$$

and where $\epsilon_1, \dots, \epsilon_{n-1}$ *are elements of* $\mathbb{F}$.

3.4 Distinguishing Knots and Links

In this section we will restrict ourselves to twisted Alexander polynomials of knots and links. Recall that given an oriented knot $K \subset S^3$ the *reverse* $\overline{K}$ is given by reversing the orientation. Given a knot $K \subset S^3$ we denote by K^* its *mirror image*, i.e. the result of reflecting K in $S^2 \subset S^3$. The mirror image is also sometimes referred to as the *obverse*.

Twisted Alexander invariants have so far been surprisingly little used to distinguish a knot from its mirror image or from its reverse (cf. [KL99b] though for a deep result showing that the knot 8_{17} and its reverse lie in different concordance classes). It is an interesting question whether Kitayama's normalized Alexander polynomial [Kiy08a] can be used to distinguish a knot from its mirror image and its reverse. We refer to [Ei07] for an interesting and very successful approach to distinguishing knots using 'knot colouring polynomials'.

In this section we are from now on only concerned with distinguishing knot types of prime knots. Here we say that two knots K_1 and K_2 are of the same knot type if there exists a homeomorphism h of the sphere with $h(K_1) = K_2$. Put differently, K_1 and K_2 are of the same knot type if they are related by an isotopy together possibly with taking the mirror image and possibly reversing the orientation.

The most common approach for distinguishing knots using twisted Alexander polynomials is to look at the set (or product) of all twisted Alexander polynomials corresponding to a 'characteristic set' of representations. Note that due to Lemma 1 we can in fact restrict ourselves to conjugacy classes of a set of characteristic representations.

The following two types of characteristic sets have been used in the literature:

1. Given a knot K consider all conjugacy classes of (all upper triangular, parabolic, metabelian, orthogonal, unitary) representations of $\pi_1(S^3 \setminus \nu K)$ of a fixed dimension over a finite ring.
2. Given K consider all conjugacy classes of homomorphisms of $\pi_1(S^3 \setminus \nu K)$ onto a finite group G composed with a fixed representation of G.

The first approach was used in Lin's original paper [Lin01] to distinguish knots with the same Alexander module. Wada [Wa94] also used the first approach to show that twisted Alexander polynomials can distinguish the Conway knot and the Kinoshita–Terasaka knot (cf. also [In00]). This shows in particular that twisted Alexander polynomials detect mutation. In fact in [FV07a] it is shown that twisted Alexander polynomials detect all mutants with 11 crossings or less. Furthermore, in [FV07a] the authors give an example of a pair of knot types of prime knots which can be distinguished using twisted Alexander polynomials, even though the HOMFLY polynomial, Khovanov homology and knot Floer homology agree.

Remark 3

1. The approach of using twisted Alexander polynomials corresponding to a characteristic set of conjugacy classes can be viewed as an extension of the approach of Riley [Ri71] who studied the first homology group corresponding to such a set of representations to distinguish the Conway knot from the Kinoshita–Terasaka knot.
2. By the work of Whitten [Wh87] and Gordon–Luecke [GL89] the knot type of a prime knot is determined by its fundamental group. It is therefore at least conceivable that twisted Alexander polynomials can distinguish any two pairs of knot types.

The following theorem shows that twisted Alexander polynomials detect the unknot and the Hopf link. The statement for knots was first proved by Silver and Williams [SW06], the extension to links was later proved in [FV07a].

Theorem 5 *Let $L \subset S^3$ be a link which is neither the unknot nor the Hopf link. Then there exists an epimorphism $\gamma : \pi_1(S^3 \setminus \nu L) \to G$ onto a finite group G such that $\Delta_L^\gamma \neq \pm 1$.*

Given a knot K we denote by $t(K)$ its tunnel number. Theorem 5 was used by Pajitnov [Pa08] to show that for any knot K there exists $\lambda > 0$ such that $t(nK) \geq \lambda n - 1$.

3.5 Twisted Alexander Polynomials and Concordance

We first recall the relevant definitions. Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be an oriented m-component link. We say that L is (topologically) *slice* if the components bound m disjointly embedded locally flat disks in D^4. Given two ordered oriented m-component links $K = K_1 \cup \cdots \cup K_m \subset S^3$ and $L = L_1 \cup \cdots \cup L_m \subset S^3$ we say that K and L are *concordant* if there exist m disjointly embedded locally flat cylinders $C_1, \ldots, C_m$ in $S^3 \times [0, 1]$ such that $\partial C_i = K_i \times 0 \cup -L_i \times 1$. (Given an oriented knot $K \subset S^3$ we write $-K = \overline{K}^*$, i.e. $-K$ is the knot obtained from the mirror image of K by reversing the orientation.) Note that two knots K_1 and K_2 are concordant if and only if $K_1\# - K_2$ is slice, and a link is slice if and only if it is concordant to the unlink.

3.5.1 Twisted Alexander Polynomials of Zero-Surgeries

Given a knot $K \subset S^3$ the zero-framed surgery N_K of S^3 along K is defined to be

$$N_K = S^3 \setminus \nu K \cup_T S^1 \times D^2$$

where $T = \partial(S^3 \setminus \nu K)$ and T is glued to $S^1 \times D^2$ by gluing the meridian to $S^1 \times \text{pt}$. The inclusion map induces an isomorphism $\mathbb{Z} \cong H_1(S^3 \setminus \nu K; \mathbb{Z}) \xrightarrow{\cong} H_1(N_K; \mathbb{Z})$. It is well-known that understanding the zero-framed surgery N_K is the key to determining whether K is slice or not (cf. e.g. [COT03], [FT05, Proposition 3.1], [CFT09, Proposition 2.1]). We therefore prefer to formulate the sliceness obstructions in terms of the twisted Alexander polynomials of the zero-framed surgery. The following lemma, together with Proposition 2, relates the twisted invariants of the zero-framed surgery with the twisted invariants of the knot complement. We refer to [KL99a, Lemma 6.3] and [Tu02a, Sect. VII] for very similar statements.

Lemma 3 *Let $K \subset S^3$ be a knot. Denote its meridian by μ. Let $\alpha : \pi_1(N_K) \to GL(k, R[t^{\pm 1}])$ be a representation such that* $\det(\alpha(\mu) - id) \neq 0$. *We denote the inclusion induced representation $\alpha : \pi_1(S^3 \setminus \nu K) \to \pi_1(N_K) \to GL(k, R[t^{\pm 1}])$ by α as well. Then*

$$\tau(S^3 \setminus \nu K, \alpha) = \tau(N_K, \alpha) \cdot \det(\alpha(\mu) - id).$$

Proof The proof of the lemma is standard and well-known. We therefore give just a quick summary. Consider the decomposition $N_K = S^3 \setminus \nu K \cup_T S$ as above, where $S = S^1 \times D^2$. Using the Mayer–Vietoris theorem for torsion we obtain that

$$\tau(N_K, \alpha) = \frac{\tau(S^3 \setminus \nu K, \alpha) \cdot \tau(S, \alpha)}{\tau(T, \alpha)}.$$

It is well-known that the torsion of the torus is trivial (c.f. e.g. [KL99a]) and that the torsion of S is given by

$$\tau(S,\alpha) = \frac{1}{\det(\alpha(\mu) - \mathrm{id})}.$$

The lemma now follows immediately. □

3.5.2 Twisted Alexander Polynomials and Knot Concordance

The first significant result in the study of slice knots is due to Fox and Milnor [FM66] who showed that if K is a slice knot, then $\Delta_K \doteq f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$.

In this section we will give an exposition and a slight generalization of the Kirk–Livingston [KL99a] sliceness obstruction theorem, which generalizes the Fox–Milnor condition. Note that we will state this obstruction using a slightly different setup, but Theorem 6 is already contained in [KL99a] and [HKL10].

Let K be a knot, as above we denote by N_K the zero-framed surgery of S^3 along K. Now suppose that K has a slice disk D. Note that $\partial(D^4 \setminus \nu D) = N_K$. Let R be a ring with (possibly trivial) involution and as usual we extend the involution to $R[t^{\pm 1}]$ by $\bar{t} := t^{-1}$. Let $\alpha : \pi_1(N_K) \to GL(R[t^{\pm 1}], k)$ be a unitary representation. Assume that α has the following two properties:

1. α factors through a representation $\pi_1(D^4 \setminus \nu D) \to GL(k, R[t^{\pm 1}])$, and
2. the induced twisted modules $H_*(D^4 \setminus \nu D; R[t^{\pm 1}]^k)$ are $R[t^{\pm 1}]$-torsion,

then using Poincaré duality for Reidemeister torsion (cf. [KL99a, Theorem 6.1]) one can show that $\tau(N_K, \alpha) \doteq f(t)\overline{f(t)}$ for some $f(t) \in R(t)$. We refer to [KL99a, Corollary 5.3] for details.

Remark 4 Note that the canonical representation $\pi_1(N_K) \to GL(1, \mathbb{Z}[t^{\pm 1}])$ extends over any slice disk complement. It can be shown that $H_*(D^4 \setminus \nu D; \mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion. We can therefore recover the Fox–Milnor theorem from this discussion.

Most of the ideas and techniques of finding representations satisfying (1) and (2) go back to the seminal work of Casson and Gordon [CG86]. We follow the approach taken in [Fr04] which is inspired by Letsche [Let00] and Kirk–Livingston [KL99a, HKL10].

In the following let K again be a knot in S^3. We denote by W_k the cyclic k-fold branched cover of K. Note that $H_1(W_k; \mathbb{Z})$ has a natural $\mathbb{Z}/k$-action, and we can therefore view $H_1(W_k; \mathbb{Z})$ as a $\mathbb{Z}[\mathbb{Z}/k]$-module. If $H_1(W_k; \mathbb{Z})$ is finite, then there exists a non-singular linking form

$$\lambda_k : H_1(W_k; \mathbb{Z}) \times H_1(W_k; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

with respect to which $\mathbb{Z}/k$ acts via isometries. We say that $M \subset H_1(W_k; \mathbb{Z})$ is a *metabolizer* of the linking form if M is a $\mathbb{Z}[\mathbb{Z}/k]$-submodule of $H_1(W_k; \mathbb{Z})$ such that $\lambda_k(M, M) = 0$ and such that $|M|^2 = |H_1(W_k; \mathbb{Z})|$. It is well-known that if K is slice, then λ_k has a metabolizer for any prime power k. We refer to [Go78] for details.

Note that $H^1(N_K; \mathbb{Z}) = \mathbb{Z}$, in particular $H^1(N_K; \mathbb{Z})$ has a unique generator (up to sign) which we denote by ϕ. We now consider the Alexander module $H_1(N_K; \mathbb{Z}[t^{\pm 1}])$ which we denote by H. Note that H is isomorphic to the usual Alexander module of K. It is well-known that given k there exists a canonical isomorphism $H/(t^k - 1) \to H_1(W_k; \mathbb{Z})$ (cf. e.g. [Fr04, Corollary 2.4] for details). Now let $\mu \in \pi_1(N_K)$ be an element with $\phi(\mu) = 1$. Note that for any $g \in \pi$ we have $\phi(\mu^{-\phi(g)} g) = 0$, in particular we can consider its image $[\mu^{-\phi(g)} g]$ in the abelianization $H_1(\ker(\phi))$, which we can identify with H. Then we have a well-defined map

$$\pi \to \mathbb{Z} \ltimes H \to \mathbb{Z} \ltimes H/(t^k - 1)$$

where the first map is given by sending $g \in \pi$ to $(\phi(g), [\mu^{-\phi(g)} g])$. Here $n \in \mathbb{Z}$ acts on H and on $H/(t^k - 1)$ via multiplication by t^n. We refer to [Fr04] and [BF08] for details.

Fix $k \in \mathbb{N}$. Let $\chi : H_1(W_k; \mathbb{Z}) \to \mathbb{Z}/q \to S^1$ be a character. We denote the induced character $H \to H/(t^k - 1) = H_1(W_k; \mathbb{Z}) \to S^1$ by χ as well. Let ζ_q be a primitive q-th root of unity. Then it is straightforward to verify that

$$\begin{aligned}
\alpha(k, \chi) : \pi \to \mathbb{Z} \ltimes H/(t^k - 1) &\to GL(k, \mathbb{Z}[\zeta_q][t^{\pm 1}]) \\
(j, h) &\mapsto \begin{pmatrix} 0 & \dots & 0 & t \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}^j \\
&\quad \times \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{k-1} h) \end{pmatrix}
\end{aligned}$$

defines a unitary representation. (Note the "t" in the upper right corner.) Also note that $\alpha(k, \chi)$ is not a tensor representation (cf. [HKL10]).

We can now formulate the following obstruction theorem which is well-known to the experts. It can be proved using the above discussion, Proposition 1, various well-known arguments going back to Casson and Gordon [CG86] and [KL99a, Lemma 6.4]. We also refer to Letsche [Let00] and [Fr04] for more information.

Theorem 6 *Let K be a slice knot. Then for any prime power k there exists a metabolizer M of λ_k such that for any odd prime power n and any character $\chi : H_1(W_k; \mathbb{Z}) \to \mathbb{Z}/n \to S^1$ vanishing on M we have*

$$\Delta_{N_K}^{\alpha(k, \chi)} \doteq f(t) \overline{f(t)}$$

for some $f(t) \in Q(\mathbb{Z}[\zeta_n])[t^{\pm 1}] = \mathbb{Q}(\zeta_n)[t^{\pm 1}]$.

Note that the original sliceness obstruction of Kirk and Livingston [KL99a, Theorem 6.2] (cf. also [HKL10, Theorem 8.1] and [Liv09, Theorem 5.4]) gives an obstruction in terms of twisted Alexander polynomials of a cyclic cover of $S^3 \setminus \nu K$ corresponding to one-dimensional representations. Using Theorem 3 and Lemma 3 (cf. also [KL99a, Lemma 6.3]) one can show that the sliceness obstruction of Theorem 6 is equivalent to the original formulation by Kirk and Livingston.

Remark 5 In Theorem 6 and later in Theorems 7 and 9 we restrict ourselves to characters of odd prime power. Similar theorems also hold for characters of even prime power order, we refer to [KL99a, Lemma 6.4] and [Liv09, Sect. 5] for more information.

Theorem 6 can be somewhat generalized using tensor representations. In the following let $k_1, \dots, k_n \in \mathbb{N}$. Assume we are given characters $\chi_i : H_1(W_{k_i}; \mathbb{Z}) \to \mathbb{Z}/q \to S^1$, $i = 1, \dots, n$ we also get a tensor representation

$$\alpha(k_1, \chi_1) \otimes \cdots \otimes \alpha(k_n, \chi_n) \to GL(k_1 \cdot \dots \cdot k_n, \mathbb{Z}[\zeta_q][t^{\pm 1}]).$$

We refer to [Fr04, Proposition 4.6] for more information.

The following theorem can be proved by modifying the proof of Theorem 6 along the lines of [Fr04, Theorem 4.7].

Theorem 7 *Let K be a slice knot. Let q be an odd prime power and let $k_1, \dots, k_n$ be coprime prime powers. Then there exist metabolizers $M_1, \dots, M_n$ of the linking forms $\lambda_{k_1}, \dots, \lambda_{k_n}$ such that for any choice of characters $\chi_i : H_1(W_{k_i}; \mathbb{Z}) \to \mathbb{Z}/q \to S^1$, $i = 1, \dots, n$ vanishing on M_i we have*

$$\Delta_{N_K}^{\alpha(k_1,\chi_1)\otimes\cdots\otimes\alpha(k_n,\chi_n)} \doteq f(t)\overline{f(t)}$$

for some $f(t) \in Q(\mathbb{Z}[\zeta_q])[t^{\pm 1}] = \mathbb{Q}(\zeta_q)[t^{\pm 1}]$.

Remark 6 It was first observed by Letsche [Let00] that non-prime power dimensional representations can give rise to sliceness obstructions. It is shown in [Fr03] that Letsche's non-prime power representations are given by the tensor representations considered in Theorem 7.

The sliceness obstruction coming from twisted Alexander polynomials is in some sense less powerful than the invariants of Casson and Gordon [CG86] (cf. [KL99a, Sect. 6] for a careful discussion) and Cochran–Orr–Teichner [COT03]. But to date twisted Alexander polynomials give the strongest sliceness obstruction for algebraically slice knots that can be computed efficiently. The Kirk–Livingston sliceness obstruction theorem has been used by various authors to produce many interesting examples, some of which we list below:

1. Kirk and Livingston [KL99b] apply twisted Alexander polynomials to show that some knots (e.g. 8_{17}) are not concordant to their inverses. This shows in particular that knots are not necessarily concordant to their mutants. In [KL99b] it is also shown that in general a knot is not even concordant to a positive mutant.
2. Tamulis [Tam02] considered knots with at most ten crossings which have algebraic concordance order two. Tamulis used twisted Alexander polynomials to show that all but one of these knots do not have order two in the knot concordance group.
3. In [HKL10] Herald, Kirk and Livingston consider all knots with up to twelve crossings. Eighteen of these knots are algebraically slice but can not be shown to be slice using elementary methods. In [HKL10] twisted Alexander polynomials are used to show that sixteen of these knots are in fact not slice and one knot is smoothly slice. Therefore among the knots with up to twelve crossings only the sliceness status of the knot 12_{a631} is unknown.
4. The concordance genus $g_c(K)$ of a knot K is defined to be the minimal genus among all knots concordant to K. Livingston [Liv09] uses twisted Alexander polynomials to show that the concordance genus of 10_{82} equals three, which is its ordinary genus.

3.5.3 Ribbon Knots and Doubly Slice Knots

A knot K is called *homotopy ribbon* if there exists a slice disk D such that $\pi_1(N_K) \to \pi_1(D^4 \setminus \nu D)$ is surjective. Note that if a knot is ribbon, then it is also homotopy ribbon. It is an open conjecture whether every knot that is slice is also homotopy ribbon.

Let $K \subset S^3$ be a knot. Recall that there exists a non-singular hermitian pairing

$$\lambda : H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \times H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

which is referred to as the *Blanchfield pairing*. We say that $M \subset H_1(N_K; \mathbb{Z}[t^{\pm 1}])$ is a metabolizer for the Blanchfield pairing if $M = M^{\perp}$, i.e. if M satisfies

$$M = \{x \in H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \,|\, \lambda(x, y) = 0 \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \text{ for all } y \in M\}.$$

If K is slice, then there exists a metabolizer for the Blanchfield pairing. Also recall from the previous section that for any k there exists a canonical isomorphism $H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^k - 1) = H_1(L_k; \mathbb{Z})$. If M is a metabolizer for the Blanchfield pairing, then for any k we have that $M/(t^k - 1) \subset H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^k - 1) = H_1(L_k; \mathbb{Z})$ is a metabolizer for the linking form λ_k. We refer to [Bl57], [Ke75], [Go78, Sect. 7], [Let00] and [Fr04, Sect. 2.3] for more information.

The proof of Theorem 6 can now be modified along well established lines (cf. e.g. [Fr04, Theorem 8.3]) to prove the following theorem:

Theorem 8 *Let $K \subset S^3$ be a ribbon knot, then there exists a metabolizer $M \subset H_1(N_K; \mathbb{Z}[t^{\pm 1}])$ for the Blanchfield pairing λ such that for any k and any non-trivial character $\chi : H_1(L_k; \mathbb{Z}) \to S^1$ of prime power vanishing on $M/(t^k - 1)$ we have*

$$\Delta_{N_K}^{\alpha(k,\chi)} \doteq f(t)\overline{f(t)}$$

for some $f(t) \in Q(\mathbb{Z}[\zeta_q])[t^{\pm 1}] = \mathbb{Q}(\zeta_q)[t^{\pm 1}]$.

Using [Fr04, Proposition 4.6] one can show that if a knot K satisfies the ribbon obstruction of Theorem 8, then it also satisfies the sliceness obstruction given by Theorem 7.

A knot $K \subset S^3$ is called *doubly slice* if there exists an unknotted locally flat two-sphere $S \subset S^4$ such that $S \cap S^3 = K$. Note that a doubly slice knot is in particular slice. The ordinary Alexander polynomial does not contain enough information to distinguish between slice and doubly slice knots. On the other hand twisted Alexander polynomials can detect the difference. The following theorem is well-known to the experts. It can be proved using the above ideas of Kirk and Livingston combined with the results of Gilmer and Livingston [GL83] (cf. also [Fr04, Sect. 8.2]). Note that many of the ideas already go back to the original paper by Sumners [Sum71].

Theorem 9 *Let K be a doubly slice knot. Then for any prime power k there exist two metabolizers M_1 and M_2 of λ_k with $M_1 \cap M_2 = \{0\}$ such that for any odd prime power n and any character $\chi : H_1(L_k; \mathbb{Z}) \to \mathbb{Z}/n \to S^1$ which vanishes either on M_1 or on M_2 we have*

$$\Delta_{N_K}^{\alpha(k,\chi)} \doteq f(t)\overline{f(t)}$$

for some $f(t) \in Q(\mathbb{Z}[\zeta_n])[t^{\pm 1}]$.

It is also possible to state and prove an analogue of Theorem 7 for doubly slice knots.

3.5.4 Twisted Invariants and Slice Links

Kawauchi [Ka78, Theorems A and B] showed that if L is a slice link with more than one component, then the ordinary Alexander module is necessarily non-torsion, in particular the corresponding Reidemeister torsion is zero. We will follow an idea of Turaev [Tu86, Sect. 5.1] to define a (twisted) invariant for links even if the (twisted) Alexander module is non-torsion. This invariant will then give rise to a sliceness obstruction for links. We refer throughout this section to [CF10] and [Tu86] for details.

Let $L \subset S^3$ be an oriented m-component link. Let $R \subset \mathbb{C}$ be a subring and let $\alpha : \pi(S^3 \setminus \nu L) \to GL(k, R)$ be a unitary representation. Suppose that $\psi : \pi_1(S^3 \setminus \nu L) \to F$ is a homomorphism to a free abelian group which is non-trivial on each meridian of L. Under these assumptions we can endow $H_1(S^3 \setminus \nu L; Q(R[F])^k)$ and $H_2(S^3 \setminus \nu L; Q(R[F])^k)$ with dual bases and using these bases we can define the Reidemeister torsion

$$\tilde{\tau}^{\alpha\otimes\psi}(L) \in Q(F)^*/N(Q(F)^*),$$

here $N(Q(F)^*)$ denotes the subgroup of norms of the multiplicative group $Q(F)^*$, i.e. $N(Q(F)^*) = \{q\overline{q} \mid q \in Q(F)^*\}$. Reidemeister torsion $\tilde{\tau}^{\alpha\otimes\psi}(L)$ viewed as an element in $Q(F)^*/N(Q(F)^*)$ is well-defined up to multiplication by an element of the form $\pm df$ where $d \in \det(\alpha(\pi_1(S^3 \setminus \nu L)))$, $f \in F$. The invariant $\tau^{\alpha\otimes\psi}(L)$ is the twisted version of an invariant first introduced by Turaev [Tu86, Sect. 5.1].

For example, if L is the m-component unlink in S^3 with meridians $\mu_1, \dots, \mu_m$, then given α and ψ as above we have

$$\tilde{\tau}^{\alpha\otimes\psi}(L) = \pm df \cdot \prod_{i=1}^{m} \det\left(\mathrm{id} - \psi(\mu_i)\alpha(\mu_i)\right)^{-1} \in Q(F)^*/N(Q(F)^*)$$

with $d \in \det(\alpha(\pi_1(S^3 \setminus \nu L)))$, $f \in F$.

In [CF10] the first author and Jae Choon Cha prove the following result.

Proposition 6 *Let L be an m-component oriented slice link with oriented meridians $\mu_1, \dots, \mu_m$. Let $R \subset \mathbb{C}$ be a subring closed under complex conjugation and let $\alpha : \pi_1(S^3 \setminus \nu L) \to GL(k, R)$ be a representation which factors through a finite group of prime power order. Let $\psi : H_1(S^3 \setminus \nu L) \to F$ be an epimorphism onto a free abelian group which is non-trivial on each meridian of L. Then*

$$\tilde{\tau}^{\alpha\otimes\psi}(L) = \pm df \cdot \prod_{i=1}^{m} \det\left(id - \psi(\mu_i)\alpha(\mu_i)\right)^{-1} \in Q(F)^*/N(Q(F))^*$$

for some $d \in \det(\alpha(\pi_1(S^3 \setminus \nu L)))$ and $f \in F$.

If α is the trivial representation over $\mathbb{Z}$, then it is shown in [Tu86, Theorem 5.1.1] that the torsion is represented by the untwisted Alexander polynomial of L corresponding to ψ. Using this observation we see that Proposition 6 generalizes earlier results of Murasugi [Mu67], Kawauchi [Ka77, Ka78] and Nakagawa [Na78]. The approach taken in [CF10] is partly inspired by Turaev's proof of the untwisted case (cf. [Tu86, Theorem 5.4.2]).

In [CF10] we will in particular use the obstruction of Proposition 6 to reprove that the Bing double of the Fig. 8 knot is not slice. This had first been shown by Cha [Ch10].

3.6 Twisted Alexander Polynomials, the Thurston Norm and Fibered Manifolds

3.6.1 Twisted Alexander Polynomials and Fibered Manifolds

Let $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ be non-trivial. We say (N,ϕ) *fibers over* S^1 if there exists a fibration $p : N \to S^1$ such that the induced map $p_* : \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$ coincides with ϕ. If K is a fibered knot, then it is a classical result of Neuwirth that Δ_K is monic and that $\deg(\Delta_K) = 2\,\mathrm{genus}(K)$. Similarly, twisted Alexander polynomials provide necessary conditions to the fiberability of a pair (N,ϕ).

In order to state the fibering obstructions for a pair (N,ϕ) we need to introduce the Thurston norm. Given (N,ϕ) the *Thurston norm* of ϕ is defined as

$$\|\phi\|_T = \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface S with connected components $S_1 \cup \cdots \cup S_k$, we define $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$. Thurston [Th86] showed that this defines a seminorm on $H^1(N;\mathbb{Z})$ which can be extended to a seminorm on $H^1(N;\mathbb{R})$. As an example consider $S^3 \setminus \nu K$, where $K \subset S^3$ is a non-trivial knot. Let $\phi \in H^1(S^3 \setminus \nu K;\mathbb{Z})$ be a generator, then $\|\phi\|_T = 2\,\mathrm{genus}(K) - 1$.

Let N be a 3-manifold N with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$. Recall that we identify the group ring $R[\mathbb{Z}]$ with the Laurent polynomial ring $R[t^{\pm 1}]$ and we will now identify $Q(R[\mathbb{Z}])$ with the field of rational functions $Q(t)$. Given a representation $\gamma : \pi_1(N) \to GL(k,R)$ we therefore view the corresponding twisted Reidemeister torsion $\tau(N,\gamma\otimes\phi)$ as an element in $Q(t)$ and we view the corresponding twisted Alexander polynomials $\Delta_{N,i}^{\gamma\otimes\phi}$ as elements in $R[t^{\pm 1}]$.

We say that the twisted Reidemeister torsion $\tau(N,\gamma\otimes\phi)$ is *monic* if there exist polynomials $p(t), q(t) \in R[t^{\pm 1}]$ with $\frac{p(t)}{q(t)} \doteq \tau(N,\gamma\otimes\phi)$ such that the top coefficients of $p(t)$ and $q(t)$ lie in

$$\{\pm\det(\gamma(g)) \mid g \in \pi_1(N)\}.$$

We also say that the twisted Alexander polynomial $\Delta_{N,i}^{\gamma\otimes\phi}$ is *monic* if one (and equivalently any) representative has a top coefficient which is a unit in R.

We recall the following basic lemma:

Lemma 4 *Let N be a 3-manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ be non-trivial and let $\gamma : \pi_1(N) \to GL(k,R)$ be a representation with R a Noetherian UFD. Then the following hold*:

1. $\Delta_{N,0}^{\gamma\otimes\phi}$ *is monic*,
2. *If $\Delta_{N,1}^{\gamma\otimes\phi}$ is non-zero, then $\Delta_{N,2}^{\gamma\otimes\phi}$ is monic.*

Proof As in Sect. 3.2.2 we pick a cell decomposition of N with one 0-cell x_0 and n 1-cells $c_1, \ldots, c_n$. We denote the corresponding elements in $\pi_1(N, x_0)$ by $c_1, \ldots, c_n$ as well. Without loss of generality we can assume that $\phi(c_1) > 0$. We then get a resolution for $H_0(N; R[t^{\pm 1}]^n)$ with presentation matrix

$$\mathrm{id}_n - (\gamma \otimes \phi)(c_1) \;\ldots\; \mathrm{id}_n - (\gamma \otimes \phi)(c_n).$$

We refer to [FK06, Proof of Proposition 6.1] for details. We have

$$\det(\mathrm{id}_n - (\gamma \otimes \phi)(c_1)) = \det(\mathrm{id}_n - t^{\phi(c_1)}\gamma(c_1)) \in R[t^{\pm 1}],$$

which is monic since the top coefficient equals $\det(\gamma(c_1))$. By definition $\Delta^{\gamma\otimes\phi}_{N,0}$ divides $\det(\mathrm{id}_n - (\gamma \otimes \phi)(c_1))$, we therefore see that $\Delta^{\gamma\otimes\phi}_{N,0}$ is monic. The claim that $\Delta^{\gamma\otimes\phi}_{N,2}$ is monic now follows from Proposition 5. □

Remark 7 Note that if $\tau(N, \gamma \otimes \phi)$ is monic, then it follows from the previous lemma and from Proposition 2 that $\Delta^{\gamma\otimes\phi}_{N,i} \in R[t^{\pm 1}]$ is monic for $i = 0, 1, 2$. Note though that the converse does not hold in general since twisted Alexander polynomials have in general a higher indeterminacy than twisted Reidemeister torsion. For example, let $\mathbb{F}$ be a field and let $\gamma : \pi_1(N) \to GL(k, \mathbb{F})$ be a representation such that $\Delta^{\gamma\otimes\phi}_{N,i} \neq 0$. It follows immediately from the definition that $\Delta^{\gamma\otimes\phi}_{N,i}$ is monic. However, $\tau(N, \gamma \otimes \psi)$ is not necessarily monic (cf. e.g. [GKM05, Example 4.2]).

We can now formulate the following fibering obstruction theorem which was proved in various levels of generality by Cha [Ch03], Kitano and Morifuji [KM05], Goda, Kitano and Morifuji [GKM05], Pajitnov [Pa07], Kitayama [Kiy08a] and [FK06].

Theorem 10 *Let N be a 3-manifold. Let $\phi \in H^1(N; \mathbb{Z})$ be non-trivial such that (N, ϕ) fibers over S^1 and such that $N \neq S^1 \times D^2, N \neq S^1 \times S^2$. Let $\gamma : \pi_1(N) \to GL(k, R)$ be a representation. Then $\tau(N, \gamma \otimes \phi) \in Q(t)$ is monic and we have*

$$k\|\phi\|_T = \deg(\tau(N, \gamma \otimes \phi)).$$

Remark 8

1. If R is a Noetherian UFD, then the last equality can be rewritten as

$$k\|\phi\|_T = \deg \Delta^{\gamma\otimes\phi}_{N,1} - \deg \Delta^{\gamma\otimes\phi}_{N,0} - \deg \Delta^{\gamma\otimes\phi}_{N,2}.$$

2. Recall that an alternating knot is fibered if and only if its ordinary Alexander polynomial is monic. In contrast to this classical result it follows from calculations by Goda and Morifuji [GM03] (cf. also [Mo08]) that there exists an alternating knot such that a twisted Reidemeister torsion is monic, but which is not fibered.

3. Theorem 10 has been generalized by Silver and Williams [SW09d] to give an obstruction for a general group to admit an epimorphism onto $\mathbb{Z}$ such that the kernel is a finitely generated free group.

Proof The condition on the degrees is proved in [FK06, Theorem 1.3] for R a Noetherian UFD. The monicness of twisted Reidemeister torsion was proved by Goda, Kitano and Morifuji [GKM05] in the case of a knot complement. The monicness of $\Delta_{N,1}^{\gamma\otimes\phi}$ was proved in [FK06, Theorem 1.3]. The general case of Theorem 10 can be obtained by a direct calculation as follows. Let S be the fiber and $f : S \to S$ the monodromy. We endow S with a CW-structure such that f is a cellular map. Denote by D_i the set of i-cells of S and denote by n_i the number of i-cells. We can then endow $N = (S \times [0, 1])/(x, 0) \sim (f(x), 1)$ with a CW-structure where the i-cells are given by D_i and $E_i := \{c \times (0, 1) \mid c \in D_{i-1}\}$. A direct calculation using [Tu01, Theorem 2.2] now shows that $\tau(N, \gamma \otimes \phi) \in Q(t)$ is monic and that

$$\deg(\tau(N, \gamma \otimes \phi)) = -k\chi(S) = k\|\phi\|_T.$$

□

The calculations in [Ch03], [GKM05] and [FK06] gave evidence that twisted Alexander polynomials are very successful at detecting non-fibered manifolds. The results of Morifuji [Mo08, p. 452] also give evidence to the conjecture that the twisted Alexander polynomial corresponding to a 'generic' representation detect fiberedness.

Using a deep result of Agol [Ag08] the authors proved in [FV08c] (see also [FV10] for an outline of the proof) the following converse to Theorem 10.

Theorem 11 *Let N be a 3-manifold with empty or toroidal boundary. Let $\phi \in H^1(N; \mathbb{Z})$ a nontrivial class. If for any epimorphism $\gamma : \pi_1(N) \to G$ onto a finite group the twisted Alexander polynomial $\Delta_N^{\gamma\otimes\phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and*

$$k\|\phi\|_T = \deg(\tau(N, \gamma \otimes \phi))$$

holds, then (N, ϕ) fibers over S^1.

Remark 9 Building on work of Taubes [Ta94, Ta95], Donaldson [Do96] and Kronheimer [Kr99] the authors also show that Theorem 11 implies the following: If N is a closed 3-manifold and if $S^1 \times N$ is symplectic, then N fibers over S^1. This provides a converse to a theorem of Thurston [Th76]. We refer to [FV06, FV08a, FV08b, FV08c] for details, and we refer to Kutluhan–Taubes [KT09], Kronheimer–Mrowka [KM08] and Ni [Ni08] for an alternative proof in the case that $b_1(N) = 1$.

It is natural to ask whether (N, ϕ) fibers if all twisted Alexander polynomials are monic. An affirmative answer would be of great interest in the study of symplectic structures of 4-manifolds with a free circle action (cf. [FV07b]). An equivalent question has also been raised as a conjecture by Goda and Pajitnov [GP05, Conjecture 13.2] in the study of Morse–Novikov numbers. We refer to [GP05] and [Pa07] for more information on the relationship between twisted Alexander polynomials, twisted Novikov homology and Morse–Novikov numbers.

Somewhat surprisingly, there is strong evidence to the following much weaker conjecture: A pair (N, ϕ) fibers if all twisted Alexander polynomials are non-zero. In fact the authors showed the following theorem (cf. [FV07a, Theorem 1.3] and [FV08b, Theorem 1, Proposition 4.6, Corollary 5.6]).

Theorem 12 *Let N be a 3-manifold with empty or toroidal boundary and $\phi \in H^1(N;\mathbb{Z})$ non-trivial. Suppose that $\Delta_N^{\gamma\otimes\phi}$ is non-zero for any epimorphism $\gamma : \pi_1(N) \to G$ onto a finite group. Furthermore suppose that one of the following holds*:

1. *$N = S^3 \setminus \nu K$ and K is a genus one knot*,
2. $\|\phi\|_T = 0$,
3. *N is a graph manifold*,
4. *ϕ is dual to a connected incompressible surface S such that $\pi_1(S) \subset \pi_1(N)$ is separable*,

then (N, ϕ) fibers over S^1.

Here we say that a subgroup A of a group π is *separable* if for any $g \in \pi \setminus A$ there exists an epimorphism $\gamma : \pi \to G$ onto a finite group G such that $\gamma(g) \notin \gamma(A)$. It is conjectured (cf. [Th82]) that given a hyperbolic 3-manifold N any finitely generated subgroup $A \subset \pi_1(N)$ is separable. In particular, if Thurston's conjecture is true, then Condition (4) of Theorem 12 is satisfied for any hyperbolic N.

The following theorem of Silver–Williams [SW09b] (cf. also [SW09a]) gives an interesting criterion for a knot to have vanishing twisted Alexander polynomial.

Theorem 13 *Let $K \subset S^3$ a knot. Then there exists an epimorphism $\gamma : \pi_1(S^3 \setminus \nu K) \to G$ to a finite group such that $\Delta_K^\gamma = 0$ if and only if the universal abelian cover of $S^3 \setminus \nu K$ has uncountably many finite covers.*

3.6.2 Twisted Alexander Polynomials and the Thurston Norm

It is a classical result of Alexander that given a knot $K \subset S^3$ the following inequality holds:

$$2\text{genus}(K) \geq \deg(\Delta_K).$$

This result was first generalized to general 3-manifolds by McMullen [McM02] and then to twisted Alexander polynomials in [FK06]. The following theorem is [FK06, Theorem 1.1]. The proof builds partly on ideas of Turaev's in [Tu02b].

Theorem 14 *Let N be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(N;\mathbb{Z})$ be non-trivial and let $\gamma : \pi_1(N) \to GL(k,R)$ be a representation such that $\Delta_N^{\gamma\otimes\phi} \neq 0$. Then*

$$\|\phi\|_T \geq \frac{1}{k}\deg(\tau(N,\gamma\otimes\phi)).$$

Equivalently,

$$\|\phi\|_T \geq \frac{1}{k}\big(\deg(\Delta_N^{\gamma\otimes\phi}) - \deg(\Delta_{N,0}^{\gamma\otimes\phi}) - \deg(\Delta_{N,2}^{\gamma\otimes\phi})\big).$$

Remark 10 In [FK06] it was furthermore shown using 'KnotTwister' (cf. [Fr09]) that twisted Alexander polynomials detect the genus of all knots with up to twelve crossings.

It seems reasonable to conjecture that given an irreducible 3-manifold N twisted Alexander polynomials detect the Thurston norm for any $\phi \in H^1(N;\mathbb{Z})$. A positive answer would have interesting consequences for 4-manifold topology as pointed out in [FV09].

If $\Delta_N^{\gamma\otimes\phi} = 0$, then we define the *torsion twisted Alexander polynomial* $\tilde{\Delta}_N^{\gamma\otimes\phi}$ to be the order of the $R[t^{\pm1}]$-module

$$\begin{aligned}&\mathrm{Tor}_{R[t^{\pm1}]}(H_1(N;R[t^{\pm1}]^k))\\ &\quad= \{v \in H_1(N;R[t^{\pm1}]^k) \mid \lambda v = 0 \text{ for some } \lambda \in R[t^{\pm1}] \setminus \{0\}\}.\end{aligned}$$

It is then shown in [FK06, Sect. 4] that the $\tilde{\Delta}_N^{\gamma\otimes\phi}$ also gives rise to give lower bounds on the Thurston norm. We point out that by [Hi02, Theorem 3.12 (3)] and [Tu01, Lemma 4.9] the torsion twisted Alexander polynomial can be computed directly from a presentation of $H_1(N;R[t^{\pm1}]^k)$.

Note that Theorem 14 gives lower bounds on the Thurston norm for a given $\phi \in H^1(N;\mathbb{Z})$. In order to give bounds for the whole Thurston norm ball at once, we will introduce twisted Alexander norms, generalizing McMullen's Alexander norm [McM02] and Turaev's torsion norm [Tu02a]. In the following let N be a 3-manifold with empty or toroidal boundary such that $b_1(N) > 1$. Let $\psi : \pi_1(N) \to F := H_1(N;\mathbb{Z})/\text{torsion}$ be the canonical projection map. Let $\gamma : \pi_1(N) \to GL(k,R)$ be a representation. If $\Delta_N^{\gamma\otimes\psi} = 0$ then we set $\|\phi\|_A^\gamma = 0$ for all $\phi \in H^1(N;\mathbb{R})$. Otherwise we write $\Delta_N^{\gamma\otimes\psi} = \sum a_i f_i$ for $a_i \in R$ and $f_i \in F$. Given $\phi \in H^1(N;\mathbb{R})$ we then define

$$\|\phi\|_A^\gamma := \max\{\phi(f_i - f_j) \mid (f_i, f_j) \text{ such that } a_i a_j \neq 0\}.$$

Note that this norm is independent of the choice of representative of $\Delta_N^{\gamma\otimes\psi}$. Clearly this defines a seminorm on $H^1(N;\mathbb{R})$ which we call the *twisted Alexander norm of* (N,γ). Note that if $\gamma : \pi_1(N) \to GL(1,\mathbb{Z})$ is the trivial representation, then we just obtain McMullen's Alexander norm.

The following is proved in [FK08a, Theorem 3.1], but we also refer to the work of McMullen [McM02], Turaev [Tu02a], [Tu02b, Sect. 6], Harvey [Ha05] and Vidussi [Vi99, Vi03]. The main idea of the proof is to combine Theorem 14 with Corollary 1.

Theorem 15 *Let N be a 3-manifold with empty or toroidal boundary such that $b_1(N) > 1$. Let $\psi : \pi_1(N) \to F := H_1(N;\mathbb{Z})/torsion$ be the canonical projection map and let $\gamma : \pi_1(N) \to GL(k,R)$ be a representation with R a Noetherian UFD. Then for any $\phi \in H^1(N;\mathbb{R})$ we have*

$$\|\phi\|_T \geq \frac{1}{k}\|\phi\|_A^{\gamma}$$

and equality holds for any ϕ in a fibered cone of the Thurston norm ball.

We refer to [McM02] and [FK08a] for some calculations, and we refer to [Du01] for more information on the relationship between the Alexander norm and the Thurston norm.

3.6.3 Normalized Twisted Reidemeister Torsion and the Free Genus of a Knot

Let $K \subset S^3$ a knot and let $\gamma : \pi_1(S^3 \setminus \nu K) \to GL(k,R)$ be a representation with R a Noetherian UFD. Let $\epsilon = \det(\gamma(\mu))$ where μ denotes a meridian of K. Kitayama [Kiy08a] introduced an invariant $\tilde{\Delta}_{K,\gamma} \in Q(R(\epsilon^{\frac{1}{2}})[t^{\pm\frac{1}{2}}])$ which is a normalized version of the twisted Reidemeister torsion $\tau(K,\gamma)$, i.e. $\tau(K,\gamma)$ has no indeterminacy and up to multiplication by an element of the form $\epsilon^r t^{\frac{s}{2}}$ it is a representative of the twisted Reidemeister torsion. (The invariant $\tilde{\Delta}_{K,\gamma}$ should not be confused with the torsion polynomial introduced in Sect. 3.5.4.)

Kitayama [Kiy08a, Theorem 6.3] studies the invariant $\tilde{\Delta}_{K,\gamma}$ for fibered knots, obtaining a refined version of the fibering obstruction which we stated in Theorem 10. Furthermore [Kiy08a, Theorem 5.8] proves a duality theorem for $\tilde{\Delta}_{K,\gamma}$, refining Proposition 4.

In the following we say that S is a *free Seifert surface* for K if $\pi_1(S^3 \setminus S)$ is a free group. Note that Seifert's algorithm produces free Seifert surfaces, in particular any knot has a free Seifert surface. Given K the *free genus* is now defined as

$$\text{free-genus}(K) = \min\{\text{genus}(S) \mid S \text{ free Seifert surface for } K\}.$$

Clearly we have free-genus$(K) \geq$ genus(K). In order to state the lower bound on the free genus coming from $\tilde{\Delta}_{K,\gamma}$ we have to make a few more definitions. Given a Laurent polynomial $p = \sum_{i=k}^{l} a_i t^i \in R[t^{\pm\frac{1}{2}}]$ with $a_k \neq 0, a_l \neq 0$ we write l-deg$(p) = k$ ('lowest degree') and h-deg$(p) = l$ ('highest degree'). Furthermore given $f \in Q(R[t^{\pm\frac{1}{2}}])$ we define

$$\text{h-deg}(f) = \text{h-deg}(p) - \text{h-deg}(q)$$

where we pick $p, q \in R[t^{\pm\frac{1}{2}}]$ with $f = \frac{p}{q}$. Kitayama [Kiy08a, Proposition 6.6] proved the following theorem:

Theorem 16 *Given a knot $K \subset S^3$ and a representation $\gamma : \pi_1(S^3 \setminus \nu K) \to GL(k, R)$ with R a Noetherian UFD we have*

$$2k\textit{free-genus}(K) \geq 2\textit{h-deg}(\tilde{\Delta}_{K,\gamma}) + k.$$

Note that if h-deg$(\tilde{\Delta}_{K,\gamma}) = -$l-deg$(\tilde{\Delta}_{K,\gamma})$ (which is the case for unitary representations) then the bound in the theorem is implied by the bound given in Theorem 14. It is a very interesting question whether Kitayama's bound can detect the difference between the genus and the free genus.

Recall that Lin's original definition of the twisted Alexander polynomial is in terms of regular Seifert surfaces, it might be worthwhile to explore the possibility that an appropriate version of Lin's twisted Alexander polynomial gives lower bounds on the 'regular genus' of a knot which can tell the regular genus apart from the free genus.

3.7 Twisted Invariants of Knots and Special Representations

Given a knot $K \subset S^3$ the representations which have been studied most are the 2-dimensional complex representations and the metabelian representations. It is therefore natural to consider special properties of twisted Alexander polynomials corresponding to such representations.

3.7.1 Parabolic Representations

Let $K \subset S^3$ be a knot. A representation $\gamma : \pi_1(S^3 \setminus \nu K) \to SL(2, \mathbb{C})$ is called *parabolic* if the image of any meridian is a matrix with trace 2. Note that Thurston [Th87] showed that given a hyperbolic knot K the discrete faithful representation $\pi_1(S^3 \setminus \nu K) \to PSL(2, \mathbb{C})$ lifts to a parabolic representation $\gamma : \pi_1(S^3 \setminus \nu K) \to SL(2, \mathbb{C})$. The twisted Reidemeister torsion corresponding to this canonical representation has been surprisingly little studied (cf. though [Sug07] and [Mo08, Corollary 4.2]).

Throughout the remainder of this section let K be a 2-bridge knot. Then the group $\pi_1(S^3 \setminus \nu K)$ has a presentation of the form $\langle x, y | Wx = yW \rangle$ where x, y are meridians of K and W is a word in $x^{\pm 1}, y^{\pm 1}$. Parabolic representations of 2-bridge knots have been extensively studied by Riley [Ri72, Sect. 3]. To a 2-bridge knot K Riley associates a monic polynomial $\Phi_K(t) \in \mathbb{Z}[t^{\pm 1}]$ such that any zero ζ of $\Phi_K(t)$ gives rise to a representation $\gamma_\zeta : \pi_1(S^3 \setminus \nu K) \to SL(2, \mathbb{C})$ of the form

$$\gamma_\zeta(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_\zeta(y) = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}.$$

Furthermore Riley shows that any parabolic representation of a 2-bridge knot is conjugate to such a representation (cf. also [SW09c, Sect. 5] for details).

Given an irreducible factor $\phi(t)$ of $\Phi_K(t)$ of degree d one can consider the representation $\oplus\gamma_{\zeta'}$ where ζ' runs over the set of all zeroes of $\phi(t)$. Silver and Williams show that this representation is conjugate to an integral representation $\gamma_{\phi(t)} : \pi_1(S^3 \setminus \nu K) \to GL(2d, \mathbb{Z})$ which is called the *total representation corresponding to* $\phi(t)$.

Twisted Alexander polynomials of 2-bridge knots corresponding to parabolic and total representations have been studied extensively by Silver and Williams [SW09c] and Hirasawa and Murasugi (cf. [Mu06] and [HM08]). In particular the following theorem is shown. It should be compared to the classical result that given a knot K we have $\Delta_K(1) = 1$.

Theorem 17 *Let K be a two-bridge knot and $\phi(t)$ an irreducible factor of $\Phi_K(t)$ of degree d. Then*

$$\left|\Delta_K^{\gamma_{\phi(t)}}(1)\right| = 2^d.$$

A special case of the theorem is shown in [HM08, Theorem A], the general case is proved in [SW09c, Theorem 6.1]. Silver and Williams also conjectured that under the assumptions of the theorem we have

$$\left|\Delta_K^{\gamma_{\phi(t)}}(-1)\right| = 2^d m^2$$

for some odd number m.

We refer to [SW09c] and [HM08] for more on twisted Alexander polynomials of 2-bridge knots. All three papers contain a wealth of interesting examples and results which we find impossible to summarize in this short survey.

Silver and Williams also considered twisted Alexander polynomials of torus knots. They showed [SW09c, Sect. 7] that for any parabolic representation γ of a torus knot K, the twisted Alexander polynomial Δ_K^γ is a product of cyclotomic polynomials.

3.7.2 Twisted Alexander Polynomials and the Space of 2-Dimensional Representations

It is a natural question to study the behavior of twisted Alexander polynomials under a change of representation. Recall that given a group π we say that two representations $\alpha, \beta : \pi \to GL(k, R)$ are *conjugate* if there exists $P \in GL(k, R)$ with $\alpha(g) = P\beta(g)P^{-1}$ for all $g \in \pi$. By Lemma 1 we can view Alexander polynomials as function on the set of conjugacy classes of representations.

Let K be a 2-bridge knot. Riley [Ri84] (cf. also [DHY09, Proposition 3]) showed that conjugacy classes of representations $\pi_1(S^3 \setminus \nu K) \to SL(2, \mathbb{C})$ correspond to the zeros of an affine algebraic curve in $\mathbb{C}^2$. The twisted Reidemeister torsion corresponding to these representations for twist knots have been studied by Morifuji [Mo08] (cf. also [GM03]). The computations show in particular that for twist knots twisted Reidemeister torsion detects fiberedness and the genus for all but finitely many conjugacy classes of non-abelian $SL(2,\mathbb{C})$-representations.

Regarding twist knots we also refer to the work of Huynh and Le [HL07, Theorem 3.3] who found an unexpected relationship between twisted Alexander polynomials and the A-polynomial. (Note thought that their definition of twisted Alexander polynomials differs somewhat from our approach.)

Finally we refer to Kitayama's work [Kiy08b] for certain symmetries when we view twisted Alexander polynomials of knots in rational homology spheres as a function on the space of 2-dimensional regular unitary representations.

3.7.3 Twisted Invariants of Hyperbolic Knots and Links

Let $L \subset S^3$ be a hyperbolic link. Then the corresponding representation $\pi_1(S^3 \setminus \nu L) \to PSL(2,\mathbb{C})$ lifts to a canonical representation $\gamma_{can} : \pi_1(S^3 \setminus \nu L) \to SL(2,\mathbb{C})$ and it induces the adjoint representation

$$\gamma_{adj} : \pi_1(S^3 \setminus \nu L) \to PSL(2,\mathbb{C}) \to \mathrm{Aut}(sl(2,\mathbb{C})) \cong SL(3,\mathbb{C}).$$

The corresponding invariant $\tau(L,\gamma_{adj})$ has been studied in great detail by Dubois and Yamaguchi [DY09]. In particular it is shown that $\tau(L,\gamma_{adj})$ is a symmetric non-zero polynomial (the non-vanishing result builds on work of Porti [Po97]). Furthermore the invariant $\tau(L,\gamma_{adj})$ is computed explicitly for many examples.

In [DFJ10] it is shown that given a knot K the invariant $\tau(K,\gamma_{can})$ gives rise to a well-defined non-zero invariant $\mathcal{T}_K(t)$. (This result builds on work of Menal-Ferrer and Porti [MP09]). This invariant is computed for all knots up to 13 crossings, for all these knots the invariant $\mathcal{T}_K(t)$ detects the genus, the fiberedness and the chirality.

3.7.4 Metabelian Representations

Given a group G the derived series of G is defined inductively by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. A representation of a group G is called metabelian if it factors through $G/G^{(2)}$. Note that if K is a knot, then the metabelian quotient $\pi_1(S^3 \setminus \nu K)/\pi_1(S^3 \setminus \nu K)^{(2)}$ is well-known to be determined by the Alexander module of K (cf. e.g. [BF08, Sect. 2]). Metabelian representations and metabelian quotients of knot groups have been studied extensively, we refer to [Fo62, Fo70, Hat79, Fr04, Je08] and [BF08] for more information.

The structure of twisted Alexander polynomials of knots corresponding to certain types of metabelian representations has been studied in detail by Hirasawa and Murasugi [HM09a, HM09b]. These papers contain several interesting conjectures regarding special properties of such twisted Alexander polynomials, furthermore these conjectures are verified for certain classes of 2-bridge knots and further evidence is given by explicit calculations.

3.8 Miscellaneous Applications of Twisted Reidemeister Torsion to Knot Theory

In this section we will summarize various applications of twisted Alexander polynomials to the study of knots and links.

3.8.1 A Partial Order on Knots

Given a knot K we write $\pi_1(S^3 \setminus \nu K) := \pi_1(S^3 \setminus K)$. For two prime knots K_1 and K_2 one defines $K_1 \geq K_2$ if there exists a surjective group homomorphism $\varphi : \pi_1(K_1) \to \pi_1(K_2)$. The relation "$\geq$" defines a partial order on the set of prime knots (cf. [KS05a]). Its study is often related to a well-known conjecture of J. Simon, that posits that for a given K_1, the set of knots K_2 s.t. $K_1 \geq K_2$ is finite.

Kitano, Suzuki and Wada [KSW05] prove the following theorem which generalizes a result of Murasugi [Mu03].

Theorem 18 *Let K_1 and K_2 be two knots in S^3 and let $\varphi : \pi_1(K_1) \to \pi_1(K_2)$ an epimorphism. Let $\gamma : \pi_1(K_2) \to GL(k, R)$ a representation with R a Noetherian UFD. Then*

$$\frac{\tau(K_1, \gamma \circ \varphi)}{\tau(K_2, \gamma)}$$

is an element in $R[t^{\pm 1}]$.

This theorem plays a crucial role in determining the partial order on the set of knots with up to eleven crossings. We refer to [KS05a, KS05b, KS08] and [HKMS09] for details.

3.8.2 Periodic and Freely Periodic Knots

A knot $K \subset S^3$ is called *periodic of period q*, if there exists a smooth transformation of S^3 of order q which leaves K invariant and such that the fixed point set is a circle A disjoint from K. Note that $A \subset S^3$ is the trivial knot by the Smith conjecture. We refer to [Ka96, Sect. 10.1] for more details on periodic knots.

Now assume that $K \subset S^3$ is a periodic knot of period q with A the fixed point set of $f : S^3 \to S^3$. Note that S^3/f is diffeomorphic to S^3. We denote by π the projection map $S^3 \to S^3/f = S^3$ and we write $\overline{K} = \pi(K)$, $\overline{A} = \pi(A)$.

The following two theorems of Hillman, Livingston and Naik [HLN06, Theorems 3 and 4] generalize results of Trotter [Tr61] and Murasugi [Mu71].

Theorem 19 *Let K be a periodic knot of period q. Let $\pi, A, \overline{A}$ and $\overline{K}$ as above. Let $\overline{\gamma} : \pi_1(S^3 \setminus \nu\overline{K}) \to GL(n, R)$ a representation with $R = \mathbb{Z}$ or $R = \mathbb{Q}$. Write $\gamma = \overline{\gamma} \circ \pi_*$. Then there exists a polynomial $F(t, s) \in R[t^{\pm 1}, s^{\pm 1}]$ such that*

$$\Delta_K^{\gamma}(t) \doteq \Delta_{\overline{K}}^{\overline{\gamma}}(t) \prod_{k=1}^{q-1} F(t, e^{2\pi i k/q}).$$

In the untwisted case the polynomial $F(t, s)$ is just the Alexander polynomial of the ordered link $\overline{K} \cup \overline{A} \subset S^3$. We refer to [HLN06, Sect. 6] for more information on $F(t, s)$. The following theorem gives an often stronger condition when the period is a prime power.

Theorem 20 *Let K be a periodic knot of period $q = p^r$ where p is a prime. Let $\pi, A, \overline{A}$ and $\overline{K}$ as above. Let $\overline{\gamma} : \pi_1(S^3 \setminus \nu\overline{K}) \to GL(n, \mathbb{Z}_p)$ be a representation. Write $\gamma = \overline{\gamma} \circ \pi_*$. If $\Delta_K^{\gamma}(t) \neq 0$, then*

$$\Delta_K^{\gamma}(t) \cdot (\Delta_{K,0}^{\gamma}(t))^{q-1} \doteq \Delta_{\overline{K}}^{\overline{\gamma}}(t)^q \, (\det(id_n - \gamma(A) t^{lk(K,A)}))^{q-1} \in \mathbb{Z}_p[t^{\pm 1}],$$

where we view A as an element in $\pi_1(S^3 \setminus \nu K)$.

(Note that Elliot [El08] gave an alternative proof for this theorem.) These theorems are applied in [HLN06, Sect. 10] to give obstructions on the periodicity of the Kinoshita–Terasaka knot and the Conway knot. Note that both knots have trivial Alexander polynomial, in particular Murasugi's obstructions are satisfied trivially.

A knot $K \subset S^3$ is called *freely periodic of period q* if there exists a free transformation f of S^3 of order q which leaves K invariant. We refer to [Ka96, Sect. 10.2] for more information on freely periodic knots. Given such a freely periodic knot K we denote by π the projection map $S^3 \to \Sigma := S^3/f$ and we write $\overline{K} = \pi(K)$.

The following theorem is [HLN06, Theorem 5]; the untwisted case was first proved by Hartley [Hat81].

Theorem 21 *Let K be a freely periodic knot of period q. Let π, § and $\overline{K}$ as above. Let $\overline{\gamma} : \pi_1(\S \setminus \overline{K}) \to GL(k, R)$ be a representation with R a Noetherian UFD. Write $\gamma = \overline{\gamma} \circ \pi_*$. Then*

$$\Delta_K^{\gamma}(t^q) \doteq \prod_{k=0}^{q-1} \Delta_{\overline{K}}^{\overline{\gamma}}(e^{2\pi i k/q} t).$$

We refer to [HLN06, Sect. 11] for an application of this theorem to a case which could not be settled with Hartley's theorem.

3.8.3 Zeroes of Twisted Alexander Polynomials and Non-abelian Representations

Let N be a 3-manifold with one boundary component and $b_1(N) = 1$. Put differently, let N be the complement of a knot in a rational homology sphere. Let $\alpha : \pi_1(N) \to \mathbb{C}^* = GL(1, \mathbb{C})$ be a one-dimensional representation. Note that α necessarily factors through a representation $H_1(N; \mathbb{Z}) \to GL(1, \mathbb{C})$ which we also denote by α. Heusener and Porti [HP05] ask when the abelian representation

$$\begin{aligned} \rho_\alpha : \pi_1(N) &\to PSL(2, \mathbb{C}) \\ g &\mapsto \pm \begin{pmatrix} \alpha(g)^{1/2} & 0 \\ 0 & \alpha(g)^{-1/2} \end{pmatrix} \end{aligned}$$

can be deformed into an irreducible representation.

Denote by $\phi \in \mathrm{Hom}(\pi_1(N), \mathbb{Z}) = H^1(N; \mathbb{Z}) \cong \mathbb{Z}$ a generator. Pick $\mu \in H_1(N; \mathbb{Z})$ with $\phi(\mu) = 1$. Denote by $\sigma(\alpha, \mu)$ the representation

$$\begin{aligned} \pi_1(N) &\to GL(1, \mathbb{C}) \\ g &\mapsto \alpha(g\mu^{-\phi(g)}). \end{aligned}$$

Heusener and Porti then give necessary and sufficient conditions for ρ_α to be deformable into an irreducible representation in terms of the order of vanishing of $\Delta_N^{\sigma(\alpha,\mu)\otimes\phi}(t) \in \mathbb{C}[t^{\pm 1}]$ at $\alpha(\mu) \in \mathbb{C}^*$. We refer to [HP05, Theorems 1.2 and 1.3] for more precise formulations and more detailed results.

Note that this result is somewhat reminiscent of the earlier results of Burde [Bu67] and de Rham [dRh68] who showed that zeroes of the (untwisted) Alexander polynomial give rise to metabelian representations of the knot group. We also refer to [SW09e] for another relationship between zeros of twisted Alexander polynomials and the representation theory of knot groups.

3.8.4 Seifert Fibered Surgeries

In [Kiy09, Sect. 3] Kitayama gives a surgery formula for twisted Reidemeister torsion. Furthermore in [Kiy09, Lemma 4.3] a formula for the Reidemeister torsion of a Seifert fibered space is given. By studying a suitable invariant derived from twisted Reidemeister torsion an obstruction for a Dehn surgery on a knot to equal a specific Seifert fibered space are given. These obstructions generalize Kadokami's obstructions given in [Ka06] and [Ka07]. Finally Kitayama applies these methods to show that there exists no Dehn surgery on the Kinoshita–Terasaka knot which is homeomorphic to any Seifert fibered space of the form $M(p_1/q_1, p_2/q_2, p_3/q_3)$ (we refer to [Kiy09, Sect. 4] for the notation).

3.8.5 Homology of Cyclic Covers

Let $K \subset S^3$ be a knot. We denote by $H := H_1(S^3 \setminus \nu K; \mathbb{Z}[t^{\pm 1}])$ its Alexander module. Given $n \in \mathbb{N}$ we denote by L_n the n-fold cyclic branched cover of K. Recall that we have a canonical isomorphism $H/(t^n - 1) \cong H_1(L_n)$. Put differently, the Alexander module determines the homology of the branched covers. In the same vein, the following formula due to Fox ([Fo56] and see also [Go78]) shows that the Alexander polynomial determines the size of the homology of a branched cover:

$$|H_1(L_n)| = \left| \prod_{j=1}^{n-1} \Delta_K(e^{2\pi i j/n}) \right|. \tag{3.1}$$

Here we write $|H_1(L_n)| = 0$ if $H_1(L_n)$ has positive rank. Gordon [Go72] used this formula to study extensively the homology of the branched covers of a knot. Gordon [Go72, p. 366] asked whether the non-zero values of the sequence $|H_1(L_n)|$ converge to infinity if there exists a zero of the Alexander polynomial which is not a root of unity.

Given a multivariable polynomial $p := p(t_1, \dots, t_n) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ the Mahler measure is defined as

$$m(p) = \exp \int_{\theta_1=0}^{1} \dots \int_{\theta_n=0}^{1} \log \left| p\left(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_1}\right) \right| d\theta_1 \dots d\theta_n.$$

Note that the integral can be singular, but one can show that the integral always converges. It is known (cf. e.g. [SW02]) that an integral one variable polynomial p always satisfies $m(p) \geq 1$, and it satisfies $m(p) = 1$ if and only if all zeroes of p are roots of unity.

Theorem 22 *Let K be any knot, then*

$$\lim_{n \to \infty} \frac{1}{n} \log |Tor\, H_1(L_n)| = \log(m(\Delta_K(t))).$$

This theorem was proved for most cases by González-Acuña and Short [GS91], the most general statement was proved by Silver and Williams [SW02, Theorem 2.1]. We also refer to [Ri90] for a related result. Note that by the above discussion this theorem in particular implies the affirmative answer to Gordon's question.

Silver and Williams also generalized this theorem to links, relating the Mahler measure multivariable Alexander polynomial to the homology growth of finite abelian covers of the link [SW02, Theorem 2.1]. Finally in [SW09c, Sect. 3] these results are extended to the twisted case for certain representations (e.g. integral representations). We refer to [SW09c, Sect. 3] for the precise formulations and to [SW09c, Sect. 5] for an interesting example.

3.8.6 Alexander Polynomials for Links in $\mathbb{R}P^3$

Given a link $L \subset \mathbb{R}P^3$ Huynh and Le [HL08, Sect. 5.3.2] use Reidemeister torsion corresponding to abelian representations to define an invariant $\nabla_L(t)$ which lies in general in $\mathbb{Z}[t^{\pm 1}, (t-t^{-1})^{-1}]$ and has no indeterminacy. Furthermore they show in [HL08, Theorem 5.7] the surprising fact that this invariant satisfies in fact a skein relation.

3.9 Twisted Alexander Polynomials of CW-complexes and Groups

3.9.1 Definitions and Basic Properties

Let X be a CW-complex with finitely many cells in each dimension. Assume we are given a non-trivial homomorphism $\psi : \pi_1(X) \to F$ to a torsion-free abelian group and a representation $\gamma : \pi_1(X) \to GL(k, R)$ where R is a Noetherian UFD. As in Sect. 3.2.4 we can define the twisted Alexander modules

$$H_i(X; R[F]^k) = H_i(C_*(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} R[F]^k)$$

where $\pi_1(X)$ acts on $C_*(\tilde{X}; \mathbb{Z})$ by deck transformations and on $R[F]^k$ by $\gamma \otimes \psi$. These modules are finitely presented and we can therefore define the twisted Alexander polynomial $\Delta_{X,i}^{\gamma\otimes\psi} \in R[F]$ to be the order of the $R[F]$-module $H_i(X; R[F]^k)$. Note that twisted Alexander polynomials are homotopy invariants, in particular given any manifold homotopy equivalent to a finite CW-complex we can define the twisted Alexander polynomials $\Delta_{X,i}^{\gamma\otimes\psi} \in R[F]$.

Let G be a finitely presented group and $X = K(G, 1)$ its Eilenberg–Maclane space. Given a non-trivial homomorphism $\psi : G \to F$ to a torsion-free abelian group and a representation $\gamma : G \to GL(k, R)$ where R is a Noetherian UFD we define

$$\Delta_{G,i}^{\gamma\otimes\psi} = \Delta_{K(G,1),i}^{\gamma\otimes\psi}.$$

Remark 11

1. For $i = 0, 1$ the Alexander polynomials $\Delta_{X,i}^{\gamma\otimes\psi} \in R[F]$ can be computed using Fox calculus as in Sect. 3.2.5.
2. Note that unless the Euler characteristic of X vanishes we can not define the Reidemeister torsion corresponding to (X, ψ, γ).
3. Most of the results of Sect. 3.3 do not hold in the general context. The only results which do generalize are Proposition 2 (2), Lemma 1 and Theorem 3.

4. Note that given a finitely presented group G and ψ, γ as above Wada [Wa94] introduced an invariant which we refer to as $W(G, \gamma \otimes \psi) \in Q(R[F])$. Using [Tu01, Lemma 4.11] one can show that

$$W(G, \gamma \otimes \psi) \doteq \frac{\Delta^{\gamma\otimes\psi}_{G,1}}{\Delta^{\gamma\otimes\psi}_{G,0}}.$$

In the literature Wada's invariant is often referred to as the twisted Alexander polynomial of a group.

3.9.2 Twisted Alexander Polynomials of Groups

The twisted Alexander polynomial has been calculated by Morifuji [Mo01, Theorem 1.1] for the braid groups B_n with $\psi : B_n \to \mathbb{Z}$ the abelianization map and together with the Burau representation. Morifuji [Mo01, Theorem 1.2] also proves a symmetry theorem for twisted Alexander polynomials of braid groups for Jones representations corresponding to dual Young diagrams.

In [Suz04] Suzuki shows that the twisted Alexander polynomial of the braid group B_4 corresponding to the Lawrence–Krammer representation and the abelianization $\phi : B_4 \to \mathbb{Z}$ is trivial. This shows in particular that the twisted Alexander polynomial of a group corresponding to a faithful representation can be trivial. It is an interesting question whether given a knot and a faithful representation the twisted Alexander polynomial can ever be trivial.

Given a 2-complex X Turaev [Tu02c] introduced a norm on $H^1(X;\mathbb{R})$ to which we refer to as the Turaev norm. The definition of the Turaev norm is inspired by the definition of the Thurston norm [Th86]. Turaev [Tu02c] uses twisted Alexander polynomials of X corresponding to one-dimensional representations to define a twisted Alexander norm similar to the one defined in Sect. 3.6.2. Turaev goes on to show that the twisted Alexander norm gives a lower bound on the Turaev norm. We refer to [Tu02b, Sect. 7.1] for more information.

3.9.3 Plane Algebraic Curves

Let $\mathcal{C} \subset \mathbb{C}^2$ be an affine algebraic curve. Denote by $P_1, \ldots, P_k$ the set of singularities and denote by $L_1, \ldots, L_k$ the links at the singularities and let L_∞ be the link at infinity (we refer to [CF07] for details). Note that $\mathbb{C}^2 \setminus \nu\mathcal{C}$ is homotopy equivalent to a finite CW-complex. By Sect. 3.9.1 we can therefore consider the twisted Alexander polynomial of $\mathbb{C}^2 \setminus \nu\mathcal{C}$. Now let $\gamma : \pi_1(\mathbb{C}^2 \setminus \nu\mathcal{C}) \to GL(k, \mathbb{F})$ be a representation where $\mathbb{F} \subset \mathbb{C}$ is a subring closed under conjugation. Let $\phi : \mathbb{C}^2 \setminus \nu\mathcal{C} \to \mathbb{Z}$ be the map given by sending each oriented meridian to one. Cogolludo and Florens [CF07, Theorem 1.1] then relate twisted Alexander polynomial of $\mathbb{C}^2 \setminus \nu\mathcal{C}$ corresponding to $\gamma \otimes \phi$ to the one-variable twisted Alexander polynomials of the links

$L_1, \dots, L_k$ and L_∞. This result generalizes a result of Libgober's regarding untwisted Alexander polynomials of affine algebraic curves (cf. [Lib82, Theorem 1]). We refer to [CF07, Sect. 6] for applications of this result. Finally we refer to [CS08] for a further application of twisted Alexander polynomials to algebraic geometry.

3.10 Alexander Polynomials and Representations over Non-commutative Rings

In the previous sections we only considered finite dimensional representations over commutative rings. One possible approach to studying invariants corresponding to infinite dimensional representations is to use the theory of L^2-invariants. We refer to [Lü02] for the definition of various L^2-invariants and for some applications to low-dimensional topology. Even though L^2-invariants are a powerful tool they have not yet been systematically studied for links and 3-manifolds. We refer to the work of Li and Zhang [LZ06a, LZ06b] for some initial work. We also would like to use this opportunity to advertise a problem stated in [FLM09, Sect. 3.2, Remark (3)].

For the remainder of this section we will now be concerned with invariants corresponding to finite dimensional representations over non-commutative rings. The study of such invariants (often referred to as higher order Alexander polynomials) was initiated by Cochran [Co04], building on ideas of Cochran, Orr and Teichner [COT03]. The notion of higher order Alexander polynomials was extended to 3-manifolds by Harvey [Ha05] and Turaev [Tu02b]. This theory is different in spirit to the fore mentioned L^2-invariants, but we refer to [Ha08] and [FLM09, Proposition 2.4] for some connections.

3.10.1 Non-commutative Alexander Polynomials

Let $\mathbb{K}$ be a (skew) field and $\gamma : \mathbb{K} \to \mathbb{K}$ a ring homomorphism. Denote by $\mathbb{K}[t^{\pm 1}]$ the corresponding skew Laurent polynomial ring over $\mathbb{K}$. The elements in $\mathbb{K}[t^{\pm 1}]$ are formal sums $\sum_{i=-r}^{s} a_i t^i$ with $a_i \in \mathbb{K}$ and multiplication in $\mathbb{K}[t^{\pm 1}]$ is given by the rule $t^i a = \gamma^i(a) t^i$ for any $a \in \mathbb{K}$.

Let N be a 3-manifold and let $\phi \in H^1(N;\mathbb{Z})$ be non-trivial. Following Turaev [Tu02b] we call a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(N)] \to \mathbb{K}[t^{\pm 1}]$ *ϕ-compatible* if for any $g \in \pi_1(N)$ we have $\varphi(g) = kt^{\phi(g)}$ for some $k \in \mathbb{K}$. Given a ϕ-compatible homomorphism $\varphi : \mathbb{Z}[\pi_1(N)] \to \mathbb{K}[t^{\pm 1}]$ we consider the $\mathbb{K}[t^{\pm 1}]$-module

$$H_i(N;\mathbb{K}[t^{\pm 1}]) = H_i(C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} \mathbb{K}[t^{\pm 1}])$$

where $\tilde{N}$ is the universal cover of N. Since $\mathbb{K}[t^{\pm 1}]$ is a principal ideal domain (PID) (cf. [Co04, Proposition 4.5]) we can decompose

$$H_i(N;\mathbb{K}[t^{\pm 1}]) \cong \bigoplus_{k=1}^{l} \mathbb{K}[t^{\pm 1}]/(p_k(t))$$

for $p_k(t) \in \mathbb{K}[t^{\pm 1}]$, $1 \le k \le l$. We define $\Delta^{\varphi}_{N,\phi,i} := \prod_{k=1}^{l} p_k(t) \in \mathbb{K}[t^{\pm 1}]$. As for twisted Alexander polynomials we write $\Delta^{\varphi}_{N,\phi} = \Delta^{\varphi}_{N,\phi,1}$. Non-commutative Alexander polynomials have in general a high indeterminacy, we refer to [Co04, p. 367] and [Fr07, Theorem 3.1] for a discussion of the indeterminacy. Note though that the degree of a non-commutative Alexander polynomial is well-defined.

The following theorem was proved for knots by Cochran [Co04] and extended to 3-manifolds by Harvey [Ha05] and Turaev [Tu02b].

Theorem 23 *Let N be a 3-manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ non-trivial. Let $\varphi : \mathbb{Z}[\pi_1(N)] \to \mathbb{K}[t^{\pm 1}]$ be a ϕ-compatible homomorphism.*

1. *If the image of $\pi_1(N) \to \mathbb{K}[t^{\pm 1}]$ is non-cyclic, then*
$$\deg(\Delta^{\varphi}_{N,\phi,0}) = 0.$$
2. *If the image of $\pi_1(N) \to \mathbb{K}[t^{\pm 1}]$ is non-cyclic and if $\Delta^{\varphi}_{N,\phi} \neq 0$, then*
$$\deg(\Delta^{\varphi}_{N,\phi,2}) = 0.$$
3. *If $\Delta^{\varphi}_{N,\phi} \neq 0$, then we have the following inequality*
$$\|\phi\|_T \ge \deg(\Delta^{\varphi}_{N,\phi}) - \deg(\Delta^{\varphi}_{N,\phi,0}) - \deg(\Delta^{\varphi}_{N,\phi,2})$$
and equality holds if ϕ is a fibered class and $N \neq S^1 \times D^2$, $N \neq S^1 \times S^2$.

We refer to [Fr07] for the definition of a twisted non-commutative Alexander polynomial and to a corresponding generalization of Theorem 23, we also refer to [Fr07] for a reinterpretation of the third statement of Theorem 23 in terms of a certain non-commutative Reidemeister torsion.

3.10.2 Higher Order Alexander Polynomials

We now recall the construction of what are arguably the most interesting examples of ϕ-compatible homomorphisms from $\pi_1(N)$ to a non-commutative Laurent polynomial ring. The ideas of this section are due to Cochran and Harvey.

Theorem 24 *Let γ be a torsion-free solvable group and let $\mathbb{F}$ be a commutative field. Then the following hold.*

1. *$\mathbb{F}[\Gamma]$ is an Ore domain, in particular it embeds in its classical right ring of quotients $\mathbb{K}(\Gamma)$.*
2. *$\mathbb{K}(\Gamma)$ is flat over $\mathbb{F}[\Gamma]$.*

Indeed, it follows from [KLM88] that $\mathbb{F}[\Gamma]$ has no zero divisors. The first part now follows from [DLMSY03, Corollary 6.3]. The second part is a well-known property of Ore localizations. We call $\mathbb{K}(\Gamma)$ the *Ore localization* of $\mathbb{F}[\Gamma]$. In [COT03] the notion of a poly-torsion-free-abelian (PTFA) group is introduced, it is well-known that these groups are torsion-free and solvable.

Remark 12

1. It follows from Higman's theorem [Hi40] that the above theorem also holds for groups which are locally indicable and amenable. We will not make use of this, but note that throughout this section 'torsion-free solvable' could be replaced by 'locally indicable and amenable'.
2. Note that the poly-torsion-free-abelian (PTFA) groups introduced in [COT03] are solvable and torsion-free.

We need the following definition.

Definition 2 Let π be a group, $\phi : \pi \to \mathbb{Z}$ an epimorphism and $\varphi : \pi \to \gamma$ an epimorphism to a torsion-free solvable group γ such that there exists a map $\phi_\Gamma : \Gamma \to \mathbb{Z}$ (which is necessarily unique) such that

$$\begin{tikzcd} \pi \arrow[r, "\varphi"] \arrow[dr, "\phi"'] & \Gamma \arrow[d, "\phi_\Gamma"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes. Following [Ha06, Definition 1.4] we call (φ, ϕ) an *admissible pair*.

Now let $(\varphi : \pi_1(N) \to \Gamma, \phi)$ be an admissible pair for $\pi_1(N)$. In the following we denote $\ker\{\phi : \Gamma \to \mathbb{Z}\}$ by $\Gamma'(\phi)$. When the homomorphism ϕ is understood we will write Γ' for $\Gamma'(\phi)$. Clearly Γ' is still solvable and torsion-free. Let $\mathbb{F}$ be any commutative field and $\mathbb{K}(\Gamma')$ the Ore localization of $\mathbb{F}[\Gamma']$. Pick an element $\mu \in \gamma$ such that $\phi(\mu) = 1$. Let $\gamma : \mathbb{K}(\Gamma') \to \mathbb{K}(\Gamma')$ be the homomorphism given by $\gamma(a) = \mu a \mu^{-1}$. Then we get a ring homomorphism

$$\begin{aligned} \mathbb{Z}[\Gamma] &\to \mathbb{K}(\Gamma')_\gamma[t^{\pm 1}] \\ g &\mapsto (g\mu^{-\phi(g)} t^{\phi(g)}), \quad \text{for } g \in \gamma. \end{aligned}$$

We denote this ring homomorphism again by φ. It is clear that φ is ϕ-compatible. Note that the ring $\mathbb{K}(\Gamma')[t^{\pm 1}]$ and hence the above representation depends on the choice of μ. We will nonetheless suppress μ in the notation since different choices of splittings give isomorphic rings. We will refer to a non-commutative Alexander polynomial corresponding to such a group homomorphism as a *higher order Alexander polynomial*.

An important example of admissible pairs is provided by Harvey's rational derived series of a group γ (cf. [Ha05, Sect. 3]). Let $\gamma_r^{(0)} = \gamma$ and define inductively

$$\gamma_r^{(n)} = \left\{ g \in \gamma_r^{(n-1)} \,|\, g^k \in \left[\gamma_r^{(n-1)}, \gamma_r^{(n-1)}\right] \text{ for some } k \in \mathbb{Z} \setminus \{0\} \right\}.$$

Note that

$$\gamma_r^{(n-1)}/\gamma_r^{(n)} \cong \left(\gamma_r^{(n-1)}/\left[\gamma_r^{(n-1)}, \gamma_r^{(n-1)}\right]\right)/\mathbb{Z}\text{-torsion}.$$

By [Ha05, Corollary 3.6] the quotients $\Gamma/\gamma_r^{(n)}$ are solvable and torsion-free for any γ and any n. If $\phi : \Gamma \to \mathbb{Z}$ is an epimorphism, then $(\Gamma \to \Gamma/\gamma_r^{(n)}, \phi)$ is an admissible pair for (Γ, ϕ) for any $n > 0$.

For example if K is a knot, $\gamma = \pi_1(S^3 \setminus \nu K)$, then it follows from [St74] that $\Gamma_r^{(n)} = \Gamma^{(n)}$, i.e. the rational derived series equals the ordinary derived series (cf. also [Co04] and [Ha05]).

Remark 13 The Achilles heel of the higher order Alexander polynomials is that they are unfortunately difficult to compute in practice. We refer to [Sa07] for some ideas on how to compute higher order Alexander polynomials in some cases.

3.10.3 Comparing Different ϕ-Compatible Maps

We now recall a definition from [Ha06].

Definition 3 Let N be a 3-manifold with empty or toroidal boundary. We write $\pi = \pi_1(N)$. Let $\phi : \pi \to \mathbb{Z}$ an epimorphism. Furthermore let $\varphi_1 : \pi \to \gamma_1$ and $\varphi_2 : \pi \to \gamma_2$ be epimorphisms to torsion-free solvable groups γ_1 and γ_2. We call $(\varphi_1, \varphi_2, \phi)$ an *admissible triple* for π if there exist epimorphisms $\varphi_2^1 : \gamma_1 \to \gamma_2$ and $\phi_2 : \gamma_2 \to \mathbb{Z}$ such that $\varphi_2 = \varphi_2^1 \circ \varphi_1$, and $\phi = \phi_2 \circ \varphi_2$.

The situation can be summarized in the following diagram

$$\begin{tikzcd} & \gamma_1 \arrow[d, "\varphi_2^1"] \\ \pi \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"] \arrow[dr, "\phi"'] & \gamma_2 \arrow[d, "\phi_2"] \\ & \mathbb{Z}. \end{tikzcd}$$

Note that in particular $(\varphi_i, \phi), i = 1, 2$ are admissible pairs for π. The following theorem is perhaps the most striking feature of higher order Alexander polynomials. In light of Theorem 23 the statement can be summarized as saying that higher order

Alexander polynomials corresponding to larger groups give better bounds on the Thurston norm.

Theorem 25 *Let N be a 3-manifold whose boundary is a (possibly empty) collection of tori. Let $(\varphi_1, \varphi_2, \phi)$ be an admissible triple for $\pi_1(N)$. Suppose that $\Delta^{\varphi_2}_{N,\phi} \neq 0$, then it follows that $\Delta^{\varphi_1}_{N,\phi} \neq 0$. We write*

$$d_i := \deg(\Delta^{\varphi_i}_{N,\phi}) - \deg(\Delta^{\varphi_i}_{N,\phi,0}) - \deg(\Delta^{\varphi_i}_{N,\phi,2}), \quad i = 1, 2$$

Then the following holds:

$$d_1 \geq d_2.$$

Furthermore, if the ordinary Alexander polynomial $\Delta^{\phi}_N \in \mathbb{Z}[t^{\pm 1}]$ is non-trivial, then $d_1 - d_2$ is an even integer.

Proof The fact that $\Delta^{\varphi_2}_{N,\phi} \neq 0$ implies that $\Delta^{\varphi_1}_{N,\phi} \neq 0$ and the inequality $d_2 \geq d_1$ were first proved for knots by Cochran [Co04]. Cochran's result were then extended to the case of 3-manifolds by Harvey [Ha06] (cf. also [Fr07]). Finally the fact that $d_2 - d_1$ is an even integer when $\Delta^{\phi}_N \neq 0$ is proved in [FK08a]. □

The strong relationship between the Thurston norm and higher order Alexander polynomials is also confirmed by the following result (cf. [FH07]).

Theorem 26 *Let N be a 3-manifold with empty or toroidal boundary, let $\varphi : \pi_1(N) \to \gamma$ be an epimorphism to a torsion-free solvable group such that the abelianization $\pi_1(N) \to F := H_1(N;\mathbb{Z})/\text{torsion}$ factors through φ. Then the map*

$$\begin{aligned} H^1(N;\mathbb{Z}) = Hom(F,\mathbb{Z}) &\to \mathbb{Z}_{\geq 0} \\ \phi &\mapsto \max\{0, \deg(\Delta^{\varphi}_{N,\phi}) - \deg(\Delta^{\varphi}_{N,\phi,0}) - \deg(\Delta^{\varphi}_{N,\phi,2})\} \end{aligned}$$

defines a seminorm on $H^1(N;\mathbb{Z})$ which gives a lower bound on the Thurston norm.

3.10.4 Miscellaneous Applications of Higher Order Alexander Polynomials

In this section we quickly summarize various applications of higher order Alexander polynomials and related invariants to various aspects of low-dimensional topology:

1. Leidy [Lei06] studied the relationship between higher order Alexander modules and non-commutative Blanchfield pairings.
2. Leidy and Maxim ([LM06] and [LM08]) studied higher order Alexander polynomials of plane curve complements.
3. Cochran and Taehee Kim [CT08] showed that given a knot with genus greater than one, the higher order Alexander polynomials do not determine the concordance class of a knot.

4. Sakasai [Sa06, Sa08] and Goda–Sakasai [GS08] studied applications of higher order Alexander invariants to homology cylinders and sutured manifolds. For example higher order Alexander invariants can be used to give obstructions to homology cylinders being products.
5. In [FK08b] it is shown that the degrees of higher order Alexander polynomials of closed 3-manifolds are even.

3.11 Open Questions and Problems

We conclude this survey paper with a list of open questions and problems.

1. Using elementary ideals one can define the twisted k-th Alexander polynomial, generalizing the k-th Alexander polynomial of a knot $K \subset S^3$. What information do these invariants contain?
2. Let $K \subset S^3$ a knot and let $\gamma : \pi_1(S^3 \setminus \nu K) \to SL(k, R)$ be a representation, where R is a Noetherian UFD with possibly trivial involution. Does it follow that Δ_K^γ is reciprocal, i.e. does it hold that $\Delta_K^\gamma \doteq \overline{\Delta_K^\gamma}$? Note that this holds for unitary representations (cf. Sect. 3.3.4, [Ki96, KL99a]) and for all calculations known to the authors.

   Added in proof: This question was answered in the negative by Hillman, Silver and Williams [HSW09], cf. also the remark after Proposition 4.
3. Can any two knots or links be distinguished using twisted Alexander polynomials?
4. If (N, ϕ) is non-fibered, does there exist a representation $\gamma : \pi_1(N) \to GL(k, R)$ such that $\Delta_N^{\gamma\otimes\phi}$ is not monic?
5. If (N, ϕ) is non-fibered, does there exist a representation $\gamma : \pi_1(N) \to SL(2, \mathbb{C})$ such that $\tau(N, \gamma \otimes \phi)$ is not monic? (cf. e.g. [GM03, Problem 1.1]).
6. If (N, ϕ) is non-fibered, does there exist a representation $\gamma : \pi_1(N) \to GL(k, R)$ such that $\Delta_N^{\gamma\otimes\phi}$ is zero?
7. Let $K \subset S^3$ be any knot, does the twisted Reidemeister torsion of [GKM05] corresponding to a generic faithful representation detect fiberedness? (cf. [Mo08, p. 452] for some calculations).
8. Let N be a 3-manifold with empty or toroidal boundary, $N \neq S^1 \times D^2, S^1 \times S^2$, let $\phi \in H^1(N; \mathbb{Z})$ and let $\gamma : \pi_1(N) \to GL(k, R)$ be a representation such that $\Delta_N^{\gamma\otimes\phi} \neq 0$. Does it follow that
$$\deg(\tau(N, \gamma \otimes \phi)) = \deg(\Delta_{N,1}^{\gamma\otimes\phi}) - \deg(\Delta_{N,0}^{\gamma\otimes\phi}) - \deg(\Delta_{N,2}^{\gamma\otimes\phi})$$
   has the parity of $k\|\phi\|_T$? Note that this holds for fibered (N, ϕ) and for the untwisted Alexander polynomials of a knot.

   Added in proof: this also holds for hyperbolic knots and the canonical $SL(2, \mathbb{C})$ representation.
9. Does the twisted Alexander polynomial detect the Thurston norm of a given $\phi \in H^1(N; \mathbb{Z})$?

10. Let K be a hyperbolic knot and $\rho : \pi_1(S^3 \setminus \nu K) \to SL(2, \mathbb{C})$ the unique discrete faithful representation.
    a. Is Δ_K^ρ non-trivial?
    b. Does $\deg(\Delta_K^\rho)$ determine the genus of K?
    c. Is K fibered if $\tau(K, \rho)$ is monic?
    Note that the unique discrete representation is over a number field which for many knots can be obtained explicitly with Snappea. These questions can therefore be answered for small crossing knots.
    Added in proof: the answer to all three questions is yes, if K has at most 13 crossings [DFJ10].
11. Does there exist a knot $K \subset S^3$ and a nonabelian representation γ such that Δ_K^γ is trivial?
12. Are there knots for which Kitayama's lower bounds on the free genus of a knot (cf. [Kiy08a]) are larger than the bound on the ordinary genus obtained in [FK06]?
13. Find a practical algorithm for computing higher order Alexander polynomials.
14. Do higher order Alexander polynomials detect mutation?
15. Does there exist a twisted version of Turaev's torsion function?
16. Use twisted Alexander polynomials to determine which knots with up to twelve crossings are doubly slice.
17. Can the results of [HK79] and [Hat80] regarding Alexander polynomials of amphichiral knots be generalized to twisted Alexander polynomials?

Acknowledgements The first author would like to thank Markus Banagl and Denis Vogel for organizing the workshop 'The Mathematics of Knots: Theory and Application' in Heidelberg in December 2008. The authors also would like to thank Takahiro Kitayama, Taehee Kim, Chuck Livingston, Takayuki Morifuji, Andrew Ranicki and Dan Silver for many helpful comments and remarks. Finally we wish to thank the referee for many helpful comments and remarks.

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Chapter 4
On Two Categorifications of the Arrow Polynomial for Virtual Knots

Heather Ann Dye, Louis Hirsch Kauffman, and Vassily Olegovich Manturov

Abstract Two categorifications are given for the arrow polynomial, an extension of the Kauffman bracket polynomial for virtual knots. The arrow polynomial extends the bracket polynomial to infinitely many variables, each variable corresponding to an integer *arrow number* calculated from each loop in an oriented state summation for the bracket. The categorifications are based on new gradings associated with these arrow numbers, and give homology theories associated with oriented virtual knots and links via extra structure on the Khovanov chain complex. Applications are given to the estimation of virtual crossing number and surface genus of virtual knots and links.

4.1 Introduction

The purpose of this paper is to give a categorification for an extension of the Kauffman bracket polynomial, giving a new categorified homology for virtual knots and links. The extension of the bracket that we work with is the *arrow polynomial* as defined in [Kau09, DK09]. This invariant was independently constructed by Miyazawa

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*, Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_4,

in [Miy08, Miy06] and so this work can also be seen as a categorification of the Miyazawa polynomial.

In [Kau09], Kauffman gives an extension of the bracket polynomial for virtual knots that is obtained by using an oriented state expansion, as indicated here in Fig. 4.1. In such an expansion there are two types of smoothing as shown in this figure. The guiding principle for the extended bracket invariant is to retain the pairing of the cusps at the reverse oriented smoothings for as long as possible. The resulting state configurations are then replaced by 4-regular virtual graphs, and the invariant is a linear combination of these graphs with polynomial coefficients. For a given state S, the corresponding graph is denoted by $[S]$. In [Kau09] this invariant is then simplified by retaining the cusps at the non-oriented smoothings but not insisting upon pairing them. In this simplified version $[S]$ is replaced by a diagram that is a union of circle graphs with (reduced) cusps and virtual crossings, modulo virtual equivalence. States are reduced via the rule that consecutive pairs of cusps on a given state curve cancel if they point to the same local side of the curve in the plane. With this caveat, each state curve can be regarded as an extra variable K_n with an index n denoting one half of the reduced number of cusps. This simplified version of the invariant is called the *arrow polynomial*. It takes the form

$$\langle K\rangle_A = \sum_S A^{\alpha(S)-\beta(S)}(-A^2 - A^{-2})^{\gamma(S)-1} \prod_{c\in c(S)} K_{n(c)}, \tag{4.1}$$

where the product is taken over all single loop components c in the state S, and $n(c)$ counts one half the number of cusps in the reduced circle graph. Here $\alpha(S)$ and $\beta(S)$ are the numbers of positively (resp., negatively) smoothed crossings, and $\gamma(S)$ is the number of loops in the state S.

H.A. Dye and L.H. Kauffman studied an equivalent version of the arrow polynomial [DK09] and used it to obtain a lower bound on the virtual crossing number for diagrams of a virtual link. In the Dye-Kauffman version the cusps are replaced by an extra orientation convention. See Fig. 4.3. Here we shall refer to $\langle K\rangle_A$ as the arrow polynomial of K. We call the reduced cusp count $n(c)$ for a state loop the *arrow number* of this loop. Thus a state loop with label K_n has arrow number n.

Both the extended bracket polynomial and the arrow polynomial $\langle K\rangle_A$ are invariant with respect to the second and the third Reidemeister moves. They can be made invariant under the first Reidemeister move by the usual normalisation by a power of $(-A^3)$. In the rest of the paper, we shall omit this normalisation. Moreover, while passing to the Khovanov homology, we shall omit the corresponding renormalisation and refer the interested reader to [BN02, Man05b] or [BN02, BN05].

The aim of the present paper is to present two categorifications of the arrow polynomial [Kau09, DK09]. We split the chain spaces of the Khovanov complex $\mathcal{C}(K)$ into subspaces $\mathcal{C}_{gr=x}(K)$ with a fixed new grading x and restrict our differential ∂ to these subspaces. Now, set $\partial = \partial' + \partial''$ where ∂' is the part of ∂ which preserves the new gradings for basic chains, and ∂'' is the remaining part of ∂. We have to define this new grading in such a way that the new differential ∂' is well defined and the corresponding homology groups $H(\mathcal{C}(K), \partial')$ are invariant with respect to Reidemeister moves.

A Khovanov homology theory for virtual knots has been constructed in a sequence of papers by Manturov. In [Man08], one gives a certain procedure for further generalization of these invariants, which deals with so-called *dotted gradings*. In working with Khovanov homology we use *enhanced states* of the Kauffman bracket polynomial. These enhanced states are collections of labelled simple closed curves obtained by smoothing crossings in the diagram. Each curve is labelled with either the algebra element X or with the number 1. The elements X and 1 belong to the algebra $k[X]/(X^2)$ where $k = Z[A, A^{-1}]$. In the dotted grading, the X and the 1 can acquire a dot in the form $\dot{X}$ and $\dot{1}$. We explain how this notation works in the discussion below.

We assume all circles in Kauffman states of a diagram can be assigned a mod $\mathbf{Z}_2$ *dotting*: every state circle is either dotted or not (the dotting should be read from the topology/combinatorics of the diagram), and the new integral grading of a chain is set to be $\#\dot{X} - \#\dot{1}$, i.e. the number of dotted circles with the element X minus the number of dotted circles carrying the element 1. If this dotting satisfies certain very simple axioms [Man08], then the complex is well defined and its homology is invariant under Reidemeister moves.

Another way to introduce the gradings for a given Khovanov homology theory is to take the coefficients like $[S]$ or $\prod_{c(S)} K_{n(s)}$ to be new (multi)gradings themselves, but for this we use $\mathbf{Z}_2$-coefficients. Possibly, this $\mathbf{Z}_2$-reduction can be avoided if we use twisted coefficients similar to those from [Man07b], but this has not been done so far.

We note that this paper makes use of enhanced states of the bracket polynomial for discussing Khovanov homology. This approach was introduced in [Viro1, Viro2]. The first categorification of link invariants in thickened surfaces, thus also of the Kauffman bracket of virtual links occurs in [APS]. Finally, two recent papers by[Caprau, CMW] can also be viewed as categorifying the arrow polynomial. although that was not the principle aim of these works. A sequel to this paper will discuss these relationships.

4.2 The Arrow Polynomial $\langle K \rangle_A$

In this section we describe the arrow polynomial invariant [Kau09, DK09]. One way to see the definition of the arrow polynomial is to begin with the extended bracket invariant [Kau09] and simplify it. The extended invariant is a sum of graphs (taken up to virtual equivalence in the plane) weighted by polynomials. In the extended bracket one uses an oriented expansion so that the smoothings consist of oriented smoothings and disoriented smoothings. At a disoriented smoothing one sees two cusps with orientation arrows going into the cusp point in one cusp and out of the cusp point for the other cusp. Rules for reducing the states of the extended bracket keep the cusps paired whenever possible. If we release the cusp pairings at the disoriented smoothings, we get simpler graphs. These are composed of disjoint collections of circle graphs that are labelled with the orientation markers and left-right distinctions that occur in the state expansion. The basic conventions for this

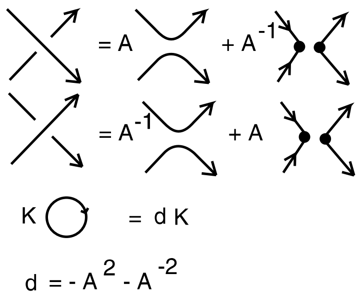


Fig. 4.1 Oriented bracket expansion

simplification are shown in Fig. 4.2. In that figure we illustrate how the disoriented smoothing is a local disjoint union of two vertices (the cusps). Each cusp is denoted by an angle with arrows either both entering the cusp or both leaving the cusp. Furthermore, the angle locally divides the plane into two parts: One part is the span of an acute angle (of size less than π); the other part is the span of an obtuse angle. We refer to the span of the acute angle as the *inside* of the cusp. In Fig. 4.2, we have labelled the insides of the cusps with the symbol $\sharp$.

Figure 4.1 illustrates the basic oriented bracket expansion formula. Figure 4.2 illustrates the reduction rule for the *arrow polynomial.* While we have indicated (above) the relationship of the arrow polynomial with the extended bracket polynomial, the reduction rule for the arrow polynomial is completely described by Fig. 4.2. We shall denote the arrow polynomial by the notation $\langle K\rangle_A$, for a virtual knot or link diagram K. The reduction rule allows the cancellation of two adjacent cusps when they have *insides on the same side* of the segment that connects them. When the insides of the cusps are on opposite sides of the connecting segment, then no cancellation is allowed. All graphs are taken up to virtual equivalence. Figure 4.2 illustrates the simplification of two circle graphs. In one case the graph reduces to a circle with no vertices. In the other case there is no further cancellation, but the graph is equivalent to one without a virtual crossing. The state expansion for $\langle K\rangle_A$ is exactly as shown in Fig. 4.1, but we use the reduction rule of Fig. 4.2 so that each state is a disjoint union of reduced circle graphs. Since such graphs are planar, each is equivalent to an embedded graph (no virtual crossings) and the reduced forms of such graphs have $2n$ cusps that alternate in type around the circle so that n are pointing inward and n are pointing outward. The circle with no cusps is evaluated as $d = -A^2 - A^{-2}$ as is usual for these expansions and the circle is removed from the graphical expansion. Let K_n denote the circle graph with $2n$ alternating vertex types as shown in Fig. 4.2 for $n = 1$ and $n = 2$. By our conventions for the extended bracket polynomial, each circle graph contributes $d = -A^2 - A^{-2}$ to the state sum and the graphs K_n (with $n \geq 1$) remain in the graphical expansion. For the arrow polynomial $\langle K\rangle_A$ we can regard each K_n as an extra variable in the polynomial. Thus a product of the K_n's denotes a state that is a disjoint union of copies of these circle graphs with multiplicities. By evaluating each circle graph as $d = -A^2 - A^{-2}$ we guarantee that the resulting polynomial will reduce to the original bracket polynomial when each of the new variables K_n is set equal to unity. Note that we continue to use the caveat that an isolated circle or circle graph (i.e. a state consisting

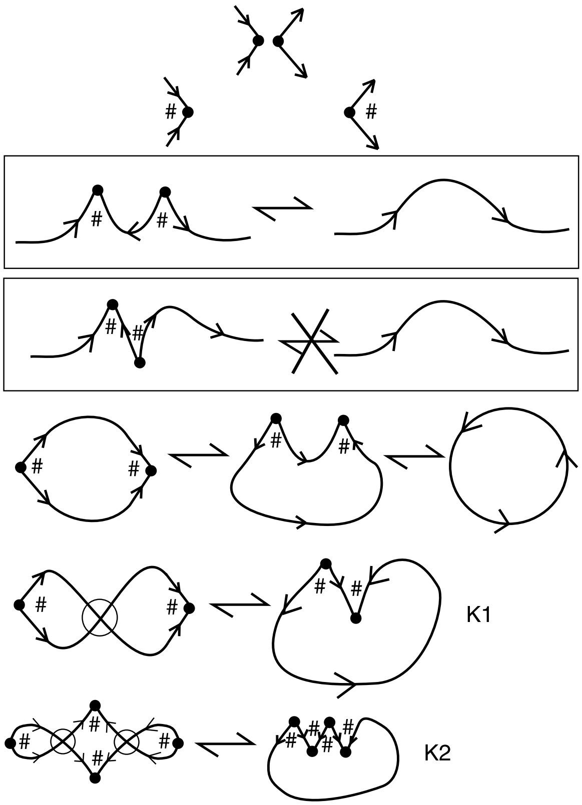


Fig. 4.2 Reduction relation for the arrow polynomial

in a single circle or single circle graph) is assigned a loop value of unity in the state sum. This assures that $\langle K \rangle_A$ is normalized so that the unknot receives the value one.

Formally, we have the following state summation for the arrow polynomial

$$\langle K \rangle_A = \sum_S \langle K|S \rangle d^{\|S\|-1} P[S]$$

where S runs over the oriented bracket states of the diagram, $\langle K|S \rangle$ is the usual product of vertex weights as in the standard bracket polynomial, $\|S\|$ is the number of circle graphs in the state S, and $P[S]$ is a product of the variables K_n associated with the non-trivial circle graphs in the state S. Note that each circle graph (trivial or not) contributes to the power of d in the state summation, but only non-trivial circle graphs contribute to $P[S]$. The regular isotopy invariance of $\langle K \rangle_A$ follows from an analysis of the behaviour of this state summation under the Reidemeister moves.

Theorem 1 *With the above conventions, the arrow polynomial $\langle K \rangle_A$ is a polynomial in A, A^{-1} and the graphical variables K_n (of which finitely many will appear for any given virtual knot or link). $\langle K \rangle_A$ is a regular isotopy invariant of virtual*

Fig. 4.3 Arrow convention

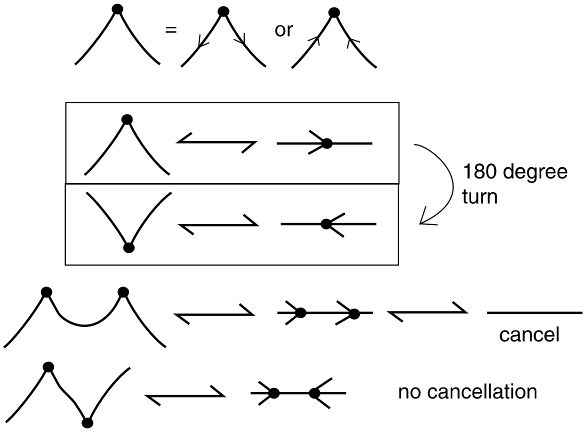
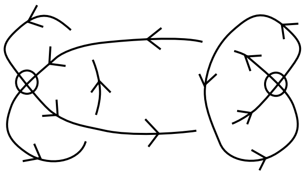


Fig. 4.4 Kishino diagram



knots and links. The normalized version

$$W[K] = (-A^3)^{-wr(K)} \langle K \rangle_A$$

is an invariant of virtual isotopy. Here $wr(K)$ denotes the writhe of the diagram K; this is the sum of the signs of all the classical crossings in the diagram. If we set $A = 1$ and $d = -A^2 - A^{-2} = -2$, then the resulting specialization

$$F[K] = \langle K \rangle_A (A = 1)$$

is an invariant of flat virtual knots and links.

Example 1 Figure 4.4 illustrates the Kishino diagram. With $d = -A^2 - A^{-2}$

$$\langle K \rangle_A = 1 + A^4 + A^{-4} - d^2 K_1^2 + 2K_2.$$

Thus the simple extended bracket shows that the Kishino is non-trivial and non-classical. In fact, note that

$$F[K] = 3 + 2K_2 - 4K_1^2.$$

Thus the invariant $F[K]$ of flat virtual diagrams proves that the flat Kishino diagram is non-trivial. This example shows the power of the arrow polynomial. See [Kau09, DK09] for the details of this calculation.

4.3 Khovanov Homology for Virtual Knots

In this section, we describe Khovanov homology for virtual knots along the lines of [Kho97, BN02, Man07b].

The bracket polynomial [Kau87] is usually described by the expansion

$$\langle \times \rangle = A \langle \,)(\, \rangle + A^{-1} \langle \asymp \rangle \tag{4.2}$$

Letting $c(K)$ denote the number of crossings in the diagram K, if we replace $\langle K \rangle$ by $A^{-c(K)}\langle K \rangle$, and then replace A^2 by $-q^{-1}$, the bracket will be rewritten in the following form:

$$\langle \times \rangle = \langle \,)(\, \rangle - q \langle \asymp \rangle \tag{4.3}$$

with $\langle \bigcirc \rangle = (q + q^{-1})$. In this form of the bracket state sum, the grading of the Khovanov homology (which is described below) appears naturally. We shall continue to refer to the smoothings labelled q (or A^{-1} in the original bracket formulation) as *B-smoothings*. We should further note that we use the well-known convention of *enhanced states* where an enhanced state has a label of 1 or X on each of its component loops. We then regard the value of the loop $(q + q^{-1})$ as the sum of the value of two circles: a circle labelled with a 1 (the value is q) and a circle labelled with an X (the value is q^{-1}).

To see how the Khovanov grading arises, consider the form of the expansion of this version of the bracket polynomial in enhanced states. We have the formula as a sum over enhanced states s:

$$\langle K \rangle = \sum_s (-1)^{n_B(s)} q^{j(s)}$$

where $n_B(s)$ is the number of B-type smoothings in s, $\lambda(s)$ is the number of loops in s labelled 1 minus the number of loops labelled X, and $j(s) = n_B(s) + \lambda(s)$. This can be rewritten in the following form:

$$\langle K \rangle = \sum_{i,j} (-1)^i q^j \left[\sum_{s:n_B(s)=i, j(s)=j} 1 \right] = \sum_{i,j} (-1)^i q^j dim(\mathcal{C}^{ij}).$$

In the Khovanov homology, the states with $n_B(s) = i$ and $j(s) = j$ form the basis for a module $\mathcal{C}^{ij}$ over the ground ring k. Thus we can write

$$dim(\mathcal{C}^{ij}) = \sum_{s:n_B(s)=i, j(s)=j} 1.$$

The bigraded complex composed of the $\mathcal{C}^{ij}$ has a differential $d : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1j}$. That is, the differential increases the *homological grading* i by 1 and preserves the *quantum grading* j. Below, we will remind the reader of the formula for the differential in the Khovanov complex. Note however that the existence of a bigraded complex of this type allows us to further write:

$$\langle K\rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j}),$$

where $\chi(\mathcal{C}^{\bullet j})$ is the Euler characteristic of the subcomplex $\mathcal{C}^{\bullet j}$ for a fixed value of j. Since j is preserved by the differential, these subcomplexes have their own Euler characteristics and homology. We can write

$$\langle K\rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j})),$$

where $H(\mathcal{C}^{\bullet j})$ denotes the homology of this complex. Thus our last formula expresses the bracket polynomial as a *graded Euler characteristic* of a homology theory associated with the enhanced states of the bracket state summation. This is the categorification of the bracket polynomial. Khovanov proves that this homology theory is an invariant of knots and links, creating a new and stronger invariant than the original Jones polynomial.

We explain the differential in this complex for mod-2 coefficients and leave it to the reader to see the references for the rest. The differential is defined via the algebra $\mathcal{A} = k[X]/(x^2)$ so that $X^2 = 0$ with coproduct $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ defined by $\Delta(X) = X \otimes X$ and $\Delta(1) = 1 \otimes X + X \otimes 1$. Partial differentials (which are defined on an enhanced state with a chosen site, whereas the differential is a sum of these mappings) are defined on each enhanced state s and a site κ of type A in that state. We consider states obtained from the given state by smoothing the given site κ. The result of smoothing κ is to produce a new state s' with one more site of type B than s. Forming s' from s we either amalgamate two loops to a single loop at κ, or we divide a loop at κ into two distinct loops. In the case of amalgamation, the new state s acquires the label on the amalgamated circle that is the product of the labels on the two circles that are its ancestors in s. That is, $m(1 \otimes X) = X$ and $m(X \otimes X) = 0$. Thus this case of the partial differential is described by the multiplication in the algebra. If one circle becomes two circles, then we apply the coproduct. Thus if the circle is labelled X, then the resultant two circles are each labelled X corresponding to $\Delta(X) = X \otimes X$. If the original circle is labelled 1 then we take the partial boundary to be a sum of two enhanced states with labels 1 and X in one case, and labels X and 1 in the other case on the respective circles. This corresponds to $\Delta(1) = 1 \otimes X + X \otimes 1$. Modulo two, the differential of an enhanced state is the sum, over all sites of type A in the state, of the partial differential at these sites. It is not hard to verify directly that the square of the differential mapping is zero and that it behaves as advertised, keeping $j(s)$ constant. There is more to say about the nature of this construction with respect to Frobenius algebras and tangle cobordisms. See [Kho97, BN02, BN05].

Here we consider bigraded complexes $\mathcal{C}^{ij}$ with *height* (homological grading) i and *quantum grading* j. In the unnormalized Khovanov complex $[[K]]$ the index i is the number of B-smoothings of the bracket, and for every enhanced state, the index

j is equal to the number of components labelled 1 minus the number of components labelled X plus the number of B-smoothings. The normalized complex differs from $[[K]]$ by an overall shift of both gradings; the differential preserves the quantum grading and increases the height by 1. The height and grading shift operations are defined as $(\mathcal{C}[k]\{l\})^{ij} = \mathcal{C}[i-k]\{j-l\}$.

This form is used as the starting point for the Khovanov homology. We now describe the formalism in a bit more detail in order to give the structure of the differential for Khovanov homology of virtual knots and links. For a diagram K of a virtual knot, we consider the *state cube* defined as follows: Enumerate all n classical crossings of K in arbitrary way and consider all Kauffman states (states as collections of loops without specific enhancement labels) as vertices of the discrete cube $\{0,1\}^n$. Each coordinate corresponds to a way of smoothing and is equal to 0 (the A-smoothing) or 1 (the B-smoothing). Thus, each vertex of the cube defines a set of circles (say, p circles), and this set of circles defines a certain vector space (module) of dimension 2^p. The module for a single circle is generated by 1 and X. The spaces together form the total chain space of the unnormalized Khovanov complex $[[K]]$ and its normalized version $\mathcal{C}[[K]]$. We omit the normalisation, which is standard, and refer the reader to [Kho97, BN02, Man07b].

We regard the loop factors for the unenhanced bracket, $(q+q^{-1})$, as graded dimensions of the module $V = Span(\{1,X\})$, $\deg 1 = 1$, $\deg X = -1$ over some ring k, and the height $i(s)$ plays the role of homological dimension. Define the chain space $[[K]]_i$ of homological dimension i to be the direct sum over all vertices of height i (defined as above) of $V^{\gamma(s)}\{i\}$ (here $\{\cdot\}$ is the quantum grading shift and $\gamma(s)$ is the number of loops in the state s). Then the alternating sum of graded dimensions of $[[K]]_i$, is precisely equal to the (modified) Kauffman bracket, as we have described above.

Thus, if one defines a differential on $[[K]]$ that preserves the grading and increases the homological dimension by 1, the Euler characteristic of that complex will be precisely the bracket.

We now consider a generalization of the Khovanov homology to virtual knots. When we pass from one state of the state cube to a neighboring state (which differs precisely at one coordinate), we get a resmoothing of the set of circles. We refer to that as a *bifurcation* of the state cube. Such a bifurcation can either merge two circles into one ($2 \to 1$-bifurcation) or split one circle into two ($1 \to 2$-bifurcation), or (in the case of virtual knots and links) transform one circle into one ($1 \to 1$-bifurcation). These bifurcations encode the information about differentials in the complex as follows.

We have defined the *state cube* consisting of state loops and carrying no information how these loops interact. For Khovanov homology, we deal with the same cube, remembering the information about the loop bifurcation. Later on, we refer to it as a *bifurcation cube*.

The chain spaces of the complex are well defined. However, the problem of finding a differential ∂ in the general case of virtual knots, is not easy. See Fig. 4.7 for a key example that we shall discuss. To define the differential, we have to pay attention to the different isomorphism classes of the chain space identified by using local bases (see below).

The differential acts on the chain space as follows: it takes a chain (regard an enhanced state as an elementary chain) corresponding to a certain vertex of the bifurcation cube to some chains corresponding to all adjacent vertices with greater homological degree. That is, the differential is a sum of *partial differentials*, each partial differential acts along an edge of the cube. Every partial differential corresponds to some direction and is associated with some classical crossing of the diagram. The total differential is the sum of these partial differentials, and so formally looks like

$$\partial = \sum_a \partial_a$$

where the summation is over all edges of the cube. In discussing differentials we shall often refer to a partial differential without indicating its subscript.

Selecting an un-enhanced Kauffman state S (consisting of loops with cusps), we choose an arbitrary order for the circles in S. and then orient each circle in S. Letting $\gamma(S) = \|S\|$ be the number of loops in S, associate the module $\Lambda^{\|S\|}(V)$ to S where this denotes the $\|S\|^{\text{th}}$ exterior power of V—the order of the factors in the exterior power depends on the choice of the ordering that was chosen. Having made this choice (of ordering and orientation), if s is an enhancement of S then label all loops in the state s with either $+X$ or $+1$ according to the enhancement. This oriented, ordered, and labelled state forms a generating chain in the complex. If the orientation of a loop in S is reversed then the label for X becomes $-X$ but the label for 1 does not change. Otherwise, signs change according to the structure of the exterior algebra.

Then for a state with l circles, we get a vector space (module) of dimension 2^l. All these chains have homological dimension $i = n_B$. We set the quantum grading j of these chains equal to i plus the number of circles marked by ± 1 minus the number of circles marked by $\pm X$.

Let us now define the partial differentials of our complex. First, we think of each classical crossing so that its edges are oriented upwards, as in Fig. 4.5, upper left picture.

Choose a certain state of a virtual link diagram $L \subset \mathcal{M}$. Choose a classical crossing U of L. We say that in a state s that a state circle γ is incident to a classical crossing U if at least one of the two local parts of smoothed crossing U belongs to γ. Consider all circles γ incident to U. Fix some orientation of these circles according to the orientation of the edge emanating in the upward-right direction and opposite to the orientation of the edge coming from the bottom left, see Fig. 4.5. Such an orientation is well defined except for the case when resmoothing one edge takes one circle to one circle. In such a situation, we shall not define the local basis $\{1, X\}$, and we set the partial differential corresponding to that edge to be zero.

In the other situations, the edge of the cube corresponding to the partial differential either increases or decreases the number of circles. This means that at the corresponding crossing the local bifurcation either takes two circles into one or takes one circle into two. If we deal with two circles incident to a crossing from opposite signs, we order them in such a way that the upper (resp., left) one is the first one; the lower (resp., right) one is the second; here the notions “left, right, upper, lower”

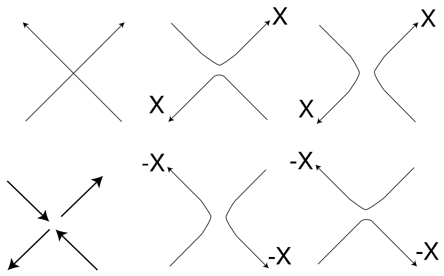


Fig. 4.5 Setting the local basis for a crossing

are chosen according to the rule for identifying the crossing neighbourhood with Fig. 4.5. Furthermore, for defining the partial differentials of types m and Δ (which correspond to decreasing/increasing the number of circles by one) we assume that the circles we deal with are in the initial positions specified in our ordered tensor product; this can always be achieved by a preliminary permutation, which, possibly leads to a sign change. Now, let us define the partial differential locally according to the prescribed choice of generators at crossings and the prescribed ordering.

Now, we describe the partial differentials ∂ from [Man07b] without new gradings. If we set $\Delta(1) = 1_1 \wedge X_2 + X_1 \wedge 1_2$; $\Delta(X) = X_1 \wedge X_2$ and $m(1_1 \wedge 1_2) = 1$; $m(X_1 \wedge 1_2) = m(1_1 \wedge X_2) = X$; $m(X_1 \wedge X_2) = 0$, define the partial differential ∂ according to the rule $\partial(\alpha \wedge \beta) = m(\alpha) \wedge \beta$ (in the case we deal with a $2 \to 1$-bifurcation, where α denotes the first two circles α) or $\partial(\alpha \wedge \beta) = \Delta(\alpha) \wedge \beta$ (when one circle marked by α bifurcates to two ones); here by β we mean an ordered set of oriented circles, not incident to the given crossings; the marks on these circles ± 1 and $\pm X$ are given.

Theorem 2 [*Man*07*b*] *Let K be a virtual knot or link. Then $[[K]]$ is a well-defined complex with respect to ∂. After a small grading shift and a height shift, the homology of $[[K]]$ is invariant under the generalised Reidemeister moves for virtual knots and links.*

4.4 Grading Considerations for the Arrow Polynomial $\langle K \rangle_A$

In order to consider gradings for Khovanov homology in relation to the structure of the arrow polynomial $\langle K \rangle_A$ we have to examine how the arrow number of state loops change under a replacement of an A-smoothing by a B-smoothing. Such replacement, when we use oriented diagrams involves the replacement of a cusp pair by an oriented smoothing or vice versa. Furthermore, we may be combining or splitting two loops. Refer to Fig. 4.6 for a depiction of the different cases. This figure shows the three basic cases.

In the first case we have two loops C_1 and C_2 sharing a disoriented site and the smoothing is a single loop C where the paired cusps of the disoriented site disappear. In this case if $n(C_1) = n$ and $n(C_2) = m$, then $n(C) = |n - m|$.

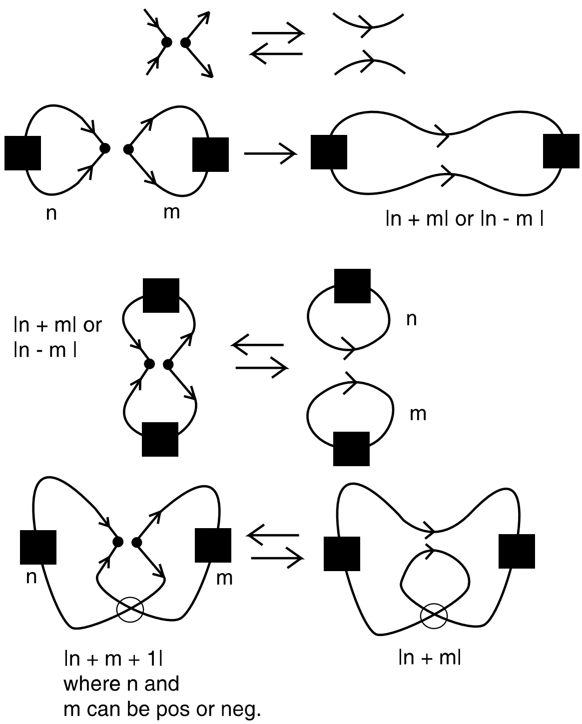


Fig. 4.6 Arrow numbers for interacting loops

In the second case, we have a single loop C with a disoriented site and a pair of cusps, and on smoothing this site we obtain two loops C_1 and C_2 whose arrow numbers are $n(C_1) = n$ and $n(C_2) = m$. The following arrow numbers for C are then possible $|n(C)| = |n - m|$ or $|n + m|$.

In the third case, we have a single loop C with a disoriented site and a pair of cusps, and on smoothing this site we obtain a single loop C'. Assuming that $n(C') = |n + m|$ as shown in the figure, we have $|n(C)| = |n + m + 1|$ where n and m can be positive or negative.

These are all the ways that loops can interact and change their respective arrow numbers. In the next section, we will apply these results to the grading in Khovanov homology.

4.5 Dotted Gradings and the Dotted Categorification

First, we introduce a concept of *dotting axiomatics* as developed in [Man08]. The purpose of this dotting axiomatics is to give general conditions under which extra decorations on the states can be used to create new gradings and hence new versions of Khovanov homology. We will apply these axiomatics to the arrow numbers on the state loops of the arrow polynomial.

For the axiomatics, assume we have some class of objects with Reidemeister moves, Kauffman bracket and the Khovanov homology (in the usual setup or in the setup of [Man05c]). Assume that there is a method, which for every diagram and every state of it associates dots to some of the circles in the bracket states in such a way that the following conditions hold:

1. The dotting of circles is additive with respect to $2 \to 1$-bifurcations and $1 \to 2$-bifurcations mod 2. This additivity means that when we merge two circles (split one circle into two), the number of dots on the circles being operated on is preserved modulo $\mathbf{Z}_2$.

   This means that the parity of the number of dots on the circles operated on is preserved whenever we merge two circles or split one circle into two.

   If the dotting is not preserved under a $1 \to 1$ bifurcation, then this bifurcation is taken to be the zero map.
2. Similar curves for corresponding smoothings of the RHS and the LHS of any Reidemeister move have the same dotting.
3. Small circles appearing for the first, the second, and the third Reidemeister moves are not dotted.

Let us call the conditions above *the dotting conditions*. With such a structure in hand, one defines a *new grading* $g(s)$ for states s by taking the difference between the number of dotted X's and the number of dotted 1's in the state.

$$g(s) = \sharp(\dot{X}) - \sharp(\dot{1})$$

We shall use this grading in the constructions that follow.

Theorem 3 *Assume there is a theory using the Khovanov complex* $([[K]], \partial)$ *such that the Kauffman states can be dotted so that the dotting conditions hold. Take* $[[K]]_g$ *to be the space* $[[K]]$ *endowed with new grading as above.*

Define ∂' *to be the composition of* ∂ *with the new grading projection and set* $\partial'' = \partial - \partial'$.

Then the homology of $[[K]]_g$ (*with respect to* ∂') *is invariant* (*up to a degree shift and a height shift*).

For any operator λ *on the ground ring, the complex* $[[K]]_g$ *is well defined with respect to the differential* $\partial' + \lambda\partial''$, *and the corresponding homology is invariant* (*up to well-known shifts*).

Moreover, if we have several forms of dotting $g_1, g_2, \ldots, g_k$ *occuring together on the same Khovanov complex so that for each of them the dotting condition holds, then the complex* $K_{g_1,\ldots,g_k}$ *with differential* $\partial_{g_1,\ldots,g_k}$ *defined to be the projection of* ∂ *to the subspace preserving all the gradings, is invariant.*

The theorem above allows one to 'raise' some additional information modulo $\mathbf{Z}_2$ to the level of gradings. Our aim is to categorify the arrow polynomial, that is, to add new gradings corresponding to the arrow count: for every state we have a set of circles labelled by a set of non-zero integers, and this set of integers should be represented in the complex as a grading. Theorem 3 shows that it is possible to do that when we consider the information of the arrow count only modulo $\mathbf{Z}_2$: the conditions of additivity and similarity under Reidemeister moves for arrow count were checked in the previous section of this paper.

In order to use the integral information about the arrow count, we have to undertake a generalization of the construction of Theorem 3. We shall do this in the next section. This section of the paper is devoted to describing a first-order categorification of the arrow polynomial.

The main idea behind the proof of Theorem 3 is as follows. Additivity of the grading can be verified and checked on a bifurcation cube. First of all, it follows from a straightforward check that ∂'' always increases the dotted grading (this is proved in [Man07b] but can be taken here as an exercise for the reader). Then, the complex is well defined because $(\partial')^2$ is nothing but a composition of $(\partial)^2$ with a "grading-preserving projection". This is guaranteed because ∂'' strictly increases the new grading. Note the mod-2 preservation of the dotting is what makes this grading increase of ∂'' work. Thus Theorem 3 depends ultimately on that parity preservation of the dotted grading.

The main idea of the invariance under Reidemeister moves is similar to the usual Khovanov idea, see for example [BN02]: we have to check that the multiplication m remains surjective after reducing ∂ to ∂' and Δ remains injective. The latter follows from the fact that "small circles are not dotted".

Now, one can easily check that the conditions of the theorem hold if we set the dotting as follows: the curve is dotted if it is marked as K_j with j odd, and it is not dotted if it is marked as K_i with i even.

Now, one checks that

1. The dotting is $\mathbf{Z}_2$-additive with respect to resmoothing (performing $1 \to 2$ or $2 \to 1$ bifurcation).

   This follows from Fig. 4.6 upper part: we see that when merging two circles with arrow count m and n, we get $\pm m \pm n$ and when splitting a circle with arrow number k, we get two circles with arrow numbers l and $\pm k \pm l$ which results in $\mathbf{Z}_2$-additivity under $2 \to 1$ and $1 \to 2$-bifurcations.

   On the other hand, if partial differentials for all $1 \to 1$ bifurcations are set to be zero, it can be checked that all faces having at least $1 \to 1$-bifurcation are anticommutative because $0 = 0$. The only non-trivial example is shown in Fig. 4.7, and the corresponding calculation is performed in [Man07b].
2. The small circles coming from Reidemeister moves are not dotted. Indeed, for the 1st Reidemeister move we have no cusps at all, and for the second move and for the third move we have two cusps of opposite signs.
3. For any Reidemeister move, the corresponding state diagrams in the LHS and RHS have the same dotting. Locally, there is no grading change for the Reidemeister moves when we use arrow counts. Again, this follows from the *invariance* under Reidemeister moves: two pictures would not get cancelled if they had different coefficients coming from cusps; this means they have the same dotting.

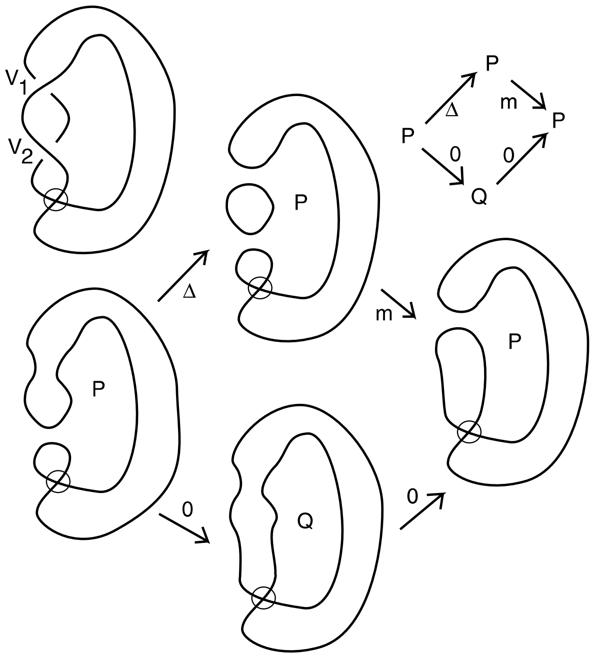


Fig. 4.7 A face for the case of the extended bracket of the cube

4.6 $\mathbb{Z}_2$-Categorification with General Gradings

4.6.1 General Setup

The aim of this section is to prove a general theorem on categorification that fits the arrow polynomial. This is an extension of the dotted grading construction, which works, however, only with $\mathbf{Z}_2$-coefficients for the homology. Later, we shall discuss whether this construction can be extended to the case of integral coefficients. For instance, we can extend this construction to the case of integral coefficients if the odd Khovanov homology theory [ORS07] can be defined for this class of knots.

Briefly, we want to start with a Khovanov homology (usual over $\mathbf{Z}_2$ or the one using twisted coefficients) and make some partial differentials equal to zero.

As the initial data for this theorem, we require that we have a well-defined bracket, and we assume that in each state of the diagram, each circle is given a non-negative integer. For the dotted conditions, we require that

1. The numbers are "plus-minus additive" with respect to $2 \to 1$-bifurcations and $1 \to 2$-bifurcations, that is, if a resmoothing of two circles labelled by non-negative integers p and q leads to one circle, the label of this circle will be $|p+q|$ or $|p-q|$.
2. Similar curves for corresponding smoothings of the RHS and the LHS of any Reidemeister move have the same numbers.
3. Small circles appearing for the first, the second, and the third Reidemeister moves are labelled by zeroes.

We call these conditions *integer labelling conditions*.

After this, our strategy will be as follows: If we attempt to make the integral arrow count the new grading and take the part of the differential preserving this, to be the new differential, we shall see that the square of this new differential will not be zero. Consider the situation when a 2-face of the bifurcation cube has arrow counts P in the left corner (both smoothings zero), P in the upper corner (both smoothings one), P in one right corner and Q in the remaining corner. See Fig. 7 for an example. Then one composition of the two differentials (going through P) survives, while the other one (going through Q) becomes zero. That is why the square of the new proposed differential, detecting the arrow count, is non-zero. On the other hand, all the information about the arrow count has to be included in order to get a faithful categorification of the arrow polynomial (that is, having a chain space with gradings one can restore the arrow count, and having the homology one can restore the arrow polynomial). In order to solve this problem, we are going to introduce two new sorts of gradings, one of which will correct the other, and make the differential well-defined.

We take the usual Khovanov differential ∂ and form two new series of gradings (called *multiple gradings* and *vector gradings*). After that, for each basic chain of the complex we have a whole collection of gradings, and we define the new differential ∂' to be the composition of ∂ with the projection to the subspace where all gradings are preserved by ∂, having the same gradings (all multiple and vector gradings) as in the preimage. That is, we let $S = \{x | gr(x) = gr(\partial x)\}$ and define $\partial' = \partial|_S$.

Now, we introduce *multiple gradings* as follows. A multiple grading is a set of strictly positive integers that is associated with a Kauffman state of the diagram. That is, the state is not yet labelled with X and 1; a *basic chain* in the state is such a labelling. With each state, we shall associate exactly one multiple grading for each basic chain in this state, independently from the particular choice of X and 1 on circles. *This multiple grading is just the set of all non-zero arrow counts on circles of the state.*

The *vector grading* is an infinite ordered collection (list) of integers (first, second, third, etc.) each of which might be either positive or negative or zero. The vector grading depends on the particular choice of 1 and X on all state circles. But before introducing the vector grading, we introduce the *vector dotting* for state circles (that have the initial labelling by arrow numbers). For a circle labelled by p we put no dots at all if $p = 0$; otherwise we represent $p = 2^{k-1}l$, where l is odd and put exactly one dot of order k over this circle (we also call it a *k-th dot*). Thus, for $p = 1$ we will have only one primary dot, for $p = 2$ we will have only one secondary dot, for $p = 4$ we will have only one ternary dot and so on. The vector dotting is an infinite vector of these *dot numbers* with one possibly non-zero coordinate for each state circle. Note that the vector dotting depends only on arrow numbers for the Kauffman state.

Now we can define the vector grading. The vector grading of a trivial circle (without dots) is the zero vector $(0, \dots, 0, \dots)$. For a non-trivial circle having one k-th dot, *the grading is set to be* $+1$ *on k-th vector position for the enhanced state carrying X and* -1 *on k-th vector position for the enhanced state carrying* 1; the other entries of the vector grading for a given enhanced state circle are set to be zero.

The vector grading of a basic chain (*enhanced state*) *is defined to be the coordinatewise sum of the vector gradings* (*these are infinite vectors*)6 *over all circles in the enhanced state.* Thus, if we have one circle labelled by 2 with element X on it and another circle labelled by 1 with element 1 on it, we get the vector grading: $(-1, 1, 0, \dots, 0, \dots)$.

The chain space of the initial Khovanov complex is split into subspaces with respect to the multiple grading and vector grading. We set the differential ∂' to be the composition of the initial differential ∂ with the projection to the subspace having the same gradings as the preimage.

Theorem 4 *If a state labelling satisfies the integer labelling conditions*, *then the complex $\mathcal{C}$ is well defined with respect to differential ∂'* (*that is*, $(\partial')^2 = 0$), *and its homology groups $H(\mathcal{C}, \partial')$ are invariant with respect to the Reidemeister moves*.

First, let us check that the arrow polynomial satisfies the integer labelling conditions. This follows from Fig. 4.6. Now, the second condition "similar curves generate similar smoothing" also follows from a direct calculation, as well as the third condition about trivial circles coming from Reidemeister moves. Indeed, for the first Reidemeister move one gets a small loop without any cusp, for the second Reidemeister move one gets either a loop without cusps or a loop with two cusps cancelling each other. The same for the third Reidemeister move: one gets at least two cusps, which should cancel each other. This proves that the *integer labelling conditions* hold for the arrow count.

Now, let us prove the main theorem. The proof will consist of the two parts: the difficult one, where we show that the complex is well defined (the square of the differential is zero) and the easy one, where we prove that the homology is invariant under Reidemeister moves. The second part will be standard and in main features it will repeat the analogous proof for the usual Khovanov homology.

Part 1. Proof that the Complex is Well Defined We first note that we work over $\mathbf{Z}_2$-coefficients. We have to prove that for every 2-face of the bifurcation cube, the two compositions corresponding to faces will coincide. This means that commutativity and anticommutativity coincide.

An *atom* is a pair (M, Γ) of a 2-manifold M and a graph Γ embedded M together with a colouring of $M \backslash \Gamma$ in a checkerboard manner. Here Γ is called the *frame* of the atom, whence by *genus* (resp., *Euler characteristic*) of the atom we mean that of the surface M.

With a virtual knot diagram (with every component having at least one classical crossing) we associate an atom as follows. (Note that the atom need not be orientable.) We take all classical crossings to be vertices of the frame. The edges of the frame correspond to branches of the diagram connecting classical crossings (we do not take into account how they intersect in virtual crossings). Moreover, the edges of the frame emanating from a vertex are naturally split into two pairs of *opposite* ones: the opposite relation (ordering of edges) is taken from the plane diagram. Thus we get two pairs of opposite edges (opposite in the sense that these edges are not adjacent in the cyclic order of edges about the vertex) and also four *angles* generated by pairs of adjacent (non-opposite) edges. Now, for the obtained four-valent graph we attach black and white cells as follows: for every crossing we indicate two pairs of adjacent edges for "pasting the black cells", and the remaining pair of angles are used for attaching black cells. Cells are attached globally to conform these local conditions. The "black angles" correspond to pairs of edges taken from the B-smoothing of the bracket. This completely defines the way for attaching black and white cells to get a 2-manifold starting from the frame.

This atomic terminology is useful in classifying virtual diagrams in terms of orientability and non-orientability of the corresponding atoms. An atom has a $1 \to 1$-bifurcation if and only if it is non-orientable [Man07b]. In the following we shall need to discuss all atoms that derive from diagrams with two crossings. The reader can easily enumerate the possible Gauss codes with two symbols and arrive at the possibilities (11)(22) (two components, four cases depending on the crossings), (12)(12) (a Hopf link configuration with four crossing possibilities), (1122) (a single unknotted component), (1212) (a non-orientable atom). These cases need to be analyzed and the reader will find them depicted in Fig. 4.8. See also Figs. 4.9 through 4.12.

Each possible 2-face of the bifurcation cube represents an atom with 2 vertices (that is, the face represents all four possibilities for smoothing a pair of crossings in the original link diagram): for each atom, there are four states AA, AB, BA, BB and four maps corresponding to partial differentials $AA \to AB$, $AA \to BA$, $AB \to BB$, $BA \to BB$. Some of them correspond to $1 \to 1$-bifurcation which means that the corresponding partial differential in the usual Khovanov complex is zero. Thus, so is the partial differential in question (it is a composition of zero map with a projection). By parity reasons, for a given atom, there may be 4, 2 or 0 partial differentials (in the initial cube) which are equal to zero.

If all four differentials are equal to zero, then we get the desired equality for the composition of the differentials as $0 = 0$. If we have 2 maps of type $1 \to 1$ then two options are possible. In one of them we have one zero map for each of the two compositions, which leads to $0 = 0$. We call such atoms *inessential*. In the other case we have 0 for the composition of the two $1 \to 1$ maps, but the other composition of maps must be analyzed.

Thus we are left with 6 essential atoms as shown in Fig. 4.8.

For each of these atoms the usual Khovanov differential produces a commutative diagram. Now, multi-gradings and multi-dotted gradings come into play. We have to show that for each atom V the equality of partial differentials $q \circ p = s \circ r$ for the usual Khovanov differentials will hold for the reduced differentials $q' \circ p' = s' \circ r'$. Here p, q, r, s denote the four partial differentials that occur in the Khovanov complex at the atom in question. Some remarks are in order.

Notation Let us denote the differential of the Khovanov complex by ∂, and denote its combination with the projection respecting the multiple grading by ∂_{multi}, its combination with the projection respecting the vector gradings by ∂_{vect} and denote the combination with both projections by ∂'. We are mostly interested in the cases when $\partial = \partial'$ or when $\partial' = 0$ for some particular element of the chain complex.

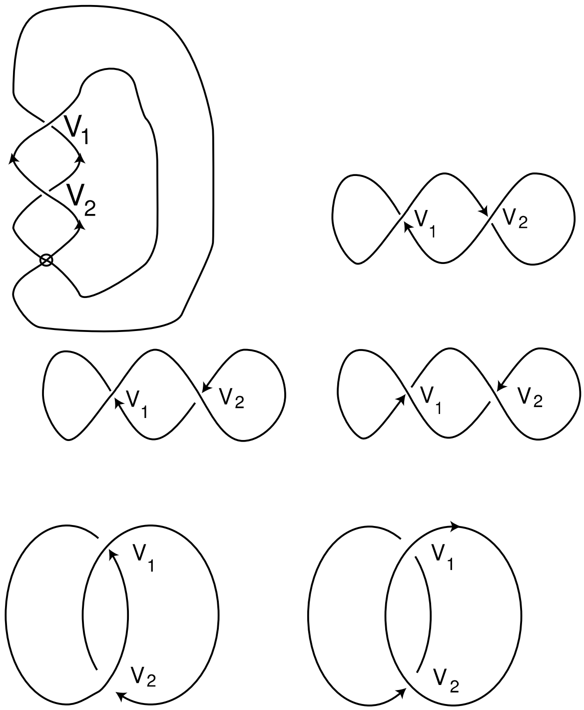


Fig. 4.8 Essential atoms with 2 vertices

We have to list all atoms with two vertices. Some of them are disconnected in the sense that there is no edge connecting one vertex to the other.

For such atoms the (anti)commutativity obviously holds.

Now, let us list all connected essential atoms. There are exactly 6 of them, one non-orientable, 3 orientable with the frame of the unlink and 2 orientable with the frame of the Hopf link, see Fig. 4.8.

For each atom, the anticommutativity of the virtual Khovanov homology over $\mathbf{Z}$ is checked in [Man07b], which leads to the (anti)commutativity over $\mathbf{Z}_2$. Our goal is to check that the multigradings and dotted multigradings preserve this (anti)commutativity.

For this sake we must consider all possible labellings of the state circles for atoms. Each labelling gives a number of integers, for which we take only absolute values and consider only non-zero ones. This leads to the following multiple gradings P, Q, R, S where P corresponds to the smoothing of the atom where both crossings have A-type of smoothing, for S both crossings have B-type of smoothing, and for each of Q, R one crossing has A-smoothing and the other one has the B-smoothing. See Fig. 4.9.

We must look at the differentials depending on P, Q, R, S. Denote the corresponding partial differentials of ∂' by f_1, f_2, f_3, f_4, respectively, see Fig. 4.9.

The following lemma holds.

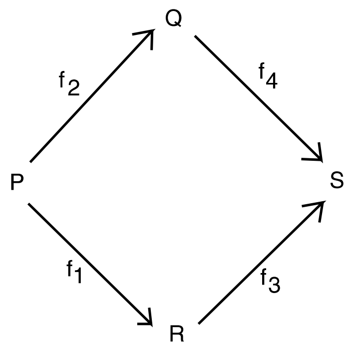


Fig. 4.9 Each atom generates a labelling and two compositions of maps

Lemma 1 *If the multiple gradings P, Q, R, S as described above are all equal* $(P = Q = R = S)$, *then for all partial differentials corresponding to the atom under discussion*, *we have* $\partial_{mult} = \partial$.

Let us now look at vector gradings. There is one case when $\partial_{vect} \neq \partial$ because of the following. Assume we have a $1 \to 2$ or $2 \to 1$-bifurcation where all three circles are dotted: two circles have dotting of order k and one circle has dotting of higher order $l > k$. This may happen, e.g., in $2 \to 1$-bifurcation, when the two circles to merge have arrow label one each (one primary dot) and one target circle has arrow label 2. In this case $\partial_{vect} \neq \partial$ because the non-trivial secondary dot leads to either $+1$ or -1 in the vector grading, hence a non-trivial higher order grading.

Note that this situation does not depend on the particular choice of chain (1 or X in a given state). It depends only on the labelling in the two neighboring states. We call this shifting in the vector grading the *odd dotting condition*.

The following Lemma follows from the definition of the vector grading.

Lemma 2 *If for an atom we have* $P = Q = R = S$ *then the odd dotting condition does not hold for any of the four edges of the bifurcation diagram.*

Now, it turns out that ∂_{vect} in some cases can play the role of the differential, that is, in some cases, $\partial_{vect}^2 = 0$.

Namely, we have the following

Lemma 3 *If the odd dotting condition fails*, *then* ∂ *does not decrease the vector grading*, *that is*, $\partial = \partial_{vect} + \tilde{\partial}$ *where* ∂_{vect} *preserves the vector grading and* $\tilde{\partial}$ *increases exactly one of the dotted gradings* (*one of the vector slots*) *by* 2.

Proof We deal with a $2 \to 1$-bifurcation or $1 \to 2$-bifurcation. We may assume that precisely two of the 3 circles are dotted; moreover, without loss of generality, we may think that these two circles have a primary dot.

Then we have to list all possible maps m and Δ to see that some of them preserve the vector grading, and the others increase the vector grading by 2. Note that all calculations occur in one vector slot since the odd dotting condition fails. In this context we can speak freely about the dotted grading and whether it increases or decreases under a differential.

Let us start with the multiplication. We see that the multiplication of 1 (without dot) with any of 1, $\dot{1}$, X, $\dot{X}$ leads to 1, $\dot{1}$, X, $\dot{X}$ and this multiplication preserves the dotted grading. Now, $\dot{1} \otimes \dot{X}$ (or $\dot{X} \otimes \dot{1}$) multiply to get X, which does not change the dotted grading. Multiplication of X (or $\dot{X}$) with another X (or $\dot{X}$) gives zero.

Finally, $\dot{1} \otimes \dot{1} \to 1$ increases the dotted grading by 2 as well as any of $\dot{1} \otimes X \to \dot{X}$ or $\dot{X} \otimes 1 \to \dot{X}$.

With comultiplication the situation is quite analogous. When none of the three circles is dotted, then the dotted grading is preserved under multiplication. If the circle in the source space and one circle in the target space is dotted, then the comultiplication looks like $\dot{1} \to \dot{1} \otimes X + \dot{X} \otimes 1$ or $\dot{X} \to \dot{X} \otimes X$. Here the only term where the dotted grading is not preserved, is $\dot{1} \to \dot{X} \otimes 1$; in this case it is increased by 2.

If the circle in the source space is not dotted and both circles in the target space are dotted then the dotting is preserved for $1 \to 1 \otimes X + X \otimes 1$, and it is increased by 2 for $X \to X \otimes X$. $\square$

Altogether Lemmas 2 and 3 lead to the following

Lemma 4 *Assume for an atom representing a face of the bifurcation cube the labellings of all four states coincide. Then the restriction of* $(\partial')^2$ *to this atom gives zero.*

Proof We see that the differentials ∂ and ∂_{mult} agree along the edges of such an atom because of Lemma 1, so the 2-face corresponding to that atom ∂_{mult} (anti)commutes. Moreover, by Lemma 2, the differential ∂ splits into the sum of two differentials, $\partial' + \partial''$, where ∂'' strictly increases the multi-dotted grading. This means that $(\partial')^2 = 0$ because $(\partial')^2$ is a composition of $(\partial^2) = 0$ with the projection to the "dotted-grading preserving subspace" $\square$

The next lemma is obvious.

Lemma 5 *Assume in the setting above* $P \neq S$. *Then both compositions for our atom are zero maps because of the multi-grading. Thus, the restriction of* $(\partial')^2$ *to this atom is zero.*

Proof This happens just because ∂' preserves the multi-grading, and so does $(\partial')^2$. $\square$

In the third case we have $P = Q = S \neq R$ or $P = R = S \neq Q$.

In this case, we must separately consider all the six atoms (the schema representing each atom depicted as in Fig. 4.9) to show that the corresponding faces of the cube anti-commute. We shall draw each atom separately in referring to the appropriate Figures in the paper.

Consider the upper left atom depicted in Fig. 4.8. We leave it to the reader to label the maps so that f_2 and f_4 correspond to $1 \to 1$-bifurcations. The composition $f_4 \circ f_2$ is then a zero-map, by definition. The remaining two maps are labelled f_1 and f_3.

Thus, we have two options. If $R \neq P$ then the other composition of differentials is zero because of multiple gradings. If $P = R = S$ then in the A-state we have only one circle labelled by P as well as in the B-state; in the intermediate state we have two circles labelled by P and 0.

The composition $f_3 \circ f_1$ behaves as follows. First, we comultiply 1, and then we multiply the result. If we start with X, we would end up with 0 because $X \to X \otimes X \to 0$ even for the usual differential ∂. If we had 1 then two options are possible. If $P = 0$ then the composition $f_3 \circ f_1$ will lead to $1 \to 1 \otimes X + X \otimes 1 \to X + X = 0$. If $P \neq 0$ then the $f_3 \circ f_1$ will take 1 to 0 as well because of the vector grading: the vector grading of 1 for a non-zero P differs from that for X by sign.

Thus, for the unique non-orientable essential atom with two vertices we have the equality $f_4 \circ f_2 = 0 = f_3 \circ f_1$, which shows the (anti)commutativity. For the other atom with the same frame (which corresponds to the Hopf link with the A-state having 2 circles) the "bad" situation does not occur, just because two single-circle states can not have different K_j's. This completes the analysis of the upper left atom in Fig. 4.8.

We now consider the remaining five essential atoms in Fig. 4.8. The atoms are all orientable, so the arrow count (labelling) is additive. Following the methodology of our previous argument, we can verify that the anticommutativity survives after the new grading is imposed for these atoms.

The unlink (bottom right in Fig. 4.8) has one circle in the opposite states and two circles in the intermediate states (see the upper part of Fig. 4.10). The Hopf link has 2 circles in the A-state, 2 circles in the B-state, and 1 circle in each of the two intermediate states, as shown in the lower part of Fig. 4.10.

Consider the three atoms having the frame of the unknot with two curls as shown in Fig. 4.8. The corresponding bifurcation cubes have a state with three circles, two states with two circles and one state with one circle (that is positioned opposite the state with three circles). The three possible bifurcation cubes depend on the number of circles in the initial state of the cube. An example of this is shown in Fig. 4.11.

For the Hopf link, assume that for both 2-circle states the multiple grading is the same as that of one of the two 1-circle states. By definition, this means that one of the two circles in one 2-circle state has arrow count zero. Denote the arrow count for the other circle by p. Consequently, the other way of merging the two circles gives us p again. This means that the labelling is $A = B = C = D = \{p\}$, and we are in the situation of Lemma 4.

If we have 1-circle in the A-state and 1-circle in the B-state, we may have a "bad" situation ($P = Q = S \neq R$) (not covered by Lemmas 4 and 5) occurring as described below.

First, note that if $P = Q = S$ then the A-state with two circles should have labelling $\{P\}$ as well as the B-state, whence the labelling for two circles corresponding to Q should be $\{P, 0\}$. We are interested in the case when the other intermediate state has labelling $R = (\alpha, \beta)$, say, $(\alpha, P \pm \alpha)$, where $\alpha \neq 0$.

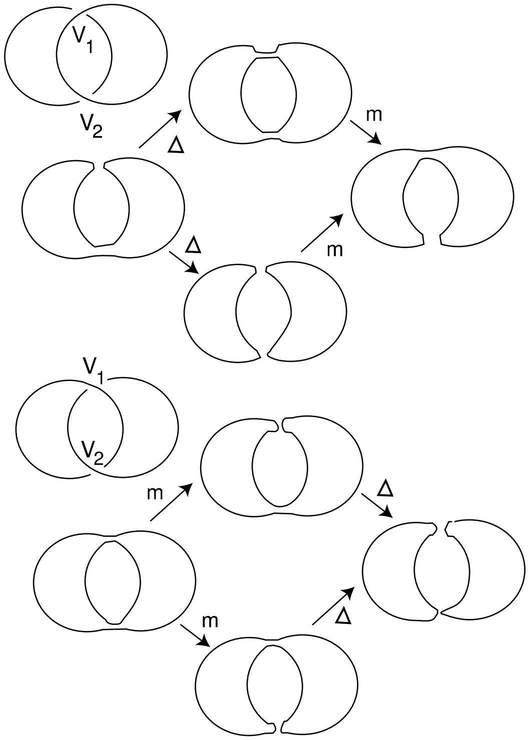


Fig. 4.10 The two atoms with the shadow of the Hopf link

In this case the composition $f_3 \circ f_1$ is zero. Let us consider the composition $f_4 \circ f_2$.

First, let us consider the partial differentials corresponding to ∂. If we apply it to X, we get 0, because the comultiplication f_1 gives us $X \otimes X$ and the further multiplication gives zero. On the other hand, the composition $f_4 \circ f_2$ takes 1 to 0 because we first get $1 \otimes X + X \otimes 1$, which is then mapped to $2X = 0$. Now, when we pass from ∂ to ∂', we see that both multiplication and comultiplication either preserve the vector grading or increase it by 2, we should compare the dotting of the initial 1 and the final X. If they both are zero, then the composition takes 1 to $2X = 0$, otherwise, 1 is taken to 0 because of the dotted gradings.

Note that this is precisely the case where we need our coefficients to be defined over $\mathbf{Z}_2$.

Finally, all 3 atoms with the frame an unknot (drawn in the middle of Fig. 4.8) are to be double-checked.

The three possibilities are: the A-state has 3 circles, or it has 2 circles or it has 1 circle, see Fig. 4.11.

Assume $P = Q = S$ (the case $P = R = S$ is analogous because of the symmetry). We claim that in this case $R = P$. Indeed, since we have 3 circles in the A-state, and one circle in the B-state, we see that the labellings of the circles are $\{p, 0, 0\}$

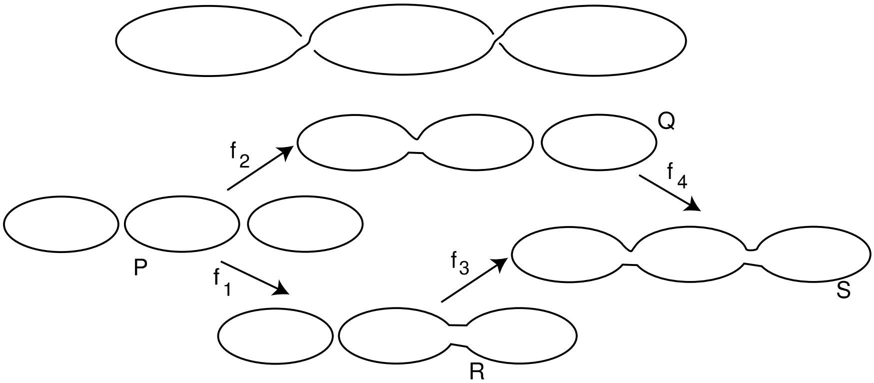


Fig. 4.11 An atom with 2 vertices

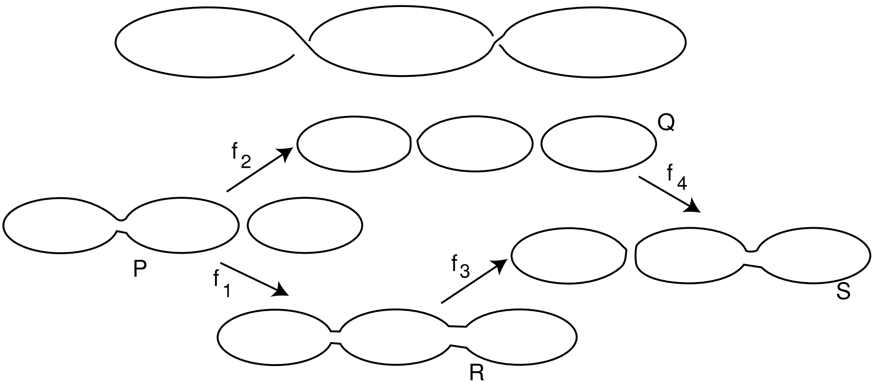


Fig. 4.12 An atom with 2 vertices

in the A-state and $\{p\}$ in the B-state. This yields that $R = \{p, 0\} = \{p\}$, and the (anti)commutativity follows from Lemma 4.

The atom when we have three circles in the B-state is analogous.

In fact, because of the symmetry, we can reduce these three cases to two cases: when we have 1 and 3 at the ends, or when we have 2 and 2 at the ends.

Now, we are left with the example shown in Fig. 4.12.

We are interested in the case when $P = S$ and either $Q \neq P$ or $R \neq P$.

Note that each of P and S consists of 2 circles. Assume $P = S = \{a, b\}$.

It is easy to see that if $R = P = S$ then $Q = P = S = R$. Indeed, if $R = S$, this means that both P and S are of the form $\{a, 0\}$ (or both are $\{b, 0\}$) which yields $Q = \{a, 0, 0\} = P = S$.

Thus we are interested in the case when $a \neq 0$, $b \neq 0$ and $Q = P = S$. This means that $Q = \{a, 0, b\}$, whence R may be of the form $\{|a \pm b|\}$. In this case the composition $f_3 \circ f_1 = 0$ because $Q \neq P$. Let us show that the composition $f_4 \circ f_2 = 0$.

Recall that both f_4 and f_2 are compositions of the partial differential ∂ with the projection map preserving the multi-grading and the multi-dotted grading.

Regardless any grading, $f_4 \circ f_2$ would take $1 \otimes 1 \to 1 \otimes X + X \otimes 1$, $1 \otimes X \to X \otimes X$, $X \otimes 1 \to X \otimes X$.

Now we note that none of these maps survives after applying the projection with respect to vector grading. Indeed, consider for instance the map from $1 \otimes 1$ to the summand $1 \otimes X$. In the source space we had 1 and 1 with vector grading coming from labelling a and b; let us denote it by $1_a + 1_b$. For $1 \otimes X$ we have either $1_a \otimes X_b$ or $1_b \otimes X_b$ depending on the circle having label a.

Here 1_a denotes the $(0, \dots, 0, -1, 0, \dots, 0)$ with the only non-trivial entry on k-th position, $a = 2^{k-1}m$ for odd m. Analogously, X_a denotes $(0, \dots, 0, +1, 0, \dots, 0)$ with the only non-trivial entry on k-th position.

It is crucially important here that neither a nor b is equal to zero. This means that $1_a + 1_b \neq 1_a + X_b$ just because $1_b \neq X_b$.

The same happens in the other cases.

This proves that $f_4 \circ f_2 = 0$, and the atom is (anti)commutative because both compositions are zeroes.

This completes the check of cases of the different atoms corresponding to faces of the bifurcation cube.

Part 2. Proof that the Homology is Invariant Under Reidemeister Moves Below, we shall sketch the outline of the main ideas of the proof. The main features mirror the invariance proof for the usual Khovanov homology along the lines of [BN02].

The invariance under the first Reidemeister move is based on the following two statements which will held when adding a small curl:

1. The mapping Δ is injective.
2. The mapping m is surjective.

In fact, the last two conditions hold when the small circle has the trivial arrow count, and this means that it does not contribute to any of the gradings.

Indeed, consider the complex

$$[[\;]] = \left([[\;]] \xrightarrow{\Delta} [[\;]]\{1\}\right). \tag{4.4}$$

The usual argument goes as follows: the complex in the right hand side contains a Δ-type partial differential, which is injective. Thus, the complex $[[\;]]$ is killed, and what remains from $[[\;]]$ is precisely (after a suitable normalisation) the homology of $[[\;]]$.

But Δ is injective because for any $l \in \{1, X\}$ we have $\Delta(l) = l \otimes X + \langle\text{other terms}\rangle$, where the second term X in $l \otimes X$ corresponds to the small circle.

But in our situation with dotted circles, this happens only if *the small circle is not dotted*. But if the small circle has non-trivial arrow count (say, it appears after splitting a circle without dots into two circles with primary dot each), it would lead, say, to $\Delta : X \to 0$, because $\dot{X} \wedge \dot{X}$ has another vector grading (which is greater by 2 than the grading of X).

An analogous situation happens with the other curl

$$[[\ \ ]] = \Big([[\ \ ]] \overset{m}{\to} [[\ \ ]]\{1\} \Big). \tag{4.5}$$

Here we need that the mapping m be surjective; actually, it would suffice that the multiplication by 1 on the small circle is the identity. But this happens if and only if the small circle has arrow count 0, that is, we have 1, not $\dot{1}$.

Quite similar things happen for the second and for the third Reidemeister moves. The necessary conditions can be summarised as follows:

The small circles which appear for the second and the third Reidemeister move should not be dotted, and similar curves for corresponding smoothings of the RHS and the LHS of any Reidemeister move have the same dotting.

The explanation comes a bit later. Now, we see that this condition is obviously satisfied when the dotting comes from a cohomology class, and not necessarily the Stiefel-Whitney cohomology class for non-orientable surface. Any homology class should do.

Thus (modulo some explanations given below) we have proved the following

Theorem 5 *Let $\mathcal{M} \to M$ be a fibration with I-fibre so that $\mathcal{M}$ is orientable and M is a 2-surface. Let h be a Z_2-cohomology class and let g be the corresponding dotting. Consider the corresponding grading on $[[K]]$. Then for a link $K \subset \mathcal{M}$ the homology of $[[K]]_g$ is invariant under isotopy of K in M (with both the orientation of M and the I-bundle structure fixed) up to some shifts of the usual (quantum) grading and height (homological grading).*

Explanation for the Second and the Third Moves We have the following picture for the Reidemeister move for $[[\ \ ]]$:

$$\begin{array}{ccc} [[\ \ ]]\{1\} & \xrightarrow{m} & [[\ \ ]]\{2\} \\ \Delta \uparrow & & \uparrow \\ [[\ \ ]] & \longrightarrow & [[\ \ ]]\{1\}. \end{array} \tag{4.6}$$

Here we use the notation $\{\cdot\}$ for the degree shifts, see p. 102.

$$\begin{array}{ccc} [[\ \ ]]\{1\} & \xrightarrow{m} & [[\ \ ]]\{2\} \\ \Delta \uparrow & & \uparrow \\ [[\ \ ]] & \longrightarrow & [[\ \ ]]\{1\}. \end{array} \tag{4.7}$$

This complex contains the subcomplex $\mathcal{C}'$:

$$\mathcal{C}' = \begin{array}{ccc} [[\ \ ]]_1\{1\} & \xrightarrow{m} & [[\ \ ]]\{2\} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} \tag{4.8}$$

if *the small circle is not dotted*.

From now on 1 denotes the mark on the small circle. Then the acyclicity of $\mathcal{C}'$ is evident. Factoring $\mathcal{C}$ by $\mathcal{C}'$, we get:

$$\begin{array}{ccc} [[\ \ ]]\{1\}/_{1=0} & \longrightarrow & 0 \\ \Delta\uparrow & & \uparrow \\ [[\ \ ]] & \longrightarrow & [[\ \ ]]\{1\}. \end{array} \tag{4.9}$$

In the last complex, the mapping Δ directed upwards, is an isomorphism (when our small circle is not dotted). Thus the initial complex has the same homology group as $[[\ \ ]]$. This proves the invariance under Ω_2.

The argument for Ω_3 is standard as well; it relies on the invariance under Ω_2 and thus we also require that the small circle is not dotted.

4.7 Applications

The complex constructed in this paper allows us to prove some properties of virtual knot diagrams coming from the Kauffman bracket, the Khovanov homology and the arrow polynomial, see [Man05b, Kau09, Man05a, Tur87, DK09].

First, the consideration of the chain spaces and arrow counts immediately leads to the following theorem.

Theorem 6 *Assume K is a virtual link diagram, and assume there is a non-trivial homology class of $[[K]]$ with multiple grading $\{k_1, \ldots, k_n\}$, such that $\sum_i |k_j| = k$. Then any diagram of K has at least k virtual crossings.*

Besides, the following generalization of the Kauffman-Murasugi Theorem says

Theorem 7 *Let K be a virtual link diagram with a connected shadow (that is, every classical crossing of K can be connected to any other classical crossing by a sequence of arcs starting and ending at classical crossings and going through virtual crossings).*

Let g be the minimal oriented atom genus for the diagram of K and let n be the number of crossings in the diagram K. Then span$\langle K\rangle \le 4n - 4g$, where span stands for the difference between the leading degree and the lowest degree of the Kauffman bracket with respect to the variable a.

The condition of Theorem 7 rules out the split link diagrams. The same argument (see [Man05b, Man05a, Man07a]) leads to

Theorem 8 *For K as in Theorem 7, the span of the arrow polynomial of K taken with respect to a does not exceed $4n - 4g$.*

On the other hand, the genus of the atom estimates from above the *thickness of the Khovanov homology*: the number of diagonals with slope two in coordinates (homological grading, quantum grading) which appear between the leftmost and the rightmost diagonal having a non-trivial homology group. The estimate in [Man07a] says that this thickness does not exceed $2 + g$. Similar considerations lead to the same estimate for the thickness of $[[\cdot]]$ (taking with respect to the *old gradings*, after forgetting all new gradings of non-trivial homology groups):

Theorem 9 *For K as in Theorem 7, the thickness of $[[\cdot]]$ does not exceed $2 + g$.*

Theorems 8 and 9 together lead to the following

Theorem 10 *Assume the diagram K represents a split virtual link (e.g. virtual knot). Then, if K having span of the arrow bracket equal to $4n - 4g$ and the thickness of the extended Khovanov homology equal to $2 + g$ then this diagram is minimal with respect to the number of classical crossings.*

It is an interesting question to determine if there exist examples where the theorems stated above give sharper estimates than the already existing invariants.

4.8 Open Questions

The methods described in the present paper allow us to extend the arrow counts in the arrow polynomial to the level of gradings of a link homology theory. We can recover the arrow polynomial from this link homology by taking the Euler characteristic, forgetting vector gradings and taking the multiple gradings as arrow counts. In this sense, our link homology theory is a true categorification of the arrow polynomial.

There is a more delicate invariant, the *extended bracket polynomial*, [Kau09], which generalizes the arrow polynomial and takes geometrical information into account (instead of just arrow counts). Can this polynomial be categorified by using techniques given in the present paper?

Another question is whether there is a categorification of the arrow polynomial (or the extended bracket polynomial) with integral coefficients. The only point where we needed the $\mathbf{Z}_2$ coefficients was the atom in Fig. 4.11 where the vector gradings and the multiple gradings together did not make the complex over $\mathbf{Z}$ well defined. However, in a similar situation one gets the commutativity of the corresponding face of the atom for *odd Khovanov homology theory*, [ORS07]. Thus, the question of generalizing odd Khovanov homology theory for virtual links gets one more motivation: it would be useful to have it for constructing a categorification of the arrow polynomial with integral coefficients.

Another issue of investigation is the notion of *parity of crossings*, developed recently by Manturov, [Man09a, Man09b] (see also [Kau04] for a precursor to this approach). The idea is to distinguish between two types of crossings, the *even ones*, and the *odd ones* according to some axioms. This approach turns out to be extremely powerful in recognizing some virtual knots and creating new virtual knot invariants. There is a natural way to generalize the arrow polynomial by using the parity argument. This, and a corresponding categorification will be discussed in a subsequent paper.

Acknowledgements The last two authors (L.H.K. and V.O.M.) express their gratitude to the Mathematisches Forschungsinstitut-Oberwolfach, where this paper was finished, for a nice creative scientific atmosphere.

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Chapter 5
An Adelic Extension of the Jones Polynomial

Jesús Juyumaya and Sofia Lambropoulou

Abstract In this paper we represent the classical braids in the Yokonuma–Hecke and the adelic Yokonuma–Hecke algebras. More precisely, we define the completion of the framed braid group and we introduce the adelic Yokonuma–Hecke algebras, in analogy to the p-adic framed braids and the p-adic Yokonuma–Hecke algebras introduced in Juyumaya and Lambropoulou (Topol. Appl. 154:1804–1826, 2007; arXiv:0905.3626v1, 2009). We further construct an adelic Markov trace, analogous to the p-adic Markov trace constructed in Juyumaya and Lambropoulou (arXiv:0905.3626v1, 2009), and using the traces in Juyumaya (J. Knot Theory Ramif. 13:25–29, 2004) and the adelic Markov trace we define topological invariants of classical knots and links, upon imposing some condition. Each invariant satisfies a cubic skein relation coming from the Yokonuma–Hecke algebra.

5.1 Introduction

The classical braid group on n strands B_n is generated by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$, under the defining *braid relations*:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{and} \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1.$$

Both authors were partially supported by Fondecyt (grant 1085002) NTUA and Dipuv.

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_5, © Springer-Verlag Berlin Heidelberg 2011

Geometrically, σ_i is a positive crossing between the ith and the $(i+1)$st strand and σ_i^{-1} is the opposite crossing. The operation in B_n corresponds to the concatenation of two braids and the braid relations reflect allowed topological moves. Closing a braid β by joining with simple arcs the corresponding top and bottom endpoints of β gives rise to an oriented knot or link, denoted $\widehat{\beta}$. By the classical Alexander theorem, an oriented knot or link can be also isotoped to the closure of a braid. *Isotopy* is the notion of topological equivalence for knots and links. Further, by the classical Markov theorem, isotopy classes of oriented knots or links are in bijective correspondence with equivalence classes of braids in $\bigcup_n B_n$ under the two moves:

1. *Conjugation in* B_n: $\alpha\beta \sim \beta\alpha$
2. *Markov move*: $\alpha \sim \alpha\sigma_n{}^{\pm 1}$, $\alpha \in B_n$

Using the above and Ocneanu's Markov trace on the Iwahori–Hecke algebra of type A, $H_n(q)$, V.F.R. Jones constructed in [Jon87] the 2-variable Jones or HOMFLYPT polynomial, a new isotopy invariant of oriented knots and links. The algebra $H_n(q)$ can be described naturally as a quotient of the group algebra $\mathbb{C}B_n$ over the quadratic relations:

$$g_i^2 = (q-1)g_i + q \quad \text{for all } i \tag{5.1}$$

The Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$, $d \in \mathbb{N}$, is a similar algebraic object and has a natural topological interpretation as quotient of the modular framed braid group algebras $\mathbb{C}\mathcal{F}_{d,n}$ (classical framed braids with framings modulo d) over certain quadratic relations, see (5.3). Originally, the algebras $\mathrm{Y}_{d,n}(u)$ were introduced by T. Yokonuma [Yok67] in the representation theory of finite Chevalley groups and they are natural generalizations of the Hecke algebras $H_n(q)$. Indeed, for $d=1$ the algebra $\mathrm{Y}_{1,n}(u)$ coincides with the algebra $H_n(q)$. In the above topological interpretation, $d=1$ means all framings zero, so the algebra $\mathrm{Y}_{1,n}(u)$ is really related to classical braids (with no framings). In [Juy04] Juyumaya constructed a Markov trace on the algebras $\mathrm{Y}_{d,n}(u)$, which for $d=1$ coincides with the Ocneanu trace. Further, in [JL07] the authors introduced the p-adic framed braids and the p-adic Yokonuma–Hecke algebras, while in [JL09] they constructed a p-adic Markov trace. This was used, together with the trace in [Juy04], in order to construct Jones-type isotopy invariants of framed links, upon imposing a certain E-condition to the trace parameters, according to the Markov braid equivalence. Finally, in [JL08] they constructed a monoid representation of the singular braid monoid to $\mathrm{Y}_{d,n}(u)$. Then the trace of [Juy04] is also a Markov trace on the singular braid monoid, so Jones-type invariants for singular knots were constructed, assuming the E-condition.

In the present paper we first relate the Yokonuma–Hecke algebras $\mathrm{Y}_{d,n}(u)$, for $d \neq 1$, to classical knots and links via a natural homomorphism of the classical braid group (5.12). We further define *the completion of the framed braid group* and we introduce *the adelic Yokonuma–Hecke algebra* (Sect. 5.2), into which the classical braid group maps also homomorphically. Then, using the Markov traces in [Juy04] (for different values of d) we construct an infinite family of 2-variable Jones-type invariants of oriented classical knots and links (Sect. 5.5), upon imposing the E-condition (Sect. 5.4). We also construct, in analogy to [JL09], an adelic

Markov trace (Sect. 5.3), which we use for defining an isotopy invariant of classical knots and links, an *adelic extension of the 2-variable Jones polynomial* (Sect. 5.5), upon imposing the E-condition. The E-condition and the E-system are discussed in Sect. 5.4. As far as the braid generators are concerned, the first 'closed' relations in the Yokonuma–Hecke algebra are *cubic relations* (5.26), which also pass to the level of each invariant in the form of a *cubic skein relation* (5.28).

The above-mentioned results are relatively new and computations with the algebras $\mathrm{Y}_{d,n}(u)$ are very complicated. We are now in the process of creating a computing package. Yet, we believe that our invariants are different from the HOMFLYPT polynomial, mainly for the following reasons. Firstly, the differences of the two algebras, $H_n(q)$ and $\mathrm{Y}_{d,n}(u)$, and of their quadratic relations. The structure of the Yokonuma–Hecke algebra is much more subtle and complicated than that of the classical Hecke algebra, even made 'framed'. In the latter case the quadratic relations (5.1) would remain the same as in $H_n(q)$, hence we would talk about the framed HOMFLYPT polynomial. Secondly, the appearance of the cubic relations on the Yokonuma–Hecke algebra level, as the first 'closed' relations of the braid generators, which induce a cubic skein relation for each invariant. Finally, the mere fact that we needed to impose the E-condition to the trace parameters in order to obtain a knot invariant, something not needed in the case of the Ocneanu trace. In fact, the trace in [Juy04] is the first Markov trace in the literature that does not rescale directly in order to yield knot invariants. (Even the fact that the complicated E-system has non-trivial solutions was a surprise to us, see Sect. 5.4 and [JL09].)

In an effort to keep this paper light we omit some technical details, which are mostly to be found in [JL09].

The Yokonuma–Hecke algebras are very versatile algebraic objects, in the sense that they can be used for completely different topological interpretations: to framed braids, to classical braids, to singular braids and, most recently, to transversal and virtual braids, and they comprise the only examples we know of algebras having this property.

5.2 An Adelic Representation of the Braid Group

5.2.1 The Yokonuma–Hecke Algebra

Fix $u \in \mathbb{C}\backslash\{0, 1\}$. Given two positive integers d and n, we denote $\mathrm{Y}_{d,n} = \mathrm{Y}_{d,n}(u)$ the Yokonuma–Hecke algebra, which is a unital associative algebra over $\mathbb{C}$, defined by the generators:

$$1, g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$\begin{array}{rll} g_i g_j = g_j g_i & \text{for } |i-j|>1 \\ g_i g_j g_i = g_j g_i g_j & \text{for } |i-j|=1 \\ t_i t_j = t_j t_i & \text{for all } i, j \\ t_j g_i = g_i t_{s_i(j)} & \text{for all } i, j \\ t_j^d = 1 & \text{for all } j \end{array} \tag{5.2}$$

where $s_i(j)$ is the result of applying the transposition $s_i = (i, i+1)$ to j (in particular $t_i g_i = g_i t_{i+1}$ and $t_{i+1} g_i = g_i t_i$), together with the extra quadratic relations:

$$g_i^2 = 1 + (u-1)e_{d,i} - (u-1)e_{d,i} g_i \quad \text{for all } i \tag{5.3}$$

where

$$e_{d,i} := \frac{1}{d} \sum_{m=0}^{d-1} t_i^m t_{i+1}^{-m}. \tag{5.4}$$

From the defining relations (5.2) and (5.3) it is easy to check that the elements $e_{d,i}$ are idempotents and that they satisfy the following relations (compare with Lemma 1 in [JL08]).

$$\begin{array}{rll} e_i e_j = e_j e_i & \\ e_i g_j = g_j e_i & \text{for } j = i \text{ and for } |i-j|>1 \\ e_j g_i g_j = g_i g_j e_i & \text{for } |i-j|=1. \end{array} \tag{5.5}$$

Remark 1 For all $1 \le i \le n$, let $C_{d,i} = \{1, t_i, t_i^2, \ldots, t_i^{d-1}\}$ denote the cyclic group containing all possible framings modulo d of the ith strand of a framed braid on n strands. Notice that $C_{d,i} \simeq \mathbb{Z}/d\mathbb{Z}$ for all i. We also define the group $H := C_{d,1} \times C_{d,2} \times \cdots \times C_{d,n} \simeq (\mathbb{Z}/d\mathbb{Z})^n$. From the defining relations among the t_i's we deduce that the groups $C_{d,i}$ and H can be regarded inside $\mathrm{Y}_{d,n}$.

The modular framed braid group $\mathcal{F}_{d,n}$ contains framed braids on n strands, but with framings modulo d. It is generated by the braiding generators σ_i and the framing generators $t_1, \ldots, t_n$, where t_j stands for the identity braid with framing one on the jth strand and framing zero on the other strands. Corresponding the braiding generators σ_i to the algebra generators g_i, relations (5.2) furnish a presentation for $\mathcal{F}_{d,n}$ and the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}$ is a quotient of the modular framed braid group algebra $\mathbb{C}\mathcal{F}_{d,n}$ over the quadratic relations (5.3). The elements $e_{d,i}$ are in the algebra $\mathbb{C}\mathcal{F}_{d,n}$ as well as in the quotient algebra $\mathrm{Y}_{d,n}$ and they are expressions of the framing generators t_i, t_{i+1}.

5.2.2 The Adelic Yokonuma–Hecke Algebra

Let $\mathbb{N}$ denote the set of positive integers regarded as a directed set with the usual order. Let also $\mathbb{N}^{\sim}$ denote the directed set of positive integers regarded with respect to the partial order defined by the divisibility relation. The notation $d|d'$ means d divides d'.

For d, $d' \in \mathbb{N}$ with $d|d'$ we consider the natural connecting ring homomorphism $\rho_d^{d'}$, defined in [JL09], (1.17):

$$\rho_d^{d'} : \mathrm{Y}_{d',n} \longrightarrow \mathrm{Y}_{d,n} \tag{5.6}$$

More precisely, we denote $\vartheta_d^{d'}$ the natural epimorphism:

$$\begin{aligned} \vartheta_d^{d'} : \mathbb{Z}/d'\mathbb{Z} &\longrightarrow \quad \mathbb{Z}/d\mathbb{Z} \\ m \quad &\mapsto \quad m \,(\mathrm{mod}\, d) \end{aligned} \tag{5.7}$$

The inverse limit $\widehat{\mathbb{Z}}$ of the inverse system of groups $(\mathbb{Z}/d\mathbb{Z}, \vartheta_d^{d'})$ indexed by $\mathbb{N}^{\sim}$ is called *the completion of* $\mathbb{Z}$:

$$\widehat{\mathbb{Z}} = \varprojlim_{d \in \mathbb{N}^{\sim}} \mathbb{Z}/d\mathbb{Z}$$

Our references for inverse limits are mainly [RZ00] and [Wil98].

By componentwise multiplication, epimorphism (5.7) defines the epimorphism:

$$\varpi_d^{d'} : (\mathbb{Z}/d'\mathbb{Z})^n \longrightarrow (\mathbb{Z}/d\mathbb{Z})^n \tag{5.8}$$

Extension to the B_n-part by the identity map yields the epimorphism:

$$\varpi_d^{d'} \cdot \mathrm{id} : \mathcal{F}_{d',n} \longrightarrow \mathcal{F}_{d,n} \tag{5.9}$$

Definition 1 The *completion* $\mathcal{F}_{\infty,n}$ *of the framed braid group* $\mathcal{F}_n$ is defined as the inverse limit of the inverse system of groups $(\mathcal{F}_{d,n}, \varpi_d^{d'} \cdot \mathrm{id})$:

$$\mathcal{F}_{\infty,n} := \varprojlim_{d \in \mathbb{N}^{\sim}} \mathcal{F}_{d,n}$$

The linear extension of map (5.9) yields an algebra epimorphism:

$$\varrho_d^{d'} : \mathbb{C}\mathcal{F}_{d',n} \longrightarrow \mathbb{C}\mathcal{F}_{d,n} \tag{5.10}$$

Remark 2 The braid group B_n acts on $\widehat{\mathbb{Z}}^n$ by permuting the factors, so we may consider the group $\widehat{\mathbb{Z}}^n \rtimes B_n$. It is easy to construct an isomorphism between the groups $\widehat{\mathbb{Z}}^n \rtimes B_n$ and $\mathcal{F}_{\infty,n}$ (proof analogous to Theorem 1 in [JL07]). We note, though, that this isomorphism does not carry through on the level of the algebras $\mathbb{C}(\widehat{\mathbb{Z}}^n \rtimes B_n)$ and $\varprojlim_{d \in \mathbb{N}^{\sim}} \mathbb{C}\mathcal{F}_{d,n}$ (see [JL09] for more details).

Passing now to the quotient algebras by relations (5.3) we obtain the algebra epimorphism (5.6).

Definition 2 The *adelic Yokonuma–Hecke* algebra $\mathrm{Y}_{\infty,n}(u) = \mathrm{Y}_{\infty,n}$ is defined as the inverse limit of the inverse system of rings $(\mathrm{Y}_{d,n}, \rho_d^{d'})$ indexed by $\mathbb{N}^{\sim}$:

$$\mathrm{Y}_{\infty,n} = \varprojlim_{d\in\mathbb{N}^{\sim}} \mathrm{Y}_{d,n}$$

Hence, elements in $\mathrm{Y}_{\infty,n}$ are infinite sequences of elements in the algebras $\mathrm{Y}_{d,n}$, for $d \in \mathbb{N}^{\sim}$, which are coherent in the sense of maps (5.7)–(5.6). Moreover, the definition of the connecting maps $\rho_d^{d'}$ do not involve the elements g_i, so we shall denote also by g_i the elements in $\mathrm{Y}_{\infty,n}$ corresponding to the infinite constant sequence (g_i).

For all $0 \le i \le n-1$ define now the groups $H_{d,i}$ as follows:

$$H_{d,i} = \{1, t_i t_{i+1}^{-1}, t_i^2 t_{i+1}^{-2}, \dots, t_i^{d-1} t_{i+1}\}$$

In this notation, the element $e_{d,i}$ in (5.4) is the average of the elements of the group $H_{d,i}$:

$$e_{d,i} = \frac{1}{d} \sum_{x\in H_{d,i}} x$$

Then $\rho_d^{d'}(H_{d',i}) = H_{d,i}$ for all $d|d'$. Hence, we deduce the following result.

Lemma 1 *For all i and for d, d' such that $d|d'$, we have*:

$$\rho_d^{d'}(e_{d',i}) = e_{d,i}.$$

We shall denote by e_i the sequence $(e_{d,i})_{d\in\mathbb{N}^{\sim}}$ in $\mathrm{Y}_{\infty,n}$:

$$e_i := (e_{d,i})_{d\in\mathbb{N}^{\sim}} \tag{5.11}$$

It follows easily from relations (5.5) and (5.3) that the adelic elements e_i satisfy the following relations (compare with Proposition 10 and Theorem 3 in [JL07]).

Proposition 1 *For all i the following relations hold in* $\mathrm{Y}_{\infty,n}$:

1. $e_i e_j = e_j e_i$
2. $e_i g_j = g_j e_i$ *for* $j = i$ *and for* $|i-j| > 1$
3. $e_j g_i g_j = g_i g_j e_i$ *for* $|i-j| = 1$
4. $g_i^2 = 1 + (u-1)e_i g_i - (u-1)g_i$.

5.2.3 Representing the Braid Group

The defining relations of $\mathrm{Y}_{d,n}$ imply that the map:

$$\begin{aligned} \flat_{d,n} : B_n &\longrightarrow \mathrm{Y}_{d,n} \\ \sigma_i &\mapsto g_i \end{aligned} \tag{5.12}$$

defines a representation of the classical braid group B_n in $\mathrm{Y}_{d,n}$. Under this representation the generators g_i of the algebra $\mathrm{Y}_{d,n}$ correspond to the braid generators σ_i. The generators t_j, though, loose their initial topological interpretation as framing generators and they are just considered formally as elements in the algebra.

Further, for all d, d', d'' such that $d|d'$ and $d'|d''$ we have the following commutative diagram:

$$\begin{tikzcd}
\cdots & B_n \arrow[l] \arrow[d, "\flat_{d,n}"'] & B_n \arrow[l, "\mathrm{Id}"'] \arrow[d, "\flat_{d',n}"'] & B_n \arrow[l, "\mathrm{Id}"'] \arrow[d, "\flat_{d'',n}"'] & \cdots \arrow[l] \\
\cdots & \mathrm{Y}_{d,n} \arrow[l] & \mathrm{Y}_{d',n} \arrow[l, "\rho_d^{d'}"'] & \mathrm{Y}_{d'',n} \arrow[l, "\rho_{d'}^{d''}"'] & \cdots \arrow[l]
\end{tikzcd} \tag{5.13}$$

By taking inverse limits in the above diagram we obtain the following representation of the classical braid group B_n in the adelic Yokonuma–Hecke algebra:

$$\flat_{\infty,n} : B_n \longrightarrow \mathrm{Y}_{\infty,n} \tag{5.14}$$

where:

$$\flat_{\infty,n} := \varprojlim_{d \in \mathbb{N}^{\sim}} \flat_{d,n}$$

5.3 An Adelic Markov Trace

5.3.1 *The Modular Markov Trace* tr_d

It is known that the Yokonuma–Hecke algebra supports a Markov trace [Juy04]. More precisely, for fixed d we consider the inductive system $(\mathrm{Y}_{d,n})_{n\in\mathbb{N}}$ associated to the natural inclusions $\mathrm{Y}_{d,n} \subset \mathrm{Y}_{d,n+1}$ for all $n \in \mathbb{N}$. Let $\mathrm{Y}_{d,\infty}$ be the corresponding inductive limit. In [Juy04] the following theorem is proved.

Theorem 1 (Juyumaya 2004) *Let $z, x_1, \ldots, x_{d-1} \in \mathbb{C}$ and let d be a positive integer. For all $n \in \mathbb{N}$ there exists a unique linear map* $\mathrm{tr}_d = (\mathrm{tr}_{d,n})_{n\in\mathbb{N}}$:

$$\mathrm{tr}_d : \mathrm{Y}_{d,\infty} \longrightarrow \mathbb{C}$$

satisfying the rules:

$$\begin{aligned}
\mathrm{tr}_{d,n}(ab) &= \mathrm{tr}_{d,n}(ba) \\
\mathrm{tr}_{d,n}(1) &= 1 \\
\mathrm{tr}_{d,n+1}(ag_n) &= z\,\mathrm{tr}_{d,n}(a) \qquad (a \in \mathrm{Y}_{d,n}) \\
\mathrm{tr}_{d,n+1}(at_{n+1}^m) &= x_m \mathrm{tr}_{d,n}(a) \quad\; (a \in \mathrm{Y}_{d,n}\,,\ 1 \le m \le d-1).
\end{aligned}$$

The proof of Theorem 1 rests on the fact that the algebra $\mathrm{Y}_{d,n+1}$ admits an inductive linear basis, where either g_n or t_{n+1}^m appears at most once. Note that, for $d = 1$ the trace restricts to the first three rules and it coincides with Ocneanu's trace on the Iwahori–Hecke algebra, which was used in [Jon87] to construct the 2-variable or HOMFLYPT Jones polynomial for oriented classical knots and links.

5.3.2 The Adelic Markov Trace $\mathbf{tr}_\infty$

Let R be the polynomial ring $\mathbb{C}[z]$ and let $R[X_d]$ be the polynomial ring with coefficients in R and variables x_a, where $a \in \mathbb{Z}/d\mathbb{Z}$. Let also $d|d'$. The natural map $x_a \mapsto x_b$ where $b := \vartheta_d^{d'}(a)$ (recall (5.7)), defines a ring epimorphism:

$$\xi_d^{d'} : R\,[X_{d'}] \longrightarrow R\,[X_d] \tag{5.15}$$

We now have the following result (compare with Lemma 7 in [JL09]).

Lemma 2 *The family* $(R[X_d], \xi_d^{d'})$ *indexed by* $\mathbb{N}^\sim$, *is an inverse system.*

We shall then consider the inverse limit:

$$\varprojlim_{d\in\mathbb{N}^\sim} R[X_d]$$

Notice that $\varprojlim_{d\in\mathbb{N}^\sim} R[X_d]$ can be regarded as the polynomial ring over $\mathbb{C}$ in the variables z and x_α, where $\alpha \in \widehat{\mathbb{Z}}$. The ring $\varprojlim_{d\in\mathbb{N}^\sim} R[X_d]$ turns out to be an integral domain.

Now, for all $n \in \mathbb{N}$ and for all d, d', d'' such that $d|d'$ and $d'|d''$, we have the following commutative diagram (compare with Lemma 6 [JL09]):

$$\begin{tikzcd} \cdots & \mathrm{Y}_{d,n} \arrow[l] \arrow[d, "\mathrm{tr}_{d,n}"'] & \mathrm{Y}_{d',n} \arrow[l, "\rho_d^{d'}"'] \arrow[d, "\mathrm{tr}_{d',n}"'] & \mathrm{Y}_{d'',n} \arrow[l, "\rho_{d'}^{d''}"'] \arrow[d, "\mathrm{tr}_{d'',n}"'] & \cdots \arrow[l] \\ \cdots & R\,[X_d] \arrow[l] & R\,[X_{d'}] \arrow[l, "\xi_d^{d'}"'] & R\,[X_{d''}] \arrow[l, "\xi_{d'}^{d''}"'] & \cdots \arrow[l] \end{tikzcd} \tag{5.16}$$

Finally, we note that there are natural inclusions $\mathrm{Y}_{\infty,n} \subset \mathrm{Y}_{\infty,n+1}$, for all $n \in \mathbb{N}$. Let

$$\mathrm{Y}_\infty := \varinjlim_{n\in\mathbb{N}} \mathrm{Y}_{\infty,n}$$

the associated inductive limit. We then have the following.

Theorem 2 *There exists a unique linear Markov trace* $\mathrm{tr}_\infty = (\mathrm{tr}_{\infty,n})_{n\in\mathbb{N}}$,

$$\mathrm{tr}_\infty : \mathrm{Y}_\infty \longrightarrow \varprojlim_{d\in\mathbb{N}^\sim} R[X_d]$$

such that

$$\begin{aligned}
\mathrm{tr}_{\infty,n}(ab) &= \mathrm{tr}_{\infty,n}(ba) \\
\mathrm{tr}_{\infty,n}(1) &= 1 \\
\mathrm{tr}_{\infty,n+1}(ag_n) &= z\,\mathrm{tr}_{\infty,n}(a) \\
\mathrm{tr}_{\infty,n+1}(ay_{n+1}) &= x_y \mathrm{tr}_{\infty,n}(a)
\end{aligned}$$

where $a, b \in \mathrm{Y}_{\infty,n}$ *and* y_{n+1} *is the element in* $\widehat{\mathbb{Z}}^{n+1}$ *with* $y \in \widehat{\mathbb{Z}}$ *in the position* $n+1$ *and* 0 *otherwise, that is*: $y_{n+1} = (0, \ldots, 0, y)$.

Proof It follows immediately from the commutative diagram (5.16) and from the existence and uniqueness of the traces tr_d. □

5.4 The *E*-Condition

5.4.1 Why E-Condition

The representations (5.12) and (5.14) of the braid group through the classical and the adelic Yokonuma–Hecke algebras composed with the Markov traces tr_d and tr_∞ of Theorems 1 and 2 map classical braids to complex polynomials. In view of the Alexander and Markov theorems for classical braids we would like to construct isotopy invariants for classical oriented knots and links. According to the Markov Theorem, such an invariant has to agree on the links $\widehat{\alpha}$, $\widehat{\alpha\sigma_n}$ and $\widehat{\alpha\sigma_n^{-1}}$, for any $\alpha \in B_n$. Following Jones' construction of the 2-variable Jones polynomial for classical knots [Jon87], we will try to define knot isotopy invariants by re-scaling and normalizing the traces tr_d and the adelic trace tr_∞. By the equation:

$$g_i^{-1} = g_i - (u^{-1} - 1)\, e_{d,i} + (u^{-1} - 1)\, e_{d,i}\, g_i \tag{5.17}$$

we have:

$$\mathrm{tr}_d(\alpha g_n^{-1}) = \mathrm{tr}_d(\alpha g_n) - (u^{-1} - 1)\mathrm{tr}_d(\alpha e_{d,n}) + (u^{-1} - 1)\mathrm{tr}_d(\alpha e_{d,n} g_n). \tag{5.18}$$

In order that the invariant agrees on the closures of the braids $\alpha\sigma_n{}^{-1}$ and $\alpha\sigma_n$ we need that $\mathrm{tr}_d(\alpha g_n^{-1})$ factorizes through $\mathrm{tr}_d(\alpha)$, just as $\mathrm{tr}_d(\alpha g_n)$ does. Indeed, for the first term we have: $\mathrm{tr}_d(\alpha g_n) = z\,\mathrm{tr}_d(\alpha)$. Further:

$$\mathrm{tr}_d(\alpha e_{d,n} g_n) = \frac{1}{d}\sum_{m=0}^{d-1} \mathrm{tr}_d(\alpha t_n^m t_{n+1}^{-m} g_n) = \frac{1}{d}\sum_{m=0}^{d-1} z\,\mathrm{tr}_d(\alpha) = z\,\mathrm{tr}_d(\alpha) \tag{5.19}$$

since $\mathrm{tr}_d(\alpha t_n^m t_{n+1}^{-m} g_n) = \mathrm{tr}_d(\alpha t_n^m g_n t_n^{-m}) = z\,\mathrm{tr}_d(\alpha t_n^m t_n^{-m}) = z\,\mathrm{tr}_d(\alpha)$. So, we need that $\mathrm{tr}_d(\alpha e_{d,n})$ also factorizes through $\mathrm{tr}_d(\alpha)$. Unfortunately, we do not have, in general, such a nice formula for $\mathrm{tr}_d(\alpha e_{d,n})$. The underlying reason on the framed braid level (which is the natural interpretation for elements in $\mathrm{Y}_{d,n}$) is that $e_{d,n}$ involves the nth strand of the braid α. Yet, by imposing some conditions on the indeterminates x_i of the trace tr_d it is possible to have this factorization.

5.4.2 The E-System

Set $X_d = \{x_0, x_1, \ldots, x_{d-1}\}$ a set of d complex numbers. We shall say that X_d satisfies the *E-condition* if the x_i's are solutions of the following non-linear system of $d-1$ equations:

$$\begin{aligned} \mathrm{E}_d^{(1)} &= \mathrm{x}_1 \mathrm{E}_d^{(0)} \\ \mathrm{E}_d^{(2)} &= \mathrm{x}_2 \mathrm{E}_d^{(0)} \\ &\vdots \\ \mathrm{E}_d^{(d-1)} &= \mathrm{x}_{d-1} \mathrm{E}_d^{(0)} \end{aligned} \tag{5.20}$$

where $\mathrm{E}_d^{(m)}$ is the polynomial in variables $\mathrm{x}_1, \ldots, \mathrm{x}_{d-1}$ defined as:

$$\mathrm{E}_d^{(m)} = \sum_{s=0}^{d-1} \mathrm{x}_{m+s} \mathrm{x}_{d-s} \tag{5.21}$$

where, by definition, $\mathrm{x}_0 = \mathrm{x}_d = 1$, and the sub-indices are regarded modulo d. We shall refer to the system above as the *(E, d)-system* or simply the *E-system*. For example, in the case $d = 3$ we have the E-system:

$$\begin{aligned} \mathrm{x}_1 + \mathrm{x}_2^2 &= 2\mathrm{x}_1^2 \mathrm{x}_2 \\ \mathrm{x}_1^2 + \mathrm{x}_2 &= 2\mathrm{x}_1 \mathrm{x}_2^2 \end{aligned}$$

We then have the following result (compare with Theorem 6 in [JL09]).

Theorem 3 *If $X_{d,S}$ is a solution of the E-system then for all $\alpha \in \mathrm{Y}_{d,n}$ we have*:

$$\mathrm{tr}_d(\alpha e_{d,n}) = \mathrm{tr}_d(\alpha)\, \mathrm{tr}_d(e_{d,n}).$$

For the proof of Theorem 3 we need to consider all different cases for α being an element in the inductive basis of $\mathrm{Y}_{d,n}(u)$. See [JL09] for details.

We still need to establish, of course, that the set of solutions of the E-system is non-empty. For $a \in \mathbb{Z}/d\mathbb{Z}$ we denote $\exp_a$ the exponential character of the group $\mathbb{Z}/d\mathbb{Z}$, that is:

$$\exp_a(k) := \cos \frac{2\pi a k}{d} + i \sin \frac{2\pi a k}{d} \quad (k \in \mathbb{Z}/d\mathbb{Z}).$$

Theorem 4 (Gérardin 2009) *The solutions of system* (5.20) *are parametrized by the non-empty subsets S of $\mathbb{Z}/d\mathbb{Z}$. More precisely, a subset S defines the solution $X_{d,S} = \{x_0, x_1, \ldots, x_{d-1}\}$, where*:

$$x_k = \frac{1}{|S|} \sum_{s \in S} \exp_s(k) \quad (0 \le k \le d-1).$$

Proof See Appendix in [JL09]. □

Let $X_{d,S}$ be a solution of the E-system. A direct computation yields that the value of the tr_d on $e_{d,i}$ (with respect to $X_{d,S}$) is:

$$\mathrm{tr}_d(e_{d,i}) = \frac{1}{|S|} \quad (1 \le i \le n-1). \tag{5.22}$$

For a thorough discussion and full proofs related to the E-condition and the E-system we refer the reader to [JL09].

5.4.3 Lifting Solutions of the E-System

For $d|d'$ we denote $\mathrm{s}_{d'}^{d}$ a section map of the natural epimorphism $\vartheta_d^{d'}$ of (5.7). By taking a section $\mathrm{s}_{d'}^{d}$ any solution of the (E,d)-system can be lifted trivially to a solution of the (E,d')-system. Indeed: If $X_{d,S}$ is a solution of the (E,d)-system, then $X_{d',S'}$ is a solution of the (E,d')-system, where $S' := \mathrm{s}_{d'}^{d}(S)$. A more interesting lifting can be constructed as follows. Define $S_{d'}^{d} = \{\mathrm{s}_{d'}^{d}(a) + b\,;\, a \in S,\ b \in \ker \vartheta_d^{d'}\}$. Then we define the lifting $X_{d,d',S}$ of $X_{d,S}$ as:

$$X_{d,d',S} := X_{d',S_{d'}^{d}} \quad (S \subseteq \mathbb{Z}/d\mathbb{Z}) \tag{5.23}$$

Notice that $|X_{d,d',S}| = |S|d'/d$ and $X_{d,d,S} = X_{d,S}$.

Lemma 3 *For $d|d'|d''$ and S non-empty subset of $\mathbb{Z}/d\mathbb{Z}$ we have*:

$$X_{d,d'',S} = X_{d',d'',S'}$$

where $S' := S_{d'}^{d}$.

Proof According to the definition of $X_{S,d}$ it is enough to prove that:

$$\left(S_{d'}^{d}\right)_{d''}^{d'} = S_{d'}^{d}$$

Now the elements in $(S_{d'}^{d})_{d''}^{d'}$ are in the form $z := \mathrm{s}_{d''}^{d'}(x) + y$, where $x \in S_{d'}^{d}$ and $y \in \ker \vartheta_{d'}^{d''}$. The element x is in the form $x = \mathrm{s}_{d'}^{d}(\mu) + \nu$, where $\mu \in S$ and $\nu \in \ker \vartheta_d^{d'}$.

So we can re-write z as:

$$z = \mathrm{s}_{d''}^{d'}\left(\mathrm{s}_{d'}^{d}(\mu) + \nu\right) + y = \mathrm{s}_{d''}^{d'}\left(\mathrm{s}_{d'}^{d}(\mu)\right) + \mathrm{s}_{d''}^{d'}(\nu) + y = \mathrm{s}_{d''}^{d}(\mu) + \mathrm{s}_{d''}^{d'}(\nu) + y$$

But $\mathrm{s}_{d''}^{d'}(\nu) + y$ belongs to the $\ker \vartheta_{d}^{d''}$; hence $z \in S_{d'}^{d}$. Thus $(S_{d'}^{d})_{d''}^{d'} = S_{d'}^{d}$. □

We showed that solutions of the E-system lift to solutions on the adelic level.

5.5 An Adelic Extension of the Jones Polynomial

5.5.1 Isotopy Invariants from tr_d

Given a solution $X_{d,S}$ of the E-system, (5.18) can be rewritten as follows, using Theorem 3:

$$\mathrm{tr}_d(\alpha g_n^{-1}) = \frac{z + (u-1)\zeta_{d,S}}{u}\,\mathrm{tr}_d(\alpha) \tag{5.24}$$

where, for all i:

$$\zeta_{d,S} := \mathrm{tr}_d(e_{d,i}) = \frac{1}{|S|}.$$

Let now $\mathcal{L}$ be the set of oriented links in S^3. Recall that by the Alexander theorem every link type may be represented by a closed braid. For the solution $X_{d,S}$ of the E-system we wish to define a link isotopy invariant $\Delta_{d,S}$. In order that $\Delta_{d,S}(\widehat{\alpha\sigma_n}) = \Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}})$, for $\alpha \in B_n$, we apply a re-scaling via the homomorphism:

$$\begin{aligned} \delta : B_n &\longrightarrow \mathrm{Y}_{d,n} \\ \sigma_i &\mapsto \sqrt{\lambda} g_i \end{aligned} \tag{5.25}$$

where:

$$\lambda := \frac{z - (1-u)\zeta_{d,S}}{uz}$$

Finally, in order that $\Delta_{d,S}(\widehat{\alpha\sigma_n}) = \Delta_{d,S}(\widehat{\alpha})$ we need to do a normalization. So, we define the following map on the set $\mathcal{L}$.

Definition 3 Let $\alpha \in B_n$, any n. We define the map $\Delta_{d,S}$ on the closure $\widehat{\alpha}$ of α as follows:

$$\Delta_{d,S}(\widehat{\alpha}) := \left(\frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}}\right)^{n-1} (\mathrm{tr}_d \circ \delta)(\alpha)$$

Equivalently, setting

$$D := \frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}}$$

we can write:

$$\Delta_{d,S}(\widehat{\alpha}) = D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}\mathrm{tr}_d(\flat_{d,n}(\alpha))$$

where $\epsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in the braid word α and where $\flat_{d,n}$ was defined in (5.12).

Theorem 5 *For any solution $X_{d,S}$ of the E-system, $\Delta_{d,S}$ is a 2-variable isotopy invariant of oriented links in S^3, depending on the variables u, z.*

Proof We need to show that $\Delta_{d,S}$ is well-defined on isotopy classes of oriented links. According to the Markov theorem, it suffices to prove that $\Delta_{d,S}$ is consistent with moves (i) and (ii). From the facts that $\epsilon(\alpha\alpha') = \epsilon(\alpha'\alpha)$ and $\mathrm{tr}_d(ab) = \mathrm{tr}_d(ba)$, it follows that $\Delta_{d,S}$ respects move (i). Let now $\alpha \in B_n$. Then $\alpha\sigma_n \in B_{n+1}$ and $\epsilon(\alpha\sigma_n) = \epsilon(\alpha) + 1$. Hence:

$$\begin{aligned}\Delta_{d,S}(\widehat{\alpha\sigma_n}) &= D^n(\sqrt{\lambda})^{\epsilon(\alpha\sigma_n)}\,\mathrm{tr}_d(\flat_{d,n}(\alpha\sigma_n)) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)+1}\,\mathrm{tr}_d(\flat_{d,n}(\alpha)g_n)\\ &= D\sqrt{\lambda}\,z\,\Delta_{d,S}(\widehat{\alpha})\end{aligned}$$

where we used that $\mathrm{tr}_d(\flat_{d,n}(\alpha)g_n) = z\,\mathrm{tr}(\flat_{d,n}(\alpha))$. Now:

$$z = \frac{(1-u)\zeta_{d,S}}{1-\lambda u},$$

so:

$$D\sqrt{\lambda}\,z = 1.$$

Therefore, $\Delta_{d,S}(\widehat{\alpha\sigma_n}) = \Delta_{d,S}(\widehat{\alpha})$. Finally, we will prove that $\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = \Delta_{d,S}(\widehat{\alpha})$. Indeed:

$$\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\alpha\sigma_n^{-1})}\mathrm{tr}_d(\flat_{d,n}(\alpha\sigma_n^{-1})) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)-1}\,\mathrm{tr}_d(\flat_{d,n}(\alpha)g_n^{-1}).$$

Resolving g_n^{-1} from (5.17) we obtain:

$$\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)-1}\left[z - (u^{-1}-1)\zeta_{d,S} + (u^{-1}-1)z\right]\mathrm{tr}_d(\flat_{d,n}(\alpha)).$$

Also, from Theorem 3 and (5.19) we have:

$$\mathrm{tr}_d(\flat_{d,n}(\alpha e_{d,n})) = \zeta_{d,S}\,\mathrm{tr}_d(\flat_{d,n}(\alpha)) \quad \text{and} \quad \mathrm{tr}_d(\flat_{d,n}(\alpha)e_{d,n}g_n) = z\,\mathrm{tr}_d(\flat_{d,n}(\alpha)).$$

Therefore:

$$\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)-1}\,\frac{z+(u-1)\zeta_{d,S}}{u}\,\mathrm{tr}_d(\flat_{d,n}(\alpha))$$

$$= \frac{D}{\sqrt{\lambda}} \frac{z + (u-1)\zeta_{d,S}}{u} \Delta_{d,S}(\widehat{\alpha})$$
$$= \Delta_{d,S}(\widehat{\alpha}).$$

Hence the proof is concluded. □

We have defined an infinite family of 2-variable isotopy invariants for oriented classical links.

5.5.2 Computations

We shall first give some formulas that are useful for computations. For powers of g_i we can deduce by induction the following formulae.

Lemma 4 *Let $m \in \mathbb{Z}, k \in \mathbb{N}$. (i) For m positive, define $\alpha_m = (u-1)\sum_{l=0}^{k-1} u^{2l}$ if $m = 2k$ and $\beta_m = u(u-1)\sum_{l=0}^{k-1} u^{2l}$ if $m = 2k+1$. Then:*

$$g_i^m = \begin{cases} 1 + \alpha_m e_{d,i} - \alpha_m e_{d,i} g_i & \text{if } m = 2k \\ g_i - \beta_m e_{d,i} + \beta_m e_{d,i} g_i & \text{if } m = 2k+1 \end{cases}$$

(ii) For m negative, define $\alpha'_m = u^{-1}(u^{-1}-1)\sum_{l=0}^{k-1} u^{-2l}$ if $m = -2k$ and $\beta'_m = (u^{-1}-1)\sum_{l=0}^{k-1} u^{-2l}$ if $m = -2k+1$. Then:

$$g_i^m = \begin{cases} 1 + \alpha'_m e_{d,i} - \alpha'_m e_{d,i} g_i & \text{if } m = -2k \\ g_i - \beta'_m e_{d,i} + \beta'_m e_{d,i} g_i & \text{if } m = -2k+1 \end{cases}$$

We now proceed with some basic computations. Clearly, for the unknot O, $\Delta_{d,S}(\mathrm{O}) = 1$. For the Hopf link and the two trefoil knots we have:

- Let $\mathrm{H} = \widehat{\sigma_1^2}$, the Hopf link. We find $\mathrm{tr}_d(g_1^2) = 1 + (u+1)(\zeta_{d,S} - z)$ and $\epsilon(\sigma_1^2) = 2$. Then:

$$\Delta_{d,S}(\mathrm{H}) = \frac{1-\lambda u}{(1-u)\zeta_{d,S}} \sqrt{\lambda}\left(1 + (u+1)(\zeta_{d,S} - z)\right)$$
$$= z^{-1}\sqrt{\lambda}\left(1 + (u+1)(\zeta_{d,S} - z)\right).$$

- Let $\mathrm{T} = \widehat{\sigma_1^3}$, the right-handed trefoil. From Lemma 4 we have $g_1^3 = g_1 - u(u-1)e_{d,1} + u(u-1)e_{d,1}g_1$. Hence: $\mathrm{tr}_d(g_1^3) = z - u(u-1)\zeta_{d,S} + u(u-1)z$. Moreover $\epsilon(\sigma_1^3) = 3$. Then, using that $1 - \lambda u = z^{-1}\zeta_{d,S}(1-u)$, we obtain:

$$\Delta_{d,S}(\mathrm{T}) = D(\sqrt{\lambda})^3\left[(u(u-1)+1)z - u(u-1)\zeta_{d,S}\right]$$
$$= \frac{\lambda}{z}\left[(u^2 - u + 1)z - (u^2 - u)\zeta_{d,S}\right].$$

- Let, finally, $-\mathrm{T} = \widehat{\sigma_1^{-3}}$, the left-handed trefoil. From Lemma 4 we have $g_1^{-3} = g_1 - (u^{-1}-1)(u^{-2}+1)e_{d,1} + (u^{-1}-1)(u^{-2}+1)e_{d,1}g_1$. Hence: $\mathrm{tr}_d(g_1^{-3}) = z - (u^{-1}-1)(u^{-2}+1)\zeta_{d,S} + (u^{-1}-1)(u^{-2}+1)z$. Moreover $\epsilon(\sigma_1^{-3}) = -3$. Then we obtain:

$$\Delta_{d,S}(-\mathrm{T}) = D(\sqrt{\lambda})^{-3}\Big[(u^{-3}-u^{-2}+u^{-1})z - (u^{-3}-u^{-2}+u^{-1}-1)\zeta_{d,S}\Big],$$

where we recall that $D = \frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}}$.

5.5.3 A Cubic Skein Relation for $\Delta_{d,S}$

Let L_+, L_-, L_0 be diagrams of oriented links, which are all identical, except near one crossing, where they differ by the ways indicated in Fig. 5.1. We shall try to establish a skein relation satisfied by the invariant $\Delta_{d,S}$. Indeed, by the Alexander theorem we may assume that L_+ is in braided form and that $L_+ = \widehat{\beta\sigma_i}$ for some $\beta \in B_n$. Also that $L_- = \widehat{\beta\sigma_i^{-1}}$ and that $L_0 = \widehat{\beta}$. Apply now relation (5.17) for the g_i^{-1} in the expression below, noting that $\epsilon(\beta\sigma_i^{-1}) = \epsilon(\beta) - 1$ and $\epsilon(\beta\sigma_i) = \epsilon(\beta) + 1$:

$$\begin{aligned}
\Delta_{d,S}(L_-) &= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta\sigma_i^{-1})}\mathrm{tr}_d(\flat_{d,n}(\beta)g_i^{-1}) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1}\big[\mathrm{tr}_d(\flat_{d,n}(\beta)g_i) - (u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}) \\
&\quad + (u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}\,g_i)\big] \\
&= \frac{1}{\lambda}\Delta_{d,S}(L_+) - D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1}(u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}) \\
&\quad + D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1}(u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}\,g_i).
\end{aligned}$$

The problem is that the algebra words $\flat_{d,n}(\beta)e_{d,i}$ and $\flat_{d,n}(\beta)e_{d,i}\,g_i$ do not have a natural lifting in the braid groups, even if we break the $e_{d,i}$'s according to (5.4). This was not the case in [JL09], where we were dealing with framed braids and all algebra generators had natural liftings in the framed braid groups. Yet, we have in the algebra $\mathrm{Y}_{d,n}$ the following 'closed' relation (compare with [Fun95]).

Lemma 5 *The generators g_i of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}$ satisfy the cubic relations*:

$$g_i^3 = -ug_i^2 + g_i + u \tag{5.26}$$

Equivalently,

$$g_i^{-1} = u^{-1}g_i^2 + g_i - u^{-1} \tag{5.27}$$

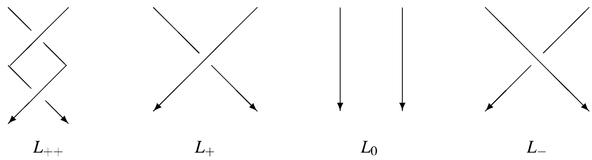

Fig. 5.1 L_{++}, L_{+}, L_0 and L_{-}

Proof From Lemma 4 we find the relation $g_i^3 = g_i + (u-1)e_{d,i}g_i - (u-1)e_{d,i}g_i^2$. Substituting (5.3) and replacing the expression $(u-1)e_{d,i} - (u-1)e_{d,i}g_i$ by the expression $g_i^2 - 1$ we arrive at the stated cubic relation. □

We then have the following result.

Proposition 2 *The invariant $\Delta_{d,S}$ satisfies the following cubic skein relation*:

$$\sqrt{\lambda}\,\Delta_{d,S}(L_-) = \frac{1}{\lambda u}\,\Delta_{d,S}(L_{++}) + \frac{1}{\sqrt{\lambda}}\,\Delta_{d,S}(L_+) - \frac{1}{u}\,\Delta_{d,S}(L_0). \tag{5.28}$$

Proof By the same reasoning as above we may assume that $L_0 = \widehat{\beta}$ for some $\beta \in B_n$. Also that $L_+ = \widehat{\beta\sigma_i}$, $L_{++} = \widehat{\beta\sigma_i^2}$ and $L_- = \widehat{\beta\sigma_i^{-1}}$. Apply now (5.27) from Lemma 5 in the expression below, noting that $\epsilon(\beta\sigma_i^{-1}) = \epsilon(\beta) - 1$, $\epsilon(\beta\sigma_i) = \epsilon(\beta) + 1$ and $\epsilon(\beta\sigma_i^2) = \epsilon(\beta) + 2$.

$$\begin{aligned}
\Delta_{d,S}(L_-) &= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta\sigma_i^{-1})}\mathrm{tr}_d(\flat_{d,n}(\beta)g_i^{-1}) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1} \\
&\quad \times \left[u^{-1}\,\mathrm{tr}_d(\flat_{d,n}(\beta)g_i^2) + \mathrm{tr}_d(\flat_{d,n}(\beta)g_i) - u^{-1}\,\mathrm{tr}_d(\flat_{d,n}(\beta))\right] \\
&= \frac{1}{(\sqrt{\lambda})^3 u}\Delta_{d,S}(L_{++}) + \frac{1}{\lambda}\Delta_{d,S}(L_+) - \frac{1}{\sqrt{\lambda}u}\Delta_{d,S}(L_0).
\end{aligned}$$

□

5.5.4 An Isotopy Invariant from tr_∞

In this subsection we extend the values of the invariants $\Delta_{d,S}$ to the adelic context. By (5.14) the braid group B_n is represented in $\mathrm{Y}_{\infty,n} = \varprojlim_{d\in\mathbb{N}^\sim} \mathrm{Y}_{d,n}$ via the map $\flat_{\infty,n} = \varprojlim_{d\in\mathbb{N}^\sim} \flat_{d,n}$. Further, by Theorem 2, elements in $\mathrm{Y}_{\infty,n}$ map, via the Markov trace $\mathrm{tr}_{\infty,n} = \varprojlim_{d\in\mathbb{N}^\sim} \mathrm{tr}_{d,n}$, in the ring $\varprojlim_{d\in\mathbb{N}^\sim} R[X_d]$, where $R = \mathbb{C}[z]$.

For any $d|d'$, now, the connecting ring epimorphism $\xi_d^{d'}$ (recall (5.15)) yields a connecting epimorphism $\Xi_d^{d'}$ from the ring of rational functions $\mathbb{C}(z, X_{d'})$ to the ring of rational functions $\mathbb{C}(z, X_d)$.

Lemma 6 *The following diagram is commutative.*

$$
\begin{tikzcd}
\mathcal{L} \arrow[r, "\Delta_{d',S}"] \arrow[d, "\mathrm{Id}"'] & \mathbb{C}(z, X_{d'}) \arrow[d, "\Xi_d^{d'}"] \\
\mathcal{L} \arrow[r, "\Delta_{d,S}"'] & \mathbb{C}(z, X_d)
\end{tikzcd}
\tag{5.29}
$$

We shall further denote by R_∞ the field of fractions of $\varprojlim_{d\in\mathbb{N}^\sim} R[\mathrm{X}_d]$. Taking now inverse limits in the diagram of Lemma 6 we obtain the map $\Delta_{\infty,S} := \varprojlim_{d\in\mathbb{N}^\sim} \Delta_{d,S}$ and we have the following.

Theorem 6 *If for all d the set X_d satisfies the E-condition, then the map*

$$
\begin{aligned}
\Delta_{\infty,S} : \mathcal{L} &\longrightarrow R_\infty \\
\widehat{\alpha} &\mapsto (\Delta_{d,S}(\widehat{\alpha}), \Delta_{d',S}(\widehat{\alpha}), \ldots)
\end{aligned}
$$

for any $\alpha \in \bigcup_n B_n$ is an isotopy invariant of oriented links in S^3. Moreover:

$$
\begin{aligned}
\Delta_{\infty,S}(\widehat{\alpha}) &= \left(\frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}}\right)^{n-1} (\sqrt{\lambda})^{\epsilon(\alpha)} \mathrm{tr}_\infty(\flat_{\infty,n}(\alpha)) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)} \mathrm{tr}_\infty(\flat_{\infty,n}(\alpha)).
\end{aligned}
$$

Proof By Lemma 3 we have non-trivial solutions of the E-system in the adelic context. Let now $\beta, \alpha \in \bigcup_n B_n$ be Markov equivalent braids. Then, any isotopy invariant agrees on the closures $\widehat{\beta}$ and $\widehat{\alpha}$. So, $\Delta_{d,S}(\widehat{\beta}) = \Delta_{d,S}(\widehat{\alpha})$, $\Delta_{d',S}(\widehat{\beta}) = \Delta_{d',S}(\widehat{\alpha})$, etc. Hence: $\Delta_{\infty,S}(\widehat{\alpha}) = \Delta_{\infty,S}(\widehat{\beta})$. Moreover, we have:

$$
\begin{aligned}
\Delta_{\infty,S}(\widehat{\alpha}) &= (\Delta_{d,S}(\widehat{\alpha}), \Delta_{d',S}(\widehat{\alpha}), \ldots) \\
&= (D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)} \mathrm{tr}_d(\flat_{d,n}(\alpha)), D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)} \mathrm{tr}_{d'}(\flat_{d',n}(\alpha)), \ldots) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)} (\mathrm{tr}_d(\flat_{d,n}(\alpha)), \mathrm{tr}_{d'}(\flat_{d',n}(\alpha)), \ldots) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)} \mathrm{tr}_\infty(\flat_{\infty,n}(\alpha)).
\end{aligned}
$$

□

The link invariant $\Delta_{\infty,S}$ is an adelic extension of the Jones polynomial.

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Chapter 6
Legendrian Grid Number One Knots and Augmentations of Their Differential Algebras

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Abstract In this article we study the differential graded algebra (DGA) invariant associated to Legendrian knots in tight lens spaces. Given a grid number one diagram for a knot in $L(p,q)$, we show how to construct a special Lagrangian diagram suitable for computing the DGA invariant for the Legendrian knot specified by the diagram. We then specialize to $L(p,p-1)$ and show that for two families of knots, the existence of an augmentation of the DGA depends solely on the value of p.

6.1 Introduction

Differential graded algebras have been associated to Legendrian knots in a variety of different contact manifolds, including the standard tight $\mathbb{R}^3$, S^3, and lens spaces $L(p,q)$ [Che02, Sab03, Lic09, NT04]. These algebras may be computed combinatorially from a Lagrangian projection of the knot, and the equivalence class of the algebra is an invariant of Legendrian knot type. These algebras are a combinatorial interpretation of the relative contact homology developed by Eliashberg, Givental, and Hofer, which is generally difficult to compute [EGH00]. In the last few years, attempts have been made to compute these algebras from the front, rather than the Lagrangian, projection, and also to extract more tractable invariants from the DGAs.

Although the Lagrangian projection is the most natural for computing the DGA, it has significant drawbacks relative to the front projection. Lagrangian projections admit only a weak Reidemeister theorem, and it is computationally intensive to determine whether or not a given picture is actually the projection of a Legendrian knot. In contrast, front projections suffer from neither of these features, but they are less geometrically natural for computing the DGA. For knots in $\mathbb{R}^3$ and the solid

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*, Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_6,

torus, this difficulty was addressed by Ng, who used "resolution" to compute the DGA directly from a front projection [Ng03]. The results in this paper are part of a program to develop an analogous technique for Legendrian knots in lens spaces.

The front projection of a Legendrian knot in a lens spaces is its projection to a Heegaard torus, and front projections are encoded combinatorially by toroidal grid diagrams as described in [BG08]. As in the Euclidean case, front projections in lens spaces are more easily manipulated than are Lagrangian projections. We focus on the case of grid number one knots (defined in Sect. 6.2.2), and we show that for these knots, a grid diagram suffices to determine a labeled Lagrangian projection with a specialized form. Grid number one knots are of particular interest because of their relationship to the Berge Conjecture, which characterizes knots in S^3 which have lens space surgeries [BGH07, Hed07, Ras07]. Topologically, a grid number one knot is a particular kind of bridge number one knot with respect to a Heegaard torus in $L(p,q)$. However, we adopt the perspective of [BG08]: any grid diagram specifies a particular Legendrian isotopy class within the topological isotopy class.

In $\mathbb{R}^3$, Chekanov used the linearized homology of the DGA to distinguish a pair of non-isotopic knots with identical classical invariants [Che02]. The existence of the linearized homology relies on whether the DGA can be augmented, and the existence of augmentations is itself an invariant of the equivalence type. Although the existence of augmentations is algorithmically decidable, the computation time is generally exponential in the number of generators.

As an application of the construction described above, we determine the (non)-existence of augmentations of the DGA for several families of grid number one knots in $L(p, p-1)$. These theorems, which are stated precisely in the next section, follow in the footsteps of other efforts to detect augmentations of Legendrian DGAs without computing the full differential. For example, the relationship between augmentations and rulings of the front projection for knots in $\mathbb{R}^3$ has been extensively studied [FI04, Fuc03, NS06, Sab05]. Although the current results apply only to particular families of knots, these examples suggest a framework for a more general approach to this problem.

6.1.1 Main Results

In order to state the main results precisely, we establish some notation that will be used throughout this article. A grid number one diagram for $L(p,q)$ can be viewed as a row of p boxes, two of which contain basepoints. For details on how such a diagram specifies a Legendrian knot, see Sect. 6.2.2. If s is the number of boxes separating the two basepoints, define $\upsilon(s)$ by

$$s + (\upsilon(s))q \equiv 0 \bmod p \quad \text{and} \quad 1 < \upsilon(s) < p-1.$$

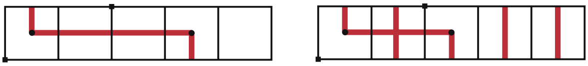

Fig. 6.1 Two diagrams for $K(5, 2, 3)$. When s is the length (measured in boxes) of the horizontal segment connecting the two basepoints, $v(s)$ is the length of the vertical segment in the rectilinear diagram with only southwest and northeast corners

Ignoring orientation, the knots associated to a pair of basepoints separated by s or $p - s$ boxes are isotopic, so define

$$h = \begin{cases} s & \text{if } s + v(s) < p \\ p - s & \text{if } s + v(p - s) > p. \end{cases}$$

Denote the knot specified by this grid diagram by $K(p, q, h)$. As shown in [Ras07], $K(p, q, h)$ is primitive if and only if $gcd(h, p) = 1$. Given a grid number one diagram for a primitive knot in $L(p, q)$ with $q \neq 1$, we construct a Lagrangian diagram for a knot in the associated Legendrian isotopy class.

Let $gcd(q - 1, p) = k$. Theorem 5, which is proved in Sect. 6.3.2, states that the crossings of the Lagrangian diagram are in one-to-one correspondence with the set of positive integers

$$\{x \mid x < h \text{ and } k|x\} \cup \{y \mid y \leq v \text{ and } k|y\}.$$

This shows that if $h, v < k$, then there is a Lagrangian projection of $K(p, q, h)$ with no crossings, so the algebra $\mathcal{A}(K(p, q, h))$ is isomorphic to the ground field $\mathbb{Z}_2$.

In contrast, the theorem implies that if $gcd(q - 1, p) = 1$, then there is knot in the specified Legendrian isotopy class whose Lagrangian projection has $h + v + 1$ crossings. It follows that the DGA $\mathcal{A}(K(p, q, h), \partial)$ is a tensor algebra on $2(h + v + 1)$ generators, and the ancillary data needed to determine the differential may also be computed from the numerical data associated to the grid diagram (Sect. 6.3.2). In Sect. 6.4, we apply this construction to show that in special cases, the existence of augmentations of the DGA can be deduced directly from the original grid diagram:

Theorem 1 *Let $K = K(p, p - 1, 1)$ be a Legendrian grid number one knot in $L(p, p - 1)$ for $gcd(p - 2, p) = 1$. Then the homology of $(\mathcal{A}(K), \partial)$ is a tensor algebra with two generators. Furthermore, the map sending both generators of $\mathcal{A}(K)$ to 0 is an augmentation.*

Theorem 2 *Let $K = K(p, p - 1, 2)$ be a Legendrian grid number one knot in $L(p, p - 1)$ for $gcd(p - 2, p) = 1$. Then $(\mathcal{A}(K), \partial)$ has an augmentation if and only if $p \equiv 3 \bmod 12$ or $p \equiv 9 \bmod 12$.*

6.1.2 Conventions and Organization

Nearly every orientation convention imaginable for lens spaces exists in the literature. Following [GS99], we view the lens space $L(p,q)$ as the result of $\frac{-p}{q}$ surgery on the unknot in S^3. With this choice, the lens spaces $L(p,1)$ are smooth S^1 bundles over S^2, and the combinatorial formulation of the DGA for Legendrian knots in these spaces is due to Sabloff [Sab03]. The invariant described in Sect. 6.2.4 applies to $L(p,q)$ with $q \neq 1$. Furthermore, we make use of the canonical correspondence between grid diagrams and toroidal front projections which is described in [BG08], but our use of "grid diagram" agrees with the authors' use of "dual grid diagram" in [BG08]. Throughout the paper, the ground field is $\mathbb{Z}_2$.

The next section has a brief introduction to augmentations, the universally tight contact structure on lens spaces, grid diagrams, and the Legendrian contact homology DGA for knots in lens spaces. In Sect. 6.3 we describe how to construct the Lagrangian projection of a knot from a grid number one diagram. Finally, in Sect. 6.3.2 we apply this construction to special classes of knots in $L(p,p-1)$ and prove Theorems 1 and 2.

6.2 Background

This section briefly reviews augmentations, grid diagrams, and the DGA for primitive knots in lens spaces. We refer the reader to [Ng03, BGH07], and [Lic09] for more details.

6.2.1 Contact Lens Spaces

View S^3 as the unit sphere in $\mathbb{C}^2$ with polar coordinates:

$$S^3 = \{(r_1,\theta_1,r_2,\theta_2) \mid r_1^2 + r_2^2 = 1\}.$$

These coordinates suggest a genus one Heegaard splitting of S^3 along the torus $r_1 = r_2 = \frac{1}{\sqrt{2}}$. Define $F_{p,q} : S^3 \to S^3$ by

$$F_{p,q}(r_1,\theta_1,r_2,\theta_2) = \left(r_1, \theta_1 + \frac{2\pi q}{p}, r_2, \theta_2 + \frac{2\pi}{p}\right).$$

The map $F_{p,q}$ is an automorphism of S^3 with order p, and the quotient of S^3 by the equivalence induced by $F_{p,q}$ is the lens space $L(p,q)$. Since $F_{p,q}$ preserves the r_i coordinates in S^3, the lens space inherits a genus one Heegaard splitting whose core curves C_1 and C_2 are the images of the curves $r_1 = 0$ and $r_2 = 0$ in S^3.

The standard tight contact structure on S^3 is induced by the one-form

$$r_1^2 d\theta_1 + r_2^2 d\theta_2.$$

For $0 < r_1 < 1$, the Reeb vector field is the constant vector field $\langle 1, 1, 0\rangle$ with respect to the basis $\{d\theta_1, d\theta_2, dr_1\}$ on $T(T^2 \times (0, 1))$. Thus the set of Reeb orbits consists of curves with slope $\frac{d\theta_2}{d\theta_1} = 1$ on each torus of fixed r_1, together with the two core curves $r_1 = 0$ and $r_2 = 0$. The *Lagrangian projection* of S^3 is the orbit space of S^3 as an S^1 bundle over S^2.

The map $F_{p,q}$ preserves the contact structure on S^3, so $L(p, q)$ inherits a tight contact structure from its universal cover. Throughout this article, we will assume that $L(p, q)$ is equipped with this universally tight contact structure, and we suppress it from the notation. The Lagrangian projection of $L(p, q)$ is again S^2, and when $q = 1$, this contact form induces a smooth S^1 bundle structure on $L(p, 1)$. In contrast, when $q > 1$, the images of the core curves C_1 and C_2 are orbifold points in the Lagrangian projection. In this case, the covering map $S^3 \to L(p, q)$ factors as a composition of cyclic covers

$$S^3 \to L(k, 1) \to L(p, q),$$

where $k = gcd(q - 1, p)$. The ramified points of the Lagrangian projection of $L(p, q)$ will have order $\frac{p}{k}$, and we identify these with the south and north poles of S^2. We will always assume that the knot K lives in $L(p, q) - (C_1 \cup C_2)$.

6.2.2 Grid Diagrams

The *front projection* of a Legendrian knot $K \subset L(p, q)$ is its projection to the genus-one Heegaard surface inherited from S^3. The knot K is completely determined by its front projection, and the Legendrian isotopy class of K may be encoded combinatorially by a grid diagram. In this section we introduce grid diagrams for links in S^3 and grid number one diagrams for knots in $L(p, q)$. The relationship between a knot $K \subset L(p, q)$ and its $F_{p,q}$-preimage $\tilde{K} \subset S^3$ plays an important role in the construction of the DGA $(\mathcal{A}(K), \partial)$, and grid diagrams are a useful tool for understanding this relationship.

In S^3, parameterize the $r_1 = \frac{1}{\sqrt{2}}$ torus by θ_i coordinates, where $0 \le \theta_i < 2\pi$ for $i = 1, 2$. Decorate this torus with curves satisfying $\theta_i = \frac{2n\pi}{p}$ for $n \in \{0, 1, \ldots, p-1\}$ and $i = 1, 2$. Cutting the torus along the curves $\theta_i = 0$ yields a square divided into p^2 boxes which are arrayed in p rows and p columns. Although the diagram is drawn as a planar object, we retain the identifications $(t, 0) \sim (t, 1)$ and $(0, t) \sim (1, t)$ in order to simultaneously view the planar grid diagram as a decorated Heegaard torus. Finally, add $2p$ basepoints to the diagram, so that each column and each row contains exactly two basepoints. The decorated square is called a *grid number p grid diagram*. See Fig. 6.2 for an example.

If a grid number p grid diagram $\tilde{\Sigma}$ is invariant under $F_{p,q}$, then its quotient under the equivalence induced by $F_{p,q}$ is called a *grid number one grid diagram* Σ. The bottom row of $\tilde{\Sigma}$ is a fundamental domain for the action of $F_{p,q}$, so we may

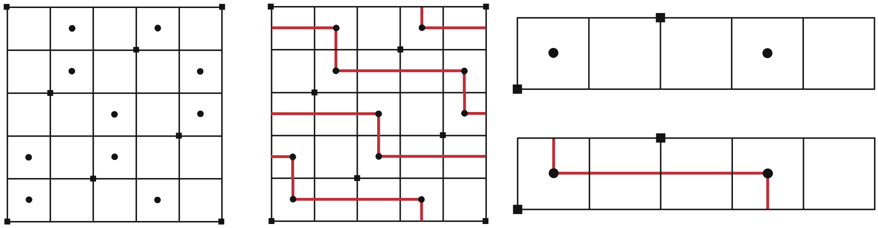

Fig. 6.2 *Left*: An $F_{5,2}$-invariant grid number 5 diagram for $\tilde{K} \subset S^3$. *Center*: A compatible rectilinear diagram for $\tilde{K} \subset S^3$. *Right*: A grid number one diagram for $K(5,2,3)$, together with a compatible rectilinear projection. In the grid number one diagrams, the rectangles on the boundary indicate the gluing which yields a Heegaard torus for $L(5,2)$

parameterize Σ by θ_i coordinates with $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_2 < \frac{2\pi}{p}$. As in the case of a grid number p diagram in S^3, a grid number one diagram is a planar depiction of a torus; to recover a Heegaard torus for $L(p,q)$, identify $(t,0)$ with $(t + \frac{2q\pi}{p}, \frac{2\pi}{p})$ and $(0,t)$ with $(1,t)$. These identifications yield a torus decorated by two connected curves which intersect p times. The complement of the images of the vertical curves from $\tilde{\Sigma}$ is connected in Σ, and it is referred to as the *column* of the grid diagram. Similarly, the complement of the images of the horizontal curves is referred to as the *row* of the grid diagram.

A grid diagram specifies a link in the associated three-manifold. Connect each pair of basepoints in the same row or same column by a linear segment. Observe that for each such pair of basepoints, one may choose between two possible line segments on the torus. The result of any set of choices is called a *rectilinear diagram compatible with the grid diagram*. (See Fig. 6.2.) Viewing the rectilinear diagram as a curve in the three-manifold, push the interior of each horizontal curve into the solid torus defined by $r_1 > \frac{1}{\sqrt{2}}$ and push the interior of each vertical curve into the solid torus defined by $r_1 < \frac{1}{\sqrt{2}}$. The resulting embedded curve intersects the Heegaard torus only at the original basepoints of the grid diagram.

The topological isotopy class of the knot constructed this way is independent of the choice of rectilinear projection, so one may refer to a *grid diagram for a knot K*. Given a grid diagram Σ for $K \subset L(p,q)$, one may construct a grid diagram for its $F_{p,q}$-preimage $\tilde{K} \subset S^3$ using p copies of Σ. This construction is indicated in Fig. 6.2, and we refer the reader to [BGH07] for a fuller treatment. Following Rasmussen, we say that a knot in $L(p,q)$ is *primitive* if it generates $H_1(L(p,q))$. The knot $K \subset L(p,q)$ is primitive if and only if $\tilde{K} \subset S^3$ has one component, and in the notation from Sect. 6.1.1, $K(p,q,h)$ is primitive if and only if h and p are relatively prime [Ras07].

A grid diagram can be interpreted as specifying not simply a topological isotopy class of knot, but rather a Legendrian isotopy class of knot in the standard contact S^3 or $L(p,q)$ [BG08, OST08]. Proposition 3.3 of [BG08] asserts that any curve on Σ with $\frac{d\theta_2}{d\theta_1}$ slope in $(-\infty, 0)$ which is smoothly embedded away from semi-cubical cusps or transverse double points is the front projection of some Legendrian knot in $L(p,q) - (C_1 \cup C_2)$. The knot can be recovered from this projection because the

Legendrian condition implies that the slope of the front projection of K determines the r_1 coordinate:

$$\frac{d\theta_2}{d\theta_1} = \frac{-r_1^2}{1 - r_1^2}.$$

Theorem 3 [BG08] *Any rectilinear diagram compatible with a grid diagram is isotopic on Σ to the front projection of some Legendrian knot K in $L(p, q)$. The Legendrian isotopy class of K is independent of the choice of compatible rectilinear diagram.*

6.2.3 Differential Graded Algebras

Let V be the vector space generated by $\{v_i\}_{i=1}^n$. Then the *tensor algebra* generated by the v_i is

$$T(v_1, \ldots, v_n) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

If V is graded by some cyclic group, extend the grading by setting $|v_1 v_2| = |v_1| + |v_2|$. If $\partial : \mathcal{A} \to \mathcal{A}$ is a graded degree -1 map which satisfies $\partial^2 = 0$, then the pair $(\mathcal{A}, \partial)$ is a *semi-free differential graded algebra* (DGA).

Definition 1 An *augmentation* of a DGA is a graded algebra homomorphism $\epsilon : \mathcal{A} \to \mathbb{Z}_2$ such that $\epsilon(1) = 1$, $\epsilon \circ \partial = 0$ and $\epsilon(x) = 0$ if $|x| \neq 0$.

The natural notion of equivalence on DGAs is that of stable tame isomorphism. (For a definition, see [Che02, Sab03], or [Lic09].) Equivalent DGAs have isomorphic homology, and the existence of augmentations of a DGA $(\mathcal{A}, \partial)$ is also an invariant of its equivalence type. Furthermore, the number of augmentations is an invariant of the equivalence type, up to a power of two.

6.2.4 The Lagrangian DGA for $K \subset L(p, q)$

In this section we introduce the Lagrangian DGA for primitive Legendrian knots in lens spaces $L(p, q)$ with $q \neq 1$. The algebra is generated by Reeb chords with both endpoints on K, so each crossing in the Lagrangian projection of K corresponds to a pair of complementary Reeb chords in $L(p, q)$.

The DGA $(\mathcal{A}(K), \partial)$ is defined in purely combinatorial terms from a labeled Lagrangian projection of K, but the reader may find it helpful to know that the proof of invariance relies on the relationship between $K \subset L(p, q)$ and a cyclic cover $\tilde{K} \subset M_k$. If $gcd(q - 1, p) = 1$, the cyclic cover M_k is the universal cover of

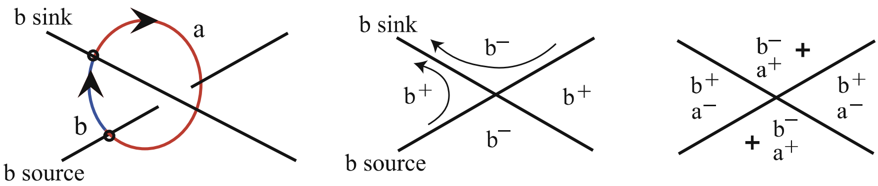

Fig. 6.3 The orientation of the chords determines the quadrant labels in the Lagrangian projection. The *left* figures shows two strands of K intersecting a fixed Reeb orbit, and the other diagrams indicated the associated labeling in the Lagrangian projection. The plus signs in the *right* figure indicate that a is the preferred chord

$L(p,q)$. If $gcd(q-1,p)=k>1$, then M_k is the lens space $L(k,1)$, which is a $\frac{p}{k}$-to-one cover of $L(p,q)$. The results in this paper are stated for primitive knots in $L(p,q)$, but they generalize to any knot which generates an order $\frac{p}{k}$ subgroup of $\pi_1(L(p,q))$. For a fuller description of this relationship, see [Lic09].

Given $K \subset L(p,q)$, denote by Γ the Lagrangian projection of K. The preimage of each double point in Γ consists of a pair of complementary Reeb chords in $L(p,q)$, and we (arbitrarily) designate one chord in each pair as *preferred*. At each crossing, Γ divides a neighborhood of the crossing into four quadrants, and we will label these quadrants so as to identify the preferred chord. (See Fig. 6.3.) Because the Reeb orbits are integral curves of the Reeb vector field, the Reeb chords are naturally oriented. At a fixed crossing, each oriented chord x assigns "source" and "sink" labels to the two arcs of K which project to the crossing, and we label a quadrant by "x^+" if traveling along the source curve to the sink curve in Γ orients the quadrant they bound positively. Similarly, we label a quadrant x^- if traveling source-to-sink orients the bounded quadrant negatively. If x is the preferred generator at a crossing, then we decorate the x^+ quadrants with an additional "+".

Suppose that the Lagrangian projection of K has m crossings, and let a_i and b_i be the complementary Reeb chords associated to the i^{th} crossing of Γ. We define $\mathcal{A}(K)$ to be the tensor algebra generated over $\mathbb{Z}_2$ by the $\{a_i\}$ and $\{b_i\}$:

$$\mathcal{A}(K) = T(a_1, b_1, \ldots, a_m, b_m).$$

The remainder of this section is devoted to defining the boundary map $\partial : \mathcal{A}(K) \to \mathcal{A}(K)$. In order to do so, we will associate a rational-valued *defect* to each component of $S^2 - \Gamma$. A Lagrangian projection which is decorated with defects in each region and "+" signs to denote the preferred chords at each crossing is called a *labeled Lagrangian diagram* for K. We defer a description of the grading on $\mathcal{A}(K)$ until Sect. 6.4.1, as it is not necessary to understand Sect. 6.3.

6.2.4.1 The Defect

We will simultaneously think of the generators of $\mathcal{A}(K)$ as formal symbols and as Reeb chords in $L(p,q)$. As such, we can assign a *length* to each generator of the algebra:

Definition 2 The *length* of a generator x is the length of the associated Reeb chord, measured as a fraction of the orbital period. Denote the length of x by $l(x) \in (0, 1)$.

As above, let $gcd(q - 1, p) = k$, and let M_k denote the $\frac{p}{k}$-fold cyclic cover of $L(p, q)$. The manifold M_k is either $L(k, 1)$, if $k > 1$, or S^3, if $k = 1$. The contact form on M_k induces a curvature 2-form on its Lagrangian base space S^2. This copy of S^2 is a $\frac{p}{k}$-to-one branched cover of the Lagrangian base space of $L(p, q)$, and the latter two-sphere inherits an area form from its covering space. Normalize the induced form so that the Lagrangian projection of $L(p, q)$ has area equal to $\frac{k^2}{p}$. Let $a(R)$ denote the area of a region $R \in S^2 - \Gamma$.

Suppose that x_m is the preferred generator at a crossing where the region R has a corner. If R fills the quadrant labeled x_m^+, define $\epsilon(m) = 1$, and if R fills the quadrant labeled x_m^-, define $\epsilon(m) = -1$.

Definition 3 Let R be a component of $S^2 - \Gamma$ with m corners. The *defect* of R is given by

$$n(R) = -a(R) + \sum_{i=1}^{m} \epsilon(i) l(x_i),$$

where the sum is taken over the preferred generators at crossings where R has a corner.

Definition 3 is closely related to Sabloff's definition of defect for Legendrian knots in smooth lens spaces, and we offer the following geometric perspective on $n(R)$ [Sab03]. The boundary of R lifts to a curve $\gamma \in L(p, q)$ which is composed of alternating Legendrian segments and preferred Reeb chords. If R is disjoint from the poles of S^2, the defect of R is the winding number of γ around the Reeb orbit with respect to an appropriate trivialization of the S^1 bundle. If R contains one of the poles of S^2, then it lifts to a region $\tilde{R}$ in the Lagrangian projection of $(M_k, \tilde{K})$. One may similarly associate a winding number $n(\tilde{R})$ to this region, and $n(R) = \frac{k}{p} n(\tilde{R})$. It follows from [Sab03] and [Lic09] that a region disjoint from the poles will have integral defect, and the defect of a region containing a pole will lie in $\frac{k}{p}\mathbb{Z}$.

If $f : (D^2, \partial D^2) \to (S^2, \Gamma)$ is smooth on the interior of D^2, extend the definition of defect to $n(f(D^2))$ by summing the defects of the regions in $f(D^2)$, counted with multiplicity.

Remark 1 In [Lic09], the defect is defined in terms of the lift of K to $\tilde{K} \subset M_k$. The area term in the present definition replaces the curvature term seen there, and the equivalence of the definitions follows from the identification of the curvature form on S^2 with the Euler class of M_k as a unit sphere bundle [Gei08].

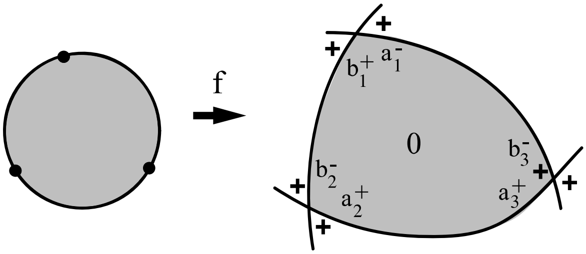

Fig. 6.4 The a_2-defect of f is $n(f(D^2)) + 1 - 1 = n(f(D^2)) = 0$, so the disc shown contributes the term $w(f, a_2) = b_3 a_1$ to ∂a_2. The same disc f also corresponds to $w(f, a_3) = a_1 b_2 \in \partial a_3$ and $w(f, b_1) = b_2 b_3 \in \partial b_1$

6.2.4.2 The Boundary Map

Definition 4 An *admissible disc* is a map $f : (D^2, \partial D^2) \to (S^2, \Gamma)$ of the disc with m marked points on its boundary which satisfies the following properties:

1. either f is an immersion or f fails to be an immersion only at points which map to the poles of S^2. In the latter case, f is diffeomorphic to $z \to z^{\frac{p}{k}}$ in a neighborhood of each singular point;
2. each marked point maps to a crossing of Γ.
3. f extends smoothly to ∂D^2 away from the marked points;
4. at each marked point, $f(D^2)$ fills exactly one quadrant.

Two admissible discs f and g are *equivalent* if there is a smooth automorphism $\phi : D^2 \to D^2$ such that $f = g \circ \phi$. As above, the defect of an admissible disc f is the sum of the defects of regions in its image, counted with multiplicity.

If f is an admissible disc which fills a quadrant marked with x^+, we associate to f the *boundary word* $w(f, x)$ and the *x-defect* $n_x(f)$:

- Moving counterclockwise around ∂D^2 from the point mapping to x^+, let y_i^- be the negative generator labels associated to the i^{th} corner of the image of D^2, where $1 \leq i \leq m - 1$. Then $w(f, x) = y_1 y_2 \ldots y_{m-1}$. See Fig. 6.4 for an example
- The x-defect $n_x(f)$ is computed from the defect of $f(D^2)$ by subtracting one for each y_i which is not a preferred generator and adding one if x is not a preferred generator.

Definition 5 Define $\partial : \mathcal{A} \to \mathcal{A}$ on generators of $\mathcal{A}$ by

$$\partial(x) = \sum_{f : n_x(f) = 0} w(f, x).$$

Extend ∂ to all of $\mathcal{A}$ via the Leibniz rule $\partial(ab) = (\partial a)b + a(\partial b)$.

Theorem 4 [Lic09] *If K is a primitive Legendrian knot in $L(p, q)$ for $q \neq 1$, the pair $(\mathcal{A}(K), \partial)$ is a DGA. The equivalence type of this DGA is an invariant of the Legendrian isotopy class of K.*

6.3 Converting Fronts to Lagrangian Projections

Given a grid diagram, Sect. 6.2.2 describes how to construct a curve which is the front projection of a knot in the Legendrian isotopy class indicated by the grid diagram. This front projection completely determines the knot, as the slope of the curve at each point recovers the coordinate lost in the projection. Abstractly, this implies that the grid diagram carries sufficient information to compute the DGA of the associated Legendrian knot, but the conversion from grid diagram to front projection to labeled Lagrangian diagram may be difficult in practice. Furthermore, although the grid diagram determines a unique Legendrian isotopy class of knot, isotopic front projections may correspond to knots whose Lagrangian projections vary tremendously; the choice of front projection therefore greatly affects the computability of the DGA. In this section we describe how to construct a relatively simple Lagrangian projection directly from a grid diagram for a Legendrian knot in $L(p,q)$.

Our approach is as follows: beginning with a grid number one diagram for $K \subset L(p,q)$, draw a special front projection compatible with the grid diagram. This front projection represents the choice of a fixed knot K_0 in the Legendrian isotopy class determined by the grid diagram. We parameterize the grid diagram for K in (θ_1, θ_2) coordinates, where $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_2 < \frac{2\pi}{p}$. The Lagrangian projection of $L(p,q)$ is a two-sphere, and we parameterize this by (ϕ, r_1), where $-\pi \leq \phi < \pi$ is the azimuthal coordinate. The r_1 coordinate corresponds to latitude on the sphere, and the north and south poles are the two ramified points of the covering map between the Lagrangian projections of $L(p,q)$ and its cyclic cover M_k. We will find it convenient to represent this S^2 as the rectangle $[-\pi, \pi] \times [0, 1]$, and we recover the sphere via the identification of each of the top and bottom edges to a point, as well as the further identification $(-\pi, t) \sim (\pi, t)$, which glues the left and right edges of the square.

In order to move between different projections, parameterize the Lagrangian projection of $L(p,q)$ so that the Reeb orbit on the Heegaard torus which passes through the point $\theta_1 = \theta_2 = 0$ maps to $\phi = 0$.

Lemma 1 *The θ_i and ϕ coordinates are related by the equation*

$$\phi = \frac{p}{k}(\theta_1 - \theta_2) \ mod\ 2\pi.$$

Proof Begin by considering a grid diagram for S^3, and recall that each Reeb orbit on the Heegaard torus is a curve with constant slope $\frac{d\theta_2}{d\theta_1} = 1$. Thus, holding $\theta_1 - \theta_2$ fixed determines a Reeb orbit, and consequently, a point in the Lagrangian projection of S^3.

Now recall that the action of $F_{p,q}$ on S^3 permutes a set of $\frac{p}{k}$ distinct Reeb orbits. The Reeb orbit on the Heegaard torus which passes through $(0,0)$ is identified in the quotient with the Reeb orbits passing through the points $\left(\frac{2ck\pi}{p}, 0\right)$ for $c \in \mathbb{Z}_k$. Since the r_1 coordinate is independent of ϕ and the θ_i, the formula follows. $\square$

Figure 6.6 provides an example illustrating Lemma 1.

The remainder of this section is divided into two parts. In the first, we describe the special front more precisely. In Sect. 6.3.2, we consider a knot K_0 with a special front projection and we show how features of the Lagrangian and front projections of K_0 correspond.

6.3.1 Special Front Projections

Label the boxes of the grid diagram from 0 to $p-1$ so that one basepoint appears in Box 0 and the other in Box h. Connect the two basepoints by a horizontal line whose length (measured in boxes) is h. Moving downward from the basepoint in Box h, draw a vertical line in the column of the grid diagram which connects the two basepoints and has length v. Add a new basepoint to the curve each time it passes through the center of a box until the curve consists of $h+v$ equal-length segments that meet at basepoints. Now allow each basepoint to slide along the anti-diagonal of its box so that the slopes of the line segments connecting successive pairs decrease strictly as one moves right from the basepoint in Box 0. See Fig. 6.5.

Observe that by keeping these perturbations small, the slopes of the formerly-horizontal segments can be held arbitrarily close to 0 and the slopes of the formerly-vertical segments can be held arbitrarily close to $-\infty$. Finally, replace a neighborhood of each basepoint with a curve that smoothly and strictly monotonically interpolates between the slopes of the line segments to either side. Since the resulting curve is smooth with negative slope, Proposition 3.3 of [BG08] implies that it is the front projection of a Legendrian knot K in $L(p,q)$.

6.3.2 The Lagrangian Projection Associated to a Special Front

Let K_0 be a Legendrian knot in $L(p,q)$ whose front projection has the form described in the previous section. For convenience, we will continue to describe the line segments as connecting basepoints except when specifically focusing on the short connecting curves introduce in the previous paragraph. Each of these line segments on the front corresponds to a subcurve of K_0 with a fixed r_1 coordinate. Thus, each subcurve maps to a horizontal curve on the Lagrangian projection. Since each of the line segments in the special front projection has a distinct slope, the corresponding horizontal curves in the Lagrangian projection are disjoint. Furthermore,

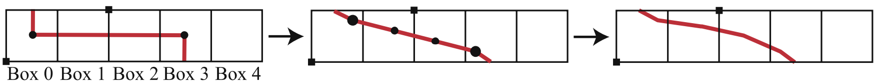

Fig. 6.5 Transforming a rectilinear diagram for $K(5,2,3)$ into a special front projection

because each line segment connects basepoints in adjacent boxes on the grid diagram, the Lagrangian projection of the corresponding subcurve satisfies

$$\frac{2c\pi}{k} + \epsilon' < \phi < \frac{2(c+1)\pi}{k} - \epsilon'$$

for some $c \in \mathbb{Z}_k$.

Each connecting curve on the special front projection lies in a small neighborhood of the anti-diagonal of some box of the grid diagram, so the Lagrangian projection of the corresponding subcurve of K_0 lies in a neighborhood of one of the vertical lines $\phi = \frac{2c\pi}{k}$ for $c \in \mathbb{Z}_k$. The connecting curve in Box 0 joins the line segment with the most negative slope to the line segment with the least negative slope; the image of this curve in the Lagrangian projection joins the $\phi = -\epsilon'$ endpoint of the bottom horizontal curve to the $\phi = \epsilon'$ endpoint of the top horizontal curve. We will refer to this as the *ascending curve*. Each of the other connecting curves on the special front joins the right endpoint of a segment to the left endpoint of a segment with a more negative slope; the corresponding *descending curve* on the Lagrangian projection joins the $\phi = \frac{2c\pi}{k} - \epsilon'$ endpoint of a horizontal line to the $\phi = \frac{2c\pi}{k} + \epsilon'$ endpoint of the horizontal line immediately below, for some $c \in \mathbb{Z}_k$. See Fig. 6.6.

From this description, we can extract the number of crossings of Γ and the number of connected components of $S^2 - \Gamma$:

Theorem 5 *The crossings of* Γ *are in one-to-one correspondence with the set of positive integers*

$$\{x \mid x < h \textit{ and } k|x\} \cup \{y \mid y \leq v \textit{ and } k|y\}.$$

The number of connected components of $S^2 - \Gamma$ *is two more than the number of crossings.*

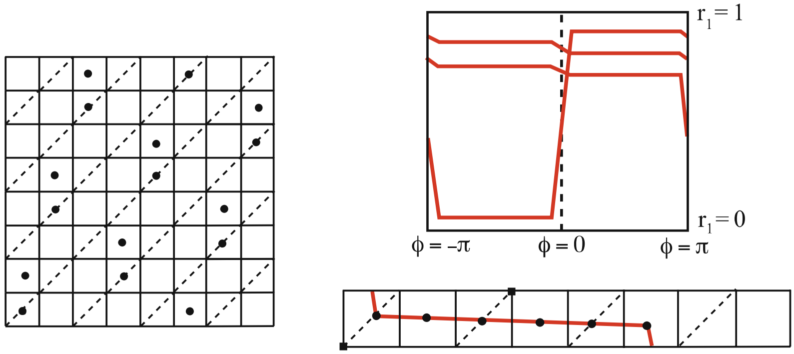

Fig. 6.6 *Left*: An $F_{8,3}$-invariant grid diagram for a knot in S^3. The *dotted lines* show 4 Reeb orbits which are identified in the quotient. *Right*: A special front projection of $K(8, 3, 5)$, together with a schematic Lagrangian projection

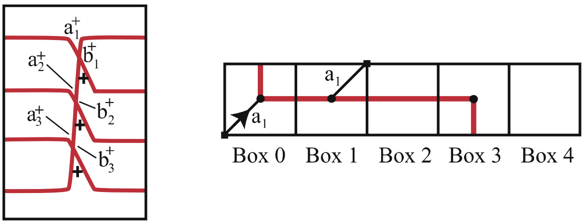

Fig. 6.7 *Left*: A schematic Lagrangian projection of $K(5,2,3)$, partially labeled. *Right*: Image of the Reeb chord a_1 on the front projection, which shows that $l(a_1) = \frac{1}{5}$

Corollary 1 When $gcd(q-1, p) = 1$, the Lagrangian projection of $K(p,q,h)$ has $h+v-1$ crossings, and there are $h+v+1$ connected components of $S^2 - \Gamma$.

Corollary 2 When $h, v < gcd(q-1, p)$, the Lagrangian projection of $K(p,q,h)$ has no crossings.

Proof of Theorem 5 In the complement of the ascending curve, we may parameterize Γ so that $dr_1 \leq 0$ and $d\phi > 0$. Thus, this portion of the Lagrangian projection of K_0 embeds in S^2 as a descending spiral. In order to determine the crossings of Γ, we count the number of times the ascending curve crosses this spiral. Observe that the ascending curve lies in a neighborhood of the line $\phi = 0$. Each crossing in the Lagrangian projection corresponds to a point on the spiral with $\phi = 0$, and this in turn corresponds to a connecting segment on the special front which appears in a box numbered ck for $c \in \mathbb{Z}_{\frac{p}{k}}$. The indexing set in the statement of Theorem 5 counts the number of times the front projection of K_0 passes through a box whose label is divisible by k. If Γ has no crossings, it divides S^2 into two regions; each time Γ spirals around S^2 adds a new crossing and a new complementary region. $\square$

The proof of Theorem 5 suggests a fuller description of the Lagrangian projection of K_0. If the diagram has no crossings, then the DGA is isomorphic to $\mathbb{Z}_2$, so consider the case when Γ has at least one crossing. Each of the regions of $S^2 - \Gamma$ which contains a pole has a single corner. Generically, each of the remaining regions of $S^2 - \Gamma$ has four corners, but this number is reduced by one for each adjacent polar region. It is also possible to completely label the Lagrangian diagram using data from the front projection.

As described in Sect. 6.2.4, each crossing in Γ corresponds to a pair of Reeb chords of $K_0 \in L(p,q)$ and, consequently, a pair of generators of $\mathcal{A}(K_0)$. We number the intersections of the Lagrangian diagram, counting from the top down. A neighborhood of each crossing is divided into north, east, south, and west quadrants by Γ. Define a_i to be the generator which corresponds to a "+" label on the north and south quadrants of the ith crossing. Similarly, let b_i be the generator which corresponds to a "+" label on the east and west quadrants of the ith crossing. We will refer to the generators a_i as *a-type* generators, and at each crossing, designate the a-type generator as preferred. See Fig. 6.7.

Assigning a defect to each region requires the lengths of the generators and the areas of the regions, both of which may be computed from a grid diagram. To compute the area of a given region, recall that the total area of the Lagrangian projection of $L(p,q)$ is normalized to be $\frac{k^2}{p}$. Assume that each connecting segment on the front projection of K_0 lies in small neighborhood of the center of its box on the grid diagram. This forces the constant-r_1 curves on the Lagrangian projection to lie in a small neighborhood of the two poles of S^2. The descending curve corresponding to the basepoint in Box h travels from a neighborhood of the north pole to a neighborhood of the south pole, whereas each of the other descending curves remains in an neighborhood of one of the poles. As a consequence, only the regions bounded to the left and right by the Box h descending curve will have more than negligible area. When $k=1$, these two regions coincide and the unique large region has area approximately $\frac{1}{p}$. When $k>1$, each of the two large regions has area approximately equal to some integral multiple of $\frac{k}{p}$. See Fig. 6.6.

To compute the length associated to each generator, it will be convenient to identify Reeb chords in $L(p,q)$ with their images on the Heegaard torus. In particular, we will denote by a_i either the Reeb chord in $L(p,q)$ or its projection to the Heegaard torus. Each Reeb chord has the same length as its front projection, and we may compute that latter by counting boxes in the grid diagram. The following proposition gives a formula for computing the length of the preferred chords.

Proposition 1 If s is the total number of crossings in the Lagrangian projection of K_0, define $B(j)$ by

$$B(j)=\begin{cases} kj & \text{if } j\le \frac{h}{k} \\ (-qk)(s+1-j) \bmod p & \text{if } j>\frac{h}{k}. \end{cases}$$

If x_j denotes the least positive integer such that

$$B(j)+(1-q)x_j\equiv 0 \bmod p,$$

then the length of the generator a_j is ϵ-close to $\frac{k}{p}x_j$. Furthermore, $l(b_j)=1-l(a_j)$.

Proof As noted above, each crossing occurs at a point where the ascending curve in the Lagrangian projection intersects a descending curve. The formula for $B(j)$ converts between two numbering systems: numbering a descending connecting segment by the crossing it projects to in Γ (its j label) and numbering it by the box it lies in on the grid diagram (its $B(j)$ label). The endpoints of the chord a_j front-project to basepoints in Box 0 and Box $B(j)$, and the orientation convention described above implies that a_j is oriented from Box $B(j)$ to Box 0. For $j\le\frac{h}{k}$, each increase in j corresponds to traveling k boxes to the right, which increases $B(j)$ by k. For larger values of j, each increase in $s+1-j$ corresponds to traveling up by k boxes, and each row change decreases the box index by q.

We may assume that each connecting segment in the front projection lies in an arbitrarily small neighborhood of the center of its respective box, so the length of a chord may be estimated by counting the number of up-one-row, right-one-column steps needed to travel from Box $B(j)$ to Box 0. The box index decreases by q with each up-one-row step, so x_j counts the number of diagonal box lengths between the two boxes.

Finally, we note that an entire Reeb orbit measures $\frac{p}{k}$ diagonal box lengths, so dividing x_j by $\frac{p}{k}$ yields the length of the generator a_j as a fraction of the orbital period. Since the chords a_j and b_j are complementary, the formula for the length of b_j follows. $\square$

We end this section with a brief comparison to Ng's resolution technique for Legendrian knots in $\mathbb{R}^3$. In [Ng03], Ng successfully reformulated Chekanov's algebra in terms of the front projection of Legendrian knot. This can be mimicked for null-homologous knots in lens spaces, but the geometric constraints imposed by representing a non-trivial homology class prevent this approach from being directly applied to the general case of knots in lens spaces. In the preceding section, we have instead tried to simplify the process of translating between different projections, identifying generators of the algebra with chords on the front projection but not computing the boundary map until after the Lagrangian projection is produced. Although it is possible to describe the loops on a grid diagram which correspond to the boundary of a disc counted by the differential, this description is not particularly useful for computational purposes. In the final section, however, we see that under special circumstances, such loops can play a useful role.

6.4 Augmentations of $(\mathcal{A}(K_0), \partial)$

In the previous section, we developed the correspondence between special front and Lagrangian projections. In this section, we apply these results to study the question of when $\mathcal{A}(K_0)$ has augmentations. Our approach relies on the grading on the DGA, which we introduce in Sect. 6.4.1. We show that when $gcd(q-1, p) = 1$, the existence of augmentations depends only on a subclass of words appearing in the boundary of the preferred generators. These words can be described in terms of certain loops on a grid diagram for $\tilde{K}_0$, the preimage of K_0 in S^3. In Sect. 6.4.4.2, we determine the existence of augmentations of $\mathcal{A}(K(p, p-1, 2))$ by analyzing the set of special loops. The DGA may still be computed from the Lagrangian projection described in Sect. 6.3.2 when $gcd(q-1, p) > 1$, but the computations in this section rely on the diagram having $h+v-1$ crossings and $h+v+1$ components of $S^2 - \Gamma$. In the remainder of this section, we restrict to the $gcd(q-1, p) = 1$ case, but we indicate which of the propositions generalize naturally.

6.4.1 The Grading

As noted above, we will restrict to the case when $gcd(q-1, p) = 1$.

A *capping path* for the generator x is a path η in Γ which is smooth away from the crossing associated to x, and which has the further property that at this crossing,

η turns a corner around a quadrant labeled by x^+. In the special Lagrangian diagrams, each a-type generator has two capping paths, and no b-type generator has a capping path. For each a_j, the capping paths positively bound discs which fill a quadrant marked a_j^+; let η denote the capping path whose associated disc lies in the complement of the south pole of S^2. Use this disc to define a rotation number $r(\eta)$ which counts the number of counter-clockwise rotations of the tangent vector η' in the disc $S^2 - \{\text{south pole}\}$. Assuming that the strands of Γ are orthogonal at each crossing, this rotation number takes values in $\mathbb{Z} - \frac{1}{4}$.

Letting $w_N(\eta)$ denote the winding number of η with respect to the north pole, a capping path is *admissible* if $w_N(\eta) \equiv 0 \bmod p$.

Definition 6 If a_j is a generator with an admissible capping path η, the grading of a_j is given by

$$|a_j| = 2\lceil r(\eta)\rceil - 2\frac{p-1}{p} w_N(\eta) - 1 + 4n(D_\eta).$$

The grading of b_j is given by

$$|b_j| = 3 - |a_j|.$$

Orient K_0, and denote by $r(K_0)$ and $n(K_0)$ the rotation number of K_0 and the defect of the disc bounded by K_0. The gradings above are well-defined up to $2(r(K_0)) - 4n(K_0)$ [Lic09, Sab03].

Remark The definition of an admissible capping path generalizes to the case $k > 1$ by replacing p with $\frac{p}{k}$; for the corresponding formulae for the grading, see [Lic09].

6.4.2 Special Boundary Discs

We will use Fuchs's characterization of augmentations [FI04]:

Given a homomorphism $\epsilon : \mathcal{A} \to \mathbb{Z}_2$, a disc which contributes a term to ∂x is *special* with respect to ϵ if $\epsilon(y_i) = 1$ for every y_i in $w(f, x)$. A graded homomorphism ϵ is an augmentation of $(\mathcal{A}, \partial)$ if for each generator x_j, the number of special boundary discs is even.

Lemma 2 *Any graded augmentation of $\mathcal{A}(K(p, q, h))$ sends every a-type generator to zero.*

We defer the proof of this lemma in order to first explain why it is helpful. The statement that every a-type generator vanishes under any graded augmentation implies that only boundary words written exclusively in b-type generators can be special. Such words only appear in the boundary of a-type generators:

Lemma 3 *If f is an admissible disc contributing a term to ∂x and $w(f, x)$ is written exclusively in b-type generators, then x is an a-type generator.*

Proof In the Lagrangian diagram for K_0, corners which preserve the orientation of K_0 are marked with a^+ and b^- labels. If γ is a path which traces out $f(\partial D^2)$, the orientation of γ relative to K_0 is preserved at every b^- corner. This implies that if $w(f, x)$ is written only in b-type generators, then x^+ must also preserve the orientation. □

Thus, the existence of an augmentation for $(\mathcal{A}, \partial)$ depends solely on the boundary of the a-type generators. Equivalently, the only discs that need to be considered are those in which every corner of $f(D^2)$ fills a corner labeled with a b^-. In Sect. 6.4.3, we will show that such discs can be identified with loops on the special front projection of $\tilde{K}_0 \subset S^3$.

We end this section with a proof of Lemma 2:

Proof Orient K_0 to bound a disc disjoint from the south pole. Then the rotation number of K_0 is $h + v$.

The defect of the disc bounded by K_0 is equal to its area, and this value may be bounded arbitrarily close to $\frac{v}{p}$ by making the connecting curves on the special front projection lie in a sufficiently small neighborhood of the centers of the boxes. According to the remark after Definition 6, this implies that $(\mathcal{A}(K_0), \partial)$ is graded by a cyclic group of order $2(h + v) - 4v = 2|h - v|$.

Recall that the a-type generators are exactly those with capping paths. It follows from Definition 6 that the grading of each a_i is the sum of an odd integer and $\frac{4v}{p}$. In order for the grading of an a-type generator to be congruent to 0, $\frac{4v}{p}$ would have to be equal to an odd integer, but this contradicts the assumption that $gcd(q-1, p) = 1$. Thus any graded augmentation sends each a_i to 0. □

6.4.3 Boundary Maps of a-Type Generators

As the discussion in Sect. 6.4.2 suggests, we will determine whether $(\mathcal{A}(K_0), \partial)$ has any augmentations by studying the summands in ∂a_i which could be special in the sense of [FI04]. The boundary of a disc counted by the differential consists of segments of the Lagrangian projection of K_0. This lifts to a loop $\gamma \subset L(p, q)$ consisting of segments of K_0 alternating with the Reeb chords which label the corners of the disc. Any such γ may be lifted further to a loop $\tilde{\gamma} \subset S^3$ which consists of segments of $\tilde{K}_0$ alternating with Reeb chords whose endpoints lie on $\tilde{K}_0$. We will characterize potential special boundary discs by studying the front projections of their associated $\tilde{\gamma}$ curves.

Just as Reeb chords in $L(p, q)$ are identified with their front projections, let $\tilde{a}_i$ denote either a lift of the chord a_i to S^3 or the front projection of that chord to the Heegaard torus. Lifting preserves chord length: $l(a_i) = l(\tilde{a}_i)$.

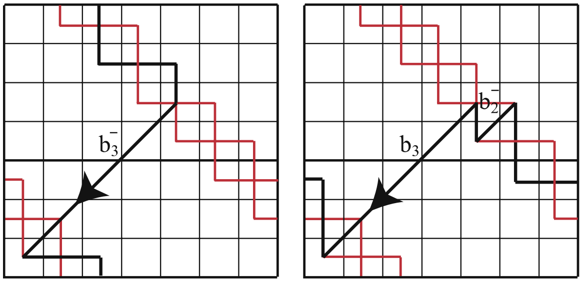

Fig. 6.8 *Left*: An N loop for $K(7,6,2)$. *Right*: An S loop for $K(7,6,2)$

Definition 7 An N-loop is an oriented simple closed curve on a special front projection for $\tilde{K}_0$ which satisfies the following:

- the curve is homotopic to the loop $\theta_1 = c$ for some $c \in [0, 2\pi]$;
- the curve alternates between $\tilde{b}_i^-$ chords and segments of K_0 traversed left to right and top to bottom.

If a single $\tilde{b}_i^-$ is replaced by its complementary $\tilde{a}_i^+$, the resulting curve is called an N_i*-loop*.

Similarly, an S-loop is a simple closed curve on a special front projection for $\tilde{K}_0$ which satisfies the following:

- the curve is homotopic to the loop $\theta_2 = c$ for some $c \in [0, 2\pi]$;
- the curve alternates between $\tilde{b}_i^-$ chords and segments of K_0 traversed bottom to top and right to left.

If a single $\tilde{b}_i^-$ is replaced by its complementary $\tilde{a}_i^+$, the resulting curve is called an S_i*-loop*. See Fig. 6.8.

Each N_i- and each S_i-loop corresponds to some $F_{p,q}$-invariant $\tilde{\gamma}$ in S^3, and equivalently, to some $\gamma \subset L(p,q)$. The Lagrangian projection of this γ is a closed curve in the Lagrangian projection of K_0, and the next proposition states that this loop bounds a disc counted by the differential.

Proposition 2 Each N_i- or S_i-loop corresponds to a summand of ∂a_i, and the associated boundary word is written in the b-type chords in the N_i or S_i loop.

Proof Consider first an S_i loop. Because $\tilde{a}_i^+$ replaced $\tilde{b}_i^-$, the S_i-loop is homotopic to the curve $\theta_1 = c$ on the grid diagram. The image of the S_i loop in the front projection of the lens space lifts to a curve $\gamma \subset L(p,q)$, and the Lagrangian projection of γ has winding number p with respect to the south pole. (This fact requires $gcd(q-1, p) = 1$.) Thus, this curve bounds an admissible disc D in the Lagrangian projection. We claim that the a_i-defect of D is zero.

Recall that the defect of a region in $S^2 - \Gamma$ is the sum of the signed lengths of the preferred chords labeling the corners, minus the area of the region. To compute the a_i-defect of D, replace each term $l(a_j)$ for $j \neq i$ in this sum by $l(a_j) - 1 = -l(b_j)$.

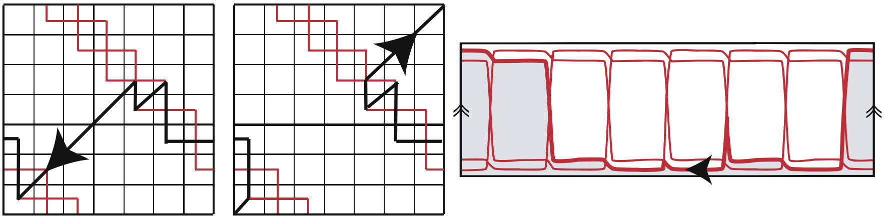

Fig. 6.9 Replacing the chord b_3^- in an S-loop by a_3^+ yields an S_3 loop which gives the boundary of a disc with a_3-defect zero. The lift of the disc with area $\frac{2}{7}$ to the Lagrangian projection of $\tilde{K} \in S^3$ is shown on the *right*

The proposition therefore follows if $l(a_i) - \sum_{j \neq i} l(b_j)$ is equal to the area of D. As usual, we assume that the connecting segments on the special front projection lie in arbitrarily small neighborhoods of the centers of the boxes, so we treat the length of each chord as an integral multiple of $\frac{1}{p}$. This multiple may be computed by counting, with sign, the number of times the chord crosses a vertical line on the grid diagram for $\tilde{K}_0$.

The area of the large region in $S^2 - \Gamma$ has area approximately equal to $\frac{1}{p}$, and the multiplicity of this region in D is equal to the number of times the boundary of D traverses a horizontal segment of Γ lying near the north pole. The number of such segments equals the number of horizontal box-lengths traversed by the S_i-loop.

Since the net horizontal displacement of the S_i loop is zero, the horizontal displacement along chords is canceled by the horizontal displacement along the front projection of $\tilde{K}_0$. This shows that the a_i-defect of D is zero, so the associated boundary word appears as a summand in ∂a_i.

The proof for an N-loop is similar, and the argument shows that if an N or S loop has k chord segments, then it will correspond to k distinct boundary discs. □

Counting S- and N-loops on the front projection of $\tilde{K}_0$ shows that exactly two admissible capping paths in the special Lagrangian diagram contribute terms to the differential.

Proposition 3 Let $j = p \bmod (h + v)$. One capping path for a_j bounds a disc disjoint from the south pole which contributes a constant term to ∂a_j, and one capping path for a_{h+v-j} bounds a disc disjoint from the north pole which contributes a constant term to ∂a_{h+v-j}.

Proof The proof requires showing first that a_j and a_{h+v-j} both have admissible capping paths, and second, that these paths correspond to an N_j- and an S_{h+v-j}-loop, respectively.

Label the horizontal segments from 1 to $h + v$, counting from the top down. An admissible capping path for a_i must complete p rotations about the sphere; starting on the top horizontal segment, each full rotation increases this index by one, counted modulo $h + v$. If an admissible capping path for a_i bounds a disc containing the

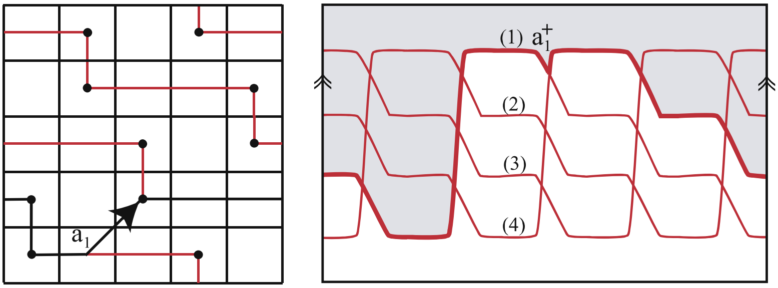

Fig. 6.10 *Left*: Replacing a_1^+ with b_1^- in the front projection of the capping path would yield an N loop. *Right*: The capping path and the disc it bounds are shown lifted to a Lagrangian projection for $\tilde{K}$; the parenthetical numbers identify the horizontal segments as they are numbered in the proof of Proposition 3

north pole, then the path leaves the ith crossing to the right on the first horizontal segment, and enters a_i from the left on the ith horizontal segment. This implies that $i = p \bmod h + v$.

On the other hand, if an admissible capping path for a_i bounds a disc containing the south pole, then the path leaves the ith crossing to the right on the $(i+1)$th segment and the path enters from the right on the $h+v$th segment. This implies that $(i+1)+(j-1) = h+v \bmod h+v$, so $i = h+v-j$.

Now lift the northern capping path for a_j to $\tilde{\gamma} \subset S^3$ and consider its front projection. On the Heegaard torus, the path traverses the chord a_j^+ and then moves right and down along K until reaching the other end of the chord. This curve forms a loop homotopic to $\theta_2 = c$ on the Heegaard torus, so replacing a_j^+ with b_j^- yields a loop homotopic to $\theta_1 = d$. (See Fig. 6.10.) This is an N-loop, so the projection of $\tilde{\gamma}$ was an N_j-loop and therefore bounded a disc counted by the differential. Since the N_j-loop has a unique Reeb chord segment, the corresponding term in ∂a_j is 1.

The proof for a_{h+v-j} is similar. □

6.4.4 Applications

In this section we apply the results about N- and S-loops to examples where the number of generators is small. Counting the number of possible loops allows us to determine whether or not $\mathcal{A}(K(p, p-1, h))$ has augmentations in the special cases $h = 1$ and $h = 2$. A similar analysis should be possible for other values of q and h, but the combinatorics involved in counting all possible N- and S-loops will be more complicated.

6.4.4.1 Augmentations for $(\mathcal{A}(K(p, p-1, 1)), \partial)$

Theorem 6 (1) *Let $K = K(p, p-1, 1)$ be a Legendrian grid number one knot in $L(p, p-1)$ for $gcd(p-2, p) = 1$. Then the homology of $(\mathcal{A}(K), \partial)$ is a tensor al-*

gebra with two generators. Furthermore, the map sending both generators of $\mathcal{A}(K)$ *to* 0 *is an augmentation.*

Proof As shown in Theorem 5, the special Lagrangian diagram for $K(p, p-1, 1)$ has only one intersection point. Studying the front projection of $\tilde{K}_0$ shows that the a chord is shorter than the b chord, so only constant terms can appear in ∂a. However, since both capping paths for the a generator contribute constant terms, they cancel modulo two and $\partial a = 0$. Similarly, $\partial b = a + a = 0$, so the entire algebra lies in the kernel of the differential. Furthermore, both $\epsilon_1(b) = 0$ and $\epsilon_2(b) = 1$ are augmentations of $(\mathcal{A}(K(p, p-1, 1)), \partial)$, with $\epsilon_i(a) = 0$ for $i = 1, 2$. □

6.4.4.2 Augmentations of $(\mathcal{A}(K(p, p-1, 2)), \partial)$

In the final section we consider the case $K = K(p, p-1, 2)$. We will show the following:

Theorem 7 (2) *Let* $K = K(p, p-1, 2)$ *be a Legendrian grid number one knot in* $L(p, p-1)$ *for gcd*$(p-2, p) = 1$. *Then* $(\mathcal{A}(K), \partial)$ *has an augmentation if and only if* $p \equiv 3 \bmod 12$ *or* $p \equiv 9 \bmod 12$.

The special Lagrangian diagram has three crossings, and Proposition 3 implies that the generators a_1 and a_3 will each have a capping path which bounds a boundary disc. One of these discs contains the north pole, and we denote the corresponding generator by a_N. Similarly, the other disc contains the south pole, and we denote the corresponding generator by a_S.

Lemma 4 *The boundary of* a_N *has no terms containing* b_S *or* b_N. *Similarly, the boundary of* a_S *has no terms containing* b_S *or* b_N.

Proof Recall that at each crossing, the a-type generator is preferred. Thus, if f is a boundary disc whose boundary word is written entirely in b-type generators, then $n_{a_N}(f) = n(f(D^2))$. The front projection of $K(p, p-1, 2)$ shows that $l(a_N) = l(b_N) - 1 = l(b_S) - 1$. Any disc with positive corner a_N^+ that had a b_N^- or b_S^- corner would therefore have a negative defect. This implies that no boundary word for a_N can contain b_S or b_N, and the argument for a_S is identical. □

Lemma 5 *If* $\epsilon : \mathcal{A} \to \mathbb{Z}_2$ *is a graded homomorphism such that* $\epsilon(b_N) = \epsilon(b_S)$, *then the generator* a_2 *will have an even number of special boundary discs.*

Proof Fix a representative of a_2^+ in the front projection of $\tilde{K}$. For each N_2-loop containing this chord, the reflection of the loop across the Reeb orbit containing the chord is an S_2-loop and vice versa. This reflection interchanges b_N and b_S, and the hypothesis that $\epsilon(b_N) = \epsilon(b_S)$ implies that the number of special boundary discs for a_2 is even. □

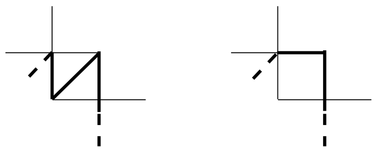

Fig. 6.11 *Left*: If $b_N = b_3$, the S loop may traverse the last b_2^-. *Right*: If $b_N = b_1$, the S loop may not traverse the last b_2^-

Since b_N and b_S do not appear in words written only in b-type generators in ∂a_N and ∂a_S, we are free to choose $\epsilon(b_S) = \epsilon(b_N)$ without affecting the number of special boundary discs of a_S and a_N. Thus, Lemmas 4 and 5 together imply that the existence of an augmentation of $(\mathcal{A}, \partial)$ depends only on the number of words in ∂a_N and ∂a_S which are written solely in b_2^-. Note that the capping paths which bound boundary discs are special with respect to any homomorphism.

Lemma 6 *The number of boundary words of a_N (respectively, a_S) associated to S-(N-) loops is odd unless* $p \equiv \pm 1 \bmod 12$.

Proof The chords b_S and b_N have the same image in the front projection of K, so they are represented by a total of p chords in the front projection of $\tilde{K}$. Pick one representative and fix this choice. Note that the S-loops containing b_N^- are reflections of the N-loops containing b_S^- and vice versa. Thus, it suffices to count only S-loops which contain b_N^-.

Trace a path on the front projection of $\tilde{K}$, beginning along b_N^-. Moving only up and left along the image of K, it is not possible to form an S-loop without traveling along any other Reeb chords; if it were, the resulting loop would contradict the choice of chord. Thus, in order to form an S-loop, the path must traverse some number of b_2^- chords. The maximal possible number of b_2^- chords is the largest odd number less than or equal to $p - l(b_N)$. To see that this is the right value, observe that if the path traverses the last b_2^- chord, it must end at b_3. However, $b_3 = b_N$ if and only if $p - l(b_N)$ is odd. See Fig. 6.11.

Locally, identify the two strands of the image of $\tilde{K}$ as A and B as indicated in Fig. 6.12. Each time the path traverses a b_2^- chord, it switches between A and B, and the total number of switches must be odd in order for the path to close up into an S-loop.

Labeling the b_2^- chords by the strand they begin on, an S-loop is defined by an odd-length sequence of chords labeled alternately by A and B. Thus each S-loop on a diagram with $2k + 1$ switching chords corresponds to an alternating subsequence of a length $2k + 1$ alternating sequence of A's and B's. Note, too, that the subsequence must start with an A. For $k = 1$, the possible paths are given by the letters in bold: $AB\mathbf{A}$, $\mathbf{ABA}$, and $\mathbf{A}BA$ (Fig. 6.12). For $k = 2$, there are eight possible paths:

$\mathbf{A}BABA$	$ABAB\mathbf{A}$
$AB\mathbf{ABA}$	$\mathbf{ABA}BA$
$\mathbf{A}BA\mathbf{BA}$	$\mathbf{AB}AB\mathbf{A}$
$AB\mathbf{A}BA$	$\mathbf{ABABA}$

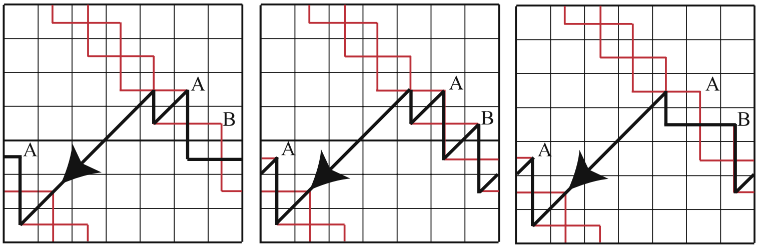

Fig. 6.12 In $L(7,6)$, the maximal number of switches is 3. The *bold curves* show the three possible S-loops which contain $b_N = b_3$. Each S-loop corresponds to a boundary disc for a_N which is disjoint from the north pole

Let $S(k)$ denote the number of S-loops which include b_N on a diagram with $2k+1$ possible switching chords. Picking the first and last A chosen in a subsequence yields the following recursive formula:

$$S(k) = k + 1 + \sum_{i=1}^{k} i\,S(k-i), \quad \text{where } S(0) = 1.$$

Expanding this yields

$$\begin{aligned}
S(k) &= k + 1 + \sum_{i=1}^{k} i\,S(k-i) \\
&= k + 1 + S(k-1) + \sum_{i=2}^{k} i\,S(k-i) \\
&= (k+1) + (k-1+1) + \sum_{j=1}^{k-1} j\,S(k-1-j) + \sum_{i=2}^{k} i\,S(k-i).
\end{aligned}$$

We then reduce the previous equation modulo 2:

$$\begin{aligned}
S(K) &\equiv 1 + \sum_{j=1}^{k-1} j\,S(k-1-j) + \sum_{i=2}^{k} i\,S(k-i) \bmod 2 \\
&\equiv 1 + \sum_{j=1}^{k-1} (2j+1) S(k-1-j) \bmod 2 \\
&\equiv 1 + \sum_{j=1}^{k-1} S(k-1-j) \bmod 2
\end{aligned}$$

Expanding the new relation and again reducing modulo two yields

$$
\begin{aligned}
S(k) &\equiv 1 + \sum_{j=1}^{k-1} S(k-1-j) \bmod 2 \\
&\equiv 1 + S(k-2) + \sum_{j=2}^{k-1} S(k-1-j) \bmod 2 \\
&\equiv 1 + 1 + \sum_{l=1}^{k-3} S(k-3-l) + \sum_{j=2}^{k-1} S(k-1-j) \bmod 2 \\
&\equiv \sum_{l=1}^{k-3} S(k-3-l) + \sum_{j=2}^{k-1} S(k-1-j) \bmod 2 \\
&\equiv S(k-3) \bmod 2
\end{aligned}
$$

Computing the first few cases directly shows that $S(k)$ is odd except when $k \equiv 2 \bmod 3$. The maximal number of switching chords is the greatest odd integer less than or equal to $p - l(b_N) = \frac{p-1}{2}$, so $k \equiv 2 \bmod 3$ if and only if $p \equiv \pm 1 \bmod 12$. □

Lemma 7 *The number of boundary words of a_N associated to N-loops is odd unless $p \equiv 5 \bmod 12$ or $p \equiv 7 \bmod 12$.*

Proof This proof is similar to the previous one. In this case the path which traverses no b_2^- chords is an N-loop, and any other N-loop also traverses an number of b_2^- chords. If the maximal number of b_2^- chords is $2k$, then $N(k)$ counts the number of even-length alternating subsequences beginning with A:

$$N(k) = 1 + \sum_{i=1}^{k} i N(k-i), \quad \text{where } N(0) = 1 \text{ and } N(1) = 2.$$

Expanding and reducing modulo two as above, this yields

$$N(k) \equiv N(k-3) \bmod 2.$$

This value is odd unless $k \equiv 1 \bmod 3$. The maximal number of possible b_2^- chords is the greatest even number less than or equal to $\frac{p-1}{2}$, so $k \equiv 1 \bmod 3$ if and only if $p \equiv 5 \bmod 12$ or $p \equiv 7 \bmod 12$. □

When $p \equiv 3 \bmod 12$ or $p \equiv 9 \bmod 12$, then the total number of N- and S-loops containing b_N is even. Similarly, the total number of S- and N-loops containing b_S is even. Thus setting $\epsilon(b_2) = 1$ implies an even number of special discs with respect to ϵ. Note that since a capping path always corresponds to a special disc, other

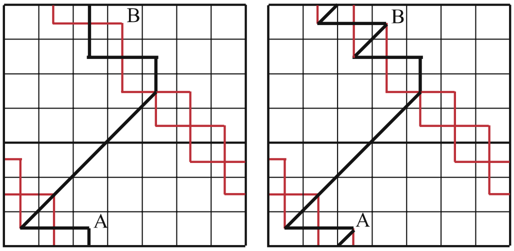

Fig. 6.13 The *bold curves* show the two possible N-loops which contain $b_N = b_3$. Each N-loop corresponds to a boundary disc for a_N which is disjoint from the south pole

values of p cannot admit an even number of special discs for any ϵ. Combining this with Lemmas 4 and 5 proves Theorem 2.

Acknowledgement I would like to thank the Max Planck Institute for their support and hospitality during the 2008–2009 year.

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Chapter 7
Embeddings of Four-valent Framed Graphs into 2-surfaces

Vassily Olegovich Manturov

Abstract It is well known that the problem of detecting the least (highest) genus of a surface where a given graph can be embedded is closely connected to the problem of embedding special *four-valent framed graphs*, i.e. 4-valent graphs with opposite edge structure at vertices specified. This problem has been studied, and some cases (e.g., recognizing planarity) are known to have a polynomial solution.

The aim of the present survey is to connect the problem above to several problems which arise in knot theory and combinatorics: Vassiliev invariants and weight systems coming from Lie algebras, Boolean matrices etc., and to give both partial solutions to the problem above and new formulations of it in the language of knot theory.

7.1 Introduction

Assume 4-valent graph Γ with each vertex endowed with opposite half-edge structure, that is, at each vertex the four half-edges are split into two pairs of *formally opposite edges*. Classify the surfaces S where Γ can be embedded in a way such that the formal opposite half-edge structure coincides with the opposite half-edge structure induced by the embedding.

A natural question is to study the highest (least) genus of the surface the graph can be embedded into. We restrict ourselves only to the case of embeddings which decompose the surface into 2-cells. We shall address this general question later in this paper. We shall start with the following partial cases of it. One of them, more general, deals with embedded graphs whose first $\mathbb{Z}_2$-homology class is orienting. As a partial case of this, we address the following

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_7,

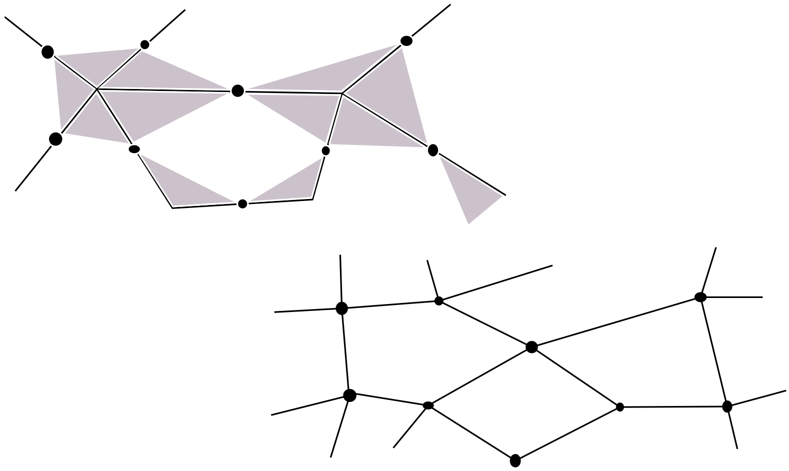


Fig. 7.1 Any embedded graph generates a 4-valent framed graph

Problem 1 Which is the least possible (highest possible) genus of a 2-surface S (closed, but not necessarily orientable) this graph can be embedded into in such a way that the embedding represents the zero homology class in the surface (alternatively, the complement to the graph is checkerboard colourable).

Embeddings of such graphs representing the $\mathbb{Z}_2$-homology class are well studied for the case of the plane (see e.g., [Ros99, LM76, RR78, Man05b]) and in the general case (see e.g., [LRS87, CR01]).

In fact, any embedding of a 4-graph in $\mathbb{R}^2$ defines a checkerboard colouring on the set of faces (we consider the infinite domain as a face of S^2, the latter being a one-point compactification of $\mathbb{R}^2$) because the plane has trivial first homology. On the other hand, any graph Γ embedded into a 2-surface S (orientable or not) can be transformed into a 4-graph by taking the medial graph Γ': the vertices of Γ' are the middle points of the edges of Γ, the edges of Γ' connect *adjacent* edges (sharing the same angle), and faces of Γ' correspond to faces (white) and vertices (black) of Γ, see Fig. 7.1.

Such 4-valent graphs appeared with many names in different problems of low-dimensional topology: as *atoms* (see rigorous definition ahead), originally due to Fomenko [Fom91], see the connection between atoms and knots in [MU05a], they are connected to Grothendieck's *dessins d'enfant*, see [LZ03] and [DFK$^+$06].

There is a nice connection between combinatorics of Vassiliev invariants and other invariants of knots and virtual knots and many well-known functions on graphs, see [CDM] and references therein.

Finally, the genus of the atom (the genus of the checkerboard surface we are interested in) is closely connected to the estimates of the thickness for Khovanov and Ozsváth-Szabó homology for classical and virtual knots, see [Man] and [Low07].

In [CR01] there was a reformulation of the problem stated above in terms of ranks of some matrices.

We give another formulation of Problem 1 in terms of ranks of matrices which is closely connected to knot theory.

Problem 2 Given a symmetric $\mathbb{Z}_2$-matrix M of size $n \times n$, find a splitting of the set of indices $\{1, \dots, n\}$ into $I \sqcup J$ such that for the corresponding square matrices M_I and M_J, the sum of ranks $rk(M_I) + rk(M_J)$ is minimal (resp., maximal).

This seems to be the easiest reformulation of the initial problem. Of course, we are looking for a solution which would either be fast (say, having polynomial time in the number of vertices) or connected to some interesting mathematical problems.

In knot theory, the study of classical knots is closely connected to the so-called *d-diagrams*, chord diagrams with 2 sets of pairwise unlinked chords (see rigorous definition ahead). It turns out that these diagrams play a special role in the chord diagram algebra having the highest possible degree of the Vassiliev invariants coming from $sl(n)$ (see [CSM04]). On the other hand, these are precisely those diagrams corresponding (in sense of [MU05a]) to planar 4-valent graphs.

This is not incidental. In fact, the generating function for such embeddings is closely connected to the $sl(n)$-weight system, and the latter weight system sometimes gives estimates for the genus of the atom where the framed graph can be embedded into.

The paper is organized as follows. In the next section, we give the definitions of atom, chord diagram, d-diagram, virtual link and establish a connection between them and embedded graphs. We also give a proof of a conjecture due to Vassiliev (stated in [Vas05]) and proved in [Man05b] saying that the only obstruction to the planar embeddability of such graphs is the existence of two cycles with no common edges with exactly one *transverse* intersection point.

Later, we also give criteria for embeddability of framed graphs to the real projective space and in the Klein bottle.

In Sect. 7.3, we define the Kauffman bracket for the virtual knots and we recall a result by Soboleva [Sob01] about the number of circles which appear after a surgery along a chord diagram. This will lead us to the reformulation of Problem 1 as Problem 2.

The approach relying on Soboleva's theorem was one of the main tools of the papers [IM09b, IM09a], where we construct *graph-link theory*. The main idea of these papers is as follows: a knot can be represented by a chord diagram, which, in turn, has an intersection graph (with some extra labeling and framing). Reidemeister moves can be translated into the language of intersection graphs, which generates new equivalence classes of all graph (not necessarily realisable by chord diagrams and knots). These equivalence classes are called *graph-links*.

Section 7.4 will be devoted to the connection between chord diagrams, weight systems and Vassiliev's invariants coming from Lie algebras in sense of Bar-Natan [BN95]. We shall prove a theorem giving an estimate in terms of $sl(n)$-invariants.

The last section will be devoted to the discussion and open problems.

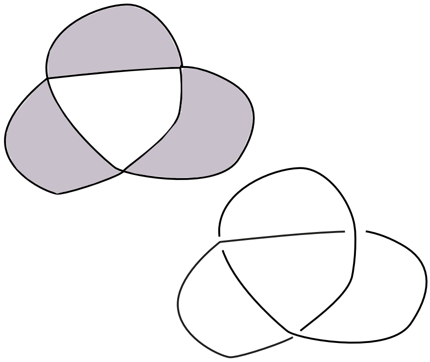


Fig. 7.2 Surface colouring and a knot diagram

7.1.1 Atoms and Knots

A four-valent planar graph Γ generates a natural checkerboard colouring of the plane by two colours (in a way such that adjacent components of the complement $\mathbb{R}^2\backslash\Gamma$ have different colours).

This construction perfectly describes the role played by *alternating diagrams* of classical knots. Recall that a link diagram is *alternating* if while walking along any component one alternates over- and underpasses. Another definition of an alternating link diagram sounds as follows: fix a checkerboard colouring of the plane (one of the two possible colourings). Then, for every vertex the colour of the region corresponding to the angle swept by going from the overpass to the underpass in the counterclockwise direction is the same.

Thus, planar graphs with natural colourings somehow correspond to alternating diagrams of knots and links on the plane: starting with a graph and a colouring, we may fix the rule for making crossings. Assume at some vertex we have four half-edges a, b, c, d in this clockwise direction, so that a is opposite to c, b is opposite to d, and the pairs (a, b) and (c, d) share black angles. Then the pair of edges (a, c) will form an overcrossing, and the pair (b, d) will form an undercrossing, see Fig. 7.2. Thus, colouring a couple of two opposite angles corresponds to a choice of a pair of opposite edges to form an overcrossing and vice versa.

Now, if we take an arbitrary link diagram and try to fix the colouring of regions by colouring angles according to the rule described above, we see that generally it is impossible unless the initial diagram is alternating: we can just get a region on the plane where colourings at two adjacent angles disagree. So, alternating diagrams perfectly match colourings of the 2-sphere (think of S^2 as a one-point compactification of $\mathbb{R}^2$). For an arbitrary link, we may try to take colours and attach cells to them in a way that the colours would agree, namely, the circuits for attaching two-cells are chosen to be closed paths on the frame which at every vertex turn inside the angle of the fixed colour.

This leads to the notion of *atom*. An *atom* is a pair (M, Γ) consisting of a 2-manifold M and a four-valent graph Γ embedded in M together with a colouring of $M\backslash\Gamma$ in a checkerboard manner. Here Γ is called the *frame* of the atom, where by *genus* (resp., *Euler characteristic*) of the atom we mean that of the surface M.

Note that the atom genus is also called the *Turaev genus*, following the paper by Turaev [Tur87].

Such a colouring exists if and only if Γ represents the trivial $\mathbb{Z}_2$-homology class in M: an obstruction to such a colouring generates a closed path intersecting the frame at odd number of points, and if the colouring exists, then the frame can be treated as a sum (over $\mathbb{Z}_2$ of boundaries of all black cells).

Thus, gluing cells to the black (white) 1-cycles, we get an atom, where the shadow of the knot plays the role of the frame. Note that the structure of opposite half-edges on the plane coincides (by construction) with that on the surface of the atom.

Now, we see that atoms on the sphere are precisely those corresponding to alternating link diagrams, whence non-alternating link diagrams lead to atoms on surfaces of higher genera.

In some sense, the genus of the atom is a measure of how far a link diagram is from an alternating one, which leads to generalisations of the celebrated Kauffman-Murasugi theorem, see [MU05a] and to some estimates concerning the Khovanov homology [Man].

Having an atom, we may try to embed its frame in $\mathbb{R}^2$ in such a way that the structure of opposite half-edges at vertices is preserved. Then we can take the "black angle" structure of the atom to restore the over/under crossing structure on the plane.

In [Man00] it is proved that the link isotopy type does not depend on the particular choice of embedding of the frame into $\mathbb{R}^2$ with the structure of opposite edges preserved. The reason is that such embeddings are unique (for "prime" graphs) up to overall colour change.

The atoms whose frame is embeddable in the plane with opposite half-edge structure preserved are called *height* or vertical.

However, not all atoms can be obtained from some classical knots. Some abstract atoms may be quite complicated for its frame to be embeddable into $\mathbb{R}^2$ with the opposite half-edges structure preserved. However, if it is impossible to *embed* a graph in $\mathbb{R}^2$, we may immerse it by marking artifacts of the immersion (intersections of images of different arcs; we assume the embedding to be generic) by small circles.

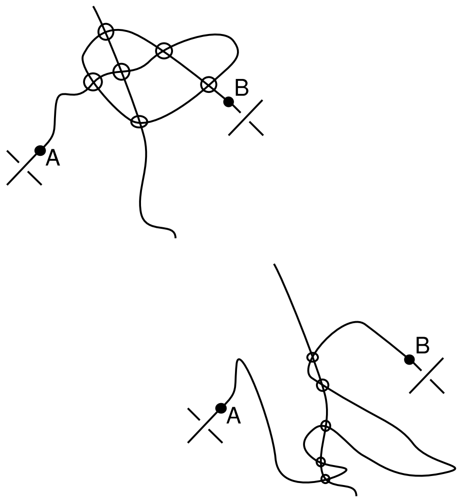
This leads to a connection between atoms and *virtual knots* which perfectly agrees with *virtual knot theory* proposed by Kauffman in [Kau99].

Definition 1 A *virtual diagram* is a 4-valent diagram in $\mathbb{R}^2$ where each crossing is either endowed with a classical crossing structure (with a choice for underpass and overpass specified) or just said to be virtual and marked by a circle.

Definition 2 The *shadow* of a virtual diagram D is a four-valent framed graph G whose vertices correspond to *classical* crossings of D, and edges correspond to arcs connecting classical crossings (in D, arcs may contain virtual crossings inside). The framing of G is taken from the plane, and D represents a generic immersion of G in $\mathbb{R}^2$.

Definition 3 A *virtual link* is an equivalence class of virtual link diagram modulo generalized Reidemeister moves. The latter consist of usual Reidemeister moves

Fig. 7.3 The detour move



referring to classical crossings and the *detour move* that replaces one arc containing only virtual intersections and self-intersection by another arc of such sort in any other place of the plane, see Fig. 7.3.

Having freedom of immersing knot diagrams into $\mathbb{R}^2$ instead of just embedding, we are able to make different virtual diagrams out of atoms. Obviously, since we disregard virtual crossings, the most we can expect is the well-definiteness of the virtual diagram corresponding to the atom up to detours. However, this allows us to get different virtual link types from the same atom, since for every vertex V of the atom with four emanating half-edges a, b, c, d (ordered cyclically on the atom) we may get two different clockwise-orderings on the plane of embedding, (a, b, c, d) and (a, d, c, b). The difference between two diagrams obtained from different immersions of the same atom leads to a move called *virtualisation*.

Definition 4 By a *virtualisation* of a classical crossing we mean a local transformation shown in Fig. 7.4.

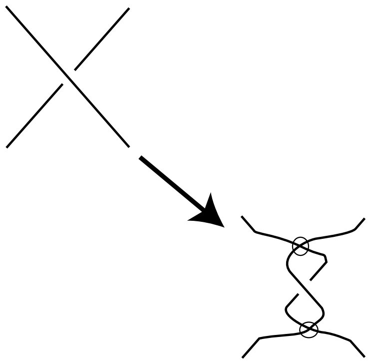
The above statements summarise as

Proposition 1 (see., e.g. [MU05a]). *Let L_1 and L_2 be two virtual links obtained from the same atom by using different immersions of its frame. Then L_1 differs from L_2 by a sequence of* (*detours and*) *virtualisations*.

Obviously, the inverse operation of obtaining an atom from a virtual diagram is well defined.

Note that many famous invariants of classical and virtual knots (Kauffman bracket, Khovanov homology, Khovanov-Rozansky homology [KR08, KR]) do not change under the virtualisation, which supports the *virtualisation conjecture*: if for

Fig. 7.4 Virtualisation



two classical links L and L' there is a sequence $L = L_0 \to \cdots \to L_n = L'$ of virtualisations and generalised Reidemeister moves then L and L' are classically equivalent (isotopic).

Note that the usual virtual equivalence implies classical equivalence for classical knots, see [GPV00].

Definition 5 By a chord diagram we mean a cubic graph with a selected oriented cycle S^1 passing through all vertices. The remaining edges are called *chords*; every vertex is incident to exactly one chord.

A chord diagram is called framed if every chord is marked by either $+1$ or -1. A chord diagram without framings is assumed to have all chords positive.

Two chords A, B of a chord diagram are *linked* if the ends A_1 and A_2 of the first chord lie in different components of $S^1 \backslash \{B_1, B_2\}$.

Analogously, one defines a chord diagram on m circles; for a cubic graph there should be m oriented cycles passing through all vertices; the edges not belonging to circles are referred to as chords.

For a given chord diagram the *intersection graph* is constructed as follows. The vertices of the intersection graph are in one-to-one correspondence with the chords. Two vertices of the graph are connected by an edge iff the corresponding chords are linked.

A chord diagram (with all chords framed positively) is a *d-diagram* if the corresponding intersection graph is bipartite.

If the chord diagram is framed then the corresponding graph obtains framings at vertices.

Now, assume we have an atom with exactly one white cell. Then the whole information about the atom can be obtained from a rotating circuit along the boundary of this cell. Namely, consider a walk along this boundary (in any direction) as a map from S^1 to the atom, and connect the preimages of vertices of the atom by chords. We thus construct a *framed chord diagram*, where the framing is positive when the orientations of the two segments locally agree or negative otherwise, see Fig. 7.5.

This way of constructing a chord diagram out of a virtual link diagram leads us to the notion of rotating circuit.

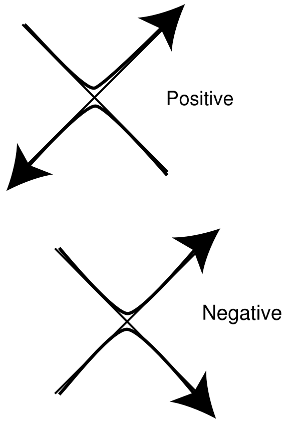


Fig. 7.5 Two types of orientations in a neighbourhood of a vertex

Definition 6 By a *rotating circuit* of a framed graph Γ we mean a map from the oriented circle $S^1 \to \Gamma$ which is homeomorphic outside preimages of vertices of Γ, and each vertex of Γ has precisely two preimages such that the corresponding neighbourhoods of them on the circle switch from one edge to an edge not opposite to it.

From the chord diagram constructed from an atom, one easily restores the atom and, thus, the corresponding virtual link up to detours and virtualisations.

Notation For a chord diagram C with one cell denote the corresponding atom by $A(C)$ and denote the corresponding virtual knot (considered up to detours and virtualisations) by $K(C)$.

7.2 Chord Diagrams, 1-dimensional Surgery and the Kauffman Bracket

The Kauffman bracket [Kau87] is a very useful model for understanding the Jones polynomial [Jon85]. The Kauffman bracket associates with a virtual knot diagram a Laurent polynomial in one variable a associated to every virtual diagram. After a small normalisation (multiplication by a power of $(-a)$) it gives an invariant for virtual links.

This invariant can be read from the atom corresponding to a knot diagram. Namely, take an atom V with n vertices corresponding to a virtual diagram L with n classical crossings and call a *state* a choice of the couple of black or white angles at every vertex of V. Every such choice gives rise to a collection of closed curves on V whose boundaries contain all the edges of V, see Fig. 7.6, and at each crossing the curves turn locally from one edge to an adjacent edge sharing the same angle of the prefixed colour.

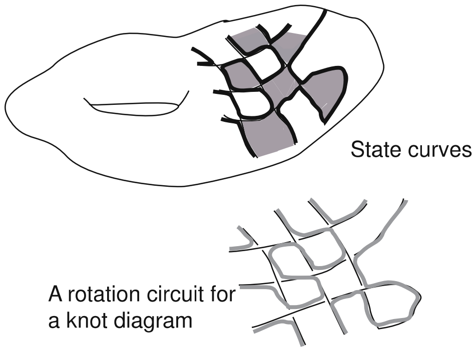


Fig. 7.6 State curves drawn on an atom

Thus, having 2^n states of the atom, we define the Kauffman bracket of it as

$$\langle V \rangle = \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\gamma(s)-1}, \tag{7.1}$$

where the sum is taken over all states s of the diagram, $\alpha(s)$ and $\beta(s)$ denote the number of white and black angles in the state (thus, $\alpha(s) + \beta(s) = n$ and $\gamma(s)$ denotes the number of curves in the state).

This formula (7.6) gives the expression of the Kauffman bracket for *any* virtual diagram corresponding to the atom V.

The Kauffman bracket is an invariant of virtual link diagrams under all Reidemeister moves except the first move. Moreover, the Kauffman bracket is invariant under the virtualisation. Thus, it is not surprising that it can be read from the corresponding atom.

If the atom A is obtained from a (framed) chord diagram C, then one can construct the Kauffman bracket $\langle C(A) \rangle$.

Thus, one obtains a function f on framed chord diagram valued in Laurent polynomials in a. We shall return to that function because it is connected to the Vassiliev invariants of knots and J-invariants of closed curves (Lando, [Lan06]).

Assume now we have a framed graph (a graph with each vertex labeled either positively or negatively).

Every rotating circuit gives rise to a framed chord diagram associated with a graph (the latter graph will have no loops or multiple edges): this chord diagram consists of the circle S^1 and chords connecting those points of S^1 having the same image in Γ. A chord is *positive* (or of framing zero) if for the corresponding vertex the two emanating edges are opposite; otherwise it is called *negative* (or of framing one).

The following statement is left to the reader:

Statement 1 *Let* Γ *be a four-valent graph and* D *be the chord diagram corresponding to some rotating circuit of* Γ. *Assume* Γ *is embedded in a* 2*-surface* S *in a checkerboard manner with framing preserved. Then* S *is orientable if and only if* D *has all chords positive.*

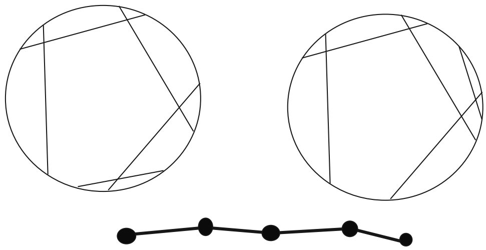


Fig. 7.7 Two chord diagrams with the same intersection graph

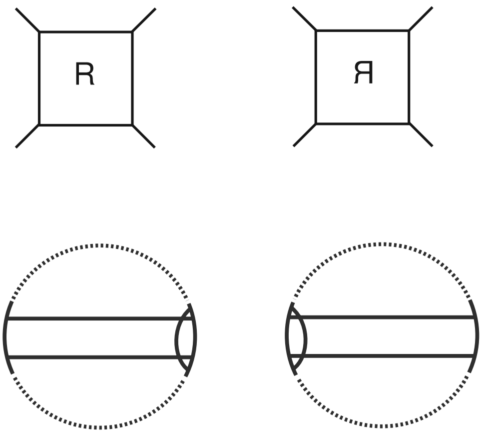


Fig. 7.8 Mutation and its chord diagram presentation

Remark 1 Note that the orientability condition holds for *one embedding* then it holds for *any embedding*, and this can be read out from *any* rotating circuit.

Consider an arbitrary simple graph G (without loops and multiple edges). It may or may not be represented as an intersection graph of a chord diagram (see [Bou94] for the details) for which it is an intersection graph. Moreover, if such a chord diagram exists, it may not be unique, see, e.g., Fig. 7.7.

This non-uniqueness usually corresponds to so called *mutations* of virtual knots.

The mutation operation (shown in the top of Fig. 7.8) cuts a piece of a knot diagram inside a box turn is by a half-twist and reglues the obtained piece to the initial position.

It turns out that the mutation operation is expressed in terms of chord diagrams in almost the same way: one cuts a piece of diagram with 4 ends and exchanges the top and the bottom part of it (see bottom picture of Fig. 7.8). Exactly this operation corresponds to the mutation from both Gauss diagram and rotating circuit points of view.

In the bottom part of Fig. 7.8 chords whose end points belong to the "dotted" area remain the same; the other chords are reflected as a whole.

Regarded from the point of view of Gauss diagrams and the Vassiliev knot invariants, this non-uniqueness corresponds to *mutations* of classical links as well. Namely, S.K. Lando and S.V. Chmutov [CL07] proved the following

Theorem 1 *Assume w is a weight system. Then w depends only on the intersection graph of the chord diagram if and only if no Vassiliev invariant having weight system w detects mutant knots.*

It is well-known that the Kauffman bracket does not detect mutations. Thus, one might guess that the corresponding Kauffman bracket can be read from the intersection graph.

Surprisingly, the Kauffman bracket can be defined in a meaningful way even for those framed graphs which can not be represented as intersection graphs of chord diagrams. This leads to the newborn theory of graph-links, [IM09b, IM09a], a far-reaching generalization of virtual and classical knot theory constructed out of intersection graphs of the corresponding chord diagrams.

Having a chord diagram, we can treat the states of the corresponding Kauffman bracket as collections of chords: we set the initial state s_0 to be the empty collection of curves (with $\alpha(s) = n$, $\beta(s) = 0$), and with each state s we associate a collection of chords corresponding to those vertices of the atom where s differs from s_0. Indeed, if the initial circle of a chord diagram corresponds to the A-state of the Kauffman bracket for some virtual knot, and chords indicate the ways of resmoothing the A-state at all classical crossings, then every state corresponds to a collection of classical crossings (chords of the diagram) where it differs from the A-state.

Now, if we are able to calculate *how many circles we have in each state*, we can apply (7.1) to calculate the Kauffman bracket.

This can be seen from a chord diagram after introducing the notion of *surgery along a chord*.

Given a chord diagram D on n circles $C_1, \ldots, C_n$. Fix a chord c of it. By *surgery along c* we mean the following operation. We delete small neighbourhoods of endpoints of c and connect the obtained endpoints by segments in the following way. There are 3 ways of pairing the four points. One of them corresponds to the disconnection we have performing. We choose one of the other two ways as follows. If c is positive then we connect the endpoints according to the orientation of circles, and if c is negative, we connect the endpoints in the way opposite manner, see Fig. 7.9.

Then we get a collection of circles, not necessarily oriented. If we choose a collection of chords $c_1, \ldots, c_k$ of C, then the surgery along these chords means the consequence of surgeries performed along all chords c_i; in each case we look at the orientations of the initial diagram C.

Assume the circle represents a boundary component of an annulus. By adding a band to the annulus circle transforms its boundary component according to a surgery along the chord corresponding to the band, see Fig. 7.10.

Thus, the number of circles in the state corresponding to the chords $d_1, \ldots, d_n$ is precisely the number of components of the manifold obtained from the initial circle by surgery along these chords.

Now, for a framed graph G on k enumerated vertices, introduce the intersection matrix of G to be $k \times k$ matrix over $\mathbb{Z}_2$ whose rows and columns correspond to vertices of G such that $M_{ij} = M_{ji} = 1$ for $i \neq j$ iff the vertices i, j are connected by an edge and $M_{ii} = 1$ iff i-th vertex is framed negatively.

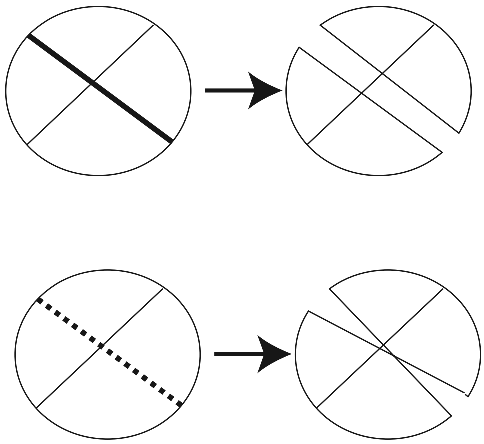


Fig. 7.9 Surgeries along positive and negative chords

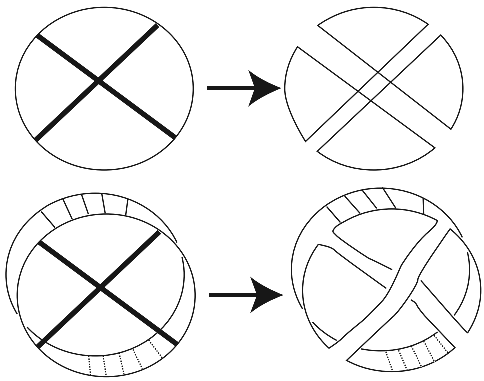


Fig. 7.10 A surgery and its band representation

Surprisingly, this number can be counted from the intersection graph even when the corresponding chord diagram does not exist, due to the following

Theorem 2 (Soboleva [Sob01]) *For a chord diagram D with an intersection graph $G = G(D)$, the number of components of the manifold obtained from D after a surgery along chords $1, \ldots, k$ is one plus the corank of $M(D)$.*

Now, we just define for a framed graph Γ the Kauffman bracket as

$$\sum_{G' \subset G} a^{2|G'|-n}(-a^2 - a^{-2})^{corank M_{\Gamma'}} \tag{7.2}$$

This formula is used in [IM09b, IM09a] to define the Kauffman bracket for graph-links.

Soboleva's theorem allows to reformulate Problem 1 as Problem 2. Indeed, let Γ be a framed 4-graph. We are looking for an embedding of Γ into a surface S of

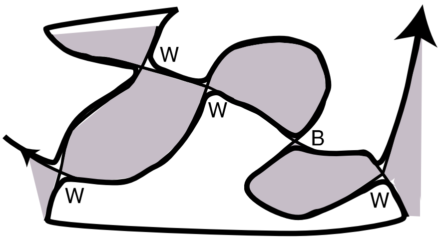


Fig. 7.11 A circuit splits chords into two sets

minimal (maximal) genus with a checkerboard face colouring. Choose a rotating circuit of Γ and a corresponding framed chord diagram $C(\Gamma)$.

Now, assume Γ is embedded in a certain surface S. Then $C(\Gamma)$ yields a mapping $S^1 \to S$ which is an embedding outside pre-images of vertices of Γ. Indeed, we just take $S^1 \to \Gamma \to C(\Gamma) \subset S$. This map can be slightly smoothed in neighbourhoods of images of vertices on S to give an embedding as shown in Fig. 7.11.

Obviously, this circle $S^1 \in S$ is *separating*: it divides the surface S into the "white part" and the "black part". Namely, no point inside any black cell can be connected by an arc to a point inside any white cell without intersecting the circle S^1. We can draw all chords of $C(\Gamma)$ as small edges on S lying in neighbourhoods of vertices of Γ. Thus, all chords of the chord diagram $C(\Gamma)$ are naturally split into two families: those lying in white regions (and connecting one white cell to another) and those lying in black regions.

Vice versa, any splitting of chords of $C(\Gamma)$ into two families (black and white) gives rise to a checkerboard colourable embedding of Γ into a certain surface S. Indeed, consider an annulus $S^1 \times I$ and let S^1 be the medial circle of this annulus. Now, we attach bands to two sides of the annulus according to the splitting. More precisely, we consider the neighbourhood of our embedded circuit: it is an annulus. Colour its boundary circles with black and white. Then we start attaching bands to different sides of this annulus: every band will correspond to a neighbourhood of some chord. Chords corresponding to the white regions are connected to the white boundary component, and chords corresponding to black regions are connected to the black boundary component. A band is overtwisted iff the corresponding chord is negative. This leads to a 2-manifold M with boundary; this boundary naturally splits into two parts corresponding to the boundary components of the annulus. Gluing the boundary components of M, we get the desired surface S without boundary, see Fig. 7.12.

Thus, the question of estimating the genus (Euler characteristic) of S is equivalent to the question of maximising (minimising) the boundary components of S. By definition, this is nothing but counting the number of components of the two 1-manifolds obtained from the sphere by a surgery along the set of chords. By Soboleva's formula, we have two subsets of chords I and J, and we should take two coranks of the adjacency matrices M_I and M_J.

Thus, we have to find a way of splitting the chords in order to maximise (minimise) the sum of ranks of the two matrices $\operatorname{rank} M_I + \operatorname{rank} M_J$.

Thus, we have proved

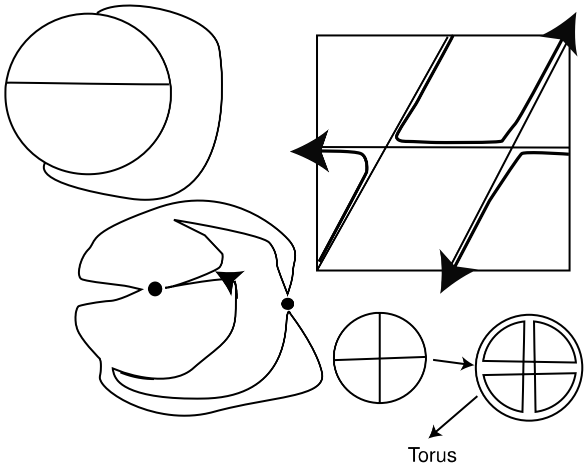


Fig. 7.12 Restoring the atom from a chord diagram with two sets of chords

Theorem 3 *Let* Γ *be a four-valent framed graph on* n *vertices, and* $M(\Gamma)$ *be the intersection matrix corresponding to some rotating circuit of* Γ. *Then* Γ *is embeddable in a surface of genus* g *in a checkerboard manner if and only if for any circuit* C *there exists a way of splitting the indices of* $M(\Gamma)$ *into two sets* I *and* J, $I \sqcup J = \{1, \ldots, n\}$ *such that* $\operatorname{rank} M_I(\Gamma) + \operatorname{rank} M_J(\Gamma) = 2g$.

Remark 2 Note that this solution does not depend on a particular choice of a rotating circuit, depending only on the initial framed graph.

The observation above leads to the following

Statement 2 *Given a framed* 4*-graph* Γ *and the chord diagram* $C(\Gamma)$ *corresponding to some rotating circuit of* Γ. *Then if all chords of* $C(\Gamma)$ *are positive then* all *checkerboard colourable embeddings of* Γ *yield orientable surface. If at least one chord of* Γ *is negative then all such surfaces are non-orientable.*

Remark 3 Note that the statement above means, in particular, that if *for some* circuit the chord diagram contains a negative chord, then so are all diagrams corresponding to all rotating circuits for the same graph.

7.2.1 The Source-sink Condition

The above condition can be reformulated in terms of some intrinsic properties of the framed 4-graph.

Definition 7 We say that a four-valent framed graph satisfies the *source-sink condition* if each edge of it can be endowed with an orientation in such a way that for each vertices some two opposite edges are emanating, and the remaining two edges are incoming.

Obviously, for a given connected graph there exists at most one source-target structure (up to overall orientation reversal of all edges). Moreover, if such a structure exists, then it agrees with *any* rotating circuit. Namely, starting with a rotating circuit one may try to orient its edges consequently in order to get a source-target structure of the whole framed graph. The only obstruction one gets in this direction corresponds to negative chords.

From the above, we get the following

Theorem 4 *A four-valent framed graph* Γ *admits a source-sink structure if and only all surfaces where it can be embedded in a checkerboard colourable manner, are orientable surface, if* Γ *does not admit such an orientation then all surfaces where it can be embedded into in a checkerboard colourable manner and with framing preserved, are non-orientable; in other words, a source-target structure means that for any rotating circuit all chords are positive (of framing zero), cf. Statement 1.*

Proof The idea of the proof goes as follows. Assume Γ admits a source-sink orientation of edges. Then, if Γ is embedded in some surface S, the boundary of each cell acquires a natural orientation from the source-sink condition. Now, assuming that the boundaries of the black cells generate the clockwise orientation of black cells, and the boundaries of white cells generate the counterclockwise orientation of the white cells, we get the desired statement.

Conversely, having an embedding of Γ into an orientable surface in a checkerboard-colourable manner, we may take the clockwise orientations for the boundaries of the black cells to generate the source-sink orientation of Γ. □

7.2.2 The Planar Case: Vassiliev's Conjecture

To see whether a 4-graph Γ is embeddable in $\mathbb{R}^2$ (or S^2), take a chord diagram $C(\Gamma)$ corresponding to some rotating circuit of Γ, and consider the adjacency matrix $M_{C(\Gamma)}$. A simple calculation shows that the corresponding sum of ranks should be *the minimal possible*, i.e., equal to zero (cf. Theorem 3). That means that all chords of Γ are positive (otherwise we would have diagonal non-zero entries giving rank at least 1). Moreover, the chords should constitute two families of non-intersecting chords (each family forming a submatrix of rank 0). This means that the corresponding intersection graph is bipartite or the diagram is a d-diagram.

Assume Γ is a 4-valent framed graph and γ_1, γ_2 are two cycles on Γ without common edges having an intersection point X. The intersection point X of γ_1, γ_2 is called *transverse* if γ_1 contains a pair of opposite edges at X and γ_2 contains the two remaining edges.

Now, one can check that for a diagram with a negative chord c there is a Vassiliev obstruction consisting of two circuits having precisely one transverse intersection point at the vertex of Γ corresponding to c.

Having proved that, we are left with the case of graphs posessing the source-sink conditions, see Theorem 4.

Besides, for a chord diagram which is not a d-diagram, one can explicitly construct a Vassiliev obstruct. Finally, if a chord diagram corresponding to the rotating circuit is a d-diagram, and the graph satisfies the source-sink condition, then the graph is embeddable in $\mathbb{R}^2$ with framing preserved.

This leads to a proof of Vassiliev's conjecture. For more details see [Man05b].

Note that the above considerations lead to a fast (quadratic on the number of chords) algorithm of planarity recognition: one takes any circuit, checks whether all chords are positive, and then checks that a diagram is a d-diagram. The latter consists of possible splitting of all chords into two disjoint sets, which is unique for chord diagrams with connected intersection graphs.

7.2.3 The Case of $\mathbb{R}P^2$

Here we shall use the adjacency matrix $M = (m_{ij})$. According to Statement 2, the adjacency matrix should have at least one non-zero diagonal element, and according to Theorem 3, the sum of two ranks of block-diagonal matrices should be equal to one. Without loss of generality, assume it is the element m_{11}. Since we are looking for a splitting of $\{1, \ldots, n\}$ in order to get rank 1, all elements entries $m_{jj} = 1$ should belong to the same set. Without loss of generality, assume $a_{11} = \cdots = m_{kk} = 1$, $m_{k+1,k+1} = \cdots = m_{nn} = 0$. Now, merge some subset $\{k+1, \ldots, n\}$ with $\{1, \ldots, n\}$ and leave the remaining part as it is in order to get the total rank 1. The remaining part should thus have rank 0, while the former should not increase rank 1 formed by the first k entries of the matrix. This can be done by the procedure similar to finding d-diagrams. The generic diagram corresponding to $\mathbb{R}P^2$ looks as follows (see Fig. 7.13): there is a family of dashed chords (all intersecting each other) and two families of pairwise disjoint chords; chords belonging to one family do not intersect dashed chords.

In Fig. 7.13 solid chords from another family are represented by thicker lines than chords belonging to the family containing all dashed chords.

Obviously, the algorithm described in the present section has quadratic complexity.

7.2.4 The Case of the Klein Bottle

The main idea of detecting the Klein bottle embeddability is the following. While seeking the minimal sum of ranks being equal to 2 (by Theorem 3) there might be

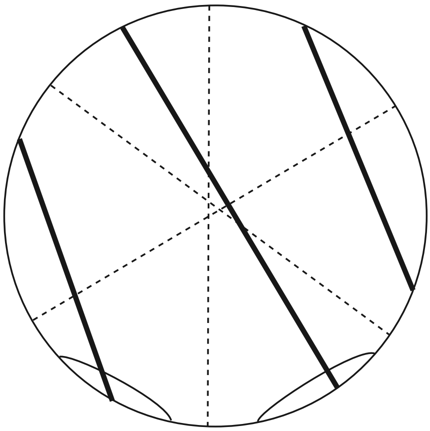


Fig. 7.13 A generic framed chord diagram corresponding to a graph in $\mathbf{R}P^3$

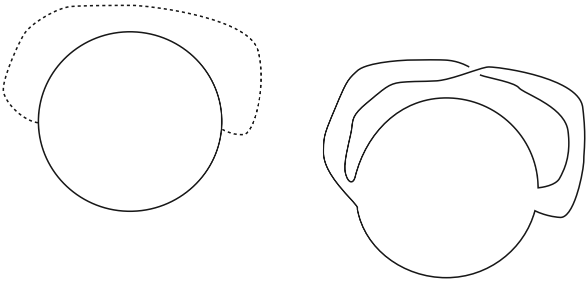


Fig. 7.14 A surgery along a *dashed chord* generates a Möbius band

two possibilities: either $2 = 1 + 1$ (which corresponds to the Klein bottle represented as a connected sum of two projective planes) or $2 = 2 + 0$, the first case is easier, and it turns out that the general case can be reduced to it).

Lemma 1 *For every four-valent framed graph embeddable in* $\mathbf{KL}^2$ *there exists a rotating circuit dividing* $\mathbf{KL}^2$ *into two copies of* $\mathbb{R}P^2$.

Proof Starting with a rotating circuit bounding a disc, one can get a desired circuit by performing surgery along a dashed chord, see Fig. 7.14. □

Then, the procedure is as follows: we take a dashed chord, perform a surgery and look whether the intersection matrix corresponding to the obtained chord diagram is splittable into two families giving the desired decomposition.

The desired decomposition should be such that inside each family any two dashed chords intersect, and any non-dashed chord is disjoint from any other chord. Thus, the procedure of finding two families is exactly the same as in the case of d-diagrams, except for the case of incidence of dashed chords.

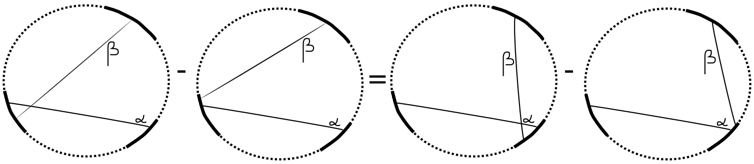


Fig. 7.15 The four-term relation

7.2.5 The Chord Diagram Algebra and the Graph Algebra

Chord diagrams play a crucial role in the study of finite-type (Vassiliev) invariants of knots [Vas90, BN95]. Roughly speaking, for every positive integer n there is a class of invariants of degree n whose "leading term" (called symbol or n-th derivative) is a function on chord diagrams satisfying a certain set of relations. Invariants of the same order having the same leading term differ by an invariant of a strictly smaller order (like polynomials of degree n whose n-th derivatives coincide).

There are two versions of the chord diagram algebra: for usual knots (with two sorts of relations, the four-term (see ahead) and the one-term relation) and for framed knots (with only the four-term relation).

The one term relation says that a chord diagram having a solitary chord (not linked with any other chord) is equivalent to zero.

We shall restrict ourselves for the case of only four-term relation which is defined as follows (see Fig. 7.15): given four diagrams on n chords for which $n-2$ chords coincide (they are not depicted in Fig. 7.15 and have endpoints in punctured areas) and the disposition of the remaining two chords, α and β is as shown in Fig. 7.15. Then for any such quadruple of chords, we set their alternating sum (as in 7.15) to be equal to zero.

Remark 4 There exists a standard "deframing" procedure which associates with each weight system satisfying only the $4T$-relation a weight system satisfying both the $4T$ and the $1T$-relation. The latter means that the diagram containing a solitary chord is equal to zero.

Definition 8 We define the linear space $\mathcal{A}_n^c$ (over $\mathbb{Q}$) to be the quotient space of all chord diagrams on n chords modulo the four-term relation.

It turns out that the space $\mathcal{A}^c = \bigoplus_{i=0}^{\infty} \mathcal{A}_n^c$ has an algebra (and even bialgebra and Hopf algebra) structure.

Let us now pass to the multiplication of chord diagrams. Given two chord diagrams C_1 and C_2, we take some points x_1 and x_2 (where x_i lies on the circle of C_i and is not a chord end), split C_i at x_i, $i=1,2$ and reconnect them to get a diagram one circle with respect to their orientation. Certainly, this operation depends on the choice of x_i.

It turns out that this operation is well-defined modulo $4T$-relation [BN95].

The comultiplication operation for a chord diagram C is defined as $\Delta(C) = \sum_{I \sqcup J} C_I \otimes C_J$, where the sum is taken over all possible ways to split the total set of chords into two subsets I and J, and we take the subdiagrams C_I and C_J formed by these sets. More precisely, we make two copies c_1 and c_2 of the circle of c and put every chord of C on exactly one of these circles. We take the tensor product of the two chord diagrams (one on c_1 and one on c_2) obtained in this way and sum over all ways of splitting the chords into two sets.

Having the intersection graph mapping, one can define the analogous operations on graphs (which is in fact, even simpler).

Indeed, a chord of a chord diagram corresponds to a vertex of the intersection graph, and two vertices are connected by an edge iff the corresponding chords are linked. Thus, the product operation on graphs corresponding to the multiplication of chord diagrams is just the disjoint sum operation. Obviously, this disjoint sum does not depend on the choice of the splitting point for chord diagram, and the product of graphs is well defined. On the other hand, one can introduce (following Lando, [CDL94]) the $4T$-relation for graphs which corresponds to the $4T$-relation on chord diagrams. It is defined as follows.

To make the definition of the graph algebra precise, we have to describe the correspondence between the four terms A, B, C, D in the graph-theoretic language. Since vertices of the graph correspond to chords, the diagrams A and B differ just by one chord: in A, the vertices α and β are connected by an edge, and in B they are not. The same for C and D. It remains only to explain how to construct the chord diagram C starting from A. In the chord diagram we moved one end of the chord β from one end of the chord α to the other while passing from A to C, see Fig. 7.15. This means that the chord β changes its incidence with all chords incident to α and does not change its incidence with those chords which are not incident to α.

Definition 9 The graph algebra $\mathcal{G}$ is the quotient algebra generated by linear combination of graphs without loops and multiple edges by the four-term relation, with multiplication operation defined to be the disconnected sum.

Remark 5 Note that the surgery over A and the surgery over C lead to the same number of circles; the same is true about B and D.

It turns out that many nice functions defined on graphs (e.g., the Tutte polynomial) satisfy the $4T$-relation. We shall touch on such functions when defining the generating function for counting genera of surfaces spanning a given graph.

It turns out that the chord diagram algebra has its meaningful "framed analogue". It is connected to so-called finite-type invariants of plane curves, see [Lan06] for details. The definition goes as follows.

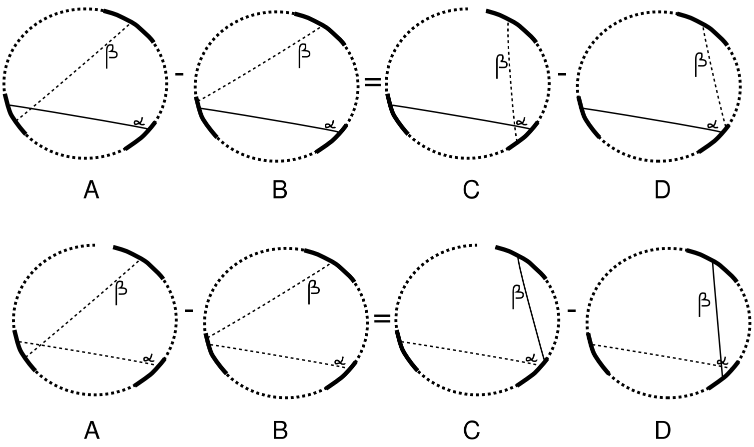


Fig. 7.16 Two types of generalized 4-term relations

Consider the set of all framed chord diagrams on n chords. Now, we set A_n^{cf} to be the $\mathbb{Q}$-linear space generated by all such chord diagrams subject to the generalised $4T$-relations depicted in Fig. 7.16.

The general rule for the defining generalized four-term relation is as follows: we consider some $n-2$ fixed chords and two chords, α and β. If α is positive then in all chord diagrams A, B, C, D, the chord β is of the same sign (in all cases positive or in all cases negative), and the relation looks just as the usual four-term relation: $A - B = C - D$. If α is negative, then while moving from A, B to C, D the chord β changes its sign; moreover, the RHS changes the overall sign: $A - B = D - C$, that means that if in the LHS we take the chord diagram with intersecting α, β with plus then in the RHS we take the diagram with intersecting α, β with minus.

One can easily check that in any special case of the generalized $4T$-relation, the surgery along A gives the same number of circles as the surgery along the diagram with plus in the right hand side (C or D), and the surgery along B gives the same number of circles as the surgery along the diagram with minus from the RHS, namely, we have $f(A) - f(B) = f(C) - f(D)$, where A, B, C, D is the quadruple of diagrams shown in Fig. 7.16.

To the best of the author's knowledge, the connected sum operation on the framed chord diagram algebra has not been proved to be well-defined. Again, one can break two chord diagrams and reconnect them together with respect to the orientation, but it is no proved that different diagrams obtained in this way for different choices of the breaking points are equal modulo generalized four-term relations.

Of course, the coalgebraic operation is well-defined.

Analogously, one defines the bialgebra of framed graphs (at the level of graphs, there is no problem to define the product, we omit the exact definition leaving it for the reader as an exercise).

7.3 The Generating Function for the Embedding Genera

7.3.1 Weight Systems Associated with Lie Algebras: a Brief Review

There is a natural way of associating a number (say, from $\mathbb{Q}$) with a given chord diagram and a given representation of a (semisimple) Lie algebra due to Bar-Natan, [BN95]. It turns out that the corresponding mapping naturally extends (for a fixed representation R of a fixed Lie algebra G) to the mapping from the algebra of chord diagram $\mathcal{A}^c$ to $\mathbb{Q}$ because of the similarity of the $4T$-relation and the Jacobi identity.

We shall deal only with the adjoint representation of Lie algebras; for a Lie algebra G we denote the corresponding mapping from $\mathcal{A}^c$ to $\mathbb{Q}$ by W_G.

The construction goes as follows. Every chord diagram is a cubic graph immersed in the plane with prefixed edge cyclic ordering at each vertex (when drawing chord diagrams on the plane we assume this rotation to be counterclockwise). We shall enlarge the construction for arbitrary cubic graphs with rotation. Namely, we take the structural tensor L_{ijk} of the Lie algebra G with all indices shifted down by using the Cartan-Killing metric. Obviously, $L_{ijk} = L_{jki} = L_{kij}$. Now, we can associate with each trivalent vertex the tensor L_{ijk} with indices corresponding to edges and going counterclockwise i, j, k. Then we contract all tensors along edges (by using the Cartan-Killing metric tensor) and get an integer.

We specify ourselves for the case of the Lie algebra $sl(n)$ and its adjoint representation.

In [Man02], see also [CSM04], we proved the following

Theorem 5 *For a given chord diagram D on n chords, $W_{sl_n}(D)$ is a polynomial in n; its degree does not exceed $k+2$; moreover, it is equal to $k+2$ only in the case when D is a d-diagram.*

As an immediate consequence from this theorem we see that each basis of the chord diagram algebra consisting of chord diagrams contains at least one d-diagram. Indeed, if we consider the weight system to be the coefficient at n^{k+2} for the W_{sl_n}, then it is non-zero only for d-diagrams, so, no d-diagram is expressible as a linear combination of chord diagrams which are not d-diagrams, in $\mathcal{A}^c$.

On the other hand, theorem 5 underlines the special role of d-diagrams amongst all chord diagrams from two points of view: as those (corresponding to graphs) embeddable in S^2 and as those having the highest possible degree of the leading term.

It turns out that this is not an accident: degrees of the W_{sl_n} polynomial are closely connected to possible embeddings of the 4-valent graph into surfaces. We shall touch on this subject in later sections.

7.3.2 Checkerboard Colourable Embeddings

As we have seen, in order to minimise (maximise) the genus of the surface the graph can be embedded into, we have to maximise (minimise) the number of circles obtained as a result of surgeries along two subsets I, J of chords of the initial chord diagram such that $I \sqcup J$ forms the complete set of chords $\{1, \ldots, n\}$.

In this section, we have reformulated this problem in terms of ranks of incidence matrices. A new formulation comes with the generating function.

Let Γ be a framed 4-valent graph on k vertices, and let D be a chord diagram corresponding to some circuit of Γ. Consider the following function

$$f(C) := f(\Gamma) = \sum_{atoms} x^{k+2-g}, \tag{7.3}$$

where the sum is taken over all atoms with the framing taken from Γ and g is the genus of the atom.

Note that here we take the genus to be $g = \frac{2-\chi}{2}$, where χ is the Euler characteristic of the surface; so g may be half-integer.

In view of Soboleva's theorem, $f(C)$ depends merely on the intersection graph of Γ.

Consider the restriction of $f(C)$ to chord diagrams with only positive chords (i.e., to graphs corresponding to orientable atoms).

Theorem 6 *$f(C)$ is a well-defined function on the algebra of chord diagrams*, *i.e.*, *it satisfies the* $4T$*-relation*.

Moreover, *$f(C)$ is multiplicative with respect to the multiplication operations in these algebras*.

Both chord diagram algebra and graph algebra have a commutative and co-commutative Hopf-algebra structure, see, e.g., [BN95, CDL94, MU05a]. By Milnor-Moore theorem, [MM65], each such algebra is isomorphic to the polynomial algebra in its primitive elements. Thus, in order to calculate $f(C)$ for a given chord diagram, one can use the algebraic structure of the Hopf algebra of chord diagrams (or graphs).

Now, we turn to the proof of Theorem 6. Consider a quadruple of chord diagrams A, B, C, D on n chords each forming a $4T$-relation $A - B = C - D$ as shown in Fig. 7.15. We can naturally identify chords from A, B, C, D: there are $n-2$ chords in common, one "fixed chord" (denoted by α in Fig. 7.15) and one "moving" chord (denoted by β).

Consider summands for $f(A), f(B), f(C), f(D)$ coming from the definition (7.3). For those summands where α and β belong to the same subset of chords (say, I, recall, that we deal with $\{1, \ldots, n\} = I \sqcup J$), the genus of surface corresponding to A is equal to the genus corresponding to C, and the genus corresponding to B is equal to the genus corresponding to D (this follows from a straightforward calculation of the number of circles). Thus, these terms give the same contribution to (7.3).

For those summands where α and β belong to different subsets I and J, the corresponding subdiagrams coincide: $A_I = B_I$, $A_J = B_J$, $C_I = D_I$, $C_J = D_J$ because when we move α and β to different subdiagrams, it does not matter whether they intersect or not.

The proof for the graph algebra is analogous.

Arguing as above, one can prove the following

Theorem 7 *The restriction of function f to $\mathcal{A}^c$ satisfies the generalised $4T$-relation.*

Corollary 1 *If for a framed graph Γ satisfying source-sink condition on k vertices and the corresponding chord diagram C we have $\deg f(C) = k$ then Γ is checkerboard-embeddable into the torus.*

Proof Indeed, the maximal possible degree $k+2$ corresponds only to planar embedding; the orientability of the surface is guaranteed by the source-sink condition, and the degree k corresponds to genus 1. □

It is important to know what sort of weight system we obtain from f. It turns out that this weight system is closely connected to W_{sl_n}; roughly speaking, W_{sl_n} can be represented as a sum of 2^{2k} summands (for k chords); some k of them give exactly the function f. Moreover, the W_{sl_n}-polynomial itself gives a "generating function" for *some more general embeddings* (see ahead), however, this generating function has signs $\pm$, so the embeddings are counted with pluses and minuses, which means that not the whole information can be restored from W_{sl_n}.

To clarify the situation, we shall need some more information about calculating W_{sl_n} (see [Man02] and [CSM04]). We will in fact work in gl_n; the result of final contraction will be the same as that for sl_n.

Given a chord diagram D, fix an arc of it and break this diagram along the arc. Then $W_{sl_n}(D) = Tr(x \to [\dots[,x]\dots])$ where by $[\dots[,]\dots]$ we mean the result of consequent commutators of x with elements of the Lie algebra, where for each chord we take α on one end of the chord and the dual element α^* on the other end of the chord and sum up when α runs over the basis of the Lie algebra. Let us be more specific. Consider the diagram shown in Fig. 7.17, upper part. Take an arbitrary point x on the circle different from a chord end.

The "long" commutator can be rewritten according to $[p,q] = pq - qp$. Thus we get 2^{2k} terms of the following form:

$$(-1)^l Tr(x \to p_1 \dots p_l x q_{2k-l} \dots q_1), \tag{7.4}$$

where $p_1, \dots p_l$ are variables corresponding to some chord ends in the usual order, and $q_{2k-l}, \dots, q_1$ are the remaining chord ends in the reversed order. For instance, for a diagram shown in Fig. 7.17, the summands of the form (7.4) looks like $(-1)Tr(x \to d^* baxcda^* b^* c^*)$.

After that, we have to simplify the trace formula by taking contraction of double occurrences of words one-by one.

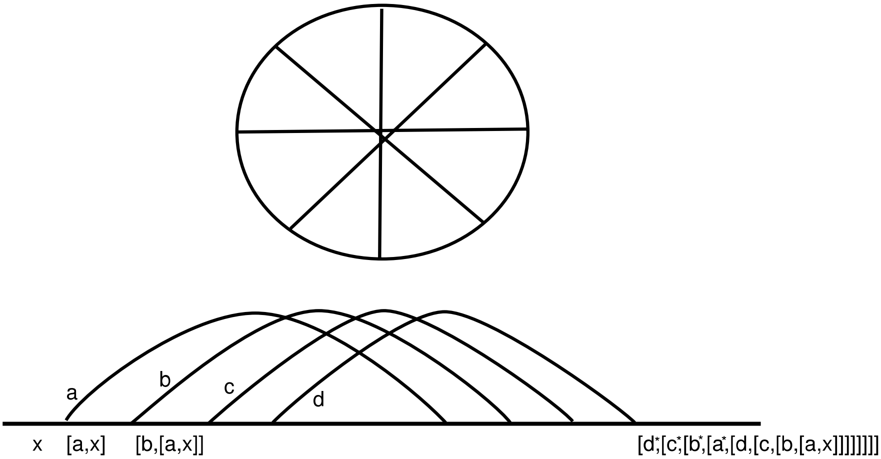


Fig. 7.17 The contraction operator

Now, two simple $gl(n)$-contraction formulae come into play:

$$Tr(A\alpha B\alpha^*) = Tr(A)Tr(B); \qquad Tr(A\alpha)Tr(B\alpha^*) = Tr(AB). \tag{7.5}$$

Here the sum is taken over α running over a basis of gl_n, whence α^* runs over the dual basis; A and B may be arbitrary $n \times n$ matrices. Looking at the formulae (7.5) and representing any factor $Tr(p_1 \cdots p_k)$ by a circle with points $p_1, \dots, p_n$ arranged in the clockwise direction, we see that the formulae (7.5) correspond to splitting (merging) of two circles into one. Here p_i correspond to some chord ends lying on those circles.

With these formulae, we may calculate any trace of the form (7.4).

This means precisely that the contraction rules in sl_n correspond to surgery operations.

We first transform the trace formula $Tr(x \to AxB) = Tr(xAx^*B)$ according to the first formula of (7.5) and get $Tr(A)Tr(B)$. Here A and B collect some variables a, b, c, d and their dual ones a^*, b^*, c^*, d^*. These variables are now inside the product of two traces, and we shall represent it schematically by taking chord ends of the chord diagram into two separate circles as shown in Fig. 7.18. One circle will contain those variables (from A) which were on the left hand side with respect to x and the other set will contain those variables (from B) which were on the right hand side with respect to x.

Schematically, it means that we have to collect 2^{2k} terms corresponding to different splittings of chord ends into two circles. For instance, $(-1)d^*baxcda^*b^*c^*$ corresponds to the diagram shown in Fig. 7.18.

Call a summand *good* if for each chord both ends belong to the same subset. Such summands contribute with sign $+$. Now, it follows from definition that the contribution of good summands gives exactly the function f.

Now, we explain the geometric meaning of the function f itself.

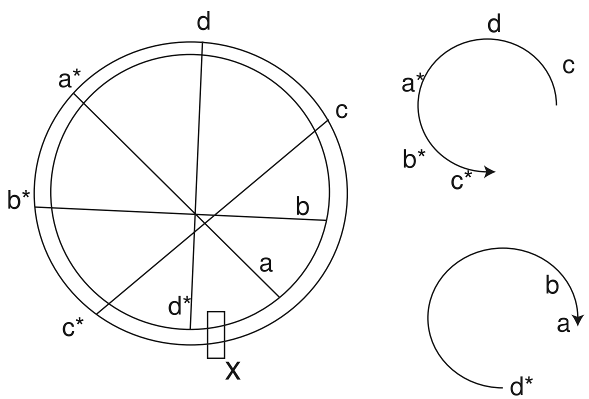


Fig. 7.18 A chord diagram on 2 circles corresponding to $(-1)d^*baxcda^*b^*c^*$

7.3.3 Embeddings with Orienting $\mathbb{Z}_2$-homology Class

As we have seen, the generating function f for all $\mathbb{Z}_2$-zero homologous embeddings can be written in a form of a function on the chord diagram algebra, which is closely connected to the Vassiliev finite type invariants of the type sl_n.

To calculate all the summands for the generating function (not necessarily good ones) we deal with arbitrary ways of splittings of chord ends into two sets. This leads to an arbitrary way of attaching bands to the annulus, and, finally, gives the generating function for arbitrary embeddings of our framed graph provided that the $\mathbb{Z}_2$-homology represented by this graph corresponds to an orienting cycle.

Let us be more specific. First of all, our weight systems corresponding to Lie algebras are defined only for the case when all chords of the chord diagram are positive (of framing zero). On the other hand, the generating function can be written down for an arbitrary graph. As we shall see, all generating functions will satisfy the generalized $4T$-relation and give a certain generalized weight system in the sense of Lando.

Analogously to f, we define the function $\tilde{f}$ as follows:

$$\tilde{f}(C) := \tilde{f}(\Gamma) = \sum_{\mathbb{Z}_2\text{-orient.emb}} x^{k+2-g}, \tag{7.6}$$

where the sum is taken over $\mathbb{Z}_2$-orientable embeddings.

Analogously to Theorem 7, one can prove

Theorem 8 *The restriction of $\tilde{f}$ to chord diagrams satisfies the $4T$-relation.*

This immediately yields the following

Corollary 2 *If $\tilde{f}(C)$ has maximal degree k then the corresponding framed 4-valent graph is checkerboard-embeddable into the torus.*

Unfortunately, the converse is not true: the graph corresponding to the chord diagram consisting of three pairwise-linked chords is embeddable into torus, though the $sl(n)$-function on this chord diagram gives zero.

Now, let us try to understand the geometric meaning of $\tilde{f}$. For a chord diagram on k chords, the function f can be represented by 2^{2k} summands which clearly correspond to the terms obtained after contracting the sl_n-sum according to the rules (7.5). These are exactly the 2^k summands corresponding to those contractions where we place both ends of each chord on the same circle.

It would be very interesting to understand the nature of contraction along chords connecting points on different circles, see Fig. 7.18.

First of all, in the expansion for a commutator, we may get a minus sign, which corresponds to a surgery along a chord diagram with two ends on different circles (indeed, the sign comes from $[a,b] = ab - ba$, and the total sign counts the number of letters on the left hand side from x in the expansion of the iterated commutator).

Now assume we perform count the $sl(n)$ weight system for a chord diagram. As we have seen, after expanding the commutators, the expressions for two circles go in the opposite order. Thus, in order to restore the real picture of embedding genera, one should perform overtwisted surgeries along chords with endpoints on different circles.

Thus, W_{sl_n}-weight system estimates the genera of the surfaces the graph can be embedded to, but:

1. It counts embeddings with signs, thus, for some embedding it does not give a real estimate for the genus. For instance, for the chord diagram on three pairwise linked chords, the value of the W_{sl_n}-function is zero.
2. The contraction corresponding to chords with ends on different circles count embeddings of the same graph, but with another framing.

So, the geometric meaning of $\tilde{f}$ is not yet completely understood.

7.3.4 The General Case

Arguing as above, one can consider the case of arbitrary embedding of four-valent graphs with opposite edge structure preserved. In this case, for a given rotating cycle C, the neighbourhood of its (smoothed) image on a surface can be either a cylinder or a Möbius band, when the type of the surface depends only on the orientability of the corresponding $\mathbb{Z}_2$-homology class.

We restrict ourselves by saying that the case of the Möbius band can be considered analogously, and when passing to the orienting double covering of the surface, one gets a generating function also satisfying the four-term relation.

7.4 Unsolved Problems

The method of matrix ranks gives an explicit polynomial solution only in a very limited number of cases. It is known that for every surface of rank g there is a solution to Problem 1 which is polynomial in the number of chords [FMR79]. It would be very interesting to get such algorithms via matrices.

Both bialgebra of chord diagrams and coalgebra of framed diagrams are well known.

However, the weight system approach is applicable only to chord diagrams with all chords having framing zero (which correspond to chord diagrams satisfying the source-sink condition). It is not yet known how to apply any similar techniques for framed chord diagrams (with chords of framing one). Possibly, one should treat positive chords by means of $sl(n)$-tensors, and negative chords by means of $so(n)$ or maybe $sp(n)$-tensors (cf. [BN95] and [CSM04]).

It is well known (see [LZ03]) that many enumerative problems in graph theory can be solved by using Gaussian integrals. However, these problems usually count generating functions for genera coming from *all possible* gluings, say, of a polygon. In our problem, we have to fix a graph (or a chord diagram), and consider the generating function for genera of surface this graph can be embedded into. Possibly, this can be done by means of a Gauss integral for all *admissible* gluings of crosses at vertices of the diagram.

Any framed four-valent graph can be represented as a shadow of a virtual link. Problem 1 is devoted to finding the minimal atom genus for a link with this shadow and some classical crossing setup. Fixing such a link with one white cell leads to a chord diagram and a rotating circuit. The Kauffman bracket of this link is very similar to the generating function for the solution f of Problem 1: it has 2^n summands. It is known [Mel00] that the Kauffman bracket (after some variable change) giving a series of weight systems integrates to give the Kauffman 2-variable polynomial of the knot. It would be interesting to know which are the knot invariants that can be obtained by integrating (in Kontsevich's sense) the generating functions for Problem 1 and which knot-theoretic properties they can detect.

In [Ros99] a new viewpoint to the planarity problem of four-valent graph is established: one uses certain vector spaces over $\mathbb{Z}_2$ calculated out of edges and vertices of the graph (together with their opposite structure), and proves that a 4-valent framed graph is realizable iff there is no homological obstruction coming from a certain scalar product.

It would be extremely interesting to understand whether this $\mathbb{Z}_2$-homological approach generalizes for embeddings of arbitrary genera.

Acknowledgements I express my gratitude for useful discussions to V.A. Vassiliev, A.T. Fomenko, L.H. Kauffman, and O.R. Campoamor-Stursberg.

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Chapter 8
Geometric Topology and Field Theory on 3-Manifolds

Kishore Marathe

Abstract In recent years the interaction between geometric topology and classical and quantum field theories has attracted a great deal of attention from both the mathematicians and physicists. This interaction has been especially fruitful in low dimensional topology. In this article We discuss some topics from the geometric topology of 3-manifolds with or without links where this has led to new viewpoints as well as new results. They include in addition to the early work of Witten, Casson, Bott, Taubes and others, the categorification of knot polynomials by Khovanov. Rozansky, Bar-Natan and Garofouladis and a special case of the gauge theory to string theory correspondence in the Euclidean version of the theories, where exact results are available. We show how the Witten-Reshetikhin-Turaev invariant in $SU(n)$ Chern-Simons theory on S^3 is related via conifold transition to the all-genus generating function of the topological string amplitudes on a Calabi-Yau manifold. This result can be thought of as an interpretation of TQFT as TQG (Topological Quantum Gravity). A brief discussion of Perelman's work on the geometrization conjecture and its relation to gravity is also included.

8.1 Introduction

This paper is based in part on my seminars given at the Max Planck Institute for Mathematics in the Sciences, and at other institutes, notably at the IIT (Mumbai), Universitá di Firenze, University of Florida at Gainsville, Inter University Center

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_8,

for Astronomy and Astrophysics, University of Pune, India and conferences given at the XXIV workshop on Geometric Methods in Physics, Poland [Mar06a] and the Blaubeuren workshop "Mathematical and Physical Aspects of Quantum Gravity" [Mar06b]. In my lectures on the mathematical and physical aspects of gauge theories in New York and Florence in the early 1980s, I began using the phrase gauge theoretic topology and geometry to describe a rapidly developing area of mathematics, where unexpected advances were made with essential use of gauge theory. By the late 1990s it was evident that in addition to gauge theory, many other parts of theoretical physics were contributing new ideas and methods to the study of topology, geometry, algebra and other fields of mathematics. I then began using the phrase "Physical Mathematics" to collectively denote the areas of mathematics benefitting from an infusion of ideas from physics. It appears in print for the first time in [MMF95] and more recently, in [Mar01] and is the theme of my forthcoming book [Mar10b] "Topics in Physical Mathematics" with Springer-Verlag.

During the past two decades a surprising number of new structures have appeared in the geometric topology of low-dimensional manifolds. Chiral, Vertex, Affine and other infinite dimensional algebras are related to 2d CFT and string theory as well as to sporadic finite groups such as the monster. In three dimensions there are the polynomial link invariants of Jones, Kaufman. HOMFLY and others, Witten-Reshetikhin-Turaev invariants of 3-manifolds, Casson invariants of homology spheres and Fukaya-Floer instanton homologies. In 4 dimensions we have the instanton invariants of Donaldson and the monopole invariants of Seiberg-Witten and the list continues to grow. These invariants may be roughly split into two groups. Those in the first group arise from combinatorial (algebraic or topological) considerations and can be computed algorithmically. Those in the second group arise from the study of moduli spaces of solutions of partial differential equations which have their origin in physical field theories. Here the computations generally depend on special conditions or extra structures. The main aim of these lectures is to study some of the relations that have been found between the invariants from the two groups and more generally, to understand the influence of ideas from field theories in geometric topology and vice versa. For example, many physicists consider supersymmetric string theory to be the most promising candidate to lead to the so called grand unification of all four fundamental forces. Unifying different string theories into a single theory (such as M-theory) would seem to be the natural first step. This goal seems distant at this time, since even the physical foundations for such unification are not yet clear. However, in mathematics it has led to new areas such as mirror symmetry, Calabi-Yau spaces, Gromov-Witten theory, and Gopakumar-Vafa invariants. The earliest and the best understood example of the relationship between invariants from the two groups is illustrated by the Casson invariant which was defined by using combinatorial topological methods. Taubes found a gauge theoretic interpretation of the Casson invariant as the Euler characteristic by using the generalized Poincaré-Hopf index which can also be obtained by using Floer's instanton homology. Yet there is no algorithm for computing the homology groups themselves.

Topological quantum field theory was ushered in by Witten in his 1989 paper [Wit89] "QFT and the Jones' polynomial". WRT invariants arose as a byproduct of the quantization of Chern-Simons theory used to characterize the Jones' polynomial. At this time, it is the only known geometric characterization of the Jones' polynomial, although the Feynman integrals used by Witten do not yet have a mathematically acceptable definition. Space-time manifolds in such theories are compact Riemannian manifolds. They are referred to as Euclidean theories in the physics literature. Their role in physically interesting theories is not clear at this time and they should be regarded as toy models.

In the last few years we have celebrated a number of special events. The Gauss' year and the 100th anniversary of Einstein's "Annus Mirabilis" (the miraculous year) are the most important among these. Indeed, Gauss' "Disquisitiones generale circa superficies curvas" was the basis and inspiration for Riemann's work which ushered in a new era in geometry. It is an extension of this geometry that is the cornerstone of relativity theory. More recently, we have witnessed the marriage between Gauge Theory and the Geometry of Fiber Bundles from the sometime warring tribes of Physics and Mathematics. Marriage brokers were none other than Chern and Simons. The 1975 paper by Wu and Yang [WY75] can be regarded as the announcement of this union. It has led to many wonderful offspring. The theories of Donaldson, Chern-Simons, Floer-Fukaya, Seiberg-Witten, and TQFT are just some of the more famous members of their extended family. Quantum Groups, CFT, Supersymmetry (SUSY), String Theory, Gromov-Witten theory and Gravity also have close ties with this family. Later in this paper we will discuss one particular relationship between gauge theory and string theory, that has recently come to light. The qualitative aspects of Chern-Simons theory as string theory were investigated by Witten [Wit95] almost ten years ago. Before recounting the main idea of this work we review the Feynman path integral method of quantization which is particularly suited for studying topological quantum field theories. For general background on gauge theory and geometric topology see, for example, [MM92, MMF95].

We now give a brief description of the contents of the paper. In Sect. 8.2 we discuss Gauss' Formula for Linking Number of knots, the earliest example of TFT (Topological Field Theory) and its recent extension to self linking invariants. Witten's fundamental work on supersymmetry and Morse theory is covered in Sect. 8.3. Chern-Simons theory is introduced in Sect. 8.6. Its relation to Casson invariant via the moduli space of flat connections is explained in Sect. 8.7. Ideas from Sects. 8.3 and 8.6 are used in Sect. 8.8 to define the Fukaya-Floer homology. This homology provides the categorification of the Casson invariant. Knot polynomials and their categorification are discussed in Sects. 8.9 and 8.10 respectively. Section 8.11 is devoted to a general discussion of TQFT and its applications to invariants of links and 3-manifolds. Atiyah-Segal axioms for TQFT are introduced in Sect. 8.11.1. In Sect. 8.11.2 we define quantum observables and introduce the Feynman path integral approach to QFT. The Euclidean version of this theory is applied in Sect. 8.11.3 to the Chern-Simons Lagrangian to obtain the skein relations for the Jones-Witten polynomial of a link in S^3. A by product of this is the family of WRT invariants of 3-manifolds. They are discussed in Sect. 8.11.4. Section 8.12 is devoted to studying the relation between WRT invariants of S^3 with gauge group $SU(n)$ and the open and closed string amplitudes in generalized Calabi-Yau manifolds. Change in geometry and topology via conifold transition which plays an important role in this study is introduced in Sect. 8.12.1 in the form needed for our specific problem. Expansion of free energy and its relation to string amplitudes is given in Sect. 8.12.2. This result is a special case of the general program introduced by Witten in [Wit95]. A realization of this program even within Euclidean field theory promises to be a rich and rewarding area of research. We have given some indication of this at the end of this section. Links between Yang-Mills, gravity and string theory are considered in the concluding Sect. 8.13. Relation of Yang-Mills equations with Einstein's equations for gravitational field in the Euclidean setting is considered in Sect. 8.13. Various formulations of Einstein's equations for gravitational field are discussed in Sect. 8.13.1. They also make a surprising appearance in Perelman's proof of Thurston's Geometrization conjecture. A brief indication of this is given in Sect. 8.13.2.

We have included some basic material and given more details than necessary to make the paper essentially self-contained. A fairly large number of references ranging from January 1833 to January 2009, when the Heidelberg conference was held, are included to facilitate further study and research in this exciting and rapidly expanding area.

8.2 Gauss' Formula for Linking Number of Knots

Knots have been known since ancient times but knot theory is of quite recent origin. One of the earliest investigations in combinatorial knot theory is contained in several unpublished notes written by Gauss between 1825 and 1844 and published posthumously as part of his Nachlaß (estate). They deal mostly with his attempts to classify "Tractfiguren" or plane closed curves with a finite number of transverse self-intersections. However, one fragment deals with a pair of linked knots. We reproduce a part of this fragment below.

> Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie $r = (x, y, z)$; der zweiten $r' = (x', y', z')$ und
>
> $$\int\int \frac{(r'-r)\cdot(dr \times dr')}{|r'-r|^3} = V$$
>
> dann ist dies Integral durch beide Linien ausgedehnt $= 4\pi m$ und m die Anzahl der Umschlingungen. Der Werth ist gegenseitig, d.i. er bleibt derselbe, wenn beide Linien gegen einander umgetauscht werden 1833. Jan. 22.

In this fragment of a note from his Nachlaß, Gauss had given an analytic formula for the linking number of a pair of knots. This number is a combinatorial topological invariant. As is quite common in Gauss's work, there is no indication of how he obtained this formula. The title of the section where the note appears, "Zur Electrodynamik" ("On Electrodynamics") and his continuing work with Weber on the properties of electric and magnetic fields leads us to guess that it originated in the study of magnetic field generated by an electric current flowing in a curved wire.

Maxwell knew Gauss's formula for the linking number and its topological significance and its origin in electromagnetic theory. In fact, before he knew of Gauss's formula, he had rediscovered it. He mentions it in a letter to Tait dated December 4, 1867. He wrote several manuscripts which study knots, links and also addressed the problem of their classification. In these and other topological problems his approach was not mathematically rigorous but was rather based on his deep understanding of physics. Indeed this situation persists today in several mathematical results obtained by physical reasoning. Like Maxwell, Tait used his physical intuition to correctly classify all knots up to seven crossings and made a number of conjectures, the last of which remained open for over hundred years.

In obtaining a topological invariant by using a physical field theory, Gauss had anticipated Topological Field Theory by almost 150 years. Even the term topology was not used then. It was introduced in 1847 by J.B. Listing, a student and protegé of Gauss, in his essay "Vorstudien zur Topologie". Gauss's linking number formula can also be interpreted as the equality of topological and analytic degree of the function λ defined by

$$\lambda(\vec{r}, \vec{r'}) := \frac{(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|}, \quad \forall(\vec{r}, \vec{r'}) \in C \times C'$$

It is well defined by the disjointness of C and C'. If ω denotes the standard volume form on S^2, then the pull back $\lambda^*(\omega)$ of ω to $C \times C'$ is precisely the integrand in the Gauss formula and $\int \omega = 4\pi$. One can check that the topological degree of λ equals the linking number m.

Recently, Bott and Taubes have used these ideas to study a self-linking invariant of knots [BT94]. It turns out that this invariant belongs to a family of knot invariants, called finite type invariants, defined by Vassiliev. Gauss forms with different normalization are used by Kontsevich [Kon94] in the formula for this invariant and it is stated that the invariant is an integer equal to the second coefficient of the Alexander-Conway polynomial of the knot. In [BC98, BC99] Bott and Cattaneo obtain invariants of rational homology 3-spheres in terms of configuration space integrals. Kontsevich views these formulas as forming a small part of a very broad program to relate the invariants of low-dimensional manifolds, homotopical algebras, and non-commutative geometry with topological field theories and the calculus of Feynman diagrams. It seems that the full realization of this program would require the best efforts of mathematicians and physicists for years to come.

8.3 Supersymmetry and Morse Theory

Classical Morse theory on a finite dimensional, compact, differentiable manifold M relates the behaviour of critical points of a suitable function on M with topological information about M. The relation is generally stated as an equality of certain polynomials as follows. Recall first that a smooth function $f : M \to \mathbb{R}$ is called a **Morse function** if its critical points are isolated and non-degenerate. If $x \in M$ is a critical point (i.e. $df(x) = 0$), then by Taylor expansion of f around x, we obtain the Hessian of f at x defined by

$$\left\{ \frac{\partial^2 f}{\partial x^i \partial x^j}(x) \right\}.$$

Then the non-degeneracy of the critical point x is equivalent to the non-degeneracy of the quadratic form determined by the Hessian. The dimension of the negative eigenspace of this form is called the **Morse index**, or simply index, of f at x and is denoted by $\mu_f(x)$ or simply $\mu(x)$ when f is understood. It can be verified that these definitions are independent of the choice of the local coordinates. Let m_k be the number of critical points with index k. Then the **Morse series** of f is the formal power series

$$\sum_k m_k t^k, \quad \text{where } m_k = 0, \forall k > dim\, M.$$

Recall that the Poincaré series of M is given by $\sum_k b_k t^k$, where $b_k \equiv b_k(M)$ is the k-th Betti number of M. The relation between the two series is given by

$$\sum_k m_k t^k = \sum_k b_k t^k + (1+t) \sum_k q_k t^k, \tag{8.1}$$

where q_k are non-negative integers. Comparing the coefficients of the powers of t in this relation leads to the well-known **Morse inequalities**

$$\sum_{k=0}^{i} m_{i-k}(-1)^k \geq \sum_{k=0}^{i} b_{i-k}(-1)^k, \quad 0 \leq i \leq n-1,$$
$$\sum_{k=0}^{n} m_{n-k}(-1)^k = \sum_{k=0}^{n} b_{n-k}(-1)^k.$$

The Morse inequalities can also be obtained from the following observation. Let C^* be the graded vector space over the set of critical points of f. Then the Morse inequalities are equivalent to the existence of a certain coboundary operator $\partial : C^* \to C^*$ so that $\partial^2 = 0$ and the cohomology of the complex (C^*, ∂) coincides with the deRham cohomology of M.

In his fundamental paper [Wit82], Witten arrives at precisely such a complex by considering a suitable supersymmetric quantum mechanical Hamiltonian. Witten showed how the standard Morse theory (see, for example, Milnor [Mil73]) can be modified by considering the gradient flow of the Morse function f between pairs of critical points of f. One may think of this as a sort of relative Morse theory. He was motivated by the phenomenon of the quantum mechanical tunneling. We now discuss this approach. From a mathematical point of view, supersymmetry may be regarded as a theory of operators on a Z_2-graded Hilbert space. In recent years this theory has attracted a great deal of interest from theoretical point of view even though as yet there is no physical evidence for its existence.

8.3.1 Graded Algebraic Structures

In this subsection we recall briefly a few important properties of graded vector spaces and graded operators in a slightly more general situation than is immediately needed. We will use this information again in studying Khovanov homology. Graded algebraic structures appear naturally in many mathematical and physical theories. We shall restrict our considerations only to $\mathbb{Z}$- and $\mathbb{Z}_2$-gradings. The most basic such structure is that of a graded vector space which we now describe. Let V be a vector space. We say that V is **$\mathbb{Z}$-graded** (resp. **$\mathbb{Z}_2$-graded**) if V is the direct sum of vector subspaces V_i, indexed by the integers (resp. integers mod. 2), i.e.

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \quad (\text{resp. } V = V_0 \oplus V_1).$$

The elements of V_i are said to be **homogeneous** of **degree** i. In the case of $\mathbb{Z}_2$-grading it is customary to call the elements of V_0 (resp. V_1) **even** (resp. **odd**). If V and W are two $\mathbb{Z}$-graded vector spaces, a linear transformation $f : V \to W$ is said to be **graded** of **degree** k if $f(V_i) \subset W_{i+k},\ \forall i \in \mathbb{Z}$. If V and W are $\mathbb{Z}_2$-graded, then a linear map $f : V \to W$ is said to be **even** if $f(V_i) \subset W_i,\ i \in \mathbb{Z}_2$ and is said to be **odd** if $f(V_i) \subset W_{i+1},\ i \in \mathbb{Z}_2$. An **algebra** A is said to be **$\mathbb{Z}$-graded** if A is $\mathbb{Z}$-graded as a vector space, i.e.

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

and $A_i A_j \subset A_{i+j},\ \forall i, j \in \mathbb{Z}$. An ideal $I \subset A$ is called a **homogeneous ideal** if

$$I = \bigoplus_{i \in \mathbb{Z}} (I \cap A_i).$$

A similar definition can be given for a $\mathbb{Z}_2$-graded algebra. In the physical literature a **$\mathbb{Z}_2$-graded algebra** is referred to as a **superalgebra**. Other algebraic structures (such as Lie, commutative etc.) have their superalgebra counterparts. An example of a $\mathbb{Z}$-graded algebra is given by the exterior algebra of differential forms $\Lambda(M)$ of a manifold M if we define $\Lambda^i(M) = 0$ for $i < 0$. The exterior differential d is a graded linear transformation of degree 1 of $\Lambda(M)$. The graded or quantum dimension of V is defined by

$$dim_q\, V = \sum_{i \in \mathbb{Z}} q^i (dim(V_i)),$$

where q is a formal variable. If we write $q = exp 2\pi i z, z \in \mathbb{C}$ then $dim_q\, V$ can be regarded as the Fourier expansion of a complex function. A spectacular application of this occurs in the study of finite groups. We discuss this briefly in the next paragraph. It is not needed in the rest of the paper. However, it has surprising connections with conformal field theory and vertex algebras. It does not deal with 3-manifolds and may be omitted without loss of continuity.

8.4 Monstrous Moonshine

It was his study of Kepler's sphere packing conjecture, that led John Conway to the discovery of his sporadic simple group. Soon thereafter the last holdouts in the complete list of the 26 finite sporadic simple groups were found. All the infinite families of finite simple groups (such as the groups $\mathbb{Z}_p$, for p a prime number and alternating groups $A_n, n > 4$ that we study in the first course in algebra) were already known. So the classification of finite simple groups was complete. It ranks as the greatest achievement of twentieth century mathematics. Hundreds of mathematicians contributed to it. The various parts of the classification together fill more than ten thousand pages. Conway's group and other sporadic simple groups are closely related to the symmetries of lattices. The study of representations of the largest of these groups (called the Friendly Giant or Fisher-Griess Monster) has led to the creation of a new field of mathematics called Vertex algebras. They turn out to be closely related to the chiral algebras in conformal field theory. These and other ideas inspired by string theory have led to a proof of Conway and Norton's Moonshine conjectures (see, for example, Borcherds [Bor92], and the book [FLM88] by Frenkel, Lepowski, Meurman). The monster Lie algebra is the simplest example of a Lie algebra of physical states of a chiral string on a 26-dimensional orbifold. This algebra can be defined by using the infinite dimensional graded representation V of the monster simple group. Its quantum dimension is related to Jacobi's $SL(2,\mathbb{Z})$ hauptmodule (elliptic modular function of genus 0) $j(q)$, where $q = e^{2\pi i z}, z \in \mathbb{H}$ by

$$dim_q V = j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots$$

The above formula is one small part in the proof of the moonshine conjectures. For more information see my review [Mar09] in the Mathematical Intelligencer.

8.5 SUSY Quantum Theory

The Hilbert space E of a supersymmetric theory is Z_2-graded, i.e. $E = E_0 \oplus E_1$, where the even (resp. odd) space E_0 (resp. E_1) is called the space of bosonic (resp. fermionic) states. These spaces are distinguished by an operator $S : E \to E$ defined by

$$Su = u, \quad \forall u \in E_0,$$
$$Sv = -v, \quad \forall v \in E_1.$$

The operator S is interpreted as counting the number of fermions modulo 2. A supersymmetric theory begins with a collection $\{Q_i \mid i = 1, \ldots, n\}$ of supercharge (or supersymmetry) operators on E which are of odd degree, i.e. anti-commute with S

$$SQ_i + Q_i S = 0, \quad \forall i \tag{8.2}$$

and satisfy the following anti-commutation relations

$$Q_i Q_j + Q_j Q_i = 0, \quad \forall i \neq j. \tag{8.3}$$

The dynamics is introduced by the Hamiltonian operator H which commutes with the supercharge operators and is usually required to satisfy additional conditions. For example, in the simplest non-relativistic theory one requires that

$$H = Q_i^2, \quad \forall i. \tag{8.4}$$

In fact this simplest supersymmetric theory has surprising connections with Morse theory which we now discuss.

Let M be a compact differentiable manifold and define E by

$$E := \Lambda(M) \otimes \mathbb{C}.$$

The natural grading on $\Lambda(M)$ induces a grading on E. We define

$$E_0 := \bigoplus_j \Lambda^{2j}(M) \otimes \mathbb{C} \quad \left(\text{resp. } E_1 := \bigoplus_j \Lambda^{2j+1}(M) \otimes \mathbb{C}\right)$$

the space of complex-valued even (resp. odd) forms on M. The exterior differential d and its formal adjoint δ have natural extension to odd operators on E and thus satisfy (8.2). We define supercharge operators $Q_j,\ j = 1, 2$, by

$$Q_1 = d + \delta, \tag{8.5}$$

$$Q_2 = i(d - \delta). \tag{8.6}$$

The Hamiltonian is taken to be the Hodge-deRham operator extended to E, i.e.

$$H = d\delta + \delta d. \tag{8.7}$$

The relations $d^2 = \delta^2 = 0$ imply the supersymmetry relations (8.3) and (8.4). We note that in this case bosonic (resp. fermionic) states correspond to even (resp. odd) forms. The relation to Morse theory arises in the following way. If f is a Morse function on M, define a one-parameter family of operators

$$d_t = e^{-ft} d e^{ft}, \qquad \delta_t = e^{ft} \delta e^{-ft}, \quad t \in \mathbb{R} \tag{8.8}$$

and the corresponding supersymmetry operators

$$Q_{1,t} = d_t + \delta_t, \qquad Q_{2,t} = i(d_t - \delta_t), \qquad H_t = d_t \delta_t + \delta_t d_t.$$

It is easy to verify that $d_t^2 = \delta_t^2 = 0$ and that $Q_{1,t},\ Q_{2,t},\ H_t$ satisfy the supersymmetry relations (8.3) and (8.4). The parameter t interpolates between the deRham cohomology and the Morse indices as t goes from 0 to $+\infty$. At $t = 0$, the number of linearly independent eigenvectors with zero eigenvalue is just the k-th Betti number

b_k when $H_0 = H$ is restricted to act on k-forms. In fact these ground states of the Hamiltonian are just the harmonic forms. On the other hand, for large t the spectrum of H_t simplifies greatly with the eigenfunctions concentrating near the critical points of the Morse function. It is in this way that the Morse indices enter into this picture. We can write H_t as a perturbation of H near the critical points. In fact, we have

$$H_t = H + t\sum_{j,k} f_{,jk}[\alpha^j, i_{X^k}] + t^2\|df\|^2,$$

where $\alpha^j = dx^j$ acts by exterior multiplication, $X^k = \partial/\partial x^k$ and i_{X^k} is the usual action of inner multiplication by X^k on forms and the norm $\|df\|$ is the norm on $\Lambda^1(M)$ induced by the Riemannian metric on M. In a suitable neighborhood of a fixed critical point taken as origin, we can approximate H_t up to quadratic terms in x^j by

$$\overline{H}_t = \sum_j \left(-\frac{\partial^2}{\partial x_j^2} + t^2\lambda_j^2 x_j^2 + t\lambda_j[\alpha^j, i_{X^j}] \right),$$

where λ_j are the eigenvalues of the Hessian of f. The first two terms correspond to the quantized Hamiltonian of a harmonic oscillator with eigenvalues

$$t\sum_j |\lambda_j|(1 + 2N_j),$$

whereas the last term defines an operator with eigenvalues $\pm\lambda_j$. It commutes with the first and thus the spectrum of $\overline{H}_t$ is given by

$$t\sum_j [|\lambda_j|(1 + 2N_j) + \lambda_j n_j],$$

where N_j's are non-negative integers and $n_j = \pm 1$. We remark that the classical harmonic oscillator was the first dynamical system that was quantized by using the canonical quantization principle. Dirac introduced his creation and annihilation operators to obtain its spectrum without solving the corresponding Schrodinger equation. Feynman used this result to test his path integral quantization method. Restricting H to act on k-forms we can find the ground states by requiring all the N_j to be 0 and by choosing n_j to be 1 whenever λ_j is negative. Thus the ground states (zero eigenvalues) of H correspond to the critical points of Morse index k. All other eigenvalues are proportional to t with positive coefficients. Starting from this observation and using standard perturbation theory, one finds that the number of k-form ground states equals the number of critical points of Morse index k. Comparing this with the ground state for $t = 0$, we obtain the weak Morse inequalities $m_k \geq b_k$. As we observed in the introduction the strong Morse inequalities are equivalent to the existence of a certain cochain complex which has cohomology isomorphic to $H^*(M)$, the cohomology of the base manifold M. Witten defines C_p, the set of p-chains of this complex, to be the free group generated by the critical points of Morse index p. He then argues that the operator d_t defined in (8.8) defines in the limit as $t \to \infty$ a coboundary operator

$$d_\infty : C_p \to C_{p+1}$$

and that the cohomology of this complex is isomorphic to the deRham cohomology of Y.

Thus we see that in establishing both the weak and strong form of Morse inequalities a fundamental role is played by the ground states of the supersymmetric quantum mechanical system (8.5), (8.6), (8.7). In a classical system the transition from one ground state to another is forbidden, but in a quantum mechanical system it is possible to have tunneling paths between two ground states. In gauge theory the role of such tunneling paths is played by instantons. Indeed, Witten uses the prescient words "instanton analysis" to describe the tunneling effects obtained by considering the gradient flow of the Morse function f between two ground states (critical points). If β (resp. α) is a critical point of f of Morse index $p+1$ (resp. p) and Γ is a gradient flow of f from β to α, then by comparing the orientation of negative eigenspaces of the Hessian of f at β and α, Witten defines the signature n_Γ of this flow. By considering the set S of all such flows from β to α, he defines

$$n(\alpha, \beta) := \sum_{\Gamma \in S} n_\Gamma .$$

Now defining δ_∞ by

$$\delta_\infty : C_p \to C_{p+1} \quad \text{by } \alpha \mapsto \sum_{\beta \in C_{p+1}} n(\alpha, \beta)\beta, \tag{8.9}$$

he shows that (C_*, δ_∞) is a cochain complex with integer coefficients. Witten conjectures that the integer-valued coboundary operator δ_∞ actually gives the integral cohomology of the manifold M. The complex (C_*, δ_∞), with the coboundary operator defined by (8.9), is referred to as the **Witten complex**. As we will see later, Floer homology is the result of such "instanton analysis" applied to the gradient flow of a suitable Morse function on the moduli space of gauge potentials on an integral homology 3-sphere. Floer has also used these ideas to study a "symplectic homology" associated to a manifold. A corollary of this theory proves the Witten conjecture for finite dimensional manifolds (see [Sal90] for further details), namely

$$H^*(C_*, \delta_\infty) = H^*(M, \mathbb{Z}).$$

A direct proof of the conjecture may be found in the appendix to K.C. Chang [Cha93]. A detailed study of the homological concepts of finite dimensional Morse theory in analogy with Floer homology may be found in M. Schwarz [Sch93]. While many basic concepts of "Morse homology" can be found in the classical investigations of Milnor, Smale and Thom, its presentation as an axiomatic homology theory in the sense of Eilenberg and Steenrod [ES52] is given for the first time in [Sch93].

One consequence of this axiomatic approach is the uniqueness result for "Morse homology" and its natural equivalence with other axiomatic homology theories defined on a suitable category of topological spaces. Witten conjecture is then a corollary of this result. A discussion of the relation of equivariant cohomolgy and supersymmetry may be found in Guillemin and Sternberg's book [GS99].

8.6 Chern-Simons Theory

Let M be a compact manifold of dimension $m = 2r+1,\ r > 0$, and let $P(M, G)$ be a principal bundle over M with a compact, semisimple Lie group G as its structure group. Let $\alpha_m(\omega)$ denote the Chern-Simons m-form on M corresponding to the gauge potential (connection) ω on P; then the Chern-Simons action $\mathcal{A}_{CS}$ is defined by

$$\mathcal{A}_{CS} = c(G) \int_M \alpha_m(\omega), \tag{8.10}$$

where $c(G)$ is a coupling constant whose normalization depends on the group G. In the rest of this paragraph we restrict ourselves to the case $r = 1$ and $G = SU(n)$. The most interesting applications of the Chern-Simons theory to low dimensional topologies are related to this case. It has been extensively studied by both physicists and mathematicians in recent years. In this case the action (8.10) takes the form

$$\mathcal{A}_{CS} = \frac{k}{4\pi} \int_M tr\left(A \wedge F - \frac{1}{3} A \wedge A \wedge A \right) \tag{8.11}$$

$$= \frac{k}{4\pi} \int_M tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{8.12}$$

where $k \in \mathbb{R}$ is a coupling constant, A denotes the pull-back to M of the gauge potential ω by a local section of P and $F = F_\omega = d^\omega A$ is the gauge field on M corresponding to the gauge potential A. A local expression for (8.11) is given by

$$\mathcal{A}_{CS} = \frac{k}{4\pi} \int_M \varepsilon^{\alpha\beta\gamma} tr\left(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma \right), \tag{8.13}$$

where $A_\alpha = A_\alpha^a T_a$ are the components of the gauge potential with respect to the local coordinates $\{x_\alpha\}$, $\{T_a\}$ is a basis of the Lie algebra $su(n)$ in the fundamental representation and $\varepsilon^{\alpha\beta\gamma}$ is the totally skew-symmetric Levi-Civita symbol with $\varepsilon^{123} = 1$. We take the basis $\{T_a\}$ with the normalization

$$tr(T_a T_b) = \frac{1}{2} \delta_{ab}, \tag{8.14}$$

where δ_{ab} is the Kronecker δ function. Let $g \in \mathcal{G}$ be a gauge transformation regarded (locally) as a function from M to $SU(n)$ and define the 1-form θ by

$$\theta = g^{-1} dg = g^{-1} \partial_\mu g dx^\mu.$$

Then the gauge transformation A^g of A by g has the local expression

$$A^g_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g. \tag{8.15}$$

In the physics literature, the connected component of the identity, $\mathcal{G}_{id} \subset \mathcal{G}$ is called the group of **small gauge transformations**. A gauge transformation not belonging to $\mathcal{G}_{id}$ is called a **large gauge transformation**. By a direct calculation, one can show that the Chern-Simons action is invariant under small gauge transformations, i.e.

$$\mathcal{A}_{CS}(A^g) = \mathcal{A}_{CS}(A), \quad \forall g \in \mathcal{G}_{id}.$$

Under a large gauge transformation g the action (8.13) transforms as follows:

$$\mathcal{A}_{CS}(A^g) = \mathcal{A}_{CS}(A) + 2\pi k \mathcal{A}_{WZ}, \tag{8.16}$$

where

$$\mathcal{A}_{WZ} := \frac{1}{24\pi^2} \int_M \varepsilon^{\alpha\beta\gamma} tr(\theta_\alpha \theta_\beta \theta_\gamma) \tag{8.17}$$

is the **Wess-Zumino action functional**. It can be shown that the Wess-Zumino functional is integer valued and hence, if the Chern-Simons coupling constant k is taken to be an integer, then we have

$$e^{i\mathcal{A}_{CS}(A^g)} = e^{i\mathcal{A}_{CS}(A)}.$$

The integer k is called the **level** of the corresponding Chern-Simons theory. It follows that the path integral quantization of the Chern-Simons model is gauge-invariant. This conclusion holds more generally for any compact simple group if the coupling constant $c(G)$ is chosen appropriately. The action is manifestly covariant since the integral involved in its definition is independent of the metric on M. It is in this sense that the Chern-Simons theory is a topological field theory. We will consider this aspect of the Chern-Simons theory later.

In general, the Chern-Simons action is defined on the space $\mathcal{A}_{P(M,G)}$ of all gauge potentials on the principal bundle $P(M,G)$. But when M is 3-dimensional P is trivial (in a non-canonical way). We fix a trivialization to write $P(M,G) = M \times G$ and write $\mathcal{A}_M$ for $\mathcal{A}_{P(M,G)}$. Then the group of gauge transformations $\mathcal{G}_P$ can be identified with the group of smooth functions from M to G and we denote it simply by $\mathcal{G}_M$. For $k \in \mathbb{N}$, the transformation law (8.16) implies that the Chern-Simons action descends to the quotient $\mathcal{B}_M = \mathcal{A}_M/\mathcal{G}_M$ as a function with values in $\mathbb{R}/\mathbb{Z}$. We denote this function by f_{CS}, i.e.

$$f_{CS} : \mathcal{B}_M \to \mathbb{R}/\mathbb{Z} \text{ is defined by } [\omega] \mapsto \mathcal{A}_{CS}(\omega), \quad \forall\, [\omega] = \omega\mathcal{G}_M \in \mathcal{B}_M. \tag{8.18}$$

The field equations of the Chern-Simons theory are obtained by setting the first variation of the action to zero as

$$\delta\mathcal{A}_{CS} = 0.$$

We shall discuss two approaches to this calculation. Consider first a one parameter family $c(t)$ of connections on P with $c(0) = \omega$ and $\dot{c}(0) = \alpha$. Differentiating the action $\mathcal{A}_{CS}(c(t))$ with respect to t and noting that differentiation commutes with integration and the tr operator, we get

$$\begin{aligned}\frac{d}{dt}\,\mathcal{A}_{CS}(c(t)) &= \frac{1}{4\pi}\int_M tr\,(2\dot{c}(t)\wedge dc(t) + 2(\dot{c}(t)\wedge c(t)\wedge c(t))) \\ &= \frac{1}{2\pi}\int_M tr\,(\dot{c}(t)\wedge (dc(t) + c(t)\wedge c(t))) \\ &= \frac{1}{2\pi}\int_M \langle \dot{c}(t),\ *F_{c(t)}\rangle\end{aligned}$$

where the inner product on the right is as defined in Definition 2.1. It follows that

$$\delta\mathcal{A}_{CS} = \frac{d}{dt}\,\mathcal{A}_{CS}(c(t))_{|t=0} = \frac{1}{2\pi}\int_M \langle\alpha,\ *F_\omega\rangle. \tag{8.19}$$

Since α can be chosen arbitrarily, the field equations are given by

$$*F_\omega = 0 \quad \text{or equivalently } F_\omega = 0. \tag{8.20}$$

Alternatively, one can start with the local coordinate expression of (8.13) as follows

$$\begin{aligned}\mathcal{A}_{CS} &= \frac{k}{4\pi}\int_M \varepsilon^{\alpha\beta\gamma}\, tr\!\left(A_\alpha\partial_\beta A_\gamma + \frac{2}{3}A_\alpha A_\beta A_\gamma\right) \\ &= \frac{k}{4\pi}\int_M \varepsilon^{\alpha\beta\gamma}\, tr\!\left(A^a_\alpha\partial_\beta A^c_\gamma T_a T_b + \frac{2}{3}A^a_\alpha A^b_\beta A^c_\gamma T_a T_b T_c\right)\end{aligned}$$

and find the field equations by using the variational equation

$$\frac{\delta\mathcal{A}_{CS}}{\delta A^a_\rho} = 0. \tag{8.21}$$

This method brings out the role of commutation relations and the structure constants of the Lie algebra $su(n)$ as well as the boundary conditions used in the integration by parts in the course of calculating the variation of the action. The result of this calculation gives

$$\frac{\delta\mathcal{A}_{CS}}{\delta A^a_\rho} = \frac{k}{2\pi}\int_M \varepsilon^{\rho\beta\gamma}\left(\partial_\beta A^a_\gamma + A^b_\beta A^c_\gamma f_{abc}\right) \tag{8.22}$$

where f_{abc} are the structure constants of $su(n)$ with respect to the basis T_a. The integrand on the right hand side of (8.22) is just the local coordinate expression of $*F_A$, the dual of the curvature, and hence leads to the same field equations.

The calculations leading to the field equations (8.20) also show that the gradient vector field of the function f_{CS} is given by

$$\text{grad } f_{CS} = \frac{1}{2\pi} * F \tag{8.23}$$

The gradient flow of f_{CS} plays a fundamental role in the definition of Floer homology. The solutions of the field equations (8.20) are called the **Chern-Simons connections**. They are precisely the flat connections. In the next paragraph we discuss flat connections on a manifold N and their relation to the homomorphisms of the fundamental group $\pi_1(N)$ into the gauge group.

8.6.1 Flat Connections

Let H be a compact Lie group and $Q(N, H)$ be a principal bundle with structure group H over a compact Riemannian manifold N. A connection ω on Q is said to be **flat** if its curvature is zero, i.e. $F_\omega = 0$. The pair (Q, ω) is called a **flat bundle**. Let $\Omega(N, x)$ be the loop space at $x \in N$. Recall that the horizontal lift h_u of $c \in \Omega(N, x)$ to $u \in \pi^{-1}(x)$ determines a unique element of H. Thus we have the map

$$h_u : \Omega(N, x) \to H.$$

It is easy to see that ω flat implies that this map h_u depends only on the homotopy class of the loop c and hence induces a map (also denoted by h_u)

$$h_u : \pi_1(N, x) \to H.$$

It is this map that is related to the Bohm-Aharonov effect. It can be shown that the map h_u is a homomorphism of groups. The group H acts on the set $Hom(\pi_1(N), H)$ by conjugation sending h_u to $g^{-1}h_u g = h_{ug}$. Thus a flat bundle (Q, ω) determines an element of the quotient $Hom(\pi_1(N), H)/H$. If $a \in \mathcal{G}(Q)$, the group of gauge transformations of Q, then $a \cdot \omega$ is also a flat connection on Q and determines the same element of $Hom(\pi_1(N), H)/H$. Conversely, let $f \in Hom(\pi_1(N), H)$ and let (U, q) be the universal covering of N. Then U is a principal bundle over N with structure group $\pi_1(N)$. Define $Q := U \times_f H$ to be the bundle associated to U by the action f with standard fiber H. It can be shown that Q admits a natural flat connection and that f and $g^{-1}fg,\ g \in H$, determine isomorphic flat bundles. Thus the moduli space $\mathcal{M}_f(N, H)$ of flat H-bundles over N can be identified with the set $Hom(\pi_1(N), H)/H$. The moduli space $\mathcal{M}_f(N, H)$ and the set $Hom(\pi_1(N), H)$ have a rich mathematical structure which has been extensively studied in the particular case when N is a compact Riemann surface [AB82].

The **flat connection deformation complex** is the generalized deRham sequence with the usual differential d replaced by the covariant differential d^ω. The fact that in this case it is a complex follows from the observation that ω flat implies $d^\omega \circ d^\omega = 0$.

By rolling up this complex, we can consider the rolled up deformation operator $d^\omega + \delta^\omega : \Lambda^{ev} \to \Lambda^{odd}$. By the index theorem, we have

$$Ind(d^\omega + \delta^\omega) = \chi(N) dim\, H$$

and hence

$$\sum_{i=0}^{n} (-1)^i b_i = \chi(N) dim\, H, \tag{8.24}$$

where b_i is the dimension of the i-th cohomology of the deformation complex. Both sides are identically zero for odd n. For even n, the formula can be used to obtain some information on the virtual dimension of $\mathcal{M}_f$ $(= b_1)$. For example, if $N = \Sigma_g$ is a Riemann surface of genus $g > 1$, then $\chi(\Sigma_g) = -2g + 2$, while, by Hodge duality, $b_0 = b_2 = 0$ at an irreducible connection. Thus, equation (8.24) gives

$$-b_1 = -(2g - 2) dim\, H.$$

From this it follows that

$$dim\, \mathcal{M}_f(\Sigma_g, H) = dim\, \mathcal{M}_f = (2g - 2) dim\, H. \tag{8.25}$$

In even dimensions greater than 2, the higher cohomology groups provide additional obstructions to smoothability of $\mathcal{M}_f$. For example, for $n = 4$, Hodge duality implies that $b_0 = b_4$ and $b_1 = b_3$ and (8.24) gives

$$b_1 = b_0 + (b_2 - \chi(N) dim\, H)/2.$$

Equation (8.25) shows that $dim\, \mathcal{M}_f$ is even. Identifying the first cohomology

$$H^1(\Lambda(M, adh), d^\omega)$$

of the deformation complex with the tangent space $T_\omega \mathcal{M}_f$ to $\mathcal{M}_f$, the intersection form defines a map $\iota_\omega : T_\omega \mathcal{M}_f \times T_\omega \mathcal{M}_f \to \mathbb{R}$ by

$$\iota(X, Y) = \int_{\Sigma_g} X \wedge Y, \quad X, Y \in T_\omega \mathcal{M}_f. \tag{8.26}$$

The map ι_ω is skew-symmetric and bilinear. The map

$$\iota : \omega \mapsto \iota_\omega, \quad \forall \omega \in \mathcal{M}_f, \tag{8.27}$$

defines a 2-form ι on $\mathcal{M}_f$. If $\mathbf{h}$ admits an H-invariant inner product, then this 2-form ι is closed and non-degenerate and hence defines a symplectic structure on $\mathcal{M}_f$. It can be shown that, for a Riemann surface with $H = PSL(2, \mathbb{R})$, the form ι, restricted to the Teichmüller space, agrees with the well-known Weil-Petersson form.

We now discuss an interesting physical interpretation of the symplectic manifold $(\mathcal{M}_f(\Sigma_g, H),\ \iota)$. Consider a Chern-Simons theory on the principal bundle

$P(M,H)$ over the $2+1$-dimensional space-time manifold $M=\Sigma_g\times\mathbb{R}$ with gauge group H and with time independent gauge potentials and gauge transformations. Let $\mathcal{A}$ (resp. $\mathcal{H}$) denote the space (resp. group) of these gauge connections (resp. transformations). It can be shown that the curvature F_ω defines an $\mathcal{H}$-equivariant moment map

$$\mu:\mathcal{A}\to\mathcal{LH}\cong\Lambda^1(M,adP),\quad \text{by } \omega\mapsto *F_\omega,$$

where $\mathcal{LH}$ is the Lie algebra of $\mathcal{H}$. The zero set $\mu^{-1}(0)$ of this map is precisely the set of flat connections and hence

$$\mathcal{M}_f\cong\mu^{-1}(0)/\mathcal{H}:=\mathcal{A}//\mathcal{H} \tag{8.28}$$

is the reduced phase space of the theory, in the sense of the Marsden-Weinstein reduction. We call $\mathcal{A}//\mathcal{H}$ the **symplectic quotient** of $\mathcal{A}$ by $\mathcal{H}$. Marsden-Weinstein reduction and symplectic quotient are fundamental constructions in geometrical mechanics and geometric quantization. They also arise in many other mathematical applications.

A situation similar to that described above, also arises in the geometric formulation of canonical quantization of field theories. One proceeds by analogy with the geometric quantization of finite dimensional systems. For example, $Q=\mathcal{A}/\mathcal{H}$ can be taken as the configuration space and T^*Q as the corresponding phase space. The associated Hilbert space is obtained as the space of L^2 sections of a complex line bundle over Q. For physical reasons this bundle is taken to be flat. Inequivalent flat $U(1)$-bundles are said to correspond to distinct sectors of the theory. Thus we see that at least formally these sectors are parametrized by the moduli space

$$\mathcal{M}_f(Q,U(1))\cong Hom(\pi_1(Q),U(1))/U(1)\cong Hom(\pi_1(Q),U(1))$$

since $U(1)$ acts trivially on $Hom(\pi_1(Q),U(1))$.

We note that the Chern-Simons theory has been extended by Witten to the cases when the gauge group is finite and when it the complexification of compact real gauge groups [DW90, Wit91]. While there are some similarities between these theories and the standard CS theory, there are major differences in the corresponding TQFTs. New invariants of some hyperbolic 3-manifolds have recently been obtained by considering the complex gauge groups leading to the concept of arithmetic TQFT by Zagier and collaborators (see [DGLZ09]). See also Dijkgraaf and Fuji [DF09] and Gukov and Witten [GW08].

8.7 Casson Invariant and Flat Connections

Let Y be a homology 3-sphere. Let D_1, D_2 be two unitary, unimodular representations of $\pi_1(Y)$ in $\mathbb{C}^2$. We say that they are equivalent if they are conjugate under the natural $SU(2)$-action on $\mathbb{C}^2$, i.e.

$$D_2(g)=S^{-1}D_1(g)S,\quad \forall g\in\pi_1(Y),\ S\in SU(2).$$

Let us denote by $\mathcal{R}(Y)$ the set of equivalence classes of such representations. It is customary to write

$$\mathcal{R}(Y) := \mathrm{Hom}\{\pi_1(Y) \to SU(2)\}/\mathrm{conj}. \tag{8.29}$$

The set $\mathcal{R}(Y)$ can be given the structure of a compact, real algebraic variety. It is called the $SU(2)$-representation variety of Y. Let $\mathcal{R}^*(Y)$ be the class of irreducible representations. Fixing an orientation of Y, Casson showed how to assign a sign $s(\alpha)$ to each element $\alpha \in \mathcal{R}^*(Y)$. He showed that the set $\mathcal{R}^*(Y)$ is 0-dimensional and compact and hence finite. Casson defined a numerical invariant of Y by counting the signed number of elements of $\mathcal{R}^*(Y)$ by

$$c(Y) := \sum_{\alpha \in \mathcal{R}^*(Y)} s(\alpha).1 \tag{8.30}$$

The integer $c(Y)$ is called the **Casson invariant** of Y.

Theorem 1 *The Casson invariant $c(Y)$ is well defined up to sign for any homology sphere Y and satisfies the following properties*:

i) $c(-Y) = -c(Y)$,
ii) $c(X\#Y) = c(X) + c(Y)$, *X a homology sphere*,
iii) $c(Y)/2 = \rho(Y) \bmod 2$, *ρ Rokhlin invariant.*

We now give a gauge theory description of $\mathcal{R}(Y)$ leading to Taubes' theorem. In [Tau90] Taubes gives a new interpretation of the Casson invariant $c(Y)$ of an oriented homology 3-sphere Y, which is defined above in terms of the signed count of equivalence classes of irreducible representations of $\pi_1(Y)$ into $SU(2)$. As indicated above, this space can be identified with the moduli space $\mathcal{M}_f(Y, SU(2))$ of flat connections in the trivial $SU(2)$-bundle over Y. Recall that this is also the space of solutions of the Chern-Simons field equations (8.20). The map $F : \omega \mapsto F_\omega$ defines a natural 1-form on $\mathcal{A}/\mathcal{G}$ and the zeros of this form are just the flat connections. We note that since $\mathcal{A}/\mathcal{G}$ is infinite dimensional, it is necessary to use suitable Fredholm perturbations to get simple zeros and to count them with appropriate signs. Let Z denote the set of zeros of the perturbed vector field and let $s(a)$ be the sign of $a \in Z$. Taubes shows that Z is contained in a compact set and that

$$c(Y) = \sum_{a \in Z} s(a).1$$

The right hand side of this equation can be interpreted as the index of a vector field in the infinite dimensional setting. The classical Poincaré-Hopf theorem can also be generalized to interprete the index as Euler characteristic. A natural question to ask is if this Euler characteristic comes from some homology theory? An affirmative answer is provided by Floer's instanton homology. We discuss it in the next section.

Another approach to Casson's invariant involves symplectic geometry and topology. We conclude this section with a brief indication of this approach. Let $Y_+ \cup_{\Sigma_g} Y_-$

be a Heegaard splitting of Y along the Riemann surface Σ_g of genus g. The space $\mathcal{R}(\Sigma_g)$ of conjugacy classes of representations of $\pi_1(\Sigma_g)$ into $SU(2)$ can be identified with the moduli space $\mathcal{M}_f(\Sigma_g, SU(2))$ of flat connections. This identification endows it with a natural symplectic structure which makes it into a $(6g-6)$-dimensional symplectic manifold. The representations which extend to Y_+ (resp. Y_-) form a $(3g-3)$-dimensional Lagrangian submanifold of $\mathcal{R}(\Sigma_g)$ which we denote by $\mathcal{R}(Y_+)$ (resp. $\mathcal{R}(Y_-)$). Casson's invariant is then obtained from the intersection number of the Lagrangian submanifolds $\mathcal{R}(Y_+)$ and $\mathcal{R}(Y_-)$ in the symplectic manifold $\mathcal{R}(\Sigma_g)$. How the Floer homology of Y fits into this scheme seems to be unknown at this time.

8.8 Fukaya-Floer Homology

The idea of instanton tunnelling and the corresponding Witten complex was extended by Floer to do Morse theory on the infinite dimensional moduli space of gauge potentials on a homology 3-sphere Y and to define new topological invariants of Y. Fukaya has generalized this work to apply to arbitrary oriented 3-manifolds. We shall refer to the invariants of Floer and Fukaya collectively as Fukaya-Floer Homology. Fukaya-Floer Homology associates to an oriented, connected, closed, smooth 3-dimensional manifold Y, a family of $\mathbb{Z}_8$-graded instanton homology groups $FF_n(Y)$, $n \in \mathbb{Z}_8$. We begin by introducing Floer's original definition, which requires Y to be a homology 3-sphere. Let $\mathcal{R}(Y)$ be the $SU(2)$-representation variety of Y as defined in (8.29) and let $\mathcal{R}^*(Y)$ be the class of irreducible representations. We say that $\alpha \in \mathcal{R}^*(Y)$ is a **regular representation** if

$$H^1(Y, ad(\alpha)) = 0. \tag{8.31}$$

We identify $\mathcal{R}(Y)$ with the space of flat or Chern-Simons connections on Y. The Chern-Simon functional has non-degenerate Hessian at α if α is regular. Fix a trivialization P of the given $SU(2)$-bundle over Y. Using the trivial connection θ on $P = Y \times SU(2)$ as a background connection on Y, we can identify the space of connections $\mathcal{A}_Y$ with the space of sections of $\Lambda^1(Y) \otimes su(2)$. In what follows we shall consider a suitable Sobolev completion of this space and continue to denote it by $\mathcal{A}_Y$.

Let $c : I \to \mathcal{A}_Y$ be a path from α to θ. The family of connections $c(t)$ on Y can be identified as a connection A on $Y \times I$. Using this connection we can rewrite the Chern-Simons action (8.11) as follows

$$\mathcal{A}_{CS} = \frac{1}{8\pi^2} \int_{Y \times I} tr(F_A \wedge F_A). \tag{8.32}$$

We note that the integrand corresponds to the second Chern class of the pull-back of the trivial $SU(2)$-bundle over Y to $Y \times I$. Recall that the critical points of the Chern-Simons action are the flat connections. The gauge group $\mathcal{G}_Y$ acts on $\mathcal{A}_{CS} : \mathcal{A} \to \mathbb{R}$

by

$$\mathcal{A}_{CS}(\alpha^g) = \mathcal{A}_{CS}(\alpha) + \deg(g), \quad g \in \mathcal{G}_Y.$$

It follows that $\mathcal{A}_{CS}$ descends to $\mathcal{B}_Y := \mathcal{A}_Y/\mathcal{G}_Y$ as a map $f_{CS} : \mathcal{B}_Y \to \mathbb{R}/\mathbb{Z}$ and we can take $\mathcal{R}(Y) \subset \mathcal{B}_Y$ as the critical set of f_{CS}. The gradient flow of this function is given by the equation

$$\frac{\partial c(t)}{\partial t} = *_Y F_{c(t)}. \tag{8.33}$$

Since Y is a homology 3-sphere, the critical points of the flow of *grad* f_{CS} and the set of reducible connections intersect at a single point, the trivial connection θ. If all the critical points of the flow are regular then it is a Morse-Smale flow. If not, one can perturb the function f_{CS} to get a Morse function.

In general the representation space $\mathcal{R}^*(Y) \subset \mathcal{B}_Y$ contains degenerate critical points of the Chern-Simons function f_{CS}. In this case Floer defines a set of perturbations of f_{CS} as follows. Let $m \in \mathbb{N}$ and let $\bigvee_{i=1}^m S_i^1$ be a bouquet of m copies of the circle S^1. Let Γ_m be the set of maps

$$\gamma : \bigvee_{i=1}^{m} S_i^1 \times D^2 \to Y$$

such that the restrictions

$$\gamma_x : \bigvee_{i=1}^{m} S_i^1 \times \{x\} \to Y \quad \text{and} \quad \gamma_i : S_i^1 \times D^2 \to Y$$

are smooth embeddings for each $x \in D^2$ and for each $i,\ 1 \le i \le m$. Let $\hat{\gamma}_x$ denote the family of holonomy maps

$$\hat{\gamma}_x : A_Y \to \underbrace{SU(2) \times \cdots \times SU(2)}_{m \text{ times}}, \quad x \in D^2.$$

The holonomy is conjugated under the action of the group of gauge transformations and we continue to denote by $\hat{\gamma}_x$ the induced map on the quotient $\mathcal{B}_Y = \mathcal{A}_Y/\mathcal{G}$. Let $\mathcal{F}_m$ denote the set of smooth functions

$$h : \underbrace{SU(2) \times \cdots \times SU(2)}_{m \text{ times}} \to \mathbb{R}$$

which are invariant under the adjoint action of $SU(2)$. Floer's set of perturbations Π is defined as

$$\Pi := \bigcup_{m \in \mathbb{N}} \Gamma_m \times \mathcal{F}_m.$$

Floer proves that for each $(\gamma, h) \in \Pi$ the function

$$h_\gamma : \mathcal{B}_Y \to \mathbb{R} \quad \text{defined by } h_\gamma(\alpha) = \int_{D^2} h(\hat{\gamma}_x(\alpha))$$

is a smooth function and that for a dense subset $\mathcal{P} \subset \mathcal{RM}(Y) \times \Pi$ the critical points of the perturbed function

$$f_{(\gamma,h)} := f_{CS} + h_\gamma$$

are non-degenerate and the corresponding moduli space decomposes into smooth, oriented manifolds of regular trajectories of the gradient flow of the function $f_{(\gamma,h)}$ with respect to a generic metric $\sigma \in \mathcal{RM}(Y)$. Furthermore, the homology groups of the perturbed chain complex are independent of the choice of perturbation in $\mathcal{P}$. We shall assume that this has been done. Let α, β be two critical points of the function f_{CS}. Considering the spectral flow (denoted by sf) from α to β we obtain the moduli space $\mathcal{M}(\alpha, \beta)$ as the moduli space of self-dual connections on $Y \times \mathbb{R}$ which are asymptotic to α and β (as $t \to \pm\infty$). Let $\mathcal{M}^j(\alpha, \beta)$ denote the component of dimension j in $\mathcal{M}(\alpha, \beta)$. There is a natural action of $\mathbb{R}$ on $\mathcal{M}(\alpha, \beta)$. Let $\hat{\mathcal{M}}^j(\alpha, \beta)$ denote the component of dimension $j-1$ in $\mathcal{M}(\alpha, \beta)/\mathbb{R}$. Let $\#\hat{\mathcal{M}}^1(\alpha, \beta)$ denote the signed sum of the number of points in $\hat{\mathcal{M}}^1(\alpha, \beta)$. Floer defines the Morse index of α by considering the spectral flow from α to the trivial connection θ. It can be shown that the spectral flow and hence the Morse index are defined modulo 8. Now define the chain groups by

$$\mathcal{R}_n(Y) = \mathbb{Z}\{\alpha \in \mathcal{R}^*(Y) \mid sf(\alpha) = n\}, \quad n \in \mathbb{Z}_8$$

and define the boundary operator ∂

$$\partial : \mathcal{R}_n(Y) \to \mathcal{R}_{n-1}(Y)$$

by

$$\partial\alpha = \sum_{\beta \in \mathcal{R}_{n-1}(Y)} \#\hat{\mathcal{M}}^1(\alpha, \beta)\beta. \tag{8.34}$$

It can be shown that $\partial^2 = 0$ and hence $(\mathcal{R}(Y), \partial)$ is a complex. This complex can be thought of as an infinite dimensional generalization [Flo89] of Witten's instanton tunnelling and we will call it the **Floer-Witten Complex** of the pair $(Y, SU(2))$. Since the spectral flow and hence the dimensions of the components of $\mathcal{M}(\alpha, \beta)$ are congruent modulo 8, this complex defines the Floer homology groups $FH_j(Y),\ j \in \mathbb{Z}_8$, where j is the spectral flow of α to θ modulo 8. If r_j denotes the rank of the Floer homology group $FH_j(Y),\ j \in \mathbb{Z}_8$, then we can define the corresponding Euler characteristic $\chi_F(Y)$ by

$$\chi_F(Y) := \sum_{j \in \mathbb{Z}_8} (-1)^j r_j.$$

Combining this with Taubes' interpretation of the Casson invariant $c(Y)$ we get

$$c(Y) = \chi_F(Y) = \sum_{j \in \mathbb{Z}_8} (-1)^j r_j. \tag{8.35}$$

An important feature of Floer's instanton homology is that it can be regarded as a functor from the category of homology 3-spheres with morphisms given by oriented cobordism, to the category of graded abelian groups. Let M be a smooth, oriented cobordism from Y_1 to Y_2 so that $\partial M = Y_2 - Y_1$. By a careful analysis of instantons on M, Floer showed [Flo88] that M induces a graded homomorphism

$$M_j : FH_j(Y_1) \to FH_{j+b(M)}(Y_2), \quad j \in \mathbb{Z}_8, \tag{8.36}$$

where

$$b(M) = 3(b_1(M) - b_2(M)). \tag{8.37}$$

Then the homomorphisms induced by cobordism has the following functorial properties.

$$(Y \times \mathbb{R})_j = id, \tag{8.38}$$

$$(MN)_j = M_{j+b(N)} N_j. \tag{8.39}$$

An algorithm for computing the Floer homology groups for Seifert-fibered homology 3-spheres with three exceptional fibers (or orbits) has been discussed in [FS90].

In addition to these invariants of 3-manifolds and the linking number, there are several other invariants of knots and links in 3-manifolds. We introduce them in the next section and study their field theory interpretations in the later sections.

8.9 Knot Polynomials

In the second half of the nineteenth century, a systematic study of knots in $\mathbb{R}^3$ was made by Tait. He was motivated by Kelvin's theory of atoms modelled on knotted vortex tubes of ether. Tait classified the knots in terms of the crossing number of a plane projection and made a number of observations about some general properties of knots which have come to be known as the "Tait conjectures". Recall that a knot κ in S^3 is an embedding of the circle S^1 and that a link is a disjoint union of knots. A **link diagram** of κ is a plane projection with crossings marked as over or under. By changing a link diagram at one crossing we can obtain three diagrams corresponding to links κ_+, κ_- and κ_0.

In the 1920s, Alexander gave an algorithm for computing a polynomial invariant $A_\kappa(q)$ of a knot κ, called the **Alexander polynomial**, by using its projection on a plane. He also gave its topological interpretation as an annihilator of a certain cohomology module associated to the knot κ. In the 1960s, Conway defined his polynomial invariant and gave its relation to the Alexander polynomial. This polynomial is

called the **Alexander-Conway polynomial** or simply the Conway polynomial. The Alexander-Conway polynomial of an oriented link L is denoted by $\nabla_L(z)$ or simply by $\nabla(z)$ when L is fixed. We denote the corresponding polynomials of L_+, L_- and L_0 by ∇_+, ∇_- and ∇_0 respectively. The Alexander-Conway polynomial is uniquely determined by the following simple set of axioms.
AC1. Let L and L' be two oriented links which are ambient isotopic. Then

$$\nabla_{L'}(z) = \nabla_L(z) \tag{8.40}$$

AC2. Let S^1 be the standard unknotted circle embedded in S^3. It is usually referred to as the **unknot** and is denoted by $\mathcal{O}$. Then

$$\nabla_{\mathcal{O}}(z) = 1. \tag{8.41}$$

AC3. The polynomial satisfies the following **skein relation**

$$\nabla_+(z) - \nabla_-(z) = z\nabla_0(z). \tag{8.42}$$

We note that the original Alexander polynomial Δ_L is related to the Alexander-Conway polynomial by the relation

$$\Delta_L(t) = \nabla_L(t^{1/2} - t^{-1/2}).$$

Despite these and other major advances in knot theory, the Tait conjectures remained unsettled for more than a century after their formulation. Then in the 1980s, Jones discovered his polynomial invariant $V_\kappa(q)$, called the **Jones polynomial**, while studying Von Neumann algebras and gave its interpretation in terms of statistical mechanics. These new polynomial invariants have led to the proofs of most of the Tait conjectures. As with the earlier invariants, Jones' definition of his polynomial invariants is algebraic and combinatorial in nature and was based on representations of the braid groups and related Hecke algebras. The Jones polynomial $V_\kappa(t)$ of κ is a Laurent polynomial in t (polynomial in t and t^{-1}) which is uniquely determined by a simple set of properties similar to the axioms for the Alexander-Conway polynomial. More generally, the Jones polynomial can be defined for any oriented link L as a Laurent polynomial in $t^{1/2}$. Reversing the orientation of all components of L leaves V_L unchanged. In particular, V_κ does not depend on the orientation of the knot κ. For a fixed link, we denote the Jones polynomial simply by V. Recall that there are 3 standard ways to change a link diagram at a crossing point. The Jones polynomials of the corresponding links are denoted by V_+, V_- and V_0 respectively. Then the Jones polynomial is characterized by the following properties:

JO1. Let κ and κ' be two oriented links which are ambient isotopic. Then

$$V_{\kappa'}(t) = V_\kappa(t) \tag{8.43}$$

JO2. Let $\mathcal{O}$ denote the unknot. Then

$$V_{\mathcal{O}}(t) = 1. \tag{8.44}$$

JO3. The polynomial satisfies the following skein relation

$$t^{-1}V_{+} - tV_{-} = (t^{1/2} - t^{-1/2})V_0. \tag{8.45}$$

An important property of the Jones polynomial that is not shared by the Alexander-Conway polynomial is its ability to sometimes distinguish between a knot and its mirror image. Let κ_m be the mirror image of the knot κ. Then

$$V_{K_m}(t) = V_K(t^{-1}). \tag{8.46}$$

It can sometimes happen that $V_K(t^{-1}) = V_K(t)$, in which case the Jones polynomial does not distinguish between the knot K and its mirror image. This can happen when K is topologically equivalent to its mirror image, and it can also happen when K is not topologically equivalent to its mirror image. Nevertheless, when $V_K(t^{-1}) \neq V_K(t)$ then one knows that K is not topologically equivalent to its mirror image. Soon after Jones' discovery a two variable polynomial generalizing V was found by several mathematicians. It is called the **HOMFLY polynomial** and is denoted by P. The HOMFLY polynomial $P(\alpha, z)$ satisfies the following skein relation

$$\alpha P_{+} - \alpha^{-1}P_{-} = zP_0. \tag{8.47}$$

If we put $\alpha = t^{-1}$ and $z = (t^{1/2} - t^{-1/2})$ in equation (8.47) we get the skein relation for the original Jones polynomial V. If we put $\alpha = 1$ we get the skein relation for the Alexander-Conway polynomial.

Knots and links in $\mathbb{R}^3$ can also be obtained by using braids. A **braid** on n strands (or with n strings or simply an n-braid) can be thought of as a set of n pairwise disjoint strings joining n distinct points in one plane with n distinct points in a parallel plane in $\mathbb{R}^3$. The set of equivalence classes of n-braids is denoted by $\mathcal{B}_n$. A braid is called elementary if only two neighboring strings cross. We denote by σ_i the elementary braid where the i-th string crosses over the $(i+1)$-th string.
Theorem (M. Artin): The set $\mathcal{B}_n$ with multiplication operation induced by concatenation of braids is a group generated by the elementary braids σ_i, $1 \leq i \leq n-1$ subject to the braid relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2 \tag{8.48}$$

and the far commutativity relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad 1 \leq i, j \leq n-1 \text{ and } |i-j| > 1. \tag{8.49}$$

The closure of a braid b obtained by gluing the endpoints is a link denoted by $c(b)$. A classical theorem of Alexander shows that the closure map from the set of braids to the set of links is surjective, i.e. any link (and, in particular, knot) is the closure of some braid. Moreover, if braids b and b' are equivalent, then the links $c(b)$ and $c(b')$ are equivalent. There are several descriptions of the braid group leading to various approaches to the study of its representations and invariants of links. For example, $\mathcal{B}_n$ is isomorphic to the fundamental group of the configuration space of n

distinct points in the plane. The action of $\mathcal{B}_n$ on the homology of the configuration space is related to the representations of certain Hecke algebras leading to invariants of links such as the Jones polynomial that we have discussed earlier. The group $\mathcal{B}_n$ is also isomorphic to the mapping class group of the n-punctured disc. This definition was recently used by Krammer and Bigelow in showing the linearity of $\mathcal{B}_n$ over the ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ of Laurent polynomials in two variables.

8.10 Categorification of Knot Polynomials

We begin by recalling that a categorification of an invariant I is the construction of a suitable (co)homology H^* such that its Euler characteristic $\chi(H^*)$ (the alternating sum of the ranks of (co)homology groups) equals I. Historically, the Euler characteristic was defined and understood well before the advent of algebraic topology. Theorema egregium of Gauss and the closely related Gauss-Bonnet theorem and its generalization by Chern give a geometric interpretation of the Euler characteristic $\chi(M)$ of a manifold M. They can be regarded as precursors of Chern-Weil theory as well as index theory. Categorification $\chi(H^*(M))$ of this Euler characteristic $\chi(M)$ by various (co)homolgy theories $H^*(M)$ came much later. A well known recent example that we have discussed is the categorification of the Casson invariant by the Fukaya-Floer homology. Categorification of quantum invariants such as Knot Polynomials requires the use of quantum Euler characteristic and multi-graded knot homologies.

Recently Khovanov [Kho00] has obtained a categorification of the Jones polynomial $V_\kappa(q)$ by constructing a bi-graded $sl(2)$-homology $H_{i,j}$ determined by the knot κ. It is called the **Khovanov homology** of the knot κ and is denoted by $KH(\kappa)$. The **Khovanov polynomial** $Kh_\kappa(t,q)$ is defined by

$$Kh_\kappa(t,q) = \sum_{i,j} t^j q^i \, dim\, H_{i,j}.$$

It can be thought of as a two variable generalization of the Poincarè polynomial. The quantum or graded Euler characteristic of the Khovanov homology equals the Jones polynomial. i.e.

$$V_\kappa(q) = \chi_q(KH(\kappa)) = \sum_{i,j} (-1)^j q^i \, dim\, H_{i,j}.$$

Khovanov's construction follows Kauffman's state-sum model [Kauf87, Kauf88, Kauf91] of the link L and his alternative definition of the Jones polynomial. Let $\hat{L}$ be a regular projection of L with $n = n_+ + n_-$ labelled crossings. At each crossing we can define two resolutions or states, the vertical or 1-state and horizontal or 0-state. Thus there are 2^n total resolutions of $\hat{L}$ which can be put into one to one correspondence with the vertices of an n-dimensional unit cube. For each vertex x let $|x|$ be the sum of its coordinates and let $c(x)$ be the number of disjoint circles in the resolution $\hat{L}_x$ of $\hat{L}$ determined by x. Kauffman's state-sum expression for the non-normalized Jones polynomial $\hat{V}(L)$ can be written as follows:

$$\hat{V}(L) = (-1)^{n_-} q^{(n_+ - 2n_-)} \sum (-q)^{|x|} (q + q^{-1})^{c(x)}. \tag{8.50}$$

Dividing this by the unknot value $(q + q^{-1})$ gives the usual normalized Jones polynomial $V(L)$. The Khovanov complex is constructed as follows. Let V be a graded vector space over a fixed ground field K, generated by two basis vectors $v_\pm$ with respective degrees ± 1. The total resolution associates to each vertex x a one dimensional manifold M_x consisting of $c(x)$ disjoint circles. We can construct a $(1+1)$-dimensional TQFT (along the lines of Atiyah-Segal axioms discussed in the next section) for each edge of the cube as follows. If xy is an edge of the cube we can get a pair of pants cobordism from M_x to M_y by noting that a circle at x can split into two at y or two circles at x can fuse into one at y. If a circle goes to a circle than the cylinder provides the cobordism. To the manifold M_x at each vertex x we associate the graded vector space

$$V_x(L) := V^{\otimes c(x)}\{|x|\}, \tag{8.51}$$

where $\{k\}$ is the degree shift by k. We define the Frobenius structure (see the book [Koc04] by Kock for Frobenius algebras and their relation to TQFT) on V as follows. Multiplication $m : V \otimes V \to V$ is defined by

$$m(v_+ \otimes v_+) = v_+, \qquad m(v_+ \otimes v_-) = v_-,$$
$$m(v_- \otimes v_+) = v_-, \qquad m(v_- \otimes v_-) = 0.$$

Co-multiplication $\Delta : V \to V \otimes V$ is defined by

$$\Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+, \qquad \Delta(v_-) = v_- \otimes v_-.$$

Thus v_+ is the unit. The co-unit $\delta \in V^*$ is defined by mapping v_+ to 0 and v_- to 1 in the base field. The r-th chain group $C_r(L)$ in the Khovanov complex is the direct sum of all vector spaces $V_x(L)$, where $|x| = r$, and the differential is defined by the Frobenius structure. Thus

$$C_r(L) := \oplus_{|x|=r} V_x(L). \tag{8.52}$$

We remark that the TQFT corresponds to the Frobenius algebra structure on V defined above. The r-th homology group of the Khovanov complex is denoted by KH_r. Khovanov has proved that the homology is independent of the various choices made in defining it. Thus we have

Theorem 2 *The homology groups KH_r are link invariants. In particular, the Khovanov polynomial*

$$Kh_L(t,q) = \sum_j t^j \, dim_q(KH_j)$$

is a link invariant that specializes to the non-normalized Jones polynomial. The Khovanov polynomial is strictly stronger than the Jones polynomial.

We note that the knots 9_{42} and 10_{125} are chiral. Their chirality is detected by the Khovanov polynomial but not by the Jones polynomial. Also there are several pairs of knots with the same Jones polynomials but different Khovanov polynomials. For example $(5_1, 10_{132})$ is such a pair.

8.10.1 Categorification of $V(3_1)$

Using equations (8.51) and (8.52) and the algebra structure on V the calculation of the Khovanov complex can be reduced to an algorithm. A computer program implementing such an algorithm is discussed in [BN02]. A table of Khovanov polynomials for knots and links up to 11 crossings is also given there. We now illustrate Khovanov's categorification of the Jones polynomial of the right handed trefoil knot 3_1. For the standard diagram of the trefoil, $n = n_+ = 3$ and $n_- = 0$. The quantum dimensions of the non-zero terms of the Khovanov complex with the shift factor included are given by

$$\begin{aligned} C_0 &= (q+q^{-1})^2, \qquad C_1 = 3q(q+q^{-1}), \\ C_2 &= 3q^2(q+q^{-1})^2, \qquad C_3 = q^3(q+q^{-1})^3. \end{aligned} \tag{8.53}$$

The non-normalized Jones polynomial can be obtained from (8.53) or directly from (8.50) giving

$$\hat{V}(L) = (q + q^3 + q^5 - q^9) \tag{8.54}$$

The normalized or standard Jones polynomial is then given by

$$V(q) = (q + q^3 + q^5 - q^9)/(q + q^{-1}) = q^2 + q^6 - q^8.$$

By direct computation or using the program in [BN02] we obtain the following formula for the Khovanov polynomial of the trefoil

$$Kh(t,q) = q + q^3 + t^2q^5 + t^3q^9, \qquad Kh(-1,q) = \chi_q = \hat{V}(L).$$

Based on computations using the program described in [BN02], Khovanov, Garofouladis and Bar-Natan (BKG) have formulated some conjectures on the structure of Khovanov polynomials over different base fields. We now state these conjectures.
The BKG Conjectures: For any prime knot κ, there exists an even integer $s = s(\kappa)$ and a polynomial $Kh'_\kappa(t,q)$ with only non-negative coefficients such that

1. Over the base field $K = \mathbb{Q}$,

$$Kh_\kappa(t,q) = q^{s-1}[1 + q^2 + (1 + tq^4)Kh'_\kappa(t,q)]$$

2. Over the base field $K = \mathbb{Z}_2$,

$$Kh_\kappa(t,q) = q^{s-1}(1+q^2)[1+(1+tq^2)Kh'_\kappa(t,q)]$$

3. Moreover, if the κ is alternating, then $s(\kappa)$ is the signature of the knot and $Kh'_\kappa(t,q)$ contains only powers of tq^2.

The conjectured results are in agreement with all the known values of the Khovanov polynomials.

If $S \subset \mathbb{R}^4$ is an oriented surface cobordism between links L_1 and L_2, then it induces a homomorphism of Khovanov homologies of links L_1 and L_2. These homomorphisms define a functor from the category of link cobordisms to the category of bigraded abelian groups. Khovanov homology extends to colored links (i.e. oriented links with components labelled by irreducible finite dimensional representations of $sl(2)$) to give a categorification of the colored Jones polynomial. Khovanov and Rozansky have defined an $sl(n)$-homology for links colored by either the defining representation or its dual. This gives categorification of the specialization of the HOMFLY polynomial $P(\alpha, q)$ with $a = q^n$. The sequence of such specializations for $n \in \mathbb{N}$ would categorify the two variable HOMFLY polynomial $P(\alpha, q)$. For $n = 0$ the theory coincides with the Heegaard Floer homology of Ozsváth and Szabo [OS03].

In the 1990s Reshetikhin, Turaev and other mathematicians obtained several quantum invariants of triples $(\mathbf{g}, L, M)$, where $\mathbf{g}$ is a simple Lie algebra, $L \subset M$ is an oriented, framed link with components labelled by irreducible representations of $\mathbf{g}$ and M is a 2-framed 3-manifold. In particular, there are polynomial invariants $\langle L\rangle$ that take values in $\mathbb{Z}[q^{-1}, q]$. Khovanov has conjectured that at least for some classes of Lie algebras (e.g. simply-laced) there exists a bigraded homology theory of labelled links such that the polynomial invariant $\langle L\rangle$ is the quantum Euler characteristic of this homology. It should define a functor from the category of framed link cobordisms to the category of bigraded abelian groups. In particular, the homology of the unknot labelled by an irreducible representation U of $\mathbf{g}$ should be a Frobenius algebra of dimension $dim(U)$.

8.11 Topological Quantum Field Theory

Quantization of classical fields is an area of fundamental importance in modern mathematical physics. Although there is no satisfactory mathematical theory of quantization of classical dynamical systems or fields, physicists have developed several methods of quantization that can be applied to specific problems. Most successful among these is QED (Quantum Electrodynamics), the theory of quantization of electromagnetic fields. The physical significance of electromagnetic fields is thus well understood at both the classical and the quantum level. Electromagnetic theory is the prototype of classical gauge theories. It is therefore, natural to try to extend the methods of QED to the quantization of other gauge field theories. The methods of quantization may be broadly classified as non-perturbative and perturbative. The literature pertaining to each of these areas is vast. See for example [DW90, Saw96, Ste84]. Our aim in this section is to discuss some aspects of a new area of research in quantum field theory, namely, topological quantum field theory (or TQFT for short). Ideas from TQFT have already led to new ways of looking at old topological invariants as well as to surprising new invariants.

8.11.1 Atiyah-Segal Axioms for TQFT

In 2 and 3 dimensional geometric topology, Conformal Field Theory (CFT) methods have proved to be useful. An attempt to put the CFT on a firm mathematical foundation was begun by Segal in [Seg89] by proposing a set of axioms for CFT. CFT is a two dimensional theory and it was necessary to modify and generalize these axioms to apply to topological field theory in any dimension. We now discuss briefly these TQFT axioms following Atiyah The Atiyah-Segal axioms for TQFT (see, for example, [Ati89, Law96]) arose from an attempt to give a mathematical formulation of the non-perturbative aspects of quantum field theory in general and to develop, in particular, computational tools for the Feynman path integrals that are fundamental in the Hamiltonian approach to Witten's topological QFT. The most spectacular application of the non-perturbative methods has been in the definition and calculation of the invariants of 3-manifolds with or without links and knots. In most physical applications however, it is the perturbative calculations that are predominantly used. Recently, perturbative aspects of the Chern-Simons theory in the context of TQFT have been considered in [BN91]. For other approaches to the invariants of 3-manifolds see [KM90, KR88, Mur93, Tur88, TV92].

Let $\mathcal{C}_n$ denote the category of compact, oriented, smooth n-dimensional manifolds with morphism given by oriented cobordism. Let $\mathcal{V}_{\mathbb{C}}$ denote the category of finite dimensional complex vector spaces. An $(n+1)$-dimensional TQFT is a functor $\mathcal{T}$ from the category $\mathcal{C}_n$ to the category $\mathcal{V}_{\mathbb{C}}$ which satisfies the following axioms.

A1. Let $-\Sigma$ denote the manifold Σ with the opposite orientation of Σ and let V^* be the dual vector space of $V \in \mathcal{V}_{\mathbb{C}}$. Then

$$\mathcal{T}(-\Sigma) = (\mathcal{T}(\Sigma))^*, \quad \forall \Sigma \in \mathcal{C}_n.$$

A2. Let $\sqcup$ denote disjoint union. Then

$$\mathcal{T}(\Sigma_1 \sqcup \Sigma_2) = \mathcal{T}(\Sigma_1) \otimes \mathcal{T}(\Sigma_2), \quad \forall \Sigma_1, \Sigma_2 \in \mathcal{C}_n.$$

A3. Let $Y_i : \Sigma_i \to \Sigma_{i+1},\ i = 1, 2$ be morphisms. Then

$$\mathcal{T}(Y_1 Y_2) = \mathcal{T}(Y_2)\mathcal{T}(Y_1) \in Hom(\mathcal{T}(\Sigma_1), \mathcal{T}(\Sigma_3)),$$

where $Y_1 Y_2$ denotes the morphism given by composite cobordism $Y_1 \cup_{\Sigma_2} Y_2$.

A4. Let $\emptyset_n$ be the empty n-dimensional manifold. Then

$$\mathcal{T}(\emptyset_n) = \mathbb{C}.$$

A5. For every $\Sigma \in \mathcal{C}_n$

$$\mathcal{T}(\Sigma \times [0,1]) : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$$

is the identity endomorphism.

We note that if Y is a compact, oriented, smooth $(n+1)$-manifold with compact, oriented, smooth boundary Σ, then

$$\mathcal{T}(Y) : \mathcal{T}(\phi_n) \to \mathcal{T}(\Sigma)$$

is uniquely determined by the image of the basis vector $1 \in \mathbb{C} \equiv \mathcal{T}(\phi_n)$. In this case the vector $\mathcal{T}(Y) \cdot 1 \in \mathcal{T}(\Sigma)$ is often denoted simply by $\mathcal{T}(Y)$ also. In particular, if Y is closed, then

$$\mathcal{T}(Y) : \mathcal{T}(\phi_n) \to \mathcal{T}(\phi_n) \quad \text{and} \quad \mathcal{T}(Y) \cdot 1 \in \mathcal{T}(\phi_n) \equiv \mathbb{C}$$

is a complex number which turns out to be an invariant of Y. Axiom A3 suggests a way of obtaining this invariant by a cut and paste operation on Y as follows. Let $Y = Y_1 \cup_\Sigma Y_2$ so that Y_1 (resp. Y_2) has boundary Σ (resp. $-\Sigma$). Then we have

$$\mathcal{T}(Y) \cdot 1 = \langle \mathcal{T}(Y_1) \cdot 1, \mathcal{T}(Y_2) \cdot 1 \rangle, \tag{8.55}$$

where $\langle , \rangle$ is the pairing between the dual vector spaces $\mathcal{T}(\Sigma)$ and $\mathcal{T}(-\Sigma) = (\mathcal{T}(\Sigma))^*$. Equation (8.55) is often referred to as a gluing formula. Such gluing formulas are characteristic of TQFT. They also arise in Fukaya-Floer homology theory of 3-manifolds, Floer-Donaldson theory of 4-manifold invariants as well as in 2-dimensional conformal field theory. For specific applications the Atiyah axioms need to be refined, supplemented and modified. For example, one may replace the category $\mathcal{V}_\mathbb{C}$ of complex vector spaces by the category of finite-dimensional Hilbert spaces. This is in fact, the situation of the $(2+1)$-dimensional Jones-Witten theory. In this case it is natural to require the following additional axiom.

A6. Let Y be a compact oriented 3-manifold with $\partial Y = \Sigma_1 \sqcup (-\Sigma_2)$. Then the linear transformations

$$\mathcal{T}(Y) : \mathcal{T}(\Sigma_1) \to \mathcal{T}(\Sigma_2) \quad \text{and} \quad \mathcal{T}(-Y) : \mathcal{T}(\Sigma_2) \to \mathcal{T}(\Sigma_1)$$

are mutually adjoint.

For a closed 3-manifold Y the axiom A6 implies that

$$\mathcal{T}(-Y) = \overline{\mathcal{T}(Y)} \in \mathbb{C}.$$

It is this property that is at the heart of the result that in general, the Jones polynomials of a knot and its mirror image are different, i.e.

$$V_\kappa(t) \neq V_{\kappa_m}(t),$$

where κ_m is the mirror image of the knot κ.

An important example of a $(3+1)$-dimensional TQFT is provided by the Floer-Donaldson theory. The functor $\mathcal{T}$ goes from the category $\mathcal{C}$ of compact, oriented Homology 3-spheres to the category of $\mathbb{Z}_8$-graded abelian groups. It is defined by

$$\mathcal{T}: Y \to HF_*(Y), \quad Y \in \mathcal{C}.$$

For a compact, oriented, 4-manifold M with $\partial M = Y$, $\mathcal{T}(M)$ is defined to be the vector $q(M, Y)$

$$q(M, Y) := (q_1(M, Y), q_2(M, Y), \ldots),$$

where the components $q_i(M, Y)$ are the relative polynomial invariants of Donaldson defined on the relative homology group $H_2(M, Y; \mathbb{Z})$.

The axioms also suggest algebraic approaches to TQFT. The most widely studied of these approaches are based on quantum groups, operator algebras, modular tensor categories and Jones' theory of subfactors. See, for example, books [Koh02, Koc04, KS01, Tur94], and articles [Tur88, TV92, TW93]. Turaev and Viro gave an algebraic construction of such a TQFT by using the quantum $6j$-symbols for the quantum group $U_q(sl_2)$ at roots of unity. Ocneanu [Ocn88] starts with a special type of subfactor to generate the data which can be used with the Turaev and Viro construction.

The correspondence between geometric (topological) and algebraic structures has played a fundamental role in the development of modern mathematics. Its roots can be traced back to the classical work of Descartes. Recent developments in low dimensional geometric topology have raised this correspodence to a new level bringing in ever more exotic algebraic structures such as quantum groups, vertex algebras, monoidal and higher categories. This broad area is now often referred to as quantum topology. See, for example, [Yet1 , Man04].

8.11.2 Quantum Observables

A **quantum field theory** may be considered as an assignment of the **quantum expectation** $\langle \Phi \rangle_\mu$ to each gauge invariant function $\Phi : \mathcal{A}(M) \to \mathbb{C}$, where $\mathcal{A}(M)$ is the space of gauge potentials for a given gauge group G and the base manifold (space-time) M. Φ is called a **quantum observable** or simply an **observable** in quantum field theory. Note that the invariance of Φ under the group of gauge transformations $\mathcal{G}$ implies that Φ descends to a function on the moduli space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ of gauge equivalence classes of gauge potentials. In the Feynman path integral approach to quantization the quantum or vacuum expectation $\langle \Phi \rangle_\mu$ of an observable is given by the following expression.

$$\langle \Phi \rangle_\mu = \frac{\int_{\mathcal{B}(M)} e^{-S_\mu(\omega)} \Phi(\omega) \mathcal{D}\mathcal{B}}{\int_{\mathcal{B}(M)} e^{-S_\mu(\omega)} \mathcal{D}\mathcal{B}}, \tag{8.56}$$

where $e^{-S_\mu}\mathcal{DB}$ is a suitably defined measure on $\mathcal{B}(M)$. It is customary to express the quantum expectation $\langle\Phi\rangle_\mu$ in terms of the **partition function** Z_μ defined by

$$Z_\mu(\Phi) := \int_{\mathcal{B}(M)} e^{-S_\mu(\omega)}\Phi(\omega)\mathcal{DB}. \tag{8.57}$$

Thus we can write

$$\langle\Phi\rangle_\mu = \frac{Z_\mu(\Phi)}{Z_\mu(1)}. \tag{8.58}$$

In the above equations we have written the quantum expectation as $\langle\Phi\rangle_\mu$ to indicate explicitly that, in fact, we have a one-parameter family of quantum expectations indexed by the coupling constant μ in the action. There are several examples of gauge invariant functions. For example, primary characteristic classes evaluated on suitable homology cycles give an important family of gauge invariant functions. The instanton number and the Yang-Mills action are also gauge invariant functions. Another important example is the Wilson loop functional well known in the physics literature.

Wilson Loop Functional Let ρ denote a representation of G on V. Let $\alpha \in \Omega(M, x_0)$ denote a loop at $x_0 \in M$. Let $\pi : P(M,G) \to M$ be the canonical projection and let $p \in \pi^{-1}(x_0)$. If ω is a connection on the principal bundle $P(M,G)$, then the parallel translation along α maps the fiber $\pi^{-1}(x_0)$ into itself. Let $\hat{\alpha}_\omega : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$ denote this map. Since G acts transitively on the fibers, $\exists g_\omega \in G$ such that $\hat{\alpha}_\omega(p) = pg_\omega$. Now define

$$\mathcal{W}_{\rho,\alpha}(\omega) := Tr[\rho(g_\omega)] \quad \forall\omega \in \mathcal{A}. \tag{8.59}$$

We note that g_ω and hence $\rho(g_\omega)$, change by conjugation if, instead of p, we choose another point in the fiber $\pi^{-1}(x_0)$, but the trace remains unchanged. We call these $\mathcal{W}_{\rho,\alpha}$ the Wilson loop functionals associated to the representation ρ and the loop α. In the particular case when $\rho = Ad$ the adjoint representation of G on $\mathbf{g}$, our constructions reduce to those considered in physics. If $L = (\kappa_1, \ldots, \kappa_n)$ is an oriented link with component knots κ_i, $1 \le i \le n$ and if ρ_i is a representation of the gauge group associated to κ_i, then we can define the quantum observable $\mathcal{W}_{\rho,L}$ associated to the pair (L, ρ), where $\rho = (\rho_1, \ldots, \rho_n)$ by

$$\mathcal{W}_{\rho,L} = \prod_{i=1}^{n} \mathcal{W}_{\rho_i,\kappa_i}.$$

8.11.3 Link Invariants

In the 1980s, Jones discovered his polynomial invariant $V_\kappa(q)$, called the **Jones polynomial**, while studying Von Neumann algebras and gave its interpretation in terms of statistical mechanics. These new polynomial invariants have led to the proofs of most of the Tait conjectures. As with most of the earlier invariants, Jones' definition of his polynomial invariants is algebraic and combinatorial in nature and was based on representations of the braid groups and related Hecke algebras. The Jones polynomial $V_\kappa(t)$ of κ is a Laurent polynomial in t (polynomial in t and t^{-1}) which is uniquely determined by a simple set of properties similar to the well known axioms for the Alexander-Conway polynomial. More generally, the Jones polynomial can be defined for any oriented link L as a Laurent polynomial in $t^{1/2}$.

A geometrical interpretation of the Jones' polynomial invariant of links was provided by Witten by applying ideas from QFT to the Chern-Simons Lagrangian constructed from the Chern-Simons action

$$\mathcal{A}_{CS} = \frac{k}{4\pi}\int_M tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right),$$

where A is the gauge potential of the $SU(n)$ connection ω. Chern-Simons action is not gauge invariant. Under a gauge transformation g the action transforms as follows:

$$\mathcal{A}_{CS}(A^g) = \mathcal{A}_{CS}(A) + 2\pi k \mathcal{A}_{WZ}, \tag{8.60}$$

where $\mathcal{A}_{WZ}$ is the **Wess-Zumino action functional**. It can be shown that the Wess-Zumino functional is integer valued and hence, if the Chern-Simons coupling constant k is taken to be an integer, then the partition function In fact, Witten's model allows us to consider the knot and link invariants in any compact 3-manifold M. Z defined by

$$Z(\Phi) := \int_{\mathcal{B}(M)} e^{-i\mathcal{A}_{CS}(\omega)}\Phi(\omega)\mathcal{D}\mathcal{B}$$

is gauge invariant. We take for Φ the Wilson loop functional $\mathcal{W}_{\rho,L}$, where ρ is a representation of $SU(n)$ and L is the link under consideration.

We denote the Jones polynomial of L simply by V. Recall that there are 3 standard ways to change a link diagram at a crossing point. The Jones polynomials of the corresponding links are denoted by V_+, V_- and V_0 respectively. To verify the defining relations for the Jones' polynomial of a link L in S^3, Witten [Wit89] starts by considering the Wilson loop functionals for the associated links L_+, L_-, L_0. For a framed link L, we denote by $\langle L \rangle$ the expectation value of the corresponding Wilson loop functional for the Chern-Simons theory of level k and gauge group $SU(n)$ and with ρ_i the fundamental representation for all i. To verify the defining relations for the Jones' polynomial of a link L in S^3, Witten considers the expectation values of the Wilson loop functionals for the associated links L_+, L_-, L_0 and obtains the relation

$$\alpha\langle L_+\rangle + \beta\langle L_0\rangle + \gamma\langle L_-\rangle = 0 \tag{8.61}$$

where the coefficients α, β, γ are given by the following expressions

$$\alpha = -\exp\left(\frac{2\pi i}{n(n+k)}\right), \tag{8.62}$$

$$\beta = -\exp\left(\frac{\pi i(2-n-n^2)}{n(n+k)}\right) + \exp\left(\frac{\pi i(2+n-n^2)}{n(n+k)}\right), \tag{8.63}$$

$$\gamma = \exp\left(\frac{2\pi i(1-n^2)}{n(n+k)}\right). \tag{8.64}$$

We note that the result makes essential use of 3-manifolds with boundary. The calculation of the coefficients α, β, γ is closely related to the Verlinde fusion rules [Ver88] and $2d$ conformal field theories. Substituting the values of α, β, γ into equation (8.61) and cancelling a common factor $\exp(\frac{\pi i(2-n^2)}{n(n+k)})$, we get

$$-t^{n/2}\langle L_+\rangle + (t^{1/2} - t^{-1/2})\langle L_0\rangle + t^{-n/2}\langle L_-\rangle = 0, \tag{8.65}$$

where we have put

$$t = \exp\left(\frac{2\pi i}{n+k}\right).$$

This is equivalent to the following skein relation for the polynomial invariant V of the link

$$t^{n/2}V_+ - t^{-n/2}V_- = (t^{1/2} - t^{-1/2})V_0 \tag{8.66}$$

For $SU(2)$ Chern-Simons theory, equation (8.66) is the skein relation that defines a variant of the original Jones' polynomial. This variant also occurs in the work of Kirby and Melvin [KM91] where the invariants are studied by using representation theory of certain Hopf algebras and the topology of framed links. It is not equivalent to the Jones polynomial. In an earlier work [Mar01] I had observed that under the transformation $\sqrt{t} \to -1/\sqrt{t}$, it goes over into the equation which is the skein relation characterizing the Jones polynomial. The Jones polynomial belongs to a different family that corresponds to the negative values of the level. Note that the coefficients in the skein relation (8.66) are defined for positive values of the level k. To extend them to negative values of the level we must also note that the shift in k by the dual Coxeter number would now change the level $-k$ to $-k-n$. If in equation (8.66) we now allow negative values of n and take t to be a formal variable, then the extended family includes both positive and negative levels.

Let $V^{(n)}$ denote the Jones-Witten polynomial corresponding to the skein relation (8.66), (with $n \in \mathbb{Z}$) then the family of polynomials $\{V^{(n)}\}$ can be shown to be equivalent to the two variable HOMFLY polynomial $P(\alpha, z)$ which satisfies the following skein relation

$$\alpha P_+ - \alpha^{-1}P_- = zP_0. \tag{8.67}$$

If we put $\alpha = t^{-1}$ and $z = (t^{1/2} - t^{-1/2})$ in equation (8.47) we get the skein relation for the original Jones polynomial V. If we put $\alpha = 1$ we get the skein relation for the Alexander-Conway polynomial.

To compare our results with those of Kirby and Melvin we note that they use q to denote our t and t to denote its fourth root. They construct a modular Hopf algebra U_t as a quotient of the Hopf algebra $U_q(sl(2,\mathbb{C}))$ which is the well known

q-deformation of the universal enveloping algebra of the Lie algebra $sl(2, \mathbb{C})$. Jones polynomial and its extensions are obtained by studying the representations of the algebras U_t and U_q.

8.11.4 WRT Invariants

Witten's TQFT invariants of 3-manifolds were given a mathematical definition by Reshetikhin and Turaev in [RT91]. In view of this and with a suggestion from Prof. Zagier, I called them Witten–Reshetikhin–Turaev or WRT invariants in [Mar01]. Several alternative approaches to WRT invariants are now available. We will discuss some of them later in this section. If $Z_k(1)$ exists, it provides a numerical invariant of M. For example, for $M = S^3$ and $G = SU(2)$, using the Chern-Simons action Witten obtains the following expression for this partition function as a function of the level k

$$Z_k(1) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right). \tag{8.68}$$

This partition function provides a new family of invariants for $M = S^3$, indexed by the level k. Such a partition function can be defined for a more general class of 3-manifolds and gauge groups. More precisely, let G be a compact, simply connected, simple Lie group and let $k \in \mathbb{Z}$. Let M be a 2-framed closed, oriented 3-manifold. We define the **Witten invariant** $\mathcal{T}_{G,k}(M)$ of the triple (M, G, k) by

$$\mathcal{T}_{G,k}(M) := Z(1) := \int_{\mathcal{B}(M)} e^{-i\mathcal{A}_{CS}} \mathcal{D}\mathcal{B}, \tag{8.69}$$

where $e^{-i\mathcal{A}_{CS}} \mathcal{D}\mathcal{B}$, is a suitable measure on $\mathcal{B}(M)$. We note that no precise definition of such a measure is available at this time and the definition is to be regarded as a formal expression. Indeed, one of the aims of TQFT is to make sense of such formal expressions. We define the **normalized Witten invariant** $\mathcal{W}_{G,k}(M)$ of a 2-framed, closed, oriented 3-manifold M by

$$\mathcal{W}_{G,k}(M) := \frac{\mathcal{T}_{G,k}(M)}{\mathcal{T}_{G,k}(S^3)}. \tag{8.70}$$

If G is a compact, simply connected, simple Lie group and M, N be two 2-framed, closed, oriented 3-manifolds. Then we have the following results:

$$\mathcal{T}_{G,k}(S^2 \times S^1) = 1 \tag{8.71}$$

$$\mathcal{T}_{SU(2),k}(S^3) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \tag{8.72}$$

$$\mathcal{W}_{G,k}(M \# N) = \mathcal{W}_{G,k}(M)\mathcal{W}_{G,k}(N) \tag{8.73}$$

If G is a compact simple group then the WRT invariant $\mathcal{T}_{G,k}(S^3)$ can be given in a closed form in terms of the root and weight lattices associated to G. In particular, for $G = SU(n)$ we get

$$\mathcal{T} = \frac{1}{\sqrt{n(k+n)^{(n-1)}}} \prod_{j=1}^{n-1} \left[2\sin\left(\frac{j\pi}{k+n}\right)\right]^{n-j}.$$

We will show later that this invariant can be expressed in terms of the generating function of topological string amplitudes in a closed string theory compactified on a suitable Calabi-Yau manifold. More generally, if a manifold M can be cut into pieces over which the CS path integral can be computed, then the gluing rules of TQFT can be applied to these pieces to find $\mathcal{T}$. Different ways of using such a cut and paste operation can lead to different ways of computing this invariant. Another method that is used in both the theoretical and experimental applications is the perturbative quantum field theory. The rules for perturbative expansion around classical solutions of field equations are well understood in physics. It is called the stationary phase approximation to the partition function. It leads to the asymptotic expansion in terms of a parameter depending on the coupling constants and the group. If $\check{c}(G)$ is the dual Coxeter number of G then the asymptotic expansion is in terms of $\hbar = 2\pi i/(k + \check{c}(G))$. This notation in TQFT is a reminder of the Planck's constant used in physical field theories. The asymptotic expansion of $\log(\mathcal{T})$ is then given by

$$\log(\mathcal{T}) = -b\log\hbar + \frac{a_0}{\hbar} + \sum_{n=1}^{\infty} a_{n+1}\hbar^n,$$

where a_i are evaluated on Feynman diagrams with i loops. The expansion may be around any flat connection and the dependence of a_i the choice of connection may be explicitly indicated if necessary. For Chern-Simons theory the above perturbative expansion is also valid for non-compact groups. In his talk at this conference, Garofouladis discussed the asymptotic expansion of the free energy associated to the LMO invariant of a 3-manifold and its many interesting properties (see Garofouladis et al. in these proceedings) I asked Stavros if he has looked at his expansion as a generating function for topological string moduli. I also asked a similar question to Don Zagier about the free energy expansion of Chern-Simons invariant with complex gauge group considered by Zagier et al. in [DGLZ09]. Both of them told me that they had not considered this aspect. It seems that the general program of relating gauge theoretic and string theoretic invariants is still far from well formulated, even in the cases where explicit asymptotic expansions are available.

8.11.5 CFT Approach to WRT Invariants

In [Koh92] Kohno defines a family of invariants $\Phi_k(M)$ of a 3-manifold M by using its Heegaard decomposition along a Riemann surface Σ_g and representations of the mapping class group of Σ_g. Kohno's work makes essential use of ideas and results from conformal field theory. We now give a brief discussion of Kohno's definition.

We begin by reviewing some information about the geometric topology of 3-manifolds and their Heegaard splittings. Recall that two compact 3-manifolds X_1, X_2 with homeomorphic boundaries can be glued together along a homeomorphism $f : \partial X_1 \to \partial X_2$ to obtain a closed 3-manifold $X = X_1 \cup_f X_2$. If X_1, X_2 are oriented with compatible orientations on the boundaries, then f can be taken to be either orientation preserving or reversing. Conversely, any closed orientable 3-manifold can be obtained by such a gluing procedure where each of the pieces is a special 3-manifold called a handlebody. Recall that a **handlebody** of **genus** g is an orientable 3-manifold obtained from gluing g copies of **1-handles** $D^2 \times [-1, 1]$ to the 3-ball D^3. The gluing homeomorphisms join the $2g$ discs $D^2 \times \{\pm 1\}$ to the $2g$ pairwise disjoint 2-discs in $\partial D^3 = S^2$ in such a way that the resulting manifold is orientable. The handlebodies H_1, H_2 have the same genus and a common boundary $H_1 \cap H_2 = \partial H_1 = \partial H_2$. Such a decomposition of a 3-manifold X is called a **Heegaard splitting** of X of genus g. We say that X has **Heegaard genus** g if it has some Heegaard splitting of genus g but no Heegaard splitting of smaller genus. Given a Heegaard splitting of genus g of X, there exists an operation called **stabilization** which gives another Heegaard splitting of X of genus $g + 1$. Two Heegaard splitting of X are called **equivalent** if there exists a homeomorphism of X onto itself taking one splitting into the other. Two Heegaard splitting of X are called **stably equivalent** if they are equivalent after a finite number of stabilizations. A proof of the following theorem is given in [Sav99].

Theorem 3 *Any two Heegaard splittings of a closed orientable* 3*-manifold X are stably equivalent.*

The **mapping class group** $\mathcal{M}(M)$ of a connected, compact, smooth surface M is the quotient group of the group of diffeomorphisms $\mathit{Diff}(M)$ of M modulo the group $\mathit{Diff}_0(M)$ of diffeomorphisms isotopic to the identity. i.e.

$$\mathcal{M}(M) := \mathit{Diff}(M)/\mathit{Diff}_0(M)$$

If M is oriented, then $\mathcal{M}(M)$ has a normal subgroup $\mathcal{M}^+(M)$ of index 2 consisting of orientation preserving diffeomorphisms of M modulo isotopies. The group $\mathcal{M}(M)$ can also be defined as $\pi_0(\mathit{Diff}(M))$. Smooth closed orientable surfaces Σ_g are classified by their genus g and in this case it is customary to denote $\mathcal{M}(\Sigma_g)$ by $\mathcal{M}_g$. In the applications that we have in mind, it is this group $\mathcal{M}_g$ that we shall use. The group $\mathcal{M}_g$ is generated by **Dehn twists** along simple closed curves in Σ_g. Let c be a simple closed curve in Σ_g which forms one of the boundaries of an annulus. In local complex coordinate z we can identify the annulus with $\{z \mid 1 \le |z| \le 2\}$ and the curve c with $\{z \mid |z| = 1\}$. Then the Dehn twist τ_c along c is an automorphism of Σ_g which is the identity outside the annulus and in the annulus, is given by the formula

$$\tau_c(re^{i\theta}) = re^{i(\theta + 2\pi(r-1))}, \quad \text{where } z = re^{i\theta},\ 1 \le r \le 2, 0 \le \theta \le 2\pi$$

Changing the curve c by an isotopic curve or changing the annulus gives isotopic twists. However, twists in opposite directions define elements of $\mathcal{M}_g$ which are the inverses of each other. Note that any two homotopic simple closed curves on Σ_g are isotopic. A useful description of $\mathcal{M}_g$ is given by the following theorem.

Theorem 4 *Let Σ_g be a smooth closed orientable surface of genus g. Then the group $\mathcal{M}_g$ is generated by the $3g-1$ Dehn twists along the curves $\alpha_i, \beta_j, \gamma_k,\ 1 \leq i, j \leq g,\ 1 \leq k < g$ which are Poincaré dual to a basis of the first integral homology of Σ_g.*

In [Koh92] Kohno obtains a representation of the mapping class group $\mathcal{M}_g$ in the space of conformal blocks which arise in conformal field theory. He then uses a special function for this representation and the stabilization to define a family of invariants $\Phi_k(M)$ of the 3-manifold M which are independent of its stable Heegaard decomposition. Kohno obtains the following formulas:

$$\Phi_k(S^2 \times S^1) = \left(\sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right)\right)^{-1}, \tag{8.74}$$

$$\Phi_k(S^3) = 1, \tag{8.75}$$

$$\Phi_k(M\#N) = \Phi_k(M) \cdot \Phi_k(N). \tag{8.76}$$

Kohno's invariant coincides with the normalized Witten invariant with the gauge group $SU(2)$. Similar results were also obtained by Crane [Cra91]. The agreement of these results with those of Witten may be regarded as strong evidence for the correctness of the TQFT calculations. In [Koh92] Kohno also obtains the Jones-Witten polynomial invariants for a framed colored link in a 3-manifold M by using representations of mapping class groups via conformal field theory. In [Koh94] the Jones-Witten polynomials are used to estimate the tunnel number of knots and the Heegaard genus of a 3-manifold. The monodromy of the Knizhnik-Zamolodchikov equation [KZ84] plays a crucial role in these calculations.

8.11.6 WRT Invariants via Quantum Groups

Shortly after the publication of Witten's paper [Wit89], Reshetikhin and Turaev [RT91] gave a precise combinatorial definition of a new invariant by using the representation theory of quantum group $U_q sl_2$ at the root of unity $q = e^{2\pi i/(k+2)}$. The parameter q coincides with Witten's $SU(n)$ Chern-Simons theory parameter t when $n = 2$ and in this case the invariant of Reshetikhin and Turaev is the same as the normalized Witten invariant. In view of this it is now customary to call the normalized Witten invariant as Witten-Reshetikhin-Turaev invariant or WRT invariant. We now discuss their construction in the form given by Kirby and Melvin in [KM91].

Let $\mathbf{U}$ denote the universal enveloping algebra of $sl(2,\mathbb{C})$ and let $\mathbf{U}_h$ denote the quantized universal enveloping algebra of formal power series in h. Recall that $\mathbf{U}$ is generated by X, Y, H subject to relations as in the algebra $sl(2,\mathbb{C})$, i.e.

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

In $\mathbf{U}_h$ the last relation is replaced by

$$[X, Y] = [H]_s := \frac{s^H - s^{-H}}{s - s^{-1}}, \quad s = e^{h/2}.$$

It can be shown that $\mathbf{U}_h$ admits a Hopf algebra structure as a module over the ring of formal power series. However, the presence of divergent series make this algebra unsuitable for representation theory. We construct a finite dimensional algebra by using $\mathbf{U}_h$. Define

$$K := e^{hH/4} \quad \text{and} \quad \bar{K} := e^{-hH/4} = K^{-1}.$$

Fix an integer $r > 1$ ($r = k + 2$ of the Witten formula) and set $q = e^h = e^{2\pi i/r}$. We restrict this to a subalgebra over the ring of convergent power series in h generated by $X, Y, K, \bar{K}$. This infinite dimensional algebra occurs in the work of Jimbo. We take its quotient by setting

$$X^r = 0, \qquad Y^r = 0, \qquad K^{4r} = 1.$$

It is the representations of this quotient algebra $\mathcal{A}$ that are used to define colored Jones polynomials and the WRT invariants. The algebra $\mathcal{A}$ is a finite dimensional complex algebra satisfying the relations

$$\bar{K} = K^{-1}, \qquad KX = sXK, \qquad KY = \bar{s}YK,$$
$$[X, Y] = \frac{K^2 - K^{-2}}{s - \bar{s}}, \quad s = e^{\pi i/r}$$

There are irreducible $\mathcal{A}$-modules V^i in each dimension $i > 0$. If we put $i = 2m + 1$, then V^i has a basis $\{e_m, \ldots, e_{-m}\}$. The action of $\mathcal{A}$ on the basis vectors is given by

$$Xe_j = [m + j + 1]_s e_{j+1}, \qquad Ye_j = [m - j + 1]_s e_{j-1}, \quad \text{and} \quad Ke_j = s^j e_j.$$

The $\mathcal{A}$-modules V^i are self dual for $0 < i < r$. The structure of their tensor products is similar to that in the classical case. The algebra $\mathcal{A}$ has the additional structure of a quasitriangular Hopf algebra with Drinfeld's universal R-matrix R satisfying the Yang-Baxter equation. One has an explicit formula for $R \in \mathcal{A} \otimes \mathcal{A}$ of the form

$$R = \sum c_{nab} X^a K^b \otimes Y^n K^b.$$

If V, W are $\mathcal{A}$-modules, then R acts on $V \otimes W$. Composing with the permutation operator we get the operator $R' : V \otimes W \to W \otimes V$. These are the operators used in the definition of our link invariants. Let L be a framed link with n components L_i colored by $\mathbf{k} = \{k_1, \ldots, k_n\}$. Let $J_{L,\mathbf{k}}$ be the corresponding colored Jones polynomial. The colors are restricted to Lie in a family of irreducible modules V^i, one for each dimension $0 < i < r$. Let σ denote the signature of the linking matrix of L. Define τ_L by

$$\tau_L = \left(\sqrt{2/r}\sin(\pi/r)\right)^n e^{3(2-r)\sigma/(8r)} \sum [\mathbf{k}] J_{L,\mathbf{k}},$$

where the sum is over all admissible colors. Every 3-manifold can be obtained by surgery on a link in S^3. Two links give isomorphic manifolds if they are related by Kirby moves. It can be shown that the invariant τ_L is preserved under Kirby moves and hence defines an invariant of the 3-manifold M_L obtained by surgery on L. With suitable normalization it coincides with the WRT invariant. WRT invariants do not belong to the class of polynomial invariants or other known 3-manifold invariants. They arose from topological quantum field theory applied to calculate the partition functions in the Chern-Simons gauge theory.

A number of other mathematicians have also obtained invariants that are closely related to the Witten invariant. The equivalence of these invariants defined by using different methods was a folk theorem until a complete proof was given by Piunikhin in [Piu93]. Another approach to WRT invariants is via Hecke algebras and related special categories. A detailed construction of modular categories from Hecke algebras at roots of unity is given in [Bla00]. For a special choice of the framing parameter, one recovers the Reshetikhin-Turaev invariants of 3-manifolds constructed from the representations of the quantum groups $U_q sl(N)$ by Reshetikhin, Turaev and Wenzl [RT91, TW93, Wen93]. These invariants were constructed by Yokota [Yok97] by using skein theory. As we have discussed earlier the quantum invariants were obtained by Witten [Wit88] by using path integral quantization of Chern-Simons theory. In "Quantum Invariants of Knots and 3-Manifolds" [Tur94], Turaev showed that the idea of modular categories is fundamental in the construction of these invariants and that it plays an essential role in extending them to a Topological Quantum Field Theory. Since these early results, WRT invariants for several other manifolds and gauge groups have been obtained. We collect together some of these results below.

Theorem 5 *The WRT invariant for the lens space $L(p, q)$ in the canonical framing is given by*

$$W_k(L(p,q)) = -\frac{i}{\sqrt{2p(k+2)}} e^{\left(\frac{6\pi i s}{k+2}\right)} \sum_{\delta \in \{-1,1\}} \sum_{n=1}^{p} \delta e^{\frac{\delta}{2p(k+2)}} e^{\frac{2\pi i q n^2 (k+2)}{p}} e^{\frac{2\pi i n(q+\delta)}{p}},$$

where $s = s(q, p)$ is the Dedekind sum defined by

$$s(q,p) := \frac{1}{4p} \sum_{k=1}^{p-1} \cot\left(\frac{\pi k}{p}\right) \cot\left(\frac{\pi k q}{p}\right).$$

In all of these the invariant is well defined only at roots of unity and perhaps near roots of unity if a perturbative expansion is possible. This situation occurs in the study of classical modular functions and Ramanujan's mock theta functions. Ramanujan had introduced his mock theta functions in a letter to Hardy in 1920 (the famous last letter) to describe some power series in variable $q = e^{2\pi i z}$, $z \in \mathbb{C}$. He also wrote down (without proof, as was usual in his work) a number of identities involving these series which were completely verified only in 1988 by Hickerson [Hic88]. Recently, Lawrence and Zagier have obtained several different formulas for the Witten invariant $\mathcal{W}_{SU(2),k}(M)$ of the Poincaré homology sphere $M = \Sigma(2,3,5)$ in [LZ99]. Using the work of Zwegers [Zwe01], they show how the Witten invariant can be extended from integral k to rational k and give its relation to the mock theta function. In particular, they obtain the following fantastic formula, a la Ramanujan, for the Witten invariant $\mathcal{W}_{SU(2),k}(M)$ of the Poincaré homology sphere

$$\mathcal{W}_{SU(2),k}(\Sigma(2,3,5)) = 1 + \sum_{n=1}^{\infty} x^{-n^2}(1+x)(1+x^2)\cdots(1+x^{n-1})$$

where $x = e^{\pi i/(k+2)}$. We note that the series on the right hand side of this formula terminates after $k+2$ terms[1].

We have not discussed the Kauffman bracket polynomial or the theory of skein modules in the study of 3-manifold invariants. An invariant that combines these two ideas has been define in the following general setting. Let R be a commutative ring and let A be a fixed invertible element of R. Then one can define a new invariant, $\mathcal{S}_{2,\infty}(M; R, A)$, of an oriented 3-manifold M called the **Kauffman bracket skein module** (see [Prz99]). The theory of skein modules is related to the theory of representations of quantum groups. This connection should prove useful in developing the theory of quantum group invariants which can be defined in terms of skein theory as well as by using the theory of representations of quantum groups.

8.12 Chern-Simons and String Theory

The general question "what is the relationship between gauge theory and string theory?" is not meaningful at this time. So I will follow the strong admonition by Galileo against[2] "disputar lungamente delle massime questioni senza conseguir verità nissuna". However, interesting special cases where such relationship can be established are emerging. For example, Witten [Wit95] has argued that Chern-Simons gauge theory on a 3-manifold M can be viewed as a string theory constructed by using a topological sigma model with target space T^*M. The perturbation theory of this string will coincide with Chern-Simons perturbation theory, in the form discussed by Axelrod and Singer [AS94]. The coefficient of k^{-r} in the perturbative

[1] I would like to thank Don Zagier for bringing this work to my attention.

[2] Lengthy discussions about the greatest questions that fail to lead to any truth whatever.

expansion of $SU(n)$ theory in powers of $1/k$ comes from Feynman diagrams with r loops. Witten shows how each diagram can be replaced by a Riemann surface Σ of genus g with h holes (boundary components) with $g = (r - h + 1)/2$. Gauge theory would then give an invariant $\Gamma_{g,h}(M)$ for every topological type of Σ. Witten shows that this invariant would equal the corresponding string partition function $Z_{g,h}(M)$. We now give an example of gauge theory to string theory correspondence relating the non-perturbative WRT invariants in Chern-Simons theory with gauge group $SU(n)$ and topological string amplitudes which generalize the GW (Gromov-Witten) invariants of Calabi-Yau 3-folds following the work in [GV99, AMnV04]. The passage from real 3 dimensional Chern-Simons theory to the 10 dimensional string theory and further onto the 11 dimensional M-theory can be schematically represented by the following:

$$\begin{aligned} 3+3 &= 6 \text{ (real symplectic 6-manifold)} \\ &= 6 \text{ (conifold in } \mathbb{C}^4\text{)} \\ &= 6 \text{ (Calabi-Yau manifold)} \\ &= 10-4 \text{ (string compactification)} \\ &= (11-1)-4 \text{ (M-theory)} \end{aligned}$$

We now discuss the significance of the various terms of the above equation array. Recall that string amplitudes are computed on a 6-dimensional manifold which in the usual setting is a complex 3-dimensional Calaby-Yau manifold obtained by string compactification. This is the most extensively studied model of passing from the 10-dimensional space of supersymmetric string theory to the usual 4-dimensional space-time manifold. However, in our work we do allow these so called extra dimensions to form an open or a symplectic Calabi-Yau manifold. We call these the generalized Calabi-Yau manifolds. The first line suggests that we consider open topological strings on such a generalized Calabi-Yau manifold, namely, the cotangent bundle T^*S^3, with Dirichlet boundary conditions on the zero section S^3. We can compute the open topological string amplitudes from the $SU(n)$ Chern-Simons theory. Conifold transition [STY02] has the effect of closing up the holes in open strings to give closed strings on the Calabi-Yau manifold obtained by the usual string compactification from 10 dimensions. Thus we recover a topological gravity result starting from gauge theory. In fact, as we discussed earlier, Witten had anticipated such a gauge theory string theory correspondance almost ten years ago. Significance of the last line is based on the conjectured equivalence of M-theory compactified on S^1 to type IIA strings compactified on a Calabi-Yau threefold. We do not consider this aspect here. The crucial step that allows us to go from a real, non-compact, symplectic 6-manifold to a compact Calabi-Yau manifold is the conifold or geometric transition. Such a change of geometry and topology is expected to play an important role in other applications of string theory as well. A discussion of this example from physical point of view may be found in [AMnV04, GV99].

8.12.1 Conifold Transition

To understand the relation of the WRT invariant of S^3 for $SU(n)$ Chern-Simons theory with open and closed topological string amplitudes on "Calabi-Yau" manifolds we need to discuss the concept of conifold transition. From the geometrical point of view this corresponds to symplectic surgery in six dimensions. It replaces a vanishing Lagrangian 3-sphere by a symplectic S^2. The starting point of the construction is the observation that T^*S^3 minus its zero section is symplectomorphic to the cone $z_1^2+z_2^2+z_3^2+z_4^2=0$ minus the origin in $\mathbb{C}^4$, where each manifold is taken with its standard symplectic structure. The complex singularity at the origin can be smoothed out by the manifold M_τ defined by $z_1^2+z_2^2+z_3^2+z_4^2=\tau$ producing a Lagrangian S^3 vanishing cycle. There are also two so called small resolutions $M^\pm$ of the singularity with exceptional set $\mathbb{C}P^1$.

They are defined by

$$M^\pm := \left\{ z \in \mathbb{C}^4 \;\middle|\; \frac{z_1+iz_2}{z_3 \pm iz_4} = \frac{-z_3 \pm iz_4}{z_1 - iz_2} \right\}.$$

Note that $M_0 \setminus \{0\}$ is symplectomorphic to each of $M^\pm \setminus \mathbb{C}P^1$. Blowing up the exceptional set $\mathbb{C}P^1 \subset M^\pm$ gives a resolution of the singularity which can be expressed as a fiber bundle F over $\mathbb{C}P^1$. Going from the fiber bundle T^*S^3 over S^3 to the fiber bundle F over $\mathbb{C}P^1$is referred to in the physics literature as the conifold transition. We note that the holomorphic automorphism of $\mathbb{C}^4$ given by $z_4 \mapsto -z_4$ switches the two small resolutions $M^\pm$ and changes the orientation of S^3. Conifold transition can also be viewed as an application of mirror symmetry to Calabi-Yau manifolds with singularities. Such an interpretation requires the notion of symplectic Calabi-Yau manifolds and the corresponding enumerative geometry. The geometric structures arising from the resolution of singularities in the conifold transition can also be interpreted in terms of the symplectic quotient construction of Marsden and Weinstein.

8.12.2 WRT Invariants and String Amplitudes

To find the relation between the large n limit of $SU(n)$ Chern-Simons theory on S^3 to a special topological string amplitude on a Calabi-Yau manifold we begin by recalling the formula for the partition function (vacuum amplitude) of the theory $\mathcal{T}_{SU(n),k}(S^3)$ or simply $\mathcal{T}$. Up to a phase, it is given by

$$\mathcal{T} = \frac{1}{\sqrt{n(k+n)^{(n-1)}}} \prod_{j=1}^{n-1} \left[2\sin\left(\frac{j\pi}{k+n}\right)\right]^{n-j}. \tag{8.77}$$

Let us denote by $F_{(g,h)}$ the amplitude of an open topological string theory on T^*S^3 of a Riemann surface of genus g with h holes. Then the generating function for the free energy can be expressed as

$$-\sum_{g=0}^{\infty}\sum_{h=1}^{\infty}\lambda^{2g-2+h}n^h F_{(g,h)} \tag{8.78}$$

This can be compared directly with the result from Chern-Simons theory by expanding the log $\mathcal{T}$ as a double power series in λ and n.

Instead of that we use the conifold transition to get the topological amplitude for a closed string on a Calabi-Yau manifold. We want to obtain the large n expansion of this amplitude in terms of parameters λ and τ which are defined in terms of the Chern-Simons parameters by

$$\lambda = \frac{2\pi}{k+n}, \quad \tau = n\lambda = \frac{2\pi n}{k+n}. \tag{8.79}$$

The parameter λ is the string coupling constant and τ is the 't Hooft coupling $n\lambda$ of the Chern-Simons theory. The parameter τ entering in the string amplitude expansion has the geometric interpretation as the Kähler modulus of a blown up S^2 in the resolved $M^{\pm}$. If $F_g(\tau)$ denotes the amplitude for a closed string at genus g then we have

$$F_g(\tau) = \sum_{h=1}^{\infty}\tau^h F_{(g,h)} \tag{8.80}$$

So summing over the holes amounts to filling them up to give the closed string amplitude.

The large n expansion of $\mathcal{T}$ in terms of parameters λ and τ is given by

$$\mathcal{T} = \exp\left[-\sum_{g=0}^{\infty}\lambda^{2g-2}F_g(\tau)\right], \tag{8.81}$$

where F_g defined in (8.80) can be interpreted on the string side as the contribution of closed genus g Riemann surfaces. For $g > 1$ the F_g can be expressed in terms of the Euler characteristic χ_g and the Chern class c_{g-1} of the Hodge bundle of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g as follows

$$F_g = \int_{\mathcal{M}_g} c_{g-1}^3 - \frac{\chi_g}{(2g-3)!}\sum_{n=1}^{\infty} n^{2g-3}e^{-n(\tau)}. \tag{8.82}$$

The integral appearing in the formula for F_g can be evaluated explicitly to give

$$\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{(-1)^{(g-1)}}{(2\pi)^{(2g-2)}}2\zeta(2g-2)\chi_g. \tag{8.83}$$

The Euler characteristic is given by the Harer-Zagier [HZ86] formula

$$\chi_g = \frac{(-1)^{(g-1)}}{(2g)(2g-2)} B_{2g}, \tag{8.84}$$

where B_{2g} is the $(2g)$-th Bernoulli number. We omit the special formulas for the genus 0 and genus 1 cases. The formulas for F_g for $g \geq 0$ coincide with those of the g-loop topological string amplitude on a suitable Calabi-Yau manifold. The change in geometry that leads to this calculation can be thought of as the result of coupling to gravity. Such a situation occurs in the quantization of Chern-Simons theory. Here the classical Lagrangian does not depend on the metric, however, coupling to the gravitational Chern-Simons term is necessary to make it TQFT.

We have mentioned the following four approaches that lead to the WRT invariants.

1. Witten's QFT calculation of the Chern-Simons partition function
2. Quantum group (or Hopf algebraic) computations initiated by Reshetikhin and Turaev
3. Kohno's special functions corresponding to representations of mapping class groups in the space of conformal blocks and a similar approach by Crane
4. Open or closed string amplitudes in suitable Calabi-Yau manifolds

These methods can also be applied to obtain invariants of links, such as the Jones polynomial. Indeed, this was the objective of Witten's original work. WRT invariants were a byproduct of this work. Their relation to topological strings came later.

The WRT to string theory correspondence has been extended by Gopakumar and Vafa (see, [GV08a, GV08b]) by using string theoretic arguments to show that the expectation value of the quantum observables defined by the Wilson loops in the Chern-Simons theory also has a similar interpretation in terms of a topological string amplitude. This leads them to conjecture a correspondence between certain knot invariants (such as the Jones polynomial) and Gromov-Witten type invariants of generalized Calabi-Yau manifolds. Gromov-Witten invariants of a Calabi-Yau 3-fold X are in general rational numbers, since one has to get the weighted count by dividing by the order of automorphism groups. Using M-theory Gopakumar and Vafa have argued that the generating series F_X of Gromov-Witten invariants in all degrees and all genera is determined by a set of integers $n(g, \beta)$. They give the following remarkable formula for F_X

$$F_X(\lambda, q) = \sum \sum_{g \geq 0} \sum_{k \geq 1} \frac{1}{k} n(g, \beta)(2\sin(k\lambda/2))^{2g-2} q^{k\beta},$$

where λ is the string coupling constant and the first sum is taken over all nonzero elements β in $H_2(X)$. We note that for a fixed genus there are only finitely many nonzero integers $n(g, \beta)$. A mathematical formulation of the Gopakumar-Vafa conjecture (GV conjecture) has been given in [Pan99]. Special cases of the conjecture have been verified (see, for example [Pan02] and references therein). In [LLZ06] a new geometric approach relating the Gromov-Witten invariants to equivariant index theory and 4-dimensional gauge theory has been used to compute the string partition functions of some local Calabi-Yau spaces and to verify the GV conjecture for them.

A knot should correspond to a Lagrangian D-brane on the string side and the knot invariant would then give a suitably defined count of compact holomorphic curves with boundary on the D-brane. To understand a proposed proof, recall first that a categorification of an invariant I is the construction of a suitable homology such that its Euler characteristic equals I. A well known example of this is Floer's categorification of the Casson invariant. We have already discussed earlier, Khovanov's categorification of the Jones polynomial $V_\kappa(q)$ by constructing a bi-graded $sl(2)$-homology $H_{i,j}$ determined by the knot κ. Its quantum or graded Euler characteristic equals the Jones polynomial. i.e.

$$V_\kappa(q) = \sum_{i,j} (-1)^j q^i \, dim \, H_{i,j}.$$

Now let L_κ be the Lagrangian submanifold corresponding to the knot κ of a fixed Calabi-Yau space X. Let r be a fixed relative integral homology class of the pair (X, L_κ). Let $\mathcal{M}_{g,r}$ denote the moduli space of pairs (Σ_g, A), where Σ_g is a compact Riemann surface in the class r with boundary S^1 and A is a flat $U(1)$ connection on Σ_g. This data together with the cohomology groups $H^k(\mathcal{M}_{g,r})$ determines a tri-graded homology. It generalizes the Khovanov homology. Its Euler characteristic is a generating function for the BPS states' invariants in string theory and these can be used to obtain the Gromov-Witten invariants. Taubes has given a construction of the Lagrangians in the Gopakumar-Vafa conjecture. We note that counting holomorphic curves with boundary on a Lagrangian manifold was introduced by Floer in his work on the Arnold conjecture.

The tri-graded homology is expected to unify knot homologies of the Khovanov type as well as knot Floer homology constructed by Ozswáth and Szabó [OS03] which provides a categorification of the Alexander polynomial. Knot Floer homology is defined by counting pseudo-holomorphic curves and has no known combinatorial description. An explicit construction of a tri-graded homology for certain torus knots has been recently given by Dunfield, Gukov and Rasmussen [DGR05].

8.13 Yang-Mills, Gravity and Strings

Recall that in string theory, an elementary particle is identified with a vibrational mode of a string. Different particles correspond to different harmonics of vibration. The Feynman diagrams of the usual QFT are replaced by fat graphs or Riemann surfaces that are generated by moving strings splitting or joining together. The particle interactions described by these Feynman diagrams are built into the basic structure of string theory. The appearance of Riemann surfaces explains the relation to conformal field theory. We have already discussed Witten's argument relating gauge and string theories. It now forms a small part of the program of relating quantum group invariants and topological string amplitudes. In general, the string states are identified with fields. The ground state of the closed string turns out to be a massless spin two field which may be interpreted as a graviton. In the large distance limit, (at least at the lower loop levels) string theory includes the vacuum equations of Einstein's general relativity theory. String theory avoids the ultraviolet divergences that appear in conventional attempts at quantizing gravity. In physically interesting string models one expects the string space to be a non-trivial bundle over a Lorentzian space-time M with compact or non-compact fibers. Relating the usual Einstein's equations with cosmological constant with the Yang-Mills equations requires the ten dimensional manifold $\Lambda^2(M)$ of differential forms of degree two. There are several differences between the Riemannian functionals used in theories of gravitation and the Yang-Mills functional used to study gauge field theories. The most important difference is that the Riemannian functionals are dependent on the bundle of frames of M or its reductions, while the Yang-Mills functional can be defined on any principal bundle over M. However, we have the following interesting theorem [Bes86].

Theorem *Let (M, g) be a compact, 4-dimensional, Riemannian manifold. Let $\Lambda^2_+(M)$ denote the bundle of self-dual 2-forms on M with induced metric G_+. Then the Levi-Civita connection λ_g on M satisfies the Euclidean gravitational instanton equations if and only if the Levi-Civita connection λ_{G_+} on $\Lambda^2_+(M)$ satisfies the Yang-Mills instanton equations.*

8.13.1 Gravitational Field Equations

A geometric formulation of gravitational field equations is generally not in the tool kit of topologists. We review them as the full Einstein equations with energy-momentum tensor corresponding to the dilaton field appear in Perelman's work on the Thurston geometrization conjecture. There are several ways of deriving Einstein's gravitational field equations. For example, we can consider natural tensors satisfying the conditions that they contain derivatives of the fundamental (pseudo-metric) tensor up to order two and depend linearly on the second order derivatives. Then we obtain the tensor

$$c_1 R^{ij} + c_2 g^{ij} S + c_3 g^{ij},$$

where R^{ij} are the components of the Ricci tensor *Ric* and S is the scalar curvature. Requiring this tensor to be divergenceless and using the Bianchi identities leads to the relation $c_1 + 2c_2 = 0$ between the constants c_1, c_2, c_3. Choosing $c_1 = 1$ and $c_3 = 0$ we obtain Einstein's equations (without the cosmological constant) which may be expressed as

$$E = -T \tag{8.85}$$

where $E := Ric - \frac{1}{2}Sg$ is the **Einstein tensor** and T is an energy-momentum tensor on the space-time manifold which acts as the source term. Now the Bianchi identities satisfied by the curvature tensor imply that

$$div\, E := \nabla_i E^{ij} = 0.$$

Hence, if Einstein's equations (8.85) are satisfied, then for consistency we must have

$$div\, T = \nabla_i T^{ij} = 0. \tag{8.86}$$

Equation (8.86) is called the differential (or local) law of conservation of energy and momentum. However, integral (or global) conservation laws can be obtained by integrating equation (8.86) only if the space-time manifold admits Killing vectors. Thus equation (8.86) has no clear physical meaning, except in special cases. An interesting discussion of this point is given by Sachs and Wu [SW77]. Einstein was aware of the tentative nature of the right hand side of equation (8.85), but he believed strongly in the expression on the left hand side of (8.85). By taking the trace of both sides of equations (8.85) we are led to the condition

$$S = t \tag{8.87}$$

where t denotes the trace of the energy-momentum tensor. The physical meaning of this condition seems even more obscure than that of condition (8.86). If we modify equation (8.85) by adding the cosmological term Λg (Λ is called the **cosmological constant**) to the left hand side of equation (8.85), we obtain Einstein's equation with cosmological constant

$$E + \Lambda g = -T. \tag{8.88}$$

This equation also leads to the consistency condition (8.86), but condition (8.87) is changed to

$$S = t + 4\Lambda. \tag{8.89}$$

Using (8.89), equation (8.88) can be rewritten in the following form

$$K = -\left(T - \frac{1}{4}tg\right), \tag{8.90}$$

where

$$K = \left(Ric - \frac{1}{4}Sg\right) \tag{8.91}$$

is the trace-free part of the Ricci tensor of g. We call equations (8.90) **generalized field equations** of gravitation. We now show that these equations arise naturally in a geometric formulation of Einstein's equations. We begin by defining a tensor of curvature type.

Let C be a tensor of type $(4,0)$ on M. We can regard C as a quadrilinear mapping (pointwise) so that for each $x \in M$, C_x can be identified with a multilinear map

$$C_x : T_x^*(M) \times T_x^*(M) \times T_x^*(M) \times T_x^*(M) \to \mathbb{R}.$$

We say that the tensor C is of curvature type if C_x satisfies the following conditions for each $x \in M$ and for all $\alpha, \beta, \gamma, \delta \in T_x^*(M)$.

1. $C_x(\alpha, \beta, \gamma, \delta) = -C_x(\beta, \alpha, \gamma, \delta)$;
2. $C_x(\alpha, \beta, \gamma, \delta) = -C_x(\alpha, \beta, \delta, \gamma)$;
3. $C_x(\alpha, \beta, \gamma, \delta) + C_x(\alpha, \gamma, \delta, \beta) + C_x(\alpha, \delta, \gamma, \beta) = 0$.

From the above definition it follows that a tensor C of curvature type also satisfies the following condition:

$$C_x(\alpha, \beta, \gamma, \delta) = C_x(\gamma, \delta, \alpha, \beta), \quad \forall x \in M.$$

We denote by $\mathcal{C}$ the space of all tensors of curvature type. The Riemann-Christoffel curvature tensor Rm is of curvature type. Indeed, the definition of tensors of curvature type is modelled after this fundamental example. Another important example of a tensor of curvature type is the tensor G defined by

$$G_x(\alpha, \beta, \gamma, \delta) = g_x(\alpha, \gamma) g_x(\beta, \delta) - g_x(\alpha, \delta) g_x(\beta, \gamma), \quad \forall x \in M \tag{8.92}$$

where g is the fundamental or metric tensor of M.

We now define the curvature product of two symmetric tensors of type $(2,0)$ on M. It was introduced by the author in [Mar71] and used in [MM89] to obtain a geometric formulation of Einstein's equations.

Let g and T be two symmetric tensors of type $(2,0)$ on M. The **curvature product** of g and T, denoted by $g \times_c T$, is a tensor of type $(4,0)$ defined by

$$\begin{aligned}(g \times_c T)_x(\alpha, \beta, \gamma, \delta) := \frac{1}{2}\big[&g(\alpha, \gamma)T(\beta, \delta) + g(\beta, \delta)T(\alpha, \gamma) \\ &- g(\alpha, \delta)T(\beta, \gamma) - g(\beta, \gamma)T(\alpha, \delta)\big],\end{aligned}$$

for all $x \in M$ and $\alpha, \beta, \gamma, \delta \in T_x^*(M)$.

In the following proposition we collect together some important properties of the curvature product and tensors of curvature type.

Proposition 1 *Let g and T be two symmetric tensors of type* $(2,0)$ *on M and let C be a tensor of curvature type on M. Then we have the following*:

1. $g \times_c T = T \times_c g$.
2. $g \times_c T$ *is a tensor of curvature type.*
3. $g \times_c g = G$, *where G is the tensor defined in* (8.92).
4. G_x *induces a pseudo-inner product on* $\Lambda_x^2(M), \forall x \in M$.
5. C_x *induces a symmetric, linear transformation of* $\Lambda_x^2(M), \forall x \in M$.

The orthogonal group $O(g)$ of the metric acts on the space $\mathcal{C}$ and splits it into three irreducible subspaces of dimensions 10, 9, and 1. Under this splitting the Riemann curvature Rm into three parts as follows:

$$Rm = W + c_1(K \times_c g) + c_2 S(g \times_c g).$$

The ten dimensional part W is the Weyl conformal curvature tensor. It splits further into its self-dual part W_+ and anti-dual part W_- under the action of $SO(g)$. The part involving the trace-free Ricci tensor K is 9 dimensional. All of these tensors occur in functionals on the space of metrics.

We denote the Hodge star operator on $\Lambda_x^2(M)$ by J_x. The fact that M is a Lorentz 4-manifold implies that J_x defines a complex structure on $\Lambda_x^2(M)$, $\forall x \in M$. Using this complex structure we can give a natural structure of a complex vector space to $\Lambda_x^2(M)$. Then we have the following proposition.

Proposition 2 *Let $U : \Lambda_x^2(M) \to \Lambda_x^2(M)$ be a real, linear transformation. Then the following are equivalent*:

1. *L commutes with J_x.*
2. *L is a complex linear transformation of the complex vector space $\Lambda_x^2(M)$.*
3. *The matrix of L with respect to a G_x-orthonormal basis of $\Lambda_x^2(M)$ is of the form*

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{8.93}$$

where A, B are real 3×3 matrices.

We now define the gravitational tensor W_{gr}, of curvature type, which includes the source term. Let M be a space-time manifold with fundamental tensor g and let T be a symmetric tensor of type $(2, 0)$ on M. Then the **gravitational tensor** W_{gr} is defined by

$$W_{gr} := Rm + g \times_c T, \tag{8.94}$$

where Rm is the Riemann-Christoffel curvature tensor of type $(4, 0)$.

We are now in a position to give a geometric formulation of the generalized field equations of gravitation.

Theorem 6 *Let W_{gr} denote the gravitational tensor defined by* (8.94) *with source tensor T. We denote by $\hat{W}_{gr}$ the linear transformation of $\Lambda_x^2(M)$ induced by W_{gr}. Then the following are equivalent*:

1. *g satisfies the generalized field equations of gravitation* (8.90);
2. *$\hat{W}_{gr}$ commutes with J_x*;
3. *$\hat{W}_{gr}$ is a complex linear transformation of the complex vector space $\Lambda_x^2(M)$.*

We shall call the triple (M, g, T) a **generalized gravitational field** if any one of the conditions of Theorem 6 is satisfied. Generalized gravitational field equations were introduced by the author in [Mar71]. Their mathematical properties have been studied in [MMF95, Mar72, Mod73]. In local coordinates, the generalized gravitational field equations can be written as

$$R^{ij} - \frac{1}{4} R g^{ij} = -\left(T^{ij} - \frac{1}{4} T g^{ij} \right). \tag{8.95}$$

We observe that the equation (8.95) does not lead to any relation between the scalar curvature and the trace of the source tensor, since both sides of equation (8.95) are trace-free. Taking divergence of both sides of equation (8.95) and using the Bianchi identities we obtain the generalized conservation condition

$$\nabla_i T^{ij} - g^{ij} \Phi_i = 0, \tag{8.96}$$

where ∇_i is the covariant derivative with respect to the vector $\frac{\partial}{\partial x^i}$,

$$\Phi = \frac{1}{4}(T - R) \tag{8.97}$$

and $\Phi_i = \frac{\partial}{\partial x^i} \Phi$. Using the function Φ defined by equation (8.97), the field equations can be written as

$$R^{ij} - \frac{1}{2} R g^{ij} - \Phi g^{ij} = -T^{ij}. \tag{8.98}$$

In this form the new field equations appear as Einstein's field equations with the cosmological constant replaced by the function Φ, which we may call the cosmological function. The cosmological function is intimately connected with the classical conservation condition expressing the divergence-free nature of the energy-momentum tensor as is shown by the following proposition.

Proposition 3 *The energy-momentum tensor satisfies the classical conservation condition*

$$\nabla_i T^{ij} = 0 \tag{8.99}$$

if and only if the cosmological function Φ is a constant. Moreover, in this case the generalized field equations reduce to Einstein's field equations with cosmological constant.

We note that, if the energy-momentum tensor is non-zero but is localized in the sense that it is negligible away from a given region, then the scalar curvature acts as a measure of the cosmological constant. By setting the energy-momentum tensor to zero in (8.95) we obtain various characterizations of the usual gravitational instanton. Solutions of the generalized gravitational field equations which are not solutions of Einstein's equations have been discussed in [Can83].

We note that the theorem (6) and the last condition in proposition (1) can be used to discuss the Petrov classification of gravitational fields (see Petrov [Pet69]).

The tensor W_{gr} can be used in place of R in the usual definition of sectional curvature to define the gravitational sectional curvature on the Grassmann manifold of non-degenerate 2-planes over M and to give a further geometric characterization of gravitational field equations. We observe that the generalized field equations of gravitation contain Einstein's equations with or without the cosmological constant as special cases. Solutions of the source-free generalized field equations are called **gravitational instantons** If the base manifold is Riemannian, then gravitational instantons correspond to Einstein spaces. A detailed discussion of the structure of Einstein spaces and their moduli spaces may be found in [Bes86]. Over a compact, 4-dimensional, Riemannian manifold (M, g), the gravitational instantons that are not solutions of the vacuum Einstein equations are critical points of the quadratic, Riemannian functional or action $\mathcal{A}_2(g)$ defined by

$$\mathcal{A}_2(g) = \int_M S^2 dv_g.$$

Furthermore, the standard **Hilbert-Einstein action**

$$\mathcal{A}_1(g) = \int_M S dv_g$$

also leads to the generalized field equations when the variation of the action is restricted to metrics of volume 1.

The generalized field equations of gravitation in the Euclidean theory can be obtained by considerations similar to those given above. It is these equations with the source the dilaton field that appear in Perelman's modification of the Ricci flow. We give a brief discussion of his work in the next section.

8.13.2 Geometrization Conjecture and Gravity

The classification problem for low dimensional manifolds is a natural question after the success of the case of surfaces by the uniformization theorem. In 1905, Poincaré formulated his famous conjecture which states in the smooth case: A closed, simply-connected 3-manifold is diffeomorphic to S^3, the standard sphere. A great deal of work in three dimensional topology in the next 100 years was motivated by this. In the 1980s Thurston studied hyperbolic manifolds. This led him to his "Geometrization Conjecture" about the existence of homogeneous metrics on all 3-manifolds. It includes the Poincaré conjecture as a special case. In the case of 4-manifolds, there is at present no analogue of the geometrization conjecture. We discuss briefly the current state of these problems in the next two subsections.

The Ricci flow equations

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

for a Riemannian metric g were introduced by Hamilton in [Ham82]. They form a system of nonlinear second order partial differential equations. Hamilton proved that this equation has a unique solution for a short time for any smooth metric on a closed manifold. The evolution equation for the metric leads to the evolution equations for the curvature and Ricci tensors and for the scalar curvature. By developing a maximum principle for tensors, Hamilton proved that the Ricci flow preserves the positivity of the Ricci tensor in dimension three and that of the curvature operator in dimension four [Ham86]. In each of these cases he proved that the evolving metrics converge to metrics of constant positive curvature (modulo scaling). These and a series of further papers led him to conjecture that the Ricci flow with surgeries could be used to prove the Thurston geometrization conjecture. In a series of e-prints Perelman developed the essential framework for implementing the Hamilton program. We would like to add that the full Einstein equations with dilaton field as source play a fundamental role in Perelman's work (see, [Per02, Per03a, Per03b]) on the geometrization conjecture. A corollary of this work is the proof of the long standing Poincaré conjecture. Perelman was awarded the Fields medal at the ICM 2006 in Madrid for his proof of the Poincaré and the geometrization conjectures. His ideas and methods have already found many applications in analysis and geometry. On March 18, 2010 Perelman was awarded the first Clay Mathematics Institute's first millenium prize of one million dollars for his resolution of the Poincaré conjecture. A complete proof of the geometrization conjecture by applying the Hamilton-Perelman theory of the Ricci flow has appeared in [CZ06] in a special issue dedicated to the memory of S.-S. Chern,[3] one of the greatest mathematicians of the twentieth century.

The Ricci flow is perturbed by a scalar field which corresponds in string theory to the dilaton. It is supposed to determine the overall strength of all interactions. The low energy effective action of the dilaton field coupled to gravity is given by the action functional

$$\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f} dv.$$

Note that when f is the constant function the action reduces to the classical Hilbert-Einstein action. The first variation can be written as

$$\begin{aligned}\delta\mathcal{F}(g, f) = \int_M \Big[& -\delta g^{ij}(R_{ij} + \nabla_i f \nabla_j f) \\ & + \left(\frac{1}{2}\delta g^{ij}(g_{ij} - \delta f)(2\Delta f - |\nabla f|^2 + R)\right)\Big] dm,\end{aligned}$$

[3] I first met Prof. Chern and his then newly arrived student S.-T. Yau in 1973 at the AMS summer workshop on differential geometry held at Stanford University. Chern was a gourmet and his conference dinners were always memorable. I attended the first one in 1973 and the last one in 2002 on the occasion of the ICM satellite conference at his institute in Tianjin. In spite of his advanced age and poor health he participated in the entire program and then continued with his duties as President of the ICM in Beijing.

where $dm = e^{-f} dv$. If $m = \int_M e^{-f} dv$ is kept fixed, then the second term in the variation is zero and then the symmetric tensor $-(R_{ij} + \nabla_i f \nabla_j f)$ is the L^2 gradient flow of the action functional $\mathcal{F}^m = \int_M (R + |\nabla f|^2) dm$. The choice of m is similar to the choice of a gauge. All choices of m lead to the same flow, up to diffeomorphism, if the flow exists. We remark that in the quantum field theory of the two-dimensional nonlinear σ-model, Ricci flow can be considered as an approximation to the renormalization group flow. This suggests gradient-flow like behaviour for the Ricci flow, from the physical point of view. Perelman's calculations confirm this result. The functional $\mathcal{F}^m$ has also a geometric interpretation in terms of the classical Bochner-Lichnerowicz formulas with the metric measure replaced by the dilaton twisted measure dm.

The corresponding variational equations are

$$R_{ij} - \frac{1}{2} R g_{ij} = -\left(\nabla_i \nabla_j f - \frac{1}{2} (\Delta f) g_{ij} \right).$$

These are the usual Einstein equations with the energy-momentum tensor of the dilaton field as source. They lead to the decoupled evolution equations

$$(g_{ij})_t = -2(R_{ij} + \nabla_i \nabla_j f), \qquad f_t = -R - \Delta f.$$

After applying a suitable diffeomorphism these equations lead to the gradient flow equations. This modified Ricci flow can be pushed through the singularities by surgery and rescaling. A detailed case by case analysis is then used to prove Thurston's geometrization conjecture. This includes as a special case the classical Poincaré conjecture.

We have seen that QFT calculations have their counterparts in string theory. One can speculate that this is a topological quantum gravity (TQG) interpretation of a result in TQFT, in the Euclidean version of the theories. If modes of vibration of a string are identified with fundamental particles, then their interactions are already built into the theory. Consistency with known physical theories requires string theory to include supersymmetry. While supersymmetry has had great success in mathematical applications, its physical verification is not yet available. However, there are indications that it may be the theory that unifies fundamental forces in the standard model at energies close to those at currently existing and planned accelerators. Perturbative supersymmetric string theory (at least up to lower loop levels) avoids the ultraviolet divergences that appear in conventional attempts at quantizing gravity. Recent work relating the Hartle-Hawking wave function to string partition function can be used to obtain a wave function for the metric fluctuations on S^3 embedded in a Calabi-Yau manifold. This may be a first step in a realistic quantum cosmology relating the entropy of certain black holes with the topological string wave function. While a string theory model unifying all fundamental forces is not yet available, a number of small results (some of which we have discussed in this paper) are emerging to suggest that supersymmetric string theory could play a fundamental role in constructing such a model. Developing a theory and phenomenology of 4-dimensional string vacua and relating them to experimental physics and cosmological data would be a major step in this direction. New mathematical ideas may be needed for the completion of this project.

Acknowledgements This work was supported in part by a research fellowship of the Max Planck Gesellschaft at the Max Planck Institute for Mathematics in the Sciences, Leipzig. I would like to thank Prof. Jost, director of the MPI-MIS for his continued interest in my work. Thanks are also due to my colleague and friend Attila Mate for his help in matters academic and otherwise. I would like to thank the organizers Markus Banagl and Denis Vogel for their hospitality at the Heidelberg Knot theory conference where I met many old friends and made some new ones.

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Chapter 9
From Goeritz Matrices to Quasi-alternating Links

Józef H. Przytycki

Abstract Knot Theory is currently a very broad field. Even a long survey can only cover a narrow area. Here we concentrate on the path from Goeritz matrices to quasi-alternating links. On the way, we often stray from the main road and tell related stories, especially if they allow us to place the main topic in a historical context. For example, we mention that the Goeritz matrix was preceded by the Kirchhoff matrix of an electrical network. The network complexity extracted from the matrix corresponds to the determinant of a link. We assume basic knowledge of knot theory and graph theory, however, we offer a short introduction under the guise of a historical perspective.

9.1 Short Historical Introduction

Combinatorics, graph theory, and knot theory have their common roots in Gottfried Wilhelm Leibniz' (1646–1716) ideas of *Ars Combinatoria*, and *Geometria Situs*. In Ars Combinatoria [Lei66], Leibniz was influenced by Ramon Llull (1232–1315) and his combinatorial machines (Fig. 9.1; [Bon85, Llu05]).

Geometria (or Analysis) Situs seems to be an invention of Leibniz. I am not aware of any Ancient or Renaissance influence (compare however [Prz98]). The first convincing example of geometria situs was proposed by Heinrich Kuhn about

The author was partially supported by the NSA grant (# H98230-08-1-0033), by the Polish Scientific Grant: Nr. N N201387034, by the GWU REF grant, and by the CCAS/UFF award.

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_9,

Fig. 9.1 Combinatorial machine of Ramon Llull from his Ars Generalis Ultima

1735 in the form of the seven bridges of Königsberg puzzle. Kuhn (1690–1769) was a Danzig (Gdańsk) mathematician born in Königsberg. He studied at the Pedagogicum there, and in 1733 settled in Danzig as a mathematics professor at the Academic Gymnasium (he was also a co-founder of the Nature Society and the first person to suggest the geometric interpretation of complex numbers [Jan01, Kue56]). Kuhn communicated to Leonard Euler (1707–1783) the puzzle of the bridges of Königsberg, suggesting that this may be an example of geometria situs. Kuhn was communicating, in fact, through his friend Carl Leonhard Gottlieb Ehler (1685–1753), correspondent of Leibniz and future mayor of Danzig. The first extant[1] letter by Ehler concerning Königsberg bridges is dated March 9, 1736. There he writes: "You would render to me and our friend Köhn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of **Calculi Situs**, worthy of your great genius. I have added a sketch of the said bridges …" In the reply of April 3, 1736 Euler writes "…Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are

[1]Very likely the bridges of Könighsberg were mentioned in previous letters of Ehler to Euler, or possibly, they discussed them when Ehler was in Petersburg. Ehler met Euler in Petersburg in late 1734 or 1735 as a member of a delegation of Danzig to Empress of Russia, asking for a reduction of reparations forced on Danzig by Russia in 1734 after the capitulation of Danzig (the city was briefly occupied by the Russians after the prolonged Siege of Danzig during the War of the Polish Succession (city capitulated June 30, 1734). The city, which supported S. Leszczyński, the losing candidate for the Polish throne, was forced to pay reparations following the siege). The delegation left Petersburg June 3, 1735; [Cz06].

solved more quickly by mathematicians than by others. In the meantime, most noble Sir, you have assigned this question to the **geometry of position**, but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way ... " [HW04]. However, when composing his famous paper on the bridges of Königsberg, Euler already agrees with Kuhn's suggestion. The geometry of position figures even in the title of the paper *Solutio problematis ad geometriam situs pertinentis*.[2]

The first paper mentioning knots from the mathematical point of view is that of Alexandre-Theophile Vandermonde (1735–1796) *Remarques sur les problèmes de situation* [Van71]. Carl Friedrich Gauss (1777–1855) had an interest in Knot Theory his whole life, starting from the 1794 drawings of knots, the drawing of a braid with complex coordinates (c. 1820), several drawings of knots with "Gaussian codes", and Gauss' linking number of 1833. He did not publish anything however; this was left to his student Johann Benedict Listing (1808–1882) who in 1847 published his monograph (Vorstudien zur Topologie, [Lis47]). The monograph is mostly devoted to knots, graphs and combinatorics.

In the XIX century Knot Theory was an experimental science. Topology (or geometria situs) had not developed enough to offer tools allowing precise definitions and proofs[3] (here Gaussian linking number is an exception). Furthermore, in the second half of that century Knot Theory was developed mostly by physicists (William Thomson (Lord Kelvin) (1824–1907), James Clerk Maxwell (1831–1879), Peter Guthrie Tait (1831–1901)) and one can argue that a high level of precision was not appreciated.[4] I outline the global history of Knot Theory in [Prz98] and in the second chapter of my book on Knot Theory [Prz12]. In the next subsection we deal with the mathematics developed in order to understand precisely the phenomenon of knotting.

[2]In the paper, Euler writes: "The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the "geometry of position" (geometria situs). This branch of geometry deals with relations dependent on position; it does not take magnitudes into considerations, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position or of the method to be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position. 2. The problem, which I am told is widely known, is as follows: in Königsberg in Prussia, there is..." [Eul36, BLW86].

[3]Listing writes in [Lis47]: *In order to reach the level of exact science, topology will have to translate facts of spatial contemplation into easier notion which, using corresponding symbols analogous to mathematical ones, we will be able to do corresponding operations following some simple rules.*

[4]This may be a controversial statement. The precision of Maxwell was different than that of Tait and both were physicists.

9.1.1 Precision Comes to Knot Theory

Throughout the XIX century knots were understood as closed curves in space up to a natural deformation, which was described as a movement in space without cutting and pasting. This understanding allowed scientists (Tait, Thomas Penyngton Kirkman, Charles Newton Little, Mary Gertrude Haseman) to build tables of knots but did not lead to precise methods allowing one to distinguish knots which could not be practically deformed into another. In a letter to O. Veblen, written in 1919, young J. Alexander expressed his disappointment:[5]

> "When looking over Tait *On Knots* among other things, He really doesn't get very far. He merely writes down all the plane projections of knots with a limited number of crossings, tries out a few transformations that he happen to think of and assumes without proof that if he is unable to reduce one knot to another with a reasonable number of tries, the two are distinct. His invariant, the generalization of the Gaussian invariant . . . for links is an invariant merely of the particular projection of the knot that you are dealing with, - the very thing I kept running up against in trying to get an integral that would apply. The same is true of his 'Beknottednes'."

In the famous Mathematical Encyclopedia Max Dehn and Poul Heegaard outlined a systematic approach to topology, in particular they precisely formulated the subject of the Knot Theory [DH07], in 1907. To bypass the notion of deformation of a curve in a space (then not yet well defined) they introduced lattice knots and the precise definition of their (lattice) equivalence. Later Reidemeister and Alexander considered more general polygonal knots in a space with equivalent knots related by a sequence of Δ-moves; they also explained Δ-moves by the elementary moves on link diagrams—Reidemeister moves (see Sect. 9.1.6). The definition of Dehn and Heegaard was long ignored and only recently lattice knots are again studied (e.g. [BL]). It is a folklore result, probably never written down in detail,[6] that the two concepts, *lattice knots* and *polygonal knots*, are equivalent.

9.1.2 Lattice Knots of Dehn and Heegaard

In this part we discuss two early XX century definitions of knots and their equivalence, one by Dehn-Heegaard and one by Reidemeister. In the XIX century knots were treated from the intuitive point of view and it was P. Heegaard in his 1898 thesis who came close to a formal proof that there are nontrivial knots.

[5] We should remember that it was written by a young revolutionary mathematician forgetting that he is "standing on the shoulders of giants" [New76]. In fact the invariant Alexander outlined in the letter is closely related to Kirchhoff matrix, and extracted numerical invariant is equivalent to complexity of a signed graph corresponding to the link via Tait translation; see Sect. 9.1.4.

[6] It is however a long routine exercise.

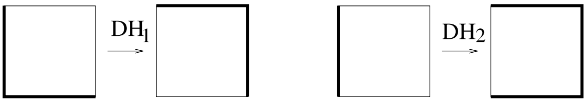


Fig. 9.2 Lattice moves DH_1 and DH_2

Dehn and Heegaard gave the following definition of a knot (or curve in their terminology) and of the equivalence of knots (which they call isotopy of curves).[7]

Definition 1 [DH07] A curve is a simple closed polygon on a cubical lattice. It has coordinates x_i, y_i, z_i. An *isotopy* of these curves is given by:

(i) Multiplication of every coordinate by a natural number.
(ii) Insertion of an elementary square, when it does not interfere with the rest of the polygon.
(iii) Deletion of an elementary square.

Elementary moves of Dehn and Heegaard can also be grouped into the following types, which are slightly different from (i)–(iii):

(DH_0) Rescaling. We show in [Prz12] that this move is a consequence of the other Dehn-Heegaard moves.
(DH_1) If a unit square intersects the lattice knot in exactly two neighboring edges then we replace these edges by two other edges of the square, as illustrated in Fig. 9.2 (DH_1).
(DH_2) If a unit square intersects the lattice knot in exactly one edge then we replace this edge by three other edges of the square, as illustrated in Fig. 9.2 (DH_2).

In this language, lattice knots (or links) and lattice isotopy are defined as follows.

Definition 2 A *lattice knot* is a simple closed polygon on a cubical lattice. Its vertices have integer coordinates x_i, y_i, z_i and edges, of length one, are parallel to one of the coordinate axis. We say that two lattice knots are *lattice isotopic* if they are related by a finite sequence of elementary lattice ("square") moves as illustrated in Fig. 9.2 (we allow the DH_1-move, the DH_2-move and its inverse the DH_2^{-1}-move). These are moves (ii) and (iii) of Dehn and Heegaard.

Below we give a few examples of lattice knots.

They can be easily coded as (cyclic) words over the alphabet $\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$. For example:

- the trivial knot can be represented by $xyx^{-1}y^{-1}$,
- the trefoil knot by $x^2z^3y^2x^{-1}z^{-2}y^{-3}zx^2y^2x^{-3}y^{-1}z^{-2}$,
- and the figure-eight knot by $y^2z^2xy^{-3}x^2y^2z^{-1}x^{-4}y^{-2}x^3yz^2x^{-2}z^{-3}$ see Fig. 9.3.

[7] Translation from German due to Chris Lamm.

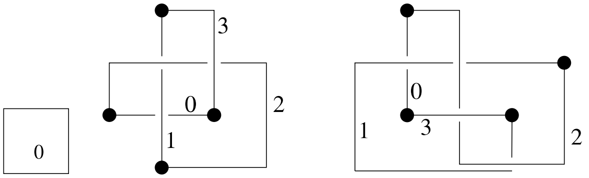


Fig. 9.3 A trivial lattice knot, with 4 edges, 4 right angles and no changes of planes. A lattice trefoil with 24 edges, 12 right angles and 8 changes of planes. A lattice figure-eight knot with 30 edges, 14 right angles and 8 changes of planes. The numbers are the z-levels and the dots are the sticks in the z-direction

9.1.3 Early Invariants of Links

The fundamental problem in knot theory is[8] to be able to distinguish non-equivalent knots. Even in the case of the unknot and the trefoil knot this was not achieved until the fundamental work of Jules Henri Poincaré (1854–1912) was used. In his seminal paper "Analysis Situs" ([Poi95] 1895) he laid the foundations for algebraic topology. According to W. Magnus [Mag78]:

> Today, it appears to be a hopeless task to assign priorities for the definition and the use of fundamental groups in the study of knots, particularly since Dehn had announced [Deh07] one of the important results of his 1910 paper (the construction of Poincaré spaces with the help of knots) already in 1907.

Wilhelm Wirtinger (1865–1945) in his lecture delivered at a meeting of the German Mathematical Society in 1905 outlined a method of finding a knot group presentation (it is called now the Wirtinger presentation of a knot group) [Wir05], but examples using his method were given after the work of Dehn.

9.1.4 Kirchhoff's Complexity of a Graph

In his fundamental paper on electrical circuits [Kir47], published in 1847, Gustav Robert Kirchhoff (1824–1887) defined the complexity of a circuit. In the language of graph theory, this complexity of a graph, $\tau(G)$, is the number of spanning trees of G, that is trees in G which contain all vertices of G. It was noted in [BSAT40] that if e is an edge of G that is not a loop then $\tau(G)$ satisfies the deleting-contracting relation:

$$\tau(G) = \tau(G - e) + \tau(G/e),$$

[8] One should rather say "was"; there are now algorithms allowing recognition of any knots, even if very slow. Modern Knot Theory looks rather for structures on a space of knots or for a mathematical or physical meanings of knot invariants.

where $G - e$ is the graph obtained from G by deleting the edge e, and G/e is obtained from G by contracting e, that is identifying endpoints of e in $G - e$. The deleting-contracting relation has an important analogue in knot theory, usually called a skein relation (e.g. Kauffman bracket skein relation). Connections were discovered only about a hundred years later (e.g. the Kirchhoff complexity of a circuit corresponds to the determinant of the knot or link yielded by the circuit, see the next subsection).

For completeness, and to be later able to see clearly the connection to Goeritz matrix in knot theory, let us define the (version of) the Kirchhoff matrix of a graph, G, the determinant of which is the complexity $\tau(G)$.

Definition 3 Consider a graph G with vertices $\{v_0, v_1, \ldots, v_n\}$ possibly with multiple edges and loops (however loops are ignored in the definitions which follow).

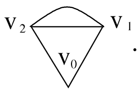
(1) The *adjacency matrix* of the graph G is the $(n+1) \times (n+1)$ matrix $A(G)$ whose entries, a_{ij} are equal to the number of edges connecting v_i with v_j; we set $v_{i,i} = 0$.
(2) The *degree matrix* $\Delta(G)$ is the diagonal $(n+1) \times (n+1)$ matrix whose ith entry is the degree of the vertex v_i (loops are ignored). Thus the ith entry is equal to $\sum_{j=0}^{n} a_{ij}$.
(3) The *Laplacian matrix* $Q'(G)$ is defined to be $\Delta(G) - A(G)$; [Big74]. Notice that the sum of rows of $Q'(G)$ is equal to zero and that $Q'(G)$ is a symmetric matrix.
(4) The *Kirchoff matrix* (or reduced Laplacian matrix) $Q(G)$ of G is obtained from $Q'(G)$ by deleting the first row and the first column from $Q'(G)$.

Theorem 1 *det*$(Q(G)) = \tau(G)$.

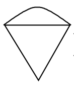
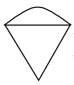
Proof The shortest proof I am aware of is by directly checking that *det*$(Q(G))$ satisfies the deleting-contracting relation for any edge e, not a loop, that is

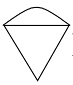
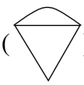
$$det(Q(G)) = det(Q(G-e)) + det(Q(G/e)).$$

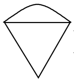
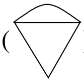
The above equation plays an important role in showing in Sect. 9.7 that an alternating link is quasi-alternating. □

Example 1 Consider the graph 

. For this graph we have:

$A($  $) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, $\quad \Delta($  $) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$Q'($  $) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix}$; $\quad Q($  $) = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$.

$det(Q($  $)) = det \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} = 5 = \tau($  $).$

As we will see in the next subsection the corresponding knot is the figure eight knot (Fig. 9.4).

9.1.5 Tait's Relation Between Knots and Graphs

Tait was the first to notice the relation between knots and planar graphs. He colored the regions of the knot diagram alternately white and black (following Listing) and constructed the graph by placing a vertex inside each white region, and then connecting vertices by edges going through the crossing points of the diagram (see Fig. 9.4) [DH07].

It is useful to mention the Tait construction going in the opposite direction, from a signed planar graph G to a link diagram $D(G)$. We replace every edge of a graph by a crossing according to the convention of Fig. 9.5 and connect endpoints along edges as in Figs. 9.6 and 9.7.

We should mention here one important observation known already to Tait (and in explicit form to Listing):

Proposition 1 *The diagram $D(G)$ of a connected graph G is alternating if and only if G is positive* (*i.e. all edges of G are positive*) *or G is negative.*

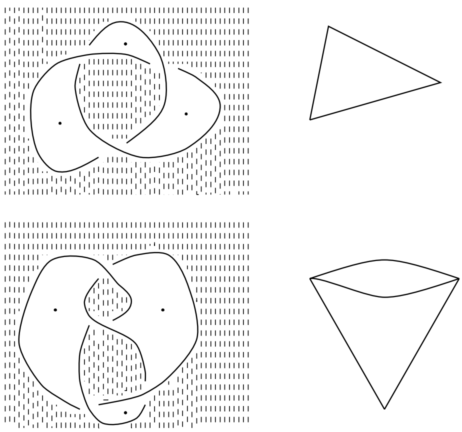


Fig. 9.4 Tait's construction of graphs from link diagrams, according to Dehn-Heegaard

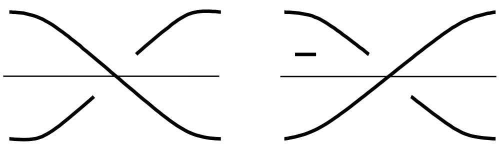


Fig. 9.5 Convention for crossings of signed edges (edges without markers are assumed to be positive)

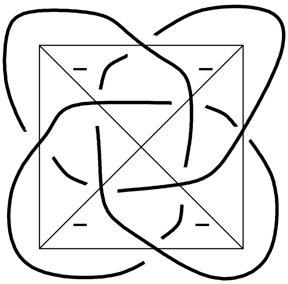


Fig. 9.6 The knot 8_{19} and its Tait graph (8_{19} is the first in tables non-alternating knot)

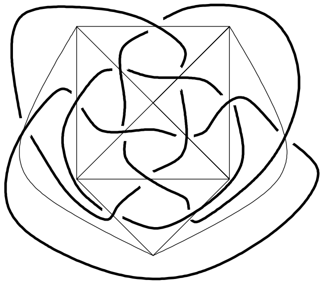


Fig. 9.7 Octahedral graph (with all positive edges) and the associated link diagram

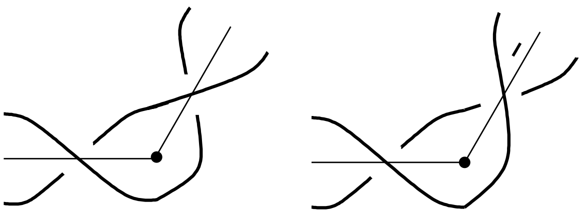


Fig. 9.8 Alternating and non-alternating parts of a diagram

A proof is illustrated in Fig. 9.8.

9.1.6 Link Diagrams and Reidemeister Moves

In this part we define, after Reidemeister, polygonal knots and links, and Δ-equivalence of knots and links. A Δ-move is an elementary deformation of a polygonal knot which intuitively agrees with the notion of "deforming without cutting and glueing," which is the first underlining principle of topology.

Definition 4 (Polygonal knot, Δ-equivalence)

(a) A *polygonal knot* is a simple closed polygonal curve in R^3.
(b) Let us assume that u is a line segment (edge) in a polygonal knot K in R^3. Let Δ be a triangle in R^3 whose boundary consists of three line segments $u,\ v,\ w$ and such that $\Delta \cap K = u$. The polygonal curve defined as $K' = (K - u) \cup v \cup w$

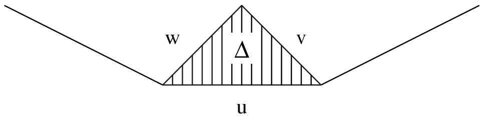


Fig. 9.9 Δ-move on a polygonal curve

is a new polygonal knot in R^3. We say that the knot K' was obtained from K by a Δ-*move*. Conversely, we say that K is obtained from K' by a Δ^{-1}-move (Fig. 9.9). We allow the triangle Δ to be degenerate so that the vertex $v \cap w$ is on the side u; in other words we allow a subdivision of the line segment u.[9]

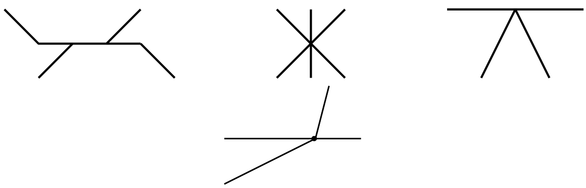
(c) We say that two polygonal knots are Δ-*equivalent* (or combinatorially equivalent) if one can be obtained from the other by a finite sequence of Δ- and Δ^{-1}-moves.

Polygonal links are usually presented by their projections to a plane. Let $p : R^3 \to R^2$ be a projection and let $L \subset R^3$ be a link. Then a point $P \in p(L)$ is called a multiple point (of p) if $p^{-1}(P)$ contains more than one point (the number of points in $p^{-1}(P)$ is called the multiplicity of P).

Definition 5 The projection p is called *regular* if

(1) p has only a finite number of multiple points and all of them are of multiplicity two,
(2) no vertex of the polygonal link is an inverse image of a multiple point of p.

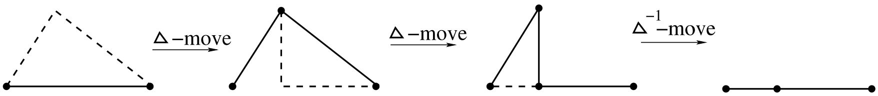
Thus for a regular projection the parts of a diagram illustrated in the figure below are not allowed.



Maxwell was the first person to consider the question of two projections representing equivalent knots. He considered some elementary moves (reminiscent of the future Reidemeister moves), but never published his findings.

The formal interpretation of Δ-equivalence of knots in terms of diagrams was done by Reidemeister [Rei27], 1927, and Alexander and Briggs [AB27], 1927.

[9] Notice that any subdivision is a combination of three non-degenerate Δ-moves, or more precisely two Δ-moves and the inverse to a Δ-move:



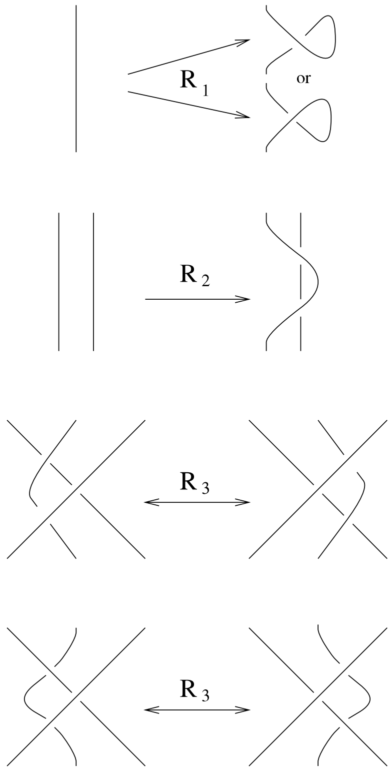


Fig. 9.10 Reidemeister moves; we draw two versions of the first and the third moves which are related by a mirror symmetry in the plane of the projection

Theorem 2 (Reidemeister theorem) *Two link diagrams are Δ-equivalent*[10] *if and only if they are connected by a finite sequence of Reidemeister moves $R_i^{\pm 1}, i = 1, 2, 3$ (see Fig. 9.10) and isotopy of the diagram inside the plane. The theorem holds also for oriented links and diagrams. One then has to take into account all possible coherent orientations of the diagrams involved in the moves.*

9.2 Goeritz Matrix and Signature of a Link

In the first half of the XX-century combinatorial methods ruled over knot theory, even though more topological approaches were available. For example, Reidemeister moves were used to prove the existence of the Alexander polynomial even though purely topological prove using the fundamental group was possible and probably well understood by Alexander himself. Later, after the Second World War, to a

[10]In modern Knot Theory, especially after the work of R. Fox, we use usually the equivalent notion of ambient isotopy in R^3 or S^3. Two links in a 3-manifold M are ambient isotopic if there is an isotopy of M sending one link into another.

great extent under the influence of Ralph Hartzler Fox (1913–1973), Knot Theory was considered to be a part of algebraic topology with the fundamental group and coverings playing an important role. The renaissance of combinatorial methods in Knot Theory can be traced back to Conway's paper [Con69] and bloomed after Jones' breakthrough [Jon85] with Conway type invariants and the Kauffman approach (compare Chap. III of [Prz12]). As we already mentioned, these had their predecessors in the 1930s [Goe33, Sei34]. The Goeritz matrix of a link can be defined purely combinatorially and is closely related to the Kirchhoff matrix of an electrical network. The Seifert matrix is a generalization of the Goeritz matrix and, even historically, its development was a mix of combinatorial and topological methods.

In this section we start from the work of L. Goeritz. He showed [Goe33] how to associate a quadratic form to a diagram of a knot and moreover how to use this form to get algebraic invariants of the knot. (The signature of this form, however, is not an invariant of the knot.) Later, H.F. Trotter [Tro62], using the Seifert form (see Sect. 9.3), introduced another quadratic form, the signature of which was an invariant of links.

C.McA. Gordon and R.A. Litherland [GL78] provided a unified approach to Goeritz and Trotter forms. They showed how to use the form of Goeritz to get (after adding a correcting factor) the signature of a link (this signature is often called the classical or Trotter, or Murasugi [Mur65] signature of a link).

We begin with a purely combinatorial description of the matrix of Goeritz and of the signature of a link. This description is based on [GL78] and [Tra85].

Definition 6 Let L be a diagram of a link. Let us checkerboard color the complement of the diagram in the projection plane R^2, that is, color in black and white the regions into which the plane is divided by the diagram.[11] We assume that the unbounded region of $R^2 \setminus L$ is colored white and it is denoted by X_0 while the other white regions are denoted by $X_1, \dots, X_n$. Now, to any crossing, p, of L we associate the number $\eta(p)$ which is either $+1$ or -1 according to the convention described in Fig. 9.11.

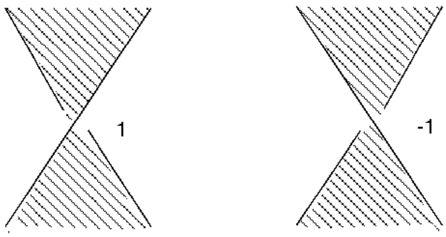


Fig. 9.11 The convention for a sign of a colored crossing

[11]This (checkerboard) coloring was first used by P.G. Tait in 1876/7, compare Chap. II of [Prz12]. However, following C. Gordon, we switched the role of white and black. We can say that Tait convention worked well with a blackboard, while our convention works better with a white-board.

Let

$$G' = \{g_{i,j}\}_{i,j=0}^{n}, \quad \text{where}$$

$$g_{i,j} = \begin{cases} -\sum_p \eta(p) & \text{for } i \neq j, \text{ where the summation extends} \\ & \text{over crossings which connect } X_i \text{ and } X_j \\ -\sum_{k=0,1,\ldots,n;k\neq i} g_{i,k} & \text{if } i = j \end{cases}$$

The matrix $G' = G'(L)$ is called the *unreduced Goeritz matrix* of the diagram L. The *reduced Goeritz matrix* (or shortly Goeritz matrix) associated to the diagram L is the matrix $G = G(L)$ obtained by removing the first row and the first column of G'.

Theorem 3 [Goe33, KP53, Kyl54] *Let us assume that L_1 and L_2 are two diagrams of a given link. Then the matrices $G(L_1)$ and $G(L_2)$ can be obtained from each other by a finite number of the following elementary operations on matrices*:

1. $G \leftrightarrow PGP^T$, *where P is a matrix with integer entries and* $\det P = \pm 1$.
2. $$G \leftrightarrow \begin{bmatrix} G & 0 \\ 0 & \pm 1 \end{bmatrix}$$
3. $$G \leftrightarrow \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}$$

Moreover, if L is a diagram of a knot,[12] *then operations* (1) *and* (2) *are sufficient.*

Corollary 1 $|\det G|$ *is an invariant of isotopy of knots called the determinant of a knot.*[13]

A sketch of a proof of Theorem 3.

We have to examine how a Goeritz matrix changes under Reidemeister moves. The matrix does not depend on the orientation of the link, let us assume, however, that the diagram L is oriented. We introduce new notation: a crossing is called of type I or II according to Fig. 9.12. Moreover, we define $\mu(L) = \sum \eta(p)$, where the summation is taken over crossings of type II.

Now let us construct a graph with vertices representing black regions (this is Tait's construction, however, the choice of black and white regions is reversed) and edges in bijection with crossings of L. Edges of the graph are in bijection with

[12] It suffices to assume that L represent a non-split link, that is a link all projections of which are connected.

[13] Often, by the determinant of a knot one understands the more delicate invariant whose absolute value is equal to $|\det G|$; see Corollary 3. This determinant can also be defined as the Alexander-Conway or Jones polynomial at $t = -1$; compare Remark 2 and Corollary 18.

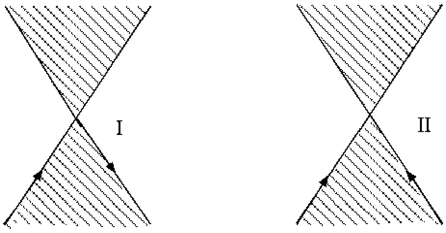


Fig. 9.12 Two types of colored oriented crossings

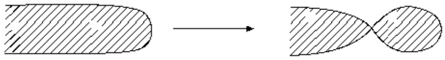


Fig. 9.13 The first coloring of R_1

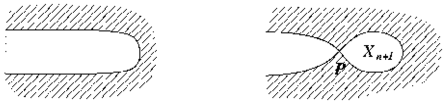


Fig. 9.14 The second coloring of R_1

crossings of L: two vertices of the graph are joined if and only if the respective regions meet in a crossing.[14] Let $B(L)$ denote the number of components of such a graph. From now on, let R be a Reidemeister move. We denote by G_1 the Goeritz matrix of L, and by G_2 the matrix of $R(L)$. Similarly we set $\mu_1 = \mu(L)$, $\mu_2 = \mu(R(L))$ and also $\beta_1 = B(L)$, $\beta_2 = B(R(L))$. We will write $G_1 \approx G_2$ if G_1 and G_2 are in relation (1) and $G_1 \sim G_2$ if G_2 can be obtained from G_1 by a sequence of relations (1)–(3).

1. Let us consider the first Reidemeister move R_1.
   a. In the case shown in Fig. 9.13 we have: $\beta_1 = \beta_2$, $\mu_1 = \mu_2$, and $G_1 \approx G_2$.
   b. In the case shown in Fig. 9.14 we have:

$$\beta_1 = \beta_2, \qquad \mu_2 = \mu_1 + \eta(p), \qquad G_2 = \begin{bmatrix} G_1 & 0 \\ 0 & \eta(p) \end{bmatrix}$$

2. Let us consider the second Reidemeister move R_2.
   a. In the case described in Fig. 9.15 we get immediately that $\beta_1 = \beta_2$ and $\mu_1 = \mu_2$ (either both crossings are of type I or of type II and always of opposite signs), $G_1 \approx G_2$.
   b. In the case described in Fig. 9.16 we have to consider two subcases. In each of them $\mu_1 = \mu_2$, since the two new crossings are either both of type I or both of type II and always of opposite signs:

[14] This construction of Tait is an important motivation for material in Chap. V of [Prz12]. The constructed graph, which we denote by $G_b(L)$, is usually called the Tait graph of L (see the first section). For an alternating diagram L this graph is the same as the graph $G_{s_+}(L)$ considered in Chap. V of [Prz12]. We often equip the edges of $G_b(L)$ with signs: the edge corresponding to a vertex p has the sign $\eta(p)$ (see Fig. 9.11). The signed graph $G_b(L)$ is considered in Chap. V of [Prz12]; compare also Definition 20.

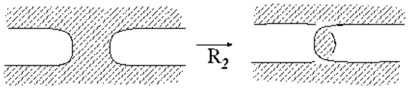


Fig. 9.15 The first coloring of R_2

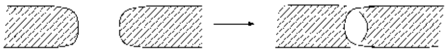


Fig. 9.16 The first coloring of R_2

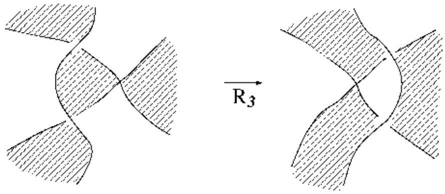


Fig. 9.17 Coloring of R_3 (Ω_2 in Fig. 9.15 should be changed to R_2

(i) $\beta_1 = \beta_2$. Then

$$G_2 \approx \begin{bmatrix} G_1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} G_2 & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} G_1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$$

We leave it for the reader to check the details, c.f. [KP53].
Both possibilities give $G_1 \sim G_2$.

(ii) $\beta_2 = \beta_1 - 1$. Then we see immediately that

$$G_2 \approx \begin{bmatrix} G_1 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}.$$

3. Let us consider the Reidemeister move R_3 (Fig. 9.17).
   We see immediately that $\beta_1 = \beta_2$. Next we should consider different orientations of arcs participating in R_3 and two possibilities for the crossing p. However, we will get always $\mu_2 = \mu_1 + \eta(p)$ and

$$G_2 \approx \begin{bmatrix} G_1 & 0 \\ 0 & \eta(p) \end{bmatrix}.$$

   We leave it for the reader to check (c.f. [Goe33] and [Rei32]).

   This concludes the proof of Theorem 3.

Corollary 2

(1) *For a link L let us define $\sigma(L) = \sigma(G(L)) - \mu(L)$, where $\sigma(G(L))$ is the signature of the Goeritz matrix of L. Then $\sigma(L)$ is an invariant of the link L, called the signature of the link*; *compare Corollary* 3 *and Definition* 17.

(2) *Let us define* $\mathrm{nul}(L) = \mathrm{nul}(G(L)) + \beta(L) - 1$, *where* $\mathrm{nul}(G(L))$ *is the nullity* (*i.e. the difference between the dimension and the rank*) *of the matrix* $G(L)$.

Then nul(L) *is an invariant of the link L and we call it the nullity* (*or defect*) *of the link*.

Proof It is enough to apply Theorem 3 to see that $\sigma(L)$ and nul(L) are invariant with respect to Reidemeister moves. □

L. Traldi [Tra85] introduced a modified matrix of an oriented link, the signature and the nullity of which are invariants of the link.

Definition 7 Let L be a diagram of an oriented link. Then we define the *generalized Goeritz matrix*

$$H(L) = \begin{bmatrix} G & & \bigcirc \\ & A & \\ \bigcirc & & B \end{bmatrix},$$

where G is a Goeritz matrix of L, and the matrices A and B are defined as follows. The matrix A is diagonal of dimension equal to the number of type II crossings and the diagonal entries equal to $-\eta(p)$, where p's are crossings of type II. The matrix B is of dimension $\beta(L) - 1$ with all entries equal to 0.

Lemma 1 [Tra85] *If L_1 and L_2 are diagrams of two isotopic oriented links then $H(L_1)$ can be obtained from $H(L_2)$ by a sequence of the following elementary equivalence operations*:

1. $H \Leftrightarrow PHP^T$, *where P is a matrix with integer entries and with det $P = \pm 1$*,
2.

$$H \Leftrightarrow \begin{bmatrix} H & & \bigcirc \\ & 1 & \\ \bigcirc & & -1 \end{bmatrix}.$$

Proof Lemma 1 follows immediately from the proof of Theorem 3. □

Corollary 3 *The determinant det*$(iH(L))$ $(i = \sqrt{-1})$ *is an isotopy invariant of a link L, called the determinant of the link, Det_L. Moreover, $\sigma(H(L)) = \sigma(L)$ and* nul$(H(L)) = $ nul(L).

The proof follows immediately from Lemma 1 and from the proof of Theorem 3.

Example 2 Consider a torus link of type $(2, k)$, we denote it by $T_{2,k}$. It is a knot for odd k and a link of two components for k even; see Fig. 9.18.

The matrix G' of $T_{2,k}$ is then equal to $\left[\begin{smallmatrix} k & -k \\ -k & k \end{smallmatrix}\right]$, and thus Goeritz matrix of the link is $G = [k]$. Moreover, $\beta = 1$ and $\mu = k$ because all crossings are of type II. Therefore, for $k \neq 0$, $\sigma(T_{2,k}) = \sigma(G) - \mu = 1 - k$ and nul$(T_{2,k}) = $ nul$(G) = 0$. The generalized Goeritz matrix H of the knot $T_{2,k}$ is of dimension $k + 1$ and it is equal

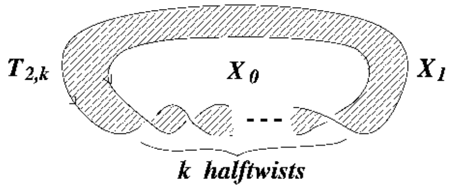


Fig. 9.18 Coloring a torus link of type $T_{2,k}$

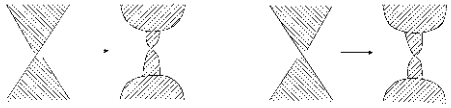


Fig. 9.19 Connecting black regions by bands

to

$$H = \begin{bmatrix} k & & & & \bigcirc \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ \bigcirc & & & & -1 \end{bmatrix}$$

Therefore $Det_L = det(iH) = (-1)^k i^{k+1} k = i^{1-k} k$. Notice also that $i^{\sigma(T_{2,k})} = \frac{Det_{T_{2,k}}}{|Det_{T_{2,k}}|}$; compare Exercise 2.

Let us note that if we connect black regions of the plane divided by the diagram of the link by half-twisted bands (as indicated in Fig. 9.19) then we get a surface in R^3 (and in S^3), the boundary of which is the given link; we denote this surface by F_b, and call it the *Tait surface* of a link diagram; compare Definition 20. If, for some checkerboard coloring of the plane, the constructed surface has an orientation which yields the given orientation of the link then this oriented diagram is called a special diagram.

Exercise 1 Prove that an oriented diagram of a link is special if and only if all crossings are of type I for some checkerboard coloring of the plane. Conclude from this that for a special diagram D, we have $\sigma(D) = \sigma(G(D))$.

Exercise 2 Show that any oriented link has a special diagram. Conclude from this that for any oriented link L one has

$$Det_L = i^{\sigma(L)} |Det_L|$$

(compare Lemma 10 and Corollary 18).

Assume now that L_0 is a sublink of an oriented link L. Let L' be an oriented link obtained from L by changing the orientation of L_0 to the opposite orientation. Let D_L be a diagram of L and the linking number $lk(L - L_0, L_0)$ be defined as $\sum_p \operatorname{sgn} p$ where the sum is taken over all crossings of the diagram of $L - L_0$ and L_0 (as subdiagrams of L_D). This definition does not depend on the choice of D_L, as

Fig. 9.20 L_n obtained from $L = L_0$ by an n-move, and L_∞

checked using Reidemeister moves, and agrees with the standard notion of linking number which is recalled in the next section.

From Corollary 2 and Corollary 3, we obtain.

Proposition 2 [Mur70]

(i) $Det_{L'} = (-1)^{lk(L-L_0,L_0)}\, Det_L$.
(ii) $\sigma(L') = \sigma(L) + 2lk(L - L_0, L_0)$.
(ii) $\sigma(L) + lk(L)$ *is independent on orientation of* L.

Proof The derivation of formulas is immediate but it is still instructive to see how Corollary 2(ii) follows from Corollary 2(i):

$$\begin{aligned}\sigma(L') &= \sigma(G(L')) - \mu(L') = \sigma(G(L)) - \mu(L') = \sigma(L) + \mu(L) - \mu(L') \\ &= \sigma(L) + 2lk(L - L_0, L_0).\end{aligned}$$

□

Recall [Prz88] that an *n-move* is a local change of an unoriented link diagram, as described in Fig. 9.20.

When computing and comparing Goeritz matrices of $L = L_0$, L_n and L_∞ we can assume that black regions are chosen as in Fig. 9.20 and that the white region X in $R^2 - L_\infty$ is divided into two regions X_0 and X_1 in $R^2 - L$.

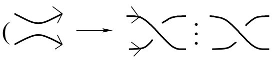
Lemma 2 $G(L_n) = \begin{bmatrix} G(L_\infty) & \alpha \\ \alpha^T & q+n \end{bmatrix}$,

Corollary 4

(i) $Det\, G(L_n) - Det\, G(L_0) = n\, Det\, G(L_\infty)$.
(ii) $\sigma(G(L_0)) \le \sigma(G(L_n)) \le \sigma(G(L_0)) + 2, n \ge 0$.
(iii) $|\sigma(G(L_n)) - \sigma(G(L_\infty))| \le 1$. *Furthermore*, $\sigma(G(L_n)) = \sigma(G(L_\infty))$ *if and only if* $rank\, G(L_n) = rank\, G(L_\infty)$ *or* $rank\, G(L_n) = rank\, G(L_\infty) + 2$.

If we orient $L = L_0$ we can use Corollary 2.13(ii) to obtain very useful properties of the signature of L and L_n.

Corollary 5 [Prz88]

(i) *Assume that* L_0 *is oriented in such a way that its strings are parallel.* L_n *is said to be obtained from* L_0 *by a* t_n-move (); *then*

$$n - 2 \le \sigma(L_0) - \sigma(L_n) \le n$$

(ii) *Assume that L_0 is oriented in such a way that its strings are anti-parallel and that $n = 2k$ is an even number. L_{2k} is said to be obtained from L_0 by a $\bar{t}_{2k}$*-move (); *then*

$$0 \le \sigma(L_{2k}) - \sigma(L_0) \le 2.$$

(iii) (*Giller* [*Gil*82])

$$0 \le \sigma(L_{\times}) - \sigma(L_{\times}) \le 2$$

Proof

(i) All new crossings of L_n are of type II (we use shading of Fig. 9.20), thus $\mu(L_n) - \mu(L_0) = n$. Therefore by Corollary 4(ii) we have $n - 2 \le \sigma(G_{L_0}) - \mu(L_0) - (\sigma(G_{L_n}) - \mu(L_n) \le n$, and Corollary 5(i) follows by Corollary 2.
(ii) In this case $\mu(L_{2k}) = \mu(L_0)$ thus (ii) follows from Corollary 4(ii). The generalization of Corollary 5(ii) to Tristram-Levine signatures is given in Corollary 14(ii).
(iii) Follows from (i), or (ii) for $n = 2$.

□

We finish the section with an example of computing a close form for the determinant of the family of links called Turk-head links. We define the nth Turk-head link, Th_n as the closure of the 3-braid $(\sigma_1\sigma_2^{-1})^n$ (see Fig. 9.21 for Th_6).[15]

Example 3 We compute that

$$Det_{Th_n} = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2,$$

or it can be written as $Det_{Th_n} = T_n(3) - 2$, where $T_i(z)$ is the Chebyshev (Tchebycheff) polynomial of the first kind:[16]

$$T_0 = 2, \qquad T_1 = z, \qquad T_i = zT_{i-1} - T_{i-2}.$$

In particular, $Det_{Th_2} = 5$, $Det_{Th_3} = 16$, $Det_{Th_4} = 45$, $Det_{Th_5} = 121$, $Det_{Th_6} = 320$, $Det_{Th_7} = 841$, and $Det_{Th_8} = 2205$; compare [Sed70, Mye71].

---

[15] Th_0 is the trivial link of 3 components, Th_1 the trivial knot, Th_2 the figure eight knot (4_1), Th_3 the Borromean rings (6^3_2), Th_4, the knot 8_{18}, Th_5 the knot 10_{123}, Th_6 the link 12^3_{474} (that is 474th link of 12 crossings and 3 components in unpublished M. Thistlethwaite tables; compare [Thi85]), and Th_7 and Th_8 are the knots 14_{a19470} and $16_{a275159}$, respectively, in Thistlethwaite (Knotscape) list.

[16] $T_n(3)$ is often named the Lucas number; more precisely $T_n(3) = \Lambda_{2n}$, where $\Lambda_0 = 2$, $\Lambda_1 = 1$ and $\Lambda_n = \Lambda_{n-1} + \Lambda_{n-2}$ as $\Lambda_n = 3\Lambda_{n-2} - \Lambda_{n-4}$.

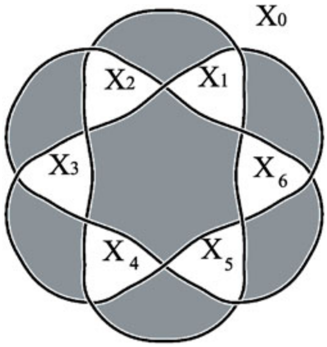


Fig. 9.21 The Turk-head link Th_6 and its checkerboard coloring

To show the above formulas, consider the (unreduced) Goeritz matrix related to the checkerboard coloring of the diagram of Th_n as shown in Fig. 9.21 (we have here $z = 3$ and we draw the case of $n = 6$).

$$G'(Th_6) = \begin{bmatrix} -n & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -z & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & -z & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -z & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -z & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -z & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & -z \end{bmatrix},$$

By crossing the first row and column of $G'(Th_n)$ we obtain the Goeritz matrix of Th_n which is also the circulant matrix with the first row $(-z, 1, 0, \ldots, 0, 1)$ ($z = 3$ and $n = 6$ in our concrete case):

$$G(Th_6) = \begin{bmatrix} -z & 1 & 0 & 0 & 0 & 1 \\ 1 & -z & 1 & 0 & 0 & 0 \\ 0 & 1 & -z & 1 & 0 & 0 \\ 0 & 0 & 1 & -z & 1 & 0 \\ 0 & 0 & 0 & 1 & -z & 1 \\ 1 & 0 & 0 & 0 & 1 & -z \end{bmatrix}.$$

To compute the determinant of the circulant matrix $CM_n(z)$ of the size $n \times n$ and the first row $(-z, 1, 0, \ldots, 0, 1)$ we treat each row as a relation and find the structure of the $Z[z]$ module generated by columns (indexed by $(e_0, e_1, \ldots, e_n)$). Thus we have n relations of the form $e_k = ze_{k-1} - e_{k-2}$, where k is taken modulo n. The relation recalls the relation of Chebyshev polynomials, and in fact we easily check that $e_k = S_{k-1}(z)e_1 - S_{k-2}(z)e_0$, where $S_k(z)$ is the Chebyshev polynomial of the second kind:

$$S_0 = 1, \qquad S_1 = z, \qquad S_i = zS_{i-1} - S_{i-2}.$$

Thus we can eliminate all vectors (columns) e_k except e_0 and e_1, and we are left with two equations $e_0 = e_n = S_{n-1}e_1 - S_{n-2}e_0$, and $e_1 = e_{n+1} = S_ne_1 - S_{n-1}e_0$.

Thus, our module can be represented by the 2×2 matrix

$$\begin{bmatrix} S_{n-1} & 1 - S_n \\ S_{n-2} + 1 & -S_{n-1} \end{bmatrix}.$$

We conclude that, up to a sign, $det\, CM_n(z)$ is equal to the determinant of our 2×2 matrix, that is $S_n - S_{n-2} - 1 - S_{n-1}^2 + S_n S_{n-2}$. To simplify this expression let us use the substitution $z = a + a^{-1}$. Then $S_n(z) = a^n + a^{n-2} + \cdots + a^{2-n} + a^{-n} = \frac{a^{n+1} - a^{-n-1}}{a - a^{-1}}$, and $T_n(z) = a^n + a^{-n}$. Therefore,

$$\begin{aligned}
&S_n - S_{n-2} - 1 - S_{n-1}^2 + S_n S_{n-2} \\
&\quad = S_n - S_{n-2} - 1 - \left(\left(\frac{a^n - a^{-n}}{a - a^{-1}} \right)^2 - \left(\frac{a^{n+1} - a^{-n-1}}{a - a^{-1}} \right) \left(\frac{a^{n-1} - a^{-n+1}}{a - a^{-1}} \right) \right) \\
&\quad = S_n - S_{n-2} - 1 - \left(\frac{(a^n - a^{-n})^2 - (a^{n+1} - a^{-n-1})(a^{n-1} - a^{-n+1})}{(a - a^{-1})^2} \right) \\
&\quad = S_n - S_{n-2} - 2 = a^n + a^{-n} - 2 = T_n(z) - 2.
\end{aligned}$$

By comparing the maximal power of z in $det\, CM_n(z)$ and $T_2(z) - 2$, we get that $det\, CM_n(z) = (-1)^n (T_n(z) - 2)$. For $z = 3$ we have $a + a^{-1} = 3$, thus $a = \frac{3 \pm \sqrt{5}}{2}$ so we can choose $a = \frac{3+\sqrt{5}}{2}$ and $a^{-1} = \frac{3-\sqrt{5}}{2}$, and thus $T_n(3) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n$.

Because, Th_n is an amphicheiral link, its signature is equal to 0 and

$$Det_{Th_n} = i^{\sigma(Th_n)} |\, det\, CM_n(3)| = T_n(3) - 2 = \left(\frac{3 + \sqrt{5}}{2} \right)^n + \left(\frac{3 - \sqrt{5}}{2} \right)^n - 2.$$

We computed the determinant of the circulant[17] matrix CM_n for a general variable z and until now used it only for $z = 3$. We see in the next exercise that the matrix has a knot theory interpretation for any rational number z.

Exercise 3 Consider the "braid like" closure of the tangle $(\sigma_2^{-\frac{1}{a}} \sigma_1^b)^n$ for any integers a and b, (see Fig. 9.22 for $(\sigma_2^{-\frac{1}{3}} \sigma_1^3)^4$). Show that the determinant of the link

[17] Recall that the *circulant* $n \times n$ matrix satisfies $a_{i,j} = a_{i-1,j-1} = \ldots a_{1,j-i+1}$, $0 \leq i, j \leq n-1$. Such a matrix has (over C) n different eigenvectors: $(1, \omega, \omega^2, \ldots, \omega^{n-1})$, where ω is any nth root of unity ($\omega^n = 1$). The corresponding eigenvalues are $\lambda_\omega = \sum_{i=0}^{n-1} \omega^i a_{1,i}$. Thus Example 3 leads to a curious identity $\Pi_{i=0}^{n-1} (\omega^i + \omega^{-i} - z) = det\, CM_n = (-1)^n (T_n(z) - 2)$ for any primitive nth root of unity ω.

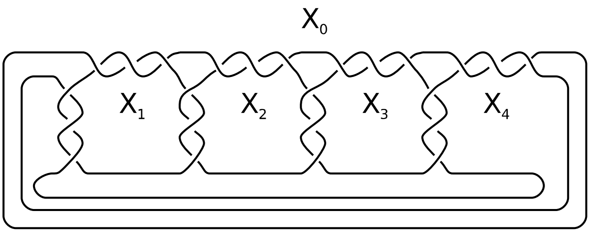


Fig. 9.22 The closure of the tangle $(\sigma_2^{-\frac{1}{3}}\sigma_1^3)^4$

satisfies the formula[18]

$$|Det_{(\sigma_2^{-\frac{1}{a}}\sigma_1^b)^n}| = \left|b^n \, det\, CM_n\left(2+\frac{a}{b}\right)\right| = \left|b^n\left(T_n\left(2+\frac{a}{b}\right)-2\right)\right|.$$

9.3 Seifert Surfaces

It was first demonstrated by P. Frankl and L. Pontrjagin in 1930 [FP30] that any knot bounds an oriented surface.[19] H. Seifert found a very simple construction of such a surface [Sei34] and developed several applications of the surface, named now *Seifert surface* (also, infrequently, Frankl-Pontrjagin surface).[20]

Definition 8 A *Seifert surface* of a link $L \subset S^3$ is a compact, connected, orientable 2-manifold $S \subset S^3$ such that $\partial S = L$.

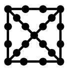
If the link L is oriented then its Seifert surface S is assumed to be oriented so that the induced orientation on the boundary agrees with that of L.

For example: a Seifert surface of a trefoil knot is pictured in Fig. 9.23.

Definition 9 The *genus of a link* $L \subset S^3$ is the minimal genus of a Seifert surface of L.

The genus is an invariant (of ambient isotopy classes) of knots and links. The following theorem provides that it is well defined.

---

[18] It is also the formula for the number of spanning trees of the generalized wheel, $W_{a,b,n}$, which is the Tait graph of the closure of $(\sigma_2^{-\frac{1}{a}}\sigma_1^b)^n$ ($W_{3,3,4} =$  ; compare Chap. V of [Prz12]).

[19] According to [FP30]: "The Theorem… [was] found by both authors independently from each other. In what follows, the Frankl's form of the proof is presented." One should add that Seifert refers in [Sei34] to the Frankl-Pontrjagin paper and says that they use a different method.

[20] Kauffman in [Kau87a, Kau83] uses the term *Seifert surface* to describes the surface obtained from an oriented link diagram by the Seifert algorithm (Construction 5), and the term spanning surface for an oriented surface bounding a link (our Seifert surface of Definition 8). In [BE82] the name *Frankl surface* is used for any oriented or unoriented spanning surface.

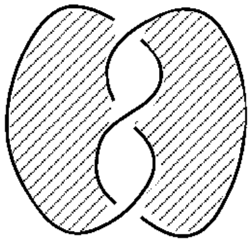


Fig. 9.23 A Seifert surface of a trefoil knot

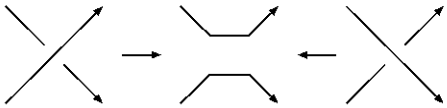


Fig. 9.24 Smoothing of a crossing

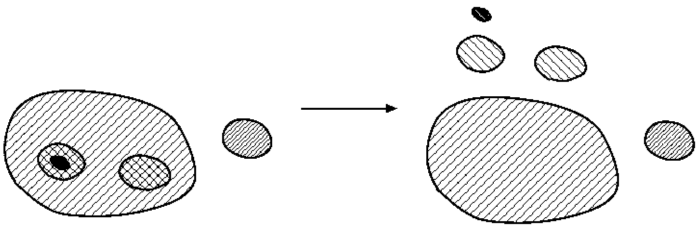


Fig. 9.25 Arranging nested disks in Seifert construction

Theorem 4 (Frankl-Pontrjagin-Seifert) *Every link in S^3 bounds a Seifert surface. If, moreover, the link is oriented then there exists a Seifert surface, an orientation of which determines the orientation of its boundary coinciding with that of L.*

Construction 5 (Seifert) Consider a fixed diagram D of an oriented link L in S^3. In the diagram there are two types of crossings, in a neighborhood of each of the crossings we make a modification of the link (called smoothing) according to Fig. 9.24.

After smoothing all crossings of D we obtain a family of disjoint oriented simple closed curves in the plane, called *Seifert circles* by R. Fox, and denoted by $D_{\vec{s}}$. Each of the curves of $D_{\vec{s}}$ bounds a disk in the plane; the disks do not have to be disjoint (they can be nested). Now we make the disks disjoint by pushing them slightly up above the projection plane. We start with the innermost disks (that is disks without any other disks inside) and proceed outwards (i.e. if $D' \subset D$ then D' is pushed above D); see Fig. 9.25.

The disks are two-sided so we can assign the signs $+$ and $-$ to each of the sides of a disk according to the following convention: the sign of the "upper" side of the disk is $+$ (respectively, $-$) if its boundary is oriented counterclockwise (respectively, clockwise), see Fig. 9.26.

Now we connect the disks together at the original crossings of the diagram D by half-twisted bands so that the 2-manifold which we obtain has L as its boundary, see Fig. 9.27.

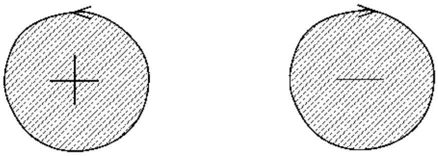


Fig. 9.26 Signs of disks in Seifert construction

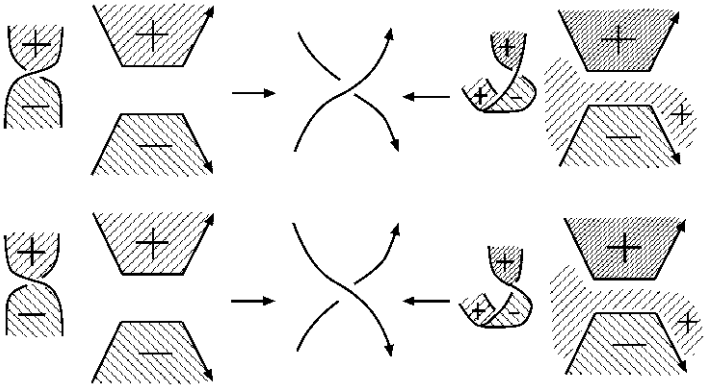


Fig. 9.27 Seifert surface around a crossing

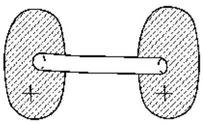


Fig. 9.28 Connecting components of a surface by a tube

Since the "+ side" is connected to another "+ side" it follows that the resulting surface is orientable. Moreover, this surface is connected if the projection of the link is connected (for example if L is a knot). If the surface is not connected then we join its components by tubes (see Fig. 9.28) in such a way that the orientation of components is preserved.

Remark 1 If the link L has more than one component then the Seifert surface which we constructed above depends on the orientation of components of L. This can be seen on the example of a torus link of type $(2, 4)$, see Fig. 9.29.

The Seifert surface from Fig. 9.29(a) has genus 1 while the surface from Fig. 9.29(b) has genus 0. Therefore the link L has genus 0 (as an unoriented link).

Corollary 6 *If a projection of a link L is connected (e.g. if L is a knot) then the surface, from the Seifert Construction 5, is unknotted, that is, its complement in S^3 is a handlebody. The genus of the handlebody is equal to $c + 1 - s$ and the Euler characteristic is equal to $s - c$, where c denotes the number of crossings of the projection and s the number of Seifert circles.*

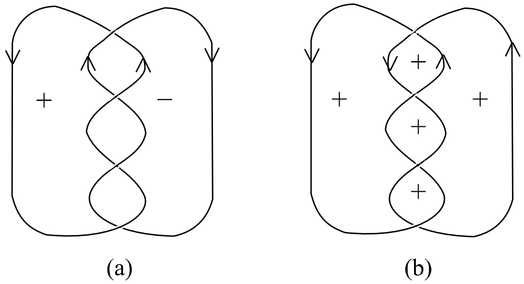


Fig. 9.29 Different orientations result in different Seifert surfaces

Proof The complement in S^3 of the plane projection of L is a 3-disk with $c+1$ handles (the projection of L cuts the projection plane (or 2-sphere) into $c+2$ regions). Furthermore adding s 2-disks in the construction of the Seifert surface we cut s of the handles thus the result remains a 3-disk with $c+1-s$ handles. The Euler characteristic of the obtained handlebody is equal to $1-(c+1-s)=s-c$. □

Corollary 7 *A knot K in S^3 is trivial if and only if its genus is equal to* 0.

Exercise 4 Let L be a link with n components and D_L its diagram. Moreover, let c denote the number of crossings in D_L and let s be the number of Seifert circles. Prove that the genus of the resulting Seifert surface is equal to:

$$\text{genus}(S) = p - \frac{s+n-c}{2},$$

where p is the number of connected components of the projection of L.

Check that, for $p=1$, the Euler characteristic of S is equal to $s-c$ so it agrees with the Euler characteristic of handlebody described in Corollary 6.

Suppose that the solid torus V_K is a closure of a regular neighborhood of a knot K in S^3 and set $M_K = S^3 - int\, V_K$ (note that M_K is homotopy equivalent to the knot complement). Let us write a Mayer-Vietoris sequence for the pair (M_K, V_K):

$$0 = H_2(S^3) \to H_1(\partial M_K) \to H_1(M_K) \oplus H_1(V_K) \to H_1(S^3) = 0.$$

For a torus ∂M_K and the solid torus V_K homology are $Z \oplus Z$ and Z, respectively. Therefore $H_1(M_K) = Z$ and it is generated by a meridian in $\partial M_K = \partial V_K$, where by the meridian we understand a simple closed curve in ∂V_k which bounds a disk in V_K. We denote the meridian by m. A simple closed curve on ∂M_K which generates $\ker(H_1(\partial M_K) \to H_1(M_K))$ is called longitude and it is denoted by l. If S^3 and K are oriented then the longitude is oriented in agreement with the orientation of K. Subsequently, the meridian is given the orientation in such a way that the pair (m, l) induces on ∂V_K the same orientation as the one induced by the solid torus V_K, which inherits its orientation from S^3. Equivalently, the linking number of m and K is equal to 1 (compare Sect. 9.5). Similar reasoning allows us also to conclude:

Proposition 3 *For any link L in S^3 the first homology of the exterior of L in S^3 is freely generated by the meridians of components of L. In particular, $H_1(S^3 - L) = Z^{com(L)}$, where $com(L)$ denotes the number of components of the link L.*

We also can use the Mayer-Vietoris sequence to find the homology of the exterior the Seifert surface in S^3. Let F_L be a Seifert surface of a link L and F' its restriction to $M_L = S^3 - int\, V_L$. Let $V_{F'}$ be a regular neighborhood of F' in M_L. Because F' is orientable $V_{F'}$ is a product $F' \times [-1, 1]$ with $F^+ = F' \times \{1\}$ and $F^- = F' \times \{-1\}$. The boundary, $\partial V_{F'}$ is homeomorphic to F^+ and F^- glued together naturally along their boundary. Now let us apply the Mayer-Vietoris sequence to $V_{F'}$ and $S^3 - int\, V_{F'}$. We get:

$$0 = H_2(S^3) \to H_1(\partial V_{F'}) \stackrel{(i_1,-i_2)}{\to} H_1(S^3 - int\, V_{F'}) \oplus H_1(V_{F'}) \to H_1(S^3) = 0,$$

where i_1 and i_2 are induced by embeddings. Clearly, $H_1(S^3 - int\, V_{F'})$ is isomorphic to the kernel of i_2. We can easily identify the elements $x^+ - x^-$ as elements of the kernel, for any x a cycle in F'. In the case of L being a knot, these elements generate the kernel.

Corollary 8 *The homology groups, $H_1(F_L)$ and $H_1(S^3 - F_L)$ are isomorphic to $Z^{2g+com(L)-1}$, where g is the genus of F_L and $com(L)$ is the number of components of L thus also the number of boundary components of F_L. Compare Theorem 6.*

Corollary 9 *Let $x_1, \ldots, x_{2g}$ be a basis of $H_1(F_K)$ where F_K is the Seifert surface of a knot K. Then $x_1^+ - x_1^-, \ldots, x_{2g}^+ - x_{2g}^-$ form a basis of $H_1(S^3 - K)$.*

With some effort we can generalize Corollary 8 to get the following result which is a version of Alexander-Lefschetz duality[21] (see [Lic97] for an elementary proof).

Theorem 6 *Let F be a Seifert surface of a link, then $H_1(S^3 - F)$ is isomorphic to $H_1(F)$ and there is a nonsingular bilinear form*

$$\beta : H_1(S^3 - F) \times H_1(F) \to Z$$

given by $\beta(a, b) = \mathrm{lk}(a, b)$, where $\mathrm{lk}(a, b)$ is defined to be the intersection number of a and a 2-chain whose boundary is b (see Sect. 9.5).

9.4 Connected Sum of Links

Definition 10 Assume that K_1 and K_2 are oriented knots in S^3. A *connected sum* of knots, $K = K_1 \# K_2$, is a knot K in S^3 obtained in the following way:

[21] Let us recall that Alexander duality gives us an isomorphism $\tilde{H}^i(S^n - X) = \tilde{H}_{n-i-1}(X)$ for a compact subcomplex X of S^n and that on the free parts of homology the Alexander isomorphism induces a nonsingular form $\beta : \tilde{H}_i(S^n - X) \times \tilde{H}_{n-i-1}(X) \to Z$, where $\tilde{H}$ denotes reduced (co)homology.

First, for $i = 1, 2$, choose a point $x_i \in K_i$ and its regular neighborhood C_i in the pair (S^3, K_i). Then, consider a pair $((S^3 - int\, C_1 \cup_\varphi S^3 - int\, C_2), (K_1 - int\, C_1 \cup_\varphi K_2 - int\, C_2))$, where φ is an orientation reversing homeomorphism $\partial C_1 \to \partial C_2$ which maps the end of $K_1 \cap (S^3 - int\, C_1)$ to the beginning of $K_2 \cap (S^3 - int\, C_2)$ (and vice versa). (Notice that notions of beginning and end are well defined because K_1 and K_2 are oriented.) We see that $(S^3 - int\, C_1) \cup_\varphi (S^3 - int\, C_2)$ is a 3-dimensional sphere and $(K_1 - int\, C_1) \cup_\varphi (K_2 - int\, C_2)$ is an oriented knot.

Lemma 3 *The connected sum of knots is a well defined, commutative and associative operation in the category of oriented knots in S^3 (up to ambient isotopy).*

A proof of the lemma follows from two theorems in PL topology which we quote without proofs.

Theorem 7 *Let (C, I) be a pair consisting of a 3-cell C and 1-cell I which is properly embedded and unknotted in C (i.e. the pair (C, I) is homeomorphic to $(\bar{B}(0, 1), [-1, 1])$ where $(\bar{B}(0, 1)$ is the closed unit ball in R^3 and $[-1, 1]$ is the interval $(x, 0, 0)$ parameterized by $x \in [-1, 1]$. Respectively, let (S^2, S^0) be a pair consisting of the 2-dimensional sphere and two points on it. Then any orientation preserving homeomorphism of C (respectively, S^2) which preserves I and is constant on ∂I (respectively, it is constant on S^0) is isotopic to the identity.*

Theorem 8 *Let K be a knot in S^3 and let C' and C'' be two regular neighborhoods in the pair (S^3, K) of two points on K. Then there exists an isotopy F of the pair (S^3, K) which is constant outside of a regular neighborhood of K and such that $F_0 = Id$ and $F_1(C') = C''$.*

Remark 2 In the definition of the connected sum of knots we assumed that the homeomorphism φ reverses the orientation. This assumption is significant as the following example shows.

Let us consider the right-handed trefoil knot K (i.e. a torus knot of type $(2, 3)$), Fig. 9.23. Let $\overline{K}$ be the mirror image of K (i.e. a torus knot of type $(2, -3)$). Then $K\#\overline{K}$ is the square knot while $K\#K$ is the knot "Granny" and these two knots are not equivalent. To distinguish them it is enough to compute their signature (as defined in Corollary 2(1). One can also use Jones polynomial, or HOMFLYPT (Jones-Conway) polynomial. For completeness let us recall their definitions, again in a historical context:
The first polynomial invariant of links was invented by James Waddell Alexander (1888–1971) in 1928 [A28]. Alexander observed also that if three oriented links, $L_+ =$ , $L_- =$ , and $L_0 =$ , have diagrams which are identical except near one crossing (as drawn) then their polynomials are linearly related [A28]. In early 1960's, J. Conway rediscovered Alexander's formula and normalized the Alexander polynomial, $\Delta_L(t) \in Z[t^{\pm 1/2}]$, defining it recursively as follows [Con69]:

(i) $\Delta_o(t) = 1$, where o denotes a knot isotopic to a simple circle

(ii)

$$\Delta_{L_+} - \Delta_{L_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\Delta_{L_0}.$$

We call the resulting polynomial the Alexander-Conway polynomial.

In the spring of 1984, Vaughan Jones discovered his invariant of links, $V_L(t)$ [Jon85]. Soon he realized that his polynomial satisfies the local relation analogous to that discovered by Alexander and Conway and established the meaning of $t = -1$. Thus the Jones polynomial is defined recursively as follows:

(i) $V_o = 1$,
(ii) $\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t)$.

In the summer and the fall of 1984, the Alexander and the Jones polynomials were generalized to the HOMFLYPT (Conway-Jones),[22] polynomial, $P_L \in Z[a^{\pm 1}, z^{\pm 1}]$, of oriented links. This polynomial is defined recursively as follows [FYHLMO85, PT87]:

(i) $P_o = 1$;
(ii) $aP_{L_+} + a^{-1}P_{L_-} = zP_{L_0}$.

In particular $\Delta_L(t) = P_L(i, i(\sqrt{t} - \frac{1}{\sqrt{t}}))$, $V_L(t) = P_L(it^{-1}, i(\sqrt{t} - \frac{1}{\sqrt{t}}))$. In August 1985 L. Kauffman found another approach to the Jones polynomial; we discuss this Kauffman bracket polynomial in Sect. 1.7.

Theorem 9 (Schubert [Sch53]) *Genus of knots in S^3 is additive, that is*

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

A proof of the Schubert theorem can be found in e.g. [JP87, Lic97].

Corollary 10 *Any knot in S^3 admits a decomposition into a finite connected sum of prime knots, i.e. knots which are not connected sums of non-trivial knots.*

In fact Schubert [Sch49] showed that the prime decomposition of knots is unique up to order of factors; in other words, knots with connected sum form a unique factorization commutative monoid.

Corollary 11 *The trefoil knot is non-trivial and prime.*

Proof The trefoil knot is non-trivial therefore its genus is positive (Corollary 7). Figure 9.23 demonstrates that the genus is equal to 1. Now primeness follows from Theorem 9 and Corollary 10. □

[22]HOMFLYPT is the acronym after the initials of the inventors: Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Traczyk [FYHLMO85, PT87].

Similarly as for knots, the notion of connected sum can be extended to oriented links. It, however, depends on the choice of the components which we glue together. The weak version of the unique factorization of links with respect to connected sum was proven by Youko Hashizume [Has58].

9.5 Linking Number; Seifert Forms and Matrices

We start this Section by introducing the linking number $\mathrm{lk}(J, K)$ for any pair of disjoint oriented knots J and K. Our definition is topological and we will show that it agrees with the diagrammatic definition considered before. We use the notation introduced right after the Exercise 4.

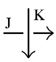
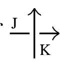
Definition 11 The linking number $\mathrm{lk}(J, K)$ is an integer such that

$$[J] = \mathrm{lk}(J, K)[m],$$

where $[J]$ and $[m]$ are homology classes in $H_1(S^3 - K)$ of the oriented curve J and the meridian m of the oriented knot K, respectively.

Lemma 4 *Let $S \subset S^3 - int\, V_K$ be a Seifert surface of a knot K (more precisely, its restriction to $S^3 - int\, V_K$), such that its orientation determines the orientation of ∂S compatible with that of the longitude l. Then $\mathrm{lk}(J, K)$ is equal to the algebraic intersection number of J and S.*

Proof First, let us recall the convention for the orientation of the boundary of an oriented manifold M. For $x \in \partial M$ we consider a basis $(v_2, \ldots, v_n)$ of the tangent space $T_x\partial M$ together with the normal $\overline{n}$ of ∂M in M which is directed outwards. Then, $v_2, \ldots, v_n$ defines orientation of $T_x\partial M$ if $\overline{n}, v_2, \ldots, v_n$ defines the orientation of T_xM. Returning to the proof of 4 we note that the meridian m intersects the Seifert surface S exactly at one point. Moreover, the algebraic intersection number of m and S is $+1$, according to our definition of the orientation of S. Thus, if the algebraic intersection number of J and S is equal to i, then $[J] = i[m]$, that is $i = \mathrm{lk}(J, M)$, which concludes the proof. $\square$

Lemma 5 *Let us consider a diagram of a link $J \cup K$ consisting of two disjoint oriented knots J and K. We assume that the orientation of $S^3 = R^3 \cup \infty$ is induced by the orientation of the plane containing the diagram of $J \cup K$ and the third axis which is directed upwards. Now, to any crossing of the diagram where J passes under K we assign $+1$ in the case of*  *and -1 in the case of* *. Then the sum of all numbers assigned to such crossings is equal to linking number $\mathrm{lk}(J, K)$.*

Proof Let us consider a Seifert surface of the knot K constructed from the diagram of K, as described in Construction 5. We may assume that the knot J is placed above this surface, except small neighborhoods of the crossings where J passes under K. We check now that the sign of the intersection of J with this surface coincides with the number that we have just assigned to such a crossing. □

Exercise 5 Show that $\mathrm{lk}(J, K) = \mathrm{lk}(K, J) = -lk(-K, J)$, where $-K$ denotes the knot K with reversed orientation.

Hint. Apply Lemma 5.

The linking number may be defined for any two disjoint 1-cycles in S^3. For example, as a definition we may use the condition from Lemma 4. That is, if α and β are disjoint 1-cycles in S^3 then $\mathrm{lk}(\alpha, \beta)$ is defined as the intersection number of α with a 2-chain in S^3 whose boundary is equal β.

Exercise 6 Prove that $\mathrm{lk}(\alpha, \beta)$ is well defined, that is, it does not depend on the 2-chain whose boundary is β.

Exercise 7 Show that lk is symmetric and bilinear, i.e. $\mathrm{lk}(\alpha, \beta) = \mathrm{lk}(\beta, \alpha)$ and $\mathrm{lk}(\alpha, n\beta) = n \cdot \mathrm{lk}(\alpha, \beta)$, and if a cycle β' is disjoint from α then $\mathrm{lk}(\alpha, \beta + \beta') = \mathrm{lk}(\alpha, \beta) + lk(\alpha, \beta')$.

Exercise 8 Prove that, if β and β' are homologous in the complement of α, then $\mathrm{lk}(\alpha, \beta) = \mathrm{lk}(\alpha, \beta')$.

Now we define a Seifert form of a knot or a link. Let S be a Seifert surface of a knot or a link K. Then S is a two-sided surface in S^3. Let $S \times [-1, 1]$ be a regular neighborhood of S in S^3. For a 1-cycle x in $\mathrm{int}\, S$ we can consider a cycle x^+ (respectively x^-) in $S \times \{1\}$ (respectively $S \times \{-1\}$) which is obtained by pushing the cycle x to $S \times \{1\}$ (respectively, to $S \times \{-1\}$). (We note that the sides of S are uniquely defined by the orientations of K and S^3.)

Definition 12 The *Seifert form* of the knot K is a function

$$f : H_1(int\, S) \times H_1(int\, S) \to Z$$

such that $f(x, y) = \mathrm{lk}(x^+, y)$. Similarly we define a Seifert form of an oriented link L using an oriented Seifert surface S of L.

Lemma 6 *The function f is a well defined bilinear form on the Z-module (i.e. abelian group) $H_1(int\, S)$.*

Proof The result follows from Exercises 7 and 8. □

Definition 13 By a *Seifert matrix* $V = \{v_{i,j}\}$ in a basis $e_1, e_2, \ldots, e_{2g+com(L)-1}$ of $H_1(S)$ we understand the matrix of f in this basis, that is

$$v_{i,j} = \mathrm{lk}(e_i^+, e_j).$$

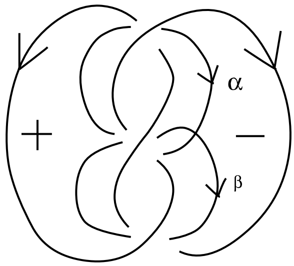


Fig. 9.30 A basis of the first homology of a Seifert surface of the right-handed trefoil knot

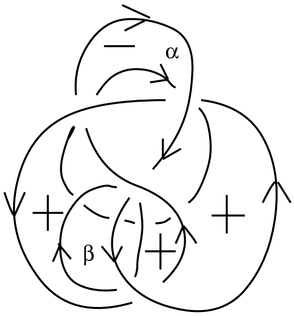


Fig. 9.31 A basis of the first homology of a Seifert surface of the figure-eight knot

Then, for $x, y \in H_1(S)$ we have $f(x, y) = x^T V y$. We use the convention that coefficients of a vector are written as a column matrix.[23]

Notice that a change of the basis in $H_1(S)$ results in the change of the matrix V to a similar matrix $P^T V P$, where $det\, P = \pm 1$.

Example 4 The Seifert matrix of a Seifert surface of the right-handed trefoil knot, computed in the basis $[\alpha], [\beta]$, is equal to $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ (see Fig. 9.30).

Example 5 The Seifert matrix of a Seifert surface of the figure-eight knot, computed in the basis $[\alpha], [\beta]$ is equal to $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ (see Fig. 9.31).

With some practice one should be able to find Seifert form efficiently and we encourage a reader to compute more examples and develop some rules; for example if α is a simple closed curve on S and on the plane then $lk(\alpha^+, \alpha) = -\frac{1}{2} \sum \operatorname{sgn} p$ where the sum is taken over all crossings of the diagram traversed by α. We illustrate it by one more example, the Seifert matrix of a pretzel knot. The computation is almost the same as in the trefoil case as the genus of the surface is equal to 1 and three crossings of the right-handed trefoil knot $\bar{3}_1$ are replaced by $2k_1 + 1$, $2k_2 + 1$, and $2k_3 + 1$, respectively.

[23] Our notation agrees with that of Kauffman [Kau87a], Kawauchi [Kaw96], and [JP87] but in the books by Burde and Zieschang [BZ85], Lickorish [Lic97], Rolfsen [Rol76], Livingston [Liv93], and Murasugi [Mur96] the convention is the opposite, that is $f(x, y) = lk(x, y^+)$.

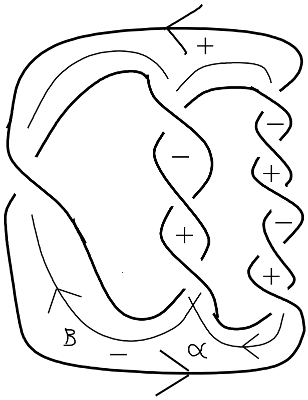


Fig. 9.32 $P_{1,3,5}$—the pretzel knot of type $(1, 3, 5)$

Example 6 Let P_{n_1,n_2,n_3} denote the pretzel link of type (n_1, n_2, n_3) (compare Fig. 9.32). The Seifert matrix of a Seifert surface of the pretzel knot $P_{2k_1+1,2k_2+1,2k_3+1}$, computed in the basis $[\alpha]$, $[\beta]$, is equal to $\left[\begin{smallmatrix} -k_1-k_2 & k_2 \\ k_2+1 & -k_1-k_2 \end{smallmatrix}\right]$ (see Fig. 9.32).

There is a classical skew-symmetric form on a homology group of an oriented surface, called an intersection form, which is related to the Seifert form f.

Definition 14 Let S be an oriented surface. For two homology classes $x, y \in H_1(S)$ represented by transversal cycles we define their *algebraic intersection number* $\tau(x, y)$ as the sum of the signed intersection points where the sign is defined in the following way: if x meet y transversally at a point p then the sign of the intersection at p is equal $+1$ if and -1 if .

Exercise 9 Prove that $\tau : H_1(S) \times H_1(S) \to Z$ is bilinear and skew-symmetric (i.e. $\tau(x, y) = -\tau(y, x)$).

Exercise 10 Prove that the determinant of a matrix of τ is equal to 1 if $\partial S = S^1$, or $\partial S = \emptyset$ and that it is equal to 0 otherwise.

Solution. Assume that S has more than one boundary component and ∂_1 is one of them. Then ∂_1 is a nontrivial element in $H_1(S)$ with trivial intersection number with any element of $H_1(S)$. Thus the matrix of τ is singular and its determinant is equal to zero.

Assume now that $\partial S = S^1$, or $\partial S = \emptyset$. Let us choose loops representing a basis of $H_1(S)$ such as in Fig. 9.33. In this basis the matrix of τ is as follows

$$\begin{bmatrix} 0 & 1 & & \bigcirc & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & \bigcirc & & -1 & 0 \end{bmatrix}$$

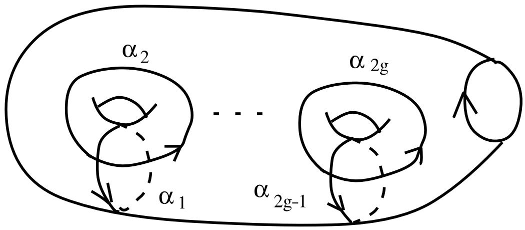


Fig. 9.33 A basis of the first homology of a surface S

Thus, its determinant is equal to 1. Notice also that the determinant of a matrix changing a basis of $H_1(S)$ is equal to 1 or -1. Thus the determinant of the form does not depend on the choice of a basis.

Exercise 11 Prove that, if S is a Seifert surface of a link then $\tau(x, y) = f(x, y) - f(y, x)$.

Solution. It follows from Lemma 5 that the crossing change between two oriented disjoint curves J and K in S^3 changes the linking number between them by 1 or -1, diagrammatically we have: $lk(\;) - lk(\;) = 1$. If J and K are two, possibly intersecting, oriented curves on an oriented surface we see that the pair (J^+, K) differs from the pair (J^-, K) by crossing changes at the crossings of J and K. Furthermore the convention we use is that $sgn(\;) = -1$.

Thus $f(J, K) - f(K, J) = \text{lk}(J^+, K) - \text{lk}(K^+, J) = \text{lk}(J^+, K) - \text{lk}(J^-, K) = \sum_{p \in J \cap K)} \text{sgn}\, p = \tau(J, K)$. The solution is completed.[24]

Corollary 12 *The Seifert matrix V of a knot K in S^3 satisfies the following equation*:

$$det(V - V^T) = 1.$$

Proof We note that $V - V^T$ is a matrix of τ (Exercise 11) and its determinant is equal to 1 (Exercise 10). □

A Seifert matrix is not an invariant of a knot or a link, but it can be used to define some well known invariants, including the Alexander polynomial.

Now we describe the relation between Seifert matrices of (possibly different) Seifert surfaces of a given link.

[24]The equality $\;-\;=\;$ is a defining relation of Vassiliev-Gusarov invariants or skein modules of links; compare Chap. IX of [Prz12]. This relation, combined with $\;\;= 0$ (that is, the value of a link with at least two singular crossings is equal to zero), leads to the (global) linking number, described as Vassiliev-Gusarov invariant of degree 1.

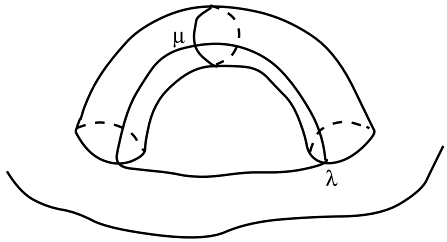


Fig. 9.34 Adding a handle to a surface

Definition 15 We call two matrices *S-equivalent* if one can be obtained from the other by a finite number of the following modifications:

(1) $A \Leftrightarrow PAP^T$ where $det\, P = \mp 1$.
(2)

$$A \Leftrightarrow \begin{bmatrix} A & \alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A \Leftrightarrow \begin{bmatrix} A & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where α is a column and β is a row.

Theorem 10 *Let us assume that L_1 and L_2 are isotopic links and F_1, respectively, F_2, are their Seifert surfaces. If A_1 and A_2 are their Seifert matrices computed in some basis B_1 and, respectively, B_2 then A_1 is S-equivalent to A_2.*

We perform the proof in two steps. Namely, we will prove the following two claims:

(1) If we attach a handle to F_1 then the resulting surface (boundary of which is again L_1) has its Seifert surface S-equivalent to the Seifert surface of F_1.
(2) We can assume that $L_1 = L_2$. Then there exists a Seifert surface for L_1 which can be reached (modulo isotopy) from both F_1 and F_2 by the operation of attaching handles.

First we prove (1).

Let A_1 be a Seifert matrix of F_1 (in some basis of $H_1(F)$). By γ and μ let us denote two new generators of $H_1(F \cup \text{ handle})$—see Fig. 9.34.

Let us recall that the Seifert form $f : H_1(F) \times H_1(F) \to Z$ was defined by the formula $f(x, y) = \mathrm{lk}(x^+, y)$, where lk denotes linking number in S^3 and x^+ is obtained by pushing the cycle x out of F in the normal direction of F.

If the pushing moves the cycle μ outside of the handle (that is μ^+ is outside the handle) then the resulting Seifert matrix is

$$\begin{bmatrix} A & \alpha & 0 \\ \beta & w_0 & 0 \\ 0 & \pm 1 & 0 \end{bmatrix}$$

which is S-equivalent to the matrix A (α and ω_0 can be converted to 0 matrices by type (1) operations; similarly, ± 1 can be converted to 1 by a type (1) operation). In the matrix, β is a row vector determined by linking numbers of λ^+ with the basis of $H_1(F)$, α is a column vector determined by linking numbers of the basis $H_1(F)$ with λ^- and $\omega_0 = lk(\lambda^+, \lambda^-)$.

Otherwise (i.e. μ^+ is inside the handle) we get the matrix:

$$\begin{bmatrix} A & \alpha & 0 \\ \beta & \omega_0 & \pm 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is S-equivalent to A as well.

Proof of (2) Assume that the Seifert surface F_1 intersects F_2 transversally (modulo the boundary L_1; in the neighborhood of L_1 they may be assumed to be disjoint outside L_1). Now we will use the following

Lemma 7 *Let M be compact connected 3-manifold and let F_1, F_2 be such submanifolds of ∂M that $\partial M = F_1 \cup F_2$ and $F_1 \cap F_2 = \partial F_1 = \partial F_2$. Then there exists a surface F in M such that $\partial F = \partial F_1$ and F can be obtained from F_1 as well as from F_2 by attaching 1-handles. More precisely: F cuts M into two 3-submanifolds M_1, containing F_1 and M_2 containing F_2. Furthermore M_i can be obtained from F_i (more precisely $F_i \times [0, 1]$) by attaching 1-handles*[25] *to $int(F_i)$. We have $F_i \cup F = \partial M_i$; in particular F is obtained from F_i by 1-surgeries.*

Sketch of the Proof. The presented proof is based on the proof of existence of Heegaard decomposition of a 3-manifold from triangulation (e.g. [Hem76, JP87]). Let X be a triangulation of (M, F_1, F_2). In particular L is in the 1-skeleton of triangulation Γ_1. Let Γ_2 denote the dual 1-skeleton. That is, Γ_2 is the maximal 1-subcomplex of the first baricentric subdivision X' of X, such that Γ_2 is disjoint with Γ_1. Let V_i $(i = 1, 2)$ be a regular neighborhood of Γ_i associated to the second baricentric subdivision of X. Then $X = V_1 \cup V_2$ and V_i is obtained from F_i by attaching (solid) 1-handles. Therefore $(F_1 \cup V_1) \cap (F_2 \cup V_2)$ is the surface F that we look for.

The proof of claim (2) is inductive with respect to the number of circles in the intersection $F_1 \cap F_2$:

(1) Suppose that $F_1 \cap F_2 = L_1$. Then we apply Lemma 7 to a part of S^3 which is bounded by the closed surface $F_1 \cup F_2$.

(n) Inductive step. Suppose that (2) holds if the number of components of $F_1 \cap F_2$ is smaller than n.

Now, assume that $F_1 \cap F_2$ consists of n circles. Then $F_1 \cup F_2$ cuts S^3 into a number of connected components and moreover different "sides" of F_1 and F_2 bound different components. Let M be a component such that $F_1' = F_1 \cap$

[25] We attach *k-handle* to an $(n+1)$-dimensional manifold M along an open subset, N, of the boundary by choosing a disk $D^{n+1} = D^k \times D^{n+1-k}$ and the embedding $\phi : \partial D^k \times D^{n+1-k} \to N$, and gluing M with D^{n+1} using ϕ. In our case $n = 2$.

∂M and $F_2' = F_2 \cap \partial M$. Now we apply Lemma 7 to the triple (M, F_1', F_2') and consequently let F_0' be the surface provided by the lemma. That is, F_0' is obtained by attaching solid 1-handles to either F_1' or F_2'.

Let F_1^0 and F_2^0 be obtained from F_1 and F_2 by replacing F_1' and F_2' by F_0'. Then by moving slightly surfaces F_1^0 and F_2^0 we can obtain a smaller number of components of their intersection and thus we can apply the inductive assumption. This concludes the proof of (2) and of Theorem 10. □

An elementary, diagrammatic proof of Theorem 10, based on Reidemeister moves and, the fact that any link has a special diagram (compare Exercises 1 and 2 or Proposition 13.15 of [BZ85]), is given in [BNFK98].

9.6 From Seifert form to Alexander Polynomial and Signatures

The Conway's potential function is defined as a normalized version of the Alexander polynomial using Seifert matrix, as follows [Kau80]:

Lemma 8 *Let A be a Seifert matrix of an oriented link L and define the potential function $\Omega_L(x) = det(xA - x^{-1}A^T)$. Then $\Omega_L(x)$ does not depend on the choice of a Seifert surface and its Seifert matrix. In particular, if T_1 is the trivial knot then $\Omega_{T_1}(x) = 1$.*

Proof. The result follows from Lemma 10. Indeed, simple computations show that if we replace the matrix A with another S-equivalent matrix then $\Omega_L(x)$ remains the same. We use the following identity

$$det\left(x\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - x^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = det\begin{bmatrix} 0 & x \\ -x^{-1} & 0 \end{bmatrix} = 1.$$

The same identity is used in the computations for the trivial knot.

If we choose $x = -\sqrt{t}$ then the potential function is the normalized Alexander polynomial (i.e. Alexander-Conway polynomial). The transposition of a matrix is preserving its determinant thus the substitution $x \to -x^{-1}$ (or $\sqrt{t} \to \frac{1}{\sqrt{t}}$) is preserving the potential function and Alexander-Conway polynomial. Furthermore, we can put $z = x^{-1} - x = \sqrt{t} - \frac{1}{\sqrt{t}}$. As follows from Theorem 11, we obtain, after the substitution, the Conway polynomial $\nabla_L(z)$ (the terminology may be sometimes confused, as $\nabla_L(z)$ is also often called Alexander-Conway polynomial).

Theorem 11 (Kauffman [Kau80]) *$\Omega_L(x) = \Delta_L(t) = \nabla_L(z)$, where $x = -\sqrt{t}$, $z = x^{-1} - x = \sqrt{t} - \frac{1}{\sqrt{t}}$.*

Proof (hint). We have to show that $\Omega_{L_+}(x) - \Omega_{L_-}(x) = (x^{-1} - x)\Omega_{L_0}(x)$. In order to demonstrate it we use the properly chosen Seifert surfaces F_+, F_-, F_0 for L_+, L_- and L_0 respectively.

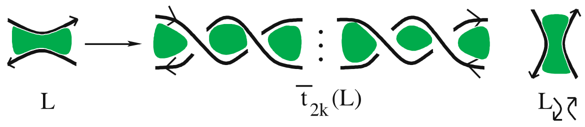


Fig. 9.35 Oriented links L, $\bar{t}_{2k}(L)$, and L_∞, and their Seifert surfaces

We give all details in the analysis of the more general case of the behavior of Seifert matrices under $\bar{t}_{2k}$-moves, which generalize the crossing change, which is $\bar{t}_{\pm 2}$-move.

Definition 16 [Prz88] The *$\bar{t}_{2k}$-move* (introducing k full twists on anti-parallel oriented arcs) is the elementary operation on an oriented diagram L resulting in $\bar{t}_{2k}(L)$ as illustrated in Fig. 9.35.

Notice that $\bar{t}_2$-move is a crossing change from a positive to negative crossing ($L_- = \bar{t}_2(L_+)$). We can choose Seifert surfaces $F(L)$, $F(\bar{t}_{2k}(L))$, and $F(L_\infty)$ for $L = L_{\asymp}$, $\bar{t}_{2k}(L)$, and $L_\infty = L_{)(}$, respectively, to look locally as in Fig. 9.35.

Let us choose a basis for $H_1(F(L_{)(}))$ and add one, standard, element, $e_{\asymp}$ to obtain a basis for $H_1(F(L_{\asymp}))$, and $e_{\bar{t}_{2k}(L)}$ to get a basis of $H_1(F(\bar{t}_{2k}(L))$. Denote the Seifert matrix of $L_{)(}$ in the chosen basis by $A_{L_{)(}}$. In these bases we have immediately:

Lemma 9

$$A_{L_{\asymp}} = \begin{bmatrix} A_{L_{)(}} & \alpha \\ \beta & q \end{bmatrix},$$

$$A_{\bar{t}_{2k}(L)} = \begin{bmatrix} A_{L_{)(}} & \alpha \\ \beta & q+k \end{bmatrix},$$

where α is a column given by linking numbers of $e^+_{\asymp}$ (or $e^+_{\bar{t}_{2k}(L)}$) with basis elements of $H_1(F(L_{)(}))$, β is a row given by linking numbers of basis elements of $H_1(F(L_{)(}))$ with $e^-_{\asymp}$ (or $e^-_{\bar{t}_{2k}(L)}$), and q is a number equal to $lk(e^+_{\asymp}, e_{\asymp})$ (compare [Kau80, PT87'] or [Prz86]).

Corollary 13

(i) *If two oriented links are $\bar{t}_{2k}$ equivalent (that is they differ by a finite number of $\bar{t}_{2k}$-moves) then their Seifert matrices are S-equivalent modulo k.*

(ii) *The potential function satisfies*:

$$\Omega_{\bar{t}_{2k}(L)} - \Omega = k(x - x^{-1})\Omega .$$

In particular the case $k = -1$ *gives*: $\Omega_{L_+}(x) - \Omega_{L_-}(x) = (x^{-1} - x)\Omega_{L_0}(x)$.

Proof (i) It follows from the fact we noted in Lemma 9 that for properly chosen Seifert surfaces and basis of their homology, the entries of Seifert matrices for $\bar{t}_{2k}$ and are congruent modulo k.

(ii) $\Omega_{\bar{t}_{2k}(L)} = det(xA_{\bar{t}_{2k}(L)} - x^{-1}A^T_{\bar{t}_{2k}(L)}) = det \begin{bmatrix} A_L & x\alpha - x^{-1}\beta^T \\ x\beta - x^{-1}\alpha^T & (x - x^{-1})(q + k) \end{bmatrix}$,

and

$$\Omega = det \begin{bmatrix} A_L & x\alpha - x^{-1}\beta^T \\ x\beta - x^{-1}\alpha^T & (x - x^{-1})q \end{bmatrix}.$$

Thus the difference is equal to $k(x - x^{-1})\Omega$. $\square$

Example 7 We can use Corollary 13 to compute the potential (and Alexander-Conway) polynomial of the pretzel link $L = P_{2k_1+1,2k_2+1,\dots,2k_m+1}$ (see Fig. 9.32 or Fig. 9.36). Namely, we apply the formula of Corollary 13(ii) for any column of a pretzel link. For $z = x^{-1} - x$ we get

$$\begin{aligned}
\Omega_L(x) = \nabla_L(z) &= \sum_{j=0}^{m-1} s_{m,j} z^j \nabla_{T_{2,m-j}}(z) \\
&= z^{m-1}\left(\binom{m-1}{0} + s_{m,1}\binom{m-2}{0} + s_{m,2}\binom{m-3}{0} + \cdots\right) \\
&\quad + z^{m-3}\left(\binom{m-2}{1} + s_{m,1}\binom{m-3}{1} + s_{m,2}\binom{m-4}{1} + \cdots\right) + \cdots \\
&= \sum_{j=0}^{\lfloor (m-1)/2 \rfloor}\left(\sum_{i=0}^{m-1-2j}\binom{m-1-j-i}{j} s_{m,j}\right) z^{m-1-2j},
\end{aligned}$$

where $s_{m,j}$ is an elementary symmetric polynomial in variables $k_1, \dots, k_m$ of degree j, that is $\Pi_{i=1}^m(z + k_i) = \sum_{j=0}^m s_{m,j} z^{m-j}$ and $\nabla_{T_{2,m-j}}(z) = \nabla_{P_{1,1,\dots,1}}(z)$ are the Alexander-Conway polynomials of the torus links of type $(2, m - j)$, in particular, it satisfies Chebyshev type[26] (compare Example 3) relations $\nabla_{T_{2,n}}(z) =$

[26] We have $\nabla_{T_{2,n}}(z) = i^{1-n} S_{n-1}(iz)$.

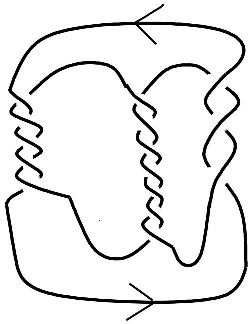


Fig. 9.36 $P_{5,7,-3}$—the pretzel knot with the trivial Alexander-Conway polynomial

$z\nabla_{T_{2,n-1}}(z) + \nabla_{T_{2,n-2}}(z)$ (with initial data $\nabla_{T_{2,0}}(z) = 0$ and $\nabla_{T_{2,1}}(z) = 1$). In particular, $\Omega_{T_{2,n}}(x) = \nabla_{T_{2,n}}(x^{-1} - x) = \frac{x^{-n}-(-1)^n x^n}{x^{-1}+x} = \binom{n-1}{0} z^{n-1} + \binom{n-2}{1} z^{n-3} + \cdots + \binom{n-1-i}{i} z^{n-1-2i} + \cdots = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-2i}{i} z^{n-1-2i}$.

9.6.1 Tristram-Levine Signature

We generalize definition of the classical (Trotter-Murasugi) signature, following Tristram and Levine (see [Gor78, Lev69, PT87', Tri69]).

Recall that a *symmetric Hermitian form* $h : C^n \times C^n \to C$ is a map which satisfies $h(a+b, c) = h(a, c) + h(b, c)$, $h(\lambda a, b) = \lambda h(a, b)$, and $h(a, b) = \overline{h(b, a)}$. The matrix H of a symmetric Hermitian form in any basis is called a Hermitian matrix (i.e. $H = \bar{H}^T$). A symmetric Hermitian form has a basis in which the matrix is diagonal with 1, -1 or 0 entries. The numbers, n_1 of 1's, n_{-1} of -1's and n_0 of 0's form a complete invariant of a symmetric Hermitian form (the Sylvester law of inertia). The number n_0 is called the nullity of the form and $\sigma = n_1 - n_{-1}$ is called the signature of the form. Recall also that if we count eigenvalues of H (with multiplicities) then n_1 is the number of positive eigenvalues of H and n_{-1} is the number of negative eigenvalues.

Definition 17 [Tri69, Lev69] Let A_L be a Seifert matrix of a link L. For each complex number ξ ($\xi \neq 1$) consider the Hermitian matrix $H_L(\xi) = (1 - \bar{\xi})A_L + (1 - \xi)A_L^T$. The signature of this matrix is called the *Tristram-Levine signature* of the link L. If the parameter ξ is considered, we denote the signature by $\sigma_L(\xi)$, if we consider $\psi = 1 - \xi$ as a parameter, we use notation $\sigma_\psi(L)$. The classical signature σ satisfies $\sigma(L) = \sigma_1(L) = \sigma_L(0) = \sigma_L(-1)$. Also, by well justified convention, we put $\sigma_L(1) = 0$ (see Remark 3).

The Tristram-Levine signature is a well defined link invariant as it is an invariant of S-equivalence of Seifert matrices. Checking this is similar to the calculation for the potential function (we leave this to the reader as an exercise).

Remark 3 The signature of a Hermitian matrix is unchanged when matrix is multiplied by a positive number,[27] we can (and will) often assume that ξ in $\sigma_L(\xi)$ and ψ in $\sigma_\psi(L)$ are of unit length. With such assumptions we have the Tristram-Levin signature functions, $\sigma_L(\xi), \sigma_\psi(L) : S^1 \to Z$. $\sigma_L(\xi)$ is the signature function tabulated in [CL], and $\sigma_\psi(L)$ is used in Examples in this book. S^1 will be usually parameterized by $arg(\psi) \in [-\pi, \pi]$.[28] Generally, we have $\sigma_L(\xi) = \sigma_{1-\xi}(L)$ but when restricted to the unit circle, we have to write $\sigma_L(\xi) = \sigma_{(1-\xi)/(|1-\xi|}(L)$. Notice that for $\psi = \frac{1-\xi}{|1-\xi|}$, we have $\psi^2 = \frac{(1-\xi)(1-\xi)}{(1-\xi)(1-\bar{\xi})} = \frac{1-\xi}{1-\bar{\xi}} = -\xi$ (and $(i\psi)^2 = \xi$). Therefore, $\sigma_\psi(L) = \sigma_L((i\psi)^2) = \sigma_L(\xi)$, for $Re(\psi) \geq 0$. As we show in Corollary 16, $\sigma_i(L) = 0$, which justifies the convention[29] that $\sigma_L(1) = 0$.

Corollary 14 [Prz88]

(i) *For any $\bar{t}_{2k}$-move and $Re(1-\xi) \geq 0$ (i.e. $|arg(\psi)| \leq \pi/2$) we have*:

$$0 \leq \sigma_{\bar{t}_{2k}(L)}(\xi) - \sigma_L(\xi) \leq 2.$$

In particular [*PT87'*], *for* $Re(1-\xi) \geq 0$, *we have* $-2 \leq \sigma_{L_+}(\xi) - \sigma_{L_-}(\xi) \leq 0$.

(ii) *Furthermore, for any ξ and k we have*:

$$0 \leq |\sigma_{L_{)(}}(\xi) - \sigma_{\bar{t}_{2k}(L)}(\xi)| \leq 1.$$

In particular, $0 \leq |\sigma_{L_+}(\xi) - \sigma_{L_0}(\xi)| \leq 1$.

Proof Applying Lemma 9 we obtain

$$H_{\bar{t}_{2k}(L}(\xi) = \begin{bmatrix} H_{L_{)(}}(\xi) & a \\ a^{-T} & m + k(2 - \xi - \bar{\xi}) \end{bmatrix},$$

$$H_L(\xi) = \begin{bmatrix} H_{L_{)(}}(\xi) & a \\ a^{-T} & m \end{bmatrix},$$

[27] The Hermitian matrix H is Hermitian similar to λH for any real positive number λ; $\lambda H = (\sqrt{\lambda} Id)H(\sqrt{\lambda} Id)$.

[28] In [CL], S^1 is parameterized by $\frac{arg\,\xi}{\pi}$.

[29] In the literature on the Tristram-Levine signature of knots, often used normalization of the Hermitian matrix $(1-\bar{\xi})A_L + (1-\xi)A_L^T$ is to take $|\xi| = 1$ ($\xi \neq 1$). When one writes the function $\sigma_K(\xi)$ then usual assumption about the parameter ξ is that it is on the unit circle. Then one has $det((1-\bar{\xi})A_L + (1-\xi)A_L^T) = det((\xi-1)(\frac{1-\bar{\xi}}{\xi-1}A - A^T)) = det((\xi-1)(\bar{\xi}A - A^T)) \doteq (\xi-1)^n \Delta(\bar{\xi})$, where $\doteq$ denotes equality up to $\pm t^i$, [Gor78] (compare Lemma 10). When dealing with links, we found more convenient (see [PT87', Prz86]) to consider $\psi = 1 - \xi$ and assume that $|\psi| = 1$. Then we have $det(i(\bar{\psi}A + \psi A^T)) = det(i\bar{\psi}A - i\psi A^T) = \Omega(i\bar{\psi}) = \Omega(i\psi) = \nabla(-i(\bar{\psi} + \psi))$ (compare Lemma 10). Therefore, for any knot $\sigma_\psi(K) = \sigma_K((i\psi)^2) = \sigma_K(\xi)$.

where $a = (1-\bar{\xi})\alpha + (1-\xi)\beta^T$ and $m = ((1-\bar{\xi}) + (1-\xi))q$. Because $2-\xi-\bar{\xi} \geq 0$, so $0 \leq \sigma(H_{\bar{t}_{2k}(L)}(\xi)) - \sigma(H_L(\xi)) \leq 2$, and the proof of (i) is finished.[30] Part (ii) follows from the easy observation that deleting the last row and column of a Hermitian matrix can change the signature at most by ± 1. □

We can use results of computations in Examples 4, 5, and 6 to find the Tristram-Levine signature for the trefoil knot, the figure eight knot, and the pretzel knot $P_{2k_1+1,2k_2+1,2k_3+1}$.

Example 8 Using the Seifert matrix for the right-handed trefoil knot ($\bar{3}_1$) computed in Example 4 we find that:

$$H_{\bar{3}_1}(\xi) = \begin{bmatrix} \xi + \bar{\xi} - 2 & 1 - \xi \\ 1 - \bar{\xi} & \xi + \bar{\xi} - 2 \end{bmatrix}$$

Therefore

$$\sigma_{\bar{3}_1}(\xi) = \begin{cases} -2 & \text{if } Re(1-\xi) > \frac{1}{2} \\ -1 & \text{if } Re(1-\xi) = \frac{1}{2} \\ 0 & \text{if } -\frac{1}{2} < Re(1-\xi) < \frac{1}{2} \\ 1 & \text{if } Re(1-\xi) = -\frac{1}{2} \\ 2 & \text{if } Re(1-\xi) < \frac{1}{2} \end{cases}$$

Part of the regularity of the Tristram-Levine signature can be explained by the observation that for $\xi_2 = 2 - \xi_1$ (i.e. $1 - \xi_2 = -(1-\xi_1)$) we have $H_L(\xi_2) = -H_L(\xi_1)$ and $\sigma_L(\xi_2) = -\sigma_L(\xi_1)$.

Example 9 Using the Seifert matrix for the figure eight knot (4_1) computed in Example 5 we find that:

$$H_{4_1}(\xi) = \begin{bmatrix} 2 - \xi - \bar{\xi} & \bar{\xi} - 1 \\ \xi - 1 & \xi + \bar{\xi} - 2 \end{bmatrix}$$

For any $\xi \neq 1$, we have $det\, H_{4_1}(\xi) = -(2-\xi-\bar{\xi})^2 - (1-\xi)(1-\bar{\xi}) < 0$, thus $\sigma_{4_1}(\xi) = 0$.

The observation that for the figure eight knot the Tristram-Levin signature is always equal to zero is not that unexpected because the figure eight knot is an amphicheiral knot ($4_1 = \bar{4}_1$) and we have:

---

[30]It holds, in general, that if two $n \times n$ Hermitian matrices H and H' differ only at one entry, $a'_{nn} > a_{nn}$ then $0 \leq \sigma(H') - \sigma(H) \leq 2$. Furthermore, if $det\, H\, det\, H' > 0$ then $\sigma(H') = \sigma(H)$ and if $det\, H\, det\, H' < 0$ then $\sigma(H') = \sigma(H) + 2$.

Corollary 15 *If $\bar{L}$ is the mirror image of a link L then the Seifert matrix $A_{\bar{L}} = -A_L$, $H_{\bar{L}}(\xi) = -H_L(\xi)$, $\sigma_{\bar{L}}(\xi) = -\sigma_L(\xi)$, and $\sigma_\psi(\bar{L}) = -\sigma_\psi(L)$. In particular, the Tristram-Levine signature of an amphicheiral link is equal to zero.*

We can also observe that i ($i = \sqrt{-1}$) times the matrix of τ from Exercise 9 is a Hermitian matrix of the signature equal to 0 thus for a knot, $\sigma_i(K) = 0$. This holds also for links as the signature is unchanged by adding to the matrix rows and columns of zeros:

Corollary 16 *For any link L we have $\sigma_i(L) = \sigma_{-i}(L) = 0$.*

It is useful to summarize our observations about the Tristram-Levin signature of links using $\psi = 1 - \xi$ and $|\psi| = 1$.

Corollary 17 *When we change ψ from 1 to i, the signature $\sigma_\psi(L)$ changes from the classical (Trotter-Murasugi) $\sigma(L)$ to 0 (equivalently, if ξ changes from 1 to -1, then $\sigma_L(\xi)$ changes from 0 to $\sigma(L)$). Furthermore, $\sigma_\psi(L) = \sigma_{\bar{\psi}}(L) = -\sigma_{-\psi}(L) = -\sigma_\psi(\bar{L})$.*

Example 10 Using the Seifert matrix of the pretzel knot $P_{2k_1+1,2k_2+1,2k_3+1}$ computed in Example 6 we find that:

$$H_{P_{2k_1+1,2k_2+1,2k_3+1}} = \begin{bmatrix} -(\psi+\bar{\psi})(k_1+k_2+1) & k_2\bar{\psi}+(k_2+1)\psi \\ (k_2+1)\bar{\psi}+k_2\psi & -(\psi+\bar{\psi})(k_2+k_3+1) \end{bmatrix}$$

Furthermore,

$$det\, H_{P_{2k_1+1,2k_2+1,2k_3+1}} = (\psi+\bar{\psi})^2(1+k_1+k_2+k_3+k_1k_2+k_1k_3+k_2k_3) - 1.$$

Therefore the Tristram-Levine signature of a pretzel knot with $1+k_1+k_2+k_3+k_1k_2+k_1k_3+k_2k_3 > 0$ (e.g. a positive pretzel knot) satisfies (in lieu of Corollary 15 we consider only $Re(\psi) \geq 0$):

$$\sigma_\psi(P_{2k_1+1,2k_2+1,2k_3+1}) = \begin{cases} -2 & \text{if } Re(\psi) > \frac{1}{2\sqrt{1+k_1+k_2+k_3+k_1k_2+k_1k_3+k_2k_3}} \\ -1 & \text{if } Re(\psi) = \frac{1}{2\sqrt{1+k_1+k_2+k_3+k_1k_2+k_1k_3+k_2k_3}} \\ 0 & \text{if } 0 \leq Re(\psi) < \frac{1}{2\sqrt{1+k_1+k_2+k_3+k_1k_2+k_1k_3+k_2k_3}} \end{cases}$$

Notice that in the example of Seifert of $P_{5,7,-3}$, Fig. 9.36, we have $det\, H_{P_{5,7,-3}} = -1$ and $\sigma_\psi(P_{5,7,-3}) \equiv 0$. We utilize the result of this calculation in [PT].

9.6.2 Potential Function and Tristram-Levine Signature

Lemma 8 and Definition 17 suggest that there is a relation between the potential function and the Tristram-Levine signature of links. In fact we have:

Lemma 10 *Assume that the potential function at $i\psi$ is different from zero. Then*

$$i^{\sigma_\psi(L)} = \frac{\Omega_L(i\psi)}{|\Omega_L(i\psi)|} = \frac{\Delta_L(t_0)}{|\Delta_L(t_0)|} = \frac{\nabla_L(-i(\psi+\bar{\psi})}{|\nabla_L(-i(\psi+\bar{\psi})|},$$

where $\Delta_L(t_0)$ is the Alexander-Conway polynomial and $t_0 = -\psi^2$ ($\sqrt{t_0} = -i\psi$). In particular, the Tristram-Levine signature is determined modulo 4 by the appropriate value of the potential function (*or Alexander-Conway polynomial*); *compare Chap. III of* [*Prz*12].

Proof The idea is to compare the formulas for the potential functions and the signature, that is:

$$\Omega_L(i\psi) = det(i\psi A_L - (i\psi)^{-1}A_L^T) = i^n\, det(\psi A_L + \bar{\psi} A_L^T) \quad \text{and}$$
$$\sigma_\psi(L) = \sigma(\bar{\psi} A_L + \psi A_L^T)$$

In more detail, we write our proof as follows:

Let H be a non-singular Hermitian matrix of dimension n and $\lambda_1, \lambda_2, \ldots, \lambda_n$ its eigenvalues (with multiplicities). Then

$$\begin{aligned} det(iH) &= i^n\, det\, H = i^n \lambda_1\lambda_2\cdots\lambda_n \\ &= i^n(-1)^{n_{-1}}|det\, H| = i^{n-2n_{-1}}|det\, H| = i^{n_1-n_{-1}}|det\, H| \\ &= i^{\sigma(H)}|det\, H|. \end{aligned}$$

Therefore, $\frac{det(iH)}{|det(iH)|} = i^{\sigma(H)}$. By applying this formula for $H = \psi A_L + \bar{\psi} A_L^T$, $|\psi| = 1$, and remembering that $\sigma(\bar{H}) = \sigma(H)$, we obtain the formula of Lemma 10. □

Example 11 We can use Lemma 10 to compute quickly the Tristram-Levine signature[31] of the torus link of type $(2, n)$, $T_{2,n}$. We use the fact that we already computed the classical signature and Alexander-Conway (and potential) polynomial to be (for $k \neq 0$):

$$\sigma(T_{2,n}) = 1-n, \qquad \Delta_{T_{2,n}}(z) = \Omega_{T_{2,n}}(x) = \frac{x^{-n} - (-1)^n x^n}{x^{-1}+x} = t^{\frac{1-n}{2}}\frac{t^n + (-1)^{n+1}}{t+1},$$

where $z = x^{-1} - x = t^{1/2} - t^{-1/2}$. In particular $\sigma_\psi(T_{2,n})$ can change only if $x = i\psi$ is a root of the potential function, and because $\Omega_{T_{2,n}}(i\psi) = i^{1-n}\frac{\psi^n - \psi^{-n}}{\psi - \psi{-1}}$, the only changes holds at ψ satisfying $\psi^{2n} = 1$ and $\psi \neq \pm 1$.

[31] It is essentially the same proof we used in Chap. III of [Prz12] to show that a signature is a skein equivalence invariant: The Alexander-Conway polynomial determines the signature modulo 4 and the Murasugi type inequalities ($|\sigma_\psi(L_+) - \sigma_\psi(L_0)| \leq 1$ and for $Re(\psi) \geq 0$, $0 \leq (\sigma_\psi(L_-) - \sigma_\psi(L_+) \leq 2$) gives the direction, and limit the size of the signature change, compare also Corollary 13.

We have for $Re\psi \geq 0, k \neq 0, 0 \leq j \leq n-1$:

$$\sigma_\psi(T_{2,n}) = \begin{cases} 1-n & \text{if } Re(\psi) > Re(e^{\pi/n}) \\ 1-n+2j & \text{if } Re(e^{j\pi/n}) > Re(\psi) > Re(e^{(j+1)\pi/n}),\ j>0 \\ 2-n+2j & \text{if } Re(\psi) = Re(e^{j\pi/n}),\ j>0. \end{cases}$$

Corollary 18 *The classical (Trotter-Murasugi) signature $\sigma(L) = \sigma_1(L) = \sigma_L(-1)$, satisfies:*

$$i^{\sigma(L)} = i^{\sigma(A_L+A_L^T)} = \frac{\Omega_L(i)}{|\Omega_L(i)} = \frac{\Delta_L(-1)}{|\Delta_L(-1)|} = \frac{Det_L}{|Det_L|} = \frac{\nabla(-2i)}{|\nabla(-2i)|},$$
assuming $Det_L \neq 0$;

here $\Delta_L(-1)$ denotes $\Delta_L(t)$ for $\sqrt{t} = -i$. Recall, that $Det_L = \Delta_L(-1) = \Omega_L(i) = det(i(A_L + A_L^T))$ is called the determinant[32] *of a link L.*

Example 12 We compute here the Tristram-Levine signature of the knot $K = 6_2$ using Lemma 10 and discuss the standard convention and notation.

We have:

$$\sigma_\psi(6_2) = \begin{cases} -2 & \text{if } Re(\psi) > \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}} \\ -1 & \text{if } Re(\psi) = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}} \approx 0.636\ldots \\ 0 & \text{if } 0 \leq Re(\psi) < \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}. \end{cases}$$

Step 1. We compute the Conway polynomial $\nabla_{6_2}(z) = 1 - z^2 + z^4$; we use resolution in Fig. 9.37 to find this value and also observe that changing a crossing at p results in the trivial knot and smoothing at p results in a connected sum of the right handed trefoil knot and the left handed Hopf link ($K_0^p = \bar{3}_1 \# H_-$). In particular the unknotting number $u(6_2) = 1$.

Step 2. $Det_K = \nabla_K(-2i) = -11$, thus $\delta(K) \equiv 2 \mod 4$, and because K can be unknotted by changing one positive crossing, thus $-2 \leq \sigma(K) \leq 0$, and finally $\sigma(K) = -2$.

Step 2. Roots of $\nabla_{6_2}(z)$ are at $z^2 = \frac{-1\pm\sqrt{5}}{2}$. Thus for $t_0 = \xi = -\psi^2$, $z = -i(\psi + \bar{\psi})$, we have $\xi + \bar{\xi} = (i\psi)^2 + (i\bar{\psi})^2 = z^2 + 2 = \frac{3\pm\sqrt{5}}{2}$. Because $|\psi| = |\xi| = 1$, therefore $-2 \leq \xi + \bar{\xi} \leq 2$ and $\xi + \bar{\xi} = \frac{3-\sqrt{5}}{2}$ ($Re(\xi) = \frac{3-\sqrt{5}}{4}$). Finally, assuming $Re(\psi) \geq 0$ we get $\psi = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$.

[32]We should mention here that $|Det_L|$ is equal to $|det(G_L)|$ where G_L is a Goeritz matrix of L. Furthermore, if D_L is a special diagram of an oriented link L then $G_{D_L} = A_L + A_L^T$ for a properly chosen basis of $H_1(S)$ where S is the Seifert surface of D_L constructed according to Seifert algorithm. Thus not only $Det_L = det(iG_{D_L}))$ but also $\sigma(L) = \sigma(G_L)$; compare Corollary 3.

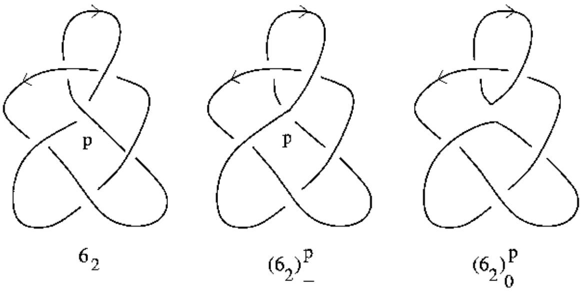
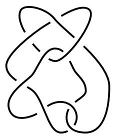


Fig. 9.37 Computing the Conway polynomial of the knot $K = 6_2$. $\nabla_{6_2}(z) = \nabla_{T_1}(z) + \nabla_{\bar{3}_1 \# H_-}(z) = 1 + (1+z^2)(-z) = 1 - z^2 - z^4$

Step 3. For $Re(\psi) \geq 0$, the value $Re(\psi) = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$ is the only place where the Tristram-Levine signature $\sigma_\psi(6_2)$ can be changing, and because we know already that $\sigma_1(6_2) = -2$ and $\sigma_i(6_2) = 0$ we conclude that $\sigma_\psi(6_2) = -2$ if $Re(\psi) > \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$ and $\sigma_\psi(6_2) = 0$ if $0 \leq Re(\psi) < \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$.

Step 4. It remains to show that $\sigma_\psi(6_2) = -1$ for $Re(\psi) = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$. Here we argue that, because the considered ψ is the singular root of the Alexander polynomial (precisely $t_0 = -\psi^2$), therefore the value of the signature at this point cannot differ by more than one from the neighboring values (so from 0 and from -2). More detailed analysis of the Hermitian matrix $\bar{\psi}A + \psi A^T$, leads to the conclusion that if $t_0 = -\psi^2$ is a singular root of the Alexander polynomial of a knot K then $\sigma_\psi(K) = \frac{\sigma_{\psi_-}(K) + \sigma_{\psi_+}(K)}{2}$, where ψ_- and ψ_+ are parameters just before ψ and just after ψ on the unit circle [Mat77].

In the convention of [Gor78, CL] one defines the Tristram-Levine signature function of variable ξ ($|\xi| = 1$) as $\sigma_L(\xi) = \sigma((1-\bar{\xi})A + (1-\xi)A^T))$. For $Re(\psi) \geq 0$, one has $\sigma_\psi(L) = \sigma_L(\xi)$, where $\xi = -\psi^2$ ($\psi = \frac{1-\xi}{|1-\xi|}$). In *knotinfo* Web page [CL], the parameter s satisfying $\xi = e^{\pi i s}$ is used. In particular, $\sigma_{6_2}(\xi) = -1$ for $Re(\xi) = \frac{3-\sqrt{5}}{4} = 1 - \cos(\pi/5) \approx 0.191$, and $s = \arccos(1 - \cos(\pi/5))/\pi \approx 0.44$ (compare Remark 3).

Example 13 The knot 9_{42} =  is the smallest knot which is not amphicheiral but the Jones, HOMFLYPT, and Kauffman polynomials are symmetric (e.g. $V_{9_{42}}(t) = V_{9_{42}}(t^{-1})$); compare [Prz12]. The non-amphicheirality of 9_{42} is detected by the signature: $\sigma(9_{42}) = -2 = -\sigma(\overline{9_{42}})$. This description can leave however an impression that the fact that 9_{42} is not ambient isotopic to its mirror image cannot be checked by the Jones polynomial alone. However, it follows from Corollary 18 (compare also Exercise 2) that $(-1)^{\sigma(K)/2} = sign(V_K(-1))$ for any knot K, thus if a knot is amphicheiral then $V_K(-1) > 0$. For 9_{42} we have $V_{9_{42}}(-1) = Det_{9_{42}} = -7 < 0$ thus 9_{42} is not amphicheiral. Furthermore, because 9_{42} can be unknotted by changing one positive crossing, we can deduce that $\sigma(9_{42}) = -2$.

In fact, the absolute value of the determinant $|Det_K| = |V_K(-1)| = |\Delta_K(-1)| = |\nabla(-2i)|$ suffices to show that the knot 9_{42} is not amphicheiral. K. Murasugi proved in [Mur65] (Theorem 5.6), the following result:

Theorem 12 *For any knot K*

$$\sigma_K \equiv |Det_K| - 1 \mod 4$$

Proof We use the fact that $Det_K = \nabla(-2i) \equiv 1 \mod 4$. Therefore, $|Det_K| \equiv 1 \mod 4$ if $Det_K > 0$ and $|Det_K| \equiv -Det_K \equiv -1 \mod 4$ if $Det_K < 0$. Furthermore, from Corollary 18 follows that $Det_K = (-1)^{\sigma(K)/2}|Det_K|$. Therefore,

$$\sigma_K \overset{\mod 4}{\equiv} \begin{cases} 0 & \text{if } |Det_K| \equiv 1 \mod 4 \\ 2 & \text{if } |Det_K| \equiv 3 \mod 4 \end{cases}$$

and Theorem 12 follows. □

Murasugi's Theorem leads to a curious formula:

Corollary 19 *For any knot K*

$$Det_K = (-1)^{(|Det_K|-1)/2}|Det_K|.$$

J. Milnor proved that the signature of a knot with the Alexander polynomial equal to one is equal to zero [Mil68]. In fact, it follows directly from Lemma 10 that the Tristram-Levin signature can change only at roots of the unit length of Alexander polynomial; therefore a link which has the Alexander polynomial without any root on the unit circle has a constant Tristram-Levin signature function. Thus:

Corollary 20 [Mil68] *If the Alexander polynomial $\Delta_L(t)$ is different from zero on the unit circle then for any ψ, ($|\psi| = 1$), we have $\sigma_\psi(L) = 0$.*

If we assume only that the determinant of a knot is equal to 1 then we get as a conclusion that the signature is divisible by eight (compare [Mur96], p. 149 after Exercise 7.5.4):

Proposition 4 *If the determinant of a knot K is equal to 1 then $\sigma(K) \equiv 0 \mod 8$.*

Proof $Det_K = 1$ means that for a Seifert matrix A of a knot K, $det(A + A^T) = 1$; The matrix/form $A + A^T$ is often called the Trotter form. The diagonal entries of the Trotter form are even because the diagonal of $A + A^T$ is twice a diagonal of A. We can summarize these conditions by saying that the Trotter form is even and unimodular; recall that unimodularity means that $det(A + A^T)$ is invertible (here equal to ± 1). The form is even if $x(A + A^T)x^T$ is always an even number. Finally, every even unimodular form over Z has its signature divisible by 8; see Theorem II.5.1 in [MH73]. □

9.7 A Combinatorial Formula for the Signature of Alternating Diagrams; Quasi-alternating Links

Corollary 18 has various interesting consequences. P. Traczyk used it back in 1987 [Tra04] to find the combinatorial formula for the signature of alternating links, starting from the analysis of the condition $\sigma(L_+) = \sigma(L_0) - 1$ (and $\sigma(L_-) = \sigma(L_0) + 1$) and observing that it holds for any essential crossing of an alternating diagram. The property was refined by Manolescu, Ozsvath, and Szabo and used to define quasi-alternating links [OS05], whose Khovanov [Kho00] and Heegaard Floer homology share with alternating links many interesting properties [MO, CK] (compare Chap. X of [Prz12]). The precise definition of quasi-alternating links is given in Sect. 9.7.1. The property of link diagrams which Manolescu, Ozsvath, and Szabo observe to be important, and which always holds for alternating links, is the following (compare Sect. 9.1.4):

$$|Det| = |Det| + |Det_{)(}|$$

The following result combines the above properties (compare [MO]):

Theorem 13 *The following two conditions are equivalent, provided that the determinants of L_0 and L_∞ are not equal to zero*[33]

(a) $|Det_{L_+}| = |Det_{L_0}| + |Det_{L_\infty}|$
(b) $\sigma(L_+) = \sigma(L_0) - 1$ *and* $\sigma(L_+) = \sigma(L_\infty) - \frac{1}{2}(w(L_0) - w(L_\infty))$.

A similar equivalence also holds for a negative crossing:

(a′) $|Det_{L_-}| = |Det_{L_0}| + |Det_{L_\infty}|$
(b′) $\sigma(L_-) = \sigma(L_0) + 1$ *and* $\sigma(L_-) = \sigma(L_\infty) + \frac{1}{2}(w(L_0) - w(L_\infty))$.

Proof ((a) $\Leftrightarrow$ (b)): We apply the formula $Det(L) = i^{\sigma(L)}|Det(L)|$ and use the relation between the Jones polynomial, and its Kauffman bracket variant, with the signature. Recall, that the Jones polynomial $V_L(t)$ of an oriented link L is normalized to be one for the trivial knot and satisfies the skein relation $t^{-1}V(t) - tV(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(t)$. For $t = -1$ (or, more precisely, $\sqrt{t} = i$) we obtain exactly the skein relation of the determinant: $Det - Det = -2i\,Det$. Thus $Det_L = V_L(-1)$; $\sqrt{t} = i$. Recall also that the Kauffman bracket polynomial of unoriented link diagrams, $\langle D\rangle \in Z[A^{\pm 1}]$, is defined by the following properties [Kau87a]:

[33] In (a) one deals with a Kauffman skein triple of unoriented links; in (b) one chooses any orientation of L_+ (e.g.) and related orientation of L_0 (), and any orientation of L_∞ (e.g. $)($ or $)($).

(i) $\langle \bigcirc \rangle = 1$
(ii) $\langle \bigcirc \sqcup D \rangle = -(A^2 + A^{-2})\langle D \rangle$
(iii) $\langle \ \rangle = A\langle \ \rangle + A^{-1}\langle \ \rangle$

Furthermore, if $\vec{D}$ is an oriented diagram with underlying unoriented diagram D then $V_{\vec{D}}(t) = (-A^3)^{w(\vec{D})}\langle D \rangle$. Thus for $A^2 = -i$ ($A^4 = -1$) we get:

$$Det(D) = (-A^3)^{-w(D)}\langle D \rangle = A^{w(D)}\langle D \rangle.$$

Recursive formula for the Kauffman bracket $\langle D_+ \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle$ leads to

$$(-A^3)^{w(D_+)} Det(D_+) = A(-A^3)^{w(D_0)} Det(D_0) + A^{-1}(-A^3)^{w(D_\infty)} Det(D_\infty)$$

then leads to $A^{-w(D_+)} Det(D_+) = A^{1-w(D_0)} Det(D_0) + A^{-1-w(D_\infty)} Det(D_\infty)$, then leads to

$$A^{-w(D_+)} i^{\sigma(D_+)} |Det(D_+)| = A^{1-w(D_0)} i^{\sigma(D_0)} |Det(D_0)| \\ + A^{-1-w(D_\infty} i^{\sigma(D_\infty} |Det(D_\infty)|$$

and eventually to

$$\begin{aligned} &|Det(D_+)| \\ &\quad = A^{w(D_+)-w(D_0)+1} i^{\sigma(D_0)-\sigma(D_+)} |Det(D_0)| \\ &\qquad + A^{w(D_+)-w(D_\infty)-1} i^{\sigma(D_\infty)-\sigma(D_+)} |Det(D_\infty)| \\ &\quad = i^{\sigma(D_0)-\sigma(D_+)-1} |Det(D_0)| + i^{\sigma(D_\infty)-\sigma(D_+)-1/2(w(D_0)-w(D_\infty))} |Det(D_\infty)|. \end{aligned}$$

When we compare this formula with that of Theorem 13(a) we see that (a) holds iff $i^{\sigma(D_0)-\sigma(D_+)-1} = 1$ and $i^{\sigma(D_\infty)-\sigma(D_+)-1/2(w(D_0)-w(D_\infty))} = 1$ and these conditions are equivalent to conditions $\sigma(D_0) - \sigma(D_+) \equiv 1 \mod 4$ and $\sigma(D_\infty) - \sigma(D_+) - \frac{1}{2}(w(D_0) - w(D_\infty)) \equiv 0 \mod 4$. These conditions are equivalent to (b) because by Corollary 5(i), we have generally that $|\sigma(D_+) - \sigma(D_0)| \leq 1)$. Furthermore, in general, we have that $|\sigma(D_+) - \sigma(D_\infty) + \frac{1}{2}(w(D_0) - w(D_\infty)| \leq 2$. The last inequality requires some explanation and consideration of two cases in which is either a mixed crossing or a self-crossing.

(m) If is a mixed crossing then let D_j be a component of D_+ such that the change of the orientation of D_j results in the link $D'_- =$. Then by Corollary 5 $|\sigma(\ ) - \sigma(\ )| \leq 1$. Further, by Proposition 2.11(ii), $|\sigma(\ ) + 2lk(D_j, D_+ - D_j) - \sigma(\ )| \leq 1$. Because $4lk(D_j, D_+ - D_j) = w(D_+) - w(D'_-) = w(\ ) - w(\ ) + 2$ we obtain $|\sigma(D_+) - \sigma(D_\infty) + \frac{1}{2}(w(D_0) - w(D_\infty)) + 1| \leq 1$ and finally $-2 \leq \sigma(D_+) - \sigma(D_\infty) - \frac{1}{2}(w(D_0) - w(D_\infty)) \leq 0$.

(s) If is a self-crossing then in $D_0 =$ the two parallel arcs belong to different link components. Let D_j be the component that contains the lower arc and let $D_0' =$ be obtained from D_0 by changing the orientation of D_j. After performing the second Reidemeister move on D_0' we obtain a diagram which has two mixed crossings. We use Corollary 5(i) on one of them to get $|\sigma(\;) - \sigma(\;)| \leq 1$. Because $\sigma(D_0') = \sigma(D_0) + 2lk(D_j, D_0 - D_j) = \sigma(D_0) - \frac{1}{2}(w(D_0) - w(D_0')) = \sigma(D_0) - \frac{1}{2}(w(D_0) - w(D_\infty))$, we obtain $|\sigma(\;) - \sigma(\;) + \frac{1}{2}(w(\;) - w(\;))| \leq 1$, then $|(\sigma(\;) - \sigma(\;)) + (\sigma(\;) - \sigma(\;) + \frac{1}{2}(w(\;) - w(\;)))| \leq 1$, and finally $|\sigma(\;) - \sigma(\;) + \frac{1}{2}(w(\;) - w(\;))| \leq 2$ as required.

The equivalence $(a') \Leftrightarrow (b')$ follows from (a) $\Leftrightarrow$ (b) by considering mirror images of diagrams from (a) and (b). In particular, for the diagram $\bar{D}$ being the mirror image of D, we always have that $\sigma(\bar{D}) = -\sigma(D)$, and $w(\bar{D}) = -w(D)$). $\square$

It is not difficult to see that any crossing of an alternating diagram satisfies properties (a), (a′) of Theorem 13. This follows from the fact that if D is an alternating diagram then also D_0 and D_∞ are alternating, and for an alternating diagram $|Det_D|$ can be interpreted as the number of spanning trees of the underlying Tait graph, $G_b(D)$, and the number of spanning trees is additive under deleting contracting rule; see Sect. 9.1.4. These ideas are developed in Chap. V of [Prz12]. Without referring to it, the properties (a) and (a′) of alternating links follow from the proof of Traczyk's formula for the signature of alternating diagrams which we present below. First, we have to recall the necessary terminology. In fact, we use this as an opportunity for introducing some basic language that unifies the notion of Tait surface and Tait graph (Footnote 14) with that of Seifert surface and Seifert graph [Cro89]. Before the general definition let us recall the definition of the Seifert graph.

Definition 18 [Cro89] The *Seifert graph* of an oriented diagram $\vec{D}$ is a signed (planar) graph $\Gamma(\vec{D})$ whose vertices correspond to Seifert circles of the diagram and whose edges correspond to crossings of the diagram. The sign of an edge is determined by the sign of the corresponding crossing.

In the more general setting we allow arbitrary smoothings of crossings of (not necessarily oriented) diagram D.

Definition 19 A *Kauffman state* s of D is a function from the set of crossings of D to the set $\{+1, -1\}$. Equivalently, we assign to each crossing of D a marker according to the following convention:

By D_s we denote the system of circles in the diagram obtained by smoothing all crossings of D according to the markers of the state s, Fig. 9.38. $|D_s|$ denotes the number of circles in D_s.

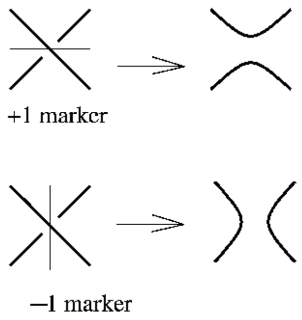


Fig. 9.38 Markers and associated smoothings

In this notation the Kauffman bracket polynomial of D is given by the state sum formula:

$$\langle D\rangle = \sum_s A^{\sigma(s)}(-A^2 - A^{-2})^{|D_s|-1},$$

where $\sigma(s) = \sum_p s(p)$ is the number of positive markers minus the number of negative markers in the state s.

The state sum formula looks like a useful but not necessarily sophisticated tool. However, state sums (and their limits) are basic and deep concepts in statistical physics and very likely the next breakthrough in Knot Theory (and more) will utilize a connection (still to be discovered) between phase transition of a physical system and Khovanov type homology based on closeness of states of the system (possibly persistent homology [EH] will play a role here).

But we are straying too far from our local goal of associating graphs and surfaces to any Kauffman state s.

Definition 20 [PPS09]

(i) Let D be a diagram of a link and s its Kauffman state. We form a *state graph*, $G_s(D)$, associated to D and s as follows. Vertices of $G_s(D)$ correspond to circles of D_s. Edges of $G_s(D)$ are in bijection with crossings of D and an edge connects given vertices if the corresponding crossing connects circles of D_s corresponding to the vertices. As in the case of the Tait graph, $G_s(D)$ is a signed graph where the sign of an edge $e(p)$ is $s(p)$, that is the sign of the marker of the Kauffman state s at the crossing p.

(ii) In the language of associated graphs we can state the definition of an s-adequate diagram as follows: the diagram D is *s-adequate* if the graph $G_s(D)$ has no loops (adequacy is studied and utilized in Chap. V of [Prz12]).

(iii) We associate with every Kauffman state s of a diagram D, a *state surface* $F_s(D)$ embedded in R^3 and with $\partial F_s(D) = D$, in a manner similar to Construction 5 of a Seifert surface. That is, we start from the collection of circles D_s. Each of the circles bounds a disk in the projection plane. We make the disks disjoint by pushing them slightly up above the plane of projection, starting from the innermost disks. We connect the disks together at the original crossings of the diagram D by half-twisted bands so that the 2-manifold which we obtain has D as its boundary, see Fig. 9.27 (we ignore orientation of the diagram, and the resulted surface can be unorientable). Equivalently, we can start a construction of $F_s(D)$ from the graph $G_s(D)$ as a spine (strong deformation retract) of the constructed surface and proceed as follows: The graph $G_s(D)$ possesses an additional structure, that is a cyclic ordering of edges at every vertex following the ordering of crossings at any circle of D_s. The sign of each edge is the label of the corresponding crossing. In short, we can assume that $G_s(D)$ is a ribbon (or framed) graph, and that with every state we associate a surface $F_s(G)$ whose core is the graph $G_s(D)$. $F_s(G)$ is naturally embedded in R^3 with $\partial F_s(G) = D$. If s is the state separating black regions of checkerboard coloring of $R^2 - D$ then $F_s(G)$ is the Tait surface of the diagram described in Exercise 2. For $s = \vec{s}$, that is, D is oriented and markers of $\vec{s}$ agree with orientation of D, $G_s(D)$ is the Seifert graph of D and $F_s(G)$ is the Seifert surface of D obtained by Seifert construction. We do not use this additional data in this survey but it may be of great use in analysis of Khovanov homology (compare [AP01] or Chap. X of [Prz12]).

The surface $F_s(G)$ is not the only surface associated with the graph $G_s(D)$. Another such surface is *Turaev surface*, $M(s)$ [Tur87], which for positive (s_+) or negative (s_-) states of an alternating diagram is a planar surface. With some justification Turaev surface can be called a background surface of a diagram. The construction of $M(s)$ for a given state s of D is illustrated, after [Tur87], in Fig. 9.39. That is, $M(s)$ is obtained from a regular neighborhood of a projection of a link by modifying (by half-twists) neighborhoods of s-wrong edges (see Fig. 9.39 and compare it to Fig. 9.8 to see that any alternating diagram has only s_+-true edges). Notice that $M(s)$ depends on s and the link projection but not on over-under information of a link diagram. Alternatively, we can say that $M(s)$ is a surface realizing the natural cobordism between circles of D_s and circles of D_{-s}. In [DFK+08] the Turaev genus of a link is defined to be the minimal genus of Turaev surface over all diagrams D of a link with $s_+(D)$ states. The immediate consequence is that the alternating link has the Turaev genus equal to zero. Notice also, that if we cup off the circles of D_s in $M(s)$ by 2-discs we obtain the surface $M^+(s)$ with boundary D_{-s} and the graph $G_s(D)$ as its spine.

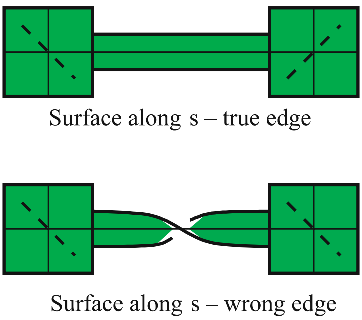


Fig. 9.39 Turaev surface $M(s)$ is composed of squares along every crossing of D connected by ribbons according to convention illustrated in this figure. s-true edge and s-wrong edge are arcs of the diagram D connecting crossings and the name depends on the label given by s to boundary crossings [Tur87]

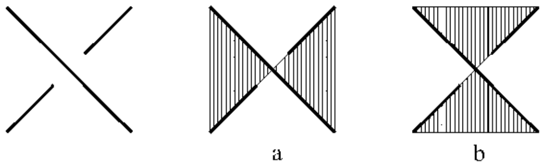


Fig. 9.40 Checkerboard shading of the plane of the projection: (**a**) Tait's, (**b**) dual to Tait's

Going back to Traczyk's combinatorial formula, we recall the convention for checkerboard shading of the projection plane. In an alternating diagram we choose the *standard shading* as in Fig. 9.40(a) complementary to the shading given in Fig. 9.40(b) (this essentially agrees with Tait's convention of checkerboard coloring, however we do not assume that the outside region is white or black).

We denote by B the number of black (shaded) areas and by W the number of white areas (for an alternating diagram D we have $B = |D_{s_-}|$ and $W = |D_{s_+}|$). Furthermore for an oriented diagram $\vec{D}$ let $\Gamma(\vec{D})$ denote its Seifert graph (Definition 18), T its (signed) spanning tree and $d_+(T)$ (resp. $d_-(T)$) the number of positive (resp. negative) edges in T. For an alternating diagram the numbers $d_+(T), d_-(T)$ do not depend on T so we can write $d_+(\vec{D})$, and $d_-(\vec{D})$ in this case[34]

Lemma 11 *If $\vec{D}$ is an oriented connected alternating diagram of a link then*

$$\frac{1}{2}(w(\vec{D}) + |D_{s_+}| - |D_{s_-}|) = d_+(\vec{D}) - d_-(\vec{D})$$

In particular, the left hand side of the equation is unchanged when one goes from $\vec{D}$ to $\vec{D}_0^p$ for a non-nugatory[35] *crossing p (in $\vec{D}_0^p$ the crossing p is smoothed according to orientation of $\vec{D}$).*

Proof One can easily prove Lemma 11 by induction on the number of non-nugatory crossings of $\vec{D}$. First one observes that if $\vec{D}$ has only nugatory crossings then $\Gamma(\vec{D})$ is a tree and $d_+(\vec{D}) = c_+(\vec{D}) = s_+(\vec{D}) - 1$ (and $d_-(\vec{D}) = c_-(\vec{D}) = s_-(\vec{D}) - 1$), thus the formula in Lemma 11 holds. In the inductive step we consider a non-nugatory crossing p of $\vec{D}$ and compare ingredients of the formula for $\vec{D}$ and $\vec{D}_0^p$, and having the formula for $\vec{D}_0^p$ deduct it for $\vec{D}$. It is worth it however to compare d_+, d_-, c_+, c_-, $|D_{s_+}|$, and $|D_{s_-}|$) in more detail. $\square$

[34]This is the case for more general class of homogeneous diagrams introduced in [Cro89] and defined as diagrams for which 2-connected components of the Seifert graph have all edges of the same sign (i.e. they are homogeneous). Alternating diagrams are special cases of homogeneous diagrams; this well known fact follows also from Lemma 11 as the lemma can be proved for a fixed choice of a spanning tree and the left side of the equation does not depend on the choice of a spanning tree.

[35]The crossing p of D is called nugatory if D_0^p has more (graph) component from D.

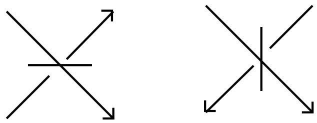


Fig. 9.41 $\vec{s}(p) = s_+(p)$ if $sgn(p) = 1$, and $\vec{s}(p) = s_-(p)$ if $sgn(p) = -1$

Lemma 12 *Let p be any crossing of an oriented diagram $\vec{D}$. Then*

(i)

$$\vec{s}(p) = \begin{cases} s_+(p) & \textit{if } p \textit{ is positive} \\ s_-(p) & \textit{if } p \textit{ is negative} \end{cases}$$

In particular if $\vec{D}$ is a positive diagram then $\vec{s} = s_+$, and if $\vec{D}$ is a negative diagram then $\vec{s} = s_-$.

(ii) $|(\vec{D}_0^p)_{\vec{s}}| = |\vec{D}_{\vec{s}}|$,

(iii)

$$|(\vec{D}_0^p)_{s_+}| = \begin{cases} |\vec{D}_{s_+}| & \textit{if } p \textit{ is positive} \\ |\vec{D}_{s_+}| - \varepsilon_+ & \textit{if } p \textit{ is negative} \end{cases}$$

$$|(\vec{D}_0^p)_{s_-}| = \begin{cases} |\vec{D}_{s_-}| - \varepsilon_- & \textit{if } p \textit{ is positive} \\ |\vec{D}_{s_-}| & \textit{if } p \textit{ is negative} \end{cases}$$

Here ε_+ and ε_- are $+1$ or -1. If p is a non-nugatory crossing of an alternating diagram then $\varepsilon_+ = \varepsilon_- = 1$.

Proof (i) The proof is illustrated in Fig. 9.41.

The other parts are equally elementary and we leave them as exercises for the reader. □

Lemma 13 *If D is a connected alternating diagram, then for a complex number A such that $A^4 = -1$, we have:*

(i) $\langle D \rangle_{A^4=-1} = A^{B-W} |\langle D \rangle_{A^4=-1}|$

(ii) *For any crossing p of an alternating diagram D one has:*

$$|\langle D \rangle_{A^4=-1}| = |\langle D_0^p \rangle_{A^4=-1}| + |\langle D_\infty^p \rangle_{A^4=-1}|$$

in other words the absolute value of the determinant of a diagram is additive under the Kauffman bracket skein triple.

Proof If all crossings of D are nugatory, then D represents the trivial knot. Choose an orientation of D. The orientation defines signs of crossings, which are independent on chosen orientation. As we noticed in the proof of Lemma 11 in this case $c_+ = W - 1$ and $c_- = B - 1$. Thus $\langle D \rangle = (-A^3)^{w(D)} = (-A^3)^{W-B}$ (for a knot $w(D)$ does not depend on the orientation of D). For $A^4 = -1$, $\langle D\rangle_{A^4=-1} = (-A^4)^{W-B}(A)^{B-W} = A^{B-W}$ as required. The inductive step follows easily: If p is a non-nugatory crossing of D, then from the Kauffman bracket skein relation

$$\langle D\rangle = A\langle D_0\rangle + A^{-1}\langle D_{0+}\rangle$$

and by the inductive assumption, for $A^4 = -1$, it follows that:

$$\begin{aligned}\langle D\rangle_{A^4=-1} &= AA^{B-W-1}|\langle D_0^p\rangle_{A^4=-1}| + A^{-1}A^{B-W+1}|\langle D_\infty^p\rangle_{A^4=-1}| \\ &= A^{B-W}\left(|\langle D_0^p\rangle_{A^4=-1}| + |\langle D_\infty^p\rangle_{A^4=-1}|\right) = A^{B-W}|\langle D\rangle_{A^4=-1}|\end{aligned}$$

which completes the proof of Lemma 13(i). It also establishes Lemma 13(ii) for a non-nugatory crossing p of a connected diagram D. If p is a nugatory crossing, then $\langle D_0^p\rangle_{A^4=-1}|$ or $|\langle D_\infty^p\rangle_{A^4=-1}|$ is equal to zero and (ii) holds immediately. If D is not a connected diagram then (ii) holds for any connected component of D and (ii) follows because Kauffman bracket (and signature) is multiplicative under disjoint sum. □

As a corollary of Theorem 13, Lemma 11, and Lemma 13, we have Traczyk's result.

Theorem 14 [*Tra*04] *If D is a reduced*[36] *alternating diagram of an oriented link, then*

(1) $\sigma(D) = -(c_+ - c_-) + d_+ - d_- = -w + d_+ - d_-$
(2) $\sigma(D) = -\frac{1}{2}(c_+ - c_-) + \frac{1}{2}(W - B) = -\frac{1}{2}w + \frac{1}{2}(W - B) = -\frac{1}{2}(w + |D_{s_+}| - |D_{s_-}|)$
(3) $\sigma(D) = \sigma(D_0^p) - sign(p)$

9.7.1 Quasi-alternating Links

Quasi-alternating links were introduced by Ozsvath, and Szabo in [OS05] and further studied by Manolescu-Ozsvath, Champanerkar-Kofman [MO, CK]. Their definition is motivated by properties (a), (a′) of Theorem 13, described in the theorem relations to signature, and applications of these properties to the thinness of Khovanov and Heegaard Floer homology:

Definition 21 [OS05] The family of *quasi-alternating links* is the smallest family of links which satisfies:

(i) The trivial knot is quasi-alternating.
(ii) If L is a link which admits a crossing such that

[36]*Reduced* means that no crossing of D is nugatory.

(1) both smoothings (L_0 and L_∞) are quasi-alternating, and
(2) $|Det_L| = |Det_{L_0}| + |Det_{L_\infty}|$,
then L is quasi-alternating.

The crossing used in the definition is called a quasi-alternating crossing of L.

Notice that a split link has its determinant equal to 0 so it cannot be quasi-alternating (the determinants of quasi-alternating links are always positive as easily follows by induction from Definition 21). Therefore, we can use condition (b) of Theorem 13 as an alternative definition of the family of quasi-alternating links.

One can ask why we choose condition (2) in the definition of quasi-alternating links and not a weaker first part of conditions (b), (b′) from the Theorem 13 ($\sigma(D_+) = \sigma(D_0) - 1$ or $\sigma(D_-) = \sigma(D_0) + 1$). The first answer is purely practical: this is exactly what is needed to have thin Khovanov (and Heegaard Floer) homology (see Chap. X of [Prz12]). One can also argue that a condition which refers only to unoriented links is sometimes a plus.

We already have proved that non-split alternating links satisfy properties which make them quasi-alternating: if D is an alternating diagram then also D_0 and D_∞ are alternating, and every non-nugatory crossing of an alternating diagram is quasi-alternating (satisfies property (ii)(2)) as long as D is a non-split link.

According to [MO] among the 85 prime knots with up to nine crossings, 82 are quasi-alternating (71 are alternating), 2 are not quasi-alternating (8_{19} and 9_{42}), and the knot 9_{46} still remains undecided. It was later showed by A. Schumakovitch using odd Khovanov homology that 9_{46} is not quasi-alternating. The classification of quasi-alternating knots up to 11 crossings was completed by J. Greene in [Gre].

It was also determined which pretzel links are quasi-alternating:

Theorem 15 ([CK, Gre] Characterization of quasi-alternating pretzel links) *The pretzel link* $P_{(1,\dots,1,p_1,\dots,p_n,-q_1,\dots,q_m)}$ *with* e 1*th*, $e+n+m \geq 3$, *and* $p_i \geq 2$, $q_i \geq 3$ *is quasi-alternating if and only if one of the conditions below holds*:

(1) $e \geq m$,
(2) $e = m - 1 > 0$,
(3) $e = 0, n = 1$, *and* $p_1 > min(q_1, \dots, q_m)$,
(4) $e = 0, m = 1$, *and* $q_1 > min(p_1, \dots, p_n)$,

The same is true on permuting parameters[37] p_i *and* q_j.

Partial classification of quasi-alternating Montesinos links is advanced in [CK, Gre, JS, Wid09].

The importance of quasi-alternating links rests on the following results of Manolescu and Ozsvath:

(1) Quasi-alternating links are Khovanov homologically σ-thin (over Z).
(2) Quasi-alternating links are Floer homologically σ-thin (over Z_2).

[37] Thus all pretzel links are covered in the theorem.

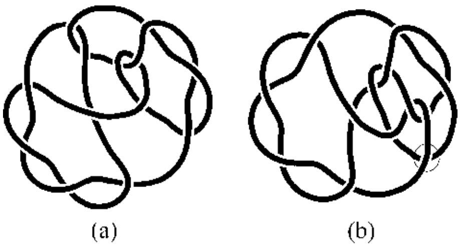


Fig. 9.42 A quasi-alternating knot $13_{n_{1659}}$ with 2 diagrams of (minimal number) 13 crossings. The first diagram is (Conway) algebraic but no crossing is quasi-alternating. In the second diagram, based on Conway's polyhedron 6*, the *circled crossing* is quasi-alternating. The determinant of $13_{n_{1659}}$ is equal to 51 while smoothings of the quasi-alternating crossing gives the trivial knot and a quasi-alternating link with the determinant equal to 50, [JS]

We explain the meaning of the first result in Chap. X of [Prz12] showing also how to generalize it to Khovanov homologically k-almost thin links.

To have some measure of complexity or depth of quasi-alternating links we introduce the *quasi-alternating computational tree index* $QACTI(L)$, defined inductively from the definition of quasi-alternating link as follows:

Definition 22 For the trivial knot T_1, $QACTI(T_1) = 0$. $QACTI(L)$ is the minimum over all quasi-alternating crossings p (of any diagram) of L of $max(QACTI(L_0^p), QACTI(L_\infty^p)) + 1$.

In other words, $QACTI(L)$ is the minimal depth of any binary computational resolving tree of L using only quasi-alternating crossings and having the trivial knot in leaves.

From Definitions 21 and 22, and Theorem 13 we get an approximation on $QACTI(L)$:

Corollary 21 *Let L be a quasi-alternating link then*:

(i) $|Det(L)| - 1 \geq QACTI(L) \geq \log_2(|Det(L)|)$
(ii) $QACTI(L) \geq |\sigma(\vec{L})|$, *for every orientation of* L.
(iii) *If* p *is a quasi-alternating crossing of* L *then*
$QACTI(L) \leq QACTI(L_0^p) + 1$, *and* $QACTI(L) \leq QACTI(L_\infty^p) + 1$.

Let us finish this survey with a nice example of a quasi-alternating knot of 13 crossings due to S. Jablan and R. Sazdanovic [JS]; see Fig. 9.42.

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Chapter 10
An Overview of Property 2R

Martin Scharlemann

Abstract The celebrated Property R Conjecture, affirmed by David Gabai (J. Differ. Geom. 26:461, 1987), can be viewed as the first stage of a sequence of conjectures culminating in what has been called the Generalized Property R Conjecture. This conjecture is relevant to the study of outstanding problems in both 3-manifolds (specifically, links in S^3) and 4-manifolds (specifically, the Schoenflies Conjecture and the smooth Poincare Conjecture). Here we give an overview of part of forthcoming work of R. Gompf, A. Thompson and the author which considers the next stage in such a progression, called the Property 2R Conjecture.

It is shown that the lowest genus counterexample (if any exists) cannot be fibered. Exploiting Andrews-Curtis type considerations on presentations of the trivial group, it is argued that one of the simplest possible candidates for a counterexample, the square knot, probably is one. This suggests there is a genus one counterexample, though we have so far been unable to identify it. Finally, we note that the counterexample need not be an obstacle to the sort of 4-manifold consequences towards which the Generalized Property R Conjecture is aimed.

10.1 Generalizing Property R

A major development in knot theory during the 1980's was David Gabai's proof of the Property R theorem [Gab87]:

Theorem 1 (Property R) *If* 0*-framed surgery on a knot* $K \subset S^3$ *yields* $S^1 \times S^2$ *then* K *is the unknot.*

Research partially supported by National Science Foundation grants.

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_10,

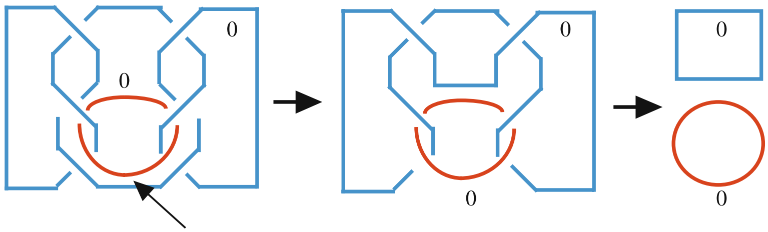


Fig. 10.1 Unlinking after a handle slide

In the ensuing quarter century some effort has been made to sensibly generalize this conjecture, though very little public progress has been made. It is a particularly provocative conjecture because it is relevant to problems in both dimension 3 and dimension 4. Here we give a brief outline of some forthcoming results by R. Gompf, A. Thompson and the author on the question [GST], results that followed a 2007 meeting arranged by Mike Freedman at Microsoft's Station Q. Proofs can be found in [GST].

There is a plausible way of trying to generalize Theorem 1 to links in S^3, but for more than one component so-called handle-slides are required. (The terminology is motivated by a related 4-dimensional picture.) Suppose U and V are two components of a framed link $L \subset S^3$. A handle-slide of U over V changes L to the link obtained by replacing U with a band sum $\overline{U}$ of U and a copy of V that has been pushed off of V by its framing.

Let $\#_n(S^1 \times S^2)$ denote the connected sum of n copies of $S^1 \times S^2$. The Generalized Property R conjecture (see [Kir97, Problem 1.82]) says this:

Conjecture 1 (Generalized Property R) Suppose L is an integrally framed link of $n \geq 1$ components in S^3, and surgery on L via the specified framing yields $\#_n(S^1 \times S^2)$. Then there is a sequence of handle slides on L that converts L into a 0-framed unlink.

Framing is not an issue: an elementary homology argument shows that any candidate must have framing 0 on all components (and also linking number 0 between any pair of components.) In the case $n = 1$ no slides are possible, so Conjecture 1 does indeed directly generalize Theorem 1. On the other hand, for $n > 1$ it is certainly necessary to include the possibility of handle slides. Figure 10.1 shows that 0-framed surgery on a certain link of the unknot with the square knot creates $\#_2(S^1 \times S^2)$. In a similar spirit, Fig. 10.2 shows that even more complicated such framed links are easily created.

There is an immediate topological restriction on the link itself (see [Hil81, Theorem 2]) a restriction that hints at the connection with 4-dimensional problems.

Proposition 1 (Hillman) *Suppose L is a framed link of $n \geq 1$ components in S^3, and surgery on L via the specified framing yields $\#_n(S^1 \times S^2)$. Then L bounds a collection of n smooth* 2*-disks in a* 4*-dimensional homotopy ball bounded by S^3.*

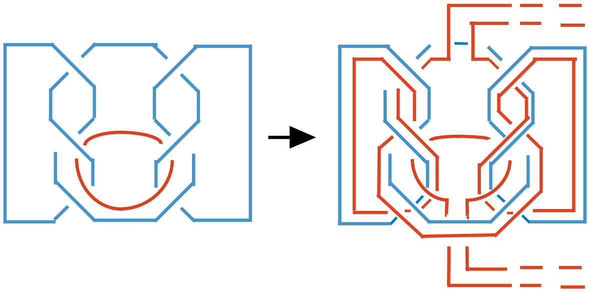


Fig. 10.2 Complicating the link by a handle slide

An equivalent way of stating the conclusion, following Freedman's proof of the 4-dimensional topological Poincare Conjecture [Fre82], is that L (and so each component of L) is topologically slice in B^4.

The Generalized Property R Conjecture is a conjecture about framed links, but if we include in the conjecture the number of components and state it somewhat obliquely, it can be viewed as a sequence of conjectures about knots:

Definition 1 A knot $K \subset S^3$ has *Property nR* if it does not appear among the components of any n-component counterexamples to the Generalized Property R conjecture.

Conjecture 2 (Property nR Conjecture) All knots have Property nR.

Thus the Generalized Property R conjecture for all n component links is equivalent to the Property nR Conjecture for all knots. Following Proposition 1 any non-slice knot has Property nR for all n. The main focus of our work has been on Property 2R.

10.2 Property 2R

To appreciate the role of handle-slides in the argument it is instructive to consider two very special cases of Property 2R. The first was shown to me by Alan Reid:

Proposition 2 (A. Reid) *Suppose $L \subset S^3$ is a 2-component link with tunnel number 1. If surgery on L gives $\#_2(S^1 \times S^2)$ then L is the unlink of two components.*

Note that handle-slides (the new and necessary ingredient for Generalized Property R) do not arise. In contrast, Fig. 10.1 shows that handle slides are needed in the proof of the following:

Proposition 3 *The unknot has Property 2R.*

That is, if surgery on a framed link of two components in which one component is the unknot gives $\#_2(S^1 \times S^2)$, then after handle-slides the link becomes the 0-framed unlink. The proof shows more: only handle-slides over the unknotted component are needed. That is, the unknotted component does not change during the sequence of handle-slides.

In contrast, the proof of the next result explicitly does require handle slides over *both* components of the link.

Theorem 2 *No smallest genus counterexample to Property* $2R$ *is fibered.*

In other words: Suppose surgery on a framed link of two components gives $\#_2(S^1 \times S^2)$, and one component of the link is a fibered knot U. Then, perhaps after handle-slides, at least one component of the link will have genus less than $genus(U)$. The proof makes use of the central result of [ST], which leads fairly directly to this preliminary observation that is interesting in its own right:

Lemma 1 *Suppose surgery on a framed link of two components* $U, V \subset S^3$ *gives* $\#_2(S^1 \times S^2)$, *and suppose* U *is a fibered knot. Then, perhaps after some slides over* U, *the component* V *lies on a fiber of* U *and the* 0*-framing of* V *in* S^3 *coincides with the framing given by the fiber.*

Following Lemma 1 it is natural to ask what properties V must have in the fiber in order that surgery on the pair U, V gives $\#_2(S^1 \times S^2)$. A surprising application of Heegaard splitting theory gives:

Proposition 4 *Suppose surgery on a framed link of two components* U, $V \subset S^3$ *gives* $\#_2(S^1 \times S^2)$. *Suppose further that*

- U *is a fibered knot*
- V *lies on a fiber* F_- *of* U *and*
- *the framing of* V *by the fiber is the* 0*-framing in* S^3.

Then, for $h : F_- \to F_-$ *the fiber monodromy* , $h(V)$ *can be isotoped off of* V *in the* closed *surface* $F = F_- \cup_\partial D^2$.

The distinction between the isotopy here taking place in the closed surface rather than the original punctured surface F_- could be crucial. For if it were not, the following proposition would guarantee that all genus two fibered knots have Property 2R, and this is regarded as highly unlikely for reasons which we will eventually discuss.

Proposition 5 *Suppose* $U \subset S^3$ *is a fibered knot, with fiber the punctured surface* $F_- \subset S^3$ *and monodromy* $h_- : F_- \to F_-$. *Suppose a knot* $V \subset F_-$ *has the property that* 0*-framed surgery on the link* $U \cup V$ *gives* $\#_2(S^1 \times S^2)$ *and* $h_-(V)$ *can be isotoped to be disjoint from* V *in* F_-. *Then either* V *is the unknot or* $genus(F_-) \neq$ $1, 2$.

In the special case of genus two fibered knots one can further show that, at the same time that $h(V)$ can be isotoped in the closed surface F to be disjoint from V, it will never be isotopic to V itself and, conversely, the properties we have shown suffice to characterize those curves V in the fiber which have the property that surgery on U, V yields $\#_2(S^1 \times S^2)$. That is:

Proposition 6 *Suppose $U \subset S^3$ is a genus two fibered knot and $V \subset S^3$ is a disjoint knot. Then 0-framed surgery on $U \cup V$ gives $\#_2(S^1 \times S^2)$ if and only if after possible handle-slides of V over U,*

1. *V lies in a fiber of U;*
2. *in the closed fiber F of the manifold M obtained by 0-framed surgery on U, $h(V)$ can be isotoped to be disjoint from V;*
3. *$h(V)$ is not isotopic to V in F; and*
4. *the framing of V given by F is the 0-framing of V in S^3.*

We turn to the specific and very simple example of the genus two fibered knot called the square knot Q. It is the connected sum of the right-hand trefoil knot and the left-hand trefoil knot. There are many 2-component links containing Q so that surgery on the link gives $\#_2(S^1 \times S^2)$. Figure 10.1 shows (by sliding Q over the unknot) that the other component could be the unknot; Fig. 10.2 shows (by instead sliding the unknot over Q) that the second component could be quite complicated. It turns out that, up to handle-slides of V over Q, there is an easy description of all two component links $Q \cup V$, so that surgery on $Q \cup V$ gives $\#_2(S^1 \times S^2)$. The critical ingredient in the characterization of V is the collection of properties listed in Proposition 6.

Let M be the 3-manifold obtained by 0-framed surgery on the square knot Q, so M fibers over the circle with fiber the closed genus 2 surface F. There is a simple picture of the monodromy $h : F \to F$ of the bundle M, obtained from a similar picture of the monodromy on the fiber of a trefoil knot, essentially by doubling it [Rol76, Sect. 10.I]:

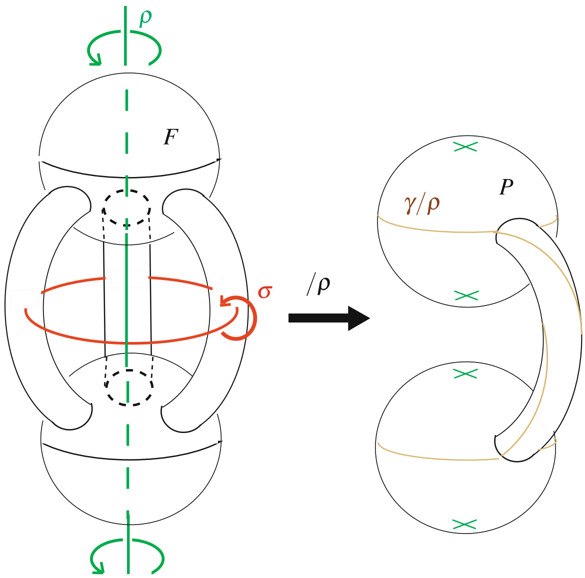
Regard F as obtained from two spheres by attaching 3 tubes between them. See Fig. 10.3. There is an obvious period 3 symmetry $\rho : F \to F$ gotten by rotating $\frac{2\pi}{3}$ around an axis intersecting each sphere in two points, and a period 2 symmetry (the hyperelliptic involution) $\sigma : F \to F$ obtained by rotating around a circular axis that intersects each tube in two points. Then $h = \rho \circ \sigma = \sigma \circ \rho$ is an automorphism of F of order $2 \times 3 = 6$.

The quotient of F under the action of ρ is a sphere with 4 branch points, each of branching index 3. Let P be the 4-punctured sphere obtained by removing the branch points. A simple closed curve in P is essential if and only if it divides P into two twice-punctured disks. It is easy to see [GST] that there is a separating simple closed curve $\gamma \subset F$ that is invariant under σ and ρ, and hence under h, that separates F into two punctured tori F_R and F_L; the restriction of h to F_R or F_L is the monodromy of the trefoil knot. The quotient of γ under ρ is shown as the brown curve in Fig. 10.3.

Here then is the characterization:

Proposition 7 *Suppose $Q \subset S^3$ is the square knot with fiber $F_- \subset S^3$ and $V \subset S^3$ is a disjoint knot. Then 0-framed surgery on $Q \cup V$ gives $\#_2(S^1 \times S^2)$ if and only*

Fig. 10.3 Action of the monodromy



if, after perhaps some handle-slides of V over Q, V lies in F_- and ρ projects V homeomorphically to an essential simple closed curve in P.

Essential simple closed curves $\overline{c}$ in P that are such homeomorphic projections are precisely those for which one branch point of F_L (or, equivalently, one branch point from F_R) lies on each side of $\overline{c}$. So another way of saying that ρ projects V homeomorphically to an essential simple closed curve in P is to say that V is the lift of an essential simple closed curve in P that separates one branch point of F_L (or, equivalently F_R) from the other.

Having established exactly what knots, combined with the square knot, can be surgered to get $\#_2(S^1 \times S^2)$, it would seem to be a straightforward task to show that these links do satisfy the Generalized Property R Conjecture. In fact the story now gets murky, as we try to integrate information from the theory of 4-manifolds.

10.3 The 4-manifold Viewpoint: A Non-standard Handle Structure on S^4

In [Gom91], R. Gompf provided unexpected examples of handle structures on homotopy 4-spheres which do not obviously simplify to give the trivial handle structure on S^4. At least one family is highly relevant to the discussion above. This is example [Gom91, Fig. 1], reproduced here as the left side of Fig. 10.4. (Setting $k = 1$ gives rise to the square knot.) A sequence of Kirby operations in [Gom91, Sect. 2] shows that the resulting 4-manifold has boundary S^3.

We will be interested in the 4-manifold that is the trace of the 2-handle surgeries, the manifold that lies between $\#_2(S^1 \times S^2)$ and S^3. If the 4-manifold is thought

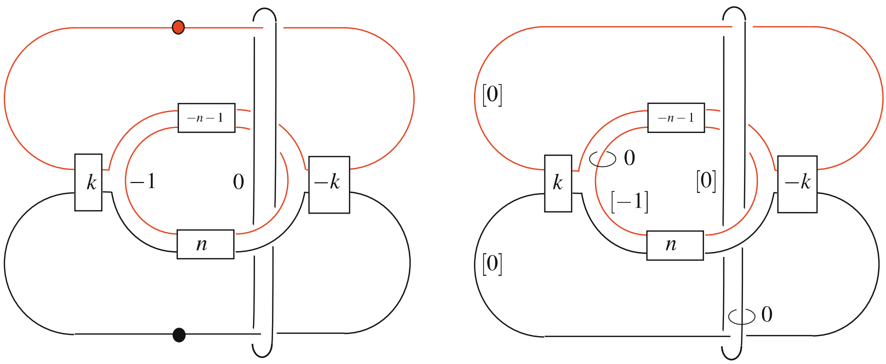


Fig. 10.4 The Gompf examples

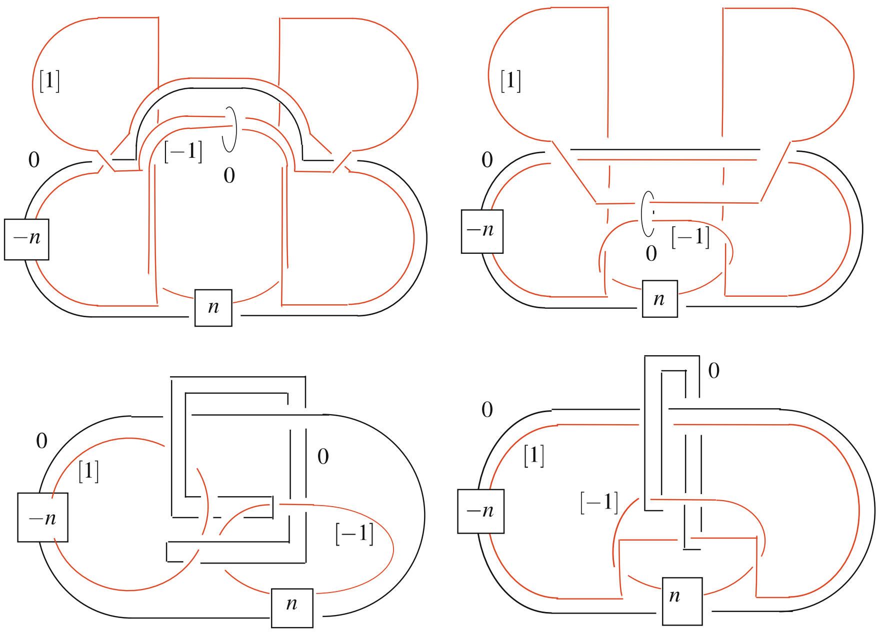


Fig. 10.5 Reconfiguring the Gompf examples

of as starting with S^3 to which two 2-handles are attached to get $\#_2(S^1 \times S^2)$ the construction is solidly in the context of this paper, for the picture becomes a link of two components, one of them the square knot.

Figures 10.5 (clockwise around the figure beginning at the upper left) and 10.6 show the end of the process; the middle 0-framed component becomes the square knot $Q \subset S^3$. (The other component becomes an interleaved connected sum of two torus knots, $V_n = T_{n,n+1} \# \overline{T_{n,n+1}}$.)

Two natural questions arise:

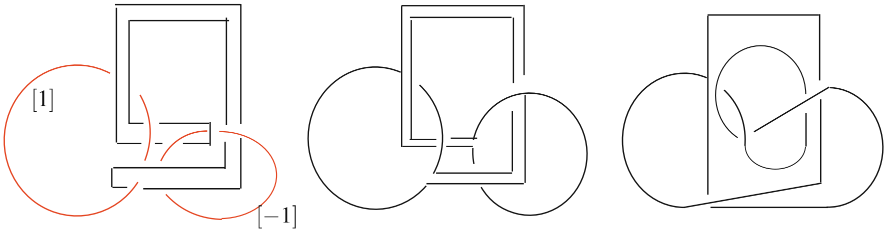


Fig. 10.6 Locating the square knot in the link

Question 1 As described, V_n does not obviously lie on a Seifert surface for Q. According to Corollary 7, some handle slides of V_n over Q should alter V_n so that it is one of the easily enumerated curves that do lie on the Seifert surface, in particular it would be among those that are lifts of (half of) the essential simple closed curves in the 4-punctured sphere P. Which curves in P represent V_n for some n?

Question 2 Is each $Q \cup V_n, n \geq 3$, a counter-example to Generalized Property R?

This second question is motivated by Fig. 10.4. As described in [Gom91], the first diagram of that figure exhibits a simply connected 2-complex, presenting the trivial group as

$$\langle x, y \mid y = w^{-1}xw,\ x^{n+1} = y^n \rangle,$$

where w is some word in $x^{\pm 1}$, $y^{\pm 1}$ depending on k and equal to yx when $k = 1$. If the 2-component link of Fig. 10.5 (after blowing down the two bracketed circles) can be changed to the unlink by handle slides, then the dual slides in Fig. 10.4 will trivialize that picture, showing that the above presentation is Andrews-Curtis trivial. For $k = 1$, for example, this is regarded as very unlikely when $n \geq 3$. Since surgery on the link is $\#_2(S^1 \times S^2)$ by construction, this suggests an affirmative answer to Question Two, which (for any one n) would imply:

Conjecture 3 The square knot does not have Property 2R.

Although this news from the world of 4-manifolds is both puzzling and perhaps unwelcome, the 4-manifold perspective also suggests a weaker but more awkward version of Generalized Property R which would still provide the sort of 4-manifold results one would hope for:

Conjecture 4 (Weak Generalized Property R) Suppose L is a framed link of $n \geq 1$ components in S^3, and surgery on L yields $\#_n(S^1 \times S^2)$. Then, perhaps after adding a distant r-component 0-framed unlink and a set of s canceling Hopf pairs to L, there is a sequence of handle-slides that creates the distant union of an $n + r$ component 0-framed unlink with a set of s canceling Hopf pairs.

Here a canceling Hopf pair is a Hopf link with one component of the link labeled with a dot and the other given framing 0. The dotted component represents a

1-handle and the 0-framed component represents the attaching circle for a canceling 2-handle. From the 4-manifold point of view adding a canceling Hopf pair makes no difference to the topology of the underlying 4-manifold since it denotes a pair of canceling 1- and 2- handles. But it can destroy the Andrews-Curtis obstruction, since adding a canceling Hopf pair introduces a new relator that is obviously trivial.

Definition 2 A knot $K \subset S^3$ has *Weak Property nR* if it does not appear among the components of any n-component counterexample to the Weak Generalized Property R conjecture.

The Weak Generalized Property R Conjecture is closely related to the Smooth (or PL) 4-Dimensional Poincaré Conjecture, that every homotopy 4-sphere is actually diffeomorphic to S^4. For a precise statement, we restrict attention to homotopy spheres that admit handle decompositions without 1-handles.

Proposition 8 *The Weak Generalized Property R Conjecture is equivalent to the Smooth* 4*-Dimensional Poincaré Conjecture for homotopy spheres that admit handle decompositions without* 1*-handles.*

While there are various known ways of constructing potential counterexamples to the Smooth 4-Dimensional Poincaré Conjecture, each method is known to produce standard 4-spheres in many special cases. (The most recent developments are [Akb, Gom].) Akbulut's recent work [Akb] has eliminated the only promising potential counterexamples currently known to admit handle decompositions without 1-handles. For 3-dimensional renderings of the full Smooth 4-Dimensional Poincaré Conjecture and other related conjectures from 4-manifold theory, see [FGMW].

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Chapter 11
DNA, Knots and Tangles

De Witt Sumners

Abstract The DNA of all organisms has a complex and essential topology. Each cell has a family of naturally occurring enzymes that manipulate cellular DNA in topologically interesting and non-trivial ways in order to mediate the vital cellular life processes of replication, transcription and recombination. In order to assay enzyme binding and mechanism, molecular biologists developed the *topological approach to enzymology*, an experimental protocol in which one reacts small artificial circular DNA substrate molecules with purified enzyme *in vitro* (in the test tube). The enzyme acts on the DNA substrate, causing changes in the geometry (*supercoiling*) and/or topology (*knotting and linking*) of the DNA molecules. Once change in topology of the DNA is known, mathematical analysis can be employed to tease out the structure of the active DNA-protein complex and the changes in that structure due to enzyme mechanism. This paper will describe the *tangle model* and apply it to the case of site-specific DNA recombination.

11.1 Introduction

The DNA of all organisms has a complex and essential topology. The genome of prokaryotes (bacteria) is a single closed duplex DNA circle; the genome of eukaryotes such as ourselves consists of a cell nucleus comprised of distinct chromosomes, and each chromosome is constructed from a single very long linear duplex DNA molecule. Each individual chromosome has an intricate interwound structure where the DNA is reduced in volume by 4–5 orders of magnitude, compared to the volume occupied by free DNA in solution. When the cell needs to access the information

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M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_11,

contained in its DNA, the DNA must be geometrically and topologically manipulated in order to expose parts of the DNA to the proteins that orchestrate vital cellular life functions. At the end of the day, the cell's DNA must be returned to its original state. Each cell has a family of naturally occurring enzymes that manipulate cellular DNA in topologically interesting and non-trivial ways in order to mediate the vital cellular life processes of replication, transcription and recombination. One of these enzymes is *topoisomerase* [Wan02, Wan09], which passes DNA through itself via an enzyme-bridged transient break in the DNA. Another enzyme is *recombinase* [SESC95, GWR06], which binds to duplex DNA at a pair of recombination sites, breaks both sites, and splices each end to the companion end at the other site. In order to assay enzyme binding and mechanism, molecular biologists developed the *topological approach to enzymology*, an experimental protocol in which one reacts small artificial circular DNA substrate molecules with purified enzyme *in vitro* (in the test tube). The enzyme acts on the DNA substrate, causing changes in the geometry (*supercoiling*) and/or topology (*knotting and linking*) of the DNA molecules. The spectrum of changes in DNA supercoiling, knotting and linking caused by enzyme action can be experimentally observed in the reaction products. In order to determine these changes, one separates the DNA reaction products by gel electrophoresis, and visualizes the products by *electron microscopy*. Once the exact topology of the reaction products is known, mathematical analysis is employed to tease out the structure of the active DNA-protein complex and the changes in that complex due to enzyme mechanism. This paper will describe the *tangle model* and apply it to the case of site-specific recombination.

Knot theory is the study of topological entanglement of flexible curves and graphs in 3-space. It is a study of embedding pathology and has proven to be fundamental as a laboratory for the development of algebraic-topological invariants and in the understanding of the topology of 3-manifolds. During the last 100 years, topologists have developed the discrete geometric language of knots to a fine mathematical art [BZ85, Kau87, Rol76]. More recently, the unexpected (coming from Von Neumann algebras and quantum mechanics) discovery by Vaughan Jones [Jon85] of new polynomial invariants that help with the knot classification problem has brought intense attention to the subject spawning that might be called the new *combinatorial* knot theory. Whatever it is, however, knot theory isn't just pure mathematics anymore. It is a prototype of what Lynn A. Steen [Ste88] calls *the science of patterns*–theory built on relations among patterns and on applications derived from the fit between pattern and observation. The precise descriptive and calculational power of knot theory has been put to work in the description and computation of molecular configurations [WC86]. Entanglement in real physical systems has consequences. Leaving aside your frustration at resolving the entanglement in the spaghetti-like mass of computer wires under your desk, unresolved entanglement of DNA in a cell is a death sentence for that cell [BZ04, LZC06, LMZC06, DMSZ07]. Most drugs for the treatment of bacterial infections or cancer work by inhibiting cellular enzymes that resolve molecular entanglement in the cell, and the target cell (a pathogen or cancer cell) dies when it is unable to disentangle its DNA. Knots in viral DNA can give information about the packing geometry of the viral DNA in the protein capsid [ATV+02, AVM+05, AVT+02, MMOS06, MMOS08, MOS+09]. Knots in linear proteins have been discovered in the protein data base [VMK06], and give information about the time course of folding and the protein function, including the role of enzymes in protein degradation. In polymer science, macroscopic properties of polymer systems often depend on microscopic intermolecular entanglement [LBHS86]; entanglement determines whether or not the polymer system is a gel or a polymer fluid, and if a solid, the entanglement has consequences in the strength of the material. In fluid dynamics, plasma and superfliud physics [Mof99, Ric99, Ric01], entanglement of magnetic and vortex filaments have important consequences for the energy of the system.

This paper describes the *tangle model for site-specific recombination* [ES90, SESC95, CWP+99, VS04, VCS05]. The binding and mechanism of many enzymes that operate on DNA involve local (near the enzyme) interaction of a pair of DNA strands and the protein. The mathematics that can be used to model this biological situation is the 2-*string tangle*. One can think of a topological enzymology experiment as happening in the 3-sphere instead of Euclidean 3-space. The globular enzyme protein is a 3-ball, and when bound to two sites in a circular DNA molecule, the spherical surface of the protein separates S^3 into two complementary 2-string tangles. Enzyme action on circular DNA can be thought of as tangle surgery, in which the action of the enzyme is to delete the protein-DNA tangle, and replace it by another protein-DNA tangle. One can regard these protein-DNA tangles as unknowns that describe enzyme binding and mechanism, and the observed topological differences between substrate and product in an equation determine equations relating these tangle unknowns. In general, solving tangle equations is a difficult task. The job is greatly simplified by the observation that most known DNA reaction products lie in the mathematically well-understood class of *4-plats* (*2-bridge knots and links*). Moreover, a great deal can be said about the decomposition of 4-plats into rational tangle summands. Rational tangles are formed by the iterated plectonemic interwinding of a pair of strands, and the structure of rational tangles looks like electron micrographs of DNA. In order to analyze DNA site-specific recombination experiments, one converts the tangle equations to equations involving the 2-fold branched cyclic covers of the tangles. In this setting, one can use the results of Dehn surgery on 3-manifolds, specifically the fact that there are relatively few Dehn surgeries on knots in S^3 that yield lens spaces.

11.2 Topological Enzymology

One of the important issues in molecular biology is the three-dimensional structure (shape) of proteins and their precursors (deoxyribonucleic acid (DNA) and ribonucleic acid (RNA)) in the cell, and the close relationship between macromolecular structure and function. Ordinarily, protein and DNA structure is determined by X-ray crystallography, electron and atomic force microscopy, and nuclear magnetic resonance imaging (NMR). Because of the close packing needed for crystallization, the manipulation required to prepare a specimen for electron or atomic force microscopy, and the lack of resolution of NMR, these methods often do not provide conclusive evidence for molecular shape in solution. Moreover, some proteins (enzymes) function as molecular machines, changing their shape as they execute their function, so one static spatial snapshot may not tell the whole story. Topology can shed light on these issues. The topological approach to enzymology (Fig. 11.5) is an experimental protocol in which the descriptive and analytical powers of topology and geometry are employed in an indirect effort to determine enzyme mechanism and the structure of active enzyme-DNA complexes *in vitro* (in a test tube) and *in vivo* (in the cell). This article will describe how results in 3-manifold topology [Con70, CGLS87, DS97, DS98, DS00, ES90, ES99, Lic81, Sum07, Sum92, Sum95, Sum09, SESC95, Vaz00, VS04, VCS05] have proven to be of use in the description and quantization of DNA structure and the action of cellular enzymes on DNA, and will draw heavily on earlier articles [ES90, Sum07, Sum92, Sum95, Sum09, Sum87, Sum90].

DNA molecules are long and thin, and the packing of DNA into the cell nucleus is very complex. DNA can be viewed as two very long curves that are intertwined millions of times, linked to other curves, and subjected to four or five successive orders of coiling to convert it into a compact form for information storage. If one scales up the cell nucleus to the size of a basketball, the DNA inside scales up to the thickness of thin fishing line, with 200 km of that line inside the nuclear basketball. Most cellular DNA is double-stranded (duplex), consisting of two linear backbones of alternating sugar and phosphorus. Each 5-carbon (pentagonal) sugar molecule in the backbone is attached to the phosphate unit on one side by the carbon atom designated $5'$, and to the phosphate unit on the other side by the carbon atom designated $3'$. Therefore, each backbone chain is endowed with a natural chemical orientation $5' \to 3'$. Attached to each sugar molecule is one of the four nucleotide bases: A = Adenine, T = Thymine, C = Cytosine, G = Guanine. A ladder is formed by hydrogen bonding between base pairs, with A bonding with T via a double hydrogen bond, and C bonding with G via a triple hydrogen bond. The base-pair sequence (or code) for a linear segment of duplex DNA is obtained by reading along one of the two backbones and is a word in the letters $\{A, C, G, T\}$. One can model the duplex DNA molecule as a ribbon R, in which the hydrogen bonds define the ruled surface of the ribbon. The edges of the ribbon R (the backbone strands) are chemically oriented opposite to each other, and the axis of the ribbon is unoriented. In the classical Crick-Watson double helix model for DNA, the backbone strands are twisted in a right-hand helical (plectonemic) fashion with an average and nearly constant pitch of approximately 10.5 base pairs per full helical twist (Fig. 11.1). Biologists often call the backbone edges of the DNA molecule C (for Crick) and W (for Watson)!

Moreover, since backbone bonds can only be formed $3'$ to $5'$, a covalently closed circular duplex DNA molecule is a twisted annulus, not a twisted Mobius band. We will ignore the natural antiparallel chemical orientation of the backbone strands, and adopt the orientation convention that the backbones of the circular DNA molecule are oriented in parallel with the axis, as in Fig. 11.2. Moreover, we will usually ignore the Crick-Watson helical interwinding of the backbone strands, and represent

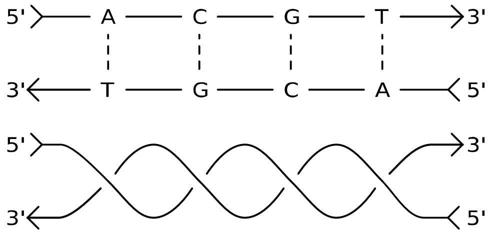


Fig. 11.1 Duplex DNA (from [Sum87])

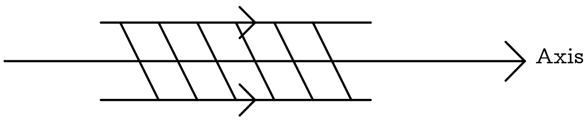


Fig. 11.2 Orientation of DNA ribbon R (from [Sum87])

duplex DNA by the axis alone. The orientation of the axis itself is often biologically determined by short non-palindromic nucleotide sequences present on the backbone strands, for example, enzyme binding sites in site-specific recombination. If no biological orientation is determined, the orientation of the molecular axis or can be arbitrarily assigned in order to facilitate computations of molecular spatial structure.

The local helical pitch of duplex DNA is a function of the base-pair sequence, and if a DNA molecule is under stress or constrained to live on a surface (bound to a protein), the helical pitch can change. Duplex DNA can exist in nature in closed circular form, where the rungs of the ladder lie on a twisted cylinder. Duplex DNA in a chromosome in a eukaryotic cell nucleus is a very long linear molecule, geometrically constrained by periodic attachment to a protein scaffold. The packing, twisting, and topological constraints all taken together mean that topological entanglement poses problems for the DNA molecules in the cell. This entanglement would interfere with, and be exacerbated by, the vital life processes of replication, transcription, translocation and recombination [DMSZ07, CCM+06]. For information retrieval and cell viability, some geometric and topological features must be introduced into the DNA, and others quickly removed [Wan02, Wan09]. For example, the Crick-Watson helical twist of duplex DNA requires local unwinding in order to make room for a protein involved in transcription to attach to the DNA. The DNA sequence in the vicinity of a gene may need to be altered to include a promoter or repressor. During replication, the daughter duplex DNA molecules become entangled and must be disentangled in order for replication to proceed to completion. After a metabolic process is finished, the original DNA conformation must be restored. Some enzymes maintain proper geometry and topology by passing one strand of DNA through another by means of a transient enzyme-bridged break in one of the DNA strands, a move performed by *topoisomerase* enzymes. Other enzymes break the DNA apart and recombine the ends by exchanging them, a move performed by *recombinase* enzymes. The description and quantization of the three-dimensional structure of DNA and the changes in DNA structure due to the action of these enzymes have required the serious use of geometry and topology in molecular biology.

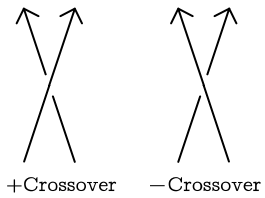


Fig. 11.3 Oriented skew lines sign convention

This use of mathematics as an analytic tool is especially important because there is no experimental way to directly observe the dynamics of enzymatic action.

One of the most useful descriptors of circular DNA ribbon structure comes from the conservation equation [Călı61, Ful71, Whi69] relating the geometric quantities *twist* (of the ribbon) and *writhe* (of the ribbon axis) to the topological quantity *link* (of the ribbon boundary). The computation of linking number and writhe depend on the oriented skew lines sign convention (right-hand rule) for a projected crossing of two oriented skew lines in space (Fig. 11.3).

Given a regular projection (one in which all crossings are transverse intersections of exactly two projected arcs) of two disjoint oriented simple closed curves $\{C, W\}$ in R^3, the linking number $Lk(C, W)$ of C and W is the sum of signed crossings where C crosses over W. The linking number is a topological invariant. Given any regular projection of the ribbon axis, compute the sum of the signed self-crossings of the axis, obtaining the *directional writhe* of the ribbon axis (the writhe in the direction of the projection). By averaging the directional writhe over all projection directions (points on the unit 2-sphere), one obtains $Wr(R)$, the *writhe* of the ribbon axis. Although the directional writhe is an integer, upon averaging, one obtains a real number for the writhe; the writhe is not a topological invariant; it is a geometric measure of non-planarity of the ribbon axis, and can change when the ribbon is moved about in space. The other geometric quantity of interest is the winding of the ribbon boundary curves around the axis of the ribbon. Given a smooth closed ribbon (a twisted annulus), choose an origin on the axis, and one of the two boundary curves C, and parameterize the axis with arc length s. Let $\tau(s)$ denote the unit tangent vector to the axis at position s, and $\nu(s)$ denote the unit perpendicular on the ribbon to $\tau(s)$ at position s, pointing toward the boundary curve C. Let $\tau(s)'\nu(s)$ denote the unit normal to the ribbon at position s. The *twist* of the ribbon R is a real number defined by the integral:

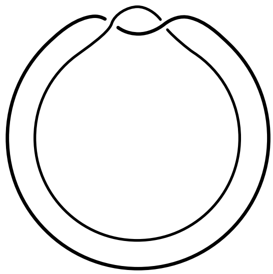
$$Tw(R) = \frac{1}{2\pi} \oint_{\text{axis}} (\tau \times \nu) \cdot \mathrm{d}\nu \tag{11.1}$$

The conservation equation [Călı61, Ful71, Whi69] relating these quantities is:

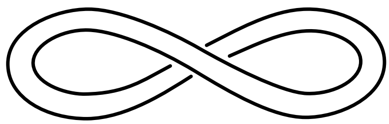
$$Lk(C, W) = Tw(R) + Wr(R) \tag{11.2}$$

Figure 11.4 shows two isotopic configurations of a ribbon. The ribbon of Fig. 11.4(a) has a planar axis, so $Wr(R) = 0$, and one left handed full twist of the ribbon yields $Lk(C, W) = Tw(R) = -1$. In Fig. 11.4(b), the local twist of the ribbon has been converted to a global writhe of the axis of the ribbon, and the ribbon has

Fig. 11.4 Link, twist and writhe of R (from [Sum95])



(a) $Lk(R) = -1, Tw(R) = -1, Wr(R) = 0$



(b) $Lk(R) = -1, Tw(R) = 0, Wr(R) = -1$

no twist, so $Tw(R) = 0$ and $Lk(C, W) = Wr(R) = -1$. The ribbon of Fig. 11.4(a) is said to be relaxed, and the ribbon of Fig. 11.4(b) is said to be negatively supercoiled. Most DNA extracted from living cells is negatively supercoiled [BCW80].

In order to describe and quantify the DNA structure and its enzyme-mediated changes, it is clear that knot theory ought to be of some help. An interesting biological development for topology has been the recent (circa 1983) emergence of a new experimental protocol, the topological approach to enzymology [WC86], which aims to exploit knot theory directly to unravel the secrets of enzyme action. Here's how it works. Focus attention on an enzyme that mediates a local DNA interaction. Because there is at present no direct observational method (either in the cell or in a laboratory) for enzyme action, one must rely on indirect methods. One can use circular DNA as a probe to deduce facts about enzyme mechanism by detecting a topological enzyme signature, the change the enzyme causes in the topological state (embedding) of the molecule upon which it is acting. In many cases, the natural substrate for the enzyme action is linear DNA. The problem for the molecular detective is that linear DNA cannot trap topological changes caused by an enzyme–there can be no interesting (observable) topology (knots) in an unconstrained linear piece of DNA. The trick is to get a particular enzyme to act on circular DNA molecules. This can be done by manufacturing (via cloning) artificial circular DNA molecules on which the enzyme will act. When an enzyme acts on circular DNA molecules, some of the enzymatic changes can be trapped in the form of DNA knots and links. One performs laboratory (*in vitro*) experiments, in which purified enzyme is reacted with a large collection of circular DNA molecules (the substrate). In such experiments, it is possible to control the amount of supercoiling (Fig. 11.4(b)), the knot type, and the linking of the family of substrate molecules. Using a biological technique (rec A coating) to enhance viewing resolution under the electron microscope, one can observe the reaction products, an enzyme-specific family of DNA knots and links (Fig. 11.5).

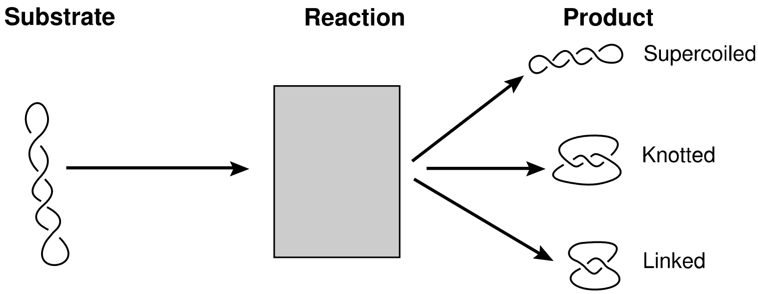


Fig. 11.5 The topological approach to enzymology (from [Sum95])

The topological approach to enzymology poses an interesting challenge for mathematics: from the observed changes in DNA geometry and topology, how can one deduce enzyme binding and mechanism? This requires the construction of mathematical models for enzyme action and the use of these models to analyze the results of topological enzymology experiments. The entangled form of the product family of DNA knots and links contains information about the enzyme that made them. In addition to utility in the analysis of experimental results, the use of mathematical models forces all of the background assumptions about the biology to be carefully laid out. At this point they can be examined and dissected, and their influence on the biological conclusions drawn from experimental results can be determined.

11.3 Site-Specific Recombination

Site-specific recombination is one of the ways in which nature alters the genetic code of an organism, either by moving a block of DNA to another position on the molecule or by integrating a block of alien DNA into a host genome. One of the biological purposes of recombination is the regulation of gene expression in the cell, because recombination can alter the relative position of the gene and its repressor and promoter sites on the genome. Site-specific recombination also plays a vital role in the life cycle of certain viruses, which utilize this process to insert and remove viral DNA into the DNA of a host organism. An enzyme that mediates site-specific recombination on DNA is called a *recombinase*. A *recombination site* is a short (10–15 base pair) segment of duplex DNA whose base pair sequence is recognized by the recombinase. Site-specific recombination can occur when a pair of sites (on the same or on different DNA molecules) become juxtaposed in the presence of the recombinase. The pair of sites is aligned through enzyme manipulation or random thermal motion (or both), and both sites (and perhaps some contiguous DNA) are bound by the enzyme. This stage of the reaction is called *synapsis*, and we will call this intermediate protein-DNA complex formed by the part of the substrate that is bound to the enzyme together with the enzyme itself the *synaptosome*. We will call the entire DNA molecule(s) involved in synapsis (including the parts of the DNA

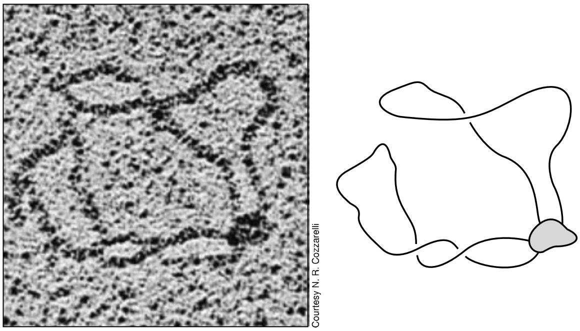


Fig. 11.6 Tn3 resolvase synaptic complex (from [Sum95])

molecule(s) not bound to the enzyme), together with the enzyme itself, the *synaptic complex*. The electron micrograph in Fig. 11.6 (courtesy of N.R. Cozzarelli) shows a synaptic complex formed by the recombination enzyme Tn3 resolvase when reacted with unknotted circular duplex DNA. In the micrograph of Fig. 11.6, the synaptosome is the black mass attached to the DNA circle, with the unbound DNA in the synaptic complex forming twisted loops in the exterior of the synaptosome. It is our intent to look behind the curtain, to deduce mathematically the path of the DNA in the black mass of the globular protein, both before and after recombination. We want to answer the question: How is DNA wound around the enzyme, and what happens during recombination? After forming the synaptosome, a single recombination event occurs: the enzyme performs two double-stranded breaks at the sites and recombines the ends by exchanging them in an enzyme-specific manner. The synaptosome can then dissociate, and the DNA is released by the enzyme. We call the pre-recombination unbound DNA molecule(s) the substrate and the post-recombination unbound DNA molecule(s) the product.

During a single productive binding encounter between enzyme and DNA, the enzyme may mediate more than one recombination event before it dissociates; this is called *processive recombination*. On the other hand, the enzyme may perform recombination in multiple productive binding encounters with the DNA, a scenario called *distributive recombination*. Some site-specific recombination enzymes mediate both distributive and processive recombination; other site-specific recombination enzymes mediate only a single recombination event, and the changed genetic sequence of the product DNA molecules cannot support further recombination events. In addition to changing the genetic sequence of the substrate DNA molecule, site-specific recombination usually changes the topology of the substrate, producing knots, links and supercoiling in the circular molecules. In order to identify these topological changes, one chooses to perform experiments on circular DNA substrate. One must perform an experiment on a large number of circular molecules in order to obtain an observable amount of product. Using cloning techniques, one can synthesize circular duplex DNA molecules, which contain two copies of a recombination site. At each recombination site, the base pair sequence is in general not palindromic hence induces a local orientation on the substrate DNA circle. If these induced orientations from a pair of sites on a singular circular molecule agree, this

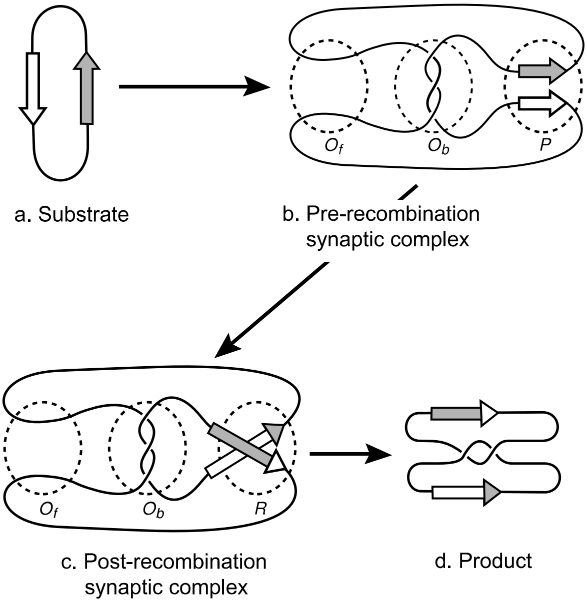


Fig. 11.7 A single recombination event: direct repeats (from [Sum95])

site configuration is called *direct repeats* (or head-to-tail), and if the induced orientations disagree, this site configuration is called *inverted repeats* (or head-to-head). If the substrate is a single DNA circle with a single pair of directly repeated sites, the recombination product is a pair of DNA circles and can form a DNA link (or catenane) (Fig. 11.7). If the substrate is a pair of DNA circles with one site each, the product is a single DNA circle (Fig. 11.7 read in reverse) and can form a DNA knot (usually with direct repeats). In processive recombination on circular substrate with direct repeats, the product of an odd number of rounds of processive recombination is a pair of DNA circles, and the product of an even number of rounds of processive recombination is a single DNA circle with the parental genotype. If the substrate is a single DNA circle with inverted repeats, the product is a single DNA circle and can form a DNA knot. In iterated site-specific recombination on a single circle with inverted repeats, and odd number of rounds of recombination produces a single DNA circle of recombinant genotype, and an even number of rounds of recombination produces a DNA circle of parental genotype. In all figures where DNA is represented by a line drawing (such as Fig. 11.7), the axis of duplex DNA is represented by a single line, and primary Crick-Watson helical structure and molecular supercoiling is omitted.

The geometry (supercoiling) and topology (knotting and linking) of circular DNA substrate molecules are experimental control variables. The geometry and topology of the recombination reaction products are observables. In vitro experiments usually proceed as follows: Circular substrate is prepared, with all of the substrate molecules representing the same knot type (usually the negatively supercoiled unknot). The amount of supercoiling of the substrate molecules is also a control variable. The substrate molecules are reacted in vitro with a high concentration

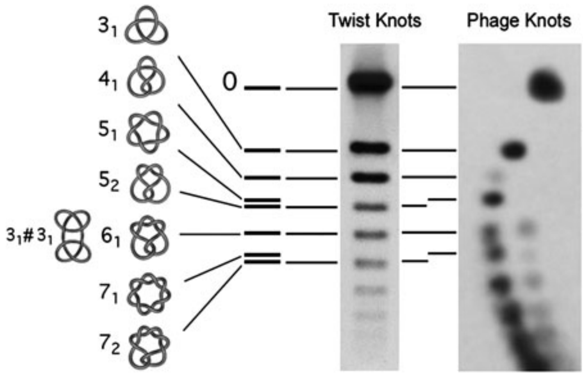


Fig. 11.8 Gel electrophoresis of DNA knots (from [AVM+05])

of purified enzyme, and the reaction products are fractionated by gel electrophoresis. DNA is a polyelectrolyte with negative charge proportional to molecular weight. An agarose gel is a sugar that provides a random obstruction field through the DNA molecules can be forced to migrate in the presence of an applied static electric field. Under the influence of the electric field, the DNA molecules migrate toward the positive electrode. Gel electrophoresis normally discriminates among DNA molecules on the basis of molecular weight, and can discriminate a difference of one base pair. Given that all molecules are exactly the same molecular weight (as is the case in these topological enzymology experiments), electrophoresis discriminates on the basis of subtle differences in the geometry (supercoiling) and/or topology of the DNA molecules. Under the proper conditions of gel density and applied voltage, circular DNA velocity in the gel is (perhaps surprisingly) determined by the crossing number of the knot or link [SKB+96]; knots and links of the same crossing number migrate with the same gel velocities (Fig. 11.8). After running the gel, the DNA molecules can be removed from the gel and coated with Rec A protein. It is this new observation technique (Rec A-enhanced electron microscopy) [KSS+83] that makes possible the detailed knot-theoretic analysis of reaction products. Rec A is an E. coli protein that binds to DNA and mediates general recombination in E. coli. The process of Rec A coating fattens, stiffens, and stretches (untwists) the DNA, but does not change the knot/link type of DNA circles. Rec A coating facilitates the unambiguous determination of crossings (nodes) in an electron micrograph of DNA, allowing precise determination of the DNA knot/link type.

11.4 Knots and Tangles

In this section, we will describe the parts of knot theory and tangle calculus that are of biological relevance. For a more rigorous mathematical treatment we refer the reader to [BZ85, Kau87, Rol76] for knot theory and [ES90] for tangle calculus. A knot K is an embedding of a single circle in R^3; a link L is an embedding of two or more circles in R^3. Unless otherwise specified, all knots and links will be unoriented; the ambient space R^3 is endowed with a fixed orientation. Two knots (links)

K_1, K_2 are *equivalent* (written $K_1 = K_2$) if there is an orientation-preserving (on R^3) homeomorphism of pairs $h : (R^3, K_1) \to (R^3, K_2)$. The homeomorphism of pairs h superimposes K_1 on K_2; in this case the knots (links) can also be made congruent by a flexible motion or flow (ambient isotopy) of space. An *ambient isotopy* is a 1-parameter family of homeomorphisms $\{H_t\}0 \le t \le 1$ of R^3 that begins with the identity and ends with the homeomorphism under consideration: $H_0 = \text{identity}$ and $H_1 = h$. An equivalence class of embeddings is called a *knot* (*link*) *type*. A knot (link) type is usually represented by drawing a diagram (projection) in a plane. This diagram is a shadow of the knot (link) cast on a plane in 3-space, coded with breaks in the undercrossing strand so that the knot (link) type can be unambiguously reconstructed in 3-space from the 2-dimensional knot (link) diagram. The *crossing number* of a knot or link type is the smallest number of crossings possible in a planar diagram. A diagram which realizes the minimum number of crossings for a knot (link) type is called a *minimal diagram*. A knot (link) diagram is alternating if, as one traverses any strand, the crossings encountered are alternately over and under. Figure 11.9 shows minimal alternating diagrams for the knots and links that turn up in Tn3 recombination experiments. In the definition of knot type, we insisted that the transformation that superimposes one knot on another must be orientation-preserving on the ambient space R^3. This restriction allows us to detect a property of great biological significance: *chirality*. If K denotes a knot (link), let $K*$ denote the mirror image. One can convert a diagram of K to a diagram of $K*$ by reversing each of the crossings in the diagram (Fig. 11.9d, e). If $K = K*$, then we say that K is *achiral*; if $K \neq K*$, then we say that K is *chiral*. In Fig. 11.9, (a) and (b) are achiral, and (c), (d) and (e) are chiral.

Fortunately for biological applications, most (if not all) of the circular DNA products produced by *in vitro* enzymology experiments fall into the mathematically well-understood and computationally tractable family of *4-plats* (*2-bridge* knots and links) [Sch56, BZ85]. This family consists of knot and link configurations produced by patterns of plectonemic interwinding of pairs of strands about each other, echoing the primary Crick-Watson helical winding in duplex DNA (Fig. 11.1). All small knots and links are members of the family of 4-plats–more precisely, all prime knots with crossing number less than 8 and all prime (two-component) links with crossing number less than 7 are 4-plats. A 4-plat is a knot or two-component link that can be formed by platting (or braiding) four strings. All of the knots and links in Fig. 11.4 are 4-plats; their standard 4-plat minimal alternating diagrams are shown in Fig. 11.10. Each standard 4-plat diagram consists of four horizontal strings, and the standard pattern of half-twists (plectonemic interwinds) of strings is encoded by an odd-length classifying vector with positive integer entries $\langle c_1, c_2, \ldots, c_{2k+1}\rangle$, as shown in Fig. 11.10. Two unoriented 4-plats are of the same knot (link) type if and only if their classifying vectors are identical, or identical upon reversal of one of the vectors. A classifying rational number can be computed from the classifying vector using the following extended fraction calculation:

$$\frac{\beta}{\alpha} = \frac{1}{c_1 + \frac{1}{c_2 + \cdots}}$$

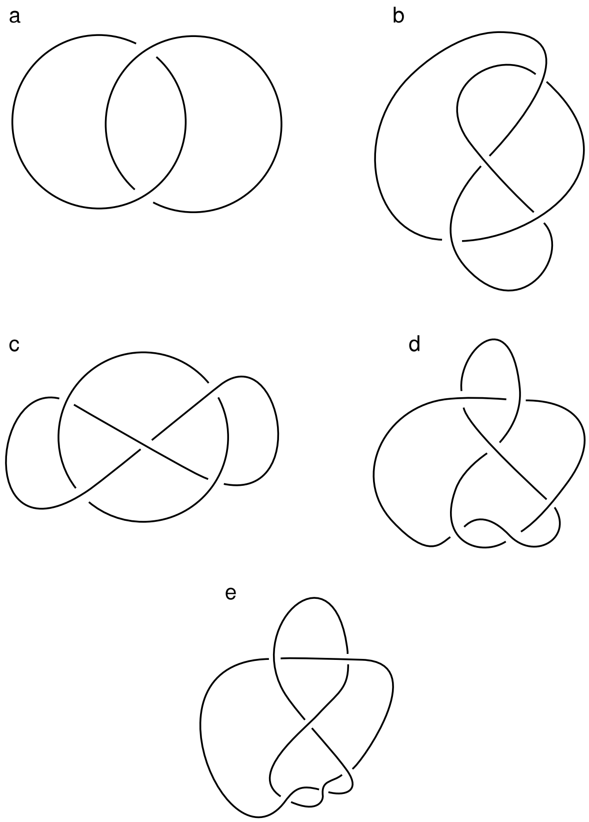


Fig. 11.9 Tn3 resolvase products: (**a**) Hopf link, (**b**) figure 8 knot, (**c**) (+) Whitehead link, (**d**) 6_2*, (**e**) 6_2 (from [Sum95])

Given the classifying rational numbers β/α and β'/α' for a pair of unoriented 4-plats, the 4-plats are the same type if and only if $\alpha + \alpha'$ and $\beta^{\pm 1} \equiv \beta' \mod \alpha$ [Sch56, BZ85].

For *in vitro* topological enzymology experiments, we can regard the enzyme mechanism as a machine that transforms some DNA 4-plats into other DNA 4-plats. We need a mathematical language for describing and computing these enzyme-mediated changes. In many enzyme-DNA reactions, a pair of sites that are distant on the substrate circle are juxtaposed in space and bound to the enzyme (Fig. 11.6) to create the synaptosome. The enzyme then performs its topological moves, and the DNA is then released.

We need a mathematical language to describe configurations of linear strings (the segments of bound DNA) in a spatially confined region (the enzyme). As is evident from the enzyme-DNA complex in Fig. 11.6, the globular protein enzyme is homeomorphic to a 3-ball, and the 2 strands of bound DNA (which contains the recombination sites) forms a protein-DNA *2-string tangle*. Tangles were introduced into knot theory by J.H. Conway [Con70]. Tangle theory is knot theory done inside a unit 3-ball (B^3) with the ends of the strings firmly glued down. On the unit 3-ball, select four points on the equator {NW;SW;SE;NE}. A 2-string tangle in the unit

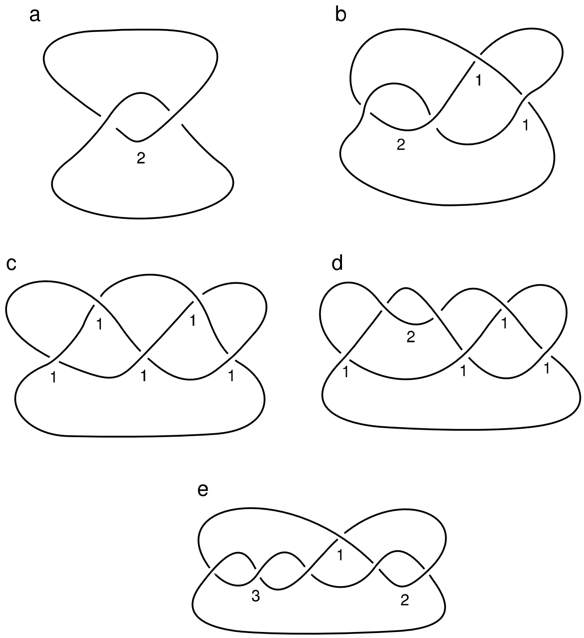


Fig. 11.10 Tn3 resolvase 4-plats: (**a**) $\langle 2 \rangle$ Hopf link, (**b**) $\langle 2, 1, 1 \rangle$ figure 8 knot, (**c**) $\langle 1, 1, 1, 1, 1 \rangle$ Whitehead link, (**d**) $\langle 1, 2, 1, 1, 1 \rangle$ $6_2{*}$, (**e**) $\langle 3, 1, 2 \rangle$ 6_2 (from [Sum95])

3-ball is a configuration of two disjoint strings in the unit 3-ball whose endpoints are the four distinguished points {NW;SW;SE;NE}. Two tangles in the unit 3-ball are *equivalent* if it is possible to ambient isotopy in the interior of the 3-ball the strings of one tangle into the strings of the other while keeping the boundary sphere fixed. A class of equivalent tangles is called a *tangle type*. Tangles are usually represented by their coded projections, called *tangle diagrams*, onto the equatorial disk in the unit 3-ball, as shown in Fig. 11.11. In all figures containing tangles, we assume that the four boundary points {NW;SW;SE;NE} are as in Fig. 11.11, and we suppress these labels.

All four of the tangles in Fig. 11.11 are pairwise inequivalent. However, if we relax the restriction that the endpoints of the strings remain fixed and allow the endpoints of the strings to move about on the boundary sphere of the 3-ball during the isotopy, then the tangle of Fig. 11.11(a) can be transformed into the trivial tangle of Fig. 11.11(d). The tangles in Figs. 11.11(b) and 11.11(c) cannot be transformed to the trivial tangle by any sequence of such turning motions of the endpoints on the boundary sphere. The family of tangles that can be converted to the trivial tangle by an ambient isotopy that is allowed to move the endpoints on the boundary sphere is the family of *rational tangles*. Equivalently, a rational tangle is one in which the strings can be transformed by ambient isotopy entirely into the boundary 2-sphere of the 3-ball. Rational tangles form a homologous family of 2-string configurations in B^3, and are formed by a pattern of plectonemic interwinding of pairs of strings. Like 4-plats, rational tangles look like DNA configurations being built up out of successive plectonemic supercoiling of pairs of strings. More specifically, enzymes are often globular in shape and are topologically equivalent to our unit-defining ball.

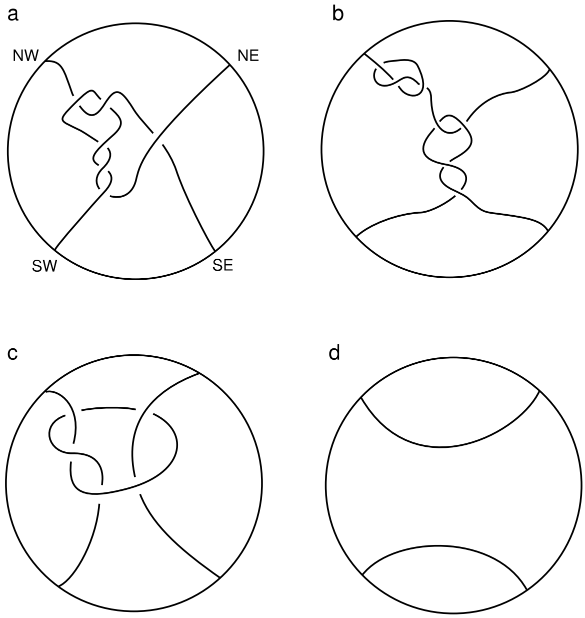


Fig. 11.11 Tangles (**a**) rational, (**b**) locally knotted, (**c**) prime, (**d**) trivial (from [Sum95])

Furthermore, we assume that any two copies of the enzyme can be superimposed by rigid motion, and this spatial congruence of the protein also respects the atomic structure of the protein (it matches corresponding atoms and endpoints of corresponding DNA strands). Thus, in an enzymatic reaction between a pair of DNA duplexes, the pair {enzyme, bound DNA} forms a well-defined 2-string tangle.

How do we know the DNA tangles are rational? I will give three arguments: 1. Since the amount of bound DNA is small, the enzyme-DNA tangle so formed admits projections with only a few nodes and therefore is rational by default. For example, all locally unknotted 2-string tangles having less than five crossings are rational. 2. In all cases studied intensively, the DNA is bound to the surface of the protein. This means that the resulting protein-DNA tangle is rational, since any tangle whose strings can be continuously deformed into the boundary of the defining ball is automatically rational. 3. We will give a mathematical proof: If the products of processive recombination are 4-plats, then one can often prove that the DNA tangles involved must be rational tangles.

There is a classification scheme for rational tangles which is based on a standard form that is a minimal alternating tangle diagram. The classifying vector for a rational tangle is an integer-entry vector $(a_1, a_2, \ldots, a_n)$ of odd or even length, with all entries (except possibly the last) nonzero and of the same sign, with $|a_1| > 1$. The integers in the classifying vector represent the left-to-right (west-to-east) alternation of vertical and horizontal windings in the standard tangle diagram, always ending with horizontal windings on the east side of the diagram. Horizontal winding is the winding between strings in the top and bottom (north and south) positions; vertical winding is the winding between strings in the left and right (west and east) positions. By convention, positive integers correspond to horizontal plectonemic right-handed supercoils and vertical left-handed plectonemic supercoils; negative integers correspond to horizontal left-handed plectonemic super-coils and vertical right-handed plectonemic supercoils (Fig. 11.12). This sign convention is opposite to that of Conway [Con70], and was chosen to agree with existing sign conventions used by biologists.

Figure 11.12 shows some standard tangle diagrams, which are minimal alternating diagrams. Two rational tangles are of the same type if and only if they have identical classifying vectors. Due to the requirement that $|a_1| > 1$ in the classifying vector convention for rational tangles, the corresponding minimal tangle diagram must have at least two crossings. There are four rational tangles $\{(0); (0,0), (1), (-1)\}$ that are exceptions to this convention $|a_1| > 1$, and are displayed in Fig. 11.12c through f. The classifying vector $(a_1, a_2, \ldots, a_n)$ can be converted to an (extended) rational number by means of the following continued fraction calculation:

$$\frac{\beta}{\alpha} = a_n + \frac{1}{a_{n-1} + \frac{1}{a_{n-2\cdots}}}$$

Two rational tangles are of the same type if and only if these rational numbers are equal [Con70, BZ85].

In order to use tangles as building blocks for knots and links and mathematically to mimic enzyme action on DNA, we now introduce the geometric operations of tangle additional and tangle closure. Given tangles A and B, one can form the tangle $(A+B)$ as shown in Fig. 11.13a. Equivalently, the tangle sum $(A+B)$ can be viewed as the decomposition of a complicated tangle into two simpler summands. The sum of two rational tangles need not be rational (Fig. 11.11c). Given any tangle A, one can form the tangle closure $N(A)$ as in Fig. 11.13b. In the closure operation on a 2-string tangle, ends NW and NE are connected outside the 3-ball, ends SW and SE are connected outside the 3-ball, and the tangle-defining ball is deleted, leaving a knot or a link of two components. Deletion of the tangle-defining 3-ball is the mathematical analogue of the biological action of deproteinization of the DNA that occurs when the synaptosome dissociates. One can combine the operations of tangle addition and tangle closure to create a tangle equation of the form $N(A+B) =$ knot (link). In such a tangle equation, the tangles A and B are said to be summands of the resulting knot (link). An example of this phenomenon is the tangle equation $N((-3,0) + (1)) = \langle 2 \rangle$ shown in Fig. 11.13c. In general, if A and B are any two rational tangles, then $N(A+B)$ is a 4-plat.

11.5 The Tangle Model for Site-Specific Recombination

The fundamental observations underlying this model are that a pair of sites bound by an enzyme (Fig. 11.6) forms a 2-string tangle and that most of the products of recombination experiments performed on unknotted substrate are 4-plats. We will

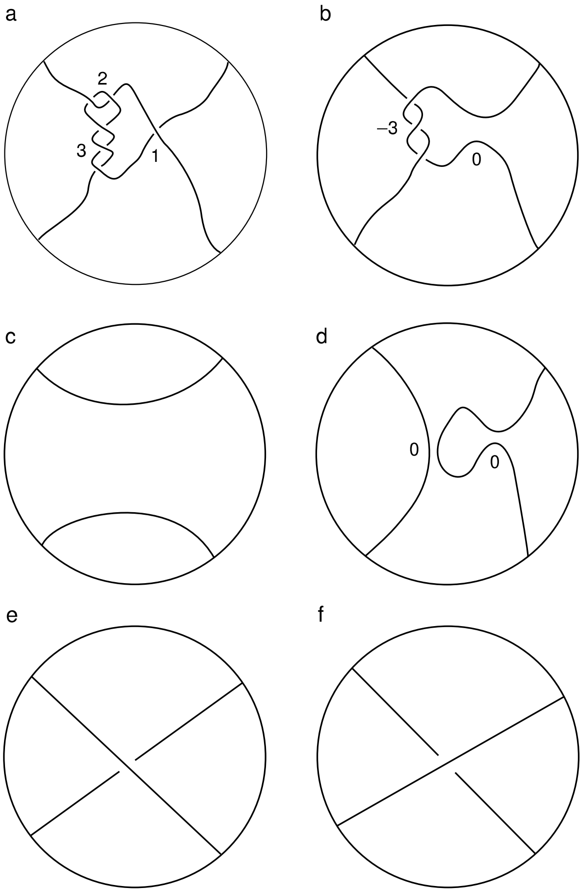


Fig. 11.12 Rational tangle diagrams: (**a**) (2, 3, 1), (**b**) (−2, 0), (**c**) (0), (**d**) (0, 0), (**e**) (+1), (**f**) (−1) (from [Sum95])

use tangles to build a model that will compute the topology of the pre- and post-recombination synaptic complex in a single recombination event, given knowledge of the topology of the substrate and product [ES90, ES99, SESC95]. In site-specific recombination on circular DNA substrate, two kinds of geometric manipulation of the DNA occur. The first is a global ambient isotopy, in which a pair of distant recombination sites are juxtaposed in space and the enzyme binds to the molecule(s), forming the synaptic complex. Once synapsis is achieved, the next move is local and due entirely to enzyme action. Within the region occupied by the enzyme, the substrate is broken at each site, and the ends are recombined. We will model this local move. Within the region controlled by the enzyme, the enzyme performs a double-stranded break in the DNA at each site and recombines the ends by exchanging them. We model the enzyme itself as a 3-ball. The synaptosome consisting of the enzyme and bound DNA forms a 2-string tangle. What follows is a list of biological and mathematical assumptions made in the tangle model [ES90, Sum92, Sum95].

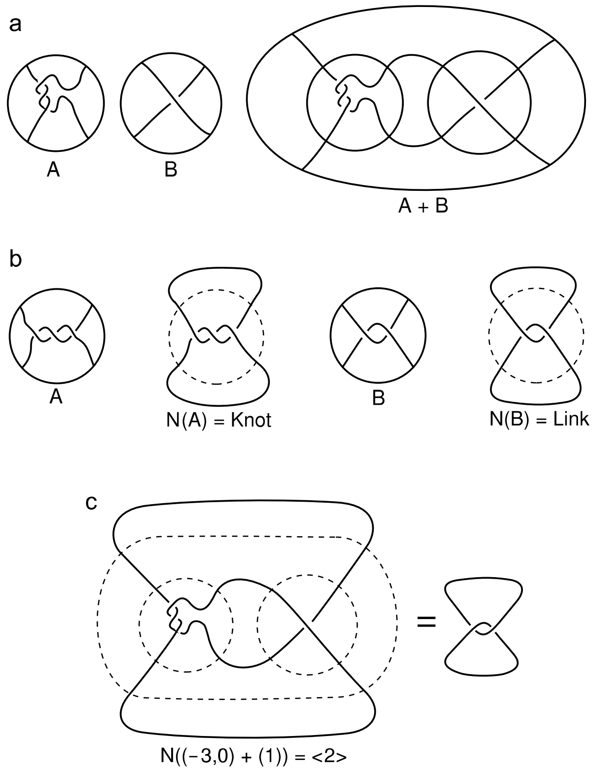


Fig. 11.13 Tangle operations: (**a**) tangle addition, (**b**) tangle closure, (**c**) tangle equation $N((-3,0)+(1)) = \langle 2\rangle$ (from [Sum95])

Assumption 1 The enzyme mechanism in a single recombination event is constant, independent of the geometry (supercoiling) and topology (knotting and catenation) of the substrate population. Moreover, recombination takes place entirely within the domain of the enzyme ball, and the substrate configuration outside the enzyme ball remains fixed while the strands are being broken and recombined inside and on the boundary of the enzyme.

That is, we assume that any two pre-recombination copies of the synaptosome are identical, meaning that we can by rotation and translation superimpose one copy on the other, with the congruence so achieved respecting the structure of both the protein and the DNA. We likewise assume that all of the copies of post-re-combination synaptosome are identical.

In a recombination event, we can mathematically divide the DNA involved into three types: (1) the DNA at and very near the sites where the DNA breakage and reunion are taking place; (2) other DNA bound to the enzyme, which is unchanged during a recombination event; and (3) the DNA in the synaptic complex that is not bound to the enzyme and consequently does not change during recombination.

We make the following mathematical assumption about DNA types (1) and (2):

Assumption 2 The synaptosome is a 2-string tangle that can be mathematically subdivided into the sum $O_b + P$ of two tangles.

One tangle, the *parental tangle* P, contains the recombination sites where strand breakage and reunion take place. The other tangle, the *outside bound tangle* O_b, is the remaining DNA in the synaptosome outside the P tangle—this is the DNA that is bound to the enzyme but that re-mains unchanged during recombination. If the enzyme must achieve a geometric footprint on the DNA in order to mediate productive synapsis, this footprint is captured in the tangle O_b. The enzyme mechanism is modeled as tangle replacement (surgery) in which the parental tangle P is removed from the smaptosome and replaced by the *recombinant tangle* R. The schematic of Fig. 11.7 shows the tangles involved in Tn3 Resolvase site-specific recombination.

Therefore, our model assumes the following:

$$\text{pre-recombination synaptosome} = (O_b + P)$$

$$\text{post-recombination synaptosome} = (O_b + R).$$

In order to accommodate nontrivial topology in the DNA of type (3), we let the *outside free tangle* O_f denote the synaptic complex DNA that is free (not bound to the enzyme) and that is unchanged during a single recombination event.

We make the following mathematical assumption:

Assumption 3 The entire synaptic complex is obtained from the tangle sum (O_f+ synaptosome) by the tangle closure construction.

If one deproteinizes the pre-recombination synaptic complex, one obtains the substrate (an experimentally controlled knot/link type); deproteinization of the post-recombination synaptic complex yields the product (an observable knot/link type). The topological structure (knot and link types) of the substrate and product yields equations in the four recombination variables $\{O_f, O_b, P, R\}$. Specifically, a single recombination event on a single circular substrate molecule produces two recombination equations in four unknowns:

Substrate Equation: $N(O_f + (O_b + P)) = \text{substrate}$,
Product Equation: $N(O_f + (O_b + R)) = \text{product}$.

The geometric meaning of these recombination equations is illustrated in Fig. 11.7. In Fig. 11.7, with the tangle unknowns evaluated as follows: $O_f = (0)$, $O_b = (-3, 0)$, $P = (0)$, $R = (1)$. With these values for the tangle variables, our recombination equations become:

Substrate Equation: $N((0) + ((-3, 0)) + (0))) = \langle 1 \rangle$,
Product Equation: $N((0) + ((-3, 0) + (1))) = \langle 2 \rangle$.

11.6 The Topology of Tn3 Resolvase

Tn3 resolvase is a site-specific recombination enzyme that reacts with negatively supercoiled circular duplex DNA substrate with directly repeated recombination sites [WDC85]. As substrate for the *in vitro* reaction, supercoiled unknotted DNA substrate is incubated with resolvase. The principal product of this reaction is known to be the DNA 4-plat (2) (the Hopf link, Figs. 11.4a and 11.5a). Resolvase is known to act dispersively in this situation–to bind to the circular DNA substrate, to mediate a single recombination event, and then to release the linked product. It is also known that resolvase and free (unbound to protein) DNA links do not react. However, once in twenty encounters, resolvase acts processively-additional recombinant strand exchanges are mediated during the single binding encounter prior to the release of the product, with yield decreasing exponentially with increasing number of strand exchanges. Two successive rounds of processive recombination produce the DNA Fig. 11.8 knot $\langle 2, 1, 1\rangle$, Fig. 11.9b and 11.10b, whose electron micrograph appears in Fig. 11.14a. Three successive rounds of processive recombination produce the DNA link $\langle 1, 1, 1, 1, 1\rangle$ (the +Whitehead link, Fig. 11.9c and 11.10c, whose electron micrograph appears in Fig. 11.14b; four successive rounds of recombination produce the DNA knot $\langle 1, 1, 1, 1, 1\rangle$, the knot 62*, Fig. 11.9d and 11.10d, whose electron micrograph appears in Fig. 11.14c.

In processive recombination, it is the synaptosome itself that repeatedly changes structure. We make the following biologically reasonable mathematical assumption in our model:

Assumption 4 In procession recombination, each additional round of recombination adds a copy of the recombinant tangle R to the synaptosome.

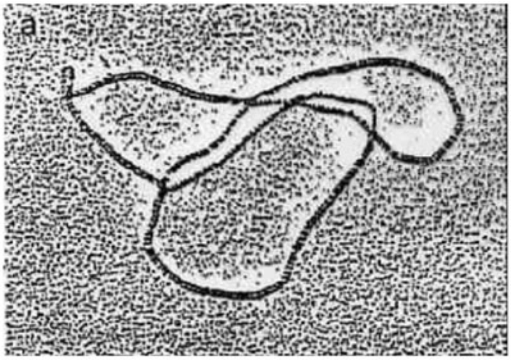
More precisely, p rounds of processive recombination at a single binding encounter generates the following system of $(p + 1)$ tangle equations in the 4 tangle unknowns $\{O_f, O_b, P, R\}$:

Substrate Equation: $N(O_f + (O_b + P)) =$ substrate
n^{th} **Round Product Equation**: $N(O_f + (O_b + nR)) = n^{\text{th}}$ round product $1 \leq n \leq p$.

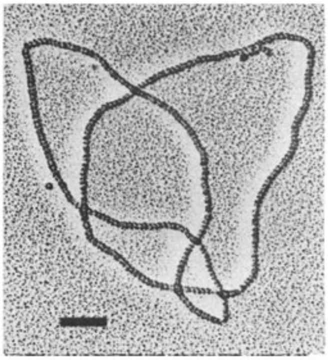
For resolvase, the electron micrograph of the synaptic complex in Fig. 11.6 reveals that $O_f = (0)$, since the DNA loops on the exterior of the synaptosome can be untwisted and are not entangled with each other. This observation from the micrograph reduces the number of variables in the tangle model by one, leaving us with three variables $\{O_b, P, R\}$.

Theorem 1 [ES90] *Suppose that tangles* $\{O_b, P, R\}$ *satisfy the following equations*:

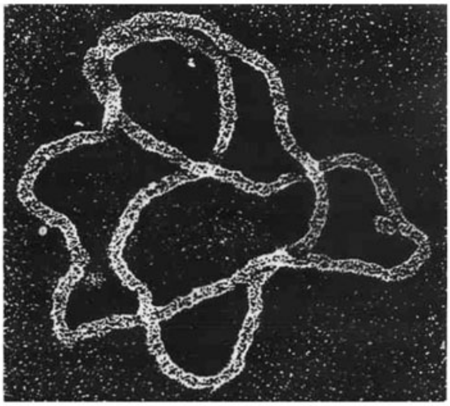
1. $N(O_b + P) = \langle 1\rangle$ (*substrate = unknot*)
2. $N(O_b + R) = \langle 2\rangle$ (1^{st} *round product = Hopf link*)
3. $N(O_b + 2R) = \langle 2, 1, 1\rangle$ (2^{nd} *round product = Fig.* 11.8 *knot*)



(a) Figure 8 Knot (from [KSS$^+$83])



(b) +Whitehead Link (from [KSS$^+$83])



(c) 6_2* (from [WDC85])

Fig. 11.14 Electron micrographs of RecA coated Tn3 resolvase recombination products

Then

$$\{O_b, R\} = \{(-3, 0), (1)\}, \{(3, 0), (-1)\}, \{(-2, -3, -1), (1)\} \text{ or } \{(2, 3, 1), (-1)\}$$

Theorem 1 gives an even number of tangle solution pairs for $\{O_b, R\}$ because each of the knots and links involved in the three equations is achiral (the unoriented Hopf link is achiral); therefore the mirror image of any solution pair is also a solution pair. Theorem 1 says that there are two mirror image solution pairs for $\{O_b, R\}$. Notice that Theorem 1 says nothing about the parental tangle P. This is because P is involved in only one tangle equation. It is known [ES90] that if the tangle X is a solution to an equation of the form $N(A + X) = K$, where A is a rational tangle and K is a 4-plat, then there are infinitely many rational solutions for X. Biologists believe that $P = (0)$; since the recombination sites are short ($\sim$15 base pairs), and duplex DNA is fairly stiff, then the DNA axis at the sites can be represented by oriented straight line segments. Given two spatially juxtaposed straight line segments in space, one can choose a projection where the segments project to a pair of parallel oriented line segments [SESC95].

The first (and most mathematically interesting) step in the proof of Theorem 1 is to argue that the solutions O_b, R must be rational tangles. We have the following facts about tangles [Lic81]: tangles come in three classes: locally knotted, prime and rational; if a tangle is locally unknotted, then it must be either prime or rational. A tangle A is rational if and only if its 2-fold branched cyclic cover A' is a solid torus. If A is a prime tangle, and A' is its 2-fold branched cyclic cover, then the inclusion induced homomorphism injects $\pi_1(\partial A') = Z \oplus Z$ into $\pi_1(A')$. Now O_b and R are locally unknotted because of equation 2, since the 1st round recombination product Hopf link has two unknotted components, and any local knot in any of the tangles would persist as a local knot in the recombination product. Also, at least one of $\{O_b, R\}$ must be rational; otherwise they are both prime, which means that the 2-fold branched cover of $N(O_b + R) = \langle 2\rangle$ (the lens space $L(2, 1)$) is obtained from O_b' and R' , glued together along their common incompressible torus boundary. This means that $L(2, 1)$ contains an incompressible torus, which is impossible.

Suppose now that O_b is rational and R is prime. Given that $N((O_b + R) + R)$ is a knot (equation 3), one can argue [ES90] that $(O_b + R)$ is also a prime tangle, which means that the 2-fold branched cyclic cover of the Fig. 11.8 knot (the lens space $L(5, 3)$) contains an incompressible torus, also impossible. We conclude that R must be a rational tangle.

The next step is to argue that O_b is rational. Otherwise, O_b is prime; in this case, since $N(O_b + P) = \rangle 1 \langle$, P must be locally unknotted and rational. The 2-fold branched cyclic cover $N(O_b + P)' = \frac{O_b' \cup P'}{\partial O_b' = \partial P' = S^1 \times S^1} = S^3$. Since P' is a solid torus, this means that O_b' is a bounded knot complement in S^3. We have that R is rational, and can argue that equation 3 implies that $(R + R)$ is also rational. Passing to the 2-fold branched cyclic covers of equations 2 and 3, we obtain the equations $N(O_b + R)' = L(2, 1)$ and $N(O_b + (R) + R))' = L(5, 3)$. Because R' and $(R + R)'$ are each a solid torus, this means that there are two Dehn fillings of the knot complement O_b', resulting in the lens spaces $L(2, 1)$ and $L(5, 3)$. The cyclic surgery theorem [CGLS87] now applies to argue that, since the orders of the cyclic fundamental groups of the lens spaces differ by more than one, this means that O_b' must be a Seifert fiber space, and that O_b' is a torus knot complement. The results of Dehn surgery on torus knots is well understood [Mos71], and one can show that in fact the torus knot in question must be the unknot, and that O_b' is a solid torus, hence O_b is a rational tangle.

The proof now amounts to computing the rational solutions to the equations in Theorem 1. Claus Ernst and I developed a “rational tangle calculus” [ES90] which uses the classifying symbols for rational tangles and 4-plats to do these calculations, obtaining the four solutions in Theorem 1. In order to decide which of these four tangle pair solutions is the biologically correct one, we must utilize more experimental evidence, and to get to a unique solution, must have a chiral recombination product. The result of 3 rounds of 3 rounds of recombination is the unoriented (+) Whitehead link, which is chiral. Using this information, we have the following:

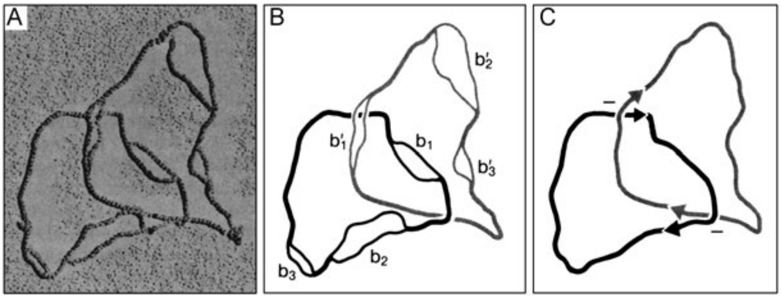


Fig. 11.15 The biologically oriented DNA Hopf link (from [WC85])

Theorem 2 [ES90] *Suppose that tangles* $\{O_b, P, R\}$ *satisfy the following equations*:

1. $N(O_b + P) = \langle 1 \rangle$ (*substrate = unknot*)
2. $N(O_b + R) = \langle 2 \rangle$ (1^{st} *round product = Hopf link*)
3. $N(O_b + 2R) = \langle 2, 1, 1 \rangle$ (2^{nd} *round product = Fig.* 11.8 *knot*)
4. $N(O_b + 3R) = \langle 1, 1, 1, 1, 1 \rangle$ (3^{rd} *round product = +Whitehead link*)

Then $\{O_b, R\} = \{(-3, 0), (1)\}$ *and the result of* 4 *rounds of processive recombination is the* 4*-plat* 62*$= \langle 1, 2, 1, 1, 1 \rangle$,

Of the four solutions produced in Theorem 1, only $\{O_b, R\} = \{(-3, 0), (1)\}$ is a solution to equation 4. The correct global topology of the first round of iterated processive Tn3 site-specific recombination is shown in Fig. 11.7. Moreover, the first 3 rounds of recombination uniquely determine the result of 4 rounds of recombination, the observed DNA knot 62* (Fig. 11.14(c)).

Is it possible to use the first two rounds of recombination to uniquely determine the enzyme binding and mechanism, and to correctly predict the results of 3 and 4 rounds of recombination? In order to do this, we need a chiral product, and fortunately experimental evidence exists which allows us to unambiguously put a recombination-induced orientation on the Hopf link product, making it chiral.

In a remarkable experiment, Steve Wassserman and Nick Cozzarelli [WC85] were able to determine the orientation induced on the 1^{st} round recombination product of resolvase acting on supercoiled unknotted substrate with directly repeated sites. In duplex DNA, the AT base pairs have a double hydrogen bond, and the CG base pairs have a triple hydrogen bond, so the AT bonding is weaker than the CG bonding. When duplex DNA is partially denatured by heating, the AT bonds break before the CG bonds. On circular substrate with directly repeated sites, the recombination sites divide the circular substrate into two domains. Using cloning techniques, one can install 3 AT-rich regions into each domain; the AT-rich regions are of 3 different lengths. Upon recombination, one obtains a linked pair of circles, each inheriting 3 AT-rich regions from the parental DNA substrate circle. In order to visualize these regions, partial denaturation of the Hopf link product revels 3 bubbles on each component circle, allowing us to unambiguously determine the orientation of each circle. With this induced orientation, the linking number of the oriented DNA Hopf link is -1 (Fig. 11.7 and Fig. 11.15), making it a chiral product. Assuming again that $P = (0)$, we have the following:

Theorem 3 [ES90] *Suppose that tangles* $\{O_b, R\}$ *satisfy the following equations*:

1. $N(O_b) = \langle 1 \rangle$ (*substrate = unknot*)
2. $N(O_b + R) = \langle 2 \rangle$ (1^{st} *round product = Hopf link, with linking number* $= -1$)
3. $N(O_b + 2R) = \langle 2, 1, 1 \rangle$ (2^{nd} *round product = Fig.* 11.8 *knot*)

Then $\{O_b, R\} = \{(-3, 0), (1)\}$ *and the result of* 3 *rounds of processive recombination is the* $+$*Whitehead link* $\langle 1, 1, 1, 1, 1 \rangle$, *and the result of* 4 *rounds of processive recombination is the* 4*-plat* $62* = \langle 1, 2, 1, 1, 1 \rangle$.

So, only the first two rounds of recombination determine enzyme mechanism and binding, and correctly predict the observed result of 3 and 4 rounds of recombination. The above analysis amounts to a mathematical proof of enzyme binding and mechanism, and is a mathematical model that is of utility in many other situations where circular DNA is used as a probe for biological activity. For example, the tangle model is useful when electron microscopy is not available. Gel electrophoresis (enhanced by radiolabelling of DNA) can be done to detect vanishingly small amounts of DNA product, and the gel velocity of relaxed circular DNA tells us the crossing number of the DNA product. Knowing only the crossing number is very useful; one can for example use crossing number information to help characterize the geometry of packing of viral DNA in phage capsids [ATV+02, AVM+05, AVT+02, TAV+01, MMOS06, MMOS08]. Since all knots and links of small crossing numbers are known, one can write down tangle equations and solve them, knowing only the crossing number of the DNA products. One goes to the knot and link tables, and for the right-hand sides of each tangle equation, plugs in all the possible knot (link) products of a given crossing number [Vaz00]. Moreover, computer programs exist to solve systems of tangle equations and visualize the answers [SV02], [http://bio.math.berkeley.edu/TangleSolve/; http://www.math.uiowa.edu/~idarcy/]. More generally, the existence of a mathematical model allows one to answer "what if" questions, and carefully investigate the utility of the assumptions that go into the model.

Acknowledgements I gratefully acknowledge research support from the National Science Foundation and the Burroughs Wellcome Fund Interfaces Grant to the Program in Mathematics and Molecular Biology at Florida State University. I would also like to thank Prof. Markus Banagl and the Mathematics Center of the University of Heidelberg for support and hospitality during the Knot Theory Semester in December 2008.

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Workshop Talks

Dorothy Buck
Department of Mathematics, Imperial College London
The Topology of DNA-Protein Interactions

Sylvain Cappell
Courant Institute of Mathematical Sciences, New York University
The Role of Knot Theory in Transformation Groups

Sergei Chmutov
Department of Mathematics, Ohio State University
Combinatorics of Gauss Diagrams and the HOMFLYPT Polynomial

Heather Dye
McKendree University
The Arrow Polynomial

Stefan Friedl
Warwick Mathematics Institute, University of Warwick
Symplectic 4-Manifolds and Fibered 3-Manifolds

Stavros Garoufalidis
School of Mathematics, Georgia Institute of Technology
The HOMFLY Polynomial of a 3-Manifold, the Trilogarithm and Painlevé I

Cameron Gordon
Department of Mathematics, University of Texas at Austin
Exceptional Dehn Filling

M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
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DOI 10.1007/978-3-642-15637-3,

Denis Ilyutko
Dept. of Mechanics And Mathematics, M.V. Lomonosov Moscow State University
Graph-links, Virtual Links, Kauffman Bracket and Simplicial Presentation of Khovanov Homology

Barbara Jablonska
Institut für Mathematik, Technische Universität Berlin
Surfaces Associated to a Knotted Space Curve

Louis Kauffman
Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago

1. *Extending the Jones Polynomial for Virtual Links*
2. *Unitary Representations of the Artin Braid Group and the Quantum Algorithms for Jones Polynomials*

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Department of Mathematics, National Technical University of Athens
Singular Knots and Yokonuma-Hecke Algebras

Joan Licata
Department of Mathematics, Stanford University
Linearized Homology for Grid Number One Knots in L(p,q)

Vassily Manturov
People's Friendship University of Russia
Which Khovanov Homology Theory Is the Right One?

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Surface Knots and Iterated Intersection Theory

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Dehn Filling and the Thurston Norm

Martin Scharlemann
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Fibered Knots and Property 2R

De Witt Sumners
Department of Mathematics, Florida State University
Random Knotting and Viral DNA Packing