An application of solvable structures to classical and non-classical similarity solutions

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Abstract

Using exterior differential systems, we extend work by Harrison and Estabrook for deriving similarity solutions of hyperbolic and parabolic partial differential equations. We use exterior calculus results to show that a symmetry (isovector) of the differential ideal corresponding to some hyperbolic or parabolic PDE can be used to generate a Cauchy characteristic vector field of a restricted exterior differential system defined on some four-dimensional regular submanifold of the first jet bundle. We then show that this restricted differential ideal has a Frobenius integrable annihilating space which can be used to yield a similarity solution of the PDE by applying results from S. Lie and É. Cartan on integrating Frobenius integrable vector field distributions via symmetry. We also give an extension to conditional symmetries.

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I. INTRODUCTION

Given a non-linear partial differential equation, a so-called 'similarity solution' is one which is invariant under some group action. Pioneered by Lie [1], techniques for using symmetries to find similarity solutions have been around for a long time, and in recent times authors such as Bluman and Cole [2], Bluman and Kumei [3], Olver [4, 5, 6] and Stephani [7] have provided modern discussions on various aspects of this similarity solution approach to PDEs.

This work considers a single second order hyperbolic or parabolic PDE of one dependent variable u and two independent variables x^1, x^2 of the form

$$f_1 \frac{\partial^2 u}{\partial (x^1)^2} + f_2 \frac{\partial^2 u}{\partial (x^2)^2} + f_3 \frac{\partial^2 u}{\partial x^1 x^2} = k, \tag{1}$$

where f_1, f_2, f_3, k are smooth functions of $x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}$. Although exterior differential systems [8, 9, 10, 11, 12] are of most use in studying systems of non-linear partial differential equations, we examine in this paper their application to similarity solutions of (1) along similar lines as Harrison and Estabrook [13]. We also give an alternative interpretation of the underlying geometric significance of such solutions.

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Since this paper is essentially concerned with algorithms based on symmetry for extracting similarity solutions of (1), we assume throughout that give a second order hyperbolic or parabolic PDE of the form in (1) and symmetry vector field, there exists a local smooth similarity solution. This also means that if we apply the Cartan-Kuranishi theorem [14], we obtain after a finite number of prolongations an involutive system of PDEs.

Our work also make use of results from Lie [15] and Cartan [16, 17] for integrating Frobenius integrable vector field distributions via symmetry that has in recent times been extended by Basarab-Horwath [18], Duzhin and Lychagin [19], Hartl and Athorne [20], and Sherring and Prince [21]. With particular emphasis on results in [21], we establish in Sections V and VII two algorithms based entirely on symmetry for generating similarity solutions of second order hyperbolic or parabolic PDEs of the type in (1), which avoids the usual requirement of having to solve some ordinary differential equation once the similarity variable is known. Finally, we briefly examine conditional symmetries. Using such symmetries we extend earlier results in this paper to give a technique for generating the so-called 'non-classical' [6, 22, 23] similarity solutions, that once again avoids the need to solve any ODE.

II. BACKGROUND

It is assumed throughout this paper that for natural numbers n and m, U^n and V^m are, respectively, some open, convex neighbourhoods of $\mathbb{R}^n$ and $\mathbb{R}^m$, with coordinates $x^1, \ldots, x^n$ and $z^1, \ldots, z^m$. On the κ-th jet bundle $J^\kappa(U^n, V^m)$, we say that the set of exterior differential p-forms $\Lambda^p(J^\kappa(U^n, V^m))$ is a section of the bundle of all homogeneous differential forms $\Lambda(J^\kappa(U^n, V^m))$. We define $\mathfrak{X}(J^\kappa(U^n, V^m))$ to be the module of smooth vector fields over $C^\infty(J^\kappa(U^n, V^m))$. Given some $\omega \in \Lambda^p(J^\kappa(U^n, V^m))$, its *kernel* is defined by $\ker(\omega) = \{X \in \mathfrak{X}(J^\kappa(U^n, V^m)) : X \lrcorner \omega = 0\}$. We assume that on their domains of definition, all vector field distributions are of constant dimension, and unless otherwise stated as in Sections VI and VII, all mappings and differential one-forms are of constant rank.

The *Cauchy characteristic* space of a differential ideal I generated by some finite collection of differential forms is denoted $A(I)$, and contains all vector fields $X \in \mathfrak{X}(J^\kappa(U^n, V^m))$ such that $X \lrcorner I \subset I$. A vector field $X \in \mathfrak{X}(J^\kappa(U^n, V^m))$ is said to be a *symmetry* (isovector) of I if it satisfies the condition involving the Lie derivative that $\mathcal{L}_X I \subset I$. A vector field $X \in \mathfrak{X}(J^\kappa(U^n, V^m))$ is a *symmetry* of a vector field distribution $D \subset \mathfrak{X}(J^\kappa(U^n, V^m))$ if $\mathcal{L}_X D \subset D$. We say that a vector field is a *non-trivial* symmetry if, in terms of a differential ideal, it is not Cauchy characteristic, or in terms of a vector field distribution, it is not in the distribution.

We also assume throughout this paper that unless otherwise stated, M^q is some open, convex q-dimensional neighbourhood of $J^\kappa(U^n, V^m)$. Since by the inverse function theorem, parameterising immersions mapping onto regular submanifolds are locally diffeomorphic, we also assume all neighbourhoods U^n, V^m and M^q are chosen such that this holds. Thus for the differential map $\Psi_* : \mathfrak{X}(M^r) \longrightarrow \mathfrak{X}(M^s)$, we can therefore assume for each $Y \in \mathfrak{X}(M^r)$ that $\Psi_* Y$ is a well-defined vector field, and the following property holds:

$$\Psi_*[Y_1, Y_2] = [\Psi_* Y_1, \Psi_* Y_2]. \tag{2}$$

for any $Y_1, Y_2 \in \mathfrak{X}(M^r)$. We also make use of the following theorem found in Sternberg [24] that we use in the next section:

Theorem 1. *Let $\Psi : M^r \longrightarrow M^s$ be a one-to-one immersion. Then for all $Y \in \mathfrak{X}(F(M^r))$ there exists $X \in \mathfrak{X}(M^r)$ such that $\Psi_* X = Y$.*

Here we write $\mathfrak{X}(\Psi(M^r))$ to mean the module of vector fields tangent to $\Psi(M^r)$. At Ψ is one-to-one, this notation is unambiguous.

The pull-back map $\Psi^* : \Lambda(M^s) \longrightarrow \Lambda(M^r)$ has the following properties:

$$(\Psi^*\omega)(Y_1, \ldots, Y_k) = \Psi^*\left(\omega(\Psi_* Y_1, \ldots, \Psi_* Y_k)\right), \tag{3}$$

for any $\omega \in \Lambda^k(M^s)$, $Y_1, \ldots, Y_k \in \mathfrak{X}(M^r)$, and

$$\Psi^* \circ d\omega^1 = d \circ \Psi^*\omega^1, \tag{4}$$

$$\Psi^*\left(\omega^1 \wedge \omega^2\right) = \left(\Psi^*\omega^1\right) \wedge \left(\Psi^*\omega^2\right), \tag{5}$$

for any $\omega^1, \omega^2 \in \Lambda(M^s)$. Given any smooth $\Phi : M^q \longrightarrow M^r$ and $\omega \in \Lambda^1(M^s)$, we also have the following composition property:

$$(\Psi \circ \Phi)^* \omega = \Phi^*(\Psi^*\omega). \tag{6}$$

III. DIFFERENTIAL IDEAL REPRESENTATION OF PDES

Working in the second jet bundle $J^2(U^2, V^1)$ with coordinates $x^1, x^2, z^1, z^1_1, z^1_2, z^1_{11}, z^1_{12}, z^1_{22}$, we define

$$F := f_1 z^1_{11} + f_2 z^1_{22} + f_3 z^1_{12} - k,$$

along with the contact forms

$$\begin{aligned} C^1 &:= dz^1 - z^1_1 dx^1 - z^1_2 dx^2, \\ C^1_1 &:= dz^1_1 - z^1_{11} dx^1 - z^1_{12} dx^2, \\ C^1_2 &:= dz^1_2 - z^1_{12} dx^1 - z^1_{22} dx^2. \end{aligned}$$

We can express a solution surface of the PDE in (1) as a two-dimensional integral manifold (immersion) of the differential ideal

$$I_F := \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, F dx^1 \wedge dx^2 \rangle,$$

such that the transverse condition $dx^1 \wedge dx^2 \neq 0$ holds on its tangent space. Note that $dC^1 \equiv 0$ mod C^1_1, C^1_2. Also, Lemma 1.1 in [25] implies

$$d\left(F dx^1 \wedge dx^2\right) \equiv 0 \quad \text{mod } C^1, C^1_1, C^1_2, dC^1_1, dC^1_2.$$

It is well-known that an integral manifold in the second jet bundle which annihilates all the contact forms that generate the second order contact system is the image of the 2-jet of some smooth map $f : U^2 \longrightarrow V^1$ if and only if $dx^1 \wedge dx^2 \neq 0$ on the tangent space of the integral manifold (see, for example, Theorem 2.3.1 in Stormark [26]). If, in addition, the integral manifold annihilates F, then the 2-jet is that of some local solution of the PDE in (1).

Our principal result of this section is the following:

Theorem 2.

$$I_F = \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, L \rangle,$$

where

$$L := f_1 dz^1_1 \wedge dx^2 - f_2 dz^1_2 \wedge dx^1 + f_3 dz^1_2 \wedge dx^2 - k dx^1 \wedge dx^2.$$

Proof.

$$F dx^1 \wedge dx^2 = \left(f_1 z^1_{11} + f_2 z^1_{22} + f_3 z^1_{12} - k\right) dx^1 \wedge dx^2.$$

Now

$$\begin{aligned} f_1 z^1_{11} dx^1 \wedge dx^2 &= f_1 (z^1_{11} dx^1 + z^1_{12} dx^2) \wedge dx^2, \\ &= f_1 (dz^1_1 - C^1_1) \wedge dx^2, \end{aligned}$$

$$\begin{aligned} f_2 z^1_{22} dx^1 \wedge dx^2 &= -f_2 (z^1_{21} dx^1 + z^1_{22} dx^2) \wedge dx^1, \\ &= -f_2 (dz^1_2 - C^1_2) \wedge dx^1, \end{aligned}$$

$$\begin{aligned} f_3 z^1_{12} dx^1 \wedge dx^2 &= f_3 (z^1_{12} dx^1 + z^1_{22} dx^2) \wedge dx^2, \\ &= f_3 (dz^1_2 - C^1_2) \wedge dx^2. \end{aligned}$$

Hence

$$\begin{aligned} F dx^1 \wedge dx^2 &\equiv f_1 dz^1_1 \wedge dx^2 - f_2 dz^1_2 \wedge dx^1 + f_3 dz^1_2 \wedge dx^2 \\ &\quad - k dx^1 \wedge dx^2 \mod C^1_1, C^1_2, \\ &\equiv L \mod C^1_1, C^1_2. \end{aligned}$$

From this we obtain

$$\begin{aligned} dL &\equiv d\left(F dx^1 \wedge dx^2\right) \mod C^1_1, C^1_2, dC^1_1, dC^1_2, \\ &\equiv 0 \mod C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, \end{aligned}$$

using Lemma 1.1 in [25]. □

Remark. In a similar fashion to above, it is easy to show that

$$I_F = \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, L^\dagger \rangle,$$

where

$$L^\dagger := f_1 dz^1_1 \wedge dx^2 - f_2 dz^1_2 \wedge dx^1 - f_3 dz^1_1 \wedge dx^1 - k dx^1 \wedge dx^2.$$

In our work, we deal mostly with L, however all results equally apply to $L^\dagger$.

We define

$$I_{\overline{F}} := \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, L \rangle.$$

Technically speaking, $I_{\overline{F}} := I_F$ (by Theorem 2), and the notation $I_{\overline{F}}$ might appear redundant. However we will use $I_{\overline{F}}$ as a brief way of referring to the particular choice of generators $C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, L$.

Now L (containing all the information specific to the PDE) does not depend on any second order terms $z^1_{11}, z^1_{12}, z^1_{22}$. Therefore, we may modify our problem to that of finding two-dimensional integral manifolds of a *reduced* differential ideal $I^r_{\overline{F}}$ defined by

$$I^r_{\overline{F}} := \langle C^1, dC^1, L, dL \rangle, \tag{7}$$

defined on the first jet bundle $J^1(U^2, V^1)$. We note that since dL is a three-form, all two-dimensional integral manifolds of $I^r_{\overline{F}}$ will trivially annihilate dL, so this differential form can therefore be ignored in all calculations.

IV. SIMILARITY SOLUTION APPROACHES

Given a Lie point symmetry $X \in \mathfrak{X}(U^2 \times V^1)$ of the PDE in (1), a similarity solution of the PDE is a local solution that remains unchanged under the one-parameter group action of the symmetry. The most well-known procedure for using X to generate a corresponding similarity solution basically involves determining the two functionally independent invariants $\gamma^1, \gamma^2 \in C^\infty(U^2 \times V^1)$ of X and finding a solution of (1) that is some function of these invariants. Doing so, one essentially obtains from (1) a second order ODE expressed in terms of γ^1, γ^2, known as the 'reduced' differential equation. In the general case for PDE problems of n independent variables, the reduced equation retains the same order of the PDE but is of $n-1$ independent variables.

An alternative and equivalent approach to finding similarity solutions is discussed by Olver in [6] where one searches for a common solution of the overdetermined system of PDEs given by (1) and the first order quasilinear PDE obtained from

$$X^{(1)} \lrcorner\, C^1 = 0, \tag{8}$$

where z^1 and z^1_1, z^1_2 are replaced with u and its respective first partial derivatives. Here we assume (8) gives a valid PDE and the Lie point symmetry X is not, for example, $\frac{\partial}{\partial z^1}$. The PDE derived from (8) is known as the *characterising invariance system* (or *invariant surface condition*) corresponding to X, and is typically solved first using invariant coordinates to give a solution in terms of an arbitrary function. Then, by inserting this solution into (1), a reduced differential equation for the arbitrary function is derived. Once this is solved, a similarity solution is obtained once more.

In this paper we do not follow either of the above procedures, but instead choose to adopt another approach formulated by Harrison and Estabrook [13] that uses exterior calculus and differential ideals. This is discussed below:

Suppose we are given some differential ideal I^r_F on $J^1(U^2, V^1)$ corresponding to some second order PDE of the form in (1). If a vector field $V \in \mathfrak{X}\left(J^1(U^2, V^1)\right)$ is a symmetry of I^r_F, then

$$\mathcal{L}_V C^1 = \lambda_1 C^1, \tag{9}$$

and

$$\mathcal{L}_V L = \alpha^1 \wedge C^1 + \lambda_2 dC^1 + \lambda_3 L, \tag{10}$$

for some $\lambda_1, \lambda_2, \lambda_3 \in C^\infty\left(J^1(U^2, V^1)\right)$ and $\alpha^1 \in \Lambda^1\left(J^1(U^2, V^1)\right)$. Applying the property that $\mathcal{L}_V\left(d\omega\right) = d\left(\mathcal{L}_V \omega\right)$ for any differential form ω, we can use (9) and (10) to derive corresponding symmetry expressions for the remaining two generators of I^r_F. A key property of the Harrison and Estabrook approach is that the symmetry algebra of I^r_F includes the Lie point symmetry algebra of (1). We state this fact without proof, however in [27] it is proved for differential ideals where the PDE is left as a 0-form generator of the ideal. Since we are dealing with PDEs of one dependent variable, the determining equations derived from (9) and (10) should also be able to establish any so-called contact symmetries of the PDE.

Suppose then that we are given some symmetry V of I^r_F (or the first prolongation of some Lie point symmetry of (1)). In the Harrison and Estabrook approach to generating similarity solutions of (1), the differential ideal I^r_F is augmented with $V \lrcorner\, C^1$, $V \lrcorner\, dC^1$, $V \lrcorner\, L$ and $V \lrcorner\, dL$. One then looks for a two-dimensional integral manifold of the augmented ideal

$$\langle C^1, dC^1, L, dL, V \lrcorner\, C^1, V \lrcorner\, C^1, V \lrcorner\, L, V \lrcorner\, dL \rangle, \tag{11}$$

defined on $J^1(U^2, V^1)$, which also satisfies the transverse condition.

The symmetry conditions in (9) and (10) can be used to easily prove that (11) is a differential ideal, and it is clear that V is a Cauchy characteristic vector field of the differential ideal. Though this obvious latter fact has also been noted by Estabrook [28], we show in Lemma 3 below that for hyperbolic and parabolic PDEs of the form in (1), there exists a more useful extension of this result.

Finally, we can simplify (11) in the following way: It is not hard to establish from using (9) and (10) that (11) is equal to

$$\langle C^1, dC^1, L, dL, V \lrcorner C^1, d(V \lrcorner C^1), V \lrcorner L, d(V \lrcorner L)\rangle. \tag{12}$$

In the next section we examine (12) more closely and show that two further reductions are possible.

V. FIRST MAIN RESULT

The class of second order PDEs we deal with is those for which L is decomposable, or equivalently, $L \wedge L = 0$ using Theorem 1.7 in Bryant *et al.* [8]. Although L defined in Theorem 2 is obviously not decomposable for some choices of f_1, f_2, f_3, and k, we will see later in Section VIII that for all hyperbolic and parabolic PDEs of the form in (1) we are able to add to L some multiple of dC^1 which is then decomposable.

Assuming then without loss that L is decomposable, we have

$$0 = Y \lrcorner (L \wedge L) = 2(Y \lrcorner L) \wedge L,$$

for any $Y \in \mathfrak{X}(J^1(U^2, V^1))$, so that if $Y \lrcorner L \neq 0$, then $L = (Y \lrcorner L) \wedge \omega$ for some $\omega \in \Lambda^1(J^1(U^2, V^1))$. Therefore, for decomposable L, any integral manifold of

$$\langle C^1, dC^1, V \lrcorner C^1, d(V \lrcorner C^1), V \lrcorner L, d(V \lrcorner L)\rangle. \tag{13}$$

is an integral manifold of (12) (the two differential ideals are equal for decomposable L). Here V is the symmetry of I^r_F described in the previous section. We shall make use of this condition on L in our two main results, Theorem 4 in this section and Theorem 9 in Section VII.

Since $V \lrcorner C^1$ is a smooth function generator of (13), we can make a further simplification to this differential ideal by pulling it back onto the regular submanifold of $J^1(U^2, V^1)$ described by $V \lrcorner C^1 = 0$, and confine our work to this region of $J^1(U^2, V^1)$. Suppose that the equation $V \lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, which we parameterise by the immersion $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$. Then denoting the pull-back of (13) onto M^4 by

$$J^r_F := \langle \Phi^* C^1, d \circ \Phi^* C^1, \Phi^*(V \lrcorner L), d \circ \Phi^*(V \lrcorner L)\rangle, \tag{14}$$

we have the following lemma:

Lemma 3. *Let $V \in \mathfrak{X}(J^1(U^2, V^1))$ be a symmetry of I^r_F. If the equation $V \lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, which we parameterise by the immersion $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$, then there exists $W \in \mathfrak{X}(M^4)$ with the property that W is a Cauchy characteristic vector field of J^r_F.*

Proof. Let $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$ be a corresponding immersion mapping onto the regular submanifold of $J^1(U^2, V^1)$ described by $V \lrcorner C^1 = 0$. It is clear that the tangent space of

$\Phi(M^4) \subset J^1(U^2, V^1)$ spans the annihilating space of $d\left(V \lrcorner C^1\right)$. From contracting the symmetry condition in (9) with V we obtain, at any point in $\Phi(M^4)$,

$$V \lrcorner d\left(V \lrcorner C^1\right) = \lambda_1 \left(V \lrcorner C^1\right) = 0.$$

Hence V is in the tangent space of $\Phi(M^4)$. Applying Theorem 1, there exists a vector field $W \in \mathfrak{X}(M^4)$ such that $\Phi_* W = V$.

We now proceed to show that W is a Cauchy characteristic vector field of $J^r_{\mathcal{F}}$ by examining each generator of the differential ideal. First,

$$W \lrcorner \Phi^* C^1 = \Phi^*\left(\Phi_* W \lrcorner C^1\right) = 0, \tag{15}$$

where for the first equality we have used the property in (3), and for the second, we have made use of the fact that the pull-back of $V \lrcorner C^1$ by Φ is zero.

Next, we have that

$$W \lrcorner \Phi^* \circ dC^1 = \Phi^*\left(\Phi_* W \lrcorner dC^1\right) = \Phi^*\left(V \lrcorner dC^1\right), \tag{16}$$

once again using (3). Now

$$\begin{aligned} \Phi^*\left(V \lrcorner dC^1\right) &= \Phi^*\left(\lambda_1 C^1 - d(V \lrcorner C^1)\right), \\ &= \left(\Phi^* \lambda_1\right) \Phi^* C^1 - d \circ \Phi^*\left(V \lrcorner C^1\right), \\ &= \left(\Phi^* \lambda_1\right) \Phi^* C^1 \in J^r_{\mathcal{F}}, \end{aligned} \tag{17}$$

where in the first line we have inserted the symmetry condition in (9), and in the second, we have used properties (4) and (5). Combining the end result in (17) with (16) and (4) then gives

$$W \lrcorner d \circ \Phi^* C^1 \in J^r_{\mathcal{F}}. \tag{18}$$

We also have from (3),

$$W \lrcorner \Phi^*(V \lrcorner L) = \Phi^*\left(\Phi_* W \lrcorner V \lrcorner L\right) = \Phi^*\left(V \lrcorner V \lrcorner L\right) = 0. \tag{19}$$

In a similar fashion,

$$W \lrcorner \Phi^* \circ d(V \lrcorner L) = \Phi^*\left(\Phi_* W \lrcorner d(V \lrcorner L)\right) = \Phi^*\left(V \lrcorner d(V \lrcorner L)\right). \tag{20}$$

The symmetry condition in (10) yields

$$\begin{aligned} V \lrcorner d\left(V \lrcorner L\right) &= V \lrcorner \left(\alpha^1 \wedge C^1 + \lambda_2 dC^1 + \lambda_3 L - V \lrcorner dL\right), \\ &= (V \lrcorner \alpha^1) C^1 - (V \lrcorner C^1)\alpha^1 + \lambda_2 (V \lrcorner dC^1) + \lambda_3 (V \lrcorner L). \end{aligned}$$

Pulling this back by Φ, then using (5) and $\Phi^*(V \lrcorner C^1) = 0$ followed by (17) gives

$$\begin{aligned} \Phi^*\left(V \lrcorner d(V \lrcorner L)\right) = &\left(\Phi^*(V \lrcorner \alpha^1)\right) \Phi^* C^1 + \left(\Phi^* \lambda_2\right) \Phi^*\left(V \lrcorner dC^1\right) \\ &+ \left(\Phi^* \lambda_3\right) \Phi^*\left(V \lrcorner L\right) \in J^r_{\mathcal{F}}, \end{aligned} \tag{21}$$

so that combining this result with (20) and (4), we obtain

$$W \lrcorner d \circ \Phi^*(V \lrcorner L) \in J^r_{\mathcal{F}}. \tag{22}$$

Therefore (15), (18), (19) and (22) imply that $W \lrcorner J^r_{\mathcal{F}} \subset J^r_{\mathcal{F}}$. $\square$

From Lemma 3 we obtain the first of our major new results:

Theorem 4. *Given some second order PDE of the form in (1) whose corresponding L is decomposable, let $V \in \mathfrak{X}\left(J^1(U^2, V^1)\right)$ be a symmetry of $I^r_{\overline{F}}$. Suppose the equation $V \lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, and denote $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$ as a corresponding immersion mapping onto this submanifold. With*

$$D^r_{\overline{F}} := \left(sp\left\{\Phi^* C^1, \Phi^*(V \lrcorner L)\right\}\right)^{\perp},$$

if $\Phi^*\left(C^1 \wedge (V \lrcorner L)\right) \neq 0$, then $\Phi_* D^r_{\overline{F}}$ generates a two-dimensional integral manifold of $I^r_{\overline{F}}$. If, in addition, $dx^1 \wedge dx^2 \neq 0$ on $\Phi_* D^r_{\overline{F}}$, then the integral manifold is the image of the 1-jet of some local solution of the PDE in (1).

Proof. We know from the proof of Lemma 3 that $V = \Phi_* W$ for some $W \in \mathfrak{X}(M^4)$. Since $\Phi^*\left(C^1 \wedge (V \lrcorner L)\right) \neq 0$, it follows that $D^r_{\overline{F}}$ is two-dimensional. From Lemma 3, W is a Cauchy characteristic vector field of the differential ideal $J^r_{\overline{F}}$ defined in (14), which implies $[W, Y] \in D^r_{\overline{F}}$ for all $Y \in D^r_{\overline{F}}$ [29, 26]. Hence $D^r_{\overline{F}}$ is Frobenius integrable. Since it is assumed Φ is diffeomorphic onto its image, $\Phi_* D^r_{\overline{F}}$ is well-defined. Now let $Z_1, Z_2 \in \Phi_* D^r_{\overline{F}}$. This means

$$Z_1 = \Phi_* P_1, \qquad Z_2 = \Phi_* P_2,$$

for some $P_1, P_2 \in D^r_{\overline{F}}$. Using (2) and the fact that $D^r_{\overline{F}}$ is Frobenius integrable, we then get

$$[Z_1, Z_2] = [\Phi_* P_1, \Phi_* P_2] = \Phi_*[P_1, P_2] \in \Phi_* D^r_{\overline{F}},$$

so $\Phi_* D^r_{\overline{F}}$ is Frobenius integrable.

Suppose that $\Psi : M^2 \longrightarrow M^4$ is an immersion mapping onto any leaf of the foliation of M^4 generated by $D^r_{\overline{F}}$. Thus $\Psi^* J^r_{\overline{F}} = 0$. Using (6),

$$(\Phi \circ \Psi)^* C^1 = \Psi^*(\Phi^* C^1) = 0, \tag{23}$$

and from (4),

$$(\Phi \circ \Psi)^*(dC^1) = d\left((\Phi \circ \Psi)^* C^1\right) = 0. \tag{24}$$

By assumption, $\Phi^*\left(C^1 \wedge (V \lrcorner L)\right) \neq 0$. This implies $V \lrcorner L \neq 0$. Since L is decomposable, we have $L = (V \lrcorner L) \wedge \omega$ for some $\omega \in \Lambda^1\left(J^1(U^2, V^1)\right)$. Concentrating on $V \lrcorner L$,

$$0 = \Psi^*\left(\Phi^*(V \lrcorner L)\right) = (\Phi \circ \Psi)^*(V \lrcorner L),$$

which gives

$$\Psi^*\left(\Phi^* L\right) = \Psi^*\left((\Phi^*(V \lrcorner L)) \wedge (\Phi^* \omega)\right) = \left((\Phi \circ \Psi)^*(V \lrcorner L)\right) \wedge \left((\Phi \circ \Psi)^* \omega\right) = 0. \tag{25}$$

Hence from (23), (24) and (25), it then follows that $(\Phi \circ \Psi)^* I^r_{\overline{F}} = 0$. If the transverse condition holds, then $\Phi \circ \Psi(M^2) = j^1 h(U^2)$ for some $h \in C^\infty(U^2, V^1)$, with h as some local solution of (1). □

Remark. In order to satisfy the transverse requirement, the symmetry V in Theorem 4 must necessarily satisfy the condition $d(V \lrcorner C^1) \wedge dx^1 \wedge dx^2 \neq 0$. If this is not the case, then $\Phi^*(dx^1 \wedge dx^2) = 0$, and hence *for all* Ψ, $(\Phi \circ \Psi)^*(dx^1 \wedge dx^2) = 0$. Consequently the transverse requirement fails.

We illustrate Theorem 4 with the following example:

Example 5. Consider the heat equation

$$\frac{\partial^2 u}{\partial (x^1)^2} = \frac{\partial u}{\partial x^2}. \tag{26}$$

Defined on $J^1(U^2, V^1)$ we have

$$I_{\overline{F}}^r = \langle C^1, dC^1, L, dL \rangle,$$

where $F = z_{11}^1 - z_2^1$ and $L = (dz_1^1 - z_2^1 dx^1) \wedge dx^2$. Now

$$V := x^1 \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^2}$$

is a Lie point symmetry of (26), and we use its first prolongation $V^{(1)}$, where

$$V^{(1)} = x^1 \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^2} - z_1^1 \frac{\partial}{\partial z_1^1} - 2z_2^1 \frac{\partial}{\partial z_2^1},$$

as our non-trivial symmetry of $I_{\overline{F}}^r$.

Applying Theorem 4, we define the four-dimensional regular submanifold $M^4 \subset J^1(U^2, V^1)$ by the locus of

$$V^{(1)} \lrcorner\, C^1 = -x^1 z_1^1 - x^2 z_2^1 = 0.$$

In a simplified manner without explicitly introducing an immersion, we assume M^4 has coordinates x^1, x^2, z^1, z_1^1 with $x^2 \neq 0$, so that on M^4,

$$\begin{aligned} C^1 &= dz^1 - z_1^1 dx^1 + \frac{z_1^1 x^1}{2x^2} dx^2, \\ V^{(1)} \lrcorner\, L &= -z_1^1 x^1 dx^1 + z_1^1 \left(\frac{(x^1)^2}{2x^2} - 1 \right) dx^2 - 2x^2 dz_1^1, \end{aligned} \tag{27}$$

with

$$J_{\overline{F}}^r = \langle C^1, dC^1, V^{(1)} \lrcorner\, L, d(V^{(1)} \lrcorner\, L) \rangle,$$

also defined on M^4. From Theorem 4 we have that $D_{\overline{F}}^r \subset \mathfrak{X}(M^4)$ generated by the annihilating space of the equations in (27) is Frobenius integrable. It is easy to show that on $D_{\overline{F}}^r$, the transverse condition $dx^1 \wedge dx^2 \neq 0$ holds, so we expect to get some local solution to the heat equation. Then applying Proposition 4.7 in Sherring and Prince [21] with a solvable structure of two symmetries, where $X_2 := \frac{\partial}{\partial z^1} \in \mathfrak{X}(M^4)$ is a non-trivial symmetry of $D_{\overline{F}}^r$, $X_1 := z_1^1 \frac{\partial}{\partial z_1^1} \in \mathfrak{X}(M^4)$ is a non-trivial symmetry of $D_{\overline{F}}^r \oplus sp\{X_2\}$, and defining

$$\Omega := \left(dz^1 - z_1^1 dx^1 + \frac{z_1^1 x^1}{2x^2} dx^2 \right) \wedge \left(-z_1^1 x^1 dx^1 + z_1^1 \left(\frac{(x^1)^2}{2x^2} - 1 \right) dx^2 - 2x^2 dz_1^1 \right),$$

we find

$$\begin{aligned} \frac{X_2 \lrcorner\, \Omega}{X_1 \lrcorner\, X_2 \lrcorner\, \Omega} &= d \left(\ln(z_1^1 \sqrt{x^2}) + \frac{(x^1)^2}{4x^2} \right), \\ \frac{X_1 \lrcorner\, \Omega}{X_2 \lrcorner\, X_1 \lrcorner\, \Omega} &\equiv d \left(z^1 - 2z_1^1 \sqrt{x^2} \exp\left(\frac{(x^1)^2}{4x^2} \right) \int \exp\left(-\xi^2\right) d\xi \right) \mod \frac{X_2 \lrcorner\, \Omega}{X_1 \lrcorner\, X_2 \lrcorner\, \Omega}, \end{aligned}$$

where $\xi := x^1/(2\sqrt{x^2})$. Putting

$$\ln(z_1^1 \sqrt{x^2}) + \frac{(x^1)^2}{4x^2} = c^1,$$

and

$$z^1 - 2z_1^1\sqrt{x^2}\exp\left(\frac{(x^1)^2}{4x^2}\right)\int \exp\left(-\xi^2\right)d\xi = c^2,$$

for any constants c^1, c^2, we obtain

$$u = 2\exp(c^1)\int \exp\left(-\xi^2\right)d\xi$$

as our local similarity solution of the heat equation corresponding to V.

We close this section with a warning that there will exist situations when applying Theorem 4 will yield a distribution $\Phi_* D_{\overline{F}}^r$ that is not transverse, even with $d(V \lrcorner C^1) \wedge dx^1 \wedge dx^2 \neq 0$. In such cases we must abandon the above approach and look to use elements of $I_{\overline{F}}^r$ that are in a sense singular. This is explained in full in the next section.

VI. A SINGULAR APPROACH

Consider a differential ideal $I := \langle \alpha^1, \alpha^2 \rangle$ defined on some open, convex neighbourhood $U^4 \subset \mathbb{R}^4$ with coordinates $x^1, \ldots, x^4$, generated by two linearly independent one-forms $\alpha^1, \alpha^2 \in \Lambda^1(U^4)$. Suppose that for each $i \in \{1,2\}$, $d\alpha^i \equiv 0 \bmod \alpha^1, \alpha^2$, i.e. $\ker(\alpha^1 \wedge \alpha^2)$ is Frobenius integrable. Here, we choose to work with a two-dimensional Pfaffian system defined on a four-dimensional space because the material in the following section on second order hyperbolic or parabolic PDEs of the type in (1) is precisely of this nature, but all results that follow in this section can easily be extended to arbitrary dimensions.

For integrating the Frobenius integrable distribution $\ker(\alpha^1 \wedge \alpha^2)$ using solvable symmetry structures, we can use Proposition 4.7 in Sherring and Prince [21] to find some functions $f_1^1, f_2^1, f_1^2, f_2^2, g^1, g^2 \in C^\infty(U^4)$ such that

$$\begin{aligned} f_1^1\alpha^1 + f_2^1\alpha^2 &= dg^1, \\ f_1^2\alpha^1 + f_2^2\alpha^2 &= dg^2. \end{aligned} \tag{28}$$

If, on U^4, the functions g^1, g^2 are of constant maximal rank two, then the equations $g^1 = c^1, g^2 = c^2$ describe a two-dimensional regular submanifold of U^4. Let $\Psi : M^2 \longrightarrow U^4$ be an immersion mapping onto this submanifold. If, in addition, the determinant

$$\Psi^* \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \neq 0$$

on M^2, then (28) and the fact that $\Psi^*(dg^1) = 0 = \Psi^*(dg^2)$ imply $\Psi^*\alpha^1 = 0 = \Psi^*\alpha^2$. Hence Ψ is a two-dimensional integral manifold of I, for arbitrary constant functions c^1, c^2.

The problem with the above 'regular' approach used in Theorem 4 for dealing with a PDE of the form in (1) is that if the submanifold generated by $\Phi_* D_{\overline{F}}^r$ is not transverse, then the method fails to give us a local solution with u as some smooth function of x^1, x^2.

Our goal in this section and the next is to provide an alternative approach for finding two-dimensional integral manifolds of I, which includes the above situation as a sub-class, as well as applies to PDE problems when $\Phi_* D_{\overline{F}}^r$ *may or may not* be transverse. We will also see that the trade-off for this extra flexibility is that there is no direct computational approach using solvable symmetry structures, however using the Frobenius integrable nature of $\ker(\alpha^1 \wedge \alpha^2)$ (or $\Phi_* D_{\overline{F}}^r$ in Theorem 4) we do come close.

Consider then the following obvious extension to the above discussion:

Theorem 6. *With α^1, α^2 and I defined as above, let there exist $f_1^1, f_2^1, f_1^2, f_2^2, g^{11}, g^{12}, g^{21}, g^{22} \in C^\infty(U^4)$ such that*

$$\begin{aligned} f_1^1\alpha^1 + f_2^1\alpha^2 &= g^{11}dg^{12}, \\ f_1^2\alpha^1 + f_2^2\alpha^2 &= g^{21}dg^{22}. \end{aligned} \tag{29}$$

Suppose that for some $p, q \in \{1,2\}$, the equations

$$g^{1p} = \begin{cases} 0 & \text{if } p = 1, \\ c^1 & \text{otherwise,} \end{cases} \qquad g^{2q} = \begin{cases} 0 & \text{if } q = 1, \\ c^2 & \text{otherwise,} \end{cases}$$

for some constants c^1, c^2 describe a two-dimensional regular submanifold of U^4, and let $\Psi : M^2 \longrightarrow U^4$ be an immersion mapping onto this submanifold. If, on M^2, the determinant

$$\Psi^* \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \neq 0, \tag{30}$$

then Ψ is a two-dimensional integral manifold of I.

For PDE problems, Theorem 6 will be used to find alternative (hopefully transverse) integral manifolds of I to those found with the usual approach reviewed at the start of this section. Unfortunately there is no algorithmic technique (without involving ODEs) for establishing (29) by means other than following direct one using Proposition 4.7 in Sherring and Prince [21] that incorporates symmetry:

Suppose then we apply Proposition 4.7 with $X_2 \in \mathfrak{X}(U^4)$ as a non-trivial symmetry of $\ker(\alpha^1 \wedge \alpha^2)$, and $X_1 \in \mathfrak{X}(U^4)$ as a non-trivial symmetry of $sp\{X_2\} \oplus \ker(\alpha^1 \wedge \alpha^2)$. We then obtain

$$\begin{aligned} \frac{X_2 \lrcorner (\alpha^1 \wedge \alpha^2)}{X_1 \lrcorner X_2 \lrcorner (\alpha^1 \wedge \alpha^2)} &= dg^{12}, \\ \frac{X_1 \lrcorner (\alpha^1 \wedge \alpha^2)}{X_2 \lrcorner X_1 \lrcorner (\alpha^1 \wedge \alpha^2)} &= dg^{22} - X_1(g^{22})dg^{12}, \end{aligned} \tag{31}$$

for some $g^{12}, g^{22} \in C^\infty(U^4)$. This gives integral manifolds of I defined by $g^{12} = c^1$, $g^{22} = c^2$ for constants c^1, c^2. Suppose these are not transverse. Rearranging the equations in (31) gives

$$\begin{aligned} \left(X_2 \lrcorner \alpha^2\right)\alpha^1 - \left(X_2 \lrcorner \alpha^1\right)\alpha^2 &= \left(X_2 \lrcorner X_1 \lrcorner (\alpha^1 \wedge \alpha^2)\right) dg^{12}, \\ \left((X_1 + X_1(g^{22})X_2) \lrcorner \alpha^2\right)\alpha^1 - &\left((X_1 + X_1(g^{22})X_2) \lrcorner \alpha^1\right)\alpha^2 \\ &= \left(X_1 \lrcorner X_2 \lrcorner (\alpha^1 \wedge \alpha^2)\right) dg^{22}. \end{aligned} \tag{32}$$

Now applying Theorem 6 with the equations in (32), we set

$$g^{11} = -g^{21} = X_2 \lrcorner X_1 \lrcorner (\alpha^1 \wedge \alpha^2).$$

We cannot choose $p = 2, q = 2$ since by assumption these integral manifolds of I are not transverse. We also cannot choose $p = 1, q = 1$ because $g^{11} = -g^{21}$ implies we do not obtain a regular two-dimensional submanifold of U^4. This is clearly due to the constant maximal rank two requirement failing. Therefore we require that at least one of the two remaining (p, q) combinations satisfy the rank two condition. Finally, the equation in (30) must also be satisfied, i.e.

$$\Psi^* \begin{vmatrix} X_2 \lrcorner \alpha^2 & -X_2 \lrcorner \alpha^1 \\ (X_1 + X_1(g^{22})X_2) \lrcorner \alpha^2 & -(X_1 + X_1(g^{22})X_2) \lrcorner \alpha^1 \end{vmatrix} \neq 0.$$

Below is a modification of Theorem 6, which shows that if we are given just one of the equations in (29) (found for example by inspection, or using Proposition 4.7 in Sherring and Prince [21] as in the above), then the other can be determined using a symmetry:

Theorem 7. *With α^1, α^2 and I defined as above, let there exist $f_1^1, f_2^1, g^{11}, g^{12} \in C^\infty(U^4)$ such that*

$$f_1^1\alpha^1 + f_2^1\alpha^2 = g^{11}dg^{12}. \tag{33}$$

Suppose that for some $p \in \{1,2\}$, the equation

$$g^{1p} = \begin{cases} 0 & \text{if } p = 1, \\ c^1 & \text{otherwise}, \end{cases} \tag{34}$$

for some constant c^1 describes a three-dimensional regular submanifold of U^4. Let $\Theta : M^3 \longrightarrow U^4$ denote an immersion mapping onto this submanifold, and let $X \in \mathfrak{X}(M^3)$ be a non-trivial symmetry of $\Theta^*\left(f_1^2\alpha^1 + f_2^2\alpha^2\right)$, for some $f_1^2, f_2^2 \in C^\infty(U^4)$. Then there exist $\overline{g}^{21}, \overline{g}^{22} \in C^\infty(M^3)$ such that

$$\Theta^*\left(f_1^2\alpha^1 + f_2^2\alpha^2\right) = \overline{g}^{21}d\overline{g}^{22}.$$

Further suppose that, for some $q \in \{1,2\}$, the equation

$$\overline{g}^{2q} = \begin{cases} 0 & \text{if } q = 1, \\ c^2 & \text{otherwise}, \end{cases}$$

for some constant c^2 describes a two-dimensional regular submanifold of M^3. With $\Psi : M^2 \longrightarrow M^3$ denoting an immersion mapping onto this submanifold, if

$$(\Theta \circ \Psi)^* \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \neq 0, \tag{35}$$

on M^2, then $\Theta \circ \Psi$ is a two-dimensional integral manifold of I.

Proof. Since for each $i \in \{1,2\}$, $d\alpha^i \equiv 0 \mod \alpha^1, \alpha^2$, it follows that with

$$\begin{aligned} \beta^1 :=& f_1^1\alpha^1 + f_2^1\alpha^2, \\ \beta^2 :=& f_1^2\alpha^1 + f_2^2\alpha^2, \end{aligned}$$

we have for each $i \in \{1,2\}$, $d\beta^i \equiv 0 \mod \beta^1, \beta^2$ for arbitrary choice of $f_1^1, f_2^1, f_1^2, f_2^2 \in C^\infty(U^4)$. Let β^1 satisfy (33) for some f_1^1, f_2^1 and some $g^{11}, g^{12} \in C^\infty(U^4)$, and for some $p \in \{1,2\}$, let the immersion $\Theta : M^3 \longrightarrow U^4$, defined as in the theorem, map onto the regular submanifold of U^4 given by (34). Then $\Theta^*\beta^1 = 0$, so that

$$d\left(\Theta^*\beta^2\right) = \Theta^*\left(d\beta^2\right) = \left(\Theta^*\mu_1\right)\Theta^*\beta^1 + \left(\Theta^*\mu_2\right)\Theta^*\beta^2 \equiv 0 \mod \Theta^*\beta^2,$$

for some $\mu_1, \mu_2 \in C^\infty(U^4)$. Let $X \in \mathfrak{X}(M^3)$ be a non-trivial symmetry of $\Theta^*\beta^2$. Hence from Proposition 4.7 in Sherring and Prince [21] (or even Theorem 2.1 in the same paper), we obtain

$$d\left(\frac{\Theta^*\beta^2}{X \lrcorner \left(\Theta^*\beta^2\right)}\right) = 0.$$

Therefore

$$\Theta^*\beta^2 = \left(X \lrcorner \left(\Theta^*\beta^2\right)\right) d\overline{g}^{22},$$

for some $g^{22} \in C^\infty(M^3)$. We set $g^{21} = X \lrcorner (\Theta^* \beta^2)$ and choose g^{2q} such that it is of constant maximal rank one on M^3. Hence with Ψ defined as in the theorem, we have

$$(\Theta \circ \Psi)^* \beta^1 = 0 = (\Theta \circ \Psi)^* \beta^2.$$

By the assumption in (35), it is then clear that $\Theta \circ \Psi$ is a two-dimensional integral manifold of I. □

Remark. The functions f_1^2, f_2^2 in Theorem 7 are not quite arbitrary: First they must be chosen so that

$$\Theta^* \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \neq 0,$$

on M^3, or else (35) fails for any Ψ. Then once Ψ is known, (35) must be checked.

Certainly the difficult part in applying Theorem 7 is in establishing (33). Once this is done however, the remaining assumptions in the theorem simply involve two maximal rank conditions, one non-zero determinant condition and one non-trivial symmetry.

Another observation we can make regarding Theorem 7 is that $\ker(\alpha^1 \wedge \alpha^2)$ must be Frobenius integrable. Of course, even if $\ker(\alpha^1 \wedge \alpha^2)$ is not Frobenius integrable, singular two-dimensional integral manifolds of I may still exist.

The following example illustrates Theorem 7:

Example 8. Suppose on some suitably chosen U^4 where $x^2 \neq 0$, $I := \langle \alpha^1, \alpha^2 \rangle$ with

$$\begin{aligned} \alpha^1 &:= dx^3 + \frac{x^1}{2x^2} dx^1 - x^4 dx^2, \\ \alpha^2 &:= \left(2x^2 x^4 - \frac{(x^1)^2}{2x^2} + 1 \right) dx^2. \end{aligned}$$

It is easy to show that for all $i \in \{1, 2\}$, $d\alpha^i \equiv 0 \mod \alpha^1, \alpha^2$, and so $\ker(\alpha^1 \wedge \alpha^2)$ is Frobenius integrable.

We begin with the 'regular' approach to integrating $\ker(\alpha^1 \wedge \alpha^2)$ reviewed at the beginning of this section. Simple inspection (or Proposition 4.7 in Sherring and Prince [21]) yields

$$\alpha^1 \wedge \alpha^2 = \left(2x^2 x^4 - \frac{(x^1)^2}{2x^2} + 1 \right) d \left(x^3 + \frac{(x^1)^2}{4x^2} \right) \wedge dx^2.$$

Hence if the equations

$$x^2 = c^1, \qquad x^3 + \frac{(x^1)^2}{4x^2} = c^2,$$

for arbitrary constants c^1, c^2 are constant maximal rank two on some suitably chosen neighbourhood of U^4, then they describe a two-dimensional foliation of the neighbourhood, where each leaf is a regular submanifold that is an integral manifold of I.

We now look to apply Theorem 7 in order to generate different two-dimensional integral manifolds of I. Applying the theorem, suppose we choose $f_1^1 := 0$, $f_2^1 := 1$, and

$$g^{11} := 2x^2 x^4 - \frac{(x^1)^2}{2x^2} + 1, \qquad g^{12} := x^2,$$

so that (33) holds. We set

$$g^{11} = 0. \tag{36}$$

We also choose $f_1^2 := 1$, $f_2^2 := 0$. Again without explicitly introducing an immersion, and pulling-back α^1 onto M^3 defined by (36) with coordinates for M^3 given by x^1, x^2, x^3, we find (on M^3)

$$\alpha^1 = dx^3 + \frac{x^1}{2x^2} dx^1 + \frac{1}{2x^2} \left(1 - \frac{(x^1)^2}{2x^2} \right) dx^2,$$

which, from Theorem 7, is closed modulo itself. Applying Theorem 2.1 in [21] with $\frac{\partial}{\partial x^3}$ as a non-trivial symmetry of α^1, we get

$$\alpha^1 = d \left(x^3 + \ln(\sqrt{x^2}) + \frac{(x^1)^2}{4x^2} \right),$$

so

$$\overline{g}^{21} = 1, \qquad \overline{g}^{22} = x^3 + \ln(\sqrt{x^2}) + \frac{(x^1)^2}{4x^2}.$$

Hence our only choice is to set

$$\overline{g}^{22} = c^3,$$

where c^3 is an arbitrary constant function. On a suitable neighbourhood of U^4 the equations

$$2x^2 x^4 - \frac{(x^1)^2}{2x^2} + 1 = 0, \qquad x^3 + \ln(\sqrt{x^2}) + \frac{(x^1)^2}{4x^2} = c^3 \tag{37}$$

are of constant maximal rank two, and it is easy to see from above that the non-zero determinant condition in (35) holds. Hence the equations in (37) describe a two-dimensional regular submanifold of the neighbourhood of U^4, that is an integral manifold of I. Note that the two-dimensional leaves described by (37) do not generate a foliation of the neighbourhood. Rather, the three-dimensional regular submanifold of the neighbourhood described by the equation on the left in (37) is foliated by the two-dimensional leaves generated by the equation on the right.

VII. A SINGULAR APPLICATION

In this section we use Theorem 7 to provide an alternative to Theorem 4 when the transverse requirement fails for $\Phi_* D^r_{\overline{F}}$. The following result is the second of our major results:

Theorem 9. *Given some second order PDE of the form in (1) whose corresponding L is decomposable, let $V \in \mathfrak{X}(J^1(U^2, V^1))$ be a symmetry of $I^r_{\overline{F}}$. Suppose the equation $V \lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, and let $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$ denote an immersion mapping onto this submanifold. Further suppose $\Phi^*(C^1 \wedge (V \lrcorner L)) \neq 0$, and we have applied Theorem 7, with $\alpha^1 := \Phi^* C^1$ and $\alpha^2 := \Phi^*(V \lrcorner L)$, thus generating some smooth $g^{1p}, \overline{g}^{2q}$ and immersions $\Theta : M^3 \longrightarrow J^1(U^2, V^1)$ and $\Psi : M^2 \longrightarrow M^3$, as in the theorem. If*

$$(\Phi \circ \Theta \circ \Psi)^*(dx^1 \wedge dx^2) \neq 0, \tag{38}$$

then $\Phi \circ \Theta \circ \Psi(M^2)$ is the image of the 1-jet of some local solution of the PDE in (1).

Proof. Using Lemma 3, we have on M^4 that

$$D^r_{\overline{F}} := \left(sp \left\{ \Phi^* C^1, \Phi^*(V \lrcorner L) \right\} \right)^\perp$$

is Frobenius integrable. Applying Theorem 7 to $J^r_{\overline{F}}$ defined in (14) then generates a two-dimensional integral manifold of $J^r_{\overline{F}}$ given by

$$\Theta \circ \Psi : M^2 \longrightarrow M^4.$$

At this point the proof becomes very similar to that of Theorem 4. As L is decomposable, we find that

$$\Phi \circ \Theta \circ \Psi : M^2 \longrightarrow J^1(U^2, V^1)$$

is a two-dimensional integral manifold of $I^r_{\overline{F}}$. The condition in (38) is a transverse requirement. It then clear that the image of $\Phi \circ \Theta \circ \Psi$ is equal to the image of the 1-jet of some local solution of the PDE in (1). □

Remark 1. Theorem 9 can obviously be modified by replacing Theorem 7 with Theorem 6.

Remark 2. While Theorem 9 does not require that $\Phi_* D^r_{\overline{F}}$ be transverse, a transverse requirement must still be introduced, but at a later stage.

The following example attempts to clarify Theorem 9:

Example 10. Consider the Potential Burgers' Equation

$$\frac{\partial u}{\partial x^2} - \frac{\partial^2 u}{\partial (x^1)^2} - \left(\frac{\partial u}{\partial x^1}\right)^2 = 0. \tag{39}$$

Defined on $J^1(U^2, V^1)$ we have

$$I^r_{\overline{F}} = \langle C^1, dC^1, L, dL \rangle,$$

where $F = z^1_2 - z^1_{11} - (z^1_1)^2$ and $L = ((z^1_2 - (z^1_1)^2)\, dx^1 - dz^1_1) \wedge dx^2$. Now

$$V := 2x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial z^1}$$

is a Lie point symmetry of (39), and we use its first prolongation $V^{(1)}$, where

$$V^{(1)} = 2x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial z^1} - \frac{\partial}{\partial z^1_1} - 2z^1_1 \frac{\partial}{\partial z^1_2},$$

as our non-trivial symmetry of $I^r_{\overline{F}}$.

Applying Theorem 9, we define M^4 to be the four-dimensional regular submanifold of $J^1(U^2, V^1)$ given by the locus of

$$V^{(1)} \lrcorner\, C^1 = -x^1 - 2x^2 z^1_1 = 0.$$

We assume M^4 has coordinates x^1, x^2, z^1, z^1_2 with $x^2 \neq 0$, so that on M^4 we have

$$C^1 = dz^1 + \frac{x^1}{2x^2} dx^1 - z^1_2 dx^2,$$

$$V^{(1)} \lrcorner\, L = \left(2x^2 z^1_2 - \frac{(x^1)^2}{2x^2} + 1\right) dx^2.$$

It is clear that the transverse condition does not hold on the two-dimensional annihilating space of $sp\{C^1, V^{(1)} \lrcorner\, L\}$ defined on M^4, so we will look to use Theorem 7. In applying this result, we refer to Example 8 which makes use of the theorem with x^3 replacing z^1 and x^4 replacing z^1_2 so that $\alpha^1 = C^1$ and $\alpha^2 = V^{(1)} \lrcorner\, L$. From the example, we then get that

$$u = -\ln(\sqrt{x^2}) - \frac{(x^1)^2}{4x^2} + c^3,$$

for any constant c^3 is a similarity solution of (39) corresponding to V.

VIII. DECOMPOSABILITY EXAMINED

Theorems 4 and 9 appear to be restricted by the requirement that L (or $L^\dagger$) be decomposable. However, since dC^1 is in $I^{r}_{\overline{F}}$, we may look to add some multiple $b \in J^1(U^2, V^1)$ of dC^1 to L so that $L + bdC^1$ is decomposable.

Without loss, working this time with $L^\dagger$, we define the following two-form

$$\Omega^\dagger := L^\dagger + bdC^1,$$

where b is for the moment any smooth function on the first jet bundle $J^1(U^2, V^1)$. The following lemma gives a simple quadratic condition on b in order that $\Omega^\dagger \wedge \Omega^\dagger = 0$, so that $\Omega^\dagger$ is decomposable by Theorem 1.7 in [8].

Lemma 11. *With $\Omega^\dagger := L^\dagger + bdC^1$, if*

$$b = \frac{-f_3 \pm \sqrt{f_3^2 - 4f_1f_2}}{2},$$

with $f_3^2 - 4f_1f_2 \geq 0$, then $\Omega^\dagger$ is decomposable.

Proof.

$$\left(L^\dagger + bdC^1\right)^2 = (L^\dagger)^2 + 2bdC^1 \wedge L^\dagger + b^2(dC^1)^2,$$

and

$$\begin{aligned}(dC^1)^2 &= 2dz_1^1 \wedge dx^1 \wedge dz_2^1 \wedge dx^2,\\ (L^\dagger)^2 &= -2f_1f_2dz_1^1 \wedge dx^2 \wedge dz_2^1 \wedge dx^1,\\ dC^1 \wedge L^\dagger &= f_3dz_2^1 \wedge dx^2 \wedge dz_1^1 \wedge dx^1.\end{aligned}$$

Hence

$$\left(L^\dagger + bdC^1\right)^2 = 2\left(b^2 + bf_3 + f_1f_2\right)dz_1^1 \wedge dx^1 \wedge dz_2^1 \wedge dx^2.$$

It follows that if

$$b = \frac{-f_3 \pm \sqrt{f_3^2 - 4f_1f_2}}{2},$$

where b is real on $J^1(U^2, V^1)$, then $\Omega^\dagger \wedge \Omega^\dagger = 0$, and therefore by Theorem 1.7 in [8], $\Omega^\dagger$ is decomposable. □

Proved in a similar way to Lemma 11, we have the following for L:

Lemma 12. *With $\Omega := L + bdC^1$, if*

$$b = \frac{f_3 \pm \sqrt{f_3^2 - 4f_1f_2}}{2},$$

with $f_3^2 - 4f_1f_2 \geq 0$, then Ω is decomposable.

The requirement that the discriminant in Lemmas 11 and 12 remains non-negative on $J^1(U^2, V^1)$ (or on some suitable neighbourhood), coincides exactly with the condition found widely in the literature that the second order PDE in (1) be hyperbolic or parabolic. Hence, if the PDE is of one of these two types, then we are always able to determine a decomposable Ω (or $\Omega^\dagger$). Thus we can apply Theorems 4 and 9 by simply replacing the L in these two theorems with Ω. We illustrate with an example:

Example 13. Consider the non-linear wave equation:

$$\frac{\partial^2 u}{\partial (x^2)^2} = u\frac{\partial^2 u}{\partial (x^1)^2}. \tag{40}$$

In terms of coordinates of $J^1(U^2, V^1)$, this equation admits the point symmetry

$$V := x^2\frac{\partial}{\partial x^2} - 2z^1\frac{\partial}{\partial z^1},$$

whose first prolongation is

$$V^{(1)} = x^2\frac{\partial}{\partial x^1} - 2z^1\frac{\partial}{\partial z^1} - 2z^1_1\frac{\partial}{\partial z^1_1} - 3z^1_2\frac{\partial}{\partial z^1_2}.$$

Working with L, we have

$$L = -z^1 dz^1_1 \wedge dx^2 - dz^1_2 \wedge dx^1,$$

which is clearly not decomposable. From Lemma 12, we find that $L \pm \sqrt{z^1}dC^1$ is decomposable. Taking the positive option gives

$$\begin{aligned} \Omega_+ &:= L + \sqrt{z^1}dC^1, \\ &= \left(dz^1_2 - \sqrt{z^1}dz^1_1\right) \wedge \left(\sqrt{z^1}dx^2 - dx^1\right). \end{aligned}$$

Applying Theorem 4, we define the four-dimensional regular submanifold $M^4 \subset J^1(U^2, V^1)$ by the locus of

$$V^{(1)} \lrcorner C^1 = -x^2 z^1_2 - 2z^1 = 0.$$

Let M^4 have coordinates x^1, x^2, z^1, z^1_1 with $x^2 \neq 0$. Then we have on M^4,

$$C^1 = dz^1 - z^1_1 dx^1 + \frac{2z^1}{x^2}dx^2,$$

$$V^{(1)} \lrcorner \Omega_+ = \left(-\frac{6z^1}{x^2} - 2\sqrt{z^1}z^1_1\right)dx^1 + \left(\frac{4(z^1)^{\frac{3}{2}}}{x^2} + 2z^1 z^1_1\right)dx^2 + 2\sqrt{z^1}dz^1 + x^2 z^1 dz^1_1.$$

It is easy to show that the transverse condition holds on the two-dimensional annihilating space of $sp\{C^1, V^{(1)} \lrcorner \Omega_+\}$ defined on M^4. By inspection,

$$X_1 := \frac{\partial}{\partial x^1} \in \mathfrak{X}(M^4)$$

is a non-trivial symmetry of $C^1 \wedge (V^{(1)} \lrcorner \Omega_+)$ (pulled-back onto M^4). Using the Lie symmetry analysis software package `DIMSYM` [30], we find

$$X_2 := -\frac{1}{(x^2)^2}\frac{\partial}{\partial z^1} \in \mathfrak{X}(M^4)$$

is another non-trivial symmetry of $C^1 \wedge (V^{(1)} \lrcorner \Omega_+)$, which also commutes with X_1. Therefore, taking advantage of this situation and applying Theorems 4.1 and 5.1 in [21] gives the two closed forms

$$\begin{aligned} \frac{X_1 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))}{X_2 \lrcorner X_1 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))} &= d\left(\frac{(x^2)^4 (z^1_1)^2}{12} - (x^2)^2 z^1\right), \\ \frac{X_2 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))}{X_1 \lrcorner X_2 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))} &= d\left(x^1 - \frac{(x^2)^2 z^1_1}{6}\right). \end{aligned}$$

Putting

$$\frac{(x^2)^4(z^1_1)^2}{12} - (x^2)^2 z^1 = c^1, \qquad x^1 - \frac{(x^2)^2 z^1_1}{6} = c^2,$$

for any constants c^1, c^2, we obtain

$$u = \frac{3\left(x^1 - c^2\right)^2 - c^1}{(x^2)^2}$$

as our similarity solution of the non-linear wave equation in (40) corresponding to V.

IX. CONDITIONAL SYMMETRIES

Following Olver [6], Stephani [7] or Bluman and Cole [23], a *conditional symmetry* $V \in \mathfrak{X}(U^2 \times V^1)$ of some second order PDE in (1) is defined as a Lie point symmetry of the overdetermined system of PDEs given by (1) and the first order quasilinear PDE obtained from

$$V^{(1)} \lrcorner\, C^1 = 0. \tag{41}$$

In this section we show that all results in the previous sections still hold true if instead of the symmetry being the first prolongation of some point symmetry of (1) it is the first prolongation of some conditional symmetry.

We define

$$\widehat{I^r_F} := \langle C^1, dC^1, L, dL, (V^{(1)} \lrcorner\, C^1) dx^1 \wedge dx^2, d(V^{(1)} \lrcorner\, C^1) \wedge dx^1 \wedge dx^2 \rangle,$$

defined on the first jet bundle $J^1(U^2, V^1)$. It is clear from Section III that the image of any two-dimensional integral manifold of $\widehat{I^r_F}$ that satisfies the transverse condition will be that of some 1-jet solution map of the overdetermined system of PDEs given by (1) and (41).

If V is a conditional symmetry of (1), then it follows from the discussion in Section IV that

$$\mathcal{L}_{V^{(1)}} \widehat{I^r_F} \subset \widehat{I^r_F}.$$

Explicitly,

$$\mathcal{L}_{V^{(1)}} C^1 = \lambda_1 C^1, \tag{42}$$

as well as

$$\mathcal{L}_{V^{(1)}} L = \alpha^1 \wedge C^1 + \lambda_2 dC^1 + \lambda_3 L + \lambda_4 \left((V^{(1)} \lrcorner\, C^1) dx^1 \wedge dx^2 \right), \tag{43}$$

and finally,

$$\begin{aligned} \mathcal{L}_{V^{(1)}} \left((V^{(1)} \lrcorner\, C^1) dx^1 \wedge dx^2 \right) = {} & \alpha^2 \wedge C^1 + \lambda_5 dC^1 + \lambda_6 L \\ & + \lambda_7 \left((V^{(1)} \lrcorner\, C^1) dx^1 \wedge dx^2 \right), \end{aligned} \tag{44}$$

for some $\lambda_1, \ldots, \lambda_7 \in C^\infty\left(J^1(U^2, V^1)\right)$ and $\alpha^1, \alpha^2 \in \Lambda^1\left(J^1(U^2, V^1)\right)$.

Suppose in terms of first jet bundle coordinates the equation in (41) describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, which we parameterise by the immersion $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$. It is then obvious that

$$\Phi^* \widehat{I^r_F} = \Phi^* I^r_F.$$

Without loss, we can assume L is decomposable, so that $L = (V^{(1)} \lrcorner\, L) \wedge \omega$ for some $\omega \in \Lambda^1(J^1(U^2, V^1))$ (assume $V^{(1)} \lrcorner\, L \neq 0$). Suppose we now wish to repeat the proof of Lemma 3, where in the lemma,

1. I^r_F is replaced by $\widehat{I^r_F}$,

2. V is replaced by the first prolongation of our conditional symmetry $V^{(1)}$,

3. The symmetry conditions in (9) and (10) are replaced by those in (42) and (43).

Now it is not hard to see that the lemma still holds true, since the pull-back of (43) by Φ forces the final term on the right to vanish. Thus when pulled-back by Φ, the two sets of equations given in item 3 above are in identical form. Hence from the lemma there exists some Cauchy characteristic vector field $W \in \mathfrak{X}(M^4)$ of J^r_F with the property that $\Phi_* W = V^{(1)}$. Consequently, with the same three substitutions given above, Theorems 4 and 9 hold.

Finally, the equation in (44) is not used in the proof of any of our results. Therefore it appears that in order for us to use symmetries of $\widehat{I^r_F}$ to derive non-classical similarity solutions, vector fields from the symmetry algebra of $\widehat{I^r_F}$ are not strictly necessary. One essentially only requires vector fields that satisfy (42) and (43).

Using a conditional symmetry, we now illustrate Theorem 4 with the following example:

Example 14. Consider the heat equation given in (26). From Stephani [7], it has the conditional symmetry

$$V := \tan(x^1)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2},$$

whose first prolongation is given by

$$V^{(1)} = \tan(x^1)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - z^1_1 \sec^2(x^1)\frac{\partial}{\partial z^1_1}.$$

From Example 5, L is decomposable. Applying Theorem 4, we define the four-dimensional regular submanifold $M^4 \subset J^1(U^2, V^1)$ by the locus of

$$V^{(1)} \lrcorner\, C^1 = -z^1_1 \tan(x^1) - z^1_2 = 0.$$

Letting M^4 have coordinates x^1, x^2, z^1, z^1_1, we pull-back C^1 and $V^{(1)} \lrcorner\, L$ so that (on M^4),

$$C^1 = dz^1 - z^1_1 dx^1 + z^1_1 \tan(x^1) dx^2,$$
$$V^{(1)} \lrcorner\, L = -z^1_1 \tan x^1 dx^1 - z^1_1 dx^2 - dz^1_1.$$

It can be shown that on M^4, $\ker\left(C^1 \wedge (V^{(1)} \lrcorner\, L)\right)$ is a two-dimensional Frobenius integrable distribution that satisfies the transverse condition. By inspection,

$$\frac{\partial}{\partial x^2}, \frac{\partial}{\partial z^1} \in \mathfrak{X}(M^4),$$

are two commuting non-trivial symmetries of $C^1 \wedge (V^{(1)} \lrcorner\, L)$. Hence by Propositions 4.1 and 5.1 in [21] we obtain the two closed forms

$$\frac{\frac{\partial}{\partial x^2} \lrcorner\, (C^1 \wedge (V^{(1)} \lrcorner\, L))}{\frac{\partial}{\partial z^1} \lrcorner\, \frac{\partial}{\partial x^2} \lrcorner\, (C^1 \wedge (V^{(1)} \lrcorner\, L))} = d\left(z^1 - z^1_1 \tan(x^1)\right),$$
$$\frac{\frac{\partial}{\partial z^1} \lrcorner\, (C^1 \wedge (V^{(1)} \lrcorner\, L))}{\frac{\partial}{\partial x^2} \lrcorner\, \frac{\partial}{\partial z^1} \lrcorner\, (C^1 \wedge (V^{(1)} \lrcorner\, L))} = d\left(\ln\left|\frac{z^1_1}{\cos(x^1)}\right| + x^2\right).$$

Putting

$$z^1 - z^1_1 \tan(x^1) = c^1, \qquad \ln\left|\frac{z^1_1}{\cos(x^1)}\right| + x^2 = c^2,$$

for any constants c^1 and c^2 yields

$$u = \sin(x^1)\exp(c^2 - x^2) + c^1$$

as our local non-classical similarity solution of the wave equation corresponding to the conditional symmetry V.

X. COMMENTS AND CONCLUSIONS

Our main results, Theorems 4 and 9, combined with Lemmas 11 and 12 show how one may use solvable symmetry structures to extract classical and non-classical similarity solutions of second order hyperbolic or parabolic PDEs of the form in (1). While the two theorems assume L (or $L^\dagger$) is decomposable, it is hardly a restriction. This is because the discriminant in the two lemmas remains non-negative on some neighbourhood precisely when the PDE is hyperbolic or parabolic. Hence, we are always able to apply Theorems 4 and 9 by replacing the given non-decomposable L with a suitable decomposable Ω, which is simply some linear combination of L and dC^1. For Theorem 4 there is a risk that the resulting two-dimensional Frobenius integrable distribution does not satisfy the transverse requirement. If this is the case, then the approach described in the theorem must be abandoned, and we are forced to use the slightly more sophisticated Theorem 9.

It is unfortunate that both Theorems 4 and 9 demand the symmetry V satisfy $d(V \lrcorner C^1) \wedge dx^1 \wedge dx^2 \neq 0$. At this stage it is not clear how to modify our work in such a way so that this restriction is avoided.

Finally, while our work has focused solely on the generation of similarity solutions in the absence of boundary conditions, there is scope for further work with such conditions. As a possible starting point, we know from Theorems 4 and 9 that given a symmetry V, we obtain uniqueness of solution up to two and one arbitrary constants respectively. We leave such research as the topic of another paper.

XI. ACKNOWLEDGMENTS

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