SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS USING SYMMETRY AND SYMBOLIC COMPUTATION

Submitted by

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Contents

Glossary of key symbols

$A(I)$	Cauchy characteristic space of I
C^j	1-st order contact structure
$C^j_{i_1 \cdots i_{k-1}}$	k-th order contact structure
$C(I)$	Cartan system of I
$C^\infty(M)$	set of smooth functions on M
$C^\infty(N,M)$	set of smooth maps from N to M
$D^\perp$	annihilating space of D
$D^r_{\overline{F}}$	annihilating space of $J^r_{\overline{F}}$
d	exterior derivative
F_*	differential (push-forward) of the map F
F^*	pull-back (reciprocal image) of the map F
$\mathcal{F}(\gamma^a)$	ring of smooth functions of $\gamma^1, \ldots, \gamma^m$
$\langle \cdot, \ldots, \cdot \rangle$	ideal generated by arguments
I	ideal; differential ideal
I_F	fundamental ideal; ideal generated by pulled-back contact structure
$I_{\overline{F}}$	ideal equal to I_F
$I^r_{\overline{F}}$	reduced ideal: $I_{\overline{F}}$ on jet bundle of order reduced by one
$\widehat{I}^r_{\overline{F}}$	reduced ideal including characteristic invariance system
$\lrcorner$	interior product (contraction)
$j^k_p(\cdot)$	k-jet of argument at the point p
$j^k(\cdot)$	k-jet extension of argument
$J^r_{\overline{F}}$	differential ideal: pull-back of $I^r_{\overline{F}}$
$J^k(U^n, V^n)$	k-th jet bundle
K	defining n-form of first order quasilinear PDE
$\ker(\cdot)$	kernel of argument
L	defining two-form of second order PDE
$L^\dagger$	alternative defining two-form of second order PDE
$\Lambda(M)$	module of homogeneneous differential forms on M
$\Lambda^k(M)$	module of differential k-forms on M

$[\cdot,\cdot]$	Lie bracket of arguments
$\mathcal{L}_X(\cdot)$	Lie derivative of argument with respect to the vector field X
M	smooth manifold of dimension m
M^k	open, convex neighbourhood of $\mathbb{R}^k$
mod	modulo
N	smooth manifold of dimension n
$\Omega^k(U^n, V^m)$	k-th order contact system
$\Omega^{\overline{k}}(U^n, V^m)$	contact system generated by only k-th order contact forms
$\{\cdot,\ldots,\cdot\}$	Pfaffian system generated by the one-form arguments
$X^{(k)}$	k-th prolongation of the vector field X
$sp\{\cdot,\ldots,\cdot\}$	span of arguments
$Sp\{\cdot,\ldots,\cdot\}$	span of linearly independent arguments
T_pM	tangent space at the point $p \in M$
TM	tangent bundle
T_p^*M	cotangent space at the point $p \in M$
T^*M	cotangent bundle
U^n	space of n independent variables
V^m	space of m dependent variables
$\wedge$	wedge (exterior) product
$\mathfrak{X}(M)$	module of smooth vector fields on M
$\mathfrak{X}^k(M)$	k-dimensional submodule of $\mathfrak{X}(M)$ over $\mathcal{F}(\gamma^a)$

Summary

The focus of this thesis is on developing symmetry techniques based on computer algebra for extracting local solutions of partial differential equations (PDEs). A class of symmetries more general than Lie point known as *solvable symmetry structures* plays a central role in our study, and we examine the classes of PDEs for which it is possible to develop systematic solution methods based on Cartan's differential geometric language of exterior differential calculus and such symmetries. Our work is significant since these symmetry structures can be found using an advanced feature of some existing Lie symmetry determination software packages such as `DIMSYM`, and moreover, we present an alternative to the usual canonical coordinates approach to symmetry solutions of differential equations as found in the literature.

The main tools in our study are Lie's solvable symmetry structure approach to integrating Frobenius integrable vector field distributions, and a simple extension in term of differential forms. From these two starting points we use solvable symmetry structures to derive the following new results: i) express a single Pfaffian equation of constant rank in 'normal form'; ii) find the coordinates for the closed two-form in Darboux's theorem; iii) give a technique for solving first order quasilinear PDEs; iv) provide two techniques for solving first order non-linear PDEs; v) develop two methods for solving a class of Cauchy problem for Pfaffian systems; vi) extract similarity solutions of a class of second order PDEs using Lie point and conditional symmetries; vii) examine how symmetries may be used to reduce certain second order PDEs to first order; and viii) extend the reduction process in vii) to higher order PDEs.

Statement of authorship

Except where reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have been qualified for or been awarded another degree or diploma.

No other person's work has been used without due acknowledgment in the main text of this thesis.

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Chapter 1

Introduction

The aim of this thesis is to use symbolic computation techniques and symmetry to present some systematic approaches for finding local solutions of linear and non-linear partial differential equations (PDEs). While symmetry methods for solving ordinary and partial differential equations can be found in books by Bluman and Kumei [22], Hill [75], Ibragimov [78], Olver [96], Ovsiannikov [100] and Stephani [117], our work hopes to give a similar discussion with several new results from the point of view of using Cartan's exterior differential calculus [29, 30]. Using symmetries known as 'isovectors', such a differential geometric approach for generating similarity solutions of PDEs can be found most notably in a landmark paper by Harrison and Estabrook [64], as well as in a significant volume of literature by Edelen [43, 44, 45, 46, 47, 48, 49]. In this thesis we make use of: i) exterior calculus; ii) a very general class of symmetries described below known as *solvable symmetry structures*; and iii) modern day computing power, to present some new and alternative symbolic computation techniques for generating explicit local solutions of several classes of PDEs, that we hope will serve to augment the work by Harrison, Estabrook and Edelen referred to above.

A solvable symmetry structure is a special finite collection of symmetries more general than Lie point that has its origin in work by Lie [90] and Cartan [29, 30] on using such symmetries for integrating by quadratures Frobenius integrable vector field distributions. Given a Frobenius integrable vector field distribution D, we say that vector fields $X_1, \dots, X_p$ form a solvable symmetry structure for D if

$$\mathcal{L}_{X_p} D \subset D,$$
$$\mathcal{L}_{X_{p-1}} (Sp\{X_p\} \oplus D) \subset Sp\{X_p\} \oplus D,$$
$$\vdots$$
$$\mathcal{L}_{X_1} (Sp\{X_2, \dots, X_p\} \oplus D) \subset Sp\{X_2, \dots, X_p\} \oplus D,$$

where $\mathcal{L}$ is the Lie derivative operator. Recent reviews and extensions of the work of Lie and Cartan have been done by authors such as Athorne [11], Basarab-Horwath [16], Dubrov and Komrakov [40], Duzhin and Lychagin [42], Hartl and Athorne [65] and Sherring and Prince [110], with some of these authors pursuing the way in which a solvable structure of symmetries can be used to solve ordinary differential equations (ODEs). For a given ODE, D is of one dimension generated by some ordinary differential equation vector field defined on some appropriate jet bundle. As shown in [65], the notion of a solvable symmetry structure for a one-dimensional vector field distribution generated by some ordinary differential equation field is tied to that of a 'hidden' symmetry of the ODE. Hidden symmetries are discussed in the literature by Abraham-Shrauner [1, 2, 3], Abraham-Shrauner and Guo [5, 6, 63], Abraham-Shrauner and Leach [7] and Abraham-Shrauner *et al.* [8], where reducing the order of the given ODE loses or gains one or more symmetries. In addition, Abraham-Shrauner and Guo have also looked at hidden symmetries that arise when the order of the ODE is *increased* [4].

In this thesis we extend the use of solvable structures to extract local general solutions of ODEs, and for the first time examine their applicability to PDEs. The advantages of working with solvable symmetry structures are two-fold: Firstly, they are considerably more general that Lie point, contact [92] or Lie-Bäcklund [9, 77] symmetries, so there is a greater likelihood that given any vector field distribution, they will exist; and secondly (and perhaps more importantly), for a Frobenius integrable vector field distribution D, the computational process simply amounts to initially finding *one* non-trivial symmetry of D, denoted by X_p, then finding *one* non-trivial symmetry of $D \oplus Sp\{X_p\}$, denoted by X_{p-1}, and so on a finite number of times until X_1 is obtained, that is a non-trivial symmetry of $D \oplus Sp\{X_2, \dots, X_p\}$. The fact that for each stage we only need one symmetry is therefore quite a weak requirement. For further details on this issue see Definition 3.2.10 in Chapter 3 and the discussion thereafter.

In the past, the main obstacle in any symmetry technique for solving differential equations has been finding the symmetries. However, the previous decade (and to some extent the decade before) has seen various authors take advantage of substantial and at the same time affordable computing power to develop computer software based symbolic programs for Lie symmetry analysis. For good surveys of the many programs that exist, including a discussion of the various programming methods, see Hereman [72, 73, 78]. Since the aim of this thesis is not to develop programming techniques for Lie symmetry analysis, we briefly mention at this point some of the programs which exist today. To begin with, working under `REDUCE` [70] there exists `CRACK` by Wolf and Brand [133, 135] which solves overdetermined systems of PDEs with polynomial terms, and a more powerful `LIEPDE` by Wolf [134] which also finds Lie point, contact and Lie-Bäcklund symmetries. Also under `REDUCE`, there exists `SPDE` by Schwarz [108] which only finds Lie point symmetries, and a larger program `DIMSYM` by Sherring and Prince [111] (influenced by the Lie symmetry package `LIE` by Head [69] operating under `muMath`) which will find Lie point, contact and Lie-Bäcklund symmetries. Under a `MACSYMA` platform, there exists `PDELIE` by Vafeades [122] and `SYMMGRP.MAX` by Champagne *et al.* [32, 71], however the latter will also find a class of symmetries known as *conditional* (also known as *non-classical*) symmetries [20]. In the language of exterior calculus as used in the Harrison and Estabrook [64] approach to finding symmetries (isovectors) of differential ideals, some early contributions exist by Edelen [43, 44] and Gragert *et al.* [60]. In more recent times, Kersten [84] has developed a series of procedures working under `REDUCE` that can be used to find symmetries more general than Lie point. There also exists `Liesymm` by Carminati *et al.* [27] operating under `Maple`. Finally, there are numerous other good packages which we have not included in this short listing, and we refer the reader to the abovementioned reviews by Hereman for a full discussion including relative merits of these as well as the packages mentioned above.

In our work we choose to use the software package `DIMSYM` [111] designed by James Sherring to generate the solvable symmetry structures described earlier. In addition to finding symmetries of differential equations, this piece of software will also find symmetries of arbitrary vector field distributions. It is this latter feature of `DIMSYM` that we use extensively throughout this thesis, and since it essentially amounts to requiring the software to solve a set of determining equations in the form of a system of first order linear PDEs, most of the other Lie symmetry analysis packages mentioned earlier are also capable in various capacities of performing this task, and hence finding such symmetry structures. In Appendix A we give full details including software input code of how one may use `DIMSYM` to find a solvable symmetry structure for an arbitrary vector field distribution. Therefore, in practice, generating such a symmetry structure amounts to nothing more than entering some data into the software program and waiting (sometimes a long time) for any symmetries in the output. Hence all of the results in this thesis will assume that given any vector field distribution, we are always able find a solvable structure of symmetries.

One of the main intentions of our study is to explore the classes of partial differential equations for which the Frobenius theorem and solvable symmetry structures from `DIMSYM` can assist in generating local solutions. At first, it is not obvious how one may even begin to attempt to apply the Frobenius theorem to the study of partial differential equations. One approach is to express the single or system of PDEs in terms of a corresponding *exterior differential system*, for which there are a number of ways. Then one is faced with the problem of looking for Frobenius integrable annihilating distributions of a dimension equal to the number of independent variables. For ordinary differential equation problems, this approach is made trivial due to the fact that a system of ODEs can be generally represented by a single vector field, which generates a trivially Frobenius integrable distribution. Hence, at least in principle, there is no difficulty in applying a solvable symmetry structure to locally solve a single n-th order non-linear ODE, and so one obtains the required first integrals for expressing the general solution. See Chapter 4 for further details on this issue.

For partial differential equations, a corresponding exterior differential system for a given system of PDEs is typically generated by the pull-back of the contact system of one-forms onto the locus of the jet bundle described by the PDEs. The task of finding a Frobenius integrable subdistribution of the annihilating space of the one-forms generating the exterior differential system is no longer a simple exercise as before, and indeed this is one of the major stumbling blocks in any study of PDEs done in this fashion. There are two obvious approaches to tackling this problem. The first is to somehow pin down a solution of the PDEs by introducing additional PDEs (not by prolongation) so that when the contact forms are pulled-back onto the locus of the jet bundle described by the augmented system of PDEs, we find that the annihilating space is Frobenius integrable and of a dimension equal to the number of independent variable. One of the most well-known examples of this technique is Darboux's method [59] on Liouvilles equation, as discussed in recent literature by Fackerell *et al.* [53], Hartley *et al.* [68], Juráš and Anderson [80] and Vassiliou [127]. The second approach is to pull-back the contact forms onto the locus described by the PDEs, and then look to augment the exterior differential system with one-forms so that the annihilating space is Frobenius integrable and of a dimension equal to the number of independent variables. A similar augmentation idea, though not with the intention of finding Frobenius integrable distributions, has been used by Gardner [54, 55] for solving the Cauchy problem for Pfaffian systems (see Chapter 6) with k*-stable* vector fields. Briefly explaining the method, if an exterior differential system generated by some finite collection of linearly independent one-forms $\alpha^1, \ldots, \alpha^p$ does not contain any non-zero Cauchy characteristic vector fields, then one looks to augment the generators of the exterior differential system with several one-forms so that the resulting exterior differential system contains such vector fields, which are said to be 1-stable. The one-forms introduced are, over all i, $X \lrcorner \, d\alpha^i$, where X is initially the most general element of the annihilating space of the span of $\alpha^1, \ldots, \alpha^p$. If a 1-stable vector field exists, then it will clearly be some X. If it is not possible to find a 1-stable vector field, then further one-forms are added, and one then looks for 2-stable vector fields. The process can be repeated until the vector field distribution annihilated by the total of number of one-form generators is Frobenius integrable, at which point *all* vector fields in the distribution are k-stable. However, one of the main objective of the process of augmentation is not to reach this final stage. For further details, see work by Gardner cited above, Fackerell *et al.* [53], Gardner and Kamran [57] and Hartley *et al.* [68].

One tool that some authors have used in the past to assist in generating integral manifolds of an exterior differential system (that may or may not correspond to some system of PDEs) is the Cartan-Kähler theorem [30, 81, 86]. In the literature, the most comprehensive modern day discussion on this theorem can be found in work by Bryant *et al.* [23], while Griffiths [62], Olver [99] and Yang [136] have given more readable but less sophisticated treatments. Other authors such as Kakié [82, 83] have studied the theorem for solving non-linear PDEs. Cianci [36] has generalised the result on a superspace for studying first order non-linear PDEs, and Hartley [67, 66] has examined the theorem from a symbolic computation point of view. Finally, Edelen [47, 48] has studied alternatives to the Cartan-Kähler theorem in order to avoid some of the 'computational horrors' that sometimes arise in its application.

The Cartan-Kähler theorem can be seen as a Cauchy problem result in the sense that if we are given some q-dimensional, real analytic, *Kähler-regular* integral manifold of some exterior differential system I, then by the Cauchy-Kowalevski theorem [115], there exists a $(q+1)$-dimensional real analytic integral manifold of I containing the one we have been given. Two drawbacks of this theorem are firstly that applying it can become computationally intensive and secondly that the result is only true in an analytic category.

The structure of this thesis is as follows: We begin in Chapter 2 by reviewing some preliminary material that is essentially an introduction to the basic elements of differential geometry, exterior calculus and symmetries. This discussion is necessarily brief. Following this, Chapter 3 gives a thorough treatment of Lie's symmetry approach to integrating Frobenius integrable distributions, as discussed in [11, 16, 40, 42, 65, 110]. The main new result of this chapter given in Theorem 3.2.14 is an extension in terms of differential forms of Theorem 3.2.13 found in Sherring and Prince [110], and throughout our work we make extensive use of both these results. In addition, since we deal specifically with solvable symmetry structures, we also give various new results concerning certain types of solvable structures that can simplify somewhat the conclusions of these two theorems. The reason we do this is simply that while generating a solvable symmetry structure using `DIMSYM` or some other Lie symmetry software package involves repeatedly finding only one symmetry of some vector field distribution, there will exist occasions when we have found more than one. Under certain conditions, the extra one(s) can be used as symmetries in some of the remaining stages of the solvable symmetry structure. Thus, in addition to simplifying the conclusions in Theorems 3.2.13 and 3.2.14, we are also able to save time by virtue of less symbolic computation.

We continue Chapter 3 by focusing on the differential form result in Theorem 3.2.14. We give several existence results concerning some necessary conditions for a given differential form to satisfy the theorem. We then examine the Pfaff problem from the perspective of how one can use a solvable structure of symmetries and Theorem 3.2.14 to express a given differential one-form of constant rank r on the domain of definition in 'normal form'. The main result we present is contained in Theorem 3.5.7, and we find that to do the job, we require one solvable symmetry structure of length $2r + 1$, with a special (but not too restrictive) requirement on the last r symmetries.

Finally, to conclude Chapter 3 we leave the Pfaff problem and turn to Darboux's theorem for closed two-forms. In particular, we formulate an algorithm adapted from the one described by Crampin and Pirani [37], where such coordinates are extracted a pair at a time.

It is Chapter 4 that begins our study of differential equations. Using Edelen's *fundamental ideal* [43, 45, 46, 49] approach to such equations, we briefly examine ODEs as a prelude to our study of PDEs. The work presented in this chapter is not new, but perhaps interpreted a little differently when compared with similar studies in [16, 42, 65, 110].

In Chapter 5 we commence our work of partial differential equations by looking at some symmetry techniques for generating local solutions of single first order PDEs. We first examine single first order quasilinear PDEs. Using fundamental ideals, we express such a PDE as an exterior differential system that essentially resides on the graph space. Applying a solvable symmetry structure, we then show how to extract local solutions. Following this, we turn to first order (possibly non-linear) PDEs. Here, we introduce the idea of a *Vessiot distribution* named after the French mathematician E. Vessiot [124, 128, 129, 130]. Briefly, for any PDE (or system of PDEs) two types of such distributions may exist, namely those *regular* and those *singular*. While Vessiot's work is most noted for using Monge characteristics of singular Vessiot distributions to extract local solutions of second order hyperbolic

PDEs, our work examines how a regular Vessiot distribution corresponding to some single first order PDE combined with solvable symmetry structures can yield local solutions of the PDE. We present two algorithms for first order PDEs of one dependent variable and two independent variables that each convert the problem to finding local solutions of a corresponding first order quasilinear PDE. The first approach requires two solvable symmetry structures, while the second only needs one solvable symmetry structure but at the expense of only applying to first order PDEs that do no involve their dependent variable of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, \frac{\partial u}{\partial x^1}\right),$$

for smooth F, where x^1, x^2 are the independent variables, and u is the dependent variable.

Chapter 6 investigates some symmetry approaches to the Cauchy problem for Pfaffian systems. Though defined more precisely in Chapter 6, if we are given a differential ideal I generated by some finite number of linearly independent one-forms and their exterior derivatives, the Cauchy problem essentially looks to extends a q-dimensional integral manifold of I (called the *Cauchy data*) to a $(q+1)$-dimensional integral manifold of I. It is well-known that the Cauchy problem may be solved using a non-zero Cauchy characteristic vector field of I (see Theorem 6.2.1), which typically involves finding the solution of a system of first order ODEs. Our work in Chapter 6 examines the extent to which solvable symmetry structures can assist in solving the problem with the intention of replacing the need to introduce any such ODEs. For a class of Pfaffian systems whose corresponding differential ideals possess one-dimensional Cauchy characteristic spaces we give two results, the first of which is only applicable to systems generated by single one-forms of rank one, while the second is a generalisation of the first for systems that are generated by a finite number of one-forms of arbitrary constant rank. For each of the two cases we also give a PDE example.

While Chapter 6 briefly touches on second order PDE problems for which there exist non-zero Cauchy characteristic vector fields, it is Chapter 7 which essentially begins our study of second order PDEs. In this chapter we use exterior differential systems and symmetry to present a new way of deriving similarity solutions of second order hyperbolic or parabolic partial differential equations of the form

$$f_1 \frac{\partial^2 u}{\partial (x^1)^2} + f_2 \frac{\partial^2 u}{\partial (x^2)^2} + f_3 \frac{\partial^2 u}{\partial x^1 x^2} = k, \tag{1.1}$$

where f_1, f_2, f_3, k are smooth functions of $x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}$. Our approach is basically an extension of the well-known method originally given by Lie [89, 91].

Given a single second order hyperbolic or parabolic PDE of the type in (1.1), we begin Chapter 7 by demonstrating that a Lie point symmetry of the exterior differential system corresponding to the PDE can be used to generate a Cauchy characteristic vector field of a restricted exterior differential system defined on some four-dimensional regular submanifold of the first jet bundle. Using the hyperbolic or parabolic nature of the PDE, we then show that this restricted exterior differential system has a Frobenius integrable annihilating space, which in turn yields a similarity solution of the PDE when it is integrated with a solvable structure of two symmetries. Finally, we close Chapter 7 by showing that all prior results in the chapter still hold true if the Lie point symmetry is replaced with a conditional symmetry.

While many texts exist on using symmetry to extract similarity solutions of PDEs [39, 76, 96, 100, 106, 117], our work in Chapter 7 is based on a technique using exterior calculus in the paper by Harrison and Estabrook [64] referred to before, which was some time later generalised by Papachristou and Harrison [101] to vector valued and Lie algebra-valued differential forms. Over the years, many other authors have applied the technique in [64] to various problems in mathematics and physics. These include, for example, non-linear diffusion (and reaction-diffusion) equations by Chowdhury [35], Suhubi and Chowdhury [121] and Waller [131], Poisson and Liouville type equations by Bhutani and Bhattacharya [18], and equations of power law creep by Delph [38].

In Chapter 8 we look at second and higher order PDEs. In particular, we discuss some necessary conditions for reducing a second order hyperbolic or parabolic PDE of the form in (1.1) to first order (generally non-linear) depending on an arbitrary smooth function. We also give various generalisations for higher order PDEs. For certain classes of such higher order PDEs we use solvable symmetry structures to develop a reduction of order process that reduces an n-th order PDE linear in its n-th order derivatives to an $(n-1)$-th order PDE plus an arbitrary smooth function. The results in Chapter 8 are significant because they not only provide some necessary conditions for reducing the order of PDEs, but also give us some direct algorithms for doing so using symmetry.

Throughout the material on PDEs in Chapters 5 to 8, we work in a smooth category, however there exist systems of PDEs in this category that are without solution, e.g. Lewy's famous example [88]. Alternatively, we could elect to work in an analytic category with 'involutive' (see Chapter 9) systems of PDEs, for which we are guaranteed local solutions by the Cauchy-Kowalevski theorem. Instead, since our main tools for these chapters are smooth results in Theorem 3.2.13 and its extension, Theorem 3.2.14, we choose to remain in a smooth category and simply assume for all our PDEs that there exist smooth local solutions. The reason we do this is also since we are not so much concerned with the existence and uniqueness of solutions of PDEs, but rather with the formulation of algorithmic approaches based on symmetry for extracting smooth local solutions.

Finally, an important feature of our work is that while the well-known symmetry techniques for solving single differential equations typically involve expressing the differential equation in terms of canonical coordinates, none of our symmetry techniques for PDEs take this path to solution. For example, all solutions of first order quasilinear and non-linear PDEs, and similarity solutions of second order PDEs are derived while always remaining in the *original coordinate system* of the differential equation. Furthermore, since we deal with a class of quite general symmetries capable of being found using existing computer software packages, our work has the potential of being very applicable to many problems in applied mathematics and physics of today.

Chapter 2

Preliminary material

In this chapter we present a number of definitions and results that will provide us with the necessary background for our study. We discuss various aspects of differential geometry, symmetries and jet bundles, most of which are well-known.

2.1 Basic differential geometry

2.1.1 Manifolds and submanifolds

Manifolds

Following Olver [96, 99], we adopt the following definition of a manifold:

Definition 2.1.1. An m-dimensional *manifold* M is a topological space which is covered by a countable collection of open subsets $W_\alpha \subset M$, called *coordinate charts*, and homeomorphisms (i.e. one-to-one and onto with a continuous inverse [37]) $\psi_\alpha : W_\alpha \longrightarrow V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$, which serve to define local coordinates on M. A manifold is *smooth* (or *differentiable*) if, given any pair of coordinate charts $W_\alpha \cap W_\beta$, the composite map

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha \left(W_\alpha \cap W_\beta\right) \longrightarrow \psi_\beta \left(W_\alpha \cap W_\beta\right)$$

is a smooth (here we mean infinitely differentiable) map.

We assume throughout that M is Hausdorff, i.e. any two distinct points $x \neq y$ in M can be separated by open subsets $x \in W$ and $y \in \overline{W}$ with $W \cap \overline{W} = \varnothing$. In our work we often omit explicit reference to coordinate maps and identify a point $p \in M$ with its image in $\mathbb{R}^m$.

Definition 2.1.2. If M, N are smooth manifolds of dimension m, n respectively, a map $F : N \longrightarrow M$ is said to be smooth if for every coordinate chart $\psi_\alpha : W_\alpha \longrightarrow V_\alpha \subset R^n$ on N and every chart $\overline{\psi}_\beta : \overline{W}_\beta \longrightarrow \overline{V}_\beta \subset R^m$ on M, the composite map

$$\overline{\psi}_\beta \circ F \circ \psi_\alpha^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is a smooth map. For smooth F we write $F \in C^\infty(N, M)$. If $m = 1$ and $f : N \longrightarrow M \subset \mathbb{R}$ is a smooth *function*, then we write $f \in C^\infty(N)$.

Our work will only deal with *smooth* manifolds and mappings, however, other authors will deal with *analytic* manifolds, usually when they require the Cauchy-Kowalevski theorem as used in, for example, the proof of the Cartan-Kähler theorem [23] discussed in Chapter 1.

Submanifolds

Following Sternberg [118],

Definition 2.1.3. Let M, N be smooth manifolds of dimension m, n respectively. A smooth map $F : N \longrightarrow M$ is called an *immersion* if it is of rank n at every point in N.

The *rank* of a smooth map at some point in N is for the moment simply taken to be the rank of its Jacobian matrix. We define this more precisely later.

Definition 2.1.4. Let M, N be smooth manifolds of dimension m, n respectively. A smooth map $F : N \longrightarrow M$ is called an *imbedding* if it is a one-to-one immersion which is homeomorphic onto its image.

For immersions, we have the following local result found in, for example, Olver [96, 99]:

Theorem 2.1.5. *For smooth manifolds M, N of dimension m, n respectively with $n \leq m$, let $F : N \longrightarrow M$ be an immersion. Then for every point in N, there exists a neighbourhood with local coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$ on M and N respectively, such that*

$$F : \left(y^1, \ldots, y^n\right) \longmapsto \left(y^1, \ldots, y^n, 0, \ldots, 0\right).$$

Next, we introduce the following definition of a submanifold:

Definition 2.1.6. Let M, N be smooth manifolds of dimension m, n respectively with $n \leq m$. An *n-dimensional submanifold* of M is a subset $\overline{N}$ and one-to-one immersion $F : N \longrightarrow \overline{N} \subset M$ such that $F(N) = \overline{N}$.

Our definition of a submanifold is commonly referred to as an *immersed* submanifold. This thesis will only deal with *regular* submanifolds, for which we have the following definition and two results:

Definition 2.1.7. [26, 96] Let M, N be smooth manifolds of dimension m, n respectively with $n \leq m$. A subset $\overline{N}$ is a *n-dimensional regular submanifold* of M if it is an n-dimensional submanifold with corresponding one-to-one immersion $F : N \longrightarrow \overline{N} \subset M$ such that for all $p \in \overline{N}$, p possesses a basis of neighbourhoods $\{U_\alpha\}$ such that for some β, $p \in U_\beta \subset M$ with the property that $F^{-1}(U_\beta \cap \overline{N})$ is a connected open subset of N.

Theorem 2.1.8. [118] *Let M be a smooth manifold of dimension m, and N be some subset of dimension n. Then N is a regular submanifold if and only if every point p of N has a neighbourhood U in M with coordinates $x^1, \ldots, x^m$ such that* $U \cap N = \{p \in M : x^1 = \cdots = x^{m-n} = 0 \text{ at } p\}$.

Theorem 2.1.9. [96] *Let M be a smooth manifold of dimension m, and $F : M \longrightarrow \mathbb{R}^n$ with $n \leq m$ be a smooth map. If F is of maximal rank on the subset $N = \{p \in M : F(p) = 0\}$, then N is an $(m-n)$-dimensional regular submanifold of M.*

2.1.2 Tangent and cotangent spaces

Tangent spaces

Consider a smooth manifold M of dimension m. A *tangent vector* to M at a point $p \in M$, can be defined as an equivalence class of curves through p where two curves are equivalent if they have the same derivatives about p [118]. Explicitly, let $\sigma : I \longrightarrow M$ be a smooth curve with $p = \sigma(0)$ and $0 \in I$. Then with coordinates $x^1, \ldots, x^m$ on M in some neighbourhood of p, we have the equivalence relation $(p, \sigma) \sim (p', \sigma')$ to mean $p = p'$ and for each $1 \leq i \leq m$,

$$\left.\frac{d\,(x^i \circ \sigma)}{dt}\right|_{t=0} = \left.\frac{d\,(x^i \circ \sigma')}{dt}\right|_{t=0} = v^i, \tag{2.1}$$

for some $v^1, \ldots, v^m \in \mathbb{R}$. Using the chain rule it is easy to show that this equivalence relation is independent of coordinate system. We define the *tangent space* as the set of all tangent vectors at $p \in M$, and is denoted by T_pM. The tangent space has the same dimension as M, given by the freedom in choosing $v^1, \ldots, v^m$. The collection of all tangent spaces is known as the *tangent bundle*, and is defined by

$$TM := \bigcup_{p \in M} T_pM.$$

TM has the natural structure of a smooth manifold of dimension $2m$.

Using a local coordinate system $x^1, \ldots, x^m$ on M, one may define a directional derivative along tangent vectors as follows: Given a tangent vector $v = (v^1, \ldots, v^m)$ at a point $p \in M$, the *directional derivative* of a function f on M along v is the real number

$$\left. \frac{d\,(f \circ \sigma)}{dt} \right|_{t=0},$$

where σ is a curve in the equivalence class such that (2.1) holds. It is easy to show that [37] (with sum)

$$\left. \frac{d\,(f \circ \sigma)}{dt} \right|_{t=0} = v^i \frac{\partial f}{\partial x^i},$$

so we remove the reference to a curve. We can identify any element $v \in T_pM$ as a directional derivative operator given by

$$v = v^i \frac{\partial}{\partial x^i},$$

and it is convenient to write the basis for T_pM as $\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m} \right\}$ evaluated at p. A *vector field* is a smoothly varying assignment of tangent vectors over the tangent bundle TM [99]. The module over $C^\infty(M)$ of vector fields is denoted $\mathfrak{X}(M)$. Any vector field $X \in \mathfrak{X}(M)$ is of the form [105]

$$X := X^i \frac{\partial}{\partial x^i},$$

for some $X^1, \ldots, X^m \in C^\infty(M)$. For any $f \in C^\infty(M)$, $X(f) \in C^\infty(M)$ given by

$$X(f) := X^i \frac{\partial f}{\partial x^i}.$$

We say that a vector field is *non-singular* on M if it does not vanish anywhere on M. In our work we will only deal with such vector fields.

A vector field *distribution* $D \subset \mathfrak{X}(M)$ generated by some $X_1, \ldots, X_q \in \mathfrak{X}(M)$ is a submodule over $C^\infty(M)$, and denoted $D := sp\{X_1, \ldots, X_p\}$. Here $X_1, \ldots, X_q$ may or may not be linearly independent at all points in M. However when we write $D := Sp\{X_1, \ldots, X_p\}$, it is assumed $X_1, \ldots, X_q$ are linearly independent on M.

Let $X, Y \in \mathfrak{X}(M)$. Then the *Lie bracket* $[X, Y]$ is given by

$$[X, Y](f) := X(Y(f)) - Y(X(f)),$$

where $f \in C^\infty(M)$. The Lie bracket of two vector fields is a vector field. Hence it displays the properties of a *derivation*, i.e.

1. $[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g)$ (Linearity over $\mathbb{R}$),

2. $[X, Y](fg) = [X, Y](f)g + [X, Y](g)f$ (Leibniz rule),

for all $X, Y \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$. The Lie bracket also has the properties:

1. $[X, Y] = -[Y, X]$ (Antisymmetric),

2. Bilinearity over $\mathbb{R}$,

3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity),

where $X, Y, Z \in \mathfrak{X}(M)$.

Cotangent spaces

For any $p \in M$, the dual space of the tangent space T_pM is called the *cotangent space*, and denoted T_p^*M. Elements of T_p^*M are differential *one-forms*, also known as *cotangent vectors*, and the *cotangent bundle* is defined by

$$T^*M := \bigcup_{p \in M} T_p^*M.$$

We denote the module of one-forms over $C^\infty(M)$ by $\Lambda^1(M)$.

Following [37, 99, 118], if we are given any $f \in C^\infty(M)$, its *differential* df can be evaluated on any vector field $X \in \mathfrak{X}(M)$ defined by the linear map

$$\langle X, df \rangle := X(f) \in C^\infty(M).$$

This may be evaluated at some point $p \in M$ to give a real result. In terms of local coordinates $x^1, \ldots, x^m$ on M, we can define a basis $\{dx^i, \ldots, dx^m\}$ for T^*M dual to $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$ using the *Kronecker delta*:

$$\left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle = \delta^i_j.$$

Thus in terms of these basis vectors, for any $f \in C^\infty(M)$, df can locally be written as

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

2.2 Exterior calculus results

2.2.1 Differential forms

We have already introduced differential one-forms on the cotangent space. At this point we introduce differential forms of higher degree. Given a smooth manifold M of dimension m, we define the *exterior calculus* [23] over M as the graded algebra

$$\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \cdots \oplus \Lambda^m(M),$$

where $\Lambda^0(M) := C^\infty(M)$. We introduce $\wedge$ as the *exterior (wedge) product* in $\Lambda(M)$. The exterior product of a k-form ω and an l-form χ is defined as a multilinear map

$$\wedge : \Lambda^k(M) \times \Lambda^l(M) \longrightarrow \Lambda^{k+l}(M),$$

where

$$(\omega \wedge \chi)(X_1, \ldots, X_{k+l}) := \frac{1}{k!l!} \sum (\mathrm{sign}\pi) \left[\omega(X_{i_1}, \ldots, X_{i_k}) \chi(X_{i_{k+1}}, \ldots, X_{i_{k+l}}) \right],$$

with summation over all distinct $1 \leq i_1, \ldots, i_{k+l} \leq k+l$, where $X_1, \ldots, X_{k+l} \in \mathfrak{X}(M)$, and $\mathrm{sign}\pi$ is the sign of the permutation $(i_1, \ldots, i_{k+l})$ [104]. The exterior produce is associative, distributive and anti-commutative. For the latter, if $\omega \in \Lambda^k(M)$ and $\chi \in \Lambda^l(M)$, then

$$\omega \wedge \chi = (-1)^{kl} \chi \wedge \omega.$$

For all $k \leq m$ we can construct the space $\Lambda^k(M)$ of *differential k-forms*. Such a form is a k-linear map:

$$\Omega : \overbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}^{k \text{ times}} \longrightarrow C^\infty(M),$$

that is anti-symmetric in its arguments.

If $\omega^1, \ldots, \omega^k \in \Lambda^1(M)$ and $\Omega := \omega^1 \wedge \cdots \wedge \omega^k$, then for any $X_1, \ldots, X_k \in \mathfrak{X}(M)$ we have

$$\Omega\left(X_1, \ldots, X_k\right) = \det\left(\langle X_i, \omega^j \rangle\right),$$

where the right hand side is the determinant of a $k \times k$ matrix. The map Ω may be evaluated at a point p in M so its arguments are in T_pM thus giving a map to $\mathbb{R}$.

If $\{\phi^1, \ldots, \phi^m\}$ is a basis for $\Lambda^1(M)$, then any $\Omega \in \Lambda^k(M)$ can be expressed in the form

$$\Omega = \Omega_{i_1 \cdots i_k} \phi^{i_1} \wedge \cdots \wedge \phi^{i_k},$$

for some $\Omega_{i_1 \cdots i_k} \in C^\infty(M)$, where summation is over all $1 \leq i_1 < \cdots < i_k \leq m$. The collection of $\phi^{i_1} \wedge \cdots \wedge \phi^{i_k}$ provide us with a basis for the $\binom{m}{k}$-dimensional space $\Lambda^k(M)$. Note that $k \leq m$ and $\dim\left(\Lambda^m(M)\right) = 1$.

Finally, from Bryant *et al.* [23] we have the following well-known result:

Theorem 2.2.1. *Let $\omega^1, \ldots, \omega^k \in \Lambda^1(M)$. Then $\omega^1, \ldots, \omega^k$ are linearly dependent at $p \in M$ if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$ at p.*

The interior product

The *interior product* (*contraction*) is a map:

$$\lrcorner : \mathfrak{X}(M) \oplus \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M),$$

defined by

$$\left(X \lrcorner\, \omega\right)\left(Y_1, \ldots, Y_{k-1}\right) := \omega\left(X, Y_1, \ldots, Y_{k-1}\right),$$

where $\omega \in \Lambda^k(M)$ and $X, Y_1, \ldots, Y_{k-1} \in \mathfrak{X}(M)$.

If $f \in \Lambda^0(M)$, then we set $X \lrcorner\, f = 0$. The interior product has the following properties [105]:

1. $(X + Y) \lrcorner\, \omega = X \lrcorner\, \omega + Y \lrcorner\, \omega$,

2. $X \lrcorner\, (\omega \wedge \chi) = (X \lrcorner\, \omega) \wedge \chi + (-1)^k \omega \wedge (X \lrcorner\, \chi)$,

3. $X \lrcorner\, X \lrcorner\, \omega = 0$,

4. $X \lrcorner Y \lrcorner \omega = -Y \lrcorner X \lrcorner \omega$,

where $\omega \in \Lambda^k(M)$, $\chi \in \Lambda^l(M)$ and $X, Y \in \mathfrak{X}(M)$.

The exterior derivative

We have already seen the exterior derivative $d : C^\infty(M) \longrightarrow \Lambda^1(M)$. Here we generalise things:

Let $\{dx^1, \ldots, dx^m\}$ be a basis for $\Lambda^1(M)$. For $k \geq 1$, the exterior derivative $d : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M)$ is defined by

$$d\left(\epsilon_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) := d\epsilon_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

It has the following properties [114, 118]: If $\omega \in \Lambda^k(M)$ and $\chi \in \Lambda^l(M)$, then

1. $d(\omega + \chi) = d\omega + d\chi$,

2. $d(\omega \wedge \chi) = d\omega \wedge \chi + (-1)^k \omega \wedge d\chi$,

3. $d\,(d\omega) = 0$.

We say that a differential form $\omega \in \Lambda^k(M)$ is *closed* if $d\omega = 0$. If there exists a $(k-1)$-form $\psi \in \Lambda^{k-1}(M)$ such that $\omega = d\psi$, then we say that ω is *exact*. For closed forms, we have the following converse of item (3) above [99]:

Lemma 2.2.2. *If $U \subset \mathbb{R}^n$ is a convex open subset, and ω is any closed k-form defined on U. Then $\omega = d\psi$ for some $(k-1)$-form ψ on U.*

In some literature, Lemma 2.2.2 (or even its converse) is referred to as the *Poincaré Lemma*. We do not use this name here, because we prefer to use it when strictly dealing with the *true* Poincaré Lemma given in books on cohomology.

Sometimes Lemma 2.2.2 is given with U being *star-shaped* [87, 93]. A set $U \subset \mathbb{R}^n$ is said to be star-shaped *with respect to a point* $p \in U$ if the line segment connecting any point $q \in U$ with p is also contained in U. For a *convex* set, by definition [25], *any* two points in the set can be connected by a line segment that is also in the set. Thus a convex set is star-shaped with respect to any point in the set.

2.2.2 Induced maps

We discuss two linear maps, the *differential* and the *pull-back*. The former is also known as the *push-forward* [17, 24], and the latter is sometimes known as the *reciprocal image* [34]. Given a smooth map $F : N \longrightarrow M$ between two smooth manifolds M, N and some point $p \in N$, the differential of F, denoted by F_*, takes tangent vectors in T_pN to tangent vectors in $T_{F(p)}M$, i.e.

$$F_* : T_pN \longrightarrow T_{F(p)}M.$$

The pull-back of F, denoted by F^*, takes differential forms on M back to differential forms on N, i.e.

$$F^* : \Lambda^k(M)\big|_{F(p)} \longrightarrow \Lambda^k(N)\big|_p .$$

The differential

Here we follow mostly [85, 96, 99]. Suppose then that $\sigma(t)$ is an arbitrary parametrised curve on N. At some $p = \sigma(t_0)$, let $v := \dot{\sigma}(t_0)$ be the tangent vector to the curve. The differential of v at the image point $q = F(p)$ is defined to be the tangent vector of the curve $F(\sigma)$ at $F(\sigma(t_0))$. Explicitly,

$$F_*(v) := \left. \frac{d}{dt} F(\sigma(t)) \right|_{t=t_0} .$$

In terms of coordinates, if $y^1, \ldots, y^n$ are local coordinates on N, $x^1, \ldots, x^m$ are local coordinates on M, and F is described in terms of these sets of coordinates by the m equations

$$x^i = f^i(y^1, \ldots, y^n),$$

for some $f^1, \ldots, f^m \in C^\infty(N)$, then with

$$X := \eta^j \frac{\partial}{\partial y^j},$$

for some $\eta^1, \ldots, \eta^n \in C^\infty(N)$, we have that

$$F_*(X|_p) = \langle X, df^i \rangle|_p \left. \frac{\partial}{\partial x^i} \right|_{F(p)} .$$

Now suppose F is diffeomorphic (i.e. one-to-one and onto with a smooth inverse [37]) onto its image. We can then remove the point p of application and say that the differential mapping vector fields on N to vector fields on $F(N)$ is well-defined and one-to-one. Thus for any vector field $X \in \mathfrak{X}(N)$, it is true that $F_*(X)$ is a well-defined vector field in $\mathfrak{X}(F(N))$. Here we write $\mathfrak{X}(F(N))$ to mean the module of vector fields tangent to $F(N)$. When F is one-to-one, this notation is unambiguous from Theorem 2.1.5.

Two vector fields $X \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ are said to be *F-related* if $Y = F_*(X)$. If $Y_1 = F_*(X_1)$ and $Y_2 = F_*(X_2)$ for some $X_1, X_2 \in \mathfrak{X}(N)$ and $Y_1, Y_2 \in \mathfrak{X}(M)$, the differential also has the following property:

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]. \tag{2.2}$$

From Theorem 8.4 in Sternberg [118] we have the following result:

Theorem 2.2.3. *With M, N as smooth manifolds of dimension m, n respectively, let $F : N \longrightarrow M$ be a one-to-one immersion. Then for all $Y \in \mathfrak{X}(F(N))$, there exists $X \in \mathfrak{X}(N)$ such that $F_*X = Y$.*

Finally, the *rank* of F at some point $p \in N$ is usually defined as the rank of the linear map F_* at p, which is by definition equal to the dimension of the image space of $F_*(T_pN)$. It is not hard to show that the rank of F at p is equal to the rank of the $m \times n$ Jacobian matrix

$$\begin{pmatrix} \left.\frac{\partial f^1}{\partial y^1}\right|_p & \cdots & \left.\frac{\partial f^1}{\partial y^n}\right|_p \\ \vdots & & \vdots \\ \left.\frac{\partial f^m}{\partial y^1}\right|_p & \cdots & \left.\frac{\partial f^m}{\partial y^n}\right|_p \end{pmatrix}.$$

The pull-back

The following material is obtained from [94, 105, 116]. With $F : N \longrightarrow M$ defined as above, if f is a smooth function defined on N, so that $f \in \Lambda^0(M)$, then the pull-back of f by F at a point $p \in N$ is defined by

$$F^* f|_p := f|_{F(p)},$$

i.e. $F^* f = f \circ F$.

For differential forms of degree ≥ 1, the pull-back can be defined as follows: For all $\omega \in \Lambda^k(N)$ and $v_1, \ldots, v_k \in T_p(M)$, we have

$$F^*(\omega)|_p (v_1, \ldots, v_k) := \omega|_{F(p)} (F_*v_1, \ldots, F_*v_k).$$

If F_* maps to well-defined vector fields, then we may write for any $Y_1, \ldots, Y_k \in \mathfrak{X}(N)$,

$$(F^*\omega)(Y_1, \ldots, Y_k) = F^*\left(\omega(F_*Y_1, \ldots, F_*Y_k)\right). \tag{2.3}$$

The pull-back also has the following properties: If $\omega \in \Lambda^k(M)$ and $\chi \in \Lambda^l(M)$, then

$$F^*(\omega + \chi) = F^*\omega + F^*\chi, \tag{2.4}$$

$$F^*(\omega \wedge \chi) = (F^*\omega) \wedge (F^*\chi), \tag{2.5}$$

$$F^*(d\omega) = d(F^*\omega). \tag{2.6}$$

Removing ω, (2.6) may be written as $F^* \circ d = d \circ F^*$.

Given any smooth $G : R \longrightarrow N$ defined on a smooth manifold R, we also have the following composition property:

$$(F \circ G)^* \omega = G^*(F^*\omega). \tag{2.7}$$

2.2.3 The Lie derivative

Consider the following definition found in Sternberg [118]:

Definition 2.2.4. A family α_t of diffeomorphisms of $M \longrightarrow M$ is called a *one parameter group* of smooth transformations on M if the map $\alpha : \mathbb{R} \times M \longrightarrow M$ sending $(t, p) \longmapsto \alpha_t(p)$ is a smooth action of the additive group of real numbers on M, i.e. if

1. The map α is smooth,
2. $\alpha_{t+s} = \alpha_t \circ \alpha_s$ for all t and s,
3. α_0 is the identity.

A one parameter group is also known as a *flow*. Every one parameter group induces a vector field on M, known as the *infinitesimal generator* of the one parameter group, which is simply the vector field generating the tangent vector of the curve $t \longmapsto \alpha_t(p)$ passing through $p \in M$. In terms of coordinates, if M has local coordinates $x^1, \ldots, x^m$ and $\alpha_t := (\alpha_t^1, \ldots, \alpha_t^m)$, where

$$\alpha_t^i := f^i(x^1, \ldots, x^m, t),$$

for some $f^1, \ldots, f^m \in C^\infty(M \times \mathbb{R})$, then

$$X_p := \left(\left. \frac{df^i}{dt} \right|_{t=0} \frac{\partial}{\partial x^i} \right) \Bigg|_p .$$

Given a vector field, one can expect to perform the reverse of the above procedure and be able to generate a one parameter group by solving a system of first order ODEs. However, as explained in Crampin and Pirani [37], every vector field generates only a *local* one parameter group on some restricted interval of values for the parameter t and on some subset of M.

Using the idea of local one parameter groups we introduce the *Lie derivative*. The Lie derivative of a linear object along some local one parameter group measures the rate of change of the object along the flow. To illustrate, if our object is some function $g \in C^\infty(M)$, then

$$\mathcal{L}_X g := \left. \frac{d}{dt} (\alpha_t^* g) \right|_{t=0}$$

defines the Lie derivative of g with respect to the flow α_t that generates the vector field $X \in \mathfrak{X}(M)$. It is a simple exercise [87, 118] to show that

$$\mathcal{L}_X g = X(g).$$

Suppose our linear object is some differential k-form $\omega \in \Lambda^k(M)$. Then its Lie derivative with respect to the flow α_t generated by the vector field $X \in \mathfrak{X}(M)$ is defined by

$$\mathcal{L}_X \omega := \left. \frac{d}{dt} (\alpha_t^* \omega) \right|_{t=0} .$$

The Lie derivative on differential forms (including 0-forms), has the following properties: If $\omega, \eta \in \Lambda(M)$ and $X, Y \in \mathfrak{X}(M)$, then

1. $\mathcal{L}_X \omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega)$,

2. $\mathcal{L}_X(\omega + \eta) = \mathcal{L}_X \omega + \mathcal{L}_X \eta$,

3. $\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X \omega) \wedge \eta + \omega \wedge (\mathcal{L}_X \eta)$,

4. $\mathcal{L}_X(d\omega) = d\,(\mathcal{L}_X \omega)$,

5. $\mathcal{L}_X(Y \lrcorner\, \omega) = [X, Y] \lrcorner\, \omega + Y \lrcorner\, (\mathcal{L}_X \omega)$,

6. $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X(\mathcal{L}_Y\omega) - \mathcal{L}_Y(\mathcal{L}_X\omega)$.

Finally, if our linear object is some vector field $Y \in \mathfrak{X}(M)$, then as in [96], its Lie derivative with respect to the flow α_t generated by the vector field $X \in \mathfrak{X}(M)$ is defined by

$$\mathcal{L}_X Y := \frac{d}{dt}\left(\alpha_{-t*}\left(Y|_{\alpha_t}\right)\right)\bigg|_{t=0}.$$

It can be shown that

$$\mathcal{L}_X Y = [X, Y].$$

2.2.4 Exterior differential systems

Exterior differential systems are useful for providing an appropriate framework for the study of PDEs. Here we give a brief introduction to some of the basic definitions and results. The following may be found in, for example, Bryant *et al.* [23], Choquet-Bruhat *et al.* [34] or Yang [136].

Definition 2.2.5. [23] Let M be some smooth manifold of dimension m, and let $\alpha^1, \ldots, \alpha^p \in \Lambda(M)$ up to some $p \in \mathbb{N}$. The (algebraic) *ideal* generated by $\alpha^1, \ldots, \alpha^p$ is the subring $I \subset \Lambda(M)$ such that for all $1 \leq i \leq p$,

1. $\alpha^i \in I$, and for all $\omega \in \Lambda(M)$, $\alpha^i \wedge \omega \in I$,

2. If $\alpha^i = \alpha^i_0 + \cdots + \alpha^i_m$ for some $\alpha^i_j \in \Lambda^j(M)$, then each $\alpha^i_j \in I$.

We write $I := \langle \alpha^1, \ldots, \alpha^p \rangle$ to mean that I is the ideal generated by the elements $\alpha^1, \ldots, \alpha^p$.

From the second property in Definition 2.2.5, the ideal is said to be *homogeneous*.

Definition 2.2.6. An *integral manifold* of an ideal is an immersion $i : N \longrightarrow M$ such that $i^*\alpha = 0$ for all $\alpha \in I$.

In general, we will look for integral manifolds that satisfy an *independence (transverse) condition*. Given some linearly independent exact one-forms $dx^1, \ldots, dx^l \in \Lambda^1(M)$ up to some $l < m$, we look for an immersion of rank l such that $dx^1 \wedge \cdots \wedge dx^l \neq 0$ on the tangent space of $i(N)$, or equivalently, $i^*(dx^1 \wedge \cdots \wedge dx^l) \neq 0$. This allows us to parameterise our integral manifold by $x^1, \ldots, x^l$, which is more appropriate for generating solutions of a PDE, where $x^1, \ldots, x^l$ are its independent variables, and the dependent variables can then be expressed in terms of these variables.

Definition 2.2.7. An ideal I is a *differential* ideal if the exterior derivative of every member of I is also in I.

The *closure* of an ideal is its corresponding differential ideal generated by the original generators of the ideal and, in addition, their exterior derivatives.

Using the fact that the exterior derivative and pull-back commute, we have the following simple result:

Theorem 2.2.8. *An ideal and its closure have the same integral manifolds.*

Definition 2.2.9. A *Pfaffian system* is a submodule over $C^\infty(M)$ generated by a finite collection of linearly independent one-forms $\alpha^1, \ldots, \alpha^p \in \Lambda^1(M)$. We denote the Pfaffian system by $\{\alpha^1, \ldots, \alpha^p\}$. The differential ideal generated the Pfaffian system is $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$.

Definition 2.2.10. A vector field $Y \in \mathfrak{X}(M)$ is called a *Cauchy characteristic* of an ideal I if $Y \lrcorner I \subset I$. Define $A(I)$ as the submodule over $C^\infty(M)$ of all Cauchy characteristic vector fields of I.

The notation for $A(I)$ as the Cauchy characteristic space of an ideal I is identical to that used in Bryant *et al.* [23], and should not be confused with the same notation given in, for example, Hartley *et al.* [68] to denote the *associated space* of an ideal. In [68], the associated space is defined as the annihilating space (see Definition 2.3.5) of the vector field distribution in Definition 2.2.10, and the vector field dual of the associated space is called the Cauchy characteristic space only if I is a differential ideal.

Using Definition 2.2.10, we have the following well-known theorem for differential ideals:

Theorem 2.2.11. *If $Y, Z \in \mathfrak{X}(M)$ are Cauchy characteristic vector fields of a differential ideal I so is their bracket [Y,Z].*

Next, we introduce the idea of a *Cartan system* of an ideal:

Definition 2.2.12. The *Cartan system* of I, denoted $C(I)$, is the submodule defined by $C(I) := \{\omega \in \Lambda^1(M) : X \lrcorner \omega = 0, \ \ for\ all \ \ X \in A(I)\}$.

For a differential ideal generated by some Pfaffian system, we have the following well-known result from Cartan [30]:

Theorem 2.2.13. *Let I be a differential ideal generated by some finitely generated Pfaffian system. If $C(I)$ is generated by some s linearly independent one-forms $d\gamma^1, \ldots, d\gamma^s \in \Lambda^1(M)$, then there exist generators for I in terms of $\gamma^1, \ldots, \gamma^s$ and their differentials.*

Note that for any ideal I not containing zero-forms, $I \cap \Lambda^1(M) \subset C(I)$.

2.3 Symmetries

We begin with two important definitions of a symmetry:

Definition 2.3.1. Let $I \subset \Lambda(M)$ be an ideal. A vector field $X \in \mathfrak{X}(M)$ is a *symmetry* (or *isovector*) of I if $\mathcal{L}_X I \subset I$.

In order to show that a vector field X is a symmetry of an ideal I, it is enough to show that the Lie derivative with respect to X of the generators of I are also in the ideal.

In dealing with a specific differential form, say Ω, we generally avoid introducing the ideal generated by the form and say that a vector field X is a symmetry of Ω if $\mathcal{L}_X \Omega \equiv 0 \mod \Omega$.

Definition 2.3.2. Let $D \subset \mathfrak{X}(M)$ be a vector field distribution. A vector field $X \in \mathfrak{X}(M)$ is a *symmetry* of D if $\mathcal{L}_X D \subset D$.

Once again, it is enough to look at simply the generators of a given vector field distribution D when determining whether a vector field X is a symmetry of the distribution.

We now present some results connecting symmetries, ideals and Cauchy characteristic spaces, some of which are not mentioned in the general literature.

Theorem 2.3.3. *Let $I \subset \Lambda(M)$ be an ideal, and suppose that $A(I)$ is not zero-dimensional. If a vector field $X \in \mathfrak{X}(M)$ is a symmetry of I then X is a symmetry of $A(I)$.*

Proof. Let X be a symmetry of the ideal I. Let $Y \in A(I)$ and $\beta \in I$. Then from rearranging the identity $\mathcal{L}_X(Y \lrcorner \beta) = [X,Y] \lrcorner \beta + Y \lrcorner (\mathcal{L}_X \beta)$, we obtain

$$[X,Y] \lrcorner \beta = \mathcal{L}_X(Y \lrcorner \beta) - Y \lrcorner (\mathcal{L}_X \beta).$$

Now the first term on the right hand side is in I since $Y \lrcorner \beta \in I$ and X is a symmetry of I. The second term is also in I since $\mathcal{L}_X \beta \in I$ and $Y \in A(I)$. Hence $[X,Y] \lrcorner \beta \in I$. Therefore $[X,Y] \in A(I)$. □

Definition 2.3.4. Let $\Omega \in \Lambda^k(M)$. Then its *kernel* (also known as *characteristic space* [37]) is defined by $\ker(\Omega) := \{X \in \mathfrak{X}(M) : X \lrcorner \Omega = 0\}$.

Definition 2.3.5. Let $\{\alpha^1, \ldots, \alpha^p\}$ be some Pfaffian system generated by p linearly independent one-forms in $\Lambda^1(M)$. Its *annihilating space* is the vector field distribution given by $(Sp\{\alpha^1, \ldots, \alpha^p\})^\perp := \{X \in \mathfrak{X}(M) : X \lrcorner \alpha^i = 0, \;\; for\ all \;\; 1 \leq i \leq p\}$. Similarly, given a vector field distribution $Sp\{X_1, \ldots, X_{m-p}\}$ generated by some $X_1, \ldots, X_{m-p} \in \mathfrak{X}(M)$, its *annihilating space* is the submodule of one-forms over $C^\infty(M)$ denoted by $(Sp\{X_1, \ldots, X_{m-p}\})^\perp := \{\alpha \in \Lambda^1(M) : X_i \lrcorner \alpha = 0, \;\; for\ all \;\; 1 \leq i \leq m-p\}$.

In Definition 2.3.5, we write $Sp\{\alpha^1, \ldots, \alpha^p\}$ to mean the submodule over $C^\infty(M)$ of $\Lambda^1(M)$ generated by linearly independent $\alpha^1, \ldots, \alpha^p$.

For linearly independent $\alpha_1, \ldots, \alpha_p \in \Lambda^1(M)$, it is not hard to show that

$$\{X \in \mathfrak{X}(M) : X \lrcorner \alpha^i = 0, \;\; for\ all \;\; 1 \leq i \leq p\} = \ker\left(\alpha^1 \wedge \cdots \wedge \alpha^p\right).$$

In terms of vector fields, we have the following result for Cauchy characteristics:

Theorem 2.3.6. *Let $\alpha^1, \ldots, \alpha^p \in \Lambda^1(M)$ be p linearly independent one-forms, and define $I := \langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$. With $D := (Sp\{\alpha^1, \ldots, \alpha^p\})^\perp$, a vector field $Y \in D$ is a Cauchy characteristic of I if and only if $[X,Y] \in D$ for all $X \in D$.*

Proof. Let Y be a Cauchy characteristic vector field of I, i.e. $Y \lrcorner \alpha^i = 0$ and $Y \lrcorner d\alpha^i \in I$ for all $1 \leq i \leq p$. This implies that for all i,

$$\mathcal{L}_Y \alpha^i = Y \lrcorner d\alpha^i \in I.$$

Hence Y is a symmetry of I. Let $X \in D$, where D is defined in the theorem. Using the property

$$\mathcal{L}_Y \left(X \lrcorner \alpha^i \right) = [Y, X] \lrcorner \alpha^i + X \lrcorner \left(\mathcal{L}_Y \alpha^i \right),$$

we know that the term on the left is zero and the second term on the right is also zero. Hence for all i, $[X, Y] \lrcorner \alpha^i = 0$, so that $[X, Y] \in D$.

Conversely, let $Y \in D$ and $[X, Y] \in D$ for all $X \in D$. We therefore have for all i,

$$Y \lrcorner \alpha^i = 0 = [X, Y] \lrcorner \alpha^i.$$

Now once again using the property

$$\mathcal{L}_Y \left(X \lrcorner \alpha^i \right) = [Y, X] \lrcorner \alpha^i + X \lrcorner \left(\mathcal{L}_Y \alpha^i \right),$$

we have

$$X \lrcorner \left(\mathcal{L}_Y \alpha^i \right) = 0. \tag{2.8}$$

Since (2.8) must hold for all $X \in D$, this means $\mathcal{L}_Y \alpha^i \in I$ for all i. Since $Y \lrcorner \alpha^i = 0$,

$$\mathcal{L}_Y \alpha^i = Y \lrcorner d\alpha^i \in I,$$

so Y is a Cauchy characteristic vector field of I. □

At this point we will introduce the idea of a *trivial symmetry*. Given a *differential* ideal I, we call all Cauchy characteristics of I trivial symmetries of I (this also means that all trivial symmetries of some differential form are in its kernel). The reason for this is contained in the next theorem.

Theorem 2.3.7. *Let I be a differential ideal, and let Y be a Cauchy characteristic vector field of I. Then Y is a symmetry of I.*

Proof. Let $\beta \in I$ and $Y \in A(I)$.

$$\mathcal{L}_Y \beta = d \left(Y \lrcorner \beta \right) + Y \lrcorner d\beta.$$

The first term on the right is in I because $Y \lrcorner \beta \in I$, and consequently $d \left(Y \lrcorner \beta \right) \in I$, since I is a differential ideal. The second term is obviously in I. □

Similarly, given a vector field distribution D, a *trivial symmetry* of D is a symmetry of D that is also in D.

A fundamental distinction between trivial and non-trivial symmetries is as follows: Given a trivial symmetry, multiplying it by any non-constant function will yield a trivial symmetry, however doing the same to a non-trivial symmetry will in general not produce a non-trivial symmetry.

For a differential ideal generated by a Pfaffian system we have the following extension of Theorem 2.3.7:

Theorem 2.3.8. *Let I be a differential ideal generated by some finite collection of linearly independent one-forms $\alpha^1, \ldots, \alpha^p \in \Lambda^1(M)$. A vector field $X \in \mathfrak{X}(M)$ is a symmetry of I in the annihilating space $D := (Sp\{\alpha^1, \ldots, \alpha^p\})^\perp$ if and only if X is a trivial symmetry (Cauchy characteristic vector field) of I.*

Proof. With X as a symmetry of I, if $X \lrcorner \alpha^i = 0$ for all $1 \leq i \leq p$, then for each i

$$I \ni \mathcal{L}_X \alpha^i = X \lrcorner d\alpha^i.$$

The converse is also obvious using Theorem 2.3.7. □

Definition 2.3.9. A differential k-form is said to be *decomposable* (or *simple*) if it can be written as the exterior product of k one-forms.

Decomposability is a local property, and a k-form defined on M is decomposable if and only if the dimension of the kernel is $m - k$ [37].

Next, consider the following two simple theorems, the first of which is proved in Sherring and Prince [110]:

Theorem 2.3.10. *A vector field $X \in \mathfrak{X}(M)$ is a symmetry of a decomposable k-form $\Omega \in \Lambda^k(M)$ if and only if X is a symmetry of* $\ker(\Omega)$.

Theorem 2.3.11. *Let $\Omega \in \Lambda^k(M)$ and $I := \langle \Omega, d\Omega \rangle$. If $d\Omega \equiv 0$ mod Ω, then* $\ker(\Omega) = A(I)$.

Proof. First suppose $\ker(\Omega)$ is not zero-dimensional, i.e. there exists a non-zero vector field $W \in \mathfrak{X}(M)$ such that $W \lrcorner \Omega = 0$. Now since $d\Omega \equiv 0 \bmod \Omega$, $W \lrcorner d\Omega = W \lrcorner (\alpha \wedge \Omega) = (W \lrcorner \alpha)\Omega$ for some $\alpha \in \Lambda^1(M)$. Therefore $W \in A(I)$.

Now suppose $A(I)$ is not zero-dimensional. This means there exists a non-zero vector field $X \in \mathfrak{X}(M)$ such that $X \lrcorner \Omega = 0$ and $X \lrcorner d\Omega = \mu\Omega$ for some $\mu \in C^\infty(M)$. Hence from the first part, $X \in \ker(\Omega)$.

If $\ker(\Omega)$ is zero-dimensional, then $Y \lrcorner \Omega \neq 0$ for all non-zero $Y \in \mathfrak{X}(M)$. This means $Y \lrcorner \Omega \notin I$, and hence $Y \notin A(I)$. Therefore $A(I)$ is zero-dimensional.

Finally, if $A(I)$ is zero-dimensional, then $Z \lrcorner \Omega \neq 0$ for all $Z \in \mathfrak{X}(M)$. Hence $\ker(\Omega)$ is zero-dimensional. □

Using the above two theorems, we obtain the following extension of Theorem 2.3.10 to differential ideals thus giving us a condition under which the converse of Theorem 2.3.3 holds true:

Theorem 2.3.12. *Let $I := \langle \Omega \rangle$ for some some $\Omega \in \Lambda^k(M)$ with $d\Omega \equiv 0$ mod Ω. Moreover, let Ω be decomposable on M and $A(I)$ not zero-dimensional. A vector field $X \in \mathfrak{X}(M)$ is a symmetry of I if and only if X is a symmetry of $A(I)$.*

Proof. From Theorem 2.3.11, $d\Omega \equiv 0 \bmod \Omega$ implies that $\ker(\Omega) = A(I)$. Hence the result follows from Theorem 2.3.10. □

Remark. If $\Omega \in \Lambda^k(M)$ with $k = m$, and I is the differential ideal generated by Ω (note $d\Omega = 0$), then *any* non-zero vector field in $\mathfrak{X}(M)$ is a symmetry of I. Moreover, $A(\langle \Omega \rangle)$ is zero-dimensional, and therefore *any* non-zero vector field in $\mathfrak{X}(M)$ is also a symmetry of a zero-dimensional $A(\langle \Omega \rangle)$.

2.4 Jet bundles and contact structures

2.4.1 Jet bundles

Comprehensive discussions on jet bundle theory may be found in Ehresmann [50], Saunders [107] and Pommaret [103]. Simplified presentations can be found in, for example, Pirani *et al.* [102], Rogers and Shadwick [105] and Steeb [116].

Define an open, simply connected neighbourhood $U^n \subset \mathbb{R}^n$ with coordinates $x^1, \ldots, x^n$, and an open, simply connected neighbourhood $V^m \subset \mathbb{R}^m$ with coordinates $z^1, \ldots, z^m$. Let $C^\infty(U^n, V^m)$ be the space of smooth maps from U^n into V^m

determined by the n coordinate functions

$$z^1 := f_1(x^1, \dots, x^n),$$
$$\vdots$$
$$z^m := f_m(x^1, \dots, x^n),$$

where $f_1, \dots, f_m \in C^\infty(U^n)$.

Definition 2.4.1. If $f, g \in C^\infty(U^n, V^m)$ are defined at some $p \in U^n$, then f and g are said to be *r-equivalent* at p if for each j, where $1 \leq j \leq m$,

$$f_j(x^1, \dots, x^n) = g_j(x^1, \dots, x^n),$$

and for each k, where $1 \leq k \leq r$,

$$\frac{\partial^k f_j}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_k}} = \frac{\partial^k g_j}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_k}},$$

where $i_1, \dots, i_k \in \{1, \dots, m\}$.

Definition 2.4.2. The equivalence class of maps $f \in C^\infty(U^n, V^m)$ at $p \in U^n$ that are r-equivalent is called the *r-jet*, and is denoted by $j_p^r f$.

Definition 2.4.3. We call the collection of all r-jets ranging over all $p \in U^n$ and all $f \in C^\infty(U^n, V^m)$ the *r-jet bundle* of maps from U^n to V^m, and we denote it by $J^r(U^n, V^m)$. Thus

$$J^r(U^n, V^m) := \bigcup_{p \in U^n, f \in C^\infty(U^n, V^m)} j_p^r f.$$

Using the natural topology for the r-jet bundle, we have the following result:

Theorem 2.4.4.

$$\dim J^r(U^n, V^m) = n + m \cdot \binom{n+r}{r}.$$

Definition 2.4.5. The map

$$\pi_{r-1}^r : j_p^r f \longmapsto j_p^{r-l} f$$

is defined as the *canonical projection map* from $J^r(U^n, V^m)$ to $J^{r-l}(U^n, V^m)$, where $l = 0, 1, \dots, r-1$. If $r = 0$, we identify $J^0(U^n, V^m)$ with $U^n \times V^m$ and

$$\pi_0^r : j_p^r f \longmapsto (p, f(p)).$$

The point p is called the *source* of $j_p^r f$, and the point $f(p)$ is called the *target* of $j_p^r f$.

Definition 2.4.6. The natural projection maps

$$\alpha : J^r(U^n, V^m) \longrightarrow U^n, \qquad \beta : J^r(U^n, V^m) \longrightarrow V^m,$$

are defined by

$$\alpha\left(j_p^r f\right) := p, \qquad \beta\left(j_p^r f\right) := f(p),$$

and known as the *source* and *target* maps, respectively.

Next, we introduce the idea of a *section* of a jet bundle as a particular example of a section of an arbitrary vector bundle:

Definition 2.4.7. A *section* of α is a map $h : U^n \longrightarrow J^r(U^n, V^m)$ satisfying

$$\alpha \circ h = \mathrm{id}_{U^n},$$

where id_{U^n} is the identity map on U^n.

Definition 2.4.8. The *r-jet extension* of $f \in C^\infty(U^n, V^m)$, denoted by $j^r f$, is defined by the section $j^r f : p \longmapsto j_p^r f$.

We have $j^0 f$ as the graph of f.

2.4.2 Contact structures

We show in this section that contact structures are useful for classifying those sections of α (where α is the natural source projection map of the previous section) that are r-jet extensions of smooth maps from U^n to V^m.

The r-jet bundle $J^r(U^n, V^m)$ is the appropriate framework for dealing with systems of PDEs of highest order r of m dependent variables and n independent variables. The bundle may be identified with the following coordinates:

- Coordinates for the independent variables: $x^1, \ldots, x^n$,
- Coordinates for the dependent variables: $z^1, \ldots, z^m$,

- First derivative coordinates: $\{z_1^1, \ldots, z_n^1\}, \ldots, \{z_1^m \ldots, z_n^m\}$,

- Second derivative coordinates:

$$\{z_{11}^1, \ldots, z_{1n}^1\}, \{z_{22}^1, \ldots, z_{2n}^1\}, \ldots, z_{nn}^1,$$
$$\vdots$$
$$\{z_{11}^m, \ldots, z_{1n}^m\}, \{z_{22}^m, \ldots, z_{2n}^m\}, \ldots, z_{nn}^m,$$

and so on until we reach the final component that contains all the r-th derivative coordinates of the form $z^j_{i_1 \cdots i_r}$ for each $1 \leq j \leq m$ and all possible choices of $1 \leq i_1 \leq \cdots \leq i_r \leq n$.

Next, for each $1 \leq j \leq m$ we define the following *contact one-forms* on $J^r(U^n, V^m)$:

$$\begin{aligned}
C^j &:= dz^j - z^j_{i_1} dx^{i_1}, \\
C^j_{i_1} &:= dz^j_{i_1} - z^j_{i_1 i_2} dx^{i_2}, \\
C^j_{i_1 i_2} &:= dz^j_{i_1 i_2} - z^j_{i_1 i_2 i_3} dx^{i_3}, \\
&\vdots \\
C^j_{i_1 \cdots i_{r-1}} &:= dz^j_{i_1 \cdots i_{r-1}} - z^j_{i_1 \cdots i_r} dx^{i_r},
\end{aligned}$$

where we are implying summation on the repeated indexes, with $1 \leq i_1 \leq \cdots \leq i_r \leq n$.

Definition 2.4.9. The *r-th order contact system*, denoted by $\Omega^r(U^n, V^m)$, is defined as the submodule over $C^\infty(J^r(U^n, V^m))$ generated by the contact forms on the r-jet bundle $J^r(U^n, V^m)$, i.e.

$$\Omega^r(U^n, V^m) := \left\{ C^j, C^j_{i_1}, C^j_{i_1 i_2}, \ldots, C^j_{i_1 \cdots i_{r-1}} : 1 \leq j \leq m, 1 \leq i_1 \leq \cdots \leq i_r \leq n \right\}.$$

It is easy to see that say, for example, we are on the 1-jet bundle $J^1(U^n, V^m)$ and given some section $g : U^n \longrightarrow J^1(U^n, V^m)$ denoted by

$$g : x \longmapsto \left(x^1, \ldots, x^n, g^1(x), \ldots, g^m(x), g^1_1(x), \ldots, g^1_n(x), \ldots, g^m_1(x), \ldots, g^m_n(x)\right),$$

where $x = (x^1, \ldots, x^n)$, then $g^* C^j = 0$ for each j if and only if

$$g^j_i = \frac{\partial g^j}{\partial x^i},$$

for each i, which holds if and only if $g = j^1 f$, where $f := \beta \circ g$.

This result can be generalised to the r-jet bundle, as summarised in Theorem 3.2 in [23] and Theorem 4.24 in [99]. This is stated in the following theorem:

Theorem 2.4.10. *A section $g : U^n \longrightarrow J^r(U^n, V^m)$ is an r-jet, i.e. $g = j^r(\beta \circ g)$ if and only if*

$$g^*\Omega^r(U^n, V^m) = 0.$$

A simple coordinate free proof of Theorem 2.4.10 can be found in Gardner and Shadwick [56]. We also have the following corollary to Theorem 2.4.10, as found in Theorem 2.3.1 in Stormark [120]:

Corollary 2.4.11. *Let $\Phi : U^n \longrightarrow J^r(U^n, V^m)$ be an immersion such that $\Phi^*\Omega^r(U^n, V^m) = 0$. Then $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$ if and only if the image of Φ is the image of some r-jet.*

Note in Corollary 2.4.11 that the right hand side means $\Phi(U^n) = j^r f(U^n)$ for some smooth $f : U^n \longrightarrow V^m$. If Φ is a section, then not only is the transverse condition automatically satisfied, but more importantly, Theorem 2.4.10 means $\Phi = j^r f$.

2.5 Differential equations on jet bundles

Following [99], consider a general κ-th order system of ρ differential equations

$$F_\nu \left(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_{i_1 \cdots i_\kappa}\right) = 0, \qquad \nu = 1, \ldots, \rho, \tag{2.9}$$

in m dependent variables $u^1, \ldots, u^m$ and n independent variables $x^1, \ldots, x^n$. The system of equations in (2.9) describes a locus of $J^\kappa(U^n, V^m)$. It is assumed that the equations $\{F_\nu = 0 : \nu = 1, \ldots, \rho\}$ are of rank ρ on $J^\kappa(U^n, V^m)$, so the partial differential equations describe a regular submanifold of $J^\kappa(U^n, V^m)$. We can parameterise the submanifold by a rank ρ immersion (which may or may not be one-to-one)

$$\Phi_F : \Sigma \longrightarrow J^\kappa(U^n, V^m).$$

Here we use the subscript F to indicate a single or system of PDEs of the form in (2.9).

From Bryant *et al.* [23], we have the following result:

Theorem 2.5.1. *A map $f : U^n \longrightarrow V^m$ is a solution map of the PDEs in (2.9) if and only if*

$$j_p^\kappa f \in \Phi_F(\Sigma),$$

for all $p \in U^n$.

In looking for a solution of the system of PDEs in (2.9), we search for a section that annihilates the submodule of contact forms $\Omega^\kappa(U^n, V^m)$ and maps into $\Phi_F(\Sigma)$. Hence by Theorems 2.4.10 and 2.5.1, this section is a κ-jet of some solution map of the PDEs in (2.9).

In terms of differential ideals, we investigate two approaches for finding solution maps of a given system of PDEs. The first is based on Pfaffian systems, where the contact forms are pulled-back onto the regular submanifold of $J^\kappa(U^n, V^m)$ described by the jet bundle coordinate representation of the PDEs, while the second uses Edelen's *fundamental ideals* [43, 45, 47, 49], where the information specific to the PDEs is encoded in some differential n-forms generating the ideal. We now proceed to discuss both methods.

2.5.1 Pfaffian system approach

Recent discussions of this approach can be found in work by, for example, Hartley *et al.* [67] and Vassiliou [124].

Suppose we are given the system of PDEs in (2.9) describing a regular submanifold of $J^\kappa(U^n, V^m)$ and some corresponding parameterising immersion

$$\Phi_F : \Sigma \longrightarrow J^\kappa(U^n, V^m).$$

We pull-back the module of contact forms $\Omega^\kappa(U^n, V^m)$ onto Σ, and denote this by

$$\Omega_F^\kappa(U^n, V^m) := \Phi_F^* \Omega^\kappa(U^n, V^m).$$

Suppose we define

$$\Omega^{\overline{\kappa}}(U^n, V^m) := \left\{ C^j_{i_1 \cdots i_{\kappa-1}} : 1 \le j \le m, 1 \le i_1 \le \cdots \le i_{\kappa-1} \le n \right\},$$

containing only the highest order contact forms. It is easy to show that the exterior derivative of all the contact forms less than κ-th order in $\Omega^\kappa(U^n, V^m)$ can be expressed as a linear combination of the contact forms in $\Omega^{\kappa}(U^n, V^m)$. This result is given in the following lemma:

Lemma 2.5.2. [49]

$$\langle \Omega^{\kappa}(U^n, V^m), d\left(\Omega^{\kappa}(U^n, V^m)\right)\rangle = \langle \Omega^{\kappa}(U^n, V^m), d\left(\Omega^{\overline{\kappa}}(U^n, V^m)\right)\rangle.$$

From this lemma it is obvious that

$$\langle \Omega^{\kappa}_F(U^n, V^m), d\left(\Omega^{\kappa}_F(U^n, V^m)\right)\rangle = \langle \Omega^{\kappa}_F(U^n, V^m), d\left(\Omega^{\overline{\kappa}}_F(U^n, V^m)\right)\rangle.$$

We therefore define the differential ideal

$$I_F := \langle \Omega^{\kappa}_F(U^n, V^m), d\left(\Omega^{\overline{\kappa}}_F(U^n, V^m)\right)\rangle,$$

on Σ. The task now becomes that of finding an n-dimensional integral manifold of I_F, i.e. an immersion

$$\Psi : U^n \longrightarrow \Sigma,$$

on which the transverse condition holds. Then from Corollary 2.4.11 and then Theorem 2.5.1, the image of the immersion given by $\Phi_F \circ \Psi$ is equal to the image of the κ-jet of some local solution of the system of PDEs in (2.9).

2.5.2 Fundamental ideal representation

Given the system of PDEs in (2.9), we define a differential ideal on $J^{\kappa}(U^n, V^m)$ in the following way:

$$I_F := \langle \Omega^{\kappa}(U^n, V^m), d\left(\Omega^{\overline{\kappa}}(U^n, V^m)\right), F_1 dx^1 \wedge \cdots \wedge dx^n, \ldots, F_\rho dx^1 \wedge \cdots \wedge dx^n\rangle.$$

The ideal I_F is referred to by Edelen as the *fundamental ideal* corresponding to the system of PDEs. We have not included each $d\left(F_\nu dx^1 \wedge \cdots \wedge dx^n\right)$ as generators for the differential ideal because of the following result which can also be found in [47, 49]:

Lemma 2.5.3. *For each* $\nu = 1, \ldots, \rho$,

$$d\left(F_\nu dx^1 \wedge \cdots \wedge dx^n\right) \equiv 0 \mod \Omega^{\kappa}(U^n, V^m), d\left(\Omega^{\overline{\kappa}}(U^n, V^m)\right).$$

Once again, if we can find an n-dimensional integral manifold of I_F, i.e. an immersion

$$\Phi : U^n \longrightarrow J^\kappa(U^n, V^m),$$

on which the transverse condition $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$ holds, then since for each ν,

$$0 = \Phi^* \left(F_\nu dx^1 \wedge \cdots \wedge dx^n \right) = \left(\Phi^* F_\nu \right) \Phi^* \left(dx^1 \wedge \cdots \wedge dx^n \right),$$

the transverse condition implies that

$$\Phi^* F_\nu = 0.$$

Now once again from Corollary 2.4.11 and then Theorem 2.5.1, $\Phi(U^n)$ is equal to the image of the κ-jet of some solution map of the system of PDEs in (2.9).

The fundamental ideal representation differs from the Pfaffian system approach in that it does not look to find n-dimensional integral manifolds of a differential ideal defined on some locus of the jet bundle $J^\kappa(U^n, V^m)$ described by the PDEs, but instead looks for integral manifolds of a differential ideal that is defined on the *whole jet bundle*. The price to be paid is that the differential ideal is no longer generated solely by one-forms and their exterior derivatives, but includes ρ n-forms.

Chapter 3

Symmetry in differential forms

3.1 Introduction

This chapter begins with some preliminary results on symmetries and Frobenius integrable distributions. Based on work by Lie [90] and Cartan [29, 30], Theorem 3.2.13 reproduces Proposition 4.7 in Sherring and Prince [110] for integrating Frobenius integrable distributions using solvable symmetry structures. While this result can also be found in Basarab-Horwath [16] using similar notation, it is an extension of Theorem 3.2.13 given in Theorem 3.2.14 that will prove to be more useful in later chapters for dealing with partial differential equations. Theorem 3.2.14 represents the first of our major new results, and shows how solvable symmetry structures can be used to give simplified expressions for a certain class of decomposable differential forms. We then consider in this chapter some types of solvable symmetry structures that simplify the conclusion of Theorem 3.2.14. Following this, we examine some necessary conditions for a given differential form to be a candidate for the theorem. Finally, we apply the theorem to finding local coordinates for the Pfaff problem and Darboux's theorem.

It is assumed throughout this chapter that our expressions apply locally on some open neighbourhood $U^n \subset \mathbb{R}^n$, with coordinates $x^1, \ldots, x^n \in C^\infty(U^n)$. One further assumption that we make on U^n is that it be convex. This allows us to use Lemma 2.2.2 on the whole of U^n.

3.2 The Frobenius theorem and symmetry

The highlights of this section are firstly a review of the well-known Frobenius theorem in Theorem 3.2.4. We then introduce the definition of a *solvable symmetry structure* in Definition 3.2.10, following which, we present in Theorem 3.2.13 an extension of Lie's symmetry approach to the integration of Frobenius integral distributions as found in Basarab-Horwath [16] and Sherring and Prince [110]. In Theorem 3.2.14 which follows, we give a differential form equivalent of Theorem 3.2.13 that we use in later chapters in conjunction with Corollary 3.2.12. Finally, in Theorem 3.2.16 and its generalisation in Theorem 3.2.17, we examine a certain type of solvable symmetry structures that can simplify the conclusion of Theorem 3.2.14.

We begin with three supporting results for the Frobenius theorem:

Lemma 3.2.1. [37] *Let $\Omega \in \Lambda^{n-m}(U^n)$ for some $m \leq n-1$. Then* $\ker(\Omega)$ *can be at most m-dimensional. Moreover,* $\ker(\Omega)$ *is precisely m-dimensional if and only if Ω is decomposable.*

From this we have the following lemma also found in Sherring and Prince [110]:

Lemma 3.2.2. *Let $\Omega \in \Lambda^m(U^n)$ for some $m > 1$ be decomposable, and let $X \in \mathfrak{X}(U^n)$ such that $X \lrcorner \Omega \neq 0$ (i.e. $X \lrcorner \Omega \neq 0$ at all $p \in U^n$). Then $X \lrcorner \Omega$ is decomposable.*

It can be shown that the converse of Lemma 3.2.2 is also true, however we will not make use of this fact.

As a corollary of Lemma 3.2.2, we have the following:

Corollary 3.2.3. *Let $D := Sp\{Y_1, \ldots, Y_m\}$ be some m-dimensional distribution in $\mathfrak{X}(U^n)$, where $m \leq n-1$. If $\Omega := Y_1 \lrcorner \ldots \lrcorner Y_m \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \in \Lambda^{n-m}(U^n)$, then Ω is decomposable and equal to the exterior product of some $n-m$ linearly independent generators of $D^\perp$.*

Proof. With D and Ω defined as in the theorem, let $Y \in \mathfrak{X}(U^n)$ be any non-zero vector field in D. Then from the definition of Ω, $Y \lrcorner \Omega = 0$. Hence $\ker(\Omega)$ is at least m-dimensional. But from Lemma 3.2.1, since Ω is an $(n-m)$-form, its kernel can not be greater than m-dimensional, and therefore Ω is decomposable.

Now we can write $\Omega = \theta^1 \wedge \cdots \wedge \theta^{n-m}$ for some linearly independent one-forms $\theta^1, \ldots, \theta^{n-m} \in \Lambda^1(U^n)$. Since for each $1 \leq i \leq m$, $Y_i \lrcorner \Omega = 0$, we then have that for each $1 \leq j \leq n-m$, $Y_i \lrcorner \theta^j = 0$. Hence $\theta^1, \ldots, \theta^{n-m}$ generate $D^\perp$. $\square$

Theorem 3.2.4. (Frobenius) *Let D be an m-dimensional distribution generated by the vector fields $Y_1, \ldots, Y_m \in \mathfrak{X}(U^n)$, where $m \leq n-1$. Define $D^\perp$ to be the submodule of all one-forms that annihilate D. Let $\Omega := Y_1 \lrcorner \ldots \lrcorner Y_m \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \in \Lambda^{n-m}(U^n)$. Then D has m-dimensional integral submanifolds on U^n if and only if either of the following two equivalent conditions are true:*

1. *For all $X, Y \in D$, $[X, Y] \in D$,*
2. *For all $\theta \in D^\perp$, $d\theta \wedge \Omega = 0$.*

For a proof of Theorem 3.2.4, see for example Bryant *et al.* [23], Chern and Wolfson [33], Hermann [74] or Warner [132].

We say that a distribution D is *Frobenius integrable* (or generates a *foliation* of U^n) if the first condition in the Frobenius theorem holds. This theorem means that D generates an m-dimensional foliation of U^n whose leaves are described by some set of $n-m$ functions $\gamma^1 = c^1, \ldots, \gamma^{n-m} = c^{n-m}$ of rank $n-m$, where $\gamma^1, \ldots, \gamma^{n-m} \in C^\infty(U^n)$ and $c^1, \ldots, c^{n-m}$ are some appropriate constant functions.

Using the previous lemma, we have the following corollary to the Frobenius theorem:

Corollary 3.2.5. *Let D be an m-dimensional distribution generated by the vector fields $Y_1, \ldots, Y_m \in \mathfrak{X}(U^n)$, where $m \leq n-1$. Let $\Omega := Y_1 \lrcorner \ldots \lrcorner Y_m \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \in \Lambda^{n-m}(U^n)$. For all $\theta \in D^\perp$, $d\theta \wedge \Omega = 0$ (i.e. D is Frobenius integrable) if and only if $d\Omega \equiv 0$ mod Ω.*

Proof. With Ω defined as in the corollary, Corollary 3.2.3 implies $\Omega = \theta^1 \wedge \cdots \wedge \theta^{n-m}$ for some linearly independent $\theta^1, \ldots, \theta^{n-m} \in \Lambda^1(U^n)$ that generate $D^\perp$. Now for each $1 \leq i \leq n-m$, the Frobenius condition $d\theta^i \wedge \Omega = 0$ is equivalent to the condition that $d\theta^i \equiv 0 \bmod \theta^1, \ldots, \theta^{n-m}$. Hence

$$\begin{aligned} d\Omega &= d\left(\theta^1 \wedge \cdots \wedge \theta^{n-m}\right), \\ &\equiv 0 \quad \bmod \Omega. \end{aligned}$$

To prove the converse, suppose $d\Omega \equiv 0 \bmod \Omega$. Now for all i,

$$d\theta^i \wedge \Omega = d\left(\theta^i \wedge \Omega\right) + \theta^i \wedge d\Omega. \tag{3.1}$$

Since $\theta^i \wedge \Omega = 0$ and Ω is closed modulo itself, we find from (3.1) that $d\theta^i \wedge \Omega = 0$. ☐

For any decomposable $\Omega \in \Lambda^m(U^n)$ such that $d\Omega \equiv 0 \bmod \Omega$, the above corollary to the Frobenius theorem essentially tells us that $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$ for some $\gamma^0, \ldots, \gamma^m \in C^\infty(U^n)$, where $\gamma^1, \ldots, \gamma^m$ are functionally independent.

Leaving the Frobenius theorem, suppose we are given some decomposable Ω with $d\Omega \equiv 0 \bmod \Omega$. In what follows, we intend to eventually show in Corollary 3.2.12 that there is a clear relationship between a solvable structure of symmetries for $A(\langle\Omega\rangle)$ and a solvable structure of symmetries for Ω (though we are yet to properly define a *solvable structure of symmetries* in both instances). We begin with the following obvious result that uses Theorems 2.2.11 and 2.3.11:

Lemma 3.2.6. *Let $\Omega \in \Lambda^m(U^n)$ such that $d\Omega \equiv 0 \bmod \Omega$. Then* $\ker(\Omega)$ *is Frobenius integrable.*

Using Lemma 3.2.2 we obtain the following theorem:

Theorem 3.2.7. *Let $\Omega \in \Lambda^m(U^n)$ for some $m > 1$ be decomposable, and let $X \in \mathfrak{X}(U^n)$ with the property $X \lrcorner \Omega \neq 0$. Then*

$$X \lrcorner \Omega = \phi^1 \wedge \cdots \wedge \phi^{m-1}, \tag{3.2}$$

for some linearly independent one-forms $\phi^1, \ldots, \phi^{m-1} \in \Lambda^1(U^n)$, and there exists $\theta \in \Lambda^1(U^n)$ such that

$$\Omega = \theta \wedge \phi^1 \wedge \cdots \wedge \phi^{m-1}.$$

Proof. Let $\Omega \in \Lambda^m(U^n)$ be decomposable, and let $X \in \mathfrak{X}(U^n)$ with $X \lrcorner \Omega \neq 0$. From Lemma 3.2.2,

$$X \lrcorner \Omega = \phi^1 \wedge \cdots \wedge \phi^{m-1} \neq 0,$$

for some linearly independent one-forms $\phi^1, \ldots, \phi^{m-1} \in \Lambda^1(U^n)$. Since $X \lrcorner X \lrcorner \Omega = 0$, we have that $X \lrcorner \phi^i = 0$ for all $1 \leq i \leq m-1$. We can always complete

$\{\phi^1, \ldots, \phi^{m-1}\}$ to a basis by adding a further $n-m+1$ linearly independent one-forms $\phi, \phi^m, \ldots, \phi^{n-1}$ such that $X \lrcorner \phi^i = 0$ for all $1 \le i \le n-1$ and $X \lrcorner \phi = 1$. If we choose $Y_1, \ldots, Y_{n-1} \in \mathfrak{X}(U^n)$ so that $\{X, Y_1, \ldots, Y_{n-1}\}$ is dual to $\{\phi, \phi^1, \ldots, \phi^{n-1}\}$, then from (3.2), $Y_j \lrcorner X \lrcorner \Omega = 0$ for all $m \le j \le n-1$. Hence (with sum)

$$\begin{aligned}\Omega &= \Omega(X, Y_1, \ldots, Y_{m-1}) \phi \wedge \phi^1 \wedge \cdots \wedge \phi^{m-1} + \Omega(Y_{k_1}, \ldots, Y_{k_m}) \phi^{k_1} \wedge \cdots \wedge \phi^{k_m}, \\ &= \phi \wedge \phi^1 \wedge \cdots \wedge \phi^{m-1} + \Omega(Y_{k_1}, \ldots, Y_{k_m}) \phi^{k_1} \wedge \cdots \wedge \phi^{k_m},\end{aligned}$$

where $1 \le k_1 < \cdots < k_m \le n-1$. Since Ω is decomposable, we have that (with sum only on p)

$$\Omega = \phi \wedge \phi^1 \wedge \cdots \wedge \phi^{m-1} + \Omega(Y_1, \ldots, Y_{m-1}, Y_p) \phi^1 \wedge \cdots \wedge \phi^{m-1} \wedge \phi^p,$$

with $m \le p \le n-1$. Hence

$$\Omega = \left(\phi + \lambda_m \phi^m + \cdots + \lambda_{n-1} \phi^{n-1}\right) \wedge \phi^1 \wedge \cdots \wedge \phi^{m-1},$$

for some $\lambda_m, \ldots, \lambda_{n-1} \in C^\infty(U^n)$. $\square$

By an obvious iteration we have the following corollary to Theorem 3.2.7:

Corollary 3.2.8. *Let $\Omega \in \Lambda^m(U^n)$ be decomposable. Let $X_1, \ldots, X_p \in \mathfrak{X}(U^n)$ up to some $p < m$ such that $X_1 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega \neq 0$. Then $X_1 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega = \phi^1 \wedge \cdots \wedge \phi^{m-p}$ for some linearly independent $\phi^1, \ldots, \phi^{m-p} \in \Lambda^1(U^n)$, and there exist $\theta^1, \ldots, \theta^p \in \Lambda^1(U^n)$ such that*

$$\begin{aligned}\Omega &= \theta^p \wedge \cdots \wedge \theta^1 \wedge \phi^1 \wedge \cdots \wedge \phi^{m-p}, \\ X_p \lrcorner \Omega &= \theta^{p-1} \wedge \cdots \wedge \theta^1 \wedge \phi^1 \wedge \cdots \wedge \phi^{m-p}, \\ &\vdots \\ X_2 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega &= \theta^1 \wedge \phi^1 \wedge \cdots \wedge \phi^{m-p}.\end{aligned}$$

Theorem 3.2.9. *Let $\Omega \in \Lambda^m(U^n)$ be decomposable, and let $X \in \mathfrak{X}(U^n)$ such that $X \lrcorner \Omega \neq 0$. Then* $\ker(X \lrcorner \Omega) = \ker(\Omega) \oplus Sp\{X\}$.

Proof. It is clear that $\ker(X \lrcorner \Omega) \supset \ker(\Omega)$. Since $X \in \ker(X \lrcorner \Omega)$, we therefore have

$$\ker(X \lrcorner \Omega) \supset \ker(\Omega) \oplus Sp\{X\}. \tag{3.3}$$

By assumption Ω is decomposable, so Lemma 3.2.1 implies $\ker(\Omega)$ has maximal dimension $n-m$. Lemma 3.2.2 now means $X \lrcorner \Omega$ is also decomposable, and moreover Lemma 3.2.1 implies $\ker(X \lrcorner \Omega)$ has maximal dimensional $n-m+1$. Hence $X \notin \ker(\Omega)$ means $\ker(\Omega) \oplus Sp\{X\}$ has dimension $n-m+1$. Thus result now follows from (3.3). ☐

Before we present the next theorem, we require the following central definition:

Definition 3.2.10. Let D be a distribution in $\mathfrak{X}(U^n)$. Then a set of p linearly independent vector fields $X_1, \ldots, X_p \in \mathfrak{X}(U^n)$ form a *solvable symmetry structure* for D if

$$\mathcal{L}_{X_p} D \subset D,$$
$$\mathcal{L}_{X_{p-1}} (Sp\{X_p\} \oplus D) \subset Sp\{X_p\} \oplus D,$$
$$\vdots$$
$$\mathcal{L}_{X_1} (Sp\{X_2, \ldots, X_p\} \oplus D) \subset Sp\{X_2, \ldots, X_p\} \oplus D.$$

Given any Frobenius integrable distribution D, a solvable structure may be found using the Lie symmetry determination software package `DIMSYM`. This is done in stages by first finding a symmetry X_p of D, then finding a symmetry X_{p-1} of D spanned with X_p, and so on until X_1 is found. For each stage we input the necessary Lie bracket relations, and let `DIMSYM` solve the linear determining equations. See Appendix A for further details. Note that while X_p is a genuine symmetry of D, X_1 is a much weaker symmetry: X_1 is a symmetry of D, modulo $X_2, \ldots, X_p$. Moreover, in dealing with PDEs on some κ-jet bundle $J^\kappa(U^n, V^m)$, $X_1, \ldots, X_p$ are quite general symmetries and not necessarily, for example, prolongations of Lie point, contact or Lie-Bäcklund symmetries. The coefficient of each basis vector in each $X_1, \ldots, X_p$ is allowed to depend on any of the coordinates of $J^\kappa(U^n, V^m)$.

Theorem 3.2.11. *Let $\Omega \in \Lambda^m(U^n)$ be decomposable and $d\Omega \equiv 0$ mod Ω. Further, let $X \in \mathfrak{X}(U^n)$ such that $A(\langle\Omega\rangle) \oplus Sp\{X\}$ is Frobenius integrable and $X \lrcorner \Omega \neq 0$. Then $d(X \lrcorner \Omega) \equiv 0$ mod $X \lrcorner \Omega$.*

Proof. Let $\Omega \in \Lambda^m(U^n)$. Since Ω decomposable and $X \lrcorner \Omega \neq 0$, from Theorem 3.2.7

we may write

$$X \lrcorner \Omega = \phi^1 \wedge \cdots \wedge \phi^{m-1},$$
$$\Omega = \phi \wedge \phi^1 \wedge \cdots \wedge \phi^{m-1},$$

for some linearly independent one-forms $\phi, \phi^1, \ldots, \phi^{m-1} \in \Lambda^1(U^n)$. Since $d\Omega \equiv 0 \mod \Omega$, Theorem 2.3.11 implies $A(\langle\Omega\rangle) = \ker(\Omega)$. Now $A(\langle\Omega\rangle) \oplus Sp\{X\}$ is Frobenius integrable, so we have from Theorem 3.2.9 that $\ker(X \lrcorner \Omega)$ is also Frobenius integrable. Hence from the Frobenius theorem,

$$(X \lrcorner \Omega) \wedge d\phi^i = 0, \tag{3.4}$$

for all $1 \leq i \leq m-1$. Since $(X \lrcorner \Omega) \wedge \phi^i = 0$ for each i, we therefore have from (3.4),

$$0 = d\left((X \lrcorner \Omega) \wedge \phi^i\right) = d(X \lrcorner \Omega) \wedge \phi^i.$$

Hence $\phi^1, \ldots, \phi^{m-1}$ are $m-1$ linearly independent factors of $d(X \lrcorner \Omega)$. This means

$$d(X \lrcorner \Omega) = \beta \wedge \phi^1 \wedge \cdots \wedge \phi^{m-1},$$
$$= \beta \wedge (X \lrcorner \Omega),$$

for some $\beta \in \Lambda^1(U^n)$. □

We have the following corollary to Theorem 3.2.11:

Corollary 3.2.12. *Let $\Omega \in \Lambda^m(U^n)$ such that Ω is decomposable and $d\Omega \equiv 0 \mod \Omega$, and suppose there exist $X_1, \ldots, X_p \in \mathfrak{X}(U^n)$ up to some $p < m$ such that $X_1 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega \neq 0$. If $A(\langle\Omega\rangle) \oplus Sp\{X_p\}$ is a Frobenius integrable distribution, and for all $1 \leq i < p$, $A(\langle\Omega\rangle) \oplus Sp\{X_i, \ldots, X_p\}$ is also Frobenius integrable, then*

$$d(X_p \lrcorner \Omega) \equiv 0 \quad \mod X_p \lrcorner \Omega,$$
$$d(X_{p-1} \lrcorner X_p \lrcorner \Omega) \equiv 0 \quad \mod X_{p-1} \lrcorner X_p \lrcorner \Omega,$$
$$\vdots$$
$$d(X_1 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega) \equiv 0 \quad \mod X_1 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega.$$

Moreover, $\{X_1, \ldots, X_p\}$ form a solvable symmetry structure for $A(\langle\Omega\rangle)$ if and only *if*

$$\begin{aligned}\mathcal{L}_{X_p}\Omega &= \lambda_p \Omega,\\ \mathcal{L}_{X_{p-1}}(X_p \lrcorner\, \Omega) &= \lambda_{p-1}(X_p \lrcorner\, \Omega),\\ &\vdots\\ \mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner\, X_p \lrcorner\, \Omega) &= \lambda_1 (X_2 \lrcorner \ldots \lrcorner\, X_p \lrcorner\, \Omega),\end{aligned} \tag{3.5}$$

for some $\lambda_1, \ldots, \lambda_p \in C^\infty(U^n)$.

The second part of Corollary 3.2.12 comes from repeatedly using Lemma 3.2.2 followed by Theorems 2.3.10 and 2.3.11, and is central to our study of PDEs in the chapters remaining since it provides a direct connection between a solvable symmetry structure for $\ker(\Omega) = A(\langle\Omega\rangle)$ and one for Ω (the equations in (3.5) will be frequently referred to as a solvable symmetry structure for Ω). Hence, by using `DIMSYM` to find a solvable symmetry structure for the former, we are able to establish a solvable symmetry structure for the latter.

The papers by Basarab-Horwath [16] and Sherring and Prince [110] (as well as those in [11, 40, 42, 65]) extend Lie's approach to integrating a Frobenius integrable distribution via a solvable structure of symmetries. In these papers, a Frobenius integrable distribution is given first. The one-form annihilating space is then generated, whose exterior product of generators is a decomposable differential form with a Frobenius integrable kernel. In our work we start with a decomposable m-form Ω with a Frobenius integrable kernel. This is achieved by also demanding that $d\Omega \equiv 0$ mod Ω. Hence by Theorems 2.2.11 and 2.3.11 respectively, the Cauchy characteristic space of the differential ideal generated by Ω is Frobenius integrable and equal to $\ker(\Omega)$. From these results, we show below in Theorem 3.2.14 how a solvable structure of symmetries for Ω (as in Corollary 3.2.12) can assist in generating a simplified expression for Ω. Theorem 3.2.14 is the key result of this chapter, and is essentially an extension of Proposition 4.7 in Sherring and Prince [110] or Proposition 3 in Basarab-Horwath [16] that will play a pivotal role in our study of PDEs in later chapters.

First we reproduce Proposition 4.7 in [110] below:

Theorem 3.2.13. [110] *Let* $D := Sp\{Y_1, \ldots, Y_q\} \subset \mathfrak{X}(U^n)$ *be a* q*-dimensional Frobenius integrable vector field distribution. Define* $\Omega := Y_1 \lrcorner \ldots \lrcorner\, Y_q \lrcorner\, (dx^1 \wedge \cdots \wedge$

$dx^n) \in \Lambda^{n-q}(U^n)$, and suppose there exists a solvable structure of linearly independent symmetries $X_1, \ldots, X_{n-q} \in \mathfrak{X}(U^n)$ such that X_{n-q} is a non-trivial symmetry of D, and that for all $1 \leq i < n-q$, X_i is a non-trivial symmetry of $D \oplus Sp\{X_{i+1}, \ldots, X_{n-q}\}$. For all $1 \leq i \leq n-q$, define ω^i by

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_{n-q} \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_{n-q} \lrcorner \Omega}.$$

Then $\{\omega^1, \ldots, \omega^{n-q}\}$ is dual to $\{X_1, \ldots, X_{n-q}\}$, and for all ω^i up to $i = n-q$,

$$\begin{aligned} \omega^1 &= d\gamma^1, \\ \omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\ \omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\ &\vdots \\ \omega^{n-q} &\equiv d\gamma^{n-q} \mod d\gamma^1, \ldots, d\gamma^{n-q-1}, \end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^{n-q} \in C^\infty(U^n)$. Moreover, on U^n, the submanifolds described by D generate a q-dimensional foliation of U^n whose leaves have $\gamma^1, \ldots, \gamma^{n-q}$ constant.

Using Theorems 2.2.11 and 2.3.11, we can modify Theorem 3.2.13 in the following way:

Theorem 3.2.14. *Let $\Omega \in \Lambda^m(U^n)$ such that Ω is decomposable and $d\Omega \equiv 0$ mod Ω. Suppose that there exists a solvable structure of linearly independent symmetries, $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ such that X_m is a non-trivial symmetry of $A(\langle\Omega\rangle)$, and that for all $1 \leq i < m$, X_i is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus Sp\{X_{i+1}, \ldots, X_m\}$. For all $1 \leq i \leq m$, define ω^i by*

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}. \tag{3.6}$$

Then $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$, and for all ω^i up to $i = m$,

$$\begin{aligned} \omega^1 &= d\gamma^1, \\ \omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\ \omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \qquad (3.7) \\ &\vdots \\ \omega^m &\equiv d\gamma^m \mod d\gamma^1, \ldots, d\gamma^{m-1}, \end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^m \in C^\infty(U^n)$. *Finally, define* $\gamma^0 := \Omega(X_1, \ldots, X_m)$. *Then* $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.

Proof. Since Ω is decomposable, we may write $\Omega = \theta^1 \wedge \cdots \wedge \theta^m$ for some linearly independent $\theta^1, \ldots, \theta^m \in \Lambda^1(U^n)$. Now $\ker(\Omega) = Sp\{Y_1, \ldots, Y_{n-m}\}$ for some $Y_1, \ldots, Y_{n-m} \in \mathfrak{X}(U^n)$. From Theorems 2.2.11 and 2.3.11, we have that $A(\langle\Omega\rangle) = \ker(\Omega)$ is Frobenius integrable. Applying Theorem 3.2.13 with the linearly independent symmetries $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ for $A(\langle\Omega\rangle)$ given in Theorem 3.2.14, we obtain that

$$\{Y_1, \ldots, Y_{n-m}, X_1, \ldots, X_m\}$$

spans $\mathfrak{X}(U^n)$ and is dual to

$$\{\phi^1, \ldots, \phi^{n-m}, \omega^1, \ldots, \omega^m\},$$

for some linearly independent $\phi^1, \ldots, \phi^{n-m} \in \Lambda^1(U^n)$ with $\omega^1, \ldots, \omega^m$ defined as in (3.6). Since $Y_j \lrcorner \Omega = 0$ for all $1 \leq j \leq n-m$, it follows that

$$\Omega = \Omega(X_1, \ldots, X_m)\omega^1 \wedge \cdots \wedge \omega^m. \tag{3.8}$$

Now Theorem 3.2.13 implies the equations in (3.7), so (3.8) simplifies to give

$$\Omega = \Omega(X_1, \ldots, X_m) d\gamma^1 \wedge \cdots \wedge d\gamma^m.$$

□

Remark 1. The fact that the symmetries in Theorem 3.2.14 are non-trivial means that the denominator is non-zero in each of the definitions for ω^i.

Remark 2. Using Corollary 3.2.12, the solvable symmetry structure condition in Theorem 3.2.14 is equivalent to having

$$\begin{aligned} \mathcal{L}_{X_m}\Omega &= \lambda_m \Omega, \\ \mathcal{L}_{X_{m-1}}(X_m \lrcorner \Omega) &= \lambda_{m-1}(X_m \lrcorner \Omega), \\ &\vdots \\ \mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) &= \lambda_1 (X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega), \end{aligned}$$

for some $\lambda_1, \ldots \lambda_m \in C^\infty(U^n)$.

We assume throughout this thesis that in practice, if we are given any Frobenius integrable vector field distribution in Theorem 3.2.13 or decomposable and closed modulo itself differential form in Theorem 3.2.14, we are always able to use `DIMSYM` to find a solvable structure of non-trivial symmetries. As mentioned earlier, Appendix A contains full discussion of the required computer input code, including examples, as well as some comment on various technical matters.

In later sections we will illustrate Theorem 3.2.14 with some applications. For now though, we have the following consequence of Theorem 3.2.14 regarding its second remark:

Theorem 3.2.15. *Given some decomposable* $\Omega \in \Lambda^m(U^n)$ *with* $d\Omega = 0 \bmod \Omega$, *and a solvable structure* $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ *for* $A(\langle\Omega\rangle)$ *as in Theorem 3.2.14, then*

$$\begin{aligned}
\mathcal{L}_{X_m}\Omega &= \{X_m \lrcorner\, d(\ln|\Omega(X_1, \ldots, X_m)|)\}\Omega, \\
\mathcal{L}_{X_{m-1}}(X_m \lrcorner\, \Omega) &= \{X_{m-1} \lrcorner\, d(\ln|\Omega(X_1, \ldots, X_m)|)\}(X_m \lrcorner\, \Omega), \\
&\vdots \\
\mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega) &= \{X_1 \lrcorner\, d(\ln|\Omega(X_1, \ldots, X_m)|)\}(X_2 \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega).
\end{aligned}$$

Proof. First we will show that for all $1 \le i \le m$, $d(\omega^1 \wedge \cdots \wedge \omega^i) = 0$. From Theorem 3.2.14 it is obvious that $d\omega^1 = 0$ and for each $1 < i \le m$ that $d\omega^i \equiv 0 \bmod \omega^1, \ldots, \omega^{i-1}$. Thus for all $i > 1$,

$$d(\omega^1 \wedge \cdots \wedge \omega^i) = 0. \tag{3.9}$$

From Theorem 3.2.14 once more, it is clear that

$$\Omega = \Omega(X_1, \ldots, X_m)\omega^1 \wedge \cdots \wedge \omega^m. \tag{3.10}$$

Hence

$$d\left(\frac{\Omega}{\Omega(X_1, \ldots, X_m)}\right) = 0. \tag{3.11}$$

Using that $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$ and contracting (3.10) with X_m, we obtain

$$\omega^1 \wedge \cdots \wedge \omega^{m-1} = \frac{X_m \lrcorner\, \Omega}{(-1)^{m-1}\Omega(X_1, \ldots, X_m)}.$$

From repeating this contraction with X_{m-1} and so on down to X_1, we obtain for all $1 \le i \le m-1$,

$$\omega^1 \wedge \cdots \wedge \omega^i = \frac{X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{(-1)^{((m-1)+\cdots+i)}\Omega(X_1,\ldots,X_m)}.$$

Hence from (3.9),

$$d\left(\frac{X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{(-1)^{((m-1)+\cdots+i)}\Omega(X_1,\ldots,X_m)}\right) = 0. \tag{3.12}$$

Equation (3.11) implies

$$d\Omega = d\left(\ln|\Omega(X_1,\ldots,X_m)|\right) \wedge \Omega, \tag{3.13}$$

while equation (3.12) means

$$d\left(X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right) = d\left(\ln|\Omega(X_1,\ldots,X_m)|\right) \wedge \left(X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right), \tag{3.14}$$

for all $1 \le i \le m-1$. Now

$$\begin{aligned}
\mathcal{L}_{X_m}\Omega &= X_m \lrcorner\, d\Omega + d\left(X_m \lrcorner \Omega\right), \\
&= X_m \lrcorner \left\{d\left(\ln|\Omega(X_1,\ldots,X_m)|\right) \wedge \Omega\right\} + d\left(\ln|\Omega(X_1,\ldots,X_m)|\right) \wedge \left(X_m \lrcorner \Omega\right), \\
&= \{X_m \lrcorner\, d(\ln|\Omega(X_1,\ldots,X_m)|)\}\Omega,
\end{aligned}$$

where in the second line we have inserted equations (3.13) and (3.14). To obtain the third line we used the identity $X \lrcorner (\omega \wedge \sigma) = (X \lrcorner \omega) \wedge \sigma + (-1)^{deg(\omega)}\omega \wedge (X \lrcorner \sigma)$ for differential forms σ, ω.

Finally, let $1 \le i \le m-1$. Then in a similar fashion to before, we get

$$\begin{aligned}
\mathcal{L}_{X_i}(X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) = X_i \lrcorner &\left\{d\left(\ln|\Omega(X_1,\ldots,X_m)|\right) \wedge \left(X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right)\right\} \\
&+ d\left(\ln|\Omega(X_1,\ldots,X_m)|\right) \wedge \left(X_i \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right),
\end{aligned}$$

which simplifies to

$$\mathcal{L}_{X_i}(X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) = \{X_i \lrcorner\, d(\ln|\Omega(X_1,\ldots,X_m)|)\}(X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega).$$

□

In general, each $\omega^2,\ldots,\omega^m$ in Theorem 3.2.14 (or Theorem 3.2.13) is not exact. Our final results for this section examine some conditions on the symmetries $X_1,\ldots,X_m$ in Theorem 3.2.14 that force at least one of $\omega^2,\ldots,\omega^m$ to be exact.

Theorem 3.2.16. *Let $\Omega \in \Lambda^m(U^n)$ for some $m \geq 3$ such that Ω is decomposable and $d\Omega \equiv 0$ mod Ω. Let there exist a solvable structure of linearly independent symmetries $X_3, \ldots, X_m \in \mathfrak{X}(U^n)$ such that X_m is a non-trivial symmetry of $A(\langle\Omega\rangle)$, and that for all $3 \leq i < m$, X_i is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus Sp\{X_{i+1}, \ldots, X_m\}$. Also, let there exist two linearly independent vector fields $X_1, X_2 \in \mathfrak{X}(U^n)$ that are non-trivial symmetries of $A(\langle\Omega\rangle) \oplus Sp\{X_3, \ldots, X_m\}$ such that*

$$[X_1, X_2] \equiv 0 \mod A(\langle\Omega\rangle) \oplus Sp\{X_3, \ldots, X_m\}. \tag{3.15}$$

For all $1 \leq i \leq m$, define ω^i by

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}.$$

Then $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$, and for all ω^i up to $i = m$,

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)d\gamma^2 - X_1(\gamma^3)d\gamma^1, \\
\omega^4 &= d\gamma^4 - X_3(\gamma^4)(d\gamma^3 - X_2(\gamma^3)d\gamma^2 - X_1(\gamma^3)d\gamma^1) - X_2(\gamma^4)d\gamma^2 - X_1(\gamma^4)d\gamma^1, \\
&\vdots \\
\omega^m &\equiv d\gamma^m \mod d\gamma^1, \ldots, d\gamma^{m-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^m \in C^\infty(U^n)$. Finally, define $\gamma^0 := \Omega(X_1, \ldots, X_m)$. Then $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.

Proof. We begin by showing that X_1 is a non-trivial symmetry of the distribution $A(\langle\Omega\rangle) \oplus Sp\{X_2, \ldots, X_m\}$. Since X_1 is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus Sp\{X_3, \ldots, X_m\}$, we have from Corollary 3.2.12 that

$$\mathcal{L}_{X_1}(X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) = \lambda (X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega),$$

for some $\lambda \in C^\infty(U^n)$. Using this fact and equation (3.15) then gives

$$\begin{aligned}
\mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) &= [X_1, X_2] \lrcorner X_3 \lrcorner \ldots X_m \lrcorner \Omega + X_2 \lrcorner \mathcal{L}_{X_1}(X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega), \\
&= \lambda (X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega).
\end{aligned}$$

From Theorem 3.2.12, our symmetries at this point satisfy Theorem 3.2.14. Therefore

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\
&\vdots \\
\omega^m &\equiv d\gamma^m \mod d\gamma^1, \dots, d\gamma^{m-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \dots, \gamma^m \in C^\infty(U^n)$. To show that $X_1(\gamma^2) = 0$, we must show that

$$d\omega^2 = d\left(\frac{X_1 \lrcorner X_3 \lrcorner \dots \lrcorner X_m \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner \dots \lrcorner X_m \lrcorner \Omega}\right) = 0. \tag{3.16}$$

This can be proved by observing that since $\ker(X_1 \lrcorner X_3 \lrcorner \dots X_m \lrcorner \Omega) = A(\langle\Omega\rangle) \oplus Sp\{X_1, X_3, \dots, X_m\}$ is a Frobenius integral distribution, we therefore have

$$d(X_1 \lrcorner X_3 \lrcorner \dots X_m \lrcorner \Omega) \equiv 0 \mod X_1 \lrcorner X_3 \lrcorner \dots \lrcorner X_m \lrcorner \Omega.$$

Then to show that X_2 is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus Sp\{X_1, X_3, \dots, X_m\}$ we use the formula

$$\begin{aligned}
\mathcal{L}_{X_2}\left(X_1 \lrcorner X_3 \lrcorner \dots \lrcorner X_m \lrcorner \Omega\right) = [X_2, X_1] \lrcorner X_3 \lrcorner \dots \lrcorner X_m \lrcorner \Omega \\
+ X_1 \lrcorner \mathcal{L}_{X_2}\left(X_3 \lrcorner \dots \lrcorner X_m \lrcorner \Omega\right).
\end{aligned}$$

Now using equation (3.15) and that X_2 is a non-trivial symmetry of $X_3 \lrcorner \dots \lrcorner X_m \lrcorner \Omega$, we get the desired result. Equation (3.16) can then be deduced from simple algebraic manipulation, or by applying Theorem 3.2.14. □

Remark. While Theorem 3.2.16 assumes $m \geq 3$, it is clear that it still holds when $m = 2$. In this situation, there is no need for symmetries other than X_1, X_2, with (3.15) reducing to $[X_1, X_2] \equiv 0 \mod A(\langle\Omega\rangle)$. Further, the expressions for ω^i in the conclusion of the theorem vanish for $i > 2$.

We can generalise Theorem 3.2.16 in the following way:

Theorem 3.2.17. *Let $\Omega \in \Lambda^m(U^n)$ for some $m \geq 3$, and suppose Ω is decomposable with $d\Omega \equiv 0 \bmod \Omega$. For some $1 \leq l < m$, let there exist a solvable structure of $m-l$ linearly independent symmetries $X_{l+1}, \ldots, X_m \in \mathfrak{X}(U^n)$ such that X_m is a non-trivial symmetry of $A(\langle\Omega\rangle)$, and that for all $l+1 \leq i < m$, X_i is a non-trivial symmetry $A(\langle\Omega\rangle) \oplus Sp\{X_{i+1}, \ldots, X_m\}$. Also, let there exist l linearly independent vector fields $X_1, \ldots, X_l \in \mathfrak{X}(U^n)$ that are non-trivial symmetries of $A(\langle\Omega\rangle) \oplus Sp\{X_{l+1}, \ldots, X_m\}$ such that*

$$[X_u, X_v] \equiv 0 \quad \bmod A(\langle\Omega\rangle) \oplus Sp\{X_{l+1}, \ldots, X_m\}, \tag{3.17}$$

for all $1 \leq u < v \leq l$. For all $1 \leq i \leq m$, define ω^i by

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}.$$

Then $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$, and for all ω^i up to $i = l$,

$$\omega^1 = d\gamma^1,$$
$$\vdots$$
$$\omega^l = d\gamma^l,$$

with for each i greater than l up to $i = m$,

$$\begin{aligned}
\omega^{l+1} &= d\gamma^{l+1} - X_l(\gamma^{l+1})d\gamma^l - X_{l-1}(\gamma^{l+1})d\gamma^{l-1} - \cdots - X_1(\gamma^{l+1})d\gamma^1, \\
\omega^{l+2} &= d\gamma^{l+2} - X_{l+1}(\gamma^{l+2})\left(d\gamma^{l+1} - X_l(\gamma^{l+1})d\gamma^l - X_{l-1}(\gamma^{l+1})d\gamma^{l-1} - \ldots \right. \\
&\qquad \left. -X_1(\gamma^{l+2})d\gamma^1\right) - X_l(\gamma^{l+2})d\gamma^l - \cdots - X_1(\gamma^{l+2})d\gamma^1, \\
&\vdots \\
\omega^m &\equiv d\gamma^m \quad \bmod d\gamma^1, \ldots, d\gamma^{m-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^m \in C^\infty(U^n)$. Finally, define $\gamma^0 := \Omega(X_1, \ldots, X_m)$. Then $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.

Proof. (Outline) The proof is similar to that of Theorem 3.2.16, and essentially involves repeating the proof of Theorem 3.2.16 $l-1$ more times. To do this, from the fact that Ω is decomposable and $d\Omega \equiv 0 \bmod \Omega$, we can then apply Corollary 3.2.12

to obtain

$$\mathcal{L}_{X_m} = \lambda_m \Omega,$$
$$\mathcal{L}_{X_{m-1}} (X_m \lrcorner \Omega) = \lambda_{m-1} (X_m \lrcorner \Omega),$$
$$\vdots$$
$$\mathcal{L}_{X_{l+1}} (X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega) = \lambda_{l+1} (X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega),$$

and also that

$$\mathcal{L}_{X_l} (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega) = \lambda_l (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega),$$
$$\mathcal{L}_{X_{l-1}} (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega) = \lambda_{l-1} (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega),$$
$$\vdots$$
$$\mathcal{L}_{X_1} (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega) = \lambda_1 (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner\ X_m \lrcorner \Omega),$$

for some $\lambda_1, \ldots, \lambda_m \in C^\infty(U^n)$. Next, using (3.17), it is easy to show that

$$\mathcal{L}_{X_m} \Omega = \lambda_m \Omega,$$
$$\mathcal{L}_{X_{m-1}}(X_m \lrcorner \Omega) = \lambda_{m-1}(X_m \lrcorner \Omega),$$
$$\vdots$$
$$\mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) = \lambda_1(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega).$$

Then we may apply Theorem 3.2.14 to obtain a set of one-forms $\{\omega^1, \ldots, \omega^m\}$ dual to $\{X_1, \ldots, X_m\}$, and that for all ω^i up to $i = m$,

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2) d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1, \\
&\vdots \\
\omega^m &\equiv d\gamma^m \mod d\gamma^1, \ldots, d\gamma^{m-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^m \in C^\infty(U^n)$. Now since we know already that $d\omega^1 = 0$, we only have to show that for each $1 < j \le l$,

$$d\omega^j = d \left(\frac{X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{X_j \lrcorner X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega} \right) = 0. \qquad (3.18)$$

The original symmetry relations for $X_1, \ldots, X_m$ tell us that for each j, $A(\langle\Omega\rangle) \oplus Sp\{X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_m\}$ is Frobenius integrable, so

$$d\left(X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right) \equiv 0 \mod X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega.$$

Finally, using (3.17), and in similar fashion to the end of the proof of Theorem 3.2.16, we get that for each j, X_j is a non-trivial symmetry of $X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega$. Simple algebraic manipulation then yields (3.18). ☐

Remark. As in Theorem 3.2.16, it is easy to see that Theorem 3.2.17 holds for all $m \geq 2$; however, here we can also say that the theorem holds if $l = m$, so (3.17) becomes $[X_u, X_v] \equiv 0 \mod A(\langle\Omega\rangle)$, for all $1 \leq u < v \leq l$. In this situation, all ω^i become exact, which is in accordance with the corollary to Proposition 2 given in [16].

The next section gives a simple application of some of the ideas presented above.

3.3 Differential forms in $\Lambda^m(\mathbb{R}^{m+1})$

In this section we show that, provided we have enough symmetries, any differential form in $\Lambda^m(\mathbb{R}^{m+1})$ can be expressed locally in terms of m functionally independent functions as in the conclusion of Theorem 3.2.14. Further details will be given in Theorem 3.3.5 below, but first, consider the following result:

Lemma 3.3.1. *Let $\Omega \in \Lambda^m(U^n)$ for some $m < n$ be non-zero, where U^n is defined as in previous sections (though the requirement that U^n be convex is not necessary here). Suppose Ω is of the form*

$$\Omega := \gamma_1 \theta^2 \wedge \theta^3 \wedge \cdots \wedge \theta^{m+1} + \gamma_2 \theta^1 \wedge \theta^3 \wedge \cdots \wedge \theta^{m+1} + \cdots + \gamma_{m+1} \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^m,$$

for some linearly independent $\theta^1, \ldots, \theta^{m+1} \in \Lambda^1(U^n)$ and $\gamma_1, \ldots, \gamma_{m+1} \in C^\infty(U^n)$. Then Ω is decomposable.

Proof. Let $\Omega \in \Lambda^m(U^n)$ be as in the theorem. We can write

$$\Omega = X \lrcorner \left(\theta^1 \wedge \cdots \wedge \theta^{m+1}\right),$$

where

$$X := \sum_{i=1}^{m+1} (-1)^{i-1} \gamma_i X_i,$$

for some $X_1, \ldots, X_{m+1} \in \mathfrak{X}(U^n)$ dual to $\theta^1, \ldots, \theta^{m+1}$. Hence from Lemma 3.2.2 the result follows. □

From Lemma 3.2.2 we obtain the following useful corollary for m-forms in $(m+1)$-dimensional spaces also found in [58] by Godbillon. Define W to be some open neighbourhood of $\mathbb{R}^{m+1}$.

Corollary 3.3.2. *Let $\Omega \in \Lambda^m(W)$. Then Ω is decomposable.*

Theorem 3.3.3. *Let $\Omega \in \Lambda^m(W)$. Then $d\Omega \equiv 0$ mod Ω.*

Proof. Let $\Omega \in \Lambda^m(W)$. Corollary 3.3.2 implies

$$\Omega = \theta^1 \wedge \cdots \wedge \theta^m,$$

for some linearly independent $\theta^1, \ldots, \theta^m \in \Lambda^1(W)$. Now $d\Omega$ is an $(m+1)$-form in $\Lambda^{m+1}(W)$, so we may complete $\theta^1, \ldots, \theta^m$ to a basis by including some linearly independent $\phi \in \Lambda^1(W)$ with the property that

$$d\Omega = \theta^1 \wedge \cdots \wedge \theta^m \wedge \phi.$$

□

Theorem 3.3.4. *Let $\Omega \in \Lambda^m(W)$. Then $\langle\Omega\rangle$ is a differential ideal and $A(\langle\Omega\rangle)$ is one-dimensional.*

Proof. From Theorem 3.3.3 we obtain the first result in the theorem. Then using Theorem 2.3.11, $A(\langle\Omega\rangle) = \ker(\Omega)$. The decomposable nature of Ω from Corollary 3.3.2 tells us immediately that $A(\langle\Omega\rangle)$ is one-dimensional. □

Theorem 3.3.5. *Let $\Omega \in \Lambda^m(W)$, where W is some open, convex neighbourhood of $\mathbb{R}^{m+1}$. If there exists a solvable structure of m symmetries for $A(\langle\Omega\rangle)$ as in Theorem 3.2.14, then we can compute functions $\gamma^0, \ldots, \gamma^m \in C^\infty(W)$ so that $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.*

Proof. We know from Corollary 3.3.2 and Theorem 2.3.11 respectively that Ω is decomposable and that $d\Omega \equiv 0 \bmod \Omega$, so Theorem 3.2.14 gives us a direct algorithm for finding $\gamma^0, \ldots, \gamma^m$. □

3.4 The suitability of a differential form

For an arbitrary differential form $\Delta \in \Lambda^m(U^n)$ we use ideas in the previous section to examine some necessary conditions for Δ to be decomposable and $d\Delta \equiv 0 \bmod \Delta$, so that we can apply Theorem 3.2.14. Of course if $m = n$, these two conditions trivially hold, and Corollary 3.3.2 and Theorem 3.3.3 mean they still hold if $m = n-1$. In this section we examine the situation when $m < n-1$. In what follows, we assume U^n is some open, convex neighbourhood of $\mathbb{R}^n$.

Theorem 3.4.1. *Let $\Delta \in \Lambda^m(U^n)$ for some $m < n-1$. If there exist $n-m-1$ linearly independent vector fields $\Gamma_1, \ldots \Gamma_{n-m-1} \in \mathfrak{X}(U^n)$ in $\ker(\Delta)$, then Δ is decomposable. Moreover, if for each $1 \leq i \leq n-m-1$,*

$$\mathcal{L}_{\Gamma_i}\Delta = \lambda_i \Delta, \tag{3.19}$$

for some $\lambda_i \in C^\infty(U^n)$, then $d\Delta \equiv 0$ mod Δ.

Proof. Let $\Delta \in \Lambda^m(U^n)$ with $m < n-1$, and let there exist linearly independent $\Gamma_1, \ldots \Gamma_{n-m-1} \in \mathfrak{X}(U^n)$ such that for all $1 \leq i \leq n-m-1$,

$$\Gamma_i \lrcorner \Delta = 0. \tag{3.20}$$

Now

$$(Sp\{\Gamma_1, \ldots, \Gamma_{n-m-1}\})^\perp = Sp\left\{\theta^1, \ldots, \theta^{m+1}\right\},$$

for some $\theta^1, \ldots, \theta^{m+1} \in \Lambda^1(U^n)$. Hence from (3.20), we must have

$$\Delta = \Delta_{j_1 \ldots j_m}\theta^{j_1} \wedge \cdots \wedge \theta^{j_m},$$

for some $\Delta_{j_1 \ldots j_m} \in C^\infty(U^n)$, with summation over $1 \leq j_1 < \cdots < j_m \leq m+1$. Therefore by Lemma 3.3.1, Δ is decomposable.

For the second part of the proof, we choose without loss,

$$\Delta = \theta^1 \wedge \cdots \wedge \theta^m.$$

We can complete $\theta^1, \ldots, \theta^{m+1}$ to a basis for $\Lambda^1(U^n)$ by adding linearly independent $\phi^1, \ldots, \phi^{n-m-1} \in \Lambda^1(U^n)$ such that

$$\left\{\phi^1, \ldots, \phi^{n-m-1}, \theta^1, \ldots, \theta^{m+1}\right\} \tag{3.21}$$

is dual to

$$\{\Gamma_1, \dots \Gamma_{n-m-1}, Y_1, \dots, Y_{m+1}\}, \tag{3.22}$$

for some linearly independent $Y_1, \dots, Y_{m+1} \in \mathfrak{X}(U^n)$. Now with summation on k over $1 \leq k \leq m$, we can write

$$d\Delta = \sigma_k \wedge \theta^1 \wedge \cdots \wedge \theta^{k-1} \wedge \theta^{k+1} \wedge \cdots \wedge \theta^m + \beta \wedge \Delta, \tag{3.23}$$

for some $\sigma_1, \dots, \sigma_m \in \Lambda^2(U^n)$ and $\beta \in \Lambda^1(U^n)$ with the property that each σ_k only depends on the basis vectors $\phi^1, \dots, \phi^{n-m-1}, \theta^{m+1}$. Hence from the dual basis property in (3.21) and (3.22), we have for each k,

$$Y_j \lrcorner\, \sigma_k = 0, \tag{3.24}$$

for all $1 \leq j \leq m$. By combining the assumptions in (3.19) and (3.20), we have for all i,

$$\Gamma_i \lrcorner\, d\Delta = \lambda_i \Delta. \tag{3.25}$$

Using the dual basis property once more, we get that for each i and $1 \leq l \leq m+1$, $\Gamma_i \lrcorner\, \theta^l = 0$. Hence substituting (3.23) into (3.25) gives (with sum),

$$(\Gamma_i \lrcorner\, \sigma_k) \wedge \theta^1 \wedge \cdots \wedge \theta^{k-1} \wedge \theta^{k+1} \wedge \cdots \wedge \theta^m + (\Gamma_i \lrcorner\, \beta) \wedge \Delta = \lambda_i \Delta, \tag{3.26}$$

for each i. Since each $\Gamma_i \lrcorner\, \sigma_k$ only depends on the basis vectors $\phi^1, \dots, \phi^{n-m-1}, \theta^{m+1}$, for (3.26) to hold we must have

$$\Gamma_i \lrcorner\, \sigma_k = 0, \tag{3.27}$$

for each i and k. Hence from (3.24) and (3.27), $\ker(\sigma_k)$ is at least $(n-1)$-dimensional. This means $\sigma_k(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(U^n)$. Thus $\sigma_k = 0$ for each k. Therefore $d\Delta = \beta \wedge \Delta$. □

Theorem 3.4.1 has the following two corollaries:

Corollary 3.4.2. *Let $\Delta \in \Lambda^m(U^n)$ such that $m < n-1$. If there exist $n-m-1$ linearly independent Cauchy characteristic vector fields of the differential ideal $\langle \Delta, d\Delta \rangle$, then Δ is decomposable and $d\Delta \equiv 0 \mod \Delta$.*

Proof. Since the Cauchy characteristic vector fields are in the kernel of Δ, Theorem 3.4.1 implies Δ is decomposable. Now it is clear that (3.19) in Theorem 3.4.1 still holds for some $\lambda_1, \ldots, \lambda_{n-m-1} \in C^\infty(U^n)$. Hence from the theorem, $d\Delta \equiv 0$ mod Δ □

Corollary 3.4.3. *Let $\Delta \in \Lambda^m(U^n)$ such that $m < n-1$. If there exist $n-m-1$ linearly independent Cauchy characteristic vector fields of the differential ideal $\langle \Delta, d\Delta \rangle$, then the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is $(n-m)$-dimensional*

Proof. From Corollary 3.4.2, Δ is decomposable, so $\ker(\Delta)$ is $(n-m)$-dimensional. The corollary also means Δ is closed modulo itself which implies $\langle \Delta \rangle = \langle \Delta, d\Delta \rangle$, and hence their Cauchy characteristic spaces are equal. From Theorem 2.3.11 the result follows. □

Now the dimension of the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is always less than or equal to that of $\ker(\Delta)$, and the maximum dimensional of $\ker(\Delta)$ is $n-m$, which occurs when Δ is decomposable. Theorem 3.4.1 therefore means that if $\ker(\Delta)$ is at least $(n-m-1)$-dimensional, then it is $(n-m)$-dimensional. Similarly, Corollary 3.4.3 means that if the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is at least $(n-m-1)$-dimensional, then it is $(n-m)$-dimensional.

Next, we illustrate Corollary 3.4.2 with the following example:

Example 3.4.4. Suppose U^4 is some suitably chosen open, convex neighbourhood of $\mathbb{R}^4$ with coordinates x^1, x^2, x^3, x^4, and

$$\begin{aligned} \Delta := \frac{2x^2x^4}{x^3} dx^3 \wedge dx^2 - \left(\frac{x^4}{x^3}\right) dx^3 \wedge dx^1 - 2dx^4 \wedge dx^1 \\ + \frac{1}{x^3x^4} dx^1 \wedge dx^2 + 4x^2 dx^4 \wedge dx^2. \end{aligned}$$

Now the vector field

$$\Gamma := 4x^2 \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{1}{x^3x^4}\frac{\partial}{\partial x^2},$$

is a Cauchy characteristic of $\langle \Delta, d\Delta \rangle$. Hence from Corollary 3.4.2, Δ is decomposable and $d\Delta \equiv 0$ mod Δ. Note from Corollary 3.4.3 that the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is two-dimensional.

We will now proceed to apply Theorem 3.2.14 to Δ. It is easy to see that $\frac{\partial}{\partial x^1}$ is a non-trivial symmetry of Δ. With

$$\frac{\partial}{\partial x^1} \lrcorner \Delta = \frac{1}{x^3 x^4} dx^2 + \frac{x^4}{x^3} dx^3 + 2dx^4,$$

it is also easy to see that $\frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $\frac{\partial}{\partial x^1} \lrcorner \Delta$. Now from Theorem 3.2.14 and Corollary 3.2.12,

$$\omega^1 := \frac{\frac{\partial}{\partial x^1} \lrcorner \Delta}{\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial x^1} \lrcorner \Delta} = dx^2 + (x^4)^2 dx^3 + 2x^3 x^4 dx^4 = d\left(x^2 + x^3 (x^4)^2\right).$$

Also, it is not hard to show that

$$\begin{aligned} \omega^2 := \frac{\frac{\partial}{\partial x^2} \lrcorner \Delta}{\frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner \Delta} &= dx^1 + 2x^2 (x^4)^2 dx^3 + 4x^2 x^3 x^4 dx^4, \\ &= d\left(x^1 - (x^2)^2\right) + 2x^2 d\left(x^2 + x^3 (x^4)^2\right). \end{aligned}$$

Hence

$$\Delta = \frac{1}{x^3 x^4} d\left(x^1 - (x^2)^2\right) \wedge d\left(x^2 + x^3 (x^4)^2\right).$$

Theorem 3.4.1 and its corollaries may be difficult to apply in practice. However suppose it is known that Δ is decomposable (i.e. $\ker(\Omega)$ is $(n-m)$-dimensional from Lemma 3.2.1) and we have an explicit expression for Δ as the exterior product of some m one-forms, so that $\Delta = \omega^1 \wedge \cdots \wedge \omega^m$. If, using for example the exterior calculus software package `EXCALC` [113] written by E. Schrüfer, one finds that

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^m = 0,$$

for each $1 \leq i \leq m$, then this implies Δ is closed modulo itself.

3.5 Pfaffian equations

In this section we examine how symmetries may be used to express a differential one-form in 'normal form' given in (3.28) below. We begin with the following definition and theorem:

Definition 3.5.1. Let $\alpha \in \Lambda^1(U^n)$. The *rank* of the Pfaffian equation $\alpha = 0$ at the point $p \in U^n$ is the non-negative integer r such that $(d\alpha)^r \wedge \alpha \neq 0$ and $(d\alpha)^{r+1} \wedge \alpha = 0$ at p.

If a one-form α is exact, i.e. $\alpha = df$ for some $f \in C^\infty(U^n)$, then it (and any linearly dependent one-form) has rank zero.

Theorem 3.5.2. *Let $\alpha \in \Lambda^1(U^n)$ and suppose the equation $\alpha = 0$ is of constant rank r on U^n. Then there exists a coordinate system $\gamma^1, \ldots, \gamma^n \in C^\infty(U^n)$, where $2r + 1 \leq n$, so that the equation becomes*

$$d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1} = 0.$$

Theorem 3.5.2 is known as the Pfaff problem. A proof of this theorem may be found in [23].

It is easy to see that multiplying any one-form of constant rank on U^n by a nowhere zero smooth function f leaves the rank unchanged, using the fact that for any $m \in \mathbb{N}$, we have $(d(f\alpha))^m \wedge (f\alpha) = f^{m+1}(d\alpha)^m \wedge \alpha$. This allows us to express any $\alpha \in \Lambda^1(U^n)$ of constant rank r on U^n as

$$\alpha = \gamma^0(d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1}), \tag{3.28}$$

for some $\gamma^0, \ldots, \gamma^{2r+1} \in C^\infty(U^n)$.

Theorem 3.5.3. *Let $\alpha \in \Lambda^1(U^n)$. Suppose α is of constant rank r on U^n, and define $\Omega := (d\alpha)^r \wedge \alpha$. Then Ω is decomposable and $d\Omega \equiv 0$ mod Ω.*

Proof. Let $\alpha \in \Lambda^1(U^n)$ with α of constant rank r on U^n. This means

$$\alpha = \gamma^0(d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1}),$$

for some $\gamma^0, \ldots, \gamma^{2r+1} \in C^\infty(U^n)$. Define

$$\overline{\alpha} := d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1}.$$

Further, define $\overline{\Omega} := (d\overline{\alpha})^r \wedge \overline{\alpha}$. We will first show that $d\overline{\Omega} = 0$. Simple computation yields

$$(d\overline{\alpha})^r = r! d\gamma^2 \wedge \cdots \wedge d\gamma^{2r+1}.$$

Hence

$$\begin{aligned} \overline{\Omega} &= \overline{\alpha} \wedge (d\overline{\alpha})^r, \\ &= r! d\gamma^1 \wedge d\gamma^2 \wedge \cdots \wedge d\gamma^{2r+1}. \end{aligned}$$

We then have $d\overline{\Omega} = 0$. Now

$$\Omega = (\gamma^0)^{r+1}(d\overline{\alpha})^r \wedge \overline{\alpha}.$$

Since $d\overline{\Omega} = 0$, we get

$$d\Omega = d((\gamma^0)^{r+1}) \wedge (d\overline{\alpha})^r \wedge \overline{\alpha}.$$

But $(d(\gamma^0\overline{\alpha}))^r \wedge ((\gamma^0)\overline{\alpha}) = (\gamma^0)^{r+1}(d\overline{\alpha})^r \wedge \overline{\alpha}$. Therefore $d\Omega \equiv 0 \mod \Omega$ as γ^0 is nowhere zero on U^n. Finally, since $\overline{\Omega}$ is decomposable and $\Omega = (\gamma^0)^{r+1}\overline{\Omega}$, Ω is therefore decomposable. ☐

Our aim is to use Theorem 3.5.3 with Theorem 3.2.14 to ultimately find some coordinates for the Pfaff problem in Theorem 3.5.2. The next theorem illustrates how this may be done for one-forms that are of constant rank one on U^n, which will be later extended to one-forms of any constant rank $r \geq 1$. The case $r = 0$ involves a trivial application of Theorem 3.2.14, and will therefore be ignored.

To assist in finding coordinates for the Pfaff problem, the following lemma will be needed:

Lemma 3.5.4. *Let $\alpha \in \Lambda^1(U^n)$ and suppose α is of constant non-zero rank r on U^n. Let $\Omega := (d\alpha)^r \wedge \alpha$ and $X \in \mathfrak{X}(U^n)$ such that $X \lrcorner \Omega = 0$. Then $X \lrcorner \alpha = 0$.*

Proof. Let $\alpha \in \Lambda^1(U^n)$. Suppose α is of constant non-zero rank r on U^n, and define Ω as in the lemma. Let $X \in \mathfrak{X}(U^n)$ with $X \lrcorner \Omega = 0$. Now

$$X \lrcorner \Omega = (X \lrcorner (d\alpha)^r) \wedge \alpha + (X \lrcorner \alpha)(d\alpha)^r.$$

By taking the exterior product with α, we obtain

$$(X \lrcorner \alpha)(d\alpha)^r \wedge \alpha = 0.$$

Since α is of rank r, $(d\alpha)^r \wedge \alpha \neq 0$, and hence $X \lrcorner \alpha = 0$. ☐

Theorem 3.5.5. *Let $\alpha \in \Lambda^1(U^n)$ such that α is of constant rank one on U^n. Let $\Omega := d\alpha \wedge \alpha$ and $\langle \Omega \rangle$ be the differential ideal generated by Ω. Suppose $X_1, X_2, X_3 \in \mathfrak{X}(U^n)$ is a solvable structure of linearly independent symmetries such that X_3 is a non-trivial symmetry of $A(\langle \Omega \rangle)$ with the extra condition that $X_3 \lrcorner \alpha = 0$, X_2 is* *a non-trivial symmetry of* $A(\langle\Omega\rangle) \oplus Sp\{X_3\}$*, and* X_1 *is a non-trivial symmetry of* $A(\langle\Omega\rangle) \oplus Sp\{X_2, X_3\}$*. Then with* $\omega^1, \omega^2, \omega^3 \in \Lambda^1(U^n)$ *defined by*

$$\omega^1 := \frac{X_2 \lrcorner X_3 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner \Omega},$$
$$\omega^2 := \frac{X_1 \lrcorner X_3 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner \Omega},$$
$$\omega^3 := \frac{X_1 \lrcorner X_2 \lrcorner \Omega}{X_3 \lrcorner X_1 \lrcorner X_1 \lrcorner \Omega},$$

we have

$$\omega^1 = d\gamma^1,$$
$$\omega^2 = d\gamma^2 - X_1(\gamma^2)d\gamma^1,$$
$$\omega^3 = d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,$$

for some functionally independent $\gamma^1, \gamma^2, \gamma^3 \in C^\infty(U^n)$*, and*

$$\alpha = (X_2 \lrcorner \alpha)\left(d\gamma^2 + \frac{(X_1 \lrcorner \alpha) - (X_2 \lrcorner \alpha)X_1(\gamma^2)}{(X_2 \lrcorner \alpha)}d\gamma^1\right). \tag{3.29}$$

Proof. With $\Omega := d\alpha \wedge \alpha$, Theorem 3.5.3 means that Ω is decomposable and $d\Omega \equiv 0$ mod Ω. Theorem 3.2.14 can be used to obtain $\{\omega^1, \omega^2, \omega^3\}$ dual to $\{X_1, X_2, X_3\}$, where

$$\omega^1 = d\gamma^1,$$
$$\omega^2 = d\gamma^2 - X_1(\gamma^2)d\gamma^1,$$
$$\omega^3 = d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,$$

for some functionally independent $\gamma^1, \gamma^2, \gamma^3 \in C^\infty(U^n)$. Now from Lemma 3.5.4, $X \lrcorner \alpha = 0$ for all $X \in A(\langle\Omega\rangle)$. And since $X_3 \lrcorner \alpha = 0$, we are left with

$$\alpha = (X_1 \lrcorner \alpha)\omega^1 + (X_2 \lrcorner \alpha)\omega^2.$$

Now $X_2 \lrcorner \alpha \neq 0$ in the neighbourhood, since α is nowhere rank zero by assumption. Hence

$$\alpha = (X_2 \lrcorner \alpha)\left(d\gamma^2 + \frac{(X_1 \lrcorner \alpha) - (X_2 \lrcorner \alpha)X_1(\gamma^2)}{(X_2 \lrcorner \alpha)}d\gamma^1\right).$$

□

Remark 1. The extra condition in Theorem 3.5.5 that the *non-trivial* symmetry X_3 satisfies $X_3 \lrcorner \alpha = 0$ implies from Theorem 2.3.10 that the symmetry is not a Cauchy characteristic vector field of $\langle \alpha, d\alpha \rangle$. Therefore $X_3 \lrcorner d\alpha$ is not some multiple of α (as α is of rank one, it is impossible that $d\alpha \equiv 0 \mod \alpha$). Such a symmetry exists since if $\gamma^1, \ldots, \gamma^n$ are coordinates for U^n and $\alpha := \gamma^0 \left(d\gamma^1 + \gamma^2 d\gamma^3 \right)$ is already in normal form for some $\gamma^0 \in C^\infty(U^n)$, then it is easy to show that Theorem 3.5.5 can be applied to such α with X_3 chosen as $\frac{\partial}{\partial \gamma^2}$ or $\frac{\partial}{\partial \gamma^3} - \gamma^2 \frac{\partial}{\partial \gamma^1}$.

Remark 2. In deriving our expression for α in (3.29), we do not need to calculate γ^3. This significantly reduces the number of algebraic manipulations required.

We illustrate Theorem 3.5.5 with an example:

Example 3.5.6. Suppose we are in some open, convex neighbourhood of $\mathbb{R}^3$, denoted by U^3, with coordinates x^1, x^2, x^3. Define on some suitably chosen U^3,

$$\alpha := -\frac{x^2 x^3}{(x^1)^2} dx^1 + \left(\frac{x^1}{x^2} + \frac{x^3}{x^1} \right) dx^2 + \frac{x^1}{x^3} dx^3.$$

By dimension, $(d\alpha)^2 \wedge \alpha = 0$, and it is easy to show that $d\alpha \wedge \alpha \neq 0$ on some region of U^3. Suppose U^3 is chosen such that $d\alpha \wedge \alpha \neq 0$ everywhere. Since *any* non-zero vector field is a non-trivial symmetry of $d\alpha \wedge \alpha \in \Lambda^3(U^3)$, we may choose any X_3 such that $X_3 \lrcorner \alpha = 0$. So let

$$X_3 := \frac{x^2 x^3}{(x^1)^2} \frac{\partial}{\partial x^3} + \frac{x^1}{x^3} \frac{\partial}{\partial x^1}$$

be the symmetry. Using the software package `DIMSYM`, we obtain that

$$X_2 := (x^3)^2 \frac{\partial}{\partial x^3}$$

is a non-trivial symmetry of $Sp\{X_3\}$ ($A(\langle d\alpha \wedge \alpha \rangle)$ is zero-dimensional), and by inspection that

$$X_1 := \frac{\partial}{\partial x^2}$$

is a non-trivial symmetry of $Sp\{X_2, X_3\}$. These yield

$$\omega^1 := \frac{X_2 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)}{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)} = dx^2,$$

and

$$\omega^2 := \frac{X_1 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)} = -\frac{x^2}{(x^1)^3} dx^1 + \frac{dx^3}{(x^3)^2},$$
$$= d\left(\frac{x^2}{2(x^1)^2} - \frac{1}{x^3}\right) - \frac{1}{2(x^1)^2} dx^2.$$

Hence a simple calculation gives

$$\alpha = x^1 x^3 \left(d\left(\frac{x^2}{2(x^1)^2} - \frac{1}{x^3}\right) + \left(\frac{1}{x^2 x^3} + \frac{1}{2(x^1)^2}\right) dx^2 \right).$$

Such expressions for α are in general not unique, and may be found by choosing different symmetries. For example, we have also obtained

$$\alpha = x^3 \left(d\left(\frac{x^2}{x^1}\right) + \frac{x^1}{x^3} d\left(\ln |x^2 x^3|\right) \right).$$

We now present a generalisation of Theorem 3.5.5:

Theorem 3.5.7. *Let $\alpha \in \Lambda^1(U^n)$ have constant rank r on U^n, and define $\Omega := (d\alpha)^r \wedge \alpha$. Let $X_1, \ldots, X_{2r+1} \in \mathfrak{X}(U^n)$ be a solvable structure of linearly independent symmetries such that X_{2r+1} is a non-trivial symmetry of $A(\langle\Omega\rangle)$, and for each $1 < i < 2r+1$, X_i is a non-trivial symmetry of of $A(\langle\Omega\rangle) \oplus Sp\{X_{i+1}, \ldots, X_{2r+1}\}$. Suppose, in addition, that for the r vector fields $X_{r+2}, \ldots, X_{2r+1}$, we have $X_{r+2} \lrcorner \alpha = 0, \ldots, X_{2r+1} \lrcorner \alpha = 0$. For all $1 \le i \le 2r+1$, define ω^i by*

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_{2r+1} \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_{2r+1} \lrcorner \Omega}.$$

Then for all ω^i up to $i = 2r+1$,

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2) d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\
&\vdots \\
\omega^{2r+1} &\equiv d\gamma^{2r+1} \mod d\gamma^1, \ldots, d\gamma^{2r},
\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^{2r+1} \in C^\infty(U^n)$, and

$$\begin{aligned}
\alpha = (X_1 \lrcorner \alpha) d\gamma^1 &+ (X_2 \lrcorner \alpha)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) \\
&+ (X_3 \lrcorner \alpha)(d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1) + \ldots \\
&\quad + (X_{r+1} \lrcorner \alpha)(d\gamma^{r+1} - X_r(\gamma^{r+1})(d\gamma^r - \cdots - X_1(\gamma^r)d\gamma^1) - \ldots \\
&\qquad - X_1(\gamma^{r+1})d\gamma^1),
\end{aligned}$$

which when rearranged give α in the form of (3.28).

Proof. The proof follows in a similar fashion to Theorem 3.5.5. The conditions $X_{r+2} \lrcorner \alpha = 0, \ldots, X_{2r+1} \lrcorner \alpha = 0$ and Lemma 3.5.4 ensure that α is a linear combination of $d\gamma^1, \ldots, d\gamma^{r+1}$. Further, since α is of constant rank r, $X_{r+1} \lrcorner \alpha \neq 0$, so we are permitted to divide by it, and hence express α in the form of (3.28). □

Remark. Both remarks for Theorem 3.5.5 may be extended to Theorem 3.5.7 as follows: Firstly, from the proof of Theorem 3.5.3 it is clear that there exist r non-trivial symmetries $X_{r+2}, \ldots, X_{2r+1}$ of $(d\alpha)^r \wedge \alpha$ in $\ker(\alpha)$, and secondly, in deriving our expression for α, we do not need to calculate any $\gamma^{r+2}, \ldots, \gamma^{2r+1}$.

3.6 Darboux systems

This section gives an algorithm based on vector fields for generating a set of coordinates in Darboux's theorem given below in Theorem 3.6.4. To begin with, we present some preliminary material. In Bryant *et al.* [23] there is the following fundamental theorem:

Theorem 3.6.1. *Let $\Omega \in \Lambda^2(U^n)$ and let r be the natural number such that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$. Then there exist $2r$ linearly independent elements $\omega^1, \ldots, \omega^{2r} \in \Lambda^1(U^n)$ such that*

$$\Omega = \omega^1 \wedge \omega^2 + \cdots + \omega^{2r-1} \wedge \omega^{2r}.$$

In what follows, we will also make use of the following lemma:

Lemma 3.6.2. *Let $\Omega \in \Lambda^2(U^n)$ and $r \in \mathbb{N}$ such that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$. Also let $X \in \mathfrak{X}(U^n)$. Then $X \lrcorner \Omega^r = 0$ if and only if $X \lrcorner \Omega = 0$.*

Proof. Let $\Omega \in \Lambda^2(U^n)$ with $X \lrcorner \Omega^r = 0$ for some vector field $X \in \mathfrak{X}(U^n)$. Then from Theorem 3.6.1 we have

$$\Omega = \omega^1 \wedge \omega^2 + \cdots + \omega^{2r-1} \wedge \omega^{2r}, \tag{3.30}$$

for some linearly independent $\omega^1, \ldots, \omega^{2r} \in \Lambda^1(U^n)$. This implies

$$\Omega^r = r!\omega^1 \wedge \cdots \wedge \omega^{2r}.$$

Now $X \lrcorner \Omega^r = 0$ implies that $X \lrcorner \omega^i = 0$ for all $1 \leq i \leq 2r$. Hence using the expression for Ω in (3.30) gives $X \lrcorner \Omega = 0$. Proving the converse is obvious since if Y is any vector field in $\mathfrak{X}(U^n)$, then $Y \lrcorner \Omega^r = r(Y \lrcorner \Omega) \wedge \Omega^{r-1}$. □

Theorem 3.6.3. *Let $\Omega \in \Lambda^2(U^n)$ be closed. Suppose r is the natural number such that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$. Further suppose we have a solvable structure of $2r$ linearly independent symmetries $X_1, \ldots, X_{2r} \in \mathfrak{X}(U^n)$ such that X_{2r} is a non-trivial symmetry of $A(\langle \Omega^r \rangle)$, and for all $1 \leq i < 2r$, X_i is a non-trivial symmetry of $A(\langle \Omega^r \rangle) \oplus Sp\{X_{i+1}, \ldots, X_{2r}\}$. Then Theorem 3.2.14 gives us an algorithm for expressing Ω solely in terms of the $2r$ functionally independent functions $\gamma^1, \ldots, \gamma^{2r} \in C^\infty(U^n)$ and their exterior derivatives.*

Proof. Let $\Omega \in \Lambda^2(U^n)$ be closed with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. Since $d\Omega = 0$ implies that $d(\Omega^r) = 0$, from Theorem 2.3.11, $\ker(\Omega^r) = A(\langle \Omega^r \rangle)$ is therefore Frobenius integrable. The fact that Ω^r is decomposable of degree $2r$ means that $A(\langle \Omega^r \rangle)$ is generated by $n - 2r$ linearly independent vector fields. Suppose we have a set of linearly independent symmetries $X_1, \ldots, X_{2r} \in \mathfrak{X}(U^n)$ such that X_{2r} is a non-trivial symmetry of $A(\langle \Omega^r \rangle)$, and for all $1 \leq i < 2r$, X_i is a non-trivial symmetry of $A(\langle \Omega^r \rangle) \oplus Sp\{X_{i+1}, \ldots, X_{2r}\}$. Then by Theorem 3.2.14 we have on U^n, $\{\omega^1, \ldots, \omega^{2r}\}$ dual to $\{X_1, \ldots, X_{2r}\}$, where for all $1 \leq j \leq 2r$,

$$\omega^j := \frac{X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_{2r} \lrcorner \Omega^r}{X_j \lrcorner X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_{2r} \lrcorner \Omega^r},$$

and

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2) d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1, \\
&\vdots \\
\omega^{2r} &\equiv d\gamma^{2r} \mod d\gamma^1, \ldots, d\gamma^{2r-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^{2r} \in C^\infty(U^n)$. Then by Lemma 3.6.2, and using the fact that $\{X_1, \ldots, X_{2r}\}$ plus any set of generators of $A(\langle \Omega^r \rangle)$ spans $\mathfrak{X}(U^n)$, we can therefore write

$$\Omega = \Omega(X_k, X_l) \omega^k \wedge \omega^l, \qquad 1 \leq k < l \leq 2r,$$

where we are implying a double summation. This means that

$$\Omega = \Omega_{kl} d\gamma^k \wedge d\gamma^l, \qquad 1 \leq k < l \leq 2r, \tag{3.31}$$

for some functions $\Omega_{kl} \in C^\infty(U^n)$. But since Ω is closed, we must have for all $\Gamma \in A(\langle \Omega^r \rangle)$,

$$\mathcal{L}_\Gamma \Omega = d(\Gamma \lrcorner \Omega) = 0,$$

also using Lemma 3.6.2. Since $\Gamma(\gamma^i) = 0$ for all i, it follows that (with sum)

$$0 = \mathcal{L}_\Gamma \Omega = \Gamma(\Omega_{kl}) d\gamma^k \wedge d\gamma^l.$$

Therefore $\Gamma(\Omega_{kl}) = 0$ for each k and l. Hence Ω only depends on the $2r$ functions $\gamma^1, \ldots, \gamma^{2r}$ and their exterior derivatives. □

Remark. In applying Theorem 3.6.3, there will exists situations when it may be difficult to express each Ω_{kl} in terms of the known $\gamma^1, \ldots, \gamma^{2r}$.

Next, consider Darboux's theorem proved in [23, 37]:

Theorem 3.6.4. (Darboux) *Let $\Omega \in \Lambda^2(U^n)$ be closed so that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. Then there exist coordinates $\gamma^1, \ldots, \gamma^n$ such that*

$$\Omega = d\gamma^1 \wedge d\gamma^2 + \cdots + d\gamma^{2r-1} \wedge d\gamma^{2r}.$$

Theorem 3.6.3 may be applied to Darboux's theorem; however, the difficulty is that Theorem 3.6.3 expresses Ω in terms of a sum of a maximum of $\binom{2r}{2}$ two-form components, which must then be simplified to r components with unit one coefficients if we wish to find a set of coordinates in Darboux's theorem.

As an alternative approach extending work in [37] by Crampin and Pirani in their proof of Darboux's theorem (though similar proofs can be found in the literature), we now look to formulate an extraction process for generating a set of coordinates in the theorem using solvable symmetry structures. The next three theorems will be useful in establishing this.

Theorem 3.6.5. *Let $\Omega \in \Lambda^2(U^n)$ with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \geq 2$. Suppose there exist $X_1, X_2 \in \mathfrak{X}(U^n)$ such that $\Omega(X_1, X_2) = 1$ and $(X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) \neq 0$. If $\overline{\Omega}$ is defined by $\overline{\Omega} := \Omega + (X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega)$, then $\overline{\Omega}^{r-1} \neq 0$ and $\overline{\Omega}^r = 0$.*

Proof. Let $\Omega \in \Lambda^2(U^n)$ such that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \geq 2$. Using the definition for $\overline{\Omega}$ in the theorem gives

$$\overline{\Omega}^r = \Omega^r + r\Omega^{r-1} \wedge (X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega). \tag{3.32}$$

Now from $\Omega(X_1, X_2) = 1$ we have

$$\begin{aligned} \Omega^r &= \Omega^r (X_2 \lrcorner X_1 \lrcorner \Omega), \\ &= X_2 \lrcorner (\Omega^r \wedge (X_1 \lrcorner \Omega)) - (X_2 \lrcorner \Omega^r) \wedge (X_1 \lrcorner \Omega), \\ &= X_2 \lrcorner (\Omega^r \wedge (X_1 \lrcorner \Omega)) - (r(X_2 \lrcorner \Omega) \wedge \Omega^{r-1}) \wedge (X_1 \lrcorner \Omega). \end{aligned} \tag{3.33}$$

In the second line we have used the property $X_2 \lrcorner (\Omega^r \wedge (X_1 \lrcorner \Omega)) = (X_2 \lrcorner \Omega^r) \wedge (X_1 \lrcorner \Omega) + (X_2 \lrcorner X_1 \lrcorner \Omega)\Omega^r$, and in the third, we have expanded $X_2 \lrcorner \Omega^r$. If we substitute the end result in (3.33) into the expression for $\overline{\Omega}^r$ in (3.32), we obtain

$$\overline{\Omega}^r = X_2 \lrcorner (\Omega^r \wedge (X_1 \lrcorner \Omega)). \tag{3.34}$$

By Theorem 3.6.1, there exist linearly independent one-forms $\omega^1, \ldots, \omega^{2r} \in \Lambda^1(U^n)$ such that

$$\Omega = \omega^1 \wedge \omega^2 + \cdots + \omega^{2r-1} \wedge \omega^{2r}.$$

Hence $X_1 \lrcorner \Omega = a_1 \omega^1 + \cdots + a_{2r} \omega^{2r}$ for some $a_1, \ldots, a_{2r} \in C^\infty(U^n)$. Since

$$\Omega^r = r! \omega^1 \wedge \cdots \wedge \omega^{2r},$$

it follows that $\Omega^r \wedge (X_1 \lrcorner \Omega) = 0$. Thus from (3.34) we get $\overline{\Omega}^r = 0$.

Now suppose $\overline{\Omega}^{r-1} = 0$. Then

$$0 = \overline{\Omega}^{r-1} = \Omega^{r-1} + (r-1)\Omega^{r-2} \wedge (X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega).$$

This implies

$$\Omega^{r-1} = (r-1)\Omega^{r-2} \wedge (X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega). \tag{3.35}$$

Taking the exterior product with Ω gives

$$\Omega^r = (r-1)\Omega^{r-1} \wedge (X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) = 0, \tag{3.36}$$

where the second equality comes from substituting Ω^{r-1} in (3.36) with its expression in (3.35). The calculations still holds for $r = 2$, and hence we reach a contradiction for all $r \geq 2$. □

Remark. Although Theorem 3.6.5 demands that X_1, X_2 be such that $\Omega(X_1, X_2) = 1$, we can relax this condition by saying that all we need is to find two vector fields

$Y_1, Y_2 \in \mathfrak{X}(U^n)$ such that $\Omega(Y_1, Y_2) \neq 0$. Then we can choose X_1, X_2 as, respectively, scaled Y_1, Y_2 such that $\Omega(X_1, X_2) = 1$.

The second theorem we require concerns the foliated exterior derivative, as explained by Vaisman [123]:

Theorem 3.6.6. *Let $\omega \in \Lambda^1(U^n)$ and $\alpha^1, \ldots, \alpha^s \in \Lambda^1(U^n)$ be s linearly independent one-forms such that for all $1 \leq i \leq s$,*

$$d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^s,$$

(i.e. the Frobenius condition holds so that $\ker(\alpha^1 \wedge \cdots \wedge \alpha^s)$ is Frobenius integrable). If

$$d\omega \equiv 0 \mod \alpha^1, \ldots, \alpha^s,$$

then

$$\omega \equiv df \mod \alpha^1, \ldots, \alpha^s,$$

for some $f \in C^\infty(U^n)$.

Using the foliated exterior derivative, we prove the following theorem:

Theorem 3.6.7. *Let $\Omega \in \Lambda^2(U^n)$ be closed. If there exists a pair of vector fields $X_1, X_2 \in \mathfrak{X}(U^n)$ such that*

1. $\mathcal{L}_{X_1}\Omega = 0$,

2. $\mathcal{L}_{X_2}\Omega \equiv 0 \mod X_1 \lrcorner\, \Omega$,

3. $(X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega) \neq 0$,

then on U^n,

$$(X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega) = df \wedge dg,$$

for some functionally independent $f, g \in C^\infty(U^n)$.

Proof. Let $\Omega \in \Lambda^2(U^n)$ be closed and let there exist vector fields $X_1, X_2 \in \mathfrak{X}(U^n)$ that satisfy the three conditions in the theorem. Now $\mathcal{L}_{X_1}\Omega = 0$ implies $d(X_1 \lrcorner \Omega) = 0$, using the property $\mathcal{L}_{X_1}\Omega = X_1 \lrcorner d\Omega + d(X_1 \lrcorner \Omega)$ and that Ω is closed. Hence $X_1 \lrcorner \Omega = df$ for some $f \in C^\infty(U^n)$.

Now suppose $\mathcal{L}_{X_2}\Omega = 0$. Then by the same argument to above, $X_2 \lrcorner \Omega = dg_1$ for some $g_1 \in C^\infty(U^n)$. If, however, $\mathcal{L}_{X_2}\Omega \neq 0$, then by assumption,

$$0 \neq \mathcal{L}_{X_2}\Omega = \alpha \wedge (X_1 \lrcorner \Omega),$$

for some $\alpha \in \Lambda^1(U^n)$. Therefore

$$(\mathcal{L}_{X_2}\Omega) \wedge (X_1 \lrcorner \Omega) = 0.$$

Using $\mathcal{L}_{X_2}\Omega = X_2 \lrcorner d\Omega + d(X_2 \lrcorner \Omega)$ and the fact that Ω is closed gives

$$d(X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega) = 0.$$

Hence

$$d(X_2 \lrcorner \Omega) \equiv 0 \mod X_1 \lrcorner \Omega.$$

Using Theorem 3.6.6, we then get

$$X_2 \lrcorner \Omega \equiv dg_2 \mod df,$$

for some $g_2 \in C^\infty(U^n)$. Hence in both cases the result is proved. □

We now present the main result of this section:

Theorem 3.6.8. *Let $\Omega \in \Lambda^2(U^n)$ be closed with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. Then the following algorithm explicitly computes a set of $2r$ functions for Ω described in Darboux's theorem:*

1. *Find vector fields $X_1, X_2 \in \mathfrak{X}(U^n)$ such that:*

   (a) $\mathcal{L}_{X_1}\Omega = 0$,

   (b) $\mathcal{L}_{X_2}\Omega \equiv 0 \mod X_1 \lrcorner \Omega$,

   (c) $(X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) \neq 0$,

   (d) $\Omega(X_1, X_2) = 1$,

2. *Let* $\Omega + (X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega)$ *be our new* Ω,

3. *Repeat steps* (1) *and* (2) *a further* $r-2$ *more times until* $\Omega^2 = 0$,

4. *Apply Theorem 3.2.14 with a solvable structure of two symmetries* $X_3, X_4 \in \mathfrak{X}(U^n)$ *for* Ω, *such that* X_3 *is a non-trivial symmetry of* Ω *and* X_4 *is a non-trivial symmetry of* $X_3 \lrcorner \Omega$ *with the property that* $\Omega(X_3, X_4) = 1$.

Proof. Let $\Omega \in \Lambda^2(U^n)$ be closed with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. From Theorem 3.6.7 and then Theorem 3.6.5, we can compute $\Omega_1 \in \Lambda^2(U^n)$, where

$$\Omega_1 = \Omega + dg_1 \wedge df_1,$$

for some $f_1, g_1 \in C^\infty(U^n)$, with $\Omega_1^{r-1} \neq 0$ and $\Omega_1^r = 0$. Then once again from Theorem 3.6.7 followed by Theorem 3.6.5, $\Omega_2 \in \Lambda^2(U^n)$ can be computed so that

$$\Omega_2 = \Omega + dg_1 \wedge df_1 + dg_2 \wedge df_2,$$

for some $f_2, g_2 \in C^\infty(U^n)$, with $\Omega_2^{r-2} \neq 0$ and $\Omega_2^{r-1} = 0$. Continuing in this way, we reach a stage when Ω_{r-1} is of the form

$$\Omega_{r-1} = \Omega + dg_1 \wedge df_1 + dg_2 \wedge df_2 + \cdots + dg_{r-1} \wedge df_{r-1},$$

such that $\Omega_{r-1} \neq 0$ and $\Omega_{r-1}^2 = 0$. Applying step (4), Ω_{r-1} is closed, and from Theorem 3.6.1, Ω_{r-1} is also decomposable. From Theorem 3.2.14 and Corollary 3.2.12, with X_3 as a non-trivial symmetry of Ω_{r-1} and X_4 as a non-trivial symmetry of $X_3 \lrcorner \Omega_{r-1}$ such that $\Omega_{r-1}(X_3, X_4) = 1$, then

$$\frac{X_3 \lrcorner \Omega_{r-1}}{X_4 \lrcorner X_3 \lrcorner \Omega_{r-1}} = dg_r,$$
$$\frac{X_4 \lrcorner \Omega_{r-1}}{X_3 \lrcorner X_4 \lrcorner \Omega_{r-1}} = df_r + \lambda dg_r,$$

for some $f_r, g_r, \lambda \in C^\infty(U^n)$, with

$$\Omega_{r-1} = \Omega_{r-1}(X_3, X_4) df_r \wedge dg_r = df_r \wedge dg_r.$$

Therefore

$$\Omega = df_1 \wedge dg_1 + df_2 \wedge dg_2 + \cdots + df_{r-1} \wedge dg_{r-1} + df_r \wedge dg_r.$$

□

Remark 1. In looking for two symmetries that satisfy the four conditions in Theorem 3.6.8, condition (d) can be relaxed a little by only requiring that $X_2 \lrcorner X_1 \lrcorner \Omega = const.$ Then X_1 or X_2 may be scaled appropriately by constants while still satisfying the other three conditions. The same holds true for the two symmetries in step (4).

Remark 2. Conditions (a) and (b) are strong requirements, and may be difficult in practice to satisfy. Since Ω is closed, they imply X_1, X_2 must be chosen such that $X_1 \lrcorner \Omega$ is closed and $X_2 \lrcorner \Omega$ is closed, modulo $X_1 \lrcorner \Omega$. Hence the result in Theorem 3.6.8 is of more theoretical significance than practical use, although it is possible to use `DIMSYM` and Theorem 2.3.10 to find X_3, X_4 in step (4).

We can provide an alternative to the requirement in step (4) in Theorem 3.6.8 as follows:

Lemma 3.6.9. *Let $\Omega \in \Lambda^2(U^n)$ be some arbitrary closed two-form. Suppose there exists some $X_3 \in \mathfrak{X}(U^n)$ not in* $\ker(\Omega)$ *such that*

$$\mathcal{L}_{X_3}\Omega = 0, \tag{3.37}$$

and $X_4 \in \mathfrak{X}(U^n)$ satisfies $\Omega(X_3, X_4) = 1$. Then

$$\mathcal{L}_{X_4}(X_3 \lrcorner \Omega) = 0.$$

Proof.

$$\mathcal{L}_{X_4}(X_3 \lrcorner \Omega) = d(X_4 \lrcorner X_3 \lrcorner \Omega) + X_4 \lrcorner d(X_3 \lrcorner \Omega) = X_4 \lrcorner (\mathcal{L}_{X_3}\Omega) = 0,$$

using that $X_4 \lrcorner X_3 \lrcorner \Omega = 1$, equation (3.37), and that Ω is closed. □

We now apply the algorithm in Theorem 3.6.8 and the modification of step (4) using Lemma 3.6.9 to the following example. It is important to realise that the difficult part in applying Theorem 3.6.8 is in finding the first $r-1$ pairs of symmetries X_1, X_2. Nevertheless, the main purposes of this example are to illustrate: i) the crucial role Theorem 3.6.5 plays in reducing the number of terms in a two-form by one; and ii) the flexibility in choosing X_4 in Lemma 3.6.9.

Example 3.6.10. Consider the following two-form $\Omega \in \Lambda^2(U^4)$, where U^4 is some suitably chosen four-dimensional, open, convex neighbourhood of $\mathbb{R}^4$ with coordinates x^1, x^2, x^3, x^4:

$$\Omega := \frac{x^1}{x^2}\left(\frac{x^3}{x^2} - 2\right) dx^1 \wedge dx^2 + \frac{x^1}{x^2} dx^1 \wedge dx^3 - \frac{2x^1}{x^4} dx^1 \wedge dx^4 - \left(\frac{x^1}{x^2}\right)^2 dx^2 \wedge dx^3.$$

It is easy to show that $d\Omega = 0$, $\Omega^2 \neq 0$ and $\Omega^3 = 0$. We may then proceed to apply Theorem 3.6.8. Define

$$X_1 := -\frac{1}{x^3}\left(\frac{x^2}{x^1}\right)^2 \frac{\partial}{\partial x^2} + \frac{x^2 x^4}{(x^1)^2 x^3} \frac{\partial}{\partial x^4}.$$

Now

$$\begin{aligned}\mathcal{L}_{X_1}\Omega &= d\left(X_1 \lrcorner \Omega\right), \\ &= d\left(\frac{1}{x^3} dx^3 + \frac{2x^2}{x^1 x^3} dx^1 + \frac{x^2}{x^1 x^3}\left(\frac{x^3}{x^2} - 2\right) dx^1\right), \\ &= d\left(\frac{1}{x^3} dx^3 + \frac{1}{x^1} dx^1\right) = 0,\end{aligned}$$

so condition (a) of step (1) in Theorem 3.6.8 is met. Hence

$$X_1 \lrcorner \Omega = d\left(\ln |x^1 x^3|\right).$$

With

$$X_2 := x^3 \frac{\partial}{\partial x^3},$$

we have $X_2 \lrcorner X_1 \lrcorner \Omega = 1$, so condition (d) is satisfied. It is not hard to show that

$$(X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) \neq 0,$$

and

$$(\mathcal{L}_{X_2}\Omega) \wedge (X_1 \lrcorner \Omega) = 0, \tag{3.38}$$

so conditions (b) and (c) are met. From (3.38),

$$d\left(X_2 \lrcorner \Omega\right) \equiv 0 \mod X_1 \lrcorner \Omega.$$

Using the foliated derivative, this implies

$$X_2 \lrcorner \Omega = dg_1 + \lambda_1 d\left(\ln|x^1x^3|\right),$$

for some $g_1, \lambda_1 \in C^\infty(U^4)$. To find these unknowns, we can perform a coordinate transformation with $\ln|x^1x^3|$ as a coordinate to yield

$$X_2 \lrcorner \Omega = -d\left(\frac{(x^1)^2x^3}{x^2}\right) + \frac{(x^1)^2x^3}{x^2} d\left(\ln|x^1x^3|\right).$$

Hence

$$(X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega) = -d\left(\frac{(x^1)^2x^3}{x^2}\right) \wedge d\left(\ln|x^1x^3|\right).$$

Observe that

$$\begin{aligned} -d\left(\frac{(x^1)^2x^3}{x^2}\right) \wedge d\left(\ln|x^1x^3|\right) &= -\frac{1}{x^1x^3} d\left(\frac{(x^1x^3)x^1}{x^2}\right) \wedge d(x^1x^3), \\ &= -d\left(\frac{x^1}{x^2}\right) \wedge d(x^1x^3). \end{aligned}$$

For other choice of X_1, X_2, we may obtain an expression for the other two-form component of Ω.

Now define $\Omega_1 := \Omega + (X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega)$ as in step (2). We then get

$$\Omega_1 = -\frac{2x^1}{x^2} dx^1 \wedge dx^2 - \frac{2x^1}{x^4} dx^1 \wedge dx^4.$$

It is clear that $d\Omega_1 = 0$ and $\Omega_1^2 = 0$ as expected, so we may proceed to apply the final step in Theorem 3.6.8 on Ω_1. It is possible at this point to use `DIMSYM` and Theorem 2.3.10 to search for a non-trivial symmetry of $\ker(\Omega_1)$ with the property that the Lie derivative of Ω_1 is zero. Defining

$$X_3 := x^1x^4 \frac{\partial}{\partial x^4},$$

we have

$$\mathcal{L}_{X_3}\Omega_1 = d\left(X_3 \lrcorner \Omega_1\right) = d\left(2(x^1)^2 dx^1\right) = 0.$$

This implies

$$X_3 \lrcorner \Omega_1 = d\left(\frac{2(x^1)^3}{3}\right).$$

Now choose

$$X_4 := \frac{1}{2(x^1)^2}\frac{\partial}{\partial x^1},$$

so that $X_4 \lrcorner X_3 \lrcorner \Omega_1 = 1$. From Lemma 3.6.9, $\mathcal{L}_{X_4}(X_3 \lrcorner \Omega_1) = 0$, and hence from Theorem 3.2.14,

$$X_4 \lrcorner \Omega_1 = df_2 + \lambda_2 d\left(\frac{2(x^1)^3}{3}\right),$$

for some $f_2, \lambda_2 \in C^\infty(U^4)$. To find f_2, it is easy to show that

$$X_4 \lrcorner \Omega_1 \equiv -d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \quad \mod dx^1.$$

Therefore

$$\Omega_1 = (X_3 \lrcorner \Omega_1) \wedge (X_4 \lrcorner \Omega_1) = d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \wedge d\left(\frac{2(x^1)^3}{3}\right).$$

Once again we may simplify this:

$$\begin{aligned} d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \wedge d\left(\frac{2(x^1)^3}{3}\right) &= 2(x^1)^2 d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \wedge dx^1, \\ &= 2x^1 d\left(\ln|x^2x^4|\right) \wedge dx^1, \\ &= d\left(\ln|x^2x^4|\right) \wedge d\left((x^1)^2\right). \end{aligned}$$

Thus

$$\begin{aligned} \Omega &= (X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) + (X_3 \lrcorner \Omega_1) \wedge (X_4 \lrcorner \Omega_1), \\ &= d\left(\frac{x^1}{x^2}\right) \wedge d\left(x^1x^3\right) + d\left(\ln|x^2x^4|\right) \wedge d\left((x^1)^2\right). \end{aligned}$$

Chapter 4

Fundamental ideals of ODEs

4.1 Introduction

As a prelude to our study of partial differential equations we examine an approach to ordinary differential equations from the perspective of *fundamental ideals* by Edelen [43, 45, 47, 49]. The material is essentially a brief review of work found in Basarab-Horwath [16], Duzhin and Lychagin [42], Hartl and Athorne [65] and Sherring and Prince [110] on solving ODEs with solvable symmetry structures. Thus it presents nothing new apart from the fact that it uses fundamental ideals.

Our study in this chapter focuses on finding the general solution of a single n-th order (possibly non-linear) ODE defined on some open, convex neighbourhood U^1 of $\mathbb{R}$ with coordinate x corresponding to the independent variable. We define V^1 to be some open, convex neighbourhood of $\mathbb{R}$ with coordinate y as the space for the dependent variable.

4.2 Ideals of ODEs

Suppose we have some n-th order ODE defined on $U^1 \times V^1$ of the form

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right), \tag{4.1}$$

for smooth F. On the n-th jet bundle $J^n(U^1, V^1)$, we define the following coordinates $x, z, z_1, z_2, \ldots, z_n$, where z_i represents the i-th derivative of y (The notation here is slightly different from that used in Chapter 2, and will only be used for this chapter).

Then from Section 2.5, a local solution of the ODE in (4.1) can be considered as a one-dimensional regular submanifold of the locus of $J^n(U^1, V^1)$ described by the jet space coordinate representation of (4.1), that in addition is the lift of some transverse curve in the graph space $U^1 \times V^1$. Equivalently, from Section 2.5.2 on the fundamental ideal representation of differential equations, a local solution of (4.1) can also be considered some one-dimensional regular submanifold of $J^n(U^1, V^1)$ whose tangent space is annihilated by the differential ideal

$$I_F := \langle dz - z_1 dx, dz_1 - z_2 dx, \ldots, dz_{n-1} - z_n dx, dz_n \wedge dx, (z_n - F)\, dx \rangle,$$

and satisfies the transverse condition that $dx \neq 0$ on the tangent space. It is this latter formulation that we use throughout this chapter.

The main result of this section is Theorem 4.2.1 given below which allows us to study the ODE given in (4.1) in a simplified framework. This is explained in greater detail following the proof of the theorem.

Theorem 4.2.1.

$$I_F = \langle dz - z_1 dx, dz_1 - z_2 dx, \ldots, dz_{n-1} - z_n dx, dz_n \wedge dx, dz_{n-1} - F dx \rangle. \qquad (4.2)$$

Proof. From Lemma 2.5.3,

$$d\left((z_n - F)\, dx\right) \equiv 0 \mod dz - z_1 dx, dz_1 - z_2 dx, \ldots, dz_{n-1} - z_n dx, dz_n \wedge dx. \quad (4.3)$$

Since

$$dz_{n-1} - F dx = (z_n - F)\, dx + (dz_{n-1} - z_n dx)\,, \qquad (4.4)$$

we obtain

$$\begin{aligned} d\left(dz_{n-1} - F dx\right) \equiv 0 \mod dz - z_1 dx,& \\ dz_1 - z_2 dx, \ldots, dz_{n-1} - z_n dx, dz_n \wedge dx.& \end{aligned} \qquad (4.5)$$

Combining (4.3), (4.4) and (4.5), we conclude (4.2). □

We define

$$I_{\overline{F}} := \langle dz - z_1 dx, dz_1 - z_2 dx, \ldots, dz_{n-1} - z_n dx, dz_n \wedge dx, dz_{n-1} - F dx \rangle.$$

Technically speaking, $I_{\overline{F}} := I_F$ (by Theorem 4.2.1), and the notation $I_{\overline{F}}$ might appear redundant. However we will use $I_{\overline{F}}$ as a brief way of referring to the particular choice of generators $dz - z_1 dx, dz_1 - z_2 dx, \ldots, dz_{n-1} - z_n dx, dz_n \wedge dx, dz_{n-1} - F dx$. Throughout this thesis we frequently make this distinction.

The final generating term in $I_{\overline{F}}$ that is specific to the ODE contains no z_n coordinate. Therefore we may disregard the highest contact form $dz_{n-1} - z_n dx$, and work in the $(n-1)$-th order jet bundle $J^{n-1}(U^1, V^1)$ with the corresponding *reduced* differential ideal

$$I^r_{\overline{F}} := \langle dz - z_1 dx, dz_1 - z_2 dx, \ldots, dz_{n-2} - z_{n-1} dx, dz_{n-1} - F dx, dF \wedge dx \rangle. \quad (4.6)$$

At this point we make the following observations regarding $I^r_{\overline{F}}$: Firstly, for $I^r_{\overline{F}}$ to be a differential ideal, (4.5) means $dF \wedge dx$ must be included as a generator. However, since differential forms in $I^r_{\overline{F}}$ of degree higher than one are trivially annihilated by the tangent space of *any* curve in $J^{n-1}(U^1, V^1)$, all such differential forms can be ignored in all our calculations. Secondly, from examining the one-form generators of $I^r_{\overline{F}}$, we can say that for ODEs, the fundamental ideal approach to differential equations coincides with the Pfaffian system approach outlined in Section 2.5.1. Finally, note from the dimension of $\Lambda^1\left(J^{n-1}(U^1, V^1)\right)$ that any one-dimensional integral manifold of $I^r_{\overline{F}}$ must necessarily satisfy the transverse condition.

4.3 A linear transformation

Concentrating on the n one-forms generators in $I^r_{\overline{F}}$, suppose we can find some $n \times n$ matrix

$$\mathbf{F} := [f_{ij}],$$

of rank n for some $f_{ij} \in C^\infty(J^{n-1}(U^1, V^1))$ over all $1 \leq i, j \leq n$, such that

$$\begin{pmatrix} dz - z_1 dx \\ \vdots \\ dz_{n-2} - z_{n-1} dx \\ dz_{n-1} - F dx \end{pmatrix} = \mathbf{F} \cdot \begin{pmatrix} d\gamma^1 \\ \vdots \\ d\gamma^n \end{pmatrix},$$

for some functionally independent $\gamma^1, \ldots, \gamma^n \in C^\infty(J^{n-1}(U^1, V^1))$. Assuming $\gamma^1, \ldots,$ γ^n are of constant rank on $J^{n-1}(U^1, V^1)$, it is then clear that since $\mathbf{F}$ is invertible, it follows that on the curve in $J^{n-1}(U^1, V^1)$ described by

$$\gamma^1 = c^1, \ldots, \gamma^n = c^n, \quad (4.7)$$

where $c^1, \ldots, c^n$ are any constant functions, all the one-forms in $I^r_{\overline{F}}$ are annihilated. Hence by Corollary 2.4.11 the curve is, in implicit form, a lifted solution of the ODE in (4.1). If we can also manipulate the n equations in (4.7) so that we are left with z solely expressed in terms of $x, c^1, \ldots, c^n$, then we obtain the (local) general solution of the ODE in $U^1 \times V^1$.

Alternatively, and slightly more generally, suppose we can find some $n \times n$ matrix

$$\widehat{\mathbf{F}} := [\hat{f}_{ij}],$$

of rank n, for some $\hat{f}_{ij} \in C^\infty(J^{n-1}(U^1, V^1))$ over all $1 \leq i, j \leq n$, such that

$$\begin{pmatrix} dz - z_1 dx \\ \vdots \\ dz_{n-2} - z_{n-1} dx \\ dz_{n-1} - F dx \end{pmatrix} = \widehat{\mathbf{F}} \cdot \begin{pmatrix} d\gamma^1 \\ d\gamma^2 \mod d\gamma^1 \\ \vdots \\ d\gamma^n \mod d\gamma^1, \ldots, d\gamma^{n-1} \end{pmatrix}, \tag{4.8}$$

for some functionally independent $\gamma^1, \ldots, \gamma^n \in C^\infty(J^{n-1}(U^1, V^1))$. Then by the same argument as before, the corresponding equations given by $\gamma^1 = c^1, \ldots, \gamma^n = c^n$, for constant functions $c^1, \ldots, c^n$, also describe a lifted solution curve of the ODE in (4.1).

Focusing on $\widehat{\mathbf{F}}$, we can summarise the above result in the following theorem:

Theorem 4.3.1. *Let $I^r_{\overline{F}}$ in (4.6) be a differential ideal on $J^{n-1}(U^1, V^1)$ corresponding to some n-th order ODE of the form in (4.1). If there exists an $n \times n$ matrix $\widehat{\mathbf{F}}$ of rank n such that (4.8) holds for some functionally independent $\gamma^1, \ldots, \gamma^n \in C^\infty(J^{n-1}(U^1, V^1))$, then the curve in $J^{n-1}(U^1, V^1)$ described by $\gamma^1 = c^1, \ldots, \gamma^n = c^n$, for any choice of constant functions $c^1, \ldots, c^n$, is a lifted solution curve of the ODE in (4.1).*

The matrix $\widehat{\mathbf{F}}$ (and $\mathbf{F}$) has the special property that it gives us a linear transformation between a column vector whose elements are expressed in terms of the $n+1$ basis vectors $dx, dz, dz_1, \ldots, dz_{n-1}$, and a column vector whose elements are linear combinations of n linearly independent exact one-forms $d\gamma^1, \ldots, d\gamma^n$. Assigning $\gamma^1 = c^1, \ldots, \gamma^n = c^n$, for constant functions $c^1, \ldots, c^n$ then yields a one-dimensional regular submanifold of $J^{n-1}(U^1, V^1)$ (assuming the functions are of constant rank), whose tangent space annihilates each one-form in $I^r_{\overline{F}}$. Therefore, finding the general solution of the ODE in (4.1) essentially amounts to finding some linear transformation that can allow us to express all the one-forms generating I^r_F in terms of a linear combination of a fewer number of linearly independent exact one-forms.

Finally, a simple calculation (expanding $dF \wedge dx$) shows that

$$\Gamma := \frac{\partial}{\partial x} + z_1 \frac{\partial}{\partial z} + \cdots + z_{n-1} \frac{\partial}{\partial z_{n-2}} + F \frac{\partial}{\partial z_{n-1}}$$

is a Cauchy characteristic vector field of I^r_F. Hence the matrix $\widehat{\mathbf{F}}$ essentially finds the functions in Theorem 2.2.13.

In the next section we show how one may use a solvable structure of symmetries to find such a matrix $\widehat{\mathbf{F}}$ and some corresponding $\gamma^1, \ldots, \gamma^n$, so that the general solution of (4.1) may be derived in the above fashion.

4.4 Solutions and symmetry

Suppose I^r_F is a differential ideal corresponding to some n-th order ODE of the form in (4.1). Define $\Omega \in \Lambda^n(J^{n-1}(U^1, V^1))$ by

$$\Omega := (dz - z_1 dx) \wedge (dz_1 - dz_2 dx) \wedge \cdots \wedge (dz_{n-2} - z_{n-1} dx) \wedge (dz_{n-1} - F dx).$$

Since Ω is an n-form defined on the $(n+1)$-dimensional space $J^{n-1}(U^1, V^1)$, from Theorem 3.3.3, we therefore have $d\Omega \equiv 0 \mod \Omega$.

Further suppose there exists a solvable structure of n linearly independent vector fields $X_1, \ldots, X_n \in \mathfrak{X}(J^{n-1}(U^1, V^1))$ such that X_n is a non-trivial symmetry of Ω, X_{n-1} is a non-trivial symmetry of $X_n \lrcorner \Omega$, and so on down to X_1 being a non-trivial symmetry of $X_2 \lrcorner \ldots \lrcorner X_n \lrcorner \Omega$. If, for all $1 \le k \le n$, we define ω^k by

$$\omega^k := \frac{X_1 \lrcorner \ldots \lrcorner X_{k-1} \lrcorner X_{k+1} \lrcorner \ldots \lrcorner X_n \lrcorner \Omega}{X_k \lrcorner X_1 \lrcorner \ldots \lrcorner X_{k-1} \lrcorner X_{k+1} \lrcorner \ldots \lrcorner X_n \lrcorner \Omega}.$$

Then from Theorem 3.2.14 and Corollary 3.2.12, we have that for all k up to n,

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2) d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1, \\
&\vdots \\
\omega^n &\equiv d\gamma^n \mod d\gamma^1, \ldots, d\gamma^{n-1},
\end{aligned} \tag{4.9}$$

for some functionally independent $\gamma^1, \ldots, \gamma^n$, and $\{\omega^1, \ldots, \omega^n\}$ is dual to $\{X_1, \ldots, X_n\}$. In other words, the equations in (4.9) are of the form

$$\mathbf{G} \cdot \begin{pmatrix} dz - z_1 dx \\ \vdots \\ dz_{n-2} - z_{n-1} dx \\ dz_{n-1} - F dx \end{pmatrix} = \begin{pmatrix} d\gamma^1 \\ d\gamma^2 - X_1(\gamma^2) d\gamma^1 \\ d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1 \\ \vdots \\ d\gamma^n \mod d\gamma^1, \ldots, d\gamma^{n-1} \end{pmatrix},$$

for some $n \times n$ matrix $\mathbf{G}$. Further, since $\{\omega^1, \ldots, \omega^n\}$ is dual to $\{X_1, \ldots, X_n\}$, $\omega^1, \ldots, \omega^n$ are linearly independent, so $\mathbf{G}$ is non-singular. Hence

$$\begin{pmatrix} dz - z_1 dx \\ \vdots \\ dz_{n-2} - z_{n-1} dx \\ dz_{n-1} - F dx \end{pmatrix} = \mathbf{G}^{-1} \cdot \begin{pmatrix} d\gamma^1 \\ d\gamma^2 - X_1(\gamma^2) d\gamma^1 \\ d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1 \\ \vdots \\ d\gamma^n \mod d\gamma^1, \ldots, d\gamma^{n-1} \end{pmatrix}.$$

Therefore we can then apply Theorem 4.3.1, with $\widehat{\mathbf{F}} := \mathbf{G}^{-1}$.

Once again we can summarise the above in the following theorem:

Theorem 4.4.1. *Suppose $I^r_{\overline{F}}$ is a differential ideal on $J^{n-1}(U^1, V^1)$ corresponding to some n-th order ODE of the form in (4.1). Define $\Omega \in \Lambda^n(J^{n-1}(U^1, V^1))$ by*

$$\Omega := (dz - z_1 dx) \wedge (dz_1 - dz_2 dx) \wedge \cdots \wedge (dz_{n-2} - z_{n-1} dx) \wedge (dz_{n-1} - F dx).$$

Further, suppose there exists a solvable structure of n linearly independent vector fields $X_1, \ldots, X_n \in \mathfrak{X}(J^{n-1}(U^1, V^1))$ such that X_n is a non-trivial symmetry of Ω, X_{n-1} is a non-trivial symmetry of $X_n \lrcorner\, \Omega$, and so on down to X_1 being a non-trivial symmetry of $X_2 \lrcorner \ldots \lrcorner X_n \lrcorner\, \Omega$. Then there exists an $n \times n$ matrix $\widehat{\mathbf{F}}$ of rank n such that (4.8) holds for some functionally independent $\gamma^1, \ldots, \gamma^n \in C^\infty(J^{n-1}(U^1, V^1))$.

Theorem 4.3.1 combined with Theorem 4.4.1 gives us an algorithm based on a solvable symmetry structure for generating the general solution of a given n-th order ODE of the form in (4.1). We illustrate these two results with the following example:

Example 4.4.2. Consider the following second order ODE (i.e. $n = 2$):

$$\frac{d^2y}{dx^2} = F_2\left(x, y, \frac{dy}{dx}\right), \tag{4.10}$$

for smooth F_2. For this example, on $J^1(U^1, V^1)$,

$$I^r_{F_2} := \langle dz - z_1 dx, dz_1 - F_2 dx, dF_2 \wedge dx \rangle.$$

Now let $\Omega \in \Lambda^2(J^1(U^1, V^1))$ defined by

$$\Omega := (dz - z_1 dx) \wedge (dz_1 - F_2 dx).$$

If we are given a solvable structure of two symmetries $X_1, X_2 \in \mathfrak{X}(J^1(U^1, V^1))$ of the form required in Theorem 4.4.1, then applying Theorem 3.2.14 in conjunction with Corollary 3.2.12, we obtain

$$\frac{X_2 \lrcorner\, \Omega}{X_1 \lrcorner\, X_2 \lrcorner\, \Omega} = d\gamma^1,$$
$$\frac{X_1 \lrcorner\, \Omega}{X_2 \lrcorner\, X_1 \lrcorner\, \Omega} = d\gamma^2 - X_1(\gamma^2) d\gamma^1,$$

for some functionally independent $\gamma^1, \gamma^2 \in C^\infty(J^1(U^1, V^1))$. Expanding these equations gives

$$\frac{1}{X_1 \lrcorner\, X_2 \lrcorner\, \Omega} \begin{pmatrix} X_2 \lrcorner\, (F_2 dx - dz_1) & X_2 \lrcorner\, (dz - z_1 dx) \\ X_1 \lrcorner\, (dz_1 - F_2 dx) & X_1 \lrcorner\, (z_1 dx - dz) \end{pmatrix} \begin{pmatrix} dz - z_1 dx \\ dz_1 - F_2 dx \end{pmatrix}$$
$$= \begin{pmatrix} d\gamma^1 \\ d\gamma^2 - X_1(\gamma^2) d\gamma^1 \end{pmatrix}.$$

Now

$$\begin{vmatrix} X_2 \lrcorner\, (F_2 dx - dz_1) & X_2 \lrcorner\, (dz - z_1 dx) \\ X_1 \lrcorner\, (dz_1 - F_2 dx) & X_1 \lrcorner\, (z_1 dx - dz) \end{vmatrix} = \left(X_2 \lrcorner\, (F_2 dx - dz_1)\right)\left(X_1 \lrcorner\, (z_1 dx - dz)\right)$$
$$- \left(X_2 \lrcorner\, (z_1 dx - dz)\right)\left(X_1 \lrcorner\, (F_2 dx - dz_1)\right),$$
$$= X_2 \lrcorner\, X_1 \lrcorner\, \Omega \neq 0.$$

Hence

$$\begin{pmatrix} dz - z_1 dx \\ dz_1 - F_2 dx \end{pmatrix} = \begin{pmatrix} X_1 \lrcorner\, (dz - z_1 dx) & X_2 \lrcorner\, (dz - z_1 dx) \\ X_1 \lrcorner\, (dz_1 - F_2 dx) & X_2 \lrcorner\, (dz_1 - F_2 dx) \end{pmatrix} \begin{pmatrix} d\gamma^1 \\ d\gamma^2 - X_1(\gamma^2) d\gamma^1 \end{pmatrix}.$$

Thus from Theorem 4.3.1, setting $\gamma^1 = c^1, \gamma^2 = c^2$ for constant functions c^1, c^2 yields a lifted solution curve of the ODE in (4.10).

Chapter 5

First order PDEs

5.1 Introduction

This chapter presents some symmetry techniques for finding local solutions of single first order partial differential equations. We begin by using solvable symmetry structures and Theorem 3.2.14 to provide a simple technique for finding local solutions of first order quasilinear PDEs of one dependent variable and n independent variables of the form

$$f_1 \frac{\partial u}{\partial x^1} + \cdots + f_n \frac{\partial u}{\partial x^n} = k,$$

for any smooth $f_1, \ldots, f_n, k$ that are functions of $u, x^1, \ldots, x^n$. It is well-known that such PDEs can be solved by the method of characteristics using ordinary differential equations (see for example Duff [41]), and our aim here is to replace this approach with an algorithm using symmetry.

Next, we give two slightly more sophisticated symmetry methods for finding local solutions of first order (possibly non-linear) PDEs of one dependent variable and two independent variables. The first approach examines such PDEs of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}\right),$$

for smooth F, while the second technique is a simplification of the first but at the expense of only applying to first order PDEs of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, \frac{\partial u}{\partial x^1}\right),$$

for smooth F not involving the dependent variable.

The first part of our work on quasilinear PDEs has been partly motivated by papers from Edelen [47, 49] on *fundamental ideals*, as used in Chapter 4 for ODEs, and following [49], we begin this chapter by presenting some necessary background material. The second part of this chapter on general first order PDEs applies work by Vessiot [128, 129, 130], and includes a detailed review of this material.

5.2 Quasilinear PDEs

Beginning with a slight generalisation to m dependent variables, suppose we have a single quasilinear PDE of the form

$$\left(f_{11}\frac{\partial u^1}{\partial x^1}+\cdots+f_{n1}\frac{\partial u^1}{\partial x^n}\right)+\cdots+\left(f_{1m}\frac{\partial u^m}{\partial x^1}+\cdots+f_{nm}\frac{\partial u^m}{\partial x^n}\right)=k, \qquad (5.1)$$

where $x^1,\ldots,x^n$ are the independent variables, $u^1,\ldots,u^m$ are the dependent variables of the PDE, and f_{ij}, for all $1\le i\le n, 1\le j\le m$, and k are smooth functions of $x^1,\ldots,x^n,u^1,\ldots,u^m$.

The dependent variables will be renamed $z^1,\ldots,z^m$ and serve as coordinates for the space V^m, which is defined as some open, convex neighbourhood of $\mathbb{R}^m$. We assume U^n, with coordinates $x^1,\ldots,x^n$ is also some open, convex neighbourhood of $\mathbb{R}^n$. On the first jet bundle $J^1(U^n,V^m)$ with coordinates $x^1,\ldots,x^n,z^1,\ldots,z^m,z^1_1,\ldots,z^m_1$ we have the first-order contact system $\Omega^1(U^n,V^m)$ generated by

$$C^j := dz^j - z^j_1dx^1-\cdots-z^j_ndx^n, \qquad 1\le j\le m.$$

Define

$$F := \left(f_{11}z^1_1+\cdots+f_{n1}z^1_n\right)+\cdots+\left(f_{1m}z^m_1+\cdots+f_{nm}z^m_n\right)-k.$$

Following Theorem 2.5.1, a local solution of the PDE in (5.1) can be thought of as a transverse immersion of rank n mapping into the locus of $J^1(U^n,V^m)$ described by the equation $F=0$, that also satisfies the nm partial derivative relations

$$z^j_i=\frac{\partial z^j}{\partial x^i}, \qquad 1\le i\le n, 1\le j\le m,$$

where the transverse nature of the immersion means that each z^j and z^j_i can be parameterised by $x^1,\ldots,x^n$.

Using Corollary 2.4.11 we have the following basic result:

Theorem 5.2.1. *Let there exist a (rank n) immersion*

$$\Phi : U^n \longrightarrow J^1(U^n, V^m),$$

satisfying the following $(m+2)$-conditions:

1. $\Phi^* C^j = 0$ *for all* $1 \leq j \leq m$,

2. $\Phi^* F = 0$,

3. $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$.

Then locally, $\Phi(U^n) = j^1 f(U^n)$ *for some smooth solution map* $f : U^n \longrightarrow V^m$ *of the PDE in (5.1).*

5.3 Ideals of quasilinear PDEs

Following Edelen [43, 45, 47, 49], and introduced in Section 2.5.2 in Chapter 2, we denote I_F as the *fundamental ideal* of the PDE in (5.1), i.e.

$$I_F := \langle C^1, \ldots, C^m, dC^1, \ldots, dC^m, F dx^1 \wedge \cdots \wedge dx^n \rangle.$$

From Lemma 2.5.3, it follows that

$$d\left(F dx^1 \wedge \cdots \wedge dx^n\right) \equiv 0 \mod C^1, \ldots, C^m, dC^1, \ldots, dC^m. \tag{5.2}$$

This means that I_F is in fact a differential ideal. Our aim is to look for an n-dimensional integral manifold of I_F, i.e. an immersion

$$\Phi : U^n \longrightarrow J^1(U^n, V^m),$$

such that $\Phi^* I_F = 0$ and $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$. Such an immersion obviously satisfies items (1) and (3) in Theorem 5.2.1. Item (2) in the theorem is seen to be satisfied if we recall that

$$0 = \Phi^*(F dx^1 \wedge \cdots \wedge dx^n) = (\Phi^* F)\Phi^*(dx^1 \wedge \cdots \wedge dx^n)$$

implies that $\Phi^* F = 0$, using item (3). Therefore, from Theorem 5.2.1 we obtain the following:

Theorem 5.3.1. *With I_F defined as above corresponding to the PDE in* (5.1)*, suppose the immersion*

$$\Phi : U^n \longrightarrow J^1(U^n, V^m),$$

is an n-dimensional integral manifold of I_F such that $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$. Then $\Phi(U^n) = j^1 f(U^n)$ for some smooth solution map $f : U^n \longrightarrow V^m$ of the PDE in (5.1).

We now show in the following theorem that the first order quasilinear nature of our PDE means the n-form $F dx^1 \wedge \cdots \wedge dx^n$ can be simplified somewhat so that, modolo $C^1, \ldots, C^m$, it does not depend on any of the first derivative coordinates $z_1^1, \ldots, z_n^1, \ldots, z_1^m, \ldots, z_n^m$.

Theorem 5.3.2.

$$I_F = \langle C^1, \ldots, C^m, dC^1, \ldots, dC^m, K \rangle,$$

where

$$\begin{aligned} K := \big(f_{11} dz^1 + \cdots + f_{1m} dz^m\big) \wedge dx^2 \wedge \cdots \wedge dx^n + \ldots \\ + dx^1 \wedge \cdots \wedge dx^{n-1} \wedge \big(f_{n1} dz^1 + \cdots + f_{nm} dz^m\big) \\ - k dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Proof. We have that

$$\begin{aligned} F dx^1 \wedge \cdots \wedge dx^n = \{ \big(f_{11} z_1^1 + \cdots + f_{n1} z_n^1\big) + \ldots \\ + (f_{1m} z_1^m + \cdots + f_{nm} z_n^m) - k \} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Now for any given $1 \leq i \leq n$ and $1 \leq j \leq m$ (no sum on i or j),

$$\begin{aligned} f_{ij} z_i^j dx^1 \wedge \cdots \wedge dx^n &= f_{ij} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge z_i^j dx^i \wedge dx^{i+1} \wedge \cdots \wedge dx^n \\ &= f_{ij} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \big(du^j - C^j\big) \wedge dx^{i+1} \wedge \cdots \wedge dx^n \\ &\equiv f_{ij} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge du^j \wedge dx^{i+1} \wedge \cdots \wedge dx^n \mod C^j, \end{aligned}$$

where in the second line we have used that $dz^j - C^j = z_1^j dx^1 + \cdots + z_i^j dx^i + \cdots + z_n^j dx^n$.

Therefore,

$$\begin{aligned}
F dx^1 \wedge \cdots \wedge dx^n \equiv \{ & \left(f_{11} dz^1 \wedge dx^2 \wedge \cdots \wedge dx^n \right) + \dots \\
& + \left(f_{n1} dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^1 \right) \} + \dots \\
& + \{ \left(f_{1m} dz^m \wedge dx^2 \wedge \cdots \wedge dx^n \right) + \dots \\
& + \left(f_{nm} dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^m \right) \} \\
& - k dx^1 \wedge \cdots \wedge dx^n \mod C^1, \dots, C^m.
\end{aligned}$$

We can collect terms, thus yielding

$$\begin{aligned}
F dx^1 \wedge \cdots \wedge dx^n \equiv & \left(f_{11} dz^1 + \cdots + f_{1m} dz^m \right) \wedge dx^2 \wedge \cdots \wedge dx^n + \dots \\
& + dx^1 \wedge \cdots \wedge dx^{n-1} \wedge \left(f_{n1} dz^1 + \cdots + f_{nm} dz^m \right) \\
& - k dx^1 \wedge \cdots \wedge dx^n \quad \mod C^1, \dots, C^m, \\
\equiv & K \quad \mod C^1, \dots, C^m.
\end{aligned}$$

To complete the proof, since

$$K \equiv F dx^1 \wedge \cdots \wedge dx^n \quad \mod C^1, \dots, C^m,$$

using (5.2) we obtain

$$\begin{aligned}
dK &\equiv d \left(F dx^1 \wedge \cdots \wedge dx^n \right) \quad \mod C^1, \dots, C^m, \\
&\equiv 0 \quad \mod C^1, \dots, C^m, dC^1, \dots, dC^m.
\end{aligned}$$

Hence the result. □

We define

$$I_{\overline{F}} := \langle C^1, \dots, C^m, dC^1, \dots, dC^m, K \rangle.$$

Once again we follow the notation introduced following Theorem 4.2.1 in Chapter 4 and use $I_{\overline{F}}$ as brief way of referring to the particular choice of generators $C^1, \dots, C^m, dC^1, \dots, dC^m, K$.

Theorem 5.3.2 now means that the task of determining solutions of (5.1) becomes that of finding n-dimensional integral manifolds of $I_{\overline{F}}$. Note that the n-form K in the ideal contains no first order derivative coordinates. In the following section we use this feature of K to show how a solvable structure of symmetries can further simplify K, so that we obtain an algorithmic approach based on symmetry for extracting local solutions of single first order quasilinear PDEs of the form in (5.1).

We end this introductory section with an obvious result that uses Theorems 5.2.1 and 5.3.2:

Theorem 5.3.3. *Let*

$$\Phi : U^n \longrightarrow U^n \times V^m,$$

be an immersion such that $\Phi^* K = 0$ *and* $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$. *Then* $\Phi(U^n) = j^0 f(U^n)$ *for some solution map* $f : U^n \longrightarrow V^m$ *of the PDE in (5.1).*

Theorem 5.3.3 means that if the pull-back of a rank n immersion mapping into the graph space satisfies the transverse condition and annihilates K, then this is enough to guarantee in this case that the 1-jet $j^1 f$ is an n-dimensional integral manifold of the differential ideal $I_{\overline{F}}$ (and hence a local solution of PDE in (5.3)). This is because K contains no first order derivative coordinates and all the contact forms are automatically annihilated.

5.4 Quasilinear PDEs of one dependent variable

This section addresses single quasilinear PDEs of one dependent variable of the form

$$f_1 \frac{\partial u}{\partial x^1} + \cdots + f_n \frac{\partial u}{\partial x^n} = k, \tag{5.3}$$

where $f_1, \ldots, f_n, k \in C^\infty(U^n \times V^1)$. For this PDE, the corresponding K in $I_{\overline{F}}$ is

$$\begin{aligned} K = & \left(f_1 dz^1 \wedge dx^2 \wedge \cdots \wedge dx^n\right) + \cdots + \left(f_n dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^1\right) \\ & - k dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Now K is an n-form in the $(n+1)$-dimensional space $U^n \times V^1$. From Corollary 3.3.2 and Theorem 3.3.3, respectively, it follows that K is decomposable and $dK \equiv 0 \bmod K$. Suppose we are given n non-trivial symmetries $X_1, \ldots, X_n \in \mathfrak{X}(U^n \times V^1)$ such that

$$\begin{aligned} \mathcal{L}_{X_n} K &= \lambda_n K, \\ \mathcal{L}_{X_{n-1}}(X_n \lrcorner K) &= \lambda_{n-1}(X_n \lrcorner K), \\ &\vdots \\ \mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K) &= \lambda_1 (X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K), \end{aligned}$$

for some $\lambda_1, \dots, \lambda_n \in C^\infty(U^n \times V^1)$. Applying Theorem 3.2.14 in conjunction with Corollary 3.2.12, we can explicitly compute some $\gamma^0, \dots, \gamma^n \in C^\infty(U^n \times V^1)$ so that

$$K = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^n.$$

Now consider the n-dimensional regular submanifold of $U^n \times V^1$ described by

$$H(\gamma^1, \dots, \gamma^n) = 0, \tag{5.4}$$

where H is any non-constant smooth function of $\gamma^1, \dots, \gamma^n$. It is assumed H is constant rank one on $U^n \times V^1$. Then

$$dH = \frac{\partial H}{\partial \gamma^1} d\gamma^1 + \cdots + \frac{\partial H}{\partial \gamma^n} d\gamma^n \overset{*}{=} 0, \tag{5.5}$$

where we use $\overset{*}{=}$ to mean equality on tangent space of the submanifold of $U^n \times V^1$ described by (5.4). We must have that at each point of this submanifold there exists some $1 \leq p \leq n$ such that

$$\frac{\partial H}{\partial \gamma^p} \neq 0.$$

Otherwise H is independent of all $\gamma^1, \dots, \gamma^n$ at some point, but it is assumed H is constant rank one. Now from inserting (5.5),

$$\begin{aligned} 0 &\overset{*}{=} dH \wedge d\gamma^1 \wedge \cdots \wedge d\gamma^{p-1} \wedge d\gamma^{p+1} \wedge \cdots \wedge d\gamma^n \\ &= \frac{\partial H}{\partial \gamma^p} d\gamma^p \wedge d\gamma^1 \wedge \cdots \wedge d\gamma^{p-1} \wedge d\gamma^{p+1} \wedge \cdots \wedge d\gamma^n. \end{aligned}$$

This implies that $K = 0$ on the submanifold described by equation (5.4).

If at some point in $U^n \times V^1$ we have

$$\frac{\partial H}{\partial z^1} \neq 0,$$

then by the implicit function theorem, we can write in some neighbourhood of the point

$$z^1 = \overline{H}(x^1, \dots, x^n), \tag{5.6}$$

for some smooth $\overline{H}$. Therefore $j^0\overline{H}^* K = 0$ (and hence $j^1\overline{H}^* I_{\overline{F}} = 0$). Since $j^0\overline{H}^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$, Theorem 5.3.3 means that equation (5.6) then represents a local solution of the quasilinear PDE in (5.3).

We summarise the above result in the following theorem:

Theorem 5.4.1. *Suppose we have a first order quasilinear PDE of the form*

$$f_1 \frac{\partial u}{\partial x^1} + \cdots + f_n \frac{\partial u}{\partial x^n} = k, \tag{5.7}$$

for some $f_1, \ldots, f_n, k \in C^\infty(U^n \times V^1)$*, with the corresponding* K *in* $I_{\overline{F}}$ *as*

$$K := \left(f_1 dz^1 \wedge dx^2 \wedge \cdots \wedge dx^n\right) + \cdots + \left(f_n dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^1\right)$$
$$- k dx^1 \wedge \cdots \wedge dx^n.$$

If there exist n *non-trivial symmetries* $X_1, \ldots, X_n \in \mathfrak{X}(U^n \times V^1)$ *such that*

$$\mathcal{L}_{X_n} K = \lambda_n K,$$
$$\mathcal{L}_{X_{n-1}}(X_n \lrcorner K) = \lambda_{n-1}(X_n \lrcorner K),$$
$$\vdots$$
$$\mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K) = \lambda_1 (X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K),$$

for some $\lambda_1, \ldots, \lambda_n \in C^\infty(U^n \times V^1)$*, then there exist some functionally independent* $\gamma^1, \ldots, \gamma^n \in C^\infty(U^n \times V^1)$ *so that*

$$K = K(X_1, \ldots, X_n) d\gamma^1 \wedge \cdots \wedge d\gamma^n.$$

Furthermore, any regular submanifold of $U^n \times V^1$ *given by*

$$H(\gamma^1, \ldots, \gamma^n) = 0,$$

for any smooth H *such that* $\frac{\partial H}{\partial z^1} \neq 0$ *on some neighbourhood, is then the graph space coordinate representation of a local solution of the PDE in (5.7).*

Remark. Since K is decomposable and closed modulo itself, from Corollary 3.2.12 and Theorem 2.3.11, we may use `DIMSYM` to generate the required symmetries in Theorem 5.4.1 by simply searching for a solvable symmetry structure for $\ker(K)$.

Before we show how Theorem 5.4.1 may be used in an example, we make an interesting comparison between the work in Chapter 4 applied to first order ODEs and this theorem in the situation when there is one independent variable. Under such circumstances, the equations in (4.1) and (5.7) are both linear in their derivatives. Consequently, it is not hard to see that Theorems 4.2.1 and 5.3.2 become very similar. In fact, for first order differential equations linear in their derivatives, the symmetry approach given in Theorems 4.4.1 using the one-form Ω is indeed a special case of the more general symmetry technique contained in Theorem 5.4.1 that involves the n-form K.

Next, we illustrate Theorem 5.4.1 with the following example:

Example 5.4.2. Consider the following first order quasilinear PDE of two independent variables and one dependent variable:

$$x^1 \frac{\partial u}{\partial x^1} - x^2 \frac{\partial u}{\partial x^2} = x^2 \exp(u). \tag{5.8}$$

Our corresponding two-form K on $U^2 \times V^1$ is

$$K = x^1 dz^1 \wedge dx^2 + x^2 dz^1 \wedge dx^1 - x^2 \exp(z^1) dx^1 \wedge dx^2.$$

Using `DIMSYM` we find that

$$X_2 := \frac{1}{x^2} \frac{\partial}{\partial x^1}$$

is a non-trivial symmetry of K. Then with

$$X_2 \lrcorner K = -dz^1 - \exp(z^1) dx^2,$$

it is easy to see that

$$X_1 := \frac{\partial}{\partial x^2}$$

is a non-trivial symmetry of $X_2 \lrcorner K$. Hence from Theorem 3.2.14 and Corollary 3.2.12,

$$\frac{X_2 \lrcorner K}{X_1 \lrcorner X_2 \lrcorner K} = d\left(x^2 - \exp(-z^1)\right), \tag{5.9}$$

and

$$\begin{aligned} \frac{X_1 \lrcorner K}{X_2 \lrcorner X_1 \lrcorner K} &= -x^1 \exp(-z^1) dz^1 + x^2 dx^1, \\ &= d(x^1 x^2) - x^1 d\left(x^2 - \exp(-z^1)\right). \end{aligned} \tag{5.10}$$

Therefore

$$K = K(X_1, X_2) d\left(x^2 - \exp(-z^1)\right) \wedge d(x^1 x^2).$$

One can then say that

$$H\left(x^1x^2, x^2-\exp(-z^1)\right)=0,$$

is, in implicit form in terms of the graph space coordinates, a local solution of (5.8) for any suitable smooth H. Thus

$$u=-\ln\left|x^2-l(x^1x^2)\right|,$$

gives local solutions for arbitrary choice of smooth l that is a function of x^1x^2.

By way of Example 5.4.2, we make the following observations: Firstly, if deducing (5.10) is not possible by simple inspection, then one can perform a coordinate transformation by making the function found in (5.9) a coordinate. Then (5.10) will take on a simpler appearance. In difficult problems, this is generally the preferred option, but quite often involves tedious algebraic manipulations.

Secondly, in determining (5.10), one is tempted to write

$$-x^1\exp(-z^1)dz^1+x^2dx^1=dg-hd\left(x^2-\exp(-z^1)\right), \tag{5.11}$$

for some choice of $g,h\in C^\infty(U^2\times V^1)$. Then once we know h, finding g becomes a simple matter. Examining this further, taking the exterior derivative of both sides of (5.11) gives

$$\exp(-z^1)dz^1\wedge dx^1-dx^1\wedge dx^2=dh\wedge dx^2-\exp(-z^1)dz^1\wedge dh.$$

Expanding this yields

$$\begin{aligned}\exp(-z^1)dz^1\wedge dx^1-dx^1\wedge dx^2=&\frac{\partial h}{\partial x^1}dx^1\wedge dx^2\\&+\frac{\partial h}{\partial z^1}dz^1\wedge dx^2-\exp(-z^1)\frac{\partial h}{\partial x^1}dz^1\wedge dx^1-\exp(-z^1)\frac{\partial h}{\partial x^2}dz^1\wedge dx^2.\end{aligned}$$

Therefore h must satisfy the following system of first order linear PDEs:

$$\frac{\partial h}{\partial x^1}=-1,\qquad \exp(-z^1)\frac{\partial h}{\partial x^2}=\frac{\partial h}{\partial z^1}. \tag{5.12}$$

All we need is one (possibly trivial) solution of (5.12) in order to obtain local solutions of (5.8). Fortunately for this particular example it is easy to see that any smooth h such that $dh\equiv -dx^1 \bmod d\left(x^2-\exp(-z^1)\right)$ is a local solution of (5.12), however in general we will not be in this fortunate position. Hence we conclude that the first idea of performing a coordinate transformation remains the best option.

While the symmetries used in Theorem 5.4.1 do not have to be Lie point symmetries, there exists a relationship between Lie point symmetries and symmetries of K that we explore below.

First, we introduce the following definition:

Definition 5.4.3. A vector field $X \in \mathfrak{X}(U^n \times V^1)$ is said to be a *Lie point symmetry* of the first order quasilinear PDE in (5.3) if

$$X^{(1)}(F) = 0,$$

whenever $F = 0$, where $X^{(1)}$ is the first prolongation of X.

Using this we obtain the following:

Theorem 5.4.4. *Given a first order quasilinear PDE of the form in (5.3), a vector field $X \in \mathfrak{X}(U^n \times V^1)$ is a symmetry of its corresponding n-form K if and only if X is a Lie point symmetry of the PDE.*

Proof. First suppose $X \in \mathfrak{X}(U^n \times V^1)$ is a symmetry of K corresponding to the quasilinear PDE in (5.3), i.e. $\mathcal{L}_X K = \lambda K$ for some $\lambda \in C^\infty(U^n \times V^1)$. Since K does not contain any first derivative coordinates, we can write

$$\mathcal{L}_{X^{(1)}} K = \lambda K, \tag{5.13}$$

with (5.13) defined on the first jet bundle $J^1(U^n, V^1)$. Now $K \equiv F dx^1 \wedge \cdots \wedge dx^n$ mod C^1, so

$$\mathcal{L}_{X^{(1)}} \left(F dx^1 \wedge \cdots \wedge dx^n \wedge C^1 - K \wedge C^1 \right) = 0. \tag{5.14}$$

It is well-known (and not hard to show) that for any point symmetry, the Lie derivative of any first order contact form with respect to the first prolongation of the symmetry is a contact form [45]. So putting $\mathcal{L}_{X^{(1)}} C^1 = \rho C^1$ for some $\rho \in C^\infty(J^1(U^n, V^1))$, we have from (5.14),

$$\begin{aligned} X^{(1)}(F) dx^1 \wedge \cdots \wedge dx^n \wedge C^1 + F \mathcal{L}_{X^{(1)}} \left(dx^1 \wedge \cdots \wedge dx^n \wedge C^1 \right) \\ = (\lambda + \rho) K \wedge C^1. \end{aligned} \tag{5.15}$$

Now $Fdx^1 \wedge \cdots \wedge dx^n \wedge C^1 = K \wedge C^1$ and $dx^1 \wedge \cdots \wedge dx^n \wedge C^1 = dx^1 \wedge \cdots \wedge dx^n \wedge dz^1$. Hence

$$X^{(1)}(F) dx^1 \wedge \cdots \wedge dx^n \wedge dz^1 = 0,$$

whenever $F = 0$. This implies that $X^{(1)}(F) = 0$ whenever $F = 0$.

Conversely, suppose that X is a Lie point symmetry of the quasilinear PDE in (5.3), i.e.

$$X^{(1)}(F) = 0, \tag{5.16}$$

whenever $F = 0$. Now

$$\begin{aligned}\mathcal{L}_{X^{(1)}}\left(K \wedge C^1\right) &= \mathcal{L}_{X^{(1)}}\left(F dx^1 \wedge \cdots \wedge dx^n \wedge C^1\right), \\ &= X^{(1)}(F) dx^1 \wedge \cdots \wedge dx^n \wedge C^1 \\ &\qquad + F\mathcal{L}_{X^{(1)}}\left(dx^1 \wedge \cdots \wedge dx^n \wedge C^1\right).\end{aligned}$$

Therefore from (5.16),

$$\mathcal{L}_{X^{(1)}}\left(K \wedge C^1\right) = 0,$$

whenever $F = 0$. Expanding, and using the fact that $\mathcal{L}_{X^{(1)}} C^1 = \rho C^1$ for some $\rho \in C^\infty(J^1(U^n, V^1))$, we obtain

$$(\mathcal{L}_{X^{(1)}} K) \wedge C^1 + \rho K \wedge C^1 = 0,$$

whenever $F = 0$. Then using $K \wedge C^1 = F dx^1 \wedge \cdots \wedge dx^n \wedge C^1$, we find

$$(\mathcal{L}_{X^{(1)}} K) \wedge C^1 = 0,$$

whenever $F = 0$. We also have

$$(\mathcal{L}_{X^{(1)}} K) \wedge C^1 = L dx^1 \wedge \cdots \wedge dx^n \wedge dz^1, \tag{5.17}$$

for some $L \in C^\infty(J^1(U^n, V^1))$ because (5.13) implies $\mathcal{L}_{X^{(1)}} K$ is an n-form expressed entirely in terms of the $n+1$ coordinates of the graph space. Furthermore, from the definition of C^1 we obtain that L is linear in z^1_1 and z^1_2. Since (5.17) is zero whenever $F = 0$ and F is also linear in z^1_1 and z^1_2, we can therefore say that $L = hF$ for some $h \in C^\infty(U^n \times V^1)$. Since $hF dx^1 \wedge \ldots dx^n \wedge dz^1 = hK \wedge C^1$, we may write

$$\mathcal{L}_{X^{(1)}} K \equiv hK \mod C^1.$$

Hence

$$\mathcal{L}_X K \equiv hK \mod C^1.$$

As h is expressed only in terms of coordinates of the graph space, we therefore have $\mathcal{L}_X K = hK$. □

Theorem 5.4.4 has the following corollary:

Corollary 5.4.5. *A vector field $X \in \mathfrak{X}(U^n \times V^1)$ is a Lie point symmetry of the first order quasilinear PDE in (5.3) if and only if $X^{(1)}$ is a symmetry of its corresponding $I_{\overline{F}}$.*

5.5 First order non-linear PDEs

In this section we examine two approaches to solving single first order non-linear PDEs of one dependent variable and two independent variables. The first involves using Vessiot theory while the second employs a simpler technique for the special case when the PDE does not explicitly involve the dependent variable. We begin with the former.

5.5.1 Vessiot theory

This section summarises the main points of Vessiot's theory [128, 129, 130] of differential equations, as reviewed by Fackerell [52], Stormark [119] and Vassiliou [124, 126].

Consider the system of ρ PDEs of n independent and m dependent variables

$$F_\nu(x^i, u^j, u^j_{i_1}, u^j_{i_1 i_2}, \ldots, u^j_{i_1 \cdots i_\kappa}) = 0, \qquad \nu = 1, \ldots, \rho, \tag{5.18}$$

where the n x^i and m u^j are, respectively, the independent and dependent variables. The subscripts $1 \leq i_1 \leq \cdots \leq i_\kappa \leq n$ are used to specify partial derivatives of u^j, where κ is the maximum order of the system.

In the κ-th jet bundle $J^\kappa(U^n, V^m)$ with coordinates $x^i, z^j, z^j_{i_1}, z^j_{i_1 i_2}, \ldots, z^j_{i_1 \cdots i_\kappa}$, we may express a solution of the system of PDEs above as a regular n-dimensional submanifold that

1. Satisfies the relations $F_\nu(x^i, z^j, z^j_{i_1}, z^j_{i_1 i_2}, \ldots, z^j_{i_1\cdots i_\kappa}) = 0$, for all $\nu = 1, \ldots, \rho$,

2. Satisfies the transverse requirement,

3. Has a tangent space that annihilates the κ-th order contact system generated by (with sum):

$$\begin{aligned} C^j &:= dz^j - z^j_{i_1} dx^{i_1}, \\ C^j_{i_1} &:= dz^j_{i_1} - z^j_{i_1 i_2} dx^{i_2}, \\ C^j_{i_1 i_2} &:= dz^j_{i_1 i_2} - z^j_{i_1 i_2 i_3} dx^{i_3}, \\ &\vdots \\ C^j_{i_1\cdots i_{\kappa-1}} &:= dz^j_{i_1\cdots i_{\kappa-1}} - z^j_{i_1\cdots i_\kappa} dx^{i_\kappa}. \end{aligned}$$

We denote the contact system by $\Omega^\kappa(U^n, V^m)$, that includes all $1 \le j \le m$.

From Corollary 2.4.11, if $\Phi : U^n \longrightarrow J^\kappa(U^n, V^m)$ is any immersion whose pull-back annihilates the contact system $\Omega^\kappa(U^n, V^m)$, and satisfies the transverse condition $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$, then $\Phi(U^n)$ is the image of some κ-jet. To incorporate the system of PDEs into the contact system, we introduce a rank ρ immersion Φ_F mapping onto the regular submanifold of the κ-th jet bundle described by the PDEs $\{F_\nu = 0 : \nu = 1, ..., \rho\}$. We then pull-back the contact forms in $\Omega^\kappa(U^n, V^m)$ by Φ_F. The *Vessiot distribution* is then defined as the vector field dual of the pulled-back contact system, i.e. $({\Phi_F}^*\Omega^\kappa(U^n, V^m))^\perp$. Using the Vessiot distribution, our task is to look for some immersion Φ of rank n that maps into the image of Φ_F and annihilates the contact system, while at the same time being transverse.

We illustrate with a simple example:

Example 5.5.1. Suppose we have a single PDE of two dependent variables u^1, u^2 and two independent variables x^1, x^2 given by

$$u^1_{22} = F(x^1, x^2, u^1, u^2, u^1_1, u^2_1, u^1_2, u^2_2, u^1_{11}, u^1_{12}, u^2_{11}, u^2_{12}, u^2_{22}). \tag{5.19}$$

Then a local solution of the PDE is a two-dimensional regular submanifold of the thirteen-dimensional locus of $J^2(U^2, V^2)$ described by the map $\Phi_F : \Sigma \longrightarrow J^2(U^2, V^2)$, where

$$\begin{aligned} \Phi_F : &(x^1, x^2, z^1, z^2, z^1_1, z^2_1, z^1_2, z^2_2, z^1_{11}, z^1_{12}, z^2_{11}, z^2_{12}, z^2_{22}) \\ &\longmapsto (x^1, x^2, z^1, z^2, z^1_1, z^2_1, z^1_2, z^2_2, z^1_{11}, z^1_{12}, F, z^2_{11}, z^2_{12}, z^2_{22}). \end{aligned}$$

Thus from the discussion immediately before this example, the image of a rank two immersion mapping into $\Phi_F(\Sigma)$ is the 2-jet image of a local solution map of the PDE if it annihilates the contact system and the transverse condition is satisfied. Explicitly,

$$\begin{aligned}\Omega^2(U^2,V^2) = Sp\{&dz^1 - z^1_1 dx^1 - z^1_2 dx^2, dz^2 - z^2_1 dx^1 - z^2_2 dx^2,\\ &dz^1_1 - z^1_{11} dx^1 - z^1_{12} dx^2, dz^2_1 - z^2_{11} dx^1 - z^2_{12} dx^2,\\ &dz^1_2 - z^1_{12} dx^1 - z^1_{22} dx^2, dz^2_2 - z^2_{12} dx^1 - z^2_{22} dx^2\}.\end{aligned}$$

Pulling this back onto the regular submanifold of $J^2(U^2, V^2)$ described by (5.19), we get

$$\begin{aligned}\Phi^*_F\Omega^2(U^2,V^2) = Sp\{&dz^1 - z^1_1 dx^1 - z^1_2 dx^2, dz^2 - z^2_1 dx^1 - z^2_2 dx^2,\\ &dz^1_1 - z^1_{11} dx^1 - z^1_{12} dx^2, dz^2_1 - z^2_{11} dx^1 - z^2_{12} dx^2,\\ &dz^1_2 - z^1_{12} dx^1 - F dx^2, dz^2_2 - z^2_{12} dx^1 - z^2_{22} dx^2\}.\end{aligned}$$

Therefore the Vessiot distribution is

$$\begin{aligned}\left(\Phi^*_F\Omega^2(U^2,V^2)\right)^{\perp} = Sp\Big\{&\frac{\partial}{\partial x^1} + z^1_1\frac{\partial}{\partial z^1} + z^2_1\frac{\partial}{\partial z^2} + z^1_{11}\frac{\partial}{\partial z^1_1} + z^2_{11}\frac{\partial}{\partial z^2_1} + z^1_{12}\frac{\partial}{\partial z^1_2}\\ &+ z^2_{12}\frac{\partial}{\partial z^2_2}, \frac{\partial}{\partial x^2} + z^1_2\frac{\partial}{\partial z^1} + z^2_2\frac{\partial}{\partial z^2} + z^1_{12}\frac{\partial}{\partial z^1_1} + z^2_{12}\frac{\partial}{\partial z^2_1}\\ &+ F\frac{\partial}{\partial z^1_2} + z^2_{22}\frac{\partial}{\partial z^2_2}, \frac{\partial}{\partial z^1_{11}}, \frac{\partial}{\partial z^1_{12}}, \frac{\partial}{\partial z^2_{11}}, \frac{\partial}{\partial z^2_{12}}, \frac{\partial}{\partial z^2_{22}}\Big\}.\end{aligned}$$

Given a Vessiot distribution for some arbitrary system of ρ PDEs of n independent and m dependent variables, we look for an n-dimensional Frobenius integrable subdistribution that satisfies the transverse condition. This is done in stages by generating a finite sequence of higher dimensional subdistributions, each containing the previous, beginning with dimensional one and ending at dimension n. We describe this below:

Definition 5.5.2. For a vector field distribution D on a smooth manifold M, the submodule E of D is said to be an *involution* if $[X, Y] \equiv 0 \bmod D$ for all $X, Y \in E$.

If E is Frobenius integrable, then it is an involution. Moreover, if E is spanned by a single vector field, say for example representing the solution curve an ordinary differential equation field, then it is trivially an involution.

Given some b-dimensional Vessiot distribution $D_F = Sp\{X_1, \ldots, X_b\}$ in $\mathfrak{X}(J^\kappa(U^n, V^m))$ corresponding to some system of PDEs in (5.18), the process of generating an n-dimensional submanifold involves first setting up a chain of lower dimensional involutions up to dimension n, where in each step, the next involution is contained in the previous. Beginning with one-dimensional involutions, since every vector field in D_F generates a one-dimensional involution, we let the distribution spanned by $Y_1 := a_1^k X_k$ generate our involution, where the a_1^k are any smooth functions defined on the $(\dim(J^\kappa(U^n, V^m)) - \rho)$-dimensional regular submanifold of $J^\kappa(U^n, V^m)$ described by the PDEs. Given any Y_1, we typically distinguish between two types of involutions, namely those *regular* and those *singular*. In determining which of the two our one-dimensional involution may be, a two-dimensional involution containing it is constructed. We do this by first defining $Y_2 := a_2^k X_k$ for some smooth a_2^k. Then the requirement that $[Y_1, Y_2] \equiv 0 \bmod D_F$ generates a system of linear algebraic equations, which in matrix form is

$$\mathbf{M}(Y_1) \cdot \mathbf{a_2} = \mathbf{0}, \tag{5.20}$$

where

$$\mathbf{a_2} := \begin{pmatrix} a_2^1 \\ \vdots \\ a_2^b \end{pmatrix}.$$

Define $s := \operatorname{rank}(\mathbf{M}(Y_1))$. In general, over all involutions of dimension one, $\mathbf{M}(Y_1)$ will have a maximal rank s_1. If $\operatorname{rank}(\mathbf{M}(Y_1)) = s_1$, then the one-dimensional involution is said to be *regular*. If, however, $\operatorname{rank}(\mathbf{M}(Y_1)) < s_1$, then we say that the involution is *singular*. If $\operatorname{rank}(\mathbf{M}(Y_1)) = 0$, then $[Y_1, Y_2] \in D_F$ for all Y_2, and here we can say further that the singular one-dimensional involution is *characteristic*. Once we have a Y_1 that generates some one-dimensional involution, we then look for all possible two-dimensional involutions of D_F containing Y_1 by solving (5.20) for some Y_2.

The process continues until we have an n-dimensional involution that may be regular or singular. To illustrate further, suppose we are give some j-dimensional involution and wish to find a $(j+1)$-dimensional involution containing it. First define $Y_{j+1} := a_{j+1}^k X_k$. Then the requirement that $[Y_i, Y_{j+1}] \equiv 0 \bmod D_F$ for all

$i = 1, \ldots, j$ generates a system of linear algebraic equations, which in matrix form is

$$\mathbf{M}(Y_1, \ldots, Y_j) \cdot \mathbf{a_{j+1}} = \mathbf{0},$$

where

$$\mathbf{a_{j+1}} := \begin{pmatrix} a^1_{j+1} \\ \vdots \\ a^b_{j+1} \end{pmatrix}.$$

Once again define $s := \text{rank}(\mathbf{M}(Y_1, \ldots, Y_j))$. Over all involutions of dimension j, let s_j be the maximal rank of the matrix. If $s = s_j$, the j-dimensional involution is regular. If $s < s_j$, then the involution is singular. If $s = 0$, then $[Y_i, Y_{j+1}] \in D_F$ for all $i = 1, \ldots, j$ and the singular involution is characteristic.

A j-dimensional involution is regular if the rank of the matrix used to determine all $(j+1)$-dimensional involutions containing it is *maximised.* In the subset of the Grassmann bundle of j-planes consisting of all j-dimensional involutions of D_F, those which are regular form a dense open subset of this space. Therefore all j-dimensional involutions of D_F in some neighbourhood of a regular j-dimensional involution are also regular. For a characteristic j-dimensional involution, any choice of vector field in D_F that is linearly independent of any vector field in the involution will generate a singular (and not necessarily characteristic) $(j+1)$-dimensional involution containing the characteristic involution. If at some stage during the process of building up a chain of higher dimensional involutions we have a singular subinvolution, then our n-dimensional involution at the end of the process will also be singular.

For any Vessiot distribution, the maximal dimension of the regular involutions in the system is defined to be the genus g. In many situations, g will be greater than or equal to the dimension of the desired involutions for the particular PDE problem at hand, which will be n, the number of independent variables. Problems arise when we are looking for n-dimensional involutions when $g < n$. One way around this is to first find a *singular* g-dimensional involution. The rank of $\mathbf{M}(Y_1, \ldots, Y_g)$ is then not at a maximum, so it will be possible to find a singular $(g+1)$-dimensional involution containing the g-dimensional involution.

Once we have an n-dimensional regular or singular involution, the final requirement that the distribution be Frobenius integrable will then give us a system of first order quasilinear PDEs where the arbitrary functions are the dependent variables. We take up this issue in the next section.

5.5.2 Application of Vessiot theory to first order PDEs

In this section we use Vessiot theory to examine symmetry solutions of single first order PDEs of one dependent variable and two independent variables. Suppose then that we are given a first order PDE of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}\right),$$

for some smooth function F. On $J^1(U^2, V^1)$ with coordinates $x^1, x^2, z^1, z^1_1, z^1_2$, our first order contact system is generated by the element

$$C^1 := dz^1 - z^1_1 dx^1 - z^1_2 dx^2.$$

Restricted to the regular submanifold $M^4 \subset J^1(U^2, V^1)$ described by

$$z^1_2 = F(x^1, x^2, z^1, z^1_1),$$

the contact system on M^4 with coordinates x^1, x^2, z^1, z^1_1 is generated by

$$C^1 = dz^1 - z^1_1 dx^1 - F dx^2.$$

The Vessiot distribution D_F is generated by

$$\begin{aligned} X_1 &:= \frac{\partial}{\partial x^1} + z^1_1 \frac{\partial}{\partial z^1}, \\ X_2 &:= \frac{\partial}{\partial x^2} + F \frac{\partial}{\partial z^1}, \\ X_3 &:= \frac{\partial}{\partial z^1_1}. \end{aligned}$$

In looking for a one-dimensional involution of D_F which is regular, let

$$Y_1 := a^k_1 X_k, \qquad Y_2 := a^k_2 X_k.$$

We have the commutator relations

$$[X_1, X_2] = X_1(F)\frac{\partial}{\partial z^1}, \qquad [X_1, X_3] = -\frac{\partial}{\partial z^1}, \qquad [X_2, X_3] = -X_3(F)\frac{\partial}{\partial z^1},$$

with all others zero. Demanding that $[Y_1, Y_2] \equiv 0 \bmod D_F$ means

$$\left(a_1^1 a_2^2 - a_1^2 a_2^1\right) X_1(F) + \left(a_1^3 a_2^2 - a_1^2 a_2^3\right) X_3(F) - a_1^1 a_2^3 + a_1^3 a_2^1 = 0.$$

In matrix form,

$$\begin{pmatrix} -a_1^2 X_1(F) + a_1^3 & a_1^1 X_1(F) + a_1^3 X_3(F) & -a_1^2 X_3(F) - a_1^1 \end{pmatrix} \cdot \begin{pmatrix} a_2^1 \\ a_2^2 \\ a_2^3 \end{pmatrix} = 0. \qquad (5.21)$$

We choose a one-dimensional involution spanned by Y_1 by letting $a_1^1 = 1$ and $a_1^2 = 0$. Then

$$\begin{pmatrix} a_1^3 & X_1(F) + a_1^3 X_3(F) & -1 \end{pmatrix} \cdot$$

is rank one, and hence in a neighbourhood of one-dimensional involutions about $Sp\{Y_1\}$, the matrix on the left in (5.21) remains rank one. Therefore $Sp\{Y_1\}$ is a regular involution.

In looking for a two-dimensional involution satisfying the transverse condition, we let $a_2^1 = 0$ and $a_2^2 = 1$ so that (5.21) holds with

$$Y_1 = X_1 + a_1^3 X_3, \qquad Y_2 = X_2 + (X_1(F) + a_1^3 X_3(F)) X_3,$$

thus generating a two-dimensional involution for arbitrary a_1^3. To see that the involution is regular, let $Y_3 = a_3^k X_k$. Requiring that $[Y_1, Y_3] \equiv 0 \bmod D_F$ and $[Y_2, Y_3] \equiv 0 \bmod D_F$ means that

$$\begin{pmatrix} a_1^3 & X_1(F) + a_1^3 X_3(F) & -1 \\ a_1^3 X_3(F) & (X_1(F) + a_1^3 X_3(F))\, X_3(F) & -X_3(F) \end{pmatrix} \cdot \begin{pmatrix} a_3^1 \\ a_3^2 \\ a_3^3 \end{pmatrix} = \mathbf{0},$$

where the matrix on the left is of rank one. The space of all possible Y_3 must contain Y_1 and Y_2, so it follows that in a neighbourhood of the two-dimensional involution $Sp\{Y_1, Y_2\}$, this rank one condition must be maintained by dimension. Therefore $Sp\{Y_1, Y_2\}$ is a regular involution.

Given a two-dimensional involution spanned by Y_1 and Y_2, we finally require that it be Frobenius integrable. We introduce the condition $[Y_1, Y_2] = 0$ which forces a_1^3 to satisfy the following first order quasilinear PDE:

$$\begin{aligned} &- X_3(F) \frac{\partial a_1^3}{\partial x^1} + \frac{\partial a_1^3}{\partial x^2} + \left(F - z_1^1 X_3(F)\right) \frac{\partial a_1^3}{\partial z^1} + X_1(F) \frac{\partial a_1^3}{\partial z_1^1} \\ &\qquad = X_1\left(X_1(F)\right) + a_1^3 X_1\left(X_3(F)\right) + a_1^3 X_3\left(X_1(F)\right) + (a_1^3)^2 X_3\left(X_3(F)\right), \end{aligned}$$

where a_1^3 is some function of x_1, x_2, z^1, z_1^1. The problem is now reduced to that of finding a solution of a first order quasilinear PDE.

We can summarise the above in the following theorem:

Theorem 5.5.3. *Consider the first order PDE*

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}\right), \tag{5.22}$$

for smooth F. On the regular submanifold of $J^1(U^2, V^1)$ described by the equation $z_2^1 = F(x^1, x^2, z^1, z_1^1)$, let

$$X_1 := \frac{\partial}{\partial x^1} + z_1^1 \frac{\partial}{\partial z^1},$$
$$X_2 := \frac{\partial}{\partial x^2} + F \frac{\partial}{\partial z^1},$$
$$X_3 := \frac{\partial}{\partial z_1^1}.$$

Define the vector fields

$$Y_1 := X_1 + a_1^3 X_3, \qquad Y_2 := X_2 + (X_1(F) + a_1^3 X_3(F))X_3,$$

with a_1^3 satisfying the first order quasilinear PDE

$$\begin{aligned} &- X_3(F)\frac{\partial a_1^3}{\partial x^1} + \frac{\partial a_1^3}{\partial x^2} + \left(F - z_1^1 X_3(F)\right)\frac{\partial a_1^3}{\partial z^1} + X_1(F)\frac{\partial a_1^3}{\partial z_1^1} \\ &\quad = X_1\left(X_1(F)\right) + a_1^3 X_1\left(X_3(F)\right) + a_1^3 X_3\left(X_1(F)\right) + (a_1^3)^2 X_3\left(X_3(F)\right), \end{aligned} \tag{5.23}$$

where a_1^3 is some smooth function of x_1, x_2, z^1, z_1^1. Then $Sp\{Y_1, Y_2\}$ generates a two-dimensional regular submanifold of $J^1(U^2, V^1)$ that is the image of the 1-jet of some local solution of the PDE in (5.22).

Remark 1. Of course, solving (5.23) using Theorem 5.4.1 will generally yield a_1^3 in terms of an arbitrary function which typically cannot be left arbitrary when integrating $Sp\{Y_1, Y_2\}$.

Remark 2. In normal applications, Theorem 5.5.3 would be used if (5.22) is non-linear. However it is obvious that the theorem still holds if the PDE is linear or quasilinear. For such situations, Theorem 5.4.1 clearly provides a simpler alternative.

In spite of the fact that our resulting first order quasilinear PDE appears much more complicated than the original (suppose non-linear) PDE, the situation is somewhat simpler because it may be solved using the symmetry technique in Theorem 5.4.1 to generate local solutions of (5.23) depending on an arbitrary function. Once we have chosen a suitable a_1^3, Theorem 3.2.13 for integrating Frobenius integrable distributions may then be applied to the vector field distribution spanned by Y_1 and Y_2.

On inspection of the manner in which Theorem 5.5.3 was established, one would like to generalise the result to single first order non-linear PDE of $n > 2$ independent variables. However the integrability conditions of the Vessiot distribution generates more than one first order quasilinear PDE, and the present technique on quasilinear PDEs described in Section 5.4 of this chapter does not address this situation.

We close this section with the following example:

Example 5.5.4. Consider the following non-linear PDE:

$$u = \frac{\partial u}{\partial x^1}\frac{\partial u}{\partial x^2}. \tag{5.24}$$

On the regular submanifold of some suitably chosen $J^1(U^2, V^1)$ specified by

$$z_2^1 = \frac{z^1}{z_1^1},$$

with coordinates x^1, x^2, z^1, z_1^1 (where $z_1^1 \neq 0$), the Vessiot distribution is generated by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x^1} + z_1^1 \frac{\partial}{\partial z^1}, \\
X_2 &= \frac{\partial}{\partial x^2} + \frac{z^1}{z_1^1}\frac{\partial}{\partial z}, \\
X_3 &= \frac{\partial}{\partial z_1^1}.
\end{aligned}$$

A two-dimensional involution satisfying the transverse condition is generated by

$$\begin{aligned}
Y_1 &= \frac{\partial}{\partial x^1} + z_1^1 \frac{\partial}{\partial z^1} + f\frac{\partial}{\partial z_1^1}, \\
Y_2 &= \frac{\partial}{\partial x^2} + \frac{z^1}{z_1^1}\frac{\partial}{\partial z^1} + \left(1 - \frac{z^1 f}{(z_1^1)^2}\right)\frac{\partial}{\partial z_1^1},
\end{aligned}$$

where f is some arbitrary smooth function of x^1, x^2, z^1, z_1^1. The integrability condition means that f must satisfy

$$\frac{z^1}{(z_1^1)^2}\frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2} + \frac{2z^1}{z_1^1}\frac{\partial f}{\partial z^1} + \frac{\partial f}{\partial z_1^1} = \frac{f}{z_1^1}\left(\frac{2z^1 f}{(z_1^1)^2} - 1\right). \tag{5.25}$$

At this point we would use Theorem 5.4.1 and `DIMSYM` to find suitable f, then integrate the distribution using Theorem 3.2.13. Quite often however, a simple observation may yield a trivial solution for f that gives a non-trivial solution to the original non-linear PDE. For example, let $f = 0$. Then integrating the resulting distribution results in the rather obvious solution to (5.24),

$$u = (c^1 + x^1)(c^2 + x^2),$$

where c^1, c^2 are arbitrary constants. We leave it to the reader to generate local solutions of (5.25) using Theorem 5.4.1. For now though, by observing from (5.25) that there exists a solution of f that is only a function of z^1 and z^1_1, we have found another suitable f to be

$$f = \frac{(z^1_1)^2}{z^1(\sqrt{z^1} + 2)}.$$

This gives

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial x^1} + z^1_1 \frac{\partial}{\partial z^1} + \frac{(z^1_1)^2}{z^1(\sqrt{z^1}+2)} \frac{\partial}{\partial z^1_1}, \\ Y_2 &= \frac{\partial}{\partial x^2} + \frac{z^1}{z^1_1} \frac{\partial}{\partial z^1} + \left(1 - \frac{1}{\sqrt{z^1}+2}\right) \frac{\partial}{\partial z^1_1}, \end{aligned}$$

as generators for our Frobenius integrable distribution. It has two obvious commuting symmetries which are

$$Z_1 := \frac{\partial}{\partial x^1}, \qquad Z_2 := \frac{\partial}{\partial x^2}.$$

They make it easier to integrate our distribution, as shown in Theorem 3.2.16 (recall the remark after the theorem discussing differential two-forms). Following Theorem 3.2.13 we can then integrate the distribution to give

$$\frac{z^1 + 2\sqrt{z^1}}{z^1_1} - x^1 = c^1, \qquad \frac{(\sqrt{z^1}+2)z^1_1}{\sqrt{z^1}} - x^2 = c^2,$$

where c^1, c^2 are arbitrary constant functions. Finally, eliminating z^1_1 and replacing z^1 with u yields the following local solution to the original non-linear PDE in (5.24):

$$u = \left(\pm\sqrt{(x^1 + c^1)(x^2 + c^2)} - 2\right)^2.$$

5.6 A class of first order non-linear PDEs

In the previous section it was shown that local solutions of a given first order non-linear PDE of one dependent variable and two independent variables could be found by generating a corresponding Vessiot distribution whose integrability condition was in the form of a first order quasilinear PDE that could be solved using Theorem 5.4.1. The major disadvantage of generating local solutions of such non-linear PDEs in this way is that even for basic examples, the resulting first order quasilinear PDE is usually quite complicated and of four independent variables, that requires a solvable structure of four symmetries to solve. In addition, a further solvable structure of two symmetries is then required to integrate the resulting Frobenius integrable distribution.

In this section we present a simpler alternative to the Vessiot integration scheme for solving single first order non-linear PDEs that also generates a corresponding first order quasilinear PDE, but which is of only two independent variables requiring a single solvable structure of just two symmetries. Unfortunately, the disadvantage here is that this technique can only be applied to first order PDEs of two independent variables and one dependent variable that do not depend on the dependent variable.

Suppose then, that our PDE is of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, \frac{\partial u}{\partial x^1}\right), \tag{5.26}$$

for smooth F, where x^1, x^2 are the independent variables, and u is the dependent variable. This gives the corresponding fundamental ideal

$$I_F = \langle dz^1 - z^1_1 dx^1 - z^1_2 dx^2, dz^1_1 \wedge dx^1 + dz^1_2 \wedge dx^2, \left(z^1_2 - F\right) dx^1 \wedge dx^2 \rangle,$$

where F is now a function of x^1, x^2, z^1_1.

The main result of this section is the following theorem:

Theorem 5.6.1. *Consider the first order PDE*

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, \frac{\partial u}{\partial x^1}\right), \tag{5.27}$$

for smooth F. In terms of coordinates of $J^1(U^2, V^1)$, set $z^1_1 = f(x^1, x^2)$ and $z^1_2 = F(x^1, x^2, f)$. Then any smooth solution $f(x^1, x^2)$ of the quasilinear PDE

$$\frac{\partial F}{\partial f}\frac{\partial f}{\partial x^1} - \frac{\partial f}{\partial x^2} = -\frac{\partial F}{\partial x^1}, \tag{5.28}$$

has the property that $z^1_1 dx^1 + z^1_2 dx^2 = dg$ *for some* $g \in C^\infty(U^2)$*. Moreover, the expression* $u = g$ *is a local solution of the PDE in (5.27).*

Proof. Let $f \in C^\infty(U^2)$ be any function. Using (5.27), set the following:

$$z^1_1 = f(x^1, x^2), \qquad z^1_2 = F(x^1, x^2, f). \tag{5.29}$$

We have

$$I_F := \langle C^1, dC^1, (z^1_2 - F) dx^1 \wedge dx^2 \rangle,$$

where $C^1 := dz^1 - z^1_1 dx^1 - z^1_2 dx^2$, and wish to look for conditions on f such that

$$-dC^1 = dz^1_1 \wedge dx^1 + dz^1_2 \wedge dx^2 = 0. \tag{5.30}$$

Supposing this, we obtain by inserting (5.29) into (5.30),

$$\frac{\partial f}{\partial x^2} dx^2 \wedge dx^1 + \left(\frac{\partial F}{\partial x^1} + \frac{\partial F}{\partial f} \frac{\partial f}{\partial x^1} \right) dx^1 \wedge dx^2 = 0. \tag{5.31}$$

Now if our f satisfies the PDE in (5.28), then from (5.31),

$$dz^1_1 \wedge dx^1 + dz^1_2 \wedge dx^2 = 0,$$

so $z^1_1 dx^1 + z^1_2 dx^2$ is closed. Therefore,

$$z^1_1 dx^1 + z^1_2 dx^2 = dg,$$

for some $g \in C^\infty(U^2)$. If we now set $z^1 = g$, then

$$C^1 := dz^1 - z^1_1 dx^1 - z^1_2 dx^2 = 0.$$

Therefore the immersion

$$j^1 g : U^2 \longrightarrow J^1(U^2, V^1),$$

maps onto the two-dimensional regular submanifold of $J^1(U^2, V^1)$ defined by the equations $z^1 = g$, $z^1_1 = f$ and $z^1_2 = F$, and has the property that $j^1 g^* I_F = 0$. Hence the expression $u = g$ is a local solution of (5.26). $\square$

Remark. The second remark for Theorem 5.5.3 is valid here. In addition, since the PDE in (5.27) is independent of u, it is obvious that $\frac{\partial}{\partial u}$ is a symmetry of (5.27), and so all local solutions may have the addition of an arbitrary constant.

Finally, we apply Theorem 5.6.1 to an example:

Example 5.6.2. Consider the following first order non-linear PDE:

$$\frac{\partial u}{\partial x^2} = \left(\frac{\partial u}{\partial x^1}\right)^{-1}. \tag{5.32}$$

Applying Theorem 5.6.1, let $f \in C^\infty(U^2)$ be non-zero on U^2, and set

$$z_1^1 = f, \qquad z_2^1 = \frac{1}{f},$$

so that

$$\begin{aligned} dz_1^1 \wedge dx^1 + dz_2^1 \wedge dx^2 &= d\left(f\,dx^1 + \frac{1}{f}dx^2\right), \\ &= df \wedge dx^1 - \frac{1}{f^2}df \wedge dx^2, \\ &= -\left(\frac{1}{f^2}\frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2}\right) dx^1 \wedge dx^2. \end{aligned}$$

In order to solve for f in the first order quasilinear PDE

$$\frac{1}{f^2}\frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2} = 0, \tag{5.33}$$

we will use Theorem 5.4.1 in Section 5.4. The corresponding two-form K is

$$K = df \wedge dx^1 - \frac{1}{f^2}df \wedge dx^2.$$

The vector field

$$X_2 := \frac{\partial}{\partial x^1}$$

is a non-trivial symmetry of K, and

$$X_1 := \frac{\partial}{\partial f}$$

is a non-trivial symmetry of $X_2 \lrcorner K = -df$. Then following Theorem 3.2.14 with Corollary 3.2.12, we obtain that

$$K = df \wedge d\left(x^1 - \frac{x^2}{f^2}\right).$$

Hence in implicit form,

$$G\left(f, x^1 - \frac{x^2}{f^2}\right) = 0,$$

is a local solution of (5.33) for any suitably defined smooth function G. Suppose we choose G so that

$$G\left(f, x^1 - \frac{x^2}{f^2}\right) = \frac{1}{f^2} + x^1 - \frac{x^2}{f^2} - c^1,$$

for any constant function c^1. Then

$$f = \sqrt{\frac{1 - x^2}{c^1 - x^1}}$$

is a local solution of (5.33), assuming that we are in some neighbourhood where $(1 - x^2)/(c^1 - x^1) > 0$. Therefore,

$$z_1^1 = \sqrt{\frac{1 - x^2}{c^1 - x^1}}, \qquad z_2^1 = \sqrt{\frac{c^1 - x^1}{1 - x^2}}.$$

From Theorem 5.6.1, these expressions for z_1^1 and z_2^1 mean that

$$d\left(z_1^1 dx^1 + z_2^1 dx^2\right) = 0.$$

So a simple integration yields

$$z_1^1 dx^1 + z_2^1 dx^2 = d\left(-2\sqrt{(c^1 - x^1)(1 - x^2)}\right).$$

Putting

$$u = -2\sqrt{(c^1 - x^1)(1 - x^2)},$$

then gives a local solution of the original non-linear PDE in (5.32). In fact,

$$u = -2\sqrt{(c^1 - x^1)(c^2 - x^2)},$$

is a local solution of the PDE for any appropriate choice of constant functions c^1, c^2.

Finally, if we suppose that

$$G\left(f, x^1 - \frac{x^2}{f^2}\right) = f\left(x^1 - \frac{x^2}{f^2}\right) - c^3,$$

for some constant c^3, then we may solve the quadratic equation

$$x^1 f^2 - c^3 f - x^2 = 0,$$

to give

$$f = \frac{c^3 \pm \sqrt{4x^1x^2 + (c^3)^2}}{2x^1}.$$

If we choose the positive option for f, and put

$$z_1^1 = \frac{c^3 + \sqrt{4x^1x^2 + (c^3)^2}}{2x^1}, \qquad z_2^1 = \frac{2x^1}{c^3 + \sqrt{4x^1x^2 + (c^3)^2}},$$

then one obtains

$$z_1^1 dx^1 + z_2^1 dx^2 = d\left(\sqrt{4x^1x^2 + (c^3)^2} + \frac{c^3}{2}\ln\left|\frac{x^1\left(\sqrt{4x^1x^2 + (c^3)^2} - c^3\right)}{x^2\left(\sqrt{4x^1x^2 + (c^3)^2} + c^3\right)}\right|\right),$$

so

$$u = \sqrt{4x^1x^2 + (c^3)^2} + \frac{c^3}{2}\ln\left|\frac{x^1\left(\sqrt{4x^1x^2 + (c^3)^2} - c^3\right)}{x^2\left(\sqrt{4x^1x^2 + (c^3)^2} + c^3\right)}\right|$$

is another local solution of the original non-linear PDE in (5.32).

Chapter 6

The Cauchy problem and symmetry

6.1 Introduction

In this chapter we investigate the extent to which solvable symmetry structures can assist in solving the Cauchy problem for Pfaffian systems. In the traditional approach, ordinary differential equation methods are used to solve the problem [128]. Our work looks to develop some computer algebra techniques using solvable symmetry structures from `DIMSYM`, that avoid introducing any such differential equations.

The plan of this chapter is to first give a brief review of the typical Cauchy problem for Pfaffian systems, as well as provide some preliminary results on solvable structures. Following this, two symmetry techniques for solving the Cauchy problem are presented. The first deals with the special situation when we are given a differential ideal that is generated by a single Pfaffian equation of rank one, while the second looks at the more general situation when we have a differential ideal that is generated by a finite collection of Pfaffian equations, each of constant and perhaps different rank on the domain of definition. The second approach is an extension of the first and is a little more sophisticated. We also provide a PDE example for each of the two techniques presented.

Since our work in this chapter makes extensive use of Theorem 3.2.14, given any solvable symmetry structure $X_1, \ldots, X_m$ for some Frobenius integrable vector field distribution D, we assume throughout that all vector fields in the symmetry structure are linearly independent, with X_m being a non-trivial symmetry of D and so on down to X_1 being a non-trivial symmetry of the distribution $D \oplus Sp\{X_2, \ldots, X_m\}$. Furthermore, we assume $\gamma^1, \ldots, \gamma^m$ are those found in Theorem 3.2.14, and denote $\mathcal{F}(\gamma^a)$ as the ring of smooth functions of $\gamma^1, \ldots, \gamma^m$. For example, $\sin(\gamma^1 - \gamma^m) \in \mathcal{F}(\gamma^a)$. Finally, as in previous chapters, if we are given a differential ideal I, we denote its Cauchy characteristic space by $A(I)$, and if we are also given a list of generators for the space, we assume throughout that these are linearly independent.

6.2 Background

Suppose we are given a differential ideal I defined on some open, convex neighbourhood $U^n \subset \mathbb{R}^n$ that is generated by a finite collection of linearly independent differential one-forms in $\Lambda^1(U^n)$. The *Cauchy problem* [54, 55] looks to extend a q-dimensional integral manifold of I to a $(q+1)$-dimensional integral manifold. To achieve this, we are given some *Cauchy data*, which is a one-to-one smooth map of maximal rank

$$\Phi : U^q \longrightarrow U^n,$$

that satisfies $\Phi^* I = 0$. We are also given a Cauchy characteristic vector field $Y \in \mathfrak{X}(U^n)$ of I that is *transverse* to Φ. By transverse we mean in this case that on U^n it is nowhere tangent to the image of Φ. In dealing with PDE problems, we demand that Y be transverse to Φ with respect to an independence condition, which means that the projections of the Cauchy data and Y onto the base manifold of the independent variables are nowhere tangent.

The usual and most basic treatment of the Cauchy problem for Pfaffian systems focuses on the following well-known theorem [54, 55, 119]:

Theorem 6.2.1. *Let $I := \langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ for some linearly independent $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U^n)$ with Cauchy data given by the one-to-one immersion*

$$\Phi : U^q \longrightarrow U^n,$$

that satisfies $\Phi^* I = 0$, and let $X \in \mathfrak{X}(U^n)$. An immersion $\Theta : U^q \times U^1 \longrightarrow U^n$

defined by

$$\Theta(u, t) := \exp_{\Phi(u)}(tX), \tag{6.1}$$

is a solution to the Cauchy problem if and only if

1. *X is a Cauchy characteristic vector field of I,*

2. *X is transverse to the Cauchy data,*

3. *$\Theta(U^q, 0) = \Phi(U^q)$.*

The standard notation given in (6.1) denotes the point in U^n at t on the integral curve given by X which passes through $\Phi(u)$ when $t = 0$. The map given by (6.1) is typically found by solving a system of first order ODEs. In the remaining sections of this chapter we examine how symmetry may be used to replace the need for solving such ODEs.

6.3 Solvable structures revisited

This section contains several results that are basically a consequence of Theorem 3.2.14.

Lemma 6.3.1. *Let $\Omega \in \Lambda^m(U^n)$ be decomposable with $d\Omega \equiv 0$ mod Ω. Suppose there exists a solvable symmetry structure $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ for $A(\langle\Omega\rangle)$, and let $Y \in A(\langle\Omega\rangle)$. Then*

1. *$X_j(\gamma^i) = 0$ for all $1 \leq i < j \leq m$,*

2. *$X_i(\gamma^i) = 1$ (no sum) for all $1 \leq i \leq m$,*

3. *$Y(\gamma^i) = 0$ for all $1 \leq i \leq m$,*

4. *$[Y, X_j] \lrcorner\, \omega^i = 0$ for all $1 \leq i \leq j \leq m$.*

Proof. From Theorem 3.2.14 $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$, and hence the expression for ω^1 in (3.7) implies $X_j(\gamma^1) = 0$ for all $j > 1$. Now this result and the expression for ω^2 in (3.7) yields $X_j(\gamma^2) = 0$ for all $j > 2$. It is clear that we can continue in this way up to ω^{m-1} to ultimately derive conclusion (1). Moreover, conclusion (1) and the expression for ω^i in (3.7) imply conclusion (2).

Now for any $Y \in A(\langle\Omega\rangle)$, we have by the definition of ω^i that $Y \lrcorner \omega^i = 0$ for each $1 \le i \le m$. Therefore from ω^1 in (3.7) we obtain $Y(\gamma^1) = 0$. Using this result, the fact that $Y \lrcorner \omega^2 = 0$, and ω^2 in (3.7) then yields $Y(\gamma^2) = 0$. Continuing in this fashion for all i up to m, we obtain conclusion (3).

Finally, it is not hard to see that the results in conclusions (1), (2) and (3) give

$$[Y, X_j] \lrcorner d\gamma^i = Y\left(X_j(\gamma^i)\right) - X_j\left(Y(\gamma^i)\right) = 0,$$

for all $1 \le i \le j \le m$. Hence using each ω^i in (3.7) we get conclusion (4). □

Using Lemma 6.3.1 we obtain the following:

Theorem 6.3.2. *Let $\Omega \in \Lambda^m(U^n)$ be decomposable with $d\Omega \equiv 0$ mod Ω. Suppose there exists a solvable symmetry structure $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ for $A(\langle\Omega\rangle)$. Then for any $Y \in A(\langle\Omega\rangle)$, we have, modulo $A(\langle\Omega\rangle)$,*

$$\begin{aligned}
[Y, X_{m-1}] &\equiv Y\left(X_{m-1}(\gamma^m)\right) X_m, \\
[Y, X_{m-2}] &\equiv Y\left(X_{m-2}(\gamma^m)\right) X_m + Y\left(X_{m-2}(\gamma^{m-1})\right)\left\{X_{m-1} - X_{m-1}(\gamma^m) X_m\right\}, \\
[Y, X_{m-3}] &\equiv Y\left(X_{m-3}(\gamma^m)\right) X_m + Y\left(X_{m-3}(\gamma^{m-1})\right)\left\{X_{m-1} - X_{m-1}(\gamma^m) X_m\right\} \\
&\quad + Y\left(X_{m-3}(\gamma^{m-2})\right)\left\{X_{m-2} - X_{m-2}(\gamma^{m-1})[X_{m-1} - X_{m-1}(\gamma^m) X_m]\right. \\
&\quad \left. - X_{m-2}(\gamma^m) X_m\right\}, \\
&\vdots \\
[Y, X_1] &\equiv Y\left(X_1(\gamma^2)\right) X_2 \mod X_3, \ldots, X_m.
\end{aligned}$$

(see the remark below for further details on this arrangement)

Proof. We begin with $[Y, X_{m-1}]$. Since $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$ and $A(\langle\Omega\rangle) \oplus Sp\{X_1, \ldots, X_m\}$ spans the tangent space of U^n, we may write (with sum)

$$[Y, X_{m-1}] \equiv \left([Y, X_{m-1}] \lrcorner \omega^i\right) X_i \mod A(\langle\Omega\rangle).$$

Conclusion (4) in Lemma 6.3.1 then gives the simplification

$$[Y, X_{m-1}] \equiv \left([Y, X_{m-1}] \lrcorner \omega^m\right) X_m \mod A(\langle\Omega\rangle).$$

We note from (3.7) that $\omega^m \equiv d\gamma^m$ mod $\omega^1, \ldots, \omega^{m-1}$. This fact and conclusion (3) in Lemma 6.3.1 then yields

$$\begin{aligned}
[Y, X_{m-1}] &\equiv \left([Y, X_{m-1}] \lrcorner d\gamma^m\right) X_m \mod A(\langle\Omega\rangle), \\
&\equiv Y\left(X_{m-1}(\gamma^m)\right) X_m \mod A(\langle\Omega\rangle),
\end{aligned}$$

which proves the first result.

Following the same argument as before, but for $[Y, X_{m-2}]$, we obtain

$$\begin{aligned}
[Y, X_{m-2}] &\equiv \left([Y, X_{m-2}] \lrcorner\, \omega^{m-1}\right) X_{m-1} + \left([Y, X_{m-2}] \lrcorner\, \omega^{m}\right) X_m \mod A(\langle\Omega\rangle), \\
&\equiv \left([Y, X_{m-2}] \lrcorner\, d\gamma^{m-1}\right) X_{m-1} \\
&\quad + \left([Y, X_{m-2}] \lrcorner \left(d\gamma^m - X_{m-1}(\gamma^m) d\gamma^{m-1}\right)\right) X_m \mod A(\langle\Omega\rangle), \\
&\equiv Y\left(X_{m-2}(\gamma^{m-1})\right) X_{m-1} + Y\left(X_{m-2}(\gamma^m)\right) X_m \\
&\quad - Y\left(X_{m-2}(\gamma^{m-1})\right) X_{m-1}(\gamma^m) X_m \mod A(\langle\Omega\rangle),
\end{aligned}$$

initially using the expressions for ω^{m-1} and ω^m in (3.7), and then several times each of conclusions (4) and (3) in Lemma 6.3.1.

In a similar way to above, we can repeat the above process to eventually prove the remaining $m-3$ results in the theorem. □

Remark. The pattern in the expressions for $[Y, X_i]$ in the conclusion of the theorem becomes obvious if we express, for example $[Y, X_{m-4}]$, modulo $A(\langle\Omega\rangle)$, in the following form:

$$\begin{aligned}
[Y, X_{m-4}] \equiv{}& Y\left(X_{m-4}(\gamma^m)\right) \overline{X}_m + Y\left(X_{m-4}(\gamma^{m-1})\right) \overline{X}_{m-1} \\
&+ Y\left(X_{m-4}(\gamma^{m-2})\right) \overline{X}_{m-2} + Y\left(X_{m-4}(\gamma^{m-3})\right) \overline{X}_{m-3},
\end{aligned}$$

where

$$\begin{aligned}
\overline{X}_m &:= X_m, \\
\overline{X}_{m-1} &:= X_{m-1} - X_{m-1}(\gamma^m)\overline{X}_m, \\
\overline{X}_{m-2} &:= X_{m-2} - X_{m-2}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-2}(\gamma^m)\overline{X}_m, \\
\overline{X}_{m-3} &:= X_{m-3} - X_{m-3}(\gamma^{m-2})\overline{X}_{m-2} - X_{m-3}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-3}(\gamma^m)\overline{X}_m.
\end{aligned}$$

Lemma 6.3.3. *Let $\Omega \in \Lambda^m(U^n)$ such that Ω is decomposable and $d\Omega \equiv 0$ mod Ω, and let $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ be a solvable symmetry structure for $A(\langle\Omega\rangle)$. If we define the vector fields $\overline{X}_1, \ldots, \overline{X}_m$ by*

$$\begin{aligned}
\overline{X}_m &:= X_m, \\
\overline{X}_{m-1} &:= X_{m-1} - X_{m-1}(\gamma^m)\overline{X}_m, \\
\overline{X}_{m-2} &:= X_{m-2} - X_{m-2}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-2}(\gamma^m)\overline{X}_m, \qquad (6.2)\\
&\;\vdots \\
\overline{X}_1 &\equiv X_1 \mod \overline{X}_2, \ldots, \overline{X}_m,
\end{aligned}$$

then $\{d\gamma^1, \ldots, d\gamma^m\}$ *is dual to* $\{\overline{X}_1, \ldots, \overline{X}_m\}$.

Proof. We work by induction. From the definition of $\overline{X}_m$ in (6.2) and conclusions (1) and (2) in Lemma 6.3.1, it is obvious that $\overline{X}_m(\gamma^m) = 1$ and $\overline{X}_m(\gamma^j) = 0$ for all $1 \leq j < m$.

Now let k be any integer such that $2 \leq k \leq m$, and assume for each $k \leq l \leq m$ and $1 \leq p \leq m$ with $p \neq l$ that

$$\overline{X}_l(\gamma^l) = 1, \tag{6.3}$$

$$\overline{X}_l(\gamma^p) = 0. \tag{6.4}$$

We wish to show (6.3) and (6.4) hold for $l = k-1$. First suppose q is any integer such that $k - 1 < q \leq m$. Consider $\overline{X}_{k-1}$ in (6.2) operated on γ^q, i.e.

$$\begin{aligned} \overline{X}_{k-1}(\gamma^q) = X_{k-1}(\gamma^q) - X_{k-1}(\gamma^k)\overline{X}_k(\gamma^q) - \cdots - X_{k-1}(\gamma^q)\overline{X}_q(\gamma^q) - \ldots \\ - X_{k-1}(\gamma^m)\overline{X}_m(\gamma^q). \end{aligned} \tag{6.5}$$

Equation (6.3) then implies

$$X_{k-1}(\gamma^q) - X_{k-1}(\gamma^q)\overline{X}_q(\gamma^q) = 0,$$

and from (6.4) all the other terms in (6.5) become zero. Thus $\overline{X}_{k-1}(\gamma^q) = 0$.

Now suppose $q = k - 1$. The expression

$$\overline{X}_{k-1}(\gamma^q) = X_{k-1}(\gamma^q) - X_{k-1}(\gamma^k)\overline{X}_k(\gamma^q) - \cdots - X_{k-1}(\gamma^m)\overline{X}_m(\gamma^q), \tag{6.6}$$

simplifies to give

$$\overline{X}_{k-1}(\gamma^{k-1}) = X_{k-1}(\gamma^{k-1}),$$

since (6.4) implies all the remaining terms in (6.6) vanish. Hence by using conclusion (2) in Lemma 6.3.1 we obtain $\overline{X}_{k-1}(\gamma^{k-1}) = 1$.

Finally suppose q is any integer such that $1 \leq q < k - 1$. Using (6.6) for such q, and then inserting (6.4) leaves

$$\overline{X}_{k-1}(\gamma^q) = X_{k-1}(\gamma^q).$$

Then from conclusion (1) in Lemma 6.3.1 we find that $\overline{X}_{k-1}(\gamma^q) = 0$. This completes the induction. □

From the definitions of $\overline{X}_1, \ldots, \overline{X}_m$ in (6.2), we have the following corollary to Theorem 6.3.2:

Corollary 6.3.4. *Let $\Omega \in \Lambda^m(U^n)$ be decomposable with $d\Omega \equiv 0 \bmod \Omega$. Given any solvable symmetry structure $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ for $A(\langle\Omega\rangle)$, let $Y \in A(\langle\Omega\rangle)$. Then with $\overline{X}_1, \ldots, \overline{X}_m$ defined as in Lemma 6.3.3, we have, modulo $A(\langle\Omega\rangle)$,*

$$
\begin{aligned}
[Y, X_{m-1}] &\equiv Y\left(X_{m-1}(\gamma^m)\right)\overline{X}_m, \\
[Y, X_{m-2}] &\equiv Y\left(X_{m-2}(\gamma^m)\right)\overline{X}_m + Y\left(X_{m-2}(\gamma^{m-1})\right)\overline{X}_{m-1}, \\
[Y, X_{m-3}] &\equiv Y\left(X_{m-3}(\gamma^m)\right)\overline{X}_m + Y\left(X_{m-3}(\gamma^{m-1})\right)\overline{X}_{m-1} \\
&\qquad + Y\left(X_{m-3}(\gamma^{m-2})\right)\overline{X}_{m-2}, \\
&\vdots \\
[Y, X_1] &\equiv Y\left(X_1(\gamma^2)\right)\overline{X}_2 \mod \overline{X}_3, \ldots, \overline{X}_m.
\end{aligned}
$$

Using Corollary 6.3.4, we have the following major result:

Theorem 6.3.5. *Let $\Omega \in \Lambda^m(U^n)$ be decomposable with $d\Omega \equiv 0 \bmod \Omega$. Given any solvable symmetry structure $X_1, \ldots, X_m \in \mathfrak{X}(U^n)$ for $A(\langle\Omega\rangle)$, let $Y \in A(\langle\Omega\rangle)$. Then with $\overline{X}_1, \ldots, \overline{X}_m$ defined as in Lemma 6.3.3, we have for each $1 \leq i \leq m$, $[Y, \overline{X}_i] \equiv 0 \bmod A(\langle\Omega\rangle)$.*

Proof. Instead of proving the result by a tedious induction, we simply examine the first three $[Y, \overline{X}_i]$ in descending order. Since $\overline{X}_m = X_m$ and X_m is a symmetry of $A(\langle\Omega\rangle)$, it is obvious that

$$[Y, \overline{X}_m] \equiv 0 \mod A(\langle\Omega\rangle). \tag{6.7}$$

Next consider $[Y, \overline{X}_{m-1}]$. By the definition of $\overline{X}_{m-1}$,

$$
\begin{aligned}
[Y, \overline{X}_{m-1}] &= [Y, X_{m-1} - X_{m-1}(\gamma^m)\overline{X}_m], \\
&= [Y, X_{m-1}] - Y\left(X_{m-1}(\gamma^m)\right)\overline{X}_m - X_{m-1}(\gamma^m)[Y, \overline{X}_m].
\end{aligned}
$$

Now using the expression for $[Y, X_{m-1}]$ in Corollary 6.3.4, we have that

$$[Y, \overline{X}_{m-1}] = -X_{m-1}(\gamma^m)[Y, \overline{X}_m].$$

Then inserting (6.7) gives our second result, namely

$$[Y, \overline{X}_{m-1}] \equiv 0 \mod A(\langle\Omega\rangle). \tag{6.8}$$

Finally, we consider $[Y, \overline{X}_{m-2}]$. Once again by definition, and then inserting the expression for $[Y, X_{m-2}]$ in Corollary 6.3.4, we get

$$\begin{aligned}
[Y, \overline{X}_{m-2}] &= [Y, X_{m-2} - X_{m-2}(\gamma^{m-1})\overline{X}_{m-1} - X_{m-2}(\gamma^m)\overline{X}_m], \\
&= [Y, X_{m-2}] - Y\left(X_{m-2}(\gamma^{m-1})\right)\overline{X}_{m-1} - Y\left(X_{m-2}(\gamma^m)\right)\overline{X}_m \\
&\quad - X_{m-2}(\gamma^{m-1})[Y, \overline{X}_{m-1}] - X_{m-2}(\gamma^m)[Y, \overline{X}_m], \\
&= -X_{m-2}(\gamma^{m-1})[Y, \overline{X}_{m-1}] - X_{m-2}(\gamma^m)[Y, \overline{X}_m].
\end{aligned}$$

Then using (6.7) and (6.8), we obtain

$$[Y, \overline{X}_{m-2}] \equiv 0 \mod A(\langle\Omega\rangle).$$

At this point, it is easy to see how induction may be used to formally prove the theorem. □

6.4 The Cauchy problem for a one-form of rank one

In this section we examine a symmetry technique for solving the Cauchy problem for a differential ideal generated by a single one-form of constant rank one on U^n. It is well-known (and illustrated later in an example) that such an ideal can be generated from a single first order PDE of one dependent variable and two independent variables by pulling back the first order contact form onto the regular submanifold of the first jet bundle described by the PDE. In this space, the Cauchy characteristic space is one-dimensional [119].

Before we present the main result, Theorem 6.4.11, we will need several preliminary results, most of which also hold for differential one-forms of constant rank higher than one.

Lemma 6.4.1. *Let $\alpha \in \Lambda^1(U^n)$ be of constant rank $r \geq 1$ on U^n and let $Y \in \mathfrak{X}(U^n)$. Then $Y \lrcorner (d\alpha \wedge \alpha) = 0$ if and only if Y is a Cauchy characteristic vector field of the differential ideal $\langle \alpha, d\alpha \rangle$.*

Proof. Suppose that $Y \lrcorner (d\alpha \wedge \alpha) = 0$ for some $Y \in \mathfrak{X}(U^n)$. This implies

$$(Y \lrcorner d\alpha) \wedge \alpha + (Y \lrcorner \alpha)\, d\alpha = 0. \tag{6.9}$$

The exterior product of (6.9) with α yields $(Y \lrcorner \alpha)\, d\alpha \wedge \alpha = 0$, but since α is of rank at least one, we must have $d\alpha \wedge \alpha \neq 0$. Therefore $Y \lrcorner \alpha = 0$. Inserting this result into (6.9) then gives that $Y \lrcorner d\alpha \equiv 0 \mod \alpha$. Hence $Y \in A(\langle \alpha, d\alpha \rangle)$. Proving the converse is obvious. □

The following lemma includes part of Lemma 6.4.1:

Lemma 6.4.2. *Let $\alpha \in \Lambda^1(U^n)$ be of constant rank $r \geq 1$ on U^n, and let $Y \in A(\langle \alpha, d\alpha \rangle)$. Then for all $1 \leq p \leq r$, $Y \lrcorner \left((d\alpha)^p \wedge \alpha\right) = 0$, and in particular $Y \in A(\langle (d\alpha)^r \wedge \alpha \rangle)$.*

Proof. Suppose α is a one-form of constant rank $r \geq 1$ on U^n, and let $Y \in A(\langle \alpha, d\alpha \rangle)$. Then for all $1 \leq p \leq r$,

$$Y \lrcorner \left((d\alpha)^p \wedge \alpha\right) = p(Y \lrcorner d\alpha) \wedge (d\alpha)^{p-1} \wedge \alpha = 0, \tag{6.10}$$

since $Y \lrcorner \alpha = 0$ and $Y \lrcorner d\alpha \equiv 0 \mod \alpha$.

Now if $p = r$, then from Theorem 3.5.3 we have that $d\left((d\alpha)^r \wedge \alpha\right) \equiv 0 \mod (d\alpha)^r \wedge \alpha$. So from Theorem 2.3.11 it follows that $\ker\left((d\alpha)^r \wedge \alpha\right) = A(\langle (d\alpha)^r \wedge \alpha \rangle)$. Hence (6.10) implies $Y \in A(\langle (d\alpha)^r \wedge \alpha \rangle)$. □

Combining Lemmas 6.4.1 and 6.4.2 gives the following theorem for one-forms of constant rank one:

Theorem 6.4.3. *Let $\alpha \in \Lambda^1(U^n)$ be of constant rank one on U^n. Then $A(\langle d\alpha \wedge \alpha \rangle) = A(\langle \alpha, d\alpha \rangle)$.*

Also for one-forms of constant rank at least one, we have the following theorem:

Theorem 6.4.4. *Let $\alpha \in \Lambda^1(U^n)$ be of constant rank $r \geq 1$ on U^n, and suppose $A(\langle \alpha, d\alpha \rangle)$ is of dimension $q \geq 1$ generated by some $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$. Then $A(\langle (d\alpha)^r \wedge \alpha \rangle) = Sp\{Y_1, \ldots, Y_q, \Gamma_1, \ldots, \Gamma_s\}$ for some $s := n - 2r - q - 1$ vector fields $\Gamma_1, \ldots, \Gamma_s \in \mathfrak{X}(U^n)$.*

Proof. From Lemma 6.4.2 we get that $A(\langle \alpha, d\alpha \rangle) \subset A(\langle (d\alpha)^r \wedge \alpha \rangle)$. Theorem 3.5.3 tells us that $(d\alpha)^r \wedge \alpha$ is decomposable and

$$d\left((d\alpha)^r \wedge \alpha\right) \equiv 0 \mod (d\alpha)^r \wedge \alpha.$$

Since $(d\alpha)^r \wedge \alpha$ is closed modulo itself, Theorem 2.3.11 implies $A(\langle (d\alpha)^r \wedge \alpha \rangle) = \ker\left((d\alpha)^r \wedge \alpha\right)$. Hence $A(\langle (d\alpha)^r \wedge \alpha \rangle)$ is of dimension $n - 2r - 1$. The theorem is now obvious. □

The remaining preparatory results given below incorporate symmetry. The precise purpose of these final few results will be made clearer in the discussion following Lemma 6.4.5.

Lemma 6.4.5. *Let $\alpha \in \Lambda^1(U^n)$ such that the dimension of $A(\langle \alpha, d\alpha \rangle)$ is greater than zero. Let $Y \in A(\langle \alpha, d\alpha \rangle)$ such that $Y \lrcorner\, d\alpha = \mu\alpha$ for some non-zero $\mu \in C^\infty(U^n)$. If $h \in C^\infty(U^n)$ satisfies*

$$Y(h) + h\mu = 0,$$

then $Y \lrcorner\, d(h\alpha) = 0$.

Proof. Let $h \in C^\infty(U^n)$. Then

$$\begin{aligned} Y \lrcorner\, d(h\alpha) &= Y \lrcorner\, (dh \wedge \alpha + h d\alpha)\,, \\ &= Y(h)\alpha + h\,(Y \lrcorner\, d\alpha)\,, \\ &= (Y(h) + h\mu)\,\alpha, \end{aligned}$$

which proves the lemma. □

For one-dimensional $A(\langle \alpha, d\alpha \rangle)$, Lemma 6.4.5 therefore allows us to find some non-zero smooth h (which can be done through ordinary differential techniques), so that the differential ideal generated by the one-form $\overline{\alpha} := h\alpha$ has the property that $Y \lrcorner\, d\overline{\alpha} = 0$. In general, we will use a better result than Lemma 6.4.5 based on symmetry, that has the advantage of including the situation when the Cauchy characteristic space is of a higher dimension than one; however, first we must examine an existence issue:

Theorem 6.4.6. *Let $\alpha \in \Lambda^1(U^n)$ such that for some $q \geq 1$, $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$ generate the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$. Then there exists some non-zero $h \in C^\infty(U^n)$ such that for all $1 \leq i \leq q$, $Y_i \lrcorner\, d(h\alpha) = 0$.*

Proof. Let the Cauchy characteristic space $A(\langle \alpha, d\alpha \rangle)$ be q-dimensional and generated by $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$. For each $1 \leq i \leq q$ we have

$$Y_i \lrcorner\, d\alpha = \mu_i \alpha,$$

for some $\mu_i \in C^\infty(U^n)$. We wish to show there exists some non-zero $h \in C^\infty(U^n)$ such that for each i,

$$Y_i(h) + h\mu_i = 0. \tag{6.11}$$

Then from Lemma 6.4.5, finding such an h will mean that for all i, $Y_i \lrcorner\, d(h\alpha) = 0$. From Theorem 2.2.11, $A(\langle \alpha, d\alpha \rangle)$ is Frobenius integrable, and therefore we can write for each $1 \leq j, k \leq q$,

$$[Y_j, Y_k] = \rho^1_{jk} Y_1 + \cdots + \rho^q_{jk} Y_q, \tag{6.12}$$

for some $\rho^1_{jk}, \ldots, \rho^q_{jk} \in C^\infty(U^n)$ with $\rho^l_{jk} = -\rho^l_{kj}$ for all $1 \leq l \leq q$. For there to exist a local solution h of the system of PDEs in (6.11), the integrability conditions that must be satisfied are

$$Y_j(\mu_k) - Y_k(\mu_j) = \rho^1_{jk}\mu_1 + \cdots + \rho^q_{jk}\mu_q. \tag{6.13}$$

To show (6.13) holds, we have that

$$\begin{aligned}
\mathcal{L}_{[Y_j, Y_k]}\alpha &= \mathcal{L}_{Y_j}\left(\mathcal{L}_{Y_k}\alpha\right) - \mathcal{L}_{Y_k}\left(\mathcal{L}_{Y_j}\alpha\right), \\
&= \mathcal{L}_{Y_j}\left(Y_k \lrcorner\, d\alpha\right) - \mathcal{L}_{Y_k}\left(Y_j \lrcorner\, d\alpha\right), \\
&= \mathcal{L}_{Y_j}\left(\mu_k\alpha\right) - \mathcal{L}_{Y_k}\left(\mu_j\alpha\right), \\
&= \left(Y_j(\mu_k) - Y_k(\mu_j)\right)\alpha.
\end{aligned}$$

But from using (6.12),

$$\begin{aligned}
\mathcal{L}_{[Y_j, Y_k]}\alpha &= \mathcal{L}_{\rho^1_{jk}Y_1 + \cdots + \rho^q_{jk}Y_q}\alpha, \\
&= \left(\rho^1_{jk}Y_1 + \cdots + \rho^q_{jk}Y_q\right) \lrcorner\, d\alpha, \\
&= \rho^1_{jk}\mu_1\alpha + \cdots + \rho^q_{jk}\mu_q\alpha.
\end{aligned}$$

Hence the conditions in (6.13) are satisfied. □

Remark. The result in Theorem 6.4.6 is expected due to the Frobenius integrability condition given in (6.12).

At this point we introduce symmetry. We begin with a simple result which we state without proof.

Lemma 6.4.7. *Let $\alpha \in \Lambda^1(U^n)$. Then $Z \in \mathfrak{X}(U^n)$ is a non-trivial symmetry of $\langle \alpha, d\alpha \rangle$ if and only if Z is a non-trivial symmetry of α.*

From this we obtain the following:

Lemma 6.4.8. *Let $\alpha \in \Lambda^1(U^n)$ and let for some $q \geq 1$, $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$ generate the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$. If $Z \in \mathfrak{X}(U^n)$ is a non-trivial symmetry of $\langle \alpha, d\alpha \rangle$, then for all $1 \leq i \leq q$,*

$$Y_i \lrcorner\, d\left(\frac{\alpha}{Z \lrcorner\, \alpha}\right) = 0.$$

Proof. Lemma 6.4.7 implies $Z \in \mathfrak{X}(U^n)$ is a non-trivial symmetry of α, so $Z \lrcorner\, \alpha \neq 0$. Now for each $1 \leq i \leq q$,

$$\begin{aligned} Y_i \lrcorner\, d\left(\frac{\alpha}{Z \lrcorner\, \alpha}\right) &= Y_i \lrcorner \left(\frac{(Z \lrcorner\, \alpha) d\alpha - d(Z \lrcorner\, \alpha) \wedge \alpha}{(Z \lrcorner\, \alpha)^2}\right), \\ &= \frac{Y_i}{(Z \lrcorner\, \alpha)^2} \lrcorner\, \left((Z \lrcorner\, \alpha) d\alpha + (Z \lrcorner\, d\alpha) \wedge \alpha\right), \\ &= \frac{1}{(Z \lrcorner\, \alpha)^2} Y_i \lrcorner\, Z \lrcorner\, (d\alpha \wedge \alpha). \end{aligned}$$

In the second line we have used the symmetry condition

$$Z \lrcorner\, d\alpha + d(Z \lrcorner\, \alpha) = \mathcal{L}_Z \alpha = \rho\alpha,$$

for some $\rho \in C^\infty(U^n)$. We have

$$Y_i \lrcorner\, (d\alpha \wedge \alpha) = (Y_i \lrcorner\, d\alpha) \wedge \alpha + (Y_i \lrcorner\, \alpha) d\alpha = 0,$$

since Y_i is a Cauchy characteristic vector field of $\langle \alpha, d\alpha \rangle$. Thus

$$\frac{1}{(Z \lrcorner\, \alpha)^2} Y_i \lrcorner\, Z \lrcorner\, (d\alpha \wedge \alpha) = 0.$$

Hence the result. □

In order to find a non-trivial symmetry of $\langle \alpha, d\alpha \rangle$ for Lemma 6.4.8, we simply use Lemma 6.4.7 and look for a non-trivial symmetry of α, or equivalently, the vector field dual space of α using `DIMSYM`. This is because from Theorem 2.3.10, the symmetries of a decomposable differential form are also the symmetries of its kernel.

We illustrate Lemma 6.4.8 with an example which we will refer to later:

Example 6.4.9. Suppose $\alpha \in \Lambda^1(U^4)$ is defined by

$$\alpha := dx^3 - x^4 dx^1 + x^3 x^4 dx^2.$$

The Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ is of dimension one and generated by

$$Y := \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^1} - (x^4)^2 \frac{\partial}{\partial x^4}.$$

Here,

$$Y \lrcorner\, d\alpha = -x^4 \alpha.$$

It is obvious that

$$Z := \frac{\partial}{\partial x^1}$$

is a non-trivial symmetry of α with $Z \lrcorner\, \alpha = -x^4$. So

$$Y \lrcorner\, d\left(\frac{\alpha}{x^4}\right) = 0,$$

on some suitably chosen subset of U^4.

The final result required is contained in the following simple lemma which can be proved in a similar way to Theorem 6.4.4, but also using Theorem 6.4.3:

Lemma 6.4.10. *Let $\alpha \in \Lambda^1(U^n)$ be of constant rank one on U^n. Then $A(\langle \alpha, d\alpha \rangle)$ is $(n-3)$-dimensional.*

At this point we are now in a position to present the main result of this section. In what follows we assume $\alpha \in \Lambda^1(U^n)$ is of constant rank one on U^n with the $(n-3)$-dimensional Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ generated by some $Y_1, \ldots, Y_{n-3} \in \mathfrak{X}(U^n)$. Further, it is assumed that we have applied Lemma 6.4.5 when $n - 3 = 1$, or Lemma 6.4.8 when $n - 3 \geq 1$, so that $Y_j \lrcorner\, d\alpha = 0$ for each $1 \leq j \leq n-3$. In Theorem 6.4.11 below, we also introduce for the first time the notation $\mathfrak{X}^m(U^n)$, where $m \leq n$, to denote an m-dimensional submodule of $\mathfrak{X}(U^n)$ over the smooth ring of functions $\mathcal{F}(\gamma^a)$. This simply means that for any $W_1, \ldots W_m \in \mathfrak{X}(U^n)$ generating $\mathfrak{X}^m(U^n)$, if $Z \in \mathfrak{X}^m(U^n)$, then

$$Z = \mu^1 W_1 + \cdots + \mu^m W_m,$$

for some $\mu^1, \ldots, \mu^m \in \mathcal{F}(\gamma^a)$.

Theorem 6.4.11, when $n = 4$, gives a symmetry approach to the Cauchy problem defined in Section 6.2 with one-dimensional Cauchy data, thus generating an integral manifold of dimension two.

Theorem 6.4.11. *Define $\Omega := d\alpha \wedge \alpha$ and let $X_1, X_2, X_3 \in \mathfrak{X}(U^n)$ be a solvable symmetry structure for $A(\langle\Omega\rangle)$. There exists a two-dimensional submodule $\mathfrak{X}^2(U^n)$ of $\mathfrak{X}(U^n)$ over $\mathcal{F}(\gamma^a)$ such that for any non-zero $Z \in \mathfrak{X}^2(U^n)$, we have that $[Y_j, Z] \equiv 0$ mod $A(\langle\alpha, d\alpha\rangle)$ for each $1 \leq j \leq n-3$. Moreover, on U^n, $Sp\{Y_1, \ldots, Y_{n-3}, Z\}$ is Frobenius integrable and generates an $(n-2)$-dimensional integral manifold of $\langle\alpha, d\alpha\rangle$.*

Proof. Theorem 3.5.3 tells us that Ω is decomposable and $d\Omega \equiv 0$ mod Ω. Then applying Theorem 3.2.14 with $\omega^1, \omega^2, \omega^3$ defined as in the theorem we have that $\{\omega^1, \omega^2, \omega^3\}$ is dual to $\{X_1, X_2, X_3\}$ with

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,
\end{aligned}$$

for some functionally independent $\gamma^1, \gamma^2, \gamma^3 \in C^\infty(U^n)$. Now from Theorem 6.4.3, it follows that $A(\langle\alpha, d\alpha\rangle) = A(\langle\Omega\rangle) = Sp\{Y_1, \ldots, Y_{n-3}\}$, so

$$\begin{aligned}
\alpha &= (X_1 \lrcorner\, \alpha)\, \omega^1 + (X_2 \lrcorner\, \alpha)\, \omega^2 + (X_3 \lrcorner\, \alpha)\, \omega^3, \\
&= \lambda_1 d\gamma^1 + \lambda_2 d\gamma^2 + \lambda_3 d\gamma^3,
\end{aligned}$$

for some $\lambda_1, \lambda_2, \lambda_3 \in C^\infty(U^n)$. Since for each $1 \leq j \leq n-3$, $Y_j \lrcorner\, \alpha = 0 = Y_j \lrcorner\, d\alpha$, we therefore have

$$\mathcal{L}_{Y_j} \alpha = 0. \tag{6.14}$$

We know from conclusion (3) in Lemma 6.3.1 that $Y_j \lrcorner\, d\gamma^i = 0$ for each $1 \leq i \leq 3$, so (6.14) implies

$$Y_j(\lambda_1)d\gamma^1 + Y_j(\lambda_2)d\gamma^2 + Y_j(\lambda_3)d\gamma^3 = 0.$$

Therefore, as $d\gamma^1, d\gamma^2, d\gamma^3$ are linearly independent, it follows that

$$Y_j(\lambda_i) = 0, \tag{6.15}$$

for all i and j. Since $Sp\{Y_1, \ldots, Y_{n-3}\}$ is Frobenius integrable with $\gamma^1, \gamma^2, \gamma^3$ as first integrals, we can then conclude from (6.15) that each λ_i is some function of $\gamma^1, \gamma^2, \gamma^3$. Hence α only depends on $\gamma^1, \gamma^2, \gamma^3$ and their exterior derivatives.

Suppose we now define the vector fields $\overline{X}_1, \overline{X}_2, \overline{X}_3$ as in (6.2), i.e.

$$\overline{X}_3 = X_3,$$
$$\overline{X}_2 = X_2 - X_2(\gamma^3)\overline{X}_3,$$
$$\overline{X}_1 = X_1 - X_1(\gamma^2)\overline{X}_2 - X_1(\gamma^3)\overline{X}_3.$$

Then from Lemma 6.3.3 we have that $\{\overline{X}_1, \overline{X}_2, \overline{X}_3\}$ is dual to $\{d\gamma^1, d\gamma^2, d\gamma^3\}$. If we choose an $1 \leq r \leq 3$ such that $\lambda_r \neq 0$, and (with a slight abuse of notation) define for each $k \in \{1, \ldots, r-1, r+1, \ldots, 3\}$,

$$W_k := \lambda_k \overline{X}_r - \lambda_r \overline{X}_k,$$

then $\ker(\alpha) = Sp\{Y_1, \ldots, Y_{n-3}, W_1, \ldots, W_{r-1}, W_{r+1}, \ldots, W_3\}$. From Theorem 6.3.5 and (6.15), it then follows that for each j and k, $[Y_j, W_k] \equiv 0 \bmod Y_1, \ldots, Y_{n-3}$.

Now let $\mathfrak{X}^2(U^n)$ be generated by $W_1, \ldots, W_{r-1}, W_{r+1}, \ldots, W_3$, and let $Z \in \mathfrak{X}^2(U^n)$. Then

$$Z = \mu^1 W_1 + \cdots + \mu^{r-1} W_{r-1} + \mu^{r+1} W_{r+1} + \cdots + \mu^3 W_3,$$

for some $\mu^1, \ldots, \mu^{r-1}, \mu^{r+1}, \ldots, \mu^3 \in \mathcal{F}(\gamma^a)$. From conclusion (3) in Lemma 6.3.1 we then have that for each j, $[Y_j, Z] \equiv 0 \bmod Y_1, \ldots, Y_{n-3}$.

Finally, it is clear that $Z \lrcorner\, \alpha = 0$, and since $Y_1, \ldots, Y_{n-3}$ are linearly independent of Z, we now have an $(n-2)$-dimensional Frobenius integrable distribution spanned by $Y_1, \ldots, Y_{n-3}$ and Z, that generates an integral manifold of $\langle \alpha \rangle$, and hence an integral manifold of $\langle \alpha, d\alpha \rangle$ using Theorem 2.2.8. $\square$

If, in applying Theorem 6.4.11 to the Cauchy problem (i.e. $n = 4$) we are given some Cauchy data in the form of a one-dimensional curve, and we are able to establish some vector field tangent to the curve that is in $\mathfrak{X}^2(U^4)$, then the theorem will generate a unique two-dimensional foliation of U^4 in which there exists a unique two-dimensional leaf containing the Cauchy data. If, however, we are only given a vector field $Z \in \mathfrak{X}^2(U^4)$ for the Cauchy data, then the theorem only guarantees *uniqueness up to foliation.* This means that we obtain, at most, a unique two-dimensional foliation of U^4, where the tangent space of each two-dimensional leaf is spanned by Z and the Cauchy characteristic vector field Y_1. Thus the Cauchy problem is 'solved' in terms of two arbitrary constants. We illustrate this further in the example in the next section.

6.5 An application

In this section we present an example showing how one may use Theorem 6.4.11 to generate local solutions of first order PDEs of one dependent variable and two independent variables of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}\right), \tag{6.16}$$

for some smooth function F. We work in the first jet bundle $J^1(U^2, V^1)$ with coordinates $x^1, x^2, z^1, z^1_1, z^1_2$, where U^2 has coordinates x^1, x^2 and V^1 has the coordinate z^1. The equation in (6.16) can be seen to describe a four-dimensional regular submanifold of the five-dimensional jet bundle, which we denote by $M^4 \subset J^1(U^2, V^1)$. Replacing the dependent variable u with the coordinate z^1, M^4 is therefore described by the locus of

$$z^1_2 = F(x^1, x^2, z^1, z^1_1).$$

With coordinates x^1, x^2, z^1, z^1_1 for M^4, the Pfaffian equation corresponding to (6.16) is

$$\alpha := dz^1 - z^1_1 dx^1 - F dx^2 = 0. \tag{6.17}$$

Before we examine a PDE example, we present the following result from Stormark [119], where it is proven for the case when there are an arbitrary number of independent variables:

Theorem 6.5.1. *With α defined as in (6.17), the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ is one-dimensional and generated by*

$$Y := \frac{\partial}{\partial x^2} - \frac{\partial F}{\partial z^1_1}\frac{\partial}{\partial x^1} + \left(F - z^1_1 \frac{\partial F}{\partial z^1_1}\right)\frac{\partial}{\partial z^1} + \left(\frac{\partial F}{\partial x^1} + z^1_1 \frac{\partial F}{\partial z^1}\right)\frac{\partial}{\partial z^1_1}.$$

Example 6.5.2. Consider the inviscid Burgers' equation:

$$\frac{\partial u}{\partial x^2} + u\frac{\partial u}{\partial x^1} = 0. \tag{6.18}$$

Here $M^4 \subset J^1(U^2, V^1)$ is described by the locus of

$$z^1_2 + z^1 z^1_1 = 0.$$

Using coordinates x^1, x^2, z^1, z^1_1 for M^4, we define

$$\alpha := dz^1 - z^1_1 dx^1 + z^1 z^1_1 dx^2.$$

By dimension, α has a maximum rank of one, so we suppose α has constant rank on M^4. The Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ is generated by

$$Y := \frac{\partial}{\partial x^2} + z^1 \frac{\partial}{\partial x^1} - (z^1_1)^2 \frac{\partial}{\partial z^1_1}.$$

It is not necessary at this point to ensure that $Y \lrcorner\, d\alpha = 0$. This can be done following the application of Theorem 3.2.14. With

$$\Omega := d\alpha \wedge \alpha,$$

we find that

$$X_3 := \frac{\partial}{\partial x^1}$$

is a non-trivial symmetry of Ω, that

$$X_2 := \frac{\partial}{\partial x^2}$$

is a non-trivial symmetry of $X_3 \lrcorner\, \Omega$, and finally that

$$X_1 := \frac{\partial}{\partial z^1}$$

is a non-trivial symmetry of $X_2 \lrcorner\, X_3 \lrcorner\, \Omega$. Applying Theorem 3.2.14 with Corollary 3.2.12 gives

$$\begin{aligned}
\omega^1 &= \frac{X_2 \lrcorner\, X_3 \lrcorner\, \Omega}{X_1 \lrcorner\, X_2 \lrcorner\, X_3 \lrcorner\, \Omega} = dz^1, \\
\omega^2 &= \frac{X_1 \lrcorner\, X_3 \lrcorner\, \Omega}{X_2 \lrcorner\, X_1 \lrcorner\, X_3 \lrcorner\, \Omega} = d\left(x^2 - \frac{1}{z^1_1}\right), \\
\omega^3 &= \frac{X_1 \lrcorner\, X_2 \lrcorner\, \Omega}{X_3 \lrcorner\, X_1 \lrcorner\, X_2 \lrcorner\, \Omega} = d\left(x^1 - \frac{z^1}{z^1_1}\right) + \frac{1}{z^1_1} dz^1,
\end{aligned}$$

on some suitably restricted domain. Next, we scale α so that $Y \lrcorner\, d\alpha = 0$. From Example 6.4.9 we obtain

$$Y \lrcorner\, d\left(\frac{\alpha}{z^1_1}\right) = 0.$$

We can therefore replace α with

$$\overline{\alpha} := \frac{\alpha}{z_1^1} = \frac{dz^1}{z_1^1} - dx^1 + z^1 dx^2.$$

Hence

$$\begin{aligned} \overline{\alpha} &= (X_1 \lrcorner \overline{\alpha})\, \omega^1 + (X_2 \lrcorner \overline{\alpha})\, \omega^2 + (X_3 \lrcorner \overline{\alpha})\, \omega^3, \\ &= z^1 d\left(x^2 - \frac{1}{z_1^1}\right) - d\left(x^1 - \frac{z^1}{z_1^1}\right). \end{aligned} \tag{6.19}$$

Now we have from Lemma 6.3.3 that

$$\left\{ dz^1, d\left(x^2 - \frac{1}{z_1^1}\right), d\left(x^1 - \frac{z^1}{z_1^1}\right) \right\}$$

is dual to

$$\left\{ \frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1} \right\}.$$

Thus from the expression for $\overline{\alpha}$ in (6.19), we let our Cauchy data be generated by

$$Z := \mu^1 \left(\frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1} \right) + \mu^2 \left(z^1 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \right),$$

where μ^1, μ^2 are any smooth functions of $z^1, x^2 - 1/z_1^1, x^1 - z^1/z_1^1$, so that $Z \lrcorner \overline{\alpha} = 0$. Note that with this construction for the Cauchy data, applying Theorem 6.4.11 will only give us uniqueness up to foliation. Therefore any solution of (6.18) will depend on two arbitrary constants.

Now from Theorem 6.4.11, $[Y, Z] \equiv 0 \bmod Y$, and hence $Sp\{Y, Z\}$ is a two-dimensional Frobenius integrable distribution that generates an integral manifold of the differential ideal $\langle \alpha, d\alpha \rangle$. To obtain a local solution to (6.18), we choose Z so that the transverse condition $dx^1 \wedge dx^2 \neq 0$ is satisfied on the distribution $Sp\{Y, Z\}$.

We now proceed to pick two different Z's, and derive a local solution to (6.18) for each choice. First choose

$$Z = \frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1}.$$

On M^4, the distribution spanned by Y and Z is annihilated by the two one-forms $\overline{\alpha}$ and $(z_1^1)^2 dx^2 + dz_1^1$. Define

$$\overline{\Omega} := \overline{\alpha} \wedge \left((z_1^1)^2 dx^2 + dz_1^1 \right).$$

Since $Sp\{Y, Z\}$ is a Frobenius integrable distribution, we have that $d\overline{\Omega} \equiv 0 \bmod \overline{\Omega}$. Hence we can proceed to once again apply Theorem 3.2.14. We find that

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$$

are two linearly independent commuting non-trivial symmetries of $\overline{\Omega}$. We can use Theorem 3.2.16 so that Theorem 3.2.14 will give us the two closed forms

$$\frac{\frac{\partial}{\partial x^1} \lrcorner \overline{\Omega}}{\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial x^1} \lrcorner \overline{\Omega}} = d\left(x^2 - \frac{1}{z_1^1}\right),$$
$$\frac{\frac{\partial}{\partial x^2} \lrcorner \overline{\Omega}}{\frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner \overline{\Omega}} = d\left(x^1 - \frac{z^1}{z_1^1}\right).$$

In general, we will not produce the same first integrals we obtained before. Put

$$x^1 - \frac{z^1}{z_1^1} = c^1, \qquad x^2 - \frac{1}{z_1^1} = c^2,$$

for any choice of constant functions c^1, c^2. Then equating the z_1^1 term and replacing z^1 with u, we get the following local solution (where $x^2 \neq c^2$) of the PDE in (6.18):

$$u = \frac{x^1 - c^1}{x^2 - c^2}.$$

For our second choice for Z, we will be a little more ambitious and suppose that

$$Z = \frac{\partial}{\partial z^1} + \frac{1}{z_1^1}\frac{\partial}{\partial x^1} + \frac{1}{z^1}\left(x^2 - \frac{1}{z_1^1}\right)\left(z^1\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\right).$$

Following the same procedure as before, and using the same notation, we define

$$\overline{\Omega} := \overline{\alpha} \wedge \left(dz_1^1 + (z_1^1)^2 dx^2 + \frac{z_1^1}{z^1}\left(1 - z_1^1 x^2\right) dz^1\right).$$

It is easy to see that $\frac{\partial}{\partial x^1}$ is a non-trivial symmetry of $\overline{\Omega}$. Using `DIMSYM` and Theorem 2.3.10, we find that $z^1\frac{\partial}{\partial z^1}$ is a non-trivial symmetry of $\frac{\partial}{\partial x^1} \lrcorner \overline{\Omega}$. Applying Theorem 3.2.14 yields

$$\frac{\frac{\partial}{\partial x^1} \lrcorner \overline{\Omega}}{z^1\frac{\partial}{\partial z^1} \lrcorner \frac{\partial}{\partial x^1} \lrcorner \overline{\Omega}} = d\left(\ln\left|\frac{z^1 z_1^1}{1 - z_1^1 x^2}\right|\right),$$
$$\frac{z^1\frac{\partial}{\partial z^1} \lrcorner \overline{\Omega}}{\frac{\partial}{\partial x^1} \lrcorner z^1\frac{\partial}{\partial z^1} \lrcorner \overline{\Omega}} \equiv d\left(x^1 + \frac{(x^2)^2 z^1 z_1^1}{2(1 - z_1^1 x^2)} - \frac{z^1}{2z_1^1(1 - z_1^1 x^2)}\right) \bmod d\left(\ln\left|\frac{z^1 z_1^1}{1 - z_1^1 x^2}\right|\right).$$

We put

$$\frac{z^1 z_1^1}{1 - z_1^1 x^2} = \pm\exp(c^1), \qquad x^1 + \frac{(x^2)^2 z^1 z_1^1}{2(1 - z_1^1 x^2)} - \frac{z^1}{2z_1^1(1 - z_1^1 x^2)} = c^2,$$

for any constant functions c^1, c^2. Removing the z_1^1 term, and replacing z^1 with u, we then get on one branch the following local solution to (6.18):

$$u = \sqrt{\exp(c^1)}\left(\sqrt{2(x^1 - c^2) + \exp(c^1)(x^2)^2} - \sqrt{\exp(c^1)}x^2\right).$$

6.6 Pfaffian systems in general

In this section we give a generalisation of Theorem 6.4.11 that will ultimately allow us to use symmetry to solve the Cauchy problem for Pfaffian systems that are generated by a finite number of linearly independent one-forms, each of arbitrary constant rank on U^n.

Before we do this we must first modify Lemmas 6.4.5 and 6.4.8, as well as provide some extra preliminary material. We begin with a slight generalisation of Lemma 6.4.5:

Lemma 6.6.1. *Let $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U^n)$ be p linearly independent one-forms such that the dimension of $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ is greater than zero. Let $Y \in \mathfrak{X}(U^n)$ be a non-zero vector field in $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ so that for each $1 \le i \le p$,*

$$Y \lrcorner\, d\alpha^i = \mu^i_1 \alpha^1 + \cdots + \mu^i_p \alpha^p,$$

for some $\mu^i_1, \ldots, \mu^i_p \in C^\infty(U^n)$. If, for each i, $h_i \in C^\infty(U^n)$ is non-zero and satisfies $Y(h_i) + h_i \mu^i_i = 0$, then (no sum)

$$Y \lrcorner\, d(h_i \alpha^i) \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

Proof. Let $Y \in A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ be non-zero with $Y \lrcorner\, d\alpha^i = \mu^i_1 \alpha^1 + \cdots + \mu^i_p \alpha^p$ for some $\mu^i_1, \ldots, \mu^i_p \in C^\infty(U^n)$. If, for all $1 \le i \le p$, h_i satisfies $Y(h_i) + h_i \mu^i_i = 0$, then (no sum)

$$\begin{aligned} Y \lrcorner\, d\left(h_i \alpha^i\right) &= Y \lrcorner\, \left(dh_i \wedge \alpha^i + h_i d\alpha^i\right), \\ &= h_i \left(\mu^i_1 \alpha^1 + \cdots + \mu^i_{i-1} \alpha^{i-1} + \mu^i_{i+1} \alpha^{i+1} + \cdots + \mu^i_p \alpha^p\right) \\ &\quad + \left(Y(h_i) + h_i \mu^i_i\right) \alpha^i. \end{aligned}$$

The result is now obvious. □

At this point, we wish to examine a generalisation of the existence result given in Theorem 6.4.6 for Cauchy characteristic spaces of a higher dimension than in Lemma 6.6.1. With $\alpha^1, \ldots, \alpha^p$ defined as in the lemma, suppose for some $q \ge 1$, $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$ generates the Cauchy characteristic space of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$. Hence for each $1 \le i \le p$ and $1 \le j \le q$,

$$Y_j \lrcorner\, d\alpha^i = \mu^i_{j1} \alpha^1 + \cdots + \mu^i_{jp} \alpha^p,$$

for some $\mu^i_{j1}, \ldots, \mu^i_{jp} \in C^\infty(U^n)$. Since $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ is Frobenius integrable, we may write for each $1 \leq k, l \leq q$,

$$[Y_k, Y_l] = \rho^1_{kl} Y_1 + \cdots + \rho^q_{kl} Y_q, \tag{6.20}$$

for some $\rho^1_{kl}, \ldots, \rho^q_{kl} \in C^\infty(U^n)$. If, for each i, there exists some non-zero $h_i \in C^\infty(U^n)$ such that for all j,

$$Y_j(h_i) + h_i \mu^i_{ji} = 0, \tag{6.21}$$

then it is clear from Lemma 6.6.1 that for each i and j, (no sum)

$$Y_j \lrcorner\, d(h_i \alpha^i) \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

For there to exist some non-zero $h_1, \ldots, h_p$, the integrability conditions for the pq partial differential equations in (6.21) can easily be shown to be

$$Y_k(\mu^i_{li}) - Y_l(\mu^i_{ki}) = \rho^1_{kl} \mu^i_{1i} + \cdots + \rho^q_{kl} \mu^i_{qi}. \tag{6.22}$$

Unfortunately, we show below that the equations in (6.22) are in general not satisfied for $q > 1$:

Using the same approach given in Theorem 6.4.6, we find that for all i,

$$\mathcal{L}_{[Y_k, Y_l]} \alpha^i = \mathcal{L}_{Y_k} \left(\mu^i_{l1} \alpha^1 + \cdots + \mu^i_{lp} \alpha^p \right) - \mathcal{L}_{Y_l} \left(\mu^i_{k1} \alpha^1 + \cdots + \mu^i_{kp} \alpha^p \right),$$

which expands to give

$$\begin{aligned} \mathcal{L}_{[Y_k, Y_l]} \alpha^i = {} & \left[Y_k(\mu^i_{l1}) - Y_l(\mu^i_{k1}) \right] \alpha^1 + \cdots + \left[Y_k(\mu^i_{li}) - Y_l(\mu^i_{ki}) \right] \alpha^i + \ldots \\ & + \left[Y_k(\mu^i_{lp}) - Y_l(\mu^i_{kp}) \right] \alpha^p + \zeta^i, \end{aligned}$$

where

$$\begin{aligned} \zeta^i := {} & \left[(\mu^i_{l1} \mu^1_{k1} + \cdots + \mu^i_{lp} \mu^p_{k1}) - (\mu^i_{k1} \mu^1_{l1} + \cdots + \mu^i_{kp} \mu^p_{l1}) \right] \alpha^1 + \ldots \\ & + \left[(\mu^i_{l1} \mu^1_{ki} + \cdots + \mu^i_{lp} \mu^p_{ki}) - (\mu^i_{k1} \mu^1_{li} + \cdots + \mu^i_{kp} \mu^p_{li}) \right] \alpha^i + \ldots \\ & + \left[(\mu^i_{l1} \mu^1_{kp} + \cdots + \mu^i_{lp} \mu^p_{kp}) - (\mu^i_{k1} \mu^1_{lp} + \cdots + \mu^i_{kp} \mu^p_{lp}) \right] \alpha^p. \end{aligned}$$

We also have using (6.20),

$$\mathcal{L}_{[Y_k, Y_l]} \alpha^i = \mathcal{L}_{\rho^1_{kl} Y_1 + \cdots + \rho^q_{kl} Y_q} \alpha^i,$$

which expands to give

$$\mathcal{L}_{[Y_k,Y_l]}\alpha^i = \left(\rho^1_{kl}\mu^i_{11} + \cdots + \rho^q_{kl}\mu^i_{q1}\right)\alpha^1 + \cdots + \left(\rho^1_{kl}\mu^i_{1i} + \cdots + \rho^q_{kl}\mu^i_{qi}\right)\alpha^i + \dots$$
$$+ \left(\rho^1_{kl}\mu^i_{1p} + \cdots + \rho^q_{kl}\mu^i_{qp}\right)\alpha^p.$$

Hence the integrability conditions in (6.22) become satisfied if and only if for each i, k and l,

$$\mu^i_{ka}\mu^a_{li} = \mu^i_{la}\mu^a_{ki}, \tag{6.23}$$

with summation only on $1 \leq a \leq p$. Of course if $q = 1$, then the conditions in (6.23) are met.

Undeterred by this inconvenience for Cauchy characteristic spaces of a higher dimension than one, in what follows we assume the integrability conditions in (6.22) are satisfied. The reason for this is that using symmetries, Lemma 6.4.8 easily generalises to give the following four results below:

Lemma 6.6.2. *Let $\alpha^1, \dots, \alpha^p \in \Lambda^1(U^n)$ be p linearly independent one-forms. Suppose $Y_1, \dots, Y_q \in \mathfrak{X}(U^n)$ generate the Cauchy characteristic space of $\langle \alpha^1, \dots, \alpha^p, d\alpha^1, \dots, d\alpha^p \rangle$. If, for each $1 \leq i \leq p$, $Z_i \in \mathfrak{X}(U^n)$ is a non-trivial symmetry of $\langle \alpha^i, d\alpha^i \rangle$, and $Z_i \lrcorner\, \alpha^j = 0$ for all $1 \leq j \leq p$ with $j \neq i$, then for each i and $1 \leq k \leq q$,*

$$Y_k \lrcorner\, d\left(\frac{\alpha^i}{Z_i \lrcorner\, \alpha^i}\right) \equiv 0 \mod \alpha^1, \dots, \alpha^{i-1}, \alpha^{i+1}, \dots, \alpha^p.$$

Proof. For each $1 \leq i \leq p$ and $1 \leq k \leq q$, we have from the proof of Lemma 6.4.8 that

$$Y_k \lrcorner\, d\left(\frac{\alpha^i}{Z_i \lrcorner\, \alpha^i}\right) = \frac{1}{(Z_i \lrcorner\, \alpha^i)^2} Y_k \lrcorner\, Z_i \lrcorner\, (d\alpha^i \wedge \alpha^i).$$

Note from Lemma 6.4.7 that Z_i is a non-trivial symmetry of α^i, so $Z_i \lrcorner\, \alpha^i \neq 0$. We have that

$$\left[Y_k \lrcorner\, d\left(\frac{\alpha^i}{Z_i \lrcorner\, \alpha^i}\right)\right] \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p$$
$$= -\frac{1}{(Z_i \lrcorner\, \alpha^i)^2}\left[Z_i \lrcorner\, Y_k \lrcorner\, (d\alpha^i \wedge \alpha^i)\right] \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p,$$
$$= -\frac{1}{(Z_i \lrcorner\, \alpha^i)^2} Z_i \lrcorner \left[\left(Y_k \lrcorner\, (d\alpha^i \wedge \alpha^i)\right) \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p\right],$$

using the assumption that $Z_i \lrcorner \alpha^j = 0$ for all $1 \leq j \leq p$ with $j \neq i$. Also by assumption, $Y_k \in A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$, so it follows that

$$\begin{aligned}\left(Y_k \lrcorner (d\alpha^i \wedge \alpha^i)\right) \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p \\ = \left(Y_k \lrcorner d\alpha^i\right) \wedge \alpha^i \wedge \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \cdots \wedge \alpha^p = 0.\end{aligned}$$

Hence for all i and k,

$$Y_k \lrcorner d\left(\frac{\alpha^i}{Z_i \lrcorner \alpha^i}\right) \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

□

Lemma 6.6.3. *Let $Y \in \mathfrak{X}(U^n)$ be a non-zero Cauchy characteristic vector field of the differential ideal $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ generated by some p linearly independent $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U^n)$. If, for each $1 \leq i \leq p$,*

$$Y \lrcorner d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p, \tag{6.24}$$

then $Y \lrcorner d(\alpha^1 \wedge \cdots \wedge \alpha^p) = 0$.

Proof. Since

$$d\left(\alpha^1 \wedge \cdots \wedge \alpha^p\right) = \left(d\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p\right) + \cdots + \left(d\alpha^p \wedge \alpha^1 \wedge \cdots \wedge \alpha^{p-1}\right),$$

and $Y \lrcorner \alpha^i = 0$ for all $1 \leq i \leq p$, we therefore have that

$$\begin{aligned}Y \lrcorner d\left(\alpha^1 \wedge \cdots \wedge \alpha^p\right) = \left((Y \lrcorner d\alpha^1) \wedge \alpha^2 \wedge \cdots \wedge \alpha^p\right) + \ldots \\ + \left((Y \lrcorner d\alpha^p) \wedge \alpha^1 \wedge \cdots \wedge \alpha^{p-1}\right).\end{aligned}$$

Then because of (6.24), the result is now obvious. □

Using Lemma 6.6.3 we have the following result:

Theorem 6.6.4. *Let $Y \in \mathfrak{X}(U^n)$ be a non-zero Cauchy characteristic vector field of the differential ideal $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ generated by some p linearly independent $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U^n)$. Suppose that for each $1 \leq i \leq p$,*

$$Y \lrcorner d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

Further suppose there exist $p-1$ linearly independent vector fields $Z_1, \ldots, Z_{p-1} \in \mathfrak{X}(U^n)$, each not in $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$, such that with $\Omega := \alpha^1 \wedge \cdots \wedge \alpha^p$, we have

$$\begin{aligned} \mathcal{L}_{Z_{p-1}} \Omega &= \lambda_{p-1} \Omega, \\ \mathcal{L}_{Z_{p-2}} (Z_{p-1} \lrcorner\, \Omega) &= \lambda_{p-2} (Z_{p-1} \lrcorner\, \Omega), \\ &\vdots \\ \mathcal{L}_{Z_1} (Z_2 \lrcorner\, \ldots \lrcorner\, Z_{p-1} \lrcorner\, \Omega) &= \lambda_1 (Z_2 \lrcorner\, \ldots \lrcorner\, Z_{p-1} \lrcorner\, \Omega), \end{aligned} \tag{6.25}$$

for some $\lambda_1, \ldots, \lambda_{p-1} \in C^\infty(U^n)$. Then

$$Z_1 \lrcorner\, \ldots \lrcorner\, Z_{p-1} \lrcorner\, \Omega = \rho_1 \alpha^1 + \cdots + \rho_p \alpha^p \neq 0,$$

for some $\rho_1, \ldots, \rho_p \in C^\infty(U^n)$ such that $Y \lrcorner\, d\left(\rho_1 \alpha^1 + \cdots + \rho_p \alpha^p\right) = 0$.

Proof. We first rewrite the equations in (6.25) in the following form:

$$\begin{aligned} Z_{p-1} \lrcorner\, d\Omega + d\left(Z_{p-1} \lrcorner\, \Omega\right) &= \lambda_{p-1} \Omega, \\ Z_{p-2} \lrcorner\, d(Z_{p-1} \lrcorner\, \Omega) + d\left(Z_{p-2} \lrcorner\, Z_{p-1} \lrcorner\, \Omega\right) &= \lambda_{p-2} (Z_{p-1} \lrcorner\, \Omega), \\ &\vdots \\ Z_1 \lrcorner\, d(X_2 \lrcorner\, \ldots \lrcorner\, Z_{p-1} \lrcorner\, \Omega) + d\left(Z_1 \lrcorner\, \ldots Z_{p-1} \lrcorner\, \Omega\right) &= \lambda_1 (Z_2 \lrcorner\, \ldots \lrcorner\, Z_{p-1} \lrcorner\, \Omega). \end{aligned} \tag{6.26}$$

Now Lemma 6.6.3 gives

$$Y \lrcorner\, Z_1 \lrcorner\, \ldots \lrcorner\, Z_{p-1} \lrcorner\, d\Omega = 0. \tag{6.27}$$

We can then use (6.27) and the first equation in (6.26) to show that

$$Y \lrcorner\, Z_1 \lrcorner\, \ldots \lrcorner\, Z_{p-1} \lrcorner\, d\Omega = Y \lrcorner\, Z_1 \lrcorner\, \ldots Z_{p-2} \lrcorner \left[\lambda_{p-1} \Omega - d\left(Z_{p-1} \lrcorner\, \Omega\right)\right] = 0.$$

Since $Y \in A\left(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle\right)$ implies $Y \lrcorner\, \Omega = 0$, we therefore have

$$Y \lrcorner\, Z_1 \lrcorner\, \ldots Z_{p-2} \lrcorner\, d\left(Z_{p-1} \lrcorner\, \Omega\right) = 0. \tag{6.28}$$

We can repeat the above procedure, but this time using (6.28) and the second equation in (6.26) to show that

$$Y \lrcorner\, Z_1 \lrcorner\, \ldots Z_{p-3} \lrcorner\, d\left(Z_{p-2} \lrcorner\, Z_{p-1} \lrcorner\, \Omega\right) = 0.$$

Continuing in this fashion we eventually obtain

$$Y \lrcorner\, d\left(Z_1 \lrcorner\, \ldots Z_{p-1} \lrcorner\, \Omega\right) = 0.$$

Since $Z_1, \ldots, Z_{p-1}$ are all linearly independent, and each not in the Cauchy characteristic space of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$, we have that

$$Z_1 \lrcorner \ldots Z_{p-1} \lrcorner \Omega \neq 0.$$

Hence

$$Z_1 \lrcorner \ldots Z_{p-1} \lrcorner \Omega = \rho_1 \alpha^1 + \cdots + \rho_p \alpha^p,$$

for some $\rho_1, \ldots, \rho_p \in C^\infty(U^n)$, with at least one of $\rho_1, \ldots, \rho_p$ being not identically zero. □

Remark. The solvable symmetry structure condition for Ω in (6.25) is not the same as the one given in Corollary 3.2.12 for the application of Theorem 3.2.14. The difference is that the former contains one less stage. Nevertheless, by repeatedly applying Theorem 2.3.11 with Lemma 3.2.2, `DIMSYM` may still be used to find the symmetries in Theorem 6.6.4.

In our work, it is the following corollary to Theorem 6.6.4 that will be useful:

Corollary 6.6.5. *Let $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U^n)$ be some p linearly independent one-forms and suppose $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ is generated by some $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$. Further suppose that for each $1 \leq i \leq p$ and $1 \leq j \leq q$,*

$$Y_j \lrcorner d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p.$$

Define $\Omega := \alpha^1 \wedge \cdots \wedge \alpha^p$. If, for each i, there exist $p-1$ linearly independent vector fields $Z^i_1, \ldots, Z^i_{p-1} \in \mathfrak{X}(U^n)$ each not in $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ that satisfy the solvable symmetry structure property

$$\mathcal{L}_{Z^i_{p-1}} \Omega = \lambda^i_{p-1} \Omega,$$
$$\mathcal{L}_{Z^i_{p-2}} (Z^i_{p-1} \lrcorner \Omega) = \lambda^i_{p-2} (Z^i_{p-1} \lrcorner \Omega),$$
$$\vdots$$
$$\mathcal{L}_{Z^i_1} (Z^i_2 \lrcorner \ldots \lrcorner Z^i_{p-1} \lrcorner \Omega) = \lambda^i_1 (Z^i_2 \lrcorner \ldots \lrcorner Z^i_{p-1} \lrcorner \Omega),$$

for some $\lambda^i_1, \ldots, \lambda^i_{p-1} \in C^\infty(U^n)$, and if our symmetries are such that the one-forms $\beta^1, \ldots, \beta^p$ defined by $\beta^i := Z^i_1 \lrcorner \ldots \lrcorner Z^i_{p-1} \lrcorner \Omega$ are linearly independent on U^n, then

$$\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle = \langle \beta^1, \ldots, \beta^p, d\beta^1, \ldots, d\beta^p \rangle,$$

such that for all i and j, $Y_j \lrcorner \beta^i = 0 = Y_j \lrcorner d\beta^i$.

We may summarise some of the above results of this section in the following way: Given $A(\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle)$ generated by some $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$, we look to first apply Lemma 6.6.1 if $q = 1$ or Lemma 6.6.2 if $q \geq 1$, so that for each $1 \leq i \leq p$ and $1 \leq j \leq q$, (no sum)

$$Y_j \lrcorner\, d(h_i \alpha^i) \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^p,$$

for some non-zero $h_1, \ldots, h_p \in C^\infty(U^n)$. Then we look to use Corollary 6.6.5 to obtain some $\beta^1, \ldots, \beta^p \in \Lambda^1(U^n)$ such that

$$\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle = \langle \beta^1, \ldots, \beta^p, d\beta^1, \ldots, d\beta^p \rangle,$$

with the property that for all i and j, $Y_j \lrcorner\, \beta^i = 0 = Y_j \lrcorner\, d\beta^i$. We will illustrate this process in the example contained in the next section.

We now present the main result of this section, Theorem 6.6.6. In what follows, we assume $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U^n)$ are p linearly independent one-forms, and the Cauchy characteristic space of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ is generated by $Y_1, \ldots, Y_q \in \mathfrak{X}(U^n)$, so that the Cartan system of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ is generated by some $n - q$ linearly independent one-forms $\sigma^1, \ldots, \sigma^{n-q} \in \Lambda^1(U^n)$. Suppose also that we have been able to apply Lemmas 6.6.1 or 6.6.2, and Corollary 6.6.5, so that for all $1 \leq i \leq p$ and $1 \leq j \leq q$, $Y_j \lrcorner\, d\alpha^i = 0$. Theorem 6.6.6, when $q = 1$, once again provides us with a symmetry approach to the Cauchy problem when the Cauchy characteristic space is one-dimensional.

Theorem 6.6.6. *Define $\Omega := \sigma^1 \wedge \cdots \wedge \sigma^{n-q}$ and let $X_1, \ldots, X_{n-q} \in \mathfrak{X}(U^n)$ be a solvable symmetry structure for $A(\langle \Omega \rangle)$. There exists a $(n - q - p)$-dimensional submodule $\mathfrak{X}^{n-q-p}(U^n)$ of $\mathfrak{X}(U^n)$ over $\mathcal{F}(\gamma^a)$ such that if we are given a v-dimensional regular submanifold of U^n whose tangent space is spanned by some v linearly independent vector fields $Z_1, \ldots, Z_v \in \mathfrak{X}^{n-q-p}(U^n)$, then for each $1 \leq j \leq q$ and $1 \leq l \leq v$, $[Y_j, Z_l] \equiv 0$ mod $Y_1, \ldots, Y_q$. Moreover, the distribution $Sp\{Y_1, \ldots, Y_q, Z_1, \ldots, Z_v\}$ is Frobenius integrable and generates a $(q + v)$-dimensional integral manifold of the differential ideal $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$.*

Proof. Since the Cauchy characteristic space of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ is Frobenius integrable, we have $d\Omega \equiv 0 \mod \Omega$. Applying Theorem 3.2.14 with $\omega^1, \ldots, \omega^{n-q}$

defined as in the theorem yields $\{\omega^1, \dots, \omega^{n-q}\}$ dual to $\{X_1, \dots, X_{n-q}\}$ such that

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\
&\vdots \\
\omega^{n-q} &\equiv d\gamma^{n-q} \mod d\gamma^1, \dots, d\gamma^{n-q-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \dots, \gamma^{n-q} \in C^\infty(U^n)$. Now from the note following Theorem 2.2.13, it follows that for all $1 \le s \le p$, $\alpha^s \in C(\langle \alpha^1, \dots, \alpha^p, d\alpha^1, \dots, d\alpha^p \rangle)$. This means that for each s,

$$\begin{aligned}
\alpha^s &= (X_1 \lrcorner\, \alpha^s)\, \omega^1 + \cdots + (X_{n-q} \lrcorner\, \alpha^s)\, \omega^{n-q}, \\
&= \lambda_1^s d\gamma^1 + \cdots + \lambda_{n-q}^s d\gamma^{n-q},
\end{aligned}$$

for some $\lambda_1^s, \dots, \lambda_{n-q}^s \in C^\infty(U^n)$. By assumption, $Y_j \lrcorner\, \alpha^s = 0 = Y_j \lrcorner\, d\alpha^s$ for each $1 \le j \le q$ and s. This gives

$$\mathcal{L}_{Y_j} \alpha^s = 0, \tag{6.29}$$

But we know from conclusion (3) in Lemma 6.3.1 that $Y_j \lrcorner\, d\gamma^i = 0$, for each $1 \le i \le n - q$ and j, so (6.29) implies

$$Y_j(\lambda_1^s)d\gamma^1 + \cdots + Y_j(\lambda_{n-q}^s)d\gamma^{n-q} = 0.$$

Since $d\gamma^1, \dots, d\gamma^{n-q}$ are linearly independent, we get

$$Y_j(\lambda_i^s) = 0,$$

for all i, j and s. By the same argument used in the proof of Theorem 6.4.11, we can then conclude that $\alpha^1, \dots, \alpha^p$ only depend on $\gamma^1, \dots, \gamma^{n-q}$ and their exterior derivatives.

The rest of the proof continues to be very similar to that of Theorem 6.4.11. Define the vector fields $\overline{X}_1, \dots, \overline{X}_{n-q}$ as in (6.2). Then from Lemma 6.3.3 we have that $\{\overline{X}_1, \dots, \overline{X}_{n-q}\}$ is dual to $\{d\gamma^1, \dots, d\gamma^{n-q}\}$. Using this and the fact that each α^s only depends on $\gamma^1, \dots, \gamma^{n-q}$ and their exterior derivatives, it is then obvious that the kernel of $\alpha^1 \wedge \cdots \wedge \alpha^p$ (which equal the distribution annihilated by each α^s)

is spanned by the Cauchy characteristic fields $Y_1, \ldots, Y_q$, and some $n-q-p$ linearly independent vector fields $W_1, \ldots, W_{n-q-p} \in \mathfrak{X}(U^n)$ that are linear combinations of $\overline{X}_1, \ldots, \overline{X}_{n-q}$ with coefficients in $\mathcal{F}(\gamma^a)$. Since we have expressed α^s solely in terms of $\gamma^1, \ldots, \gamma^{n-q}$ and their exterior derivatives, Lemma 6.3.3 now means that calculating $W_1, \ldots, W_{n-q-p}$ is a straightforward exercise. From Theorem 6.3.5 and conclusion (3) in Lemma 6.3.1, it now follows that for each j and $1 \leq k \leq n-q-p$, $[Y_j, W_k] \equiv 0 \bmod Y_1, \ldots, Y_q$.

Now let $\mathfrak{X}^{n-q-p}(U^n)$ be generated by $W_1, \ldots, W_{n-q-p}$, and suppose we are given some v-dimensional regular submanifold of U^n whose tangent space is spanned by v linearly independent vector fields $Z_1, \ldots, Z_v \in \mathfrak{X}^{n-q-p}(U^n)$. Then for each $1 \leq l \leq v$,

$$Z_l = \mu_l^1 W_1 + \cdots + \mu_l^{n-q-p} W_{n-q-p},$$

for some $\mu_l^1, \ldots, \mu_l^{n-q-p} \in \mathcal{F}(\gamma^a)$. From conclusion (3) in Lemma 6.3.1 we then get that for each j and l, $[Y_j, Z_l] \equiv 0 \bmod Y_1, \ldots, Y_q$. Since for each l, $Z_l \lrcorner\, \alpha = 0$, and for each j, Y_j is linearly independent of $Z_1, \ldots, Z_v$, it follows that $Sp\{Y_1, \ldots, Y_q, Z_1, \ldots, Z_v\}$ is a $(q+v)$-dimensional Frobenius integrable distribution that generates a $(q+v)$-dimensional integral manifold of $\langle \alpha^1, \ldots, \alpha^p \rangle$, and hence an integral manifold of $\langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$ using Theorem 2.2.8. $\square$

As in Theorem 6.4.11 earlier, if in applying Theorem 6.6.6 to the Cauchy problem (i.e. $q = 1$) we are given the Cauchy data only in the form of some $Z_1, \ldots, Z_v \in \mathfrak{X}^{n-1-p}(U^n)$ that are closed under the Lie bracket, then the theorem only guarantees uniqueness up to foliation. Therefore we obtain a unique $(v+1)$-dimensional foliation of U^n, where the $(v+1)$-dimensional leaves depend on $n-v-1$ constants.

In the next section we discuss two PDE applications of Theorem 6.6.6 and give one example.

6.7 Two applications

This section examines two types of PDE problems for which there exist non-zero Cauchy characteristic vector fields. The first of these considers a single first order (possibly non-linear) PDE of one dependent variable and an arbitrary number of independent variables, while the second deals with a special class of systems of two second order PDEs of one dependent variable and two independent variables.

Beginning with the first situation, suppose we are given a first order PDE of one dependent variable and n independent variables of the form

$$\frac{\partial u}{\partial x^n} = F\left(x^1, \ldots, x^n, u, \frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^{n-1}}\right), \tag{6.30}$$

for smooth F. We work in the first jet bundle $J^1(U^n, V^1)$, where U^n has coordinates $x^1, \ldots, x^n$ and V^1 has the coordinate z^1. We express the PDE in (6.30) as

$$z^1_n = F(x^1, \ldots, x^n, z^1, z^1_1, \ldots, z^1_{n-1}), \tag{6.31}$$

replacing the dependent variable u with the coordinate z^1. The Pfaffian equation corresponding to (6.30) on the regular submanifold $M^{2n} \subset J^1(U^n, V^1)$ defined by (6.31), with coordinates $x^1, \ldots, x^n, z^1, z^1_1, \ldots, z^1_{n-1}$ is

$$\alpha := dz^1 - z^1_1 dx^1 - \cdots - z^1_{n-1} dx^{n-1} - F dx^n = 0. \tag{6.32}$$

The generalisation of Theorem 6.5.1 as in [119] is the following:

Theorem 6.7.1. *With α defined as in (6.32), the Cauchy characteristic space of $\langle \alpha, d\alpha \rangle$ is one-dimensional and generated by*

$$Y := \frac{\partial}{\partial x^n} - \sum_{i=1}^{n-1} \frac{\partial F}{\partial z^1_i} \frac{\partial}{\partial x^i} + \left(F - \sum_{i=1}^{n-1} z^1_i \frac{\partial F}{\partial z^1_i}\right) \frac{\partial}{\partial z^1} + \sum_{i=1}^{n-1} \left(\frac{\partial F}{\partial x^i} + z^1_i \frac{\partial F}{\partial z^1}\right) \frac{\partial}{\partial z^1_i}.$$

Using Theorem 6.7.1 we may therefore apply Theorem 6.6.6 to find local solutions of the first order PDE in (6.30). We will not provide an example of this. Instead, we will give an example of the following material that deals with second order PDEs:

Consider the following pair of second order PDEs:

$$\begin{aligned} \frac{\partial^2 u}{\partial (x^1)^2} &= F_1\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2} \frac{\partial^2 u}{\partial x^1 \partial x^2}\right), \\ \frac{\partial^2 u}{\partial (x^2)^2} &= F_2\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2} \frac{\partial^2 u}{\partial x^1 \partial x^2}\right), \end{aligned} \tag{6.33}$$

for some smooth functions F_1 and F_2. This time, we work in the second jet bundle $J^2(U^2, V^1)$, and represent the PDEs in (6.33) by

$$\begin{aligned} z^1_{11} &= F_1(x^1, x^2, z^1, z^1_1, z^1_2, z^1_{12}), \\ z^1_{22} &= F_2(x^1, x^2, z^1, z^1_1, z^1_2, z^1_{12}), \end{aligned} \tag{6.34}$$

replacing the dependent variable u with the coordinate z^1. Define $M^6 \subset J^2(U^2, V^1)$ as the six-dimensional regular submanifold defined by (6.34). On M^6 with coordinates $x^1, x^2, z^1, z_1^1, z_2^1, z_{12}^1$, the Pfaffian system corresponding to (6.33) is generated by the one-forms

$$\begin{aligned} \alpha^1 &:= dz^1 - z_1^1 dx^1 - z_2^1 dx^2, \\ \alpha^2 &:= dz_1^1 - F_1 dx^1 - z_{12}^1 dx^2, \\ \alpha^3 &:= dz_2^1 - z_{12}^1 dx^1 - F_2 dx^2. \end{aligned} \tag{6.35}$$

Note that $d\alpha^1 \equiv 0 \mod \alpha^2, \alpha^3$. We will use the following result also found in [119]:

Theorem 6.7.2. *With $\alpha^1, \alpha^2, \alpha^3$ defined as in (6.35), the Cauchy characteristic space of $\langle \alpha^1, \alpha^2, \alpha^3, d\alpha^2, d\alpha^3 \rangle$ is one-dimensional and generated by*

$$\begin{aligned} Y := \frac{\partial}{\partial x^1} - \frac{\partial F_1}{\partial z_{12}^1}\frac{\partial}{\partial x^2} + \left(z_1^1 - z_2^1 \frac{\partial F_1}{\partial z_{12}^1}\right)\frac{\partial}{\partial z^1} + \left(F_1 - z_{12}^1 \frac{\partial F_1}{\partial z_{12}^1}\right)\frac{\partial}{\partial z_1^1} \\ + \left(z_{12}^1 - F_2 \frac{\partial F_1}{\partial z_{12}^1}\right)\frac{\partial}{\partial z_2^1} + \left(\frac{\partial F_1}{\partial x^2} + z_2^1 \frac{\partial F_1}{\partial z^1} + z_{12}^1 \frac{\partial F_1}{\partial z_1^1} + F_2 \frac{\partial F_1}{\partial z_2^1}\right)\frac{\partial}{\partial z_{12}^1}, \end{aligned}$$

if and only if

$$\frac{\partial F_1}{\partial z_{12}^1}\frac{\partial F_2}{\partial z_{12}^1} = 1,$$

and

$$\begin{aligned} \frac{\partial F_1}{\partial x^2} + z_2^1 \frac{\partial F_1}{\partial z^1} + z_{12}^1 \frac{\partial F_1}{\partial z_1^1} + F_2 \frac{\partial F_1}{\partial z_2^1} \\ + \frac{\partial F_1}{\partial z_{12}^1}\left(\frac{\partial F_2}{\partial x^1} + z_1^1 \frac{\partial F_2}{\partial z^1} + F_1 \frac{\partial F_2}{\partial z_1^1} + z_{12}^1 \frac{\partial F_2}{\partial z_2^1}\right) = 0. \end{aligned}$$

We close this section with the following PDE example:

Example 6.7.3. Consider the linear second order PDE:

$$\frac{\partial^2 u}{\partial (x^1)^2} = \frac{\partial^2 u}{\partial (x^2)^2} + \frac{\partial u}{\partial x^1} - \frac{\partial u}{\partial x^2}, \tag{6.36}$$

for which we now use Theorem 6.6.6 to find a local solution. We can put (6.36) in the form of (6.33) and restrict our solution to one which satisfies

$$\begin{aligned} \frac{\partial^2 u}{\partial (x^1)^2} &= \frac{\partial^2 u}{\partial x^1 \partial x^2} + \frac{\partial u}{\partial x^1} - \frac{\partial u}{\partial x^2}, \\ \frac{\partial^2 u}{\partial (x^2)^2} &= \frac{\partial^2 u}{\partial x^1 \partial x^2}. \end{aligned} \tag{6.37}$$

On $M^6 \subset J^2(U^2, V^1)$ defined by the locus of

$$z^1_{11} = z^1_{12} + z^1_1 - z^1_2,$$
$$z^1_{22} = z^1_{12},$$

we let the differential ideal I defined on M^6 be generated by the one-forms

$$\alpha^1 := dz^1 - z^1_1 dx^1 - z^1_2 dx^2,$$
$$\alpha^2 := dz^1_1 - \left(z^1_{12} + z^1_1 - z^1_2\right) dx^1 - z^1_{12} dx^2,$$
$$\alpha^3 := dz^1_2 - z^1_{12} dx^1 - z^1_{12} dx^2,$$

and the exterior derivatives of α^2 and α^3. Theorem 6.7.2 can be applied to (6.37). Doing so, we find that the Cauchy characteristic space of $I := \langle \alpha^1, \alpha^2, \alpha^3, d\alpha^2, d\alpha^3 \rangle$ is generated by

$$Y = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} + \left(z^1_1 - z^1_2\right) \frac{\partial}{\partial z^1} + \left(z^1_1 - z^1_2\right) \frac{\partial}{\partial z^1_1}. \tag{6.38}$$

Our first step is to ensure that for all $1 \le i \le 3$,

$$Y \lrcorner\, d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^3.$$

We have

$$Y \lrcorner\, d\alpha^1 = \alpha^2 - \alpha^3,$$
$$Y \lrcorner\, d\alpha^2 = \alpha^2 - \alpha^3,$$
$$Y \lrcorner\, d\alpha^3 = 0,$$

so we only need to scale α^2 by some non-zero function. Using Lemma 6.6.1 we obtain by inspection

$$Y \lrcorner\, d\left(\exp\left(x^2\right)\alpha^2\right) = -\exp\left(x^2\right)\alpha^3.$$

Therefore we define

$$\overline{\alpha}^1 := \alpha^1, \qquad \overline{\alpha}^2 := \exp\left(x^2\right)\alpha^2, \qquad \overline{\alpha}^3 := \alpha^3.$$

This gives

$$Y \lrcorner\, d\overline{\alpha}^1 = \exp\left(-x^2\right)\overline{\alpha}^2 - \overline{\alpha}^3,$$
$$Y \lrcorner\, d\overline{\alpha}^2 = -\exp\left(x^2\right)\overline{\alpha}^3,$$
$$Y \lrcorner\, d\overline{\alpha}^3 = 0.$$

Our second step is to look to apply Corollary 6.6.5. It is easy to see that $\frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3$, and that $\frac{\partial}{\partial x^1}$ is a non-trivial symmetry of $\frac{\partial}{\partial x^2} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^2\right)$. Thus,

$$\frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right) = -\exp(x^2) z^1_{12} \left(z^1_1 - z^1_2\right) \overline{\alpha}^1 + z^1_{12}(z^1_1 - z^1_2)\overline{\alpha}^2 + \exp(x^2) \left(z^1_2 z^1_{12} + z^1_1 z^1_2 - (z^1_2)^2 - z^1_1 z^1_{12}\right) \overline{\alpha}^3,$$

with the property that

$$Y \lrcorner\, d\left(\frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right)\right) = 0.$$

We need another pair of symmetries. We have that $\frac{\partial}{\partial z^1}$ is a non-trivial symmetry of $\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3$, and that $\frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $\frac{\partial}{\partial z^1} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^2\right)$. Hence

$$\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial z^1} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right) = z^1_{12}\overline{\alpha}^2 - \exp(x^2) z^1_{12}\overline{\alpha}^3,$$

with

$$Y \lrcorner\, d\left(\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial z^1} \lrcorner \left(\overline{\alpha}^1 \wedge \overline{\alpha}^2 \wedge \overline{\alpha}^3\right)\right) = 0.$$

If we now make the following non-singular linear transformation,

$$\begin{aligned}
\widehat{\alpha}^1 &:= -\exp(x^2) z^1_{12} \left(z^1_1 - z^1_2\right) \overline{\alpha}^1 + z^1_{12}(z^1_1 - z^1_2)\overline{\alpha}^2 \\
&\quad + \exp(x^2) \left(z^1_2 z^1_{12} + z^1_1 z^1_2 - (z^1_2)^2 - z^1_1 z^1_{12}\right) \overline{\alpha}^3, \\
\widehat{\alpha}^2 &:= z^1_{12}\overline{\alpha}^2 - \exp(x^2) z^1_{12}\overline{\alpha}^3, \\
\widehat{\alpha}^3 &:= \overline{\alpha}^3,
\end{aligned}$$

then $Y \lrcorner\, \widehat{\alpha}^i = 0 = Y \lrcorner\, d\widehat{\alpha}^i$ for all $1 \leq i \leq 3$. Explicitly,

$$\begin{aligned}
\widehat{\alpha}^1 &= \exp(x^2)(z^1_2 - z^1_1) \left(z^1_{12} dz^1 - z^1_{12} dz^1_1 - (z^1_2 - z^1_{12}) dz^1_2\right), \\
\widehat{\alpha}^2 &= \exp(x^2) z^1_{12} \left(dz^1_1 - dz^1_2 - (z^1_1 - z^1_2) dx^1\right), \\
\widehat{\alpha}^3 &= dz^1_2 - z^1_{12} dx^1 - z^1_{12} dx^2.
\end{aligned}$$

Applying Theorem 6.6.6, it is easy to show that on M^6, the Cartan system of I is generated by $\widehat{\alpha}^1, \widehat{\alpha}^2, \widehat{\alpha}^3, dz^1_2, dz^1_{12}$. We define

$$\Omega := \widehat{\alpha}^1 \wedge \widehat{\alpha}^2 \wedge \widehat{\alpha}^3 \wedge dz^1_2 \wedge dz^1_{12}.$$

Since the Cauchy characteristic space of I is Frobenius integrable, it follows that $d\Omega \equiv 0 \mod \Omega$. Therefore, using Theorem 3.2.14 with the following solvable symmetry structure $X_1, \ldots, X_5 \in \mathfrak{X}(M^6)$ for Ω found using `DIMSYM`, where

$$X_1 := \frac{\partial}{\partial z_1^1} + \frac{\partial}{\partial z_2^1}, \qquad X_2 := \frac{1}{(z_{12}^1)^3}\frac{\partial}{\partial z_{12}^1}, \qquad X_3 := \frac{\partial}{\partial x^1},$$
$$X_4 := \frac{\partial}{\partial x^2}, \qquad X_5 := \frac{\partial}{\partial z^1},$$

we find

$$\begin{aligned}
\omega^1 &= \frac{X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega} = dz_2^1, \\
\omega^2 &= \frac{X_1 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega} = d\left(\frac{(z_{12}^1)^4}{4}\right), \\
\omega^3 &= \frac{X_1 \lrcorner X_2 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega}{X_3 \lrcorner X_1 \lrcorner X_2 \lrcorner X_4 \lrcorner X_5 \lrcorner \Omega} = d\left(x^1 - \ln|z_2^1 - z_1^1|\right), \\
\omega^4 &= \frac{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_5 \lrcorner \Omega}{X_4 \lrcorner X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_5 \lrcorner \Omega} = d\left(x^2 + \ln|z_2^1 - z_1^1|\right), \\
\omega^5 &= \frac{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner \Omega}{X_5 \lrcorner X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner X_4 \lrcorner \Omega} = d(z^1 - z_1^1 + z_2^1).
\end{aligned}$$

Putting

$$\gamma^1 := z_2^1, \qquad \gamma^2 := \frac{(z_{12}^1)^4}{4}, \qquad \gamma^3 := x^1 - \ln|z_2^1 - z_1^1|,$$
$$\gamma^4 := x^2 + \ln|z_2^1 - z_1^1|, \qquad \gamma^5 := z^1 - z_1^1 + z_2^1,$$

and assuming we are working in a neighbourhood where $z_2^1 > z_1^1$ and $z_{12}^1 > 0$ (the other three cases – excluding $z_1^1 = z_2^1$ and $z_{12}^1 = 0$ – can be treated similarly), we obtain

$$\begin{aligned}
\widehat{\alpha}^1 &= \left(4\gamma^2\right)^{\frac{1}{4}} \exp\left(\gamma^4\right) d\gamma^5 - \gamma^1 \exp\left(\gamma^4\right) d\gamma^1, \\
\widehat{\alpha}^2 &= \left(4\gamma^2\right)^{\frac{1}{4}} \exp\left(\gamma^4\right) d\gamma^3, \\
\widehat{\alpha}^3 &= d\gamma^1 - \left(4\gamma^2\right)^{\frac{1}{4}} d\gamma^4 - \left(4\gamma^2\right)^{\frac{1}{4}} d\gamma^3.
\end{aligned} \tag{6.39}$$

Following the procedure in Theorem 6.6.6, we find that for all $1 \leq i \leq 5$, $\overline{X}_i = X_i$, because $d\omega^i = 0$. Therefore using the annihilating space of (6.39), we require the vector field Z describing our Cauchy data to be of the form

$$Z := \lambda_1\left(\left(4\gamma^2\right)^{\frac{1}{4}} X_1 + X_4 + \gamma^1 X_5\right) + \lambda_2 X_2,$$

for any choice of λ_1, λ_2 that are smooth functions of $\gamma^1, \ldots, \gamma^5$. We suppose that $\lambda_1 = 1$ and $\lambda_2 = 0$, so that

$$Z = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial z^1} + z_{12}^1\frac{\partial}{\partial z_1^1} + z_{12}^1\frac{\partial}{\partial z_1^1}.$$

On M^6, the annihilating space of $Sp\{Y, Z\}$ is spanned by $\alpha^1, \alpha^2, \alpha^3, dz^1_{12}$. Defining $\overline{\Omega} := \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge dz^1_{12}$, and with the solvable symmetry structure $\Gamma_1, \ldots, \Gamma_4 \in \mathfrak{X}(M^6)$ for $\overline{\Omega}$ (partly found using `DIMSYM`), where

$$\Gamma_1 := \frac{1}{z^1_{12}} \frac{\partial}{\partial z^1_{12}}, \qquad \Gamma_2 := \frac{\partial}{\partial x^1}, \qquad \Gamma_3 := \frac{\partial}{\partial x^2}, \qquad \Gamma_4 := \frac{\partial}{\partial z^1},$$

we obtain using Theorem 3.2.14,

$$\begin{aligned}
\overline{\omega}^1 &:= \frac{\Gamma_2 \lrcorner \Gamma_3 \lrcorner \Gamma_4 \lrcorner \overline{\Omega}}{\Gamma_1 \lrcorner \Gamma_2 \lrcorner \Gamma_3 \lrcorner \Gamma_4 \lrcorner \overline{\Omega}} = d\left(\frac{(z^1_{12})^2}{2}\right), \\
\overline{\omega}^2 &:= \frac{\Gamma_1 \lrcorner \Gamma_3 \lrcorner \Gamma_4 \lrcorner \overline{\Omega}}{\Gamma_2 \lrcorner \Gamma_1 \lrcorner \Gamma_3 \lrcorner \Gamma_4 \lrcorner \overline{\Omega}} = d\left(x^1 - \ln(z^1_2 - z^1_1)\right), \\
\overline{\omega}^3 &:= \frac{\Gamma_1 \lrcorner \Gamma_2 \lrcorner \Gamma_4 \lrcorner \overline{\Omega}}{\Gamma_3 \lrcorner \Gamma_1 \lrcorner \Gamma_2 \lrcorner \Gamma_4 \lrcorner \overline{\Omega}} \equiv d\left(x^2 + \ln(z^1_2 - z^1_1) - \frac{z^1_2}{z^1_{12}}\right) \quad \mathrm{mod}\ d\left(\frac{(z^1_{12})^2}{2}\right), \\
\overline{\omega}^4 &:= \frac{\Gamma_1 \lrcorner \Gamma_2 \lrcorner \Gamma_3 \lrcorner \overline{\Omega}}{\Gamma_4 \lrcorner \Gamma_1 \lrcorner \Gamma_2 \lrcorner \Gamma_3 \lrcorner \overline{\Omega}} \equiv d\left(z^1 - z^1_1 + z^1_2 - \frac{(z^1_2)^2}{2z^1_{12}}\right) \quad \mathrm{mod}\ d\left(\frac{(z^1_{12})^2}{2}\right).
\end{aligned}$$

Putting

$$z^1_{12} = c^1, \qquad x^1 - \ln(z^1_2 - z^1_1) = c^2, \qquad x^2 + \ln(z^1_2 - z^1_1) - \frac{z^1_2}{z^1_{12}} = c^3,$$
$$z^1 - z^1_1 + z^1_2 - \frac{(z^1_2)^2}{2z^1_{12}} = c^4,$$

for any constant functions c^1, c^2, c^3, c^4 (where we have assumed that $c^1 > 0$), yields the following local solution to the system of PDEs in (6.37), and hence the PDE in (6.36):

$$u = c^5 + c^6 \exp(x^1) + c^7(x^1 + x^2) + c^8(x^1 + x^2)^2, \tag{6.40}$$

for any constants c^5, c^6, c^7, c^8. It can be shown that the assumption $c^1 > 0$ now means that $c^8 > 0$, but the solution in (6.40) also holds for zero and negative c^8. Finally, note that

$$u = c^5 + c^6 \exp(x^1) + f(x^1 + x^2),$$

is also a solution to (6.36) (and (6.37)), where f is any smooth function of $x^1 + x^2$. Of course, f will depend on the Cauchy data.

Chapter 7

Similarity solutions

7.1 Introduction

Given a non-linear partial differential equation, a so-called 'similarity solution' is one which is invariant under some group action. Pioneered by Lie [89, 91], techniques for using symmetry to find similarity solutions have been around for a long time, and in recent times authors such as Bluman and Cole [21], Bluman and Kumei [22], Olver [96, 97, 98] and Stephani [117] have provided modern discussions on various aspects of this similarity solution approach to PDEs.

This chapter considers a single second order hyperbolic or parabolic PDE of one dependent variable u and two independent variables x^1, x^2 of the form

$$f_1 \frac{\partial^2 u}{\partial (x^1)^2} + f_2 \frac{\partial^2 u}{\partial (x^2)^2} + f_3 \frac{\partial^2 u}{\partial x^1 x^2} = k, \tag{7.1}$$

where f_1, f_2, f_3, k are smooth functions of $x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}$. We use exterior calculus to study similarity solutions of (7.1) along similar lines as Harrison and Estabrook [64], however our treatment gives an alternative interpretation of the underlying geometric significance of such solutions. We also make use of Theorem 3.2.13 for integrating Frobenius integrable vector field distributions to establish an algorithm based entirely on symmetry for generating similarity solutions of second order hyperbolic or parabolic PDEs of the type in (7.1). This avoids the usual requirement of having to solve some ordinary differential equation once the similarity variable is known. Finally, we briefly examine conditional symmetries. Using such symmetries we extend earlier results in this chapter to give a technique for generating the so-called 'non-classical' [10, 20, 98] similarity solutions, that once again avoids the need to solve any ODE.

7.2 A preliminary result

It is assumed throughout this chapter that unless otherwise stated, M^q is some open, convex neighbourhood of $\mathbb{R}^q$. Moreover, since the inverse function theorem [17] means immersions are locally diffeomorphic onto their images, we also assume the domains of all our immersions are suitably chosen so that this diffeomorphic property holds. Hence from Section 2.2.2, the differential maps of all our immersions map vector fields to well-defined vector fields, and so (2.2) can be applied.

Working in the second jet bundle $J^2(U^2, V^1)$ with coordinates $x^1, x^2, z^1, z^1_1, z^1_2, z^1_{11}$, z^1_{12}, z^1_{22}, we define

$$F := f_1 z^1_{11} + f_2 z^1_{22} + f_3 z^1_{12} - k,$$

along with the contact forms

$$C^1 := dz^1 - z^1_1 dx^1 - z^1_2 dx^2,$$
$$C^1_1 := dz^1_1 - z^1_{11} dx^1 - z^1_{12} dx^2,$$
$$C^1_2 := dz^1_2 - z^1_{12} dx^1 - z^1_{22} dx^2.$$

We can express a solution surface of the PDE in (7.1) as a two-dimensional integral manifold (immersion) of the differential ideal

$$I_F := \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, F dx^1 \wedge dx^2 \rangle,$$

such that the transverse condition $dx^1 \wedge dx^2 \neq 0$ holds on its tangent space. Note that $dC^1 \equiv 0 \mod C^1_1, C^1_2$. Recall from Corollary 2.4.11 in Chapter 2 that an integral manifold in the second jet bundle which annihilates all the contact forms that generate the second order contact system is the image of the 2-jet of some smooth map $f : U^2 \longrightarrow V^1$ if and only if $dx^1 \wedge dx^2 \neq 0$ on the tangent space of the integral manifold. If, in addition, the integral manifold annihilates F, then by Theorem 2.5.1 the 2-jet is that of some local solution of the PDE in (7.1).

Our principal result of this section is the following:

Theorem 7.2.1.

$$I_F = \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, L\rangle,$$

where

$$L := f_1 dz^1_1 \wedge dx^2 - f_2 dz^1_2 \wedge dx^1 + f_3 dz^1_2 \wedge dx^2 - k dx^1 \wedge dx^2.$$

Proof.

$$F dx^1 \wedge dx^2 = \left(f_1 z^1_{11} + f_2 z^1_{22} + f_3 z^1_{12} - k\right) dx^1 \wedge dx^2.$$

Now

$$\begin{aligned} f_1 z^1_{11} dx^1 \wedge dx^2 &= f_1 (z^1_{11} dx^1 + z^1_{12} dx^2) \wedge dx^2, \\ &= f_1 (dz^1_1 - C^1_1) \wedge dx^2, \end{aligned}$$

$$\begin{aligned} f_2 z^1_{22} dx^1 \wedge dx^2 &= -f_2 (z^1_{21} dx^1 + z^1_{22} dx^2) \wedge dx^1, \\ &= -f_2 (dz^1_2 - C^1_2) \wedge dx^1, \end{aligned}$$

$$\begin{aligned} f_3 z^1_{12} dx^1 \wedge dx^2 &= f_3 (z^1_{12} dx^1 + z^1_{22} dx^2) \wedge dx^2, \\ &= f_3 (dz^1_2 - C^1_2) \wedge dx^2, \end{aligned}$$

Hence

$$\begin{aligned} F dx^1 \wedge dx^2 &\equiv f_1 dz^1_1 \wedge dx^2 - f_2 dz^1_2 \wedge dx^1 + f_3 dz^1_2 \wedge dx^2 \\ &\quad - k dx^1 \wedge dx^2 \mod C^1_1, C^1_2, \\ &\equiv L \mod C^1_1, C^1_2. \end{aligned}$$

From this we obtain

$$\begin{aligned} dL &\equiv d\left(F dx^1 \wedge dx^2\right) \mod C^1_1, C^1_2, dC^1_1, dC^1_2, \\ &\equiv 0 \mod C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, \end{aligned}$$

using Lemma 2.5.3. □

Remark. In a similar fashion to above, it is easy to show that

$$I_F = \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, L^\dagger\rangle,$$

where

$$L^\dagger := f_1 dz_1^1 \wedge dx^2 - f_2 dz_2^1 \wedge dx^1 - f_3 dz_1^1 \wedge dx^1 - k dx^1 \wedge dx^2.$$

In our work we deal mostly with L, however all results equally apply to $L^\dagger$.

We define

$$I_{\overline{F}} := \langle C^1, C_1^1, C_2^1, dC_1^1, dC_2^1, L \rangle.$$

Now L (containing all the information specific to the PDE) does not depend on any second order terms $z_{11}^1, z_{12}^1, z_{22}^1$. Therefore, we may modify our problem to that of finding two-dimensional integral manifolds of a *reduced* differential ideal $I_{\overline{F}}^r$ defined by

$$I_{\overline{F}}^r := \langle C^1, dC^1, L, dL \rangle, \tag{7.2}$$

defined on the first jet bundle $J^1(U^2, V^1)$.

7.3 Similarity solution approaches

Given a Lie point symmetry $X \in \mathfrak{X}(U^2 \times V^1)$ of the PDE in (7.1), a similarity solution of the PDE is a local solution that remains unchanged under the one-parameter group action of the symmetry. The most well-known procedure for using X to generate a corresponding similarity solution basically involves determining the two functionally independent invariants $\gamma^1, \gamma^2 \in C^\infty(U^2 \times V^1)$ of X and finding a solution of (7.1) that is some function of these invariants. Doing so, one essentially obtains from (7.1) a second order ODE expressed in terms of γ^1, γ^2, known as the 'reduced' differential equation. In the general case for PDE problems of n independent variables, the reduced equation retains the same order of the PDE but is of $n-1$ independent variables.

An alternative and equivalent approach to finding similarity solutions is discussed by Olver in [98] where one searches for a common solution of the overdetermined system of PDEs given by (7.1) and the first order quasilinear PDE obtained from

$$X^{(1)} \lrcorner C^1 = 0, \tag{7.3}$$

where z^1 and z^1_1, z^1_2 are replaced with u and its respective first partial derivatives. Here we assume (7.3) gives a valid PDE and the Lie point symmetry X is not, for example, $\frac{\partial}{\partial z^1}$. The PDE derived from (7.3) is known as the *characterising invariance system* [98] (or *invariant surface condition* [95]) corresponding to X, and is typically solved first using invariant coordinates to give a solution in terms of an arbitrary function. Then, by inserting this solution into (7.1), a reduced differential equation for the arbitrary function is derived. Once this is solved, a similarity solution is obtained once more.

In this chapter we do not follow either of the above procedures, but instead choose to adopt another approach formulated by Harrison and Estabrook [64] that uses exterior calculus and differential ideals. This is discussed below:

Suppose we are given some differential ideal I^r_F on $J^1(U^2, V^1)$ corresponding to some second order PDE of the form in (7.1). If a vector field $V \in \mathfrak{X}\left(J^1(U^2, V^1)\right)$ is a symmetry of I^r_F, then

$$\mathcal{L}_V C^1 = \lambda_1 C^1, \tag{7.4}$$

and

$$\mathcal{L}_V L = \alpha^1 \wedge C^1 + \lambda_2 dC^1 + \lambda_3 L, \tag{7.5}$$

for some $\lambda_1, \lambda_2, \lambda_3 \in C^\infty\left(J^1(U^2, V^1)\right)$ and $\alpha^1 \in \Lambda^1\left(J^1(U^2, V^1)\right)$. Applying the property that $\mathcal{L}_V\left(d\omega\right) = d\left(\mathcal{L}_V \omega\right)$ for any differential form ω, we can use (7.4) and (7.5) to derive corresponding symmetry expressions for the remaining two generators of I^r_F. A key property of the Harrison and Estabrook approach is that the symmetry algebra of I^r_F includes the Lie point symmetry algebra of (7.1) obtained from using Definition 5.4.3. We state this fact without proof, however in [31] it is proved for differential ideals where the PDE is left as a 0-form generator of the ideal. Since we are dealing with PDEs of one dependent variable, the determining equations derived from (7.4) and (7.5) should also be able to establish any so-called contact symmetries of the PDE.

Suppose then that we are given some symmetry V of I^r_F (or the first prolongation of some Lie point symmetry of (7.1)). In the Harrison and Estabrook approach to generating similarity solutions of (7.1), the differential ideal I^r_F is augmented with

$V \lrcorner C^1$, $V \lrcorner dC^1$, $V \lrcorner L$ and $V \lrcorner dL$. One then looks for a two-dimensional integral manifold of the augmented ideal

$$\langle C^1, dC^1, L, dL, V \lrcorner C^1, V \lrcorner dC^1, V \lrcorner L, V \lrcorner dL \rangle, \tag{7.6}$$

defined on $J^1(U^2, V^1)$, which also satisfies the transverse condition.

The symmetry conditions in (7.4) and (7.5) can be used to easily prove that (7.6) is a differential ideal, and it is clear that V is a Cauchy characteristic vector field of the differential ideal. Though this obvious latter fact has also been noted by Estabrook [51], we show in Lemma 7.4.1 below that for hyperbolic and parabolic PDEs of the form in (7.1), there exists a more useful extension of this result.

Finally, we can simplify (7.6) in the following way: It is not hard to establish from using (7.4) and (7.5) that (7.6) is equal to

$$\langle C^1, dC^1, L, dL, V \lrcorner C^1, d(V \lrcorner C^1), V \lrcorner L, d(V \lrcorner L) \rangle. \tag{7.7}$$

In the next section we examine (7.7) more closely and show that two further reductions are possible.

7.4 Main results

The class of second order PDEs we deal with is those for which L is decomposable, or equivalently, $L \wedge L = 0$ using Theorem 3.6.1. Although L defined in Theorem 7.2.1 is obviously not decomposable for some choices of f_1, f_2, f_3 and k, we will see later in Section 7.7 that for all hyperbolic and parabolic PDEs of the form in (7.1) we are able to add to L some multiple of dC^1 which is then decomposable.

Assuming then without loss that L is decomposable, we have

$$0 = Y \lrcorner (L \wedge L) = 2(Y \lrcorner L) \wedge L,$$

for any $Y \in \mathfrak{X}\left(J^1(U^2, V^1)\right)$, so that if $Y \lrcorner L \neq 0$, then $L = (Y \lrcorner L) \wedge \omega$ for some $\omega \in \Lambda^1\left(J^1(U^2, V^1)\right)$. Therefore, for decomposable L, any integral manifold of

$$\langle C^1, dC^1, V \lrcorner C^1, d(V \lrcorner C^1), V \lrcorner L, d(V \lrcorner L) \rangle, \tag{7.8}$$

is an integral manifold of (7.7) (the two differential ideals are equal for decomposable L). Here V is the symmetry of $I^r_{\overline{F}}$ described in the previous section. We shall make use of this condition on L in our two main results, Theorems 7.4.2 and 7.6.1.

Since $V \lrcorner C^1$ is a smooth function generator of (7.8), we can make a further simplification to this differential ideal by pulling it back onto the regular submanifold of $J^1(U^2, V^1)$ described by $V \lrcorner C^1 = 0$, and confine our work to this region of $J^1(U^2, V^1)$. Suppose that the equation $V \lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, which we parameterise by the immersion $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$. Then denoting the pull-back of (7.8) onto M^4 by

$$J_{\overline{F}}^{r} := \langle \Phi^* C^1, d \circ \Phi^* C^1, \Phi^*(V \lrcorner L), d \circ \Phi^*(V \lrcorner L) \rangle, \tag{7.9}$$

we have the following lemma:

Lemma 7.4.1. *Let $V \in \mathfrak{X}(J^1(U^2, V^1))$ be a symmetry of $I_{\overline{F}}^{r}$. If the equation $V \lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, which we parameterise by the immersion $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$, then there exists $W \in \mathfrak{X}(M^4)$ with the property that W is a Cauchy characteristic vector field of $J_{\overline{F}}^{r}$.*

Proof. Let $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$ be a corresponding immersion mapping onto the regular submanifold of $J^1(U^2, V^1)$ described by $V \lrcorner C^1 = 0$. It is clear that the tangent space of $\Phi(M^4) \subset J^1(U^2, V^1)$ spans the annihilating space of $d(V \lrcorner C^1)$. From contracting the symmetry condition in (7.4) with V we obtain, at any point in $\Phi(M^4)$,

$$V \lrcorner d \left(V \lrcorner C^1\right) = \lambda_1 \left(V \lrcorner C^1\right) = 0.$$

Hence V is in the tangent space of $\Phi(M^4)$. Applying Theorem 2.2.3, there exists a vector field $W \in \mathfrak{X}(M^4)$ such that $\Phi_* W = V$.

We now proceed to show that W is a Cauchy characteristic vector field of $J_{\overline{F}}^{r}$ by examining each generator of the differential ideal. First,

$$W \lrcorner \Phi^* C^1 = \Phi^* \left(\Phi_* W \lrcorner C^1\right) = 0, \tag{7.10}$$

where for the first equality we have used the property of the pull-back in (2.3), and for the second, we have made use of the fact that the pull-back of $V \lrcorner C^1$ by Φ is zero.

Next, we have that

$$W \lrcorner \Phi^* \circ dC^1 = \Phi^* \left(\Phi_* W \lrcorner dC^1\right) = \Phi^* \left(V \lrcorner dC^1\right), \tag{7.11}$$

once again using (2.3). Now

$$\begin{aligned}\Phi^*\left(V\lrcorner dC^1\right) &= \Phi^*\left(\lambda_1 C^1 - d(V\lrcorner C^1)\right), \\ &= \left(\Phi^*\lambda_1\right)\Phi^* C^1 - d\circ\Phi^*\left(V\lrcorner C^1\right), \\ &= \left(\Phi^*\lambda_1\right)\Phi^* C^1 \in J^r_{\overline{F}},\end{aligned} \tag{7.12}$$

where in the first line we have inserted the symmetry condition in (7.4), and in the second, we have used the pull-back properties in (2.5) and (2.6). Combining the end result in (7.12) with (7.11) and (2.6) then gives

$$W\lrcorner d\circ\Phi^* C^1 \in J^r_{\overline{F}}. \tag{7.13}$$

We also have from (2.3),

$$W\lrcorner\Phi^*(V\lrcorner L) = \Phi^*\left(\Phi_* W\lrcorner V\lrcorner L\right) = \Phi^*\left(V\lrcorner V\lrcorner L\right) = 0. \tag{7.14}$$

In a similar fashion,

$$W\lrcorner\Phi^*\circ d(V\lrcorner L) = \Phi^*\left(\Phi_* W\lrcorner d(V\lrcorner L)\right) = \Phi^*\left(V\lrcorner d(V\lrcorner L)\right). \tag{7.15}$$

The symmetry condition in (7.5) yields

$$\begin{aligned}V\lrcorner d\left(V\lrcorner L\right) &= V\lrcorner\left(\alpha^1\wedge C^1 + \lambda_2 dC^1 + \lambda_3 L - V\lrcorner dL\right), \\ &= (V\lrcorner\alpha^1)C^1 - (V\lrcorner C^1)\alpha^1 + \lambda_2(V\lrcorner dC^1) + \lambda_3(V\lrcorner L).\end{aligned}$$

Pulling this back by Φ, then using (2.5) and $\Phi^*(V\lrcorner C^1) = 0$ followed by (7.12) gives

$$\begin{aligned}\Phi^*\left(V\lrcorner d(V\lrcorner L)\right) = \left(\Phi^*(V\lrcorner\alpha^1)\right)\Phi^* C^1 + \left(\Phi^*\lambda_2\right)\Phi^*\left(V\lrcorner dC^1\right) \\ + \left(\Phi^*\lambda_3\right)\Phi^*\left(V\lrcorner L\right) \in J^r_{\overline{F}},\end{aligned} \tag{7.16}$$

so that combining this result with (7.15) and (2.6), we obtain

$$W\lrcorner d\circ\Phi^*(V\lrcorner L) \in J^r_{\overline{F}}. \tag{7.17}$$

Therefore (7.10), (7.13), (7.14) and (7.17) imply that $W\lrcorner J^r_{\overline{F}} \subset J^r_{\overline{F}}$. □

From Lemma 7.4.1 we obtain the first of our major new results for this chapter:

Theorem 7.4.2. *Given some second order PDE of the form in (7.1) whose corresponding L is decomposable, let $V \in \mathfrak{X}\left(J^1(U^2, V^1)\right)$ be a symmetry of $I^r_{\overline{F}}$. Suppose the equation $V\lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of*

$J^1(U^2,V^1)$, and denote $\Phi : M^4 \longrightarrow J^1(U^2,V^1)$ as a corresponding immersion mapping onto this submanifold. With

$$D^r_{\overline{F}} := \left(Sp\left\{\Phi^* C^1, \Phi^*(V \lrcorner L)\right\}\right)^{\perp},$$

if $\Phi^*\left(C^1 \wedge (V \lrcorner L)\right) \neq 0$, then $\Phi_* D^r_{\overline{F}}$ generates a two-dimensional integral manifold of $I^r_{\overline{F}}$. If, in addition, $dx^1 \wedge dx^2 \neq 0$ on $\Phi_* D^r_{\overline{F}}$, then the integral manifold is the image of the 1-jet of some local solution of the PDE in (7.1).

Proof. We know from the proof of Lemma 7.4.1 that $V = \Phi_* W$ for some $W \in \mathfrak{X}(M^4)$. Since $\Phi^*\left(C^1 \wedge (V \lrcorner L)\right) \neq 0$, it follows that $D^r_{\overline{F}}$ is two-dimensional, and from Lemma 7.4.1, that W is a Cauchy characteristic vector field of the differential ideal $J^r_{\overline{F}}$ defined in (7.9). Hence $D^r_{\overline{F}}$ is, from Theorem 2.3.6, Frobenius integrable. Since it is assumed Φ is diffeomorphic onto its image, $\Phi_* D^r_{\overline{F}}$ is well-defined. Now let $Z_1, Z_2 \in \Phi_* D^r_{\overline{F}}$. This means

$$Z_1 = \Phi_* P_1, \qquad Z_2 = \Phi_* P_2,$$

for some $P_1, P_2 \in D^r_{\overline{F}}$. Using (2.2) and the fact that $D^r_{\overline{F}}$ is Frobenius integrable, we then get

$$[Z_1, Z_2] = [\Phi_* P_1, \Phi_* P_2] = \Phi_*[P_1, P_2] \in \Phi_* D^r_{\overline{F}},$$

so $\Phi_* D^r_{\overline{F}}$ is Frobenius integrable.

Suppose that $\Psi : M^2 \longrightarrow M^4$ is an immersion mapping onto any leaf of the foliation of M^4 generated by $D^r_{\overline{F}}$. Thus $\Psi^* J^r_{\overline{F}} = 0$. Now

$$(\Phi \circ \Psi)^* C^1 = \Psi^*(\Phi^* C^1) = 0, \tag{7.18}$$

and

$$(\Phi \circ \Psi)^*(dC^1) = d\left((\Phi \circ \Psi)^* C^1\right) = 0. \tag{7.19}$$

By assumption, $\Phi^*\left(C^1 \wedge (V \lrcorner L)\right) \neq 0$. This implies $V \lrcorner L \neq 0$. Since L is decomposable, we have $L = (V \lrcorner L) \wedge \omega$ for some $\omega \in \Lambda^1\left(J^1(U^2,V^1)\right)$. Concentrating on $V \lrcorner L$,

$$0 = \Psi^*\left(\Phi^*(V \lrcorner L)\right) = (\Phi \circ \Psi)^*(V \lrcorner L),$$

which gives

$$\Psi^*(\Phi^* L) = \Psi^*((\Phi^*(V \lrcorner L)) \wedge (\Phi^*\omega)) = ((\Phi \circ \Psi)^*(V \lrcorner L)) \wedge ((\Phi \circ \Psi)^*\omega) = 0. \quad (7.20)$$

Hence from (7.18), (7.19) and (7.20), it then follows that $(\Phi \circ \Psi)^* I^r_{\overline{F}} = 0$. If the transverse condition holds, then $\Phi \circ \Psi(M^2) = j^1 h(U^2)$ for some $h \in C^\infty(U^2, V^1)$, with h as some local solution of (7.1). □

Remark. In order to satisfy the transverse requirement, the symmetry V in Theorem 7.4.2 must necessarily satisfy the condition $d(V \lrcorner C^1) \wedge dx^1 \wedge dx^2 \neq 0$. If this is not the case, then $\Phi^*(dx^1 \wedge dx^2) = 0$, and hence *for all* Ψ, $(\Phi \circ \Psi)^*(dx^1 \wedge dx^2) = 0$. Consequently the transverse requirement fails.

We illustrate Theorem 7.4.2 with the following example:

Example 7.4.3. Consider the heat equation

$$\frac{\partial^2 u}{\partial (x^1)^2} = \frac{\partial u}{\partial x^2}. \quad (7.21)$$

Defined on $J^1(U^2, V^1)$ we have

$$I^r_{\overline{F}} = \langle C^1, dC^1, L, dL \rangle,$$

where $F = z^1_{11} - z^1_2$ and $L = (dz^1_1 - z^1_2 dx^1) \wedge dx^2$. Now

$$V := x^1 \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^2}$$

is a Lie point symmetry of (7.21), and we use its first prolongation $V^{(1)}$, where

$$V^{(1)} = x^1 \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^2} - z^1_1 \frac{\partial}{\partial z^1_1} - 2z^1_2 \frac{\partial}{\partial z^1_2},$$

as our non-trivial symmetry of $I^r_{\overline{F}}$.

Applying Theorem 7.4.2, we define the four-dimensional regular submanifold $M^4 \subset J^1(U^2, V^1)$ by the locus of

$$V^{(1)} \lrcorner C^1 = -x^1 z^1_1 - x^2 z^1_2 = 0.$$

In a simplified manner without explicitly introducing an immersion, we assume M^4 has coordinates x^1, x^2, z^1, z^1_1 with $x^2 \neq 0$, so that on M^4,

$$\begin{aligned} C^1 &= dz^1 - z^1_1 dx^1 + \frac{z^1_1 x^1}{2x^2} dx^2, \\ V^{(1)} \lrcorner L &= -z^1_1 x^1 dx^1 + z^1_1 \left(\frac{(x^1)^2}{2x^2} - 1 \right) dx^2 - 2x^2 dz^1_1, \end{aligned} \quad (7.22)$$

with

$$J^r_{\overline{F}} = \langle C^1, dC^1, V^{(1)} \lrcorner L, d(V^{(1)} \lrcorner L) \rangle,$$

also defined on M^4. From Theorem 7.4.2 we have that $D^r_{\overline{F}} \subset \mathfrak{X}(M^4)$ generated by the annihilating space of equations in (7.22) is Frobenius integrable. It is easy to show that on $D^r_{\overline{F}}$, the transverse condition $dx^1 \wedge dx^2 \neq 0$ holds, so we expect to get some local solution to the heat equation. Then applying Theorem 3.2.13 (we could equally choose to use Theorem 3.2.14) with a solvable structure of two symmetries, where $X_2 := \frac{\partial}{\partial z^1} \in \mathfrak{X}(M^4)$ is a non-trivial symmetry of $D^r_{\overline{F}}$, $X_1 := z^1_1 \frac{\partial}{\partial z^1_1} \in \mathfrak{X}(M^4)$ is a non-trivial symmetry of $D^r_{\overline{F}} \oplus Sp\{X_2\}$, and defining

$$\Omega := \left(dz^1 - z^1_1 dx^1 + \frac{z^1_1 x^1}{2x^2} dx^2 \right) \wedge \left(-z^1_1 x^1 dx^1 + z^1_1 \left(\frac{(x^1)^2}{2x^2} - 1 \right) dx^2 - 2x^2 dz^1_1 \right),$$

we find

$$\frac{X_2 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner \Omega} = d\left(\ln(z^1_1 \sqrt{x^2}) + \frac{(x^1)^2}{4x^2} \right),$$

$$\frac{X_1 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner \Omega} \equiv d\left(z^1 - 2z^1_1 \sqrt{x^2} \exp\left(\frac{(x^1)^2}{4x^2} \right) \int \exp\left(-\xi^2\right) d\xi \right) \mod \frac{X_2 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner \Omega},$$

where $\xi := x^1/(2\sqrt{x^2})$. Putting

$$\ln(z^1_1 \sqrt{x^2}) + \frac{(x^1)^2}{4x^2} = c^1,$$

and

$$z^1 - 2z^1_1 \sqrt{x^2} \exp\left(\frac{(x^1)^2}{4x^2} \right) \int \exp\left(-\xi^2\right) d\xi = c^2,$$

for any constants c^1, c^2, we obtain

$$u = 2\exp(c^1) \int \exp\left(-\xi^2\right) d\xi$$

as our local similarity solution of the heat equation corresponding to V.

We close this section with a warning that there will exist situations when applying Theorem 7.4.2 will yield a distribution $\Phi_* D^r_{\overline{F}}$ that is not transverse, even with $d(V \lrcorner C^1) \wedge dx^1 \wedge dx^2 \neq 0$. In such cases we must abandon the above approach and look to use elements of $I^r_{\overline{F}}$ that are in a sense singular. This is explained in full in the next section.

7.5 A singular approach

Consider a differential ideal $I := \langle \alpha^1, \alpha^2 \rangle$ defined on some open, convex neighbourhood $U^4 \subset \mathbb{R}^4$, with coordinates $x^1, \ldots, x^4$ generated by two linearly independent one-forms $\alpha^1, \alpha^2 \in \Lambda^1(U^4)$. Suppose that for each $i \in \{1, 2\}$, $d\alpha^i \equiv 0 \bmod \alpha^1, \alpha^2$, i.e. $\ker(\alpha^1 \wedge \alpha^2)$ is Frobenius integrable. Here, we choose to work with a two-dimensional Pfaffian system defined on a four-dimensional space because the material in the following section on second order hyperbolic or parabolic PDEs of the type in (7.1) is precisely of this nature, but all results that follow in this section can easily be extended to arbitrary dimensions.

In the usual treatment in this thesis for integrating the Frobenius integrable distribution $\ker(\alpha^1 \wedge \alpha^2)$, we use Theorem 3.2.13 to find some functions $f_1^1, f_2^1, f_1^2, f_2^2, g^1$, $g^2 \in C^\infty(U^4)$ such that

$$\begin{aligned} f_1^1 \alpha^1 + f_2^1 \alpha^2 &= dg^1, \\ f_1^2 \alpha^1 + f_2^2 \alpha^2 &= dg^2. \end{aligned} \tag{7.23}$$

If, on U^4, the functions g^1, g^2 are of constant maximal rank two, then the equations $g^1 = c^1, g^2 = c^2$ describe a two-dimensional regular submanifold of U^4. Let $\Psi : M^2 \longrightarrow U^4$ be an immersion mapping onto this submanifold. If, in addition, the determinant

$$\Psi^* \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \neq 0$$

on M^2, then (7.23) and the fact that $\Psi^*(dg^1) = 0 = \Psi^*(dg^2)$ imply $\Psi^*\alpha^1 = 0 = \Psi^*\alpha^2$. Hence Ψ is a two-dimensional integral manifold of I, for arbitrary constant functions c^1, c^2.

The problem with the above 'regular' approach used in Theorem 7.4.2 for dealing with a PDE of the form in (7.1) is that if the submanifold generated by $\Phi_* D^r_{\overline{F}}$ is not transverse, then the method fails to give us a local solution with u as some smooth function of x^1, x^2.

Our goal in this section and the next is to provide an alternative approach for finding two-dimensional integral manifolds of I, which includes the above situation as a sub-class, as well as applies to PDE problems when $\Phi_* D^r_{\overline{F}}$ *may or may not* be transverse. We will also see that the trade-off for this extra flexibility is that there is no direct computational approach using solvable symmetry structures, however using the Frobenius integrable nature of $\ker(\alpha^1 \wedge \alpha^2)$ (or $\Phi_* D^r_{\overline{F}}$ in Theorem 7.4.2) we do come close.

Consider then the following obvious extension to the above discussion:

Theorem 7.5.1. *With α^1, α^2 and I defined as above, let there exist $f^1_1, f^1_2, f^2_1, f^2_2$, $g^{11}, g^{12}, g^{21}, g^{22} \in C^\infty(U^4)$ such that*

$$\begin{aligned} f^1_1\alpha^1 + f^1_2\alpha^2 &= g^{11}dg^{12}, \\ f^2_1\alpha^1 + f^2_2\alpha^2 &= g^{21}dg^{22}. \end{aligned} \tag{7.24}$$

Suppose that for some $p, q \in \{1, 2\}$, the equations

$$g^{1p} = \begin{cases} 0 & \text{if } p = 1, \\ c^1 & \text{otherwise}, \end{cases} \qquad g^{2q} = \begin{cases} 0 & \text{if } q = 1, \\ c^2 & \text{otherwise}, \end{cases}$$

for some constants c^1, c^2 describe a two-dimensional regular submanifold of U^4, and let $\Psi : M^2 \longrightarrow U^4$ be an immersion mapping onto this submanifold. If, on M^2, the determinant

$$\Psi^* \begin{vmatrix} f^1_1 & f^1_2 \\ f^2_1 & f^2_2 \end{vmatrix} \neq 0, \tag{7.25}$$

then Ψ is a two-dimensional integral manifold of I.

For PDE problems, Theorem 7.5.1 will be used to find alternative (hopefully transverse) integral manifolds of I to those found with the usual approach reviewed at the start of this section. Unfortunately there is no algorithmic technique (without involving ODEs) for establishing (7.24) by means other than following direct one using Theorem 3.2.13 that incorporates symmetry:

Suppose then we apply Theorem 3.2.13 with $X_2 \in \mathfrak{X}(U^4)$ as a non-trivial symmetry of $\ker(\alpha^1 \wedge \alpha^2)$, and $X_1 \in \mathfrak{X}(U^4)$ as a non-trivial symmetry of $Sp\{X_2\} \oplus \ker(\alpha^1 \wedge \alpha^2)$. We then obtain

$$\begin{aligned} \frac{X_2 \lrcorner (\alpha^1 \wedge \alpha^2)}{X_1 \lrcorner X_2 \lrcorner (\alpha^1 \wedge \alpha^2)} &= dg^{12}, \\ \frac{X_1 \lrcorner (\alpha^1 \wedge \alpha^2)}{X_2 \lrcorner X_1 \lrcorner (\alpha^1 \wedge \alpha^2)} &= dg^{22} - X_1(g^{22})dg^{12}, \end{aligned} \tag{7.26}$$

for some $g^{12}, g^{22} \in C^\infty(U^4)$. This gives integral manifolds of I defined by $g^{12} = c^1$, $g^{22} = c^2$ for constants c^1, c^2. Suppose these are not transverse. Rearranging the equations in (7.26) gives

$$\begin{aligned}
\left(X_2 \lrcorner\, \alpha^2\right) \alpha^1 - \left(X_2 \lrcorner\, \alpha^1\right) \alpha^2 &= \left(X_2 \lrcorner\, X_1 \lrcorner\, (\alpha^1 \wedge \alpha^2)\right) dg^{12}, \\
\left((X_1 + X_1(g^{22})X_2) \lrcorner\, \alpha^2\right) \alpha^1 - &\left((X_1 + X_1(g^{22})X_2) \lrcorner\, \alpha^1\right) \alpha^2 \\
&= \left(X_1 \lrcorner\, X_2 \lrcorner\, (\alpha^1 \wedge \alpha^2)\right) dg^{22}.
\end{aligned} \tag{7.27}$$

Now applying Theorem 7.5.1 with the equations in (7.27), we set

$$g^{11} = -g^{21} = X_2 \lrcorner\, X_1 \lrcorner\, (\alpha^1 \wedge \alpha^2).$$

We cannot choose $p = 2, q = 2$ since by assumption these integral manifolds of I are not transverse. We also cannot choose $p = 1, q = 1$ because $g^{11} = -g^{21}$ implies we do not obtain a regular two-dimensional submanifold of U^4. This is clearly due to the constant maximal rank two requirement failing. Therefore we require that at least one of the two remaining (p, q) combinations satisfy the rank two condition. Finally, the equation in (7.25) must also be satisfied, i.e.

$$\Psi^* \begin{vmatrix} X_2 \lrcorner\, \alpha^2 & -X_2 \lrcorner\, \alpha^1 \\ (X_1 + X_1(g^{22})X_2) \lrcorner\, \alpha^2 & -(X_1 + X_1(g^{22})X_2) \lrcorner\, \alpha^1 \end{vmatrix} \neq 0.$$

Below is a modification of Theorem 7.5.1, which shows that if we are given just one of the equations in (7.24) (found for example by inspection, or using Theorem 3.2.13 as in the above), then the other can be determined using a symmetry:

Theorem 7.5.2. *With α^1, α^2 and I as defined as above, let $f^1_1, f^1_2, g^{11}, g^{12} \in C^\infty(U^4)$ such that*

$$f^1_1 \alpha^1 + f^1_2 \alpha^2 = g^{11} dg^{12}. \tag{7.28}$$

Suppose that for some $p \in \{1, 2\}$, the equation

$$g^{1p} = \begin{cases} 0 & \text{if } p = 1, \\ c^1 & \text{otherwise,} \end{cases} \tag{7.29}$$

for some constant c^1 describes a three-dimensional regular submanifold of U^4. Let $\Theta : M^3 \longrightarrow U^4$ denote an immersion mapping onto this submanifold, and let $X \in$

$\mathfrak{X}(M^3)$ be a non-trivial symmetry of $\Theta^*(f_1^2\alpha^1 + f_2^2\alpha^2)$, for some $f_1^2, f_2^2 \in C^\infty(U^4)$. Then there exist $\overline{g}^{21}, \overline{g}^{22} \in C^\infty(M^3)$ such that

$$\Theta^*\left(f_1^2\alpha^1 + f_2^2\alpha^2\right) = \overline{g}^{21}d\overline{g}^{22}.$$

Further suppose that, for some $q \in \{1,2\}$, the equation

$$\overline{g}^{2q} = \begin{cases} 0 & \textit{if } q = 1, \\ c^2 & \textit{otherwise}, \end{cases}$$

for some constant c^2 describes a two-dimensional regular submanifold of M^3. With $\Psi : M^2 \longrightarrow M^3$ denoting an immersion mapping onto this submanifold, if

$$(\Theta \circ \Psi)^* \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \neq 0, \tag{7.30}$$

on M^2, then $\Theta \circ \Psi$ is a two-dimensional integral manifold of I.

Proof. Since for each $i \in \{1,2\}$, $d\alpha^i \equiv 0 \mod \alpha^1, \alpha^2$, it follows that with

$$\beta^1 := f_1^1\alpha^1 + f_2^1\alpha^2,$$
$$\beta^2 := f_1^2\alpha^1 + f_2^2\alpha^2,$$

we have for each $i \in \{1,2\}$, $d\beta^i \equiv 0 \mod \beta^1, \beta^2$ for arbitrary choice of $f_1^1, f_2^1, f_1^2, f_2^2 \in C^\infty(U^4)$. Let β^1 satisfy (7.28) for some f_1^1, f_2^1 and some $g^{11}, g^{12} \in C^\infty(U^4)$, and for some $p \in \{1,2\}$, let the immersion $\Theta : M^3 \longrightarrow U^4$, defined as in the theorem, map onto the regular submanifold of U^4 given by (7.29). Then $\Theta^*\beta^1 = 0$, so that

$$d\left(\Theta^*\beta^2\right) = \Theta^*\left(d\beta^2\right) = \left(\Theta^*\mu_1\right)\Theta^*\beta^1 + \left(\Theta^*\mu_2\right)\Theta^*\beta^2 \equiv 0 \mod \Theta^*\beta^2,$$

for some $\mu_1, \mu_2 \in C^\infty(U^4)$. Let $X \in \mathfrak{X}(M^3)$ be a non-trivial symmetry of $\Theta^*\beta^2$. Hence, applying Theorem 3.2.14 combined with Theorems 2.3.10 and 2.3.11, we obtain

$$d\left(\frac{\Theta^*\beta^2}{X \lrcorner \left(\Theta^*\beta^2\right)}\right) = 0.$$

Therefore

$$\Theta^*\beta^2 = \left(X \lrcorner \left(\Theta^*\beta^2\right)\right) d\overline{g}^{22},$$

for some $\overline{g}^{22} \in C^\infty(M^3)$. We set $\overline{g}^{21} = X\lrcorner(\Theta^*\beta^2)$ and choose $\overline{g}^{2q}$ such that it is of constant maximal rank one on M^3. Hence with Ψ defined as in the theorem, we have

$$(\Theta \circ \Psi)^*\beta^1 = 0 = (\Theta \circ \Psi)^*\beta^2.$$

By the assumption in (7.30), it is then clear that $\Theta \circ \Psi$ is a two-dimensional integral manifold of I. $\square$

Remark. The functions f_1^2, f_2^2 in Theorem 7.5.2 are not quite arbitrary: First they must be chosen so that

$$\Theta^* \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \neq 0,$$

on M^3, or else (7.30) fails for any Ψ. Then once Ψ is known, (7.30) must be checked.

Certainly the difficult part in applying Theorem 7.5.2 is in establishing (7.28). Once this is done however, the remaining assumptions in the theorem simply involve two maximal rank conditions, one non-zero determinant condition and one non-trivial symmetry.

Another observation we can make regarding Theorem 7.5.2 is that $\ker(\alpha^1 \wedge \alpha^2)$ must be Frobenius integrable. Of course, even if $\ker(\alpha^1 \wedge \alpha^2)$ is not Frobenius integrable, singular two-dimensional integral manifolds of I may still exist.

The following example illustrates Theorem 7.5.2:

Example 7.5.3. Suppose on some suitably chosen U^4 where $x^2 \neq 0$, $I := \langle \alpha^1, \alpha^2 \rangle$ with

$$\alpha^1 := dx^3 + \frac{x^1}{2x^2} dx^1 - x^4 dx^2,$$
$$\alpha^2 := \left(2x^2x^4 - \frac{(x^1)^2}{2x^2} + 1\right) dx^2.$$

It is easy to show that for all $i \in \{1, 2\}$, $d\alpha^i \equiv 0 \bmod \alpha^1, \alpha^2$, and so $\ker(\alpha^1 \wedge \alpha^2)$ is Frobenius integrable.

We begin with the 'regular' approach to integrating $\ker(\alpha^1 \wedge \alpha^2)$ reviewed at the beginning of this section. Simple inspection (or Theorem 3.2.14) yields

$$\alpha^1 \wedge \alpha^2 = \left(2x^2x^4 - \frac{(x^1)^2}{2x^2} + 1\right) d\left(x^3 + \frac{(x^1)^2}{4x^2}\right) \wedge dx^2.$$

Hence if the equations

$$x^2 = c^1, \qquad x^3 + \frac{(x^1)^2}{4x^2} = c^2,$$

for arbitrary constants c^1, c^2 are constant maximal rank two on some suitably chosen neighbourhood of U^4, then they describe a two-dimensional foliation of the neighbourhood, where each leaf is a regular submanifold that is an integral manifold of I.

We now look to apply Theorem 7.5.2 in order to generate different two-dimensional integral manifolds of I. Applying the theorem, suppose we choose $f_1^1 := 0$, $f_2^1 := 1$, and

$$g^{11} := 2x^2x^4 - \frac{(x^1)^2}{2x^2} + 1, \qquad g^{12} := x^2,$$

so that (7.28) holds. We set

$$g^{11} = 0. \tag{7.31}$$

We also choose $f_1^2 := 1$, $f_2^2 := 0$. Again without explicitly introducing an immersion, and pulling-back α^1 onto M^3 defined by (7.31) with coordinates for M^3 given by x^1, x^2, x^3, we find (on M^3)

$$\alpha^1 = dx^3 + \frac{x^1}{2x^2}dx^1 + \frac{1}{2x^2}\left(1 - \frac{(x^1)^2}{2x^2}\right)dx^2,$$

which, from Theorem 7.5.2, is closed modulo itself. Applying Theorem 3.2.14 with $\frac{\partial}{\partial x^3}$ as a non-trivial symmetry of α^1, we get

$$\alpha^1 = d\left(x^3 + \ln(\sqrt{x^2}) + \frac{(x^1)^2}{4x^2}\right),$$

so

$$\overline{g}^{21} = 1, \qquad \overline{g}^{22} = x^3 + \ln(\sqrt{x^2}) + \frac{(x^1)^2}{4x^2}.$$

Hence our only choice is to set

$$\overline{g}^{22} = c^3,$$

where c^3 is an arbitrary constant function. On a suitable neighbourhood of U^4 the equations

$$2x^2x^4 - \frac{(x^1)^2}{2x^2} + 1 = 0, \qquad x^3 + \ln(\sqrt{x^2}) + \frac{(x^1)^2}{4x^2} = c^3 \tag{7.32}$$

are of constant maximal rank two, and it is easy to see from above that the non-zero determinant condition in (7.30) holds. Hence the equations in (7.32) describe a two-dimensional regular submanifold of the neighbourhood of U^4, that is an integral manifold of I. Note that the two-dimensional leaves described by (7.32) do not generate a foliation of the neighbourhood. Rather, the three-dimensional regular submanifold of the neighbourhood described by the equation on the left in (7.32) is foliated by the two-dimensional leaves generated by the equation on the right.

7.6 A singular application

In this section we use Theorem 7.5.2 to provide an alternative to Theorem 7.4.2 when the transverse requirement fails for $\Phi_* D^r_{\overline{F}}$. The following result is the second of our major new results for this chapter:

Theorem 7.6.1. *Given some second order PDE of the form in (7.1) whose corresponding L is decomposable, let $V \in \mathfrak{X}\left(J^1(U^2, V^1)\right)$ be a symmetry of $I^r_{\overline{F}}$. Suppose the equation $V \lrcorner C^1 = 0$ describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, and let $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$ denote an immersion mapping onto this submanifold. Further suppose $\Phi^*(C^1 \wedge (V \lrcorner L)) \neq 0$, and we have applied Theorem 7.5.2 with $\alpha^1 := \Phi^* C^1$ and $\alpha^2 := \Phi^*(V \lrcorner L)$, thus generating some smooth $g^{1p}, \overline{g}^{2q}$ and corresponding immersions $\Theta : M^3 \longrightarrow J^1(U^2, V^1)$ and $\Psi : M^2 \longrightarrow M^3$, as in the theorem. If*

$$(\Phi \circ \Theta \circ \Psi)^*(dx^1 \wedge dx^2) \neq 0, \tag{7.33}$$

then $\Phi \circ \Theta \circ \Psi(M^2)$ is the image of the 1-jet of some local solution of the PDE in (7.1).

Proof. Using Lemma 7.4.1, we have on M^4 that

$$D^r_{\overline{F}} := \left(Sp\{\Phi^* C^1, \Phi^*(V \lrcorner L)\}\right)^\perp$$

is Frobenius integrable. Applying Theorem 7.5.2 to $J^r_{\overline{F}}$ defined in (7.9) then generates a two-dimensional integral manifold of $J^r_{\overline{F}}$ given by

$$\Theta \circ \Psi : M^2 \longrightarrow M^4.$$

At this point the proof becomes very similar to that of Theorem 7.4.2. As L is decomposable, we find that

$$\Phi \circ \Theta \circ \Psi : M^2 \longrightarrow J^1(U^2, V^1)$$

is a two-dimensional integral manifold of I^r_F. The condition in (7.33) is a transverse requirement. It is then clear that the image of $\Phi \circ \Theta \circ \Psi$ is equal to the image of the 1-jet of some local solution of the PDE in (7.1). □

Remark 1. Theorem 7.6.1 can obviously be modified by replacing Theorem 7.5.2 with Theorem 7.5.1.

Remark 2. While Theorem 7.6.1 does not require that $\Phi_* D^r_F$ be transverse, a transverse requirement must still be introduced, but at a later stage.

The following example attempts to clarify Theorem 7.6.1:

Example 7.6.2. Consider the potential Burgers' equation:

$$\frac{\partial u}{\partial x^2} - \frac{\partial^2 u}{\partial (x^1)^2} - \left(\frac{\partial u}{\partial x^1}\right)^2 = 0. \tag{7.34}$$

Defined on $J^1(U^2, V^1)$ we have

$$I^r_F = \langle C^1, dC^1, L, dL \rangle,$$

where $F = z^1_2 - z^1_{11} - (z^1_1)^2$ and $L = ((z^1_2 - (z^1_1)^2)\, dx^1 - dz^1_1) \wedge dx^2$. Now

$$V := 2x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial z^1}$$

is a Lie point symmetry of (7.34), and we use its first prolongation $V^{(1)}$, where

$$V^{(1)} = 2x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial z^1} - \frac{\partial}{\partial z^1_1} - 2z^1_1 \frac{\partial}{\partial z^1_2},$$

as our non-trivial symmetry of I^r_F.

Applying Theorem 7.6.1, we define M^4 to be the four-dimensional regular submanifold of $J^1(U^2, V^1)$ given by the locus of

$$V^{(1)} \lrcorner\, C^1 = -x^1 - 2x^2 z^1_1 = 0.$$

We assume M^4 has coordinates x^1, x^2, z^1, z^1_2 with $x^2 \neq 0$, so that on M^4 we have

$$C^1 = dz^1 + \frac{x^1}{2x^2}dx^1 - z^1_2 dx^2,$$
$$V^{(1)} \lrcorner L = \left(2x^2 z^1_2 - \frac{(x^1)^2}{2x^2} + 1\right) dx^2.$$

It is clear that the transverse condition does not hold on the two-dimensional annihilating space of $Sp\{C^1, V^{(1)} \lrcorner L\}$ defined on M^4, so we will look to use Theorem 7.5.2. In applying this result, we refer to Example 7.5.3 which makes use of the theorem with x^3 replacing z^1 and x^4 replacing z^1_2 so that $\alpha^1 = C^1$ and $\alpha^2 = V^{(1)} \lrcorner L$. From the example, we then get that

$$u = -\ln(\sqrt{x^2}) - \frac{(x^1)^2}{4x^2} + c^3,$$

for any constant c^3 is a similarity solution of (7.34) corresponding to V.

7.7 Decomposability examined

Theorems 7.4.2 and 7.6.1 appear to be restricted by the requirement that L (or $L^\dagger$) be decomposable. However, since dC^1 is in $I^r_{\overline{F}}$, we may look to add some multiple $b \in J^2(U^2, V^1)$ of dC^1 to L so that $L + bdC^1$ is decomposable.

Without loss, working this time with $L^\dagger$, we define the following two-form

$$\Omega^\dagger := L^\dagger + bdC^1,$$

where b is for the moment any smooth function on the first jet bundle $J^1(U^2, V^1)$. The following theorem gives a simple quadratic condition on b in order that $\Omega^\dagger \wedge \Omega^\dagger = 0$ (i.e. $\Omega^\dagger$ is decomposable by Theorem 3.6.1).

Theorem 7.7.1. *With $\Omega^\dagger := L^\dagger + bdC^1$, if*

$$b = \frac{-f_3 \pm \sqrt{f_3^2 - 4f_1 f_2}}{2},$$

with $f_3^2 - 4f_1 f_2 \geq 0$, then $\Omega^\dagger$ is decomposable.

Proof.

$$\left(L^\dagger + bdC^1\right)^2 = (L^\dagger)^2 + 2bdC^1 \wedge L^\dagger + b^2(dC^1)^2,$$

and

$$(dC^1)^2 = 2dz_1^1 \wedge dx^1 \wedge dz_2^1 \wedge dx^2,$$
$$(L^\dagger)^2 = -2f_1f_2dz_1^1 \wedge dx^2 \wedge dz_2^1 \wedge dx^1,$$
$$dC^1 \wedge L^\dagger = f_3dz_2^1 \wedge dx^2 \wedge dz_1^1 \wedge dx^1.$$

Hence

$$\left(L^\dagger + bdC^1\right)^2 = 2\left(b^2 + bf_3 + f_1f_2\right) dz_1^1 \wedge dx^1 \wedge dz_2^1 \wedge dx^2.$$

It follows that if

$$b = \frac{-f_3 \pm \sqrt{f_3^2 - 4f_1f_2}}{2},$$

where b is real on $J^1(U^2, V^1)$, then $\Omega^\dagger \wedge \Omega^\dagger = 0$, and therefore by Theorem 3.6.1, $\Omega^\dagger$ is decomposable. □

Proved in a similar way to Theorem 7.7.1, we have the following for L:

Theorem 7.7.2. *With* $\Omega := L + bdC^1$, *if*

$$b = \frac{f_3 \pm \sqrt{f_3^2 - 4f_1f_2}}{2},$$

with $f_3^2 - 4f_1f_2 \geq 0$, *then* Ω *is decomposable.*

The requirement that the discriminant in Theorems 7.7.1 and 7.7.2 remains non-negative on $J^2(U^2, V^1)$ (or on some suitable neighbourhood), coincides exactly with the condition found widely in the literature [19, 28, 61] that the second order PDE in (7.1) be hyperbolic or parabolic. Hence, if the PDE is of one of these two types, we are always able to determine a decomposable Ω (or $\Omega^\dagger$). Thus we can apply Theorems 7.4.2 and 7.6.1 by simply replacing the L in these two theorems with Ω. We illustrate with an example:

Example 7.7.3. Consider the non-linear wave equation:

$$\frac{\partial^2 u}{\partial (x^2)^2} = u\frac{\partial^2 u}{\partial (x^1)^2}. \tag{7.35}$$

In terms of coordinates of $J^1(U^2, V^1)$, this equation admits the point symmetry

$$V := x^2\frac{\partial}{\partial x^2} - 2z^1\frac{\partial}{\partial z^1},$$

whose first prolongation is

$$V^{(1)} = x^2 \frac{\partial}{\partial x^1} - 2z^1 \frac{\partial}{\partial z^1} - 2z^1_1 \frac{\partial}{\partial z^1_1} - 3z^1_2 \frac{\partial}{\partial z^1_2}.$$

Working with L we have

$$L = -z^1 dz^1_1 \wedge dx^2 - dz^1_2 \wedge dx^1,$$

which is clearly not decomposable. From Theorem 7.7.2 we find that $L \pm \sqrt{z^1} dC^1$ is decomposable. Taking the positive option gives

$$\begin{aligned} \Omega_+ &:= L + \sqrt{z^1} dC^1, \\ &= \left(dz^1_2 - \sqrt{z^1} dz^1_1 \right) \wedge \left(\sqrt{z^1} dx^2 - dx^1 \right). \end{aligned}$$

Applying Theorem 7.4.2, we define the four-dimensional regular submanifold $M^4 \subset J^1(U^2, V^1)$ by the locus of

$$V^{(1)} \lrcorner C^1 = -x^2 z^1_2 - 2z^1 = 0.$$

Let M^4 have coordinates x^1, x^2, z^1, z^1_1 with $x^2 \neq 0$. Then we have on M^4,

$$C^1 = dz^1 - z^1_1 dx^1 + \frac{2z^1}{x^2} dx^2,$$

$$V^{(1)} \lrcorner \Omega_+ = \left(-\frac{6z^1}{x^2} - 2\sqrt{z^1} z^1_1 \right) dx^1 + \left(\frac{4(z^1)^{\frac{3}{2}}}{x^2} + 2z^1 z^1_1 \right) dx^2 + 2\sqrt{z^1} dz^1 + x^2 z^1 dz^1_1.$$

It is easy to show that the transverse condition holds on the two-dimensional annihilating space of $Sp\{C^1, V^{(1)} \lrcorner \Omega_+\}$ defined on M^4. By inspection,

$$X_1 := \frac{\partial}{\partial x^1} \in \mathfrak{X}(M^4)$$

is a non-trivial symmetry of $C^1 \wedge (V^{(1)} \lrcorner \Omega_+)$ (pulled-back onto M^4). Using `DIMSYM`,

$$X_2 := -\frac{1}{(x^2)^2} \frac{\partial}{\partial z^1} \in \mathfrak{X}(M^4)$$

is another non-trivial symmetry of $C^1 \wedge (V^{(1)} \lrcorner \Omega_+)$, which also commutes with X_1. Therefore, taking advantage of this situation and applying Theorem 3.2.16 with Corollary 3.2.12 gives the two closed forms

$$\begin{aligned} \frac{X_1 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))}{X_2 \lrcorner X_1 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))} &= d\left(\frac{(x^2)^4 (z^1_1)^2}{12} - (x^2)^2 z^1 \right), \\ \frac{X_2 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))}{X_1 \lrcorner X_2 \lrcorner (C^1 \wedge (V^{(1)} \lrcorner \Omega_+))} &= d\left(x^1 - \frac{(x^2)^2 z^1_1}{6} \right). \end{aligned}$$

Putting

$$\frac{(x^2)^4(z_1^1)^2}{12} - (x^2)^2 z^1 = c^1, \qquad x^1 - \frac{(x^2)^2 z_1^1}{6} = c^2,$$

for any constants c^1, c^2, we obtain

$$u = \frac{3\left(x^1 - c^2\right)^2 - c^1}{(x^2)^2}$$

as our similarity solution of the non-linear wave equation in (7.35) corresponding to V.

7.8 Conditional symmetries

Following Bluman and Cole [20], Olver [98] or Stephani [117], a *conditional symmetry* $V \in \mathfrak{X}(U^2 \times V^1)$ of some second order PDE in (7.1) is defined as a Lie point symmetry of the overdetermined system of PDEs given by (7.1) and the first order quasilinear PDE obtained from

$$V^{(1)} \lrcorner\, C^1 = 0. \tag{7.36}$$

In this section we show that all results in the previous sections still hold true if instead of the symmetry being the first prolongation of some point symmetry of (7.1) it is the first prolongation of some conditional symmetry.

We define

$$\widehat{I^r_F} := \langle C^1, dC^1, L, dL, (V^{(1)} \lrcorner\, C^1) dx^1 \wedge dx^2, d(V^{(1)} \lrcorner\, C^1) \wedge dx^1 \wedge dx^2 \rangle,$$

defined on the first jet bundle $J^1(U^2, V^1)$. It is clear from Section 2.5.2 in Chapter 2 that the image of any two-dimensional integral manifold of $\widehat{I^r_F}$ that satisfies the transverse condition will be that of some 1-jet solution map of the overdetermined system of PDEs given by (7.1) and (7.36).

If V is a conditional symmetry of (7.1), then it follows from the discussion in Section 7.3 that

$$\mathcal{L}_{V^{(1)}} \widehat{I^r_F} \subset \widehat{I^r_F}.$$

Explicitly,

$$\mathcal{L}_{V^{(1)}} C^1 = \lambda_1 C^1, \tag{7.37}$$

as well as

$$\mathcal{L}_{V^{(1)}} L = \alpha^1 \wedge C^1 + \lambda_2 dC^1 + \lambda_3 L + \lambda_4 \left((V^{(1)} \lrcorner C^1) dx^1 \wedge dx^2 \right), \tag{7.38}$$

and finally,

$$\begin{aligned} \mathcal{L}_{V^{(1)}} \left((V^{(1)} \lrcorner C^1) dx^1 \wedge dx^2 \right) = \alpha^2 \wedge C^1 + \lambda_5 dC^1 + \lambda_6 L \\ + \lambda_7 \left((V^{(1)} \lrcorner C^1) dx^1 \wedge dx^2 \right), \end{aligned} \tag{7.39}$$

for some $\lambda_1, \ldots, \lambda_7 \in C^\infty \left(J^1(U^2, V^1) \right)$ and $\alpha^1, \alpha^2 \in \Lambda^1 \left(J^1(U^2, V^1) \right)$.

Suppose in terms of first jet bundle coordinates the equation in (7.36) describes a four-dimensional regular submanifold of $J^1(U^2, V^1)$, which we parameterise by the immersion $\Phi : M^4 \longrightarrow J^1(U^2, V^1)$. It is then obvious that

$$\Phi^* \widehat{I}^r_F = \Phi^* I^r_F.$$

Without loss, we can assume L is decomposable, so that $L = (V^{(1)} \lrcorner L) \wedge \omega$ for some $\omega \in \Lambda^1 \left(J^1(U^2, V^1) \right)$ (assume $V^{(1)} \lrcorner L \neq 0$). Suppose we now wish to repeat the proof of Lemma 7.4.1, where in the lemma,

1. I^r_F is replaced by $\widehat{I}^r_F$,
2. V is replaced by the first prolongation of our conditional symmetry $V^{(1)}$,
3. The symmetry conditions in (7.4) and (7.5) are replaced by those in (7.37) and (7.38).

Now it is not hard to see that the lemma still holds true, since the pull-back of (7.38) by Φ forces the final term on the right to vanish. Thus when pulled-back by Φ, the two sets of equations given in item (3) above are in identical form. Hence from the lemma there exists some Cauchy characteristic vector field $W \in \mathfrak{X}(M^4)$ of J^r_F with the property that $\Phi_* W = V^{(1)}$. Consequently, with the same three substitutions given above, Theorems 7.4.2 and 7.6.1 hold.

Finally, the equation in (7.39) is not used in the proof of any of our results. Therefore it appears that in order for us to use symmetries of $\widehat{I}^r_F$ to derive non-classical similarity solutions, vector fields from the symmetry algebra of $\widehat{I}^r_F$ are not strictly necessary. One essentially only requires vector fields that satisfy (7.37) and (7.38).

Using a conditional symmetry, we now illustrate Theorem 7.4.2 with the following example:

Example 7.8.1. Consider the heat equation given in (7.21). From Stephani [117], it has the conditional symmetry

$$V := \tan(x^1)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2},$$

whose first prolongation is given by

$$V^{(1)} = \tan(x^1)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - z^1_1 \sec^2(x^1)\frac{\partial}{\partial z^1_1}.$$

From Example 7.4.3, L is decomposable. Applying Theorem 7.4.2, we define the four-dimensional regular submanifold $M^4 \subset J^1(U^2, V^1)$ by the locus of

$$V^{(1)} \lrcorner C^1 = -z^1_1 \tan(x^1) - z^1_2 = 0.$$

Letting M^4 have coordinates x^1, x^2, z^1, z^1_1, we pull-back C^1 and $V^{(1)} \lrcorner L$ so that (on M^4),

$$C^1 = dz^1 - z^1_1 dx^1 + z^1_1 \tan(x^1) dx^2,$$
$$V^{(1)} \lrcorner L = -z^1_1 \tan x^1 dx^1 - z^1_1 dx^2 - dz^1_1.$$

It can be shown that on M^4, $\ker\left(C^1 \wedge (V^{(1)} \lrcorner L)\right)$ is a two-dimensional Frobenius integrable distribution that satisfies the transverse condition. By inspection,

$$\frac{\partial}{\partial x^2}, \frac{\partial}{\partial z^1} \in \mathfrak{X}(M^4),$$

are two commuting non-trivial symmetries of $C^1 \wedge (V^{(1)} \lrcorner L)$. Hence by Theorem 3.2.16 we obtain the two closed forms

$$\frac{\frac{\partial}{\partial x^2} \lrcorner (C^1 \wedge (V^{(1)} \lrcorner L))}{\frac{\partial}{\partial z^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner (C^1 \wedge (V^{(1)} \lrcorner L))} = d\left(z^1 - z^1_1 \tan(x^1)\right),$$
$$\frac{\frac{\partial}{\partial z^1} \lrcorner (C^1 \wedge (V^{(1)} \lrcorner L))}{\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial z^1} \lrcorner (C^1 \wedge (V^{(1)} \lrcorner L))} = d\left(\ln\left|\frac{z^1_1}{\cos(x^1)}\right| + x^2\right).$$

Putting

$$z^1 - z^1_1 \tan(x^1) = c^1, \qquad \ln\left|\frac{z^1_1}{\cos(x^1)}\right| + x^2 = c^2,$$

for any constants c^1 and c^2 yields

$$u = \sin(x^1)\exp(c^2 - x^2) + c^1$$

as our local non-classical similarity solution of the wave equation corresponding to the conditional symmetry V.

Chapter 8

Symmetry reduction of second and higher order PDEs

8.1 Introduction

Consider a second order PDE of one dependent variable u and two independent variables x^1, x^2 of the following form:

$$f_1 \frac{\partial^2 u}{\partial (x^1)^2} + f_2 \frac{\partial^2 u}{\partial (x^2)^2} + f_3 \frac{\partial^2 u}{\partial x^1 x^2} = k, \tag{8.1}$$

where f_1, f_2, f_3, k are smooth functions of $x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}$.

This chapter partly examines conditions under which a certain class of PDEs of the form in (8.1) can be 'integrated' to first order (generally non-linear). This is achieved by firstly expressing a given second order PDE in terms of the differential two-form L (or $L^\dagger$), as in Chapter 7. If the differential form is decomposable and closed modulo itself, it is then shown that a solvable structure of two symmetries can reduce the order of the PDE to first order depending on one arbitrary function.

The remainder of this chapter attempts to generalise the result above for second order PDEs to single r-th order PDEs of n independent variables and m dependent variables linear in their r-th order derivatives. It is shown that if the defining differential n-form for the PDE is decomposable and closed modulo itself, then it is possible to use a solvable structure of n symmetries to again reduce the PDE to $(r-1)$-th order depending on one arbitrary function.

8.2 Review

Reviewing Section 7.2 of Chapter 7, we can express solution surfaces of the PDE in (8.1) as two-dimensional submanifolds of the locus described by $F = 0$ in $J^2(U^2, V^1)$ (with coordinates $x^1, x^2, z^1, z^1_1, z^1_2, z^1_{11}, z^1_{22}, z^1_{12}$), where

$$F := f_1 z^1_{11} + f_2 z^1_{22} + f_3 z^1_{12} - k.$$

Following Theorem 7.2.1, we use the contact forms

$$\begin{aligned} C^1 &:= dz^1 - z^1_1 dx^1 - z^1_2 dx^2, \\ C^1_1 &:= dz^1_1 - z^1_{11} dx^1 - z^1_{12} dx^2, \\ C^1_2 &:= dz^1_2 - z^1_{12} dx^1 - z^1_{22} dx^2, \end{aligned}$$

and look for a two-dimensional Frobenius integrable distribution that is an integral manifold of the differential ideal

$$I_{\overline{F}} := \langle C^1, C^1_1, C^1_2, dC^1_1, dC^1_2, L \rangle,$$

where

$$L := f_1 dz^1_1 \wedge dx^2 - f_2 dz^1_2 \wedge dx^1 + f_3 dz^1_2 \wedge dx^2 - k dx^1 \wedge dx^2,$$

such that the transverse condition $dx^1 \wedge dx^2 \neq 0$ also holds on the distribution.

8.3 Second order symmetry reduction

The main result of this section is the following:

Theorem 8.3.1. *Suppose we are given some second order PDE of the form in (8.1) such that*

1. *L is locally decomposable,*

2. *$dL \equiv 0$ mod L.*

Let $X_1, X_2 \in \mathfrak{X}(J^1(U^2, V^1))$ be a solvable structure of linearly independent symmetries such that X_2 is a non-trivial symmetry of $\ker(L)$ *and X_1 is a non-trivial* *symmetry of* $\ker(L) \oplus Sp\{X_2\}$. *Then there exist some functionally independent* $\gamma^1, \gamma^2 \in C^\infty(J^1(U^2, V^1))$, *so that*

$$L = L(X_1, X_2) d\gamma^1 \wedge d\gamma^2.$$

Moreover, suppose H *is any smooth function of* γ^1, γ^2 *such that* $H(\gamma^1, \gamma^2) = 0$ *is, in terms of its first jet bundle coordinates, some first order PDE. Then any local solution of this PDE is also a local solution of (8.1).*

Proof. Theorem 3.2.14 in conjunction with Theorem 2.3.11 means we can find some functionally independent $\gamma^1, \gamma^2 \in C^\infty(J^1(U^2, V^1))$ so that

$$L = L(X_1, X_2) d\gamma^1 \wedge d\gamma^2. \tag{8.2}$$

Now we choose H, so that the expression

$$H(\gamma^1, \gamma^2) = 0, \tag{8.3}$$

gives us some first order PDE of the form

$$\overline{H}\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}\right) = 0, \tag{8.4}$$

for some smooth $\overline{H}$. From the discussion preceding Theorem 5.4.1, it follows that on the regular submanifold of $J^1(U^2, V^1)$ given by (8.3), L in (8.2) is annihilated. Thus if $u = h(x^1, x^2)$ is any local solution of the PDE in (8.4), it is then clear that its 2-jet

$$j^2 h : U^2 \longrightarrow J^2(U^2, V^1),$$

has the property that

$$j^2 h^* L = 0.$$

Further, the pull-back of all the contact forms in $I_{\overline{F}}$ is also zero. Therefore $j^2 h^* I_{\overline{F}} = 0$, so $u = h(x^1, x^2)$ is a local solution of the original second order PDE. ☐

Remark 1. While we exclusively deal with L, Theorem 8.3.1 can be applied to $L^\dagger$ as defined in Section 7.2 of Chapter 7.

Remark 2. It is possible to choose H so that we are left with some equation in (8.4) that is defined entirely on the graph space, and consequently does not give us a first order PDE. If this equation can locally be expressed as $u = h(x^1, x^2)$ for some smooth h, then it is clear that it will be some local solution of (8.1). This chapter ignores such pathological situations.

In order to solve the resulting first order PDE in Theorem 8.3.1 we may look to use Theorem 5.4.1 in Chapter 5 if it is quasilinear, Theorem 5.5.3 (or Theorem 5.6.1 if the PDE does not contain u) if it is non-linear, or one of the more complicated techniques in Theorems 6.4.11 and 6.6.6 in Chapter 6.

8.3.1 Examples

In this section we apply Theorem 8.3.1 to two examples:

Example 8.3.2. Consider the following second order PDE:

$$\frac{\partial^2 u}{\partial (x^1)^2} + \frac{2}{x^2}\frac{\partial u}{\partial x^2}\frac{\partial^2 u}{\partial x^1 \partial x^2} = x^2.$$

Our corresponding L is

$$L = \left(dz_1^1 + \frac{2z_2^1}{x^2} dz_2^1 - x^2 dx^1 \right) \wedge dx^2.$$

Now L is locally decomposable, and it is obvious that $dL = 0$, so we may proceed to apply Theorem 8.3.1.

$\frac{\partial}{\partial z_1^1}$ is a non-trivial (and not of Lie point type) symmetry of L. Further,

$$\frac{\partial}{\partial z_1^1} \lrcorner L = dx^2,$$

and $\frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $\frac{\partial}{\partial z_1^1} \lrcorner L$. From Theorem 3.2.14 and Corollary 3.2.12,

$$\frac{\frac{\partial}{\partial z_1^1} \lrcorner L}{\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial z_1^1} \lrcorner L} = dx^2,$$

and

$$\begin{aligned} \frac{\frac{\partial}{\partial x^2} \lrcorner L}{\frac{\partial}{\partial z_1^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner L} &= dz_1^1 + \frac{2z_2^1}{x^2} dz_2^1 - x^2 dx^1, \\ &= d\left(z_1^1 + \frac{(z_2^1)^2}{x^2} - x^2 x^1 \right) - \left(x^1 + \frac{(z_2^1)^2}{(x^2)^2} \right) dx^2. \end{aligned}$$

Hence

$$L = d\left(z_1^1 + \frac{(z_2^1)^2}{x^2} - x^2x^1\right) \wedge dx^2.$$

Thus our first order non-linear PDE is

$$\frac{\partial u}{\partial x^1} + \frac{1}{x^2}\left(\frac{\partial u}{\partial x^2}\right)^2 - x^1x^2 = g(x^2),$$

for any choice of smooth function g that is a function of x^2.

From the example, it is no coincidence that

$$\frac{\partial}{\partial x^1}\left(\frac{\partial u}{\partial x^1} + \frac{1}{x^2}\left(\frac{\partial u}{\partial x^2}\right)^2 - x^1x^2 - g(x^2)\right) = 0$$

simplifies to give the original second order PDE that we began with. Theorem 8.3.1 uses symmetries to find a first order PDE with such a property, which usually cannot be found by simple inspection. This fact is reinforced by the next example:

Example 8.3.3. Consider the following second order PDE:

$$\frac{x^2}{x^1}\frac{\partial^2 u}{\partial (x^2)^2} + \frac{\partial^2 u}{\partial x^1 \partial x^2} = \left(\frac{\partial u}{\partial x^2}\right)^{-1}. \tag{8.5}$$

The corresponding L is

$$L = -\frac{x^2}{x^1}dz_2^1 \wedge dx^1 + dz_2^1 \wedge dx^2 - \frac{1}{z_2^1}dx^1 \wedge dx^2.$$

Since $L \wedge L = 0$, by Theorem 3.6.1, L is decomposable on a suitably chosen domain. Since

$$d\left(\frac{z_2^1}{x^2}L\right) = 0,$$

it follows that $dL \equiv 0 \bmod L$. Now $x^2\frac{\partial}{\partial x^2}$ and $\frac{1}{z_2^1}\frac{\partial}{\partial z_2^1}$ are two commuting non-trivial symmetries of L, so by Theorem 3.2.16 we would expect two closed forms. We find

$$\frac{x^2\frac{\partial}{\partial x^2} \lrcorner L}{\frac{1}{z_2^1}\frac{\partial}{\partial z_2^1} \lrcorner x^2\frac{\partial}{\partial x^2} \lrcorner L} = d\left(\frac{(z_2^1)^2}{2} - x^1\right),$$

$$\frac{\frac{1}{z_2^1}\frac{\partial}{\partial z_2^1} \lrcorner L}{x^2\frac{\partial}{\partial x^2} \lrcorner \frac{1}{z_2^1}\frac{\partial}{\partial z_2^1} \lrcorner L} = d\left(\ln\left|\frac{x^2}{x^1}\right|\right).$$

So our first order non-linear PDE defined on a suitable domain is

$$\frac{1}{2}\left(\frac{\partial u}{\partial x^2}\right)^2 - x^1 = g\left(\ln\left|\frac{x^2}{x^1}\right|\right), \tag{8.6}$$

for any smooth g.

Note that the partial derivative of (8.6) with respect to x^1 yields

$$\frac{\partial u}{\partial x^2}\frac{\partial^2 u}{\partial x^1 x^2} - 1 + \frac{1}{x^1}g'\left(\ln\left|\frac{x^2}{x^1}\right|\right) = 0, \tag{8.7}$$

and the partial derivative of (8.6) with respect to x^2 gives

$$\frac{\partial u}{\partial x^2}\frac{\partial^2 u}{\partial (x^2)^2} - \frac{1}{x^2}g'\left(\ln\left|\frac{x^2}{x^1}\right|\right) = 0. \tag{8.8}$$

It is then easy to see that a combination of (8.7) and (8.8) produces (8.5).

8.3.2 Generalisation to higher order

Suppose we are given a single r-th order PDE of $n > 2$ independent variables linear in its r-th order derivatives. In a similar approach to that given in the proof of Theorem 7.2.1 in Chapter 7, it is not hard to obtain a corresponding differential n-form L such that it is expressed solely in terms of $(r-1)$-th jet bundle coordinates and their differentials. Using this fact we can generalise Theorem 8.3.1 to such r-th order PDEs in the following way:

Theorem 8.3.4. *Suppose we are given some r-th order PDE of m dependent variables and n independent variables that is linear in its r-th order derivatives. With F defined to be the PDE in terms of local coordinates of the r-th jet bundle $J^r(U^n, V^m)$, let the equation $F = 0$ describe a regular submanifold of $J^r(U^n, V^m)$. Then the differential n-form $F dx^1 \wedge \cdots \wedge dx^n$ can be expressed, modulo the highest order contact forms, solely in term of the local coordinates of $J^{r-1}(U^n, V^m)$. Call this differential form L_r. If L_r is locally decomposable and $dL_r \equiv 0 \bmod L_r$, and if we have a solvable symmetry structure $X_1, \ldots, X_n \in \mathfrak{X}(J^{r-1}(U^n, V^m))$ for $\ker(L_r)$, then there exist some functionally independent $\gamma^1, \ldots, \gamma^n \in C^\infty(J^{r-1}(U^n, V^m))$ so that*

$$L_r = L_r(X_1, \ldots, X_n) d\gamma^1 \wedge \cdots \wedge d\gamma^n.$$

Moreover, any local solution of the $(r-1)$-th order PDE of m dependent variables and n independent variables given by $H(\gamma^1, \ldots, \gamma^n) = 0$, is a local solution of the original r-th order PDE.

From Lemma 3.2.1, if $\ker(L_r)$ has codimension n in some region, then L_r is locally decomposable; however determining whether $dL_r \equiv 0 \bmod L_r$ is the major obstacle in applying Theorem 8.3.4 (similarly for L in Theorem 8.3.1). As mentioned earlier in Chapter 3, if it is known that L_r is decomposable with $L_r = \omega^1 \wedge \cdots \wedge \omega^r$ for some r one-forms, then we may use the exterior calculus package `EXCALC` to see if for all $1 \leq i \leq r$,

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^r = 0.$$

This will then imply L_r is closed modulo itself.

8.4 Extended reduction approaches

This section attempts to tackle a wider class of second order PDEs of the form in (8.1) than those examined in the previous section. Given a second order PDE of the type in (8.1) we will once again derive a corresponding first order PDE, but this time in a more general setting than that in Theorem 8.3.1.

Using Theorem 7.7.2 we begin with the following result for hyperbolic and parabolic PDEs of the form in (8.1):

Theorem 8.4.1. *Consider the following second order PDE:*

$$f_1 \frac{\partial^2 u}{\partial (x^1)^2} + f_2 \frac{\partial^2 u}{\partial (x^2)^2} + f_3 \frac{\partial^2 u}{\partial x^1 x^2} = k, \tag{8.9}$$

where f_1, f_2, f_3, k are smooth functions of $x^1, x^1, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}$. With

$$L := f_1 dz_1^1 \wedge dx^2 - f_2 dz_2^1 \wedge dx^1 + f_3 dz_2^1 \wedge dx^2 - k dx^1 \wedge dx^2,$$

define $\Omega := L + b dC^1$, where

$$b := \frac{f_3 \pm \sqrt{f_3^2 - 4 f_1 f_2}}{2}.$$

with $f_3^2 - 4f_1 f_2 \geq 0$. Suppose $d\Omega \equiv 0 \bmod \Omega$, and let there exist a solvable structure of two symmetries $X_1, X_2 \in \mathfrak{X}(J^1(U^2, V^1))$ for $\ker(\Omega)$. Then there exist some functionally independent $\gamma^1, \gamma^2 \in C^\infty(J^1(U^2, V^1))$ so that

$$\Omega = \Omega(X_1, X_2) d\gamma^1 \wedge d\gamma^2.$$

Moreover, any local solution of the first order PDE given by $H(\gamma^1, \gamma^2) = 0$, is then a local solution of (8.9).

Proof. From Theorem 7.7.2, Ω is decomposable. By assumption, $d\Omega \equiv 0 \bmod \Omega$. Applying Theorem 3.2.14 with Theorem 2.3.11, we are able to determine functionally independent $\gamma^1, \gamma^2 \in C^\infty(J^1(U^2, V^1))$ such that $\Omega = \Omega(X_1, X_2)d\gamma^1 \wedge d\gamma^2$. The rest of the proof is virtually identical to that of Theorem 8.3.1. □

We illustrate Theorem 8.4.1 with the following simple example:

Example 8.4.2. Consider the wave equation:

$$\frac{\partial^2 u}{\partial (x^1)^2} = \frac{\partial^2 u}{\partial (x^2)^2}. \tag{8.10}$$

In this example we generate the general solution of (8.10) using Theorem 8.4.1 combined with the work on first order PDEs in Chapter 5.

Applying Theorem 8.4.1, we have

$$L = dz^1_1 \wedge dx^2 + dz^1_2 \wedge dx^1,$$

which is not decomposable. Taking the positive option for Ω we obtain by inspection

$$\begin{aligned}\Omega_+ :=& L + dC^1,\\ =& d\left(z^1_1 + z^1_2\right) \wedge d\left(x^1 + x^2\right).\end{aligned}$$

Normally we would also apply Theorem 3.2.14 to reach this stage. Therefore our resulting first order PDE is

$$\frac{\partial u}{\partial x^1} + \frac{\partial u}{\partial x^2} = g(x^1 + x^2), \tag{8.11}$$

for arbitrary smooth g that is a function of $x^1 + x^2$. Now since (8.11) is first order linear, using Theorem 5.4.1 we have

$$K = dz^1 \wedge dx^2 + dx^1 \wedge dz^1 - g(x^1 + x^2)dx^1 \wedge dx^2.$$

defined on $U^2 \times V^1$. It is clear that

$$\frac{\partial}{\partial z^1}, \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} \in \mathfrak{X}(U^2 \times V^1)$$

are two commuting non-trivial symmetries of K. Then applying Theorem 3.2.16 with Corollary 3.2.12 we obtain

$$K = d\left(x^1 - x^2\right) \wedge d\left(z^1 - \frac{1}{2}\int g(\xi)d\xi\right),$$

where $\xi = x^1 + x^2$. We put

$$u - \frac{1}{2}\int g(\xi)d\xi = h(x^1 - x^2),$$

for arbitrary smooth h that is a function of $x^1 - x^2$. From Theorem 5.4.1 this implies

$$u = h(x^1 - x^2) + l(x^1 + x^2),$$

for arbitrary smooth l is a local solution of (8.11). Hence from Theorem 8.4.1 we obtain the general solution of (8.10).

In typical problems, solving the resulting first order PDE may not be possible with the introduced function left arbitrary. Fortunately in the example above we are able to do this for (8.11). For first order quasilinear PDEs this may be feasable, although finding the solvable symmetry structure for K in Theorem 5.4.1 can potentially be quite difficult.

We present below two results that in a sense attempt to generalise Theorem 8.4.1. Unfortunately these results are difficult to apply in practice and essentially have more theoretical interest at this point in time.

Theorem 8.4.3. *Consider the following second order PDE:*

$$f_1\frac{\partial^2 u}{\partial(x^1)^2} + f_2\frac{\partial^2 u}{\partial(x^2)^2} + f_3\frac{\partial^2 u}{\partial x^1 x^2} = k, \tag{8.12}$$

where f_1, f_2, f_3, k are smooth functions of $x^1, x^1, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}$. With

$$L := f_1 dz^1_1 \wedge dx^2 - f_2 dz^1_2 \wedge dx^1 + f_3 dz^1_2 \wedge dx^2 - k dx^1 \wedge dx^2,$$

and $C^1 := dz^1 - z^1_2 dx^1 - z^1_2 dx^2$, suppose there exist some $b \in C^\infty(J^1(U^2, V^1))$ and $\beta \in \Lambda^1(J^1(U^2, V^1))$, such that

$$\Omega := L + \beta \wedge C^1 + b dC^1$$

is decomposable and $d\Omega \equiv 0$ mod Ω. Let there exist a solvable structure of two non-trivial symmetries $X_1, X_2 \in \mathfrak{X}(J^1(U^2, V^1))$ for $\ker(\Omega)$. Then there exist some functionally independent $\gamma^1, \gamma^2 \in C^\infty(J^1(U^2, V^1))$ so that

$$\Omega = \Omega(X_1, X_2) d\gamma^1 \wedge d\gamma^2.$$

Further, any local solution of the first order PDE given by $H(\gamma^1, \gamma^2) = 0$, is then a local solution of (8.12).

Theorem 8.4.3 is similar to Theorem 3.1 in [125]. We can extend Theorem 8.4.3 to give us something more sophisticated than Theorem 8.3.4 as follows:

Consider the contact one-forms on the r-th jet bundle $J^r(U^n, V^m)$:

$$\begin{aligned}
C^j &:= dz^j - z^j_{i_1} dx^{i_1}, \\
C^j_{i_1} &:= dz^j_{i_1} - z^j_{i_1 i_2} dx^{i_2}, \\
C^j_{i_1 i_2} &:= dz^j_{i_1 i_2} - z^j_{i_1 i_2 i_3} dx^{i_3}, \\
&\vdots \\
C^j_{i_1 \ldots i_{r-1}} &:= dz^j_{i_1 \ldots i_{r-1}} - z^j_{i_1 \ldots i_r} dx^{i_r},
\end{aligned}$$

for all $1 \leq j \leq m$, where summation is implied on the repeated indexes with $1 \leq i_1 \leq \cdots \leq i_r \leq n$. Using these one-forms, we have the following result:

Theorem 8.4.4. *Suppose we are given some r-th order PDE of m dependent variables and n independent variables that is linear in its r-th order derivatives. With F defined to be the PDE in terms of coordinates of $J^r(U^n, V^m)$, let the equation $F = 0$ describe a regular submanifold of $J^r(U^n, V^m)$. Then the differential n-form $F dx^1 \wedge \cdots \wedge dx^n$ can be expressed, modulo the highest order contact forms, solely in term of the local coordinates of $J^{r-1}(U^n, V^m)$. Call this differential form L_r. Further suppose the differential n-form (with sum)*

$$\Omega_r := L_r + \alpha_j \wedge C^j + \alpha^{i_1}_j \wedge C^j_{i_1} + \cdots + \alpha^{i_1 \ldots i_{r-2}}_j \wedge C^j_{i_1 \ldots i_{r-2}} + \beta^{i_1 \ldots i_{r-2}}_j \wedge dC^j_{i_1 \ldots i_{r-2}},$$

with $1 \leq j \leq m$ and $1 \leq i_1 \leq \cdots \leq i_{r-2} \leq n$, is locally decomposable and $dL_r \equiv 0 \bmod L_r$ for some choice of $\alpha_j, \alpha^{i_1}_j, \ldots, \alpha^{i_1 \ldots i_{r-2}}_j \in \Lambda^{n-1}(J^{r-1}(U^n, V^m))$ and $\beta^{i_1 \ldots i_{r-2}}_j \in \Lambda^{n-2}(J^{r-1}(U^n, V^m))$. If we have a solvable symmetry structure $X_1, \ldots, X_n \in \mathfrak{X}(J^{r-1}(U^n, V^m))$ for $\ker(\Omega_r)$, then there exist some functionally independent $\gamma^1, \ldots, \gamma^n \in C^\infty(J^{r-1}(U^n, V^m))$ so that

$$L_r = L_r(X_1, \ldots, X_n) d\gamma^1 \wedge \cdots \wedge d\gamma^n,$$

and hence any local solution of the $(r-1)$-th order PDE of m dependent variables and n independent variables given by $H(\gamma^1, \ldots, \gamma^n) = 0$, is a local solution of the original r-th order PDE.

Chapter 9

Summary and concluding remarks

9.1 Introduction

In this thesis we have attempted to develop some systematic methods for finding local solutions of partial differential equations using solvable symmetry structures. Our work has been motivated by the fact that while there exists a significant amount of modern literature on using such symmetries for solving ordinary differential equations, it has generally been lacking for PDEs. Hence a fundamental aim has been to examine the classes of PDEs for which we can use solvable symmetry structures to extract local solutions.

In this final chapter we begin with a review of some of the important new results contained in Chapters 3 to 8. We then discuss some areas for further research. Following this, we end by giving an overall conclusion expressing what we feel is a true evaluation of what this thesis has achieved.

9.2 Review

In this section we give some discussion on the most significant new results that were established in Chapters 3 to 8.

After introducing some background material in Chapter 2, our work essentially began in Chapter 3 by presenting in Theorem 3.2.13 an extension to Lie's solvable symmetry structure approach to integrating Frobenius integrable distributions. While this is not a new result, we used it to obtain a new result in Theorem 3.2.14

on using solvable symmetry structures to find simplifying expressions of differential forms that are decomposable and closed modulo themselves. We then showed in Theorems 3.2.16 and 3.2.17 that the conclusion of Theorem 3.2.14 (and also Theorem 3.2.13) may be simplified somewhat when there exist certain solvable symmetry structures that force more than one of the ω^i in Theorem 3.2.14 to become exact. Theorem 3.2.17 is a generalisation of Theorem 3.2.16 and gives a symmetry structure condition for the first l ω^i to become exact.

Chapter 3 then examined in Theorems 3.4.1 (and its corollaries) some necessary conditions for a given differential form to be decomposable and closed modulo itself, in order to be able to apply Theorem 3.2.14. It was noted from Corollaries 3.4.2 and 3.4.3 that for a given $\Delta \in \Lambda^m(U^n)$ (where $m < n-1$), if one finds $n-m-1$ linearly independent Cauchy characteristic vector fields of $\langle \Delta, d\Delta \rangle$, then the Cauchy characteristic space is $(n-m)$-dimensional with Δ decomposable and closed modulo itself.

Next, as an application of Theorem 3.2.14, we demonstrated in Theorem 3.5.7 how it is possible to apply Theorem 3.2.14 to find normal form coordinates for the Pfaff problem. Given a one-form α of constant rank r on its domain, we showed that one solvable symmetry structure of length $2r+1$ for $(d\alpha)^r \wedge \alpha$ is enough to yield normal form, provided the last r symmetries are in the kernel of α.

We closed Chapter 3 by looking at closed differential two-forms. The main result contained in Theorem 3.6.8 is a method based on vector fields for finding the coordinates in Darboux's theorem. The technique is derived from a well-known iterative scheme, where a pair of new coordinates are extracted each time with the last stage using Theorem 3.2.14.

Chapter 4, as an introduction to PDEs, briefly examined ODEs. We gave an approach to finding the general solution of a single ODE using a solvable symmetry structure. Based on work by Edelen on the fundamental ideal representation of differential equations, Theorem 4.3.1 combined with Theorem 4.4.1 present an extension of Theorem 3.2.14 that can be used to generate the general local solution of single n-th order non-linear ODEs.

Next, in Chapter 5 we gave several algorithms based on solvable symmetry structures for finding local solutions of various types of first order PDEs. Firstly, we examined single first order quasilinear PDEs of one dependent variable and n independent variables. Given such a PDE, we showed in Theorem 5.4.1 that using its fundamental ideal representation and a solvable structure of n symmetries we are able to derive a local solution of the PDE in terms of an arbitrary function of n 'invariants'. The symmetry approach in Theorem 5.4.1 replaces the usual method of characteristics, where a parameterising variable is introduced, a system of first order ODEs is solved, and finally the parameterising variable is removed.

Next, in Theorem 5.5.3 we looked at the problem of finding local solutions of single first order non-linear PDEs of one dependent variable and two independent variables. The theorem uses Vessiot theory, to transform the problem to that of finding a local solution of some corresponding quasilinear PDE of precisely the form described before. This generally involves a two stage process requiring two separate solvable symmetry structures, one of four symmetries and the other of two. It is important to realise here that using Vessiot theory, we are restricted to single first order PDEs of two independent variables. This is because the integrability conditions for $n > 2$ independent variables typically involve more than one quasilinear PDE, for which we do not give a symmetry approach for generating local solutions.

Finally, we closed Chapter 5 with the situation when we are given a first order non-linear PDE of one dependent variable and two independent variables that does not involve the dependent variable. For such PDEs, we presented in Theorem 5.6.1 an alternative approach to using Vessiot theory, which is simpler in that it requires only one solvable structure of two symmetries.

While one of the aims of Chapter 5 was to avoid using Cauchy characteristics and the associated equations of Charpit and Lagrange to solve single first order PDEs, it was in Chapter 6 where we formally introduced and focused on such characteristics. In this chapter we examined some symmetry approaches to the Cauchy problem for Pfaffian systems. Our objective was to determine the extent to which symbolic computation techniques using solvable symmetry structures can be applied to the Cauchy problem. We managed to develop two techniques in Theorems 6.4.11 and 6.6.6 that solve the problem for the situation when the Cauchy characteristic space is strictly one-dimensional. Although these two results appear to be just algorithms for using solvable structures to find the functions in Theorem 2.2.13 and then express the Pfaffian system in terms of these functions (which is a significant new result in itself), it is Lemma 6.3.3 and Theorem 6.3.5 which allow use to apply the results to the Cauchy problem. Furthermore, the symmetry results given in Lemmas 6.4.8 and 6.6.2 and Corollary 6.6.5 (all of which can be applied using DIMSYM) are equally crucial to Theorems 6.4.11 and 6.6.6. It is significant to note that the symmetries required in Lemma 6.4.8 and Corollary 6.6.5 are not restricted by the condition in Lemma 6.6.2 that $Z_i \lrcorner \alpha^j = 0$ for all $j \neq i$. Therefore in using DIMSYM to generate symmetries for these results, it is only those for Lemma 6.6.2 that may be more difficult to find.

An inherent deficiency in applying the methods in Theorems 6.4.11 and 6.6.6 to solving the Cauchy problem is that the tangent space of the Cauchy data must be specified by vector fields of the precise form in the theorems: In generating the data, one must pick vector fields that are linear combinations of W_i, whose coefficients are smooth functions of $\gamma^1, \ldots, \gamma^m$. For one-dimensional data, this is all that is required; however in Theorem 6.6.6, where $Z_1, \ldots, Z_v \in \mathfrak{X}^{n-1-p}(U^n)$ must form a Frobenius integrable distribution, this may be difficult (or may not even exist) if $v > 1$. Nevertheless, if we are given the Cauchy data only in the form of some Frobenius integrable vector field distribution spanned by some $Z_1, \ldots, Z_v$, then we at least obtain uniqueness up to foliation. We used this situation in Sections 6.5 and 6.7 to show how to apply Theorems 6.4.11 and 6.6.6 to extract local solutions of single first order PDEs and a class of systems of two second order PDEs, that depend on some arbitrary constants. In solving the Cauchy problem for such PDEs with uniqueness only up to foliation, the failure in the uniqueness of the PDE solution is simply due to the arbitrariness of several constants. Thus if the Cauchy data is given only in form of some Frobenius integrable vector field distribution, then all solutions in the PDE solution space corresponding to the data are still functionally dependent.

Chapter 7 then looked at several symmetry approaches for finding similarity solutions of second order PDEs of the form

$$f_1 \frac{\partial^2 u}{\partial (x^1)^2} + f_2 \frac{\partial^2 u}{\partial (x^2)^2} + f_3 \frac{\partial^2 u}{\partial x^1 x^2} = k, \tag{9.1}$$

where f_1, f_2, f_3, k are arbitrary smooth functions of $x^1, x^1, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}$. Our main results, Theorems 7.4.2 and 7.6.1, combined with Theorems 7.7.1 and 7.7.2 show how one may use solvable symmetry structures to extract similarity solutions of a given second order hyperbolic or parabolic PDE of the form in (9.1). While the former two theorems assume L (or $L^\dagger$) is decomposable, it is hardly a restriction. This is because the discriminant in the latter two theorems remains non-negative on some neighbourhood precisely when the PDE is hyperbolic or parabolic. Hence, we are always able to apply Theorems 7.4.2 and 7.6.1 by replacing the given non-decomposable L with a suitable decomposable Ω, which is simply some linear combination of L and dC^1. For Theorem 7.4.2 there is a risk that the resulting two-dimensional Frobenius integrable distribution does not satisfy the transverse requirement. For such situations, Theorem 7.6.1 is an alternative to Theorem 7.4.2 that uses a non-regular feature of differential forms.

Finally, we ended Chapter 7 with an extension of earlier results using conditional symmetries. We showed that all previous results in the chapter are still true if I^r_F is replaced with the differential ideal $\widehat{I^r_F}$, where the latter is the fundamental ideal corresponding to the system of PDEs given by (9.1) and the first order quasilinear characterising PDE obtained from $V^{(1)} \lrcorner C^1 = 0$, where V is a conditional point symmetry of (9.1). We observed in Chapter 7 that since all our results are defined on the regular submanifold of the first jet bundle described by the locus of $V^{(1)} \lrcorner C^1 = 0$, on this region the difference between a conditional and a Lie point symmetry vanishes.

Chapter 8 initially presented two symmetry methods in Theorems 8.3.1 and 8.4.1 for reducing a second order hyperbolic or parabolic PDE of the form in (9.1) to first order. Using solvable symmetry structures, these theorems essentially give us algorithms for 'integrating' such a PDE to first order (generally non-linear) plus an arbitrary function. In applying the two techniques, it is necessary that the defining decomposable two-form L in Theorem 8.3.1 (or Ω in Theorem 8.4.1) corresponding to the PDE is closed modulo itself. While this condition may be difficult to test in practice, Theorem 3.4.1 and one of its corollaries in Corollary 3.4.2 at least offer us some hope of determining whether a differential two-form is closed modulo itself.

We also examined in Chapter 8 higher order PDEs linear in their highest order derivatives and of an arbitrary number of independent variables. Given such a PDE, Theorems 8.3.4 and 8.4.4 present two symmetry approaches for reducing its order by one that depends on one arbitrary function. Again, the assumption is that the defining n-form is decomposable and closed modulo itself. The results in Chapter 8 are a useful reformulation of the linearisation process by Bluman and Kumei [22].

For second order hyperbolic or parabolic PDEs of the form in (9.1) we may summarise some of the results in Chapters 5, 6, 7 and 8 in the following way: Since for such PDEs the corresponding L (or Ω) is locally decomposable, then we can at least generate some similarity solutions using results in Chapter 7. If, in addition to decomposability we have $dL \equiv 0 \bmod L$ (similarly for Ω), then as well as yielding similarity solutions, we can 'integrate' the PDE to first order plus an arbitrary smooth function using results in Chapter 8. For dealing with such first order PDEs, if the PDE is quasilinear then we may apply Theorem 5.4.1 in Chapter 5; however if it is non-linear then we have several choices: i) Vessiot theory in Theorem 5.5.3 followed by Theorem 5.4.1; ii) if the PDE does not contain u, then Theorem 5.6.1 followed once again by Theorem 5.4.1; or iii) one of the more complicated techniques in Theorems 6.4.11 and 6.6.6 in Chapter 6.

9.3 Areas for further work

In this section we examine some interesting areas arising from our work that either we have not had the time to pursue or have been unable to make any significant progress.

i) Solvable structure types

Examining Theorems 3.2.13 and 3.2.14, it would be of interest to be able to classify the classes of solvable symmetry structures that display $\omega^1, \ldots, \omega^m$ in (3.7) in particular rank configurations. The types given in Theorem 3.2.16 and its extension in Theorem 3.2.17 are what we consider to be the most obvious, and we found it very difficult to find others, partly because the algebraic manipulations became large and unmanageable. We would like to prove the following conjecture, which is essentially an extension of Theorem 3.2.17:

Suppose we are in the process of generating a solvable symmetry structure for some $(n-m)$-dimensional Frobenius integrable vector field distribution D defined on

U^n, and at some stage $s := m-p+1$ for some $1 \leq p \leq m$ we have found q commuting modulo $D \oplus Sp\{X_{p+1}, \ldots X_m\}$ non-trivial symmetries of $D \oplus Sp\{X_{p+1}, \ldots X_m\}$, which we denote by $X_{p-q+1}, \ldots, X_p$. Further suppose these vector fields are used as symmetries for the next q stages in the structure (this is possible using their commuting modulo $D \oplus Sp\{X_{p+1}, \ldots X_m\}$ property). Now ω^{p-q+1} can have, at most, a rank of $p-q$. Then one would hope to show that each $\omega^{p-q+2}, \ldots, \omega^p$ is of rank less than or equal to $p-q$. Theorem 3.2.17 proves this situation when $p = q = l$.

Ideally, given *any* solvable symmetry structure, it would be desirable to know *a priori* the rank of each ω^i. At this point in time, this seems quite a difficult task.

ii) The Pfaff problem

In the algorithm given in Theorem 3.5.7 in Chapter 3 for using Theorem 3.2.14 to find normal form coordinates for the Pfaff problem, it is not clear at this stage how to remove the condition on the last r symmetries in the solvable structure that they be in the kernel of the given one-form. However the following discussion provides a possible alternative approach:

For a given one-form α of constant rank r, suppose we remove the requirement in Theorem 3.5.7 that the last r symmetries be in the kernel of α, and are given an arbitrary solvable structure. Then one way to tackle the Pfaff problem is to look to classify the solvable symmetry structures that give α in the required normal form. For example, in the situation when $r = 1$ discussed in Theorem 3.5.5, suppose $X_3 \lrcorner\, \alpha$ is not necessarily zero. Now it is clear from repeating the proof of Theorem 3.5.5 that we eventually obtain

$$\alpha = (X_1 \lrcorner\, \alpha)\omega^1 + (X_2 \lrcorner\, \alpha)\omega^2 + (X_3 \lrcorner\, \alpha)\omega^3.$$

Expanding this, it can be shown that the coefficient of $d\gamma^3$ is $X_3 \lrcorner\, \alpha$, that of $d\gamma^2$ is

$$X_2 \lrcorner\, \alpha - (X_3 \lrcorner\, \alpha) X_2(\gamma^3),$$

and the coefficient of $d\gamma^1$ is

$$X_1 \lrcorner\, \alpha - (X_2 \lrcorner\, \alpha) X_1(\gamma^2) + (X_3 \lrcorner\, \alpha) X_2(\gamma^3) X_1(\gamma^2) - (X_3 \lrcorner\, \alpha) X_1(\gamma^3).$$

If any one of these three coefficients is zero (it is impossible for more than one to be zero since α is of rank one), then we are once again able to express α in normal form.

Hence for a given α of arbitrary constant rank $r \geq 1$, if any r of the coefficients of $d\gamma^1, \ldots, d\gamma^{2r+1}$ are zero, then normal form is achieved. We can therefore say that Theorem 3.5.7 is indeed a special case of this situation where the coefficients of $d\gamma^{r+2}, \ldots, d\gamma^{2r+1}$ become zero by demanding that $X_i \lrcorner\, \alpha = 0$ for all $r+2 \leq i \leq 2r+1$.

iii) Systems of PDEs

For 'involutive' systems of PDEs, i.e. in the sense of those that can be solved using the Cartan-Kähler theorem as a finite sequence of Cauchy problems, the most obvious area for further work is to develop symmetry approaches that are applicable to PDE problems with specific types of *Cartan characters* [23, 79, 86, 109]. Using such characters gives an indication of the number of arbitrary functions in the Cauchy data of the Cauchy problems. For example, in the classical Cauchy problem, the characters are of the form $(s^0, \ldots, s^n) = (0, \ldots, 0, s, 0)$, as the Cauchy data contains s arbitrary functions of $n-1$ variables. Of course the restriction to involutive systems of PDEs poses no obstruction since by the Cartan-Kuranishi theorem [86], any non-involutive system of PDEs becomes involutive after a finite number of prolongations of the system. Unfortunately to study such involutive systems we can no longer work in a smooth category. Instead, we must rely on the Cauchy-Kowalevski theorem and hence work in a real analytic category.

iv) The Cauchy problem for Pfaffian systems

Our treatment of the Cauchy problem in Chapter 6 is based on using Cauchy characteristic vector fields of the given differential ideal corresponding to the Pfaffian system, however it is possible to extend our work to a class of higher order vector fields introduced in Chapter 1 known as being *k-stable* [54, 55, 57]. Given a differential ideal

$$I := \langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle,$$

generated by a finite number of linearly independent one-forms $\alpha^1, \ldots, \alpha^p$ and their exterior derivatives, we remind the reader from Chapter 1 that X is a 1-*stable* vector field of I if it is a Cauchy characteristic of the augmented differential ideal

$$I_1 := \langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p, X \lrcorner\, d\alpha^1, \ldots, X \lrcorner\, d\alpha^p, d\left(X \lrcorner\, d\alpha^1\right), \ldots, d\left(X \lrcorner\, d\alpha^p\right) \rangle.$$

Finding such an X typically involves solving some awkward system of first order quasilinear PDEs, although the system may be simplified somewhat by looking for an $X \in (Sp\{\alpha^1, \ldots, \alpha^p\})^\perp$ that is singular in the sense that $X \lrcorner\, d\alpha^1, \ldots, X \lrcorner\, d\alpha^p$ is less than maximal rank [53, 57]. Once a 1-stable vector field is known, the fact that any integral manifold of I_1 is an integral manifold of I means that the Cauchy problem can be solved for I_1 using Theorem 6.2.1 given at the beginning of Chapter 6 while taking particular care that the Cauchy data is an integral manifold of I_1. While it is obvious how the symmetry approach to the Cauchy problem given in Theorem 6.6.6 may be applied to I_1, the main task would be to incorporate symmetry into finding a 1-stable vector field, or at least reducing the difficulty of the quasilinear PDEs that must be solved.

v) Similarity solutions

The work in Chapter 7 on similarity solutions of hyperbolic and parabolic PDEs of the form in (9.1) can be extended as follows:

Using the decomposability property of L in Theorem 7.2.1 (or Ω in Theorem 7.7.2 when L is not decomposable), if Y is a vector field such that $Y \lrcorner\, L \neq 0$, then we may write $L = (Y \lrcorner\, L) \wedge \omega$ for some one-form ω. This fact means that any integral manifold of

$$\langle C^1, dC^1, V \lrcorner\, L, d(V \lrcorner\, L) \rangle \tag{9.2}$$

is an integral manifold of $I_{\overline{F}}^r$, though the converse is not true as the ideals are not equal. Suppose we look for a symmetry of the differential ideal in (9.2). We may replace the symmetry requirement for V given in (7.4) with

$$\mathcal{L}_V C^1 = \lambda_1 C^1 + \lambda_2 (V \lrcorner\, L), \tag{9.3}$$

and the requirement in (7.5) with

$$\mathcal{L}_V (V \lrcorner\, L) = V \lrcorner\, (\mathcal{L}_V L) = \lambda_3 C^1 + \lambda_4 (V \lrcorner\, L), \tag{9.4}$$

for some smooth functions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Now if V satisfies (9.3) and (9.4) for some decomposable L, it is easy to show that Lemma 7.4.1 (and hence Theorems 7.4.2 and 7.6.1) in Chapter 7 still hold for such V. Moreover, since we are dealing with a very broad class of second order hyperbolic or parabolic PDEs of the form in (9.1) which includes the heat equation, examining the symmetry algebra of (9.2) modulo the symmetry algebra of I^r_F may be a worthwhile exercise in the hope of generating new similarity solutions.

Finally, consider the work on conditional symmetries in Section 7.8 of Chapter 7. In that section, Lie point symmetries of the overdetermined system of PDEs given by the original second order PDE and the characterising invariance system are used, but the issue of determining any differences between the symmetry algebra of the pair of equations given by (7.37) and (7.38) and that of $\widehat{I}^r_F$ is left unanswered. It would be useful to examine whether any symmetries of (7.37) and (7.38) not contained in $\widehat{I}^r_F$ can once gain yield any new similarity solutions.

vi) Computer algebra

From a computational perspective, our work throughout this thesis has been confined to using the Lie symmetry determination software package `DIMSYM` to finding symmetries of differential forms that are *decomposable*. This limitation has been largely due to Theorem 2.3.10. In order to obtain symmetries of a given *non-decomposable* form, say Δ, it is necessary to generate by hand the determining equations derived from

$$\mathcal{L}_X \Delta = \lambda \Delta, \tag{9.5}$$

and then feed them into `DIMSYM` in order to find some vector field X. It would be useful to write a module for `DIMSYM` that automatically generates and solves the determining equations of (9.5) for any given differential form, or even systems of differential forms, as in the discussion just given above on similarity solutions. Such software would also help with finding vector fields X_1 and X_2 in step (1) in Theorem 3.6.8 for generating coordinates in Darboux's theorem.

9.4 Final conclusion

In conclusion, our main tools in this thesis have been a result in Theorem 3.2.13 for integrating Frobenius integrable distributions using solvable symmetry structures, and an extension in Theorem 3.2.14 for simplifying differential forms that are decomposable and closed modulo themselves. Since both of these tasks can be performed without symmetries using ordinary differential equation techniques, it comes as no surprise (and with a little disappointment) that the types of PDEs for which we have been able to apply solvable symmetry structures have essentially been those that can be solved using ordinary differential equations. Nevertheless, for such classes of PDEs, we at least have provided direct computational approaches based on symmetry which replace the need for solving any differential equations at all. We believe that this is a significant contribution to the mathematical community. While our methods are really only limited by the capabilities of the particular software package used to solve the linear determining equations, this is perhaps only a temporary limitation since computing is perpetually becoming both cheaper and faster, thus attracting an increasing number of applied mathematicians to the field of solving problems in applied mathematics with symbolic computation methods that was only in its infancy a decade ago.

Appendix A

Solvable structures using DIMSYM

Here we illustrate using Examples 5.4.2 and 6.7.3 how to use DIMSYM to find a solvable structure of an arbitrary vector field distribution.

In the first example we are given a one dimensional vector field distribution corresponding to the Cauchy characteristic space of the differential ideal $\langle K \rangle$, where K is defined as in the example, and we wish to find solvable symmetry structure of two non-trivial symmetries. Working in REDUCE (with x(1),x(2),x(3) representing, respectively, coordinates x^1, x^2, z^1), we load the DIMSYM package and begin by entering the Cauchy characteristic vector field like so:

```
Y1 := x(1)*@x(1) - x(2)*@x(2) + x(2)*exp(x(3))*@x(3);
```

We then define some arbitrary functions:

```
for i := 1:3 do
xi(i) := newarb(x(1),x(2),x(3));
```

and also a symmetry vector:

```
symvec := xi(1)*@x(1) + xi(2)*@x(2) + xi(3)*@x(3);
```

Now we define the following arbitrary function:

```
a(1) := newarb(x(1),x(2),x(3));
```

and set up the Lie bracket symmetry relation:

```
zvec1 := comm(symvec,Y1) - a(1)*Y1;
```

Next, we introduce determining equations:

```
for i := 1:3 do
deteqn(i) := vecder(zvec1,x(i));
```

where `vecder` is a `DIMSYM` command which in this case simply extracts from the vector field in the first argument, the coefficient of the basis vector in the second argument. We then read the determining equations into `DIMSYM` using the following command:

```
readdets();
```

Next, we ask `DIMSYM` to solve the determining equations using the standard algorithm:

```
solvedets(std);
```

Any unsolved determining equations that were too difficult for `DIMSYM` to solve will be shown as follows:

```
showdets();
```

Finally, the infinitesimal generators of all the trivial as well as nontrivial symmetries that `DIMSYM` has been able to find are displayed in the following way:

```
mkgens();
```

```
end;
```

Although Example 5.4.2 uses simple observation to obtain a symmetry of $X_2 \lrcorner K$ (or equivalently, a symmetry of the span of `symvec` and `Y1`), for completeness, we give a brief indication below of how the code above may be extended to find this additional symmetry.

First we include the symmetry of the distribution spanned by `Y1` and call this `X2`:

```
X2 := 1/x(2)*@x(1);
```

While the definition for `symvec` stays the same, we make the following changes:

```
for i := 1:4 do
a(i) := newarb(x(1),x(2),x(3));

zvec1 := comm(symvec,Y1) - a(1)*Y1 - a(2)*X2;
zvec2 := comm(symvec,X2) - a(3)*Y1 - a(4)*X2;

for i := 1:3 do
begin
deteqn(i) := vecder(zvec1,x(i));
deteqn(i+3) := vecder(zvec2,x(i));
end;
```

The remainder of the code is the same.

In performing the above routines, we would ideally like no unsolved determining equations to exist when we execute `showdets()` each time; however in most cases, there will be unsolved determining equations if the `symvec` coefficients `xi(1),xi(2),xi(3),xi(4),xi(5)` are left purely arbitrary as they are above. Typically, for at least one of these coefficients, it is necessary to reduce the number of arguments of the corresponding `newarb`, or even define the coefficient to be zero.

Once a symmetry of a distribution is obtained, testing that it is non-trivial can simply be done by observation, or by using the exterior calculus package EXCALC to check that the interior product of the symmetry with the wedge product of the annihilating one-forms of the distribution is non-zero.

In generating solvable structures, it frequently occurs that at least one symmetry in the structure can be obtained by observation without the use of DIMSYM. In Example 6.7.3, X_3, X_4, X_5 are obvious symmetries found in this way. However to obtain the symmetry X_2, we are forced to used DIMSYM. Suppose then that we are given Y and symmetries X_3, X_4, X_5 in the example, and wish to find X_2. Then working in REDUCE with x(1), x(2), x(3), x(4), x(5), x(6) representing $x^1, x^2, z^1, z^1_1, z^1_2, z^1_{12}$, the required input code for DIMSYM is:

```
Y := @x(1) - @x(2) + (x(4)-x(5))*@x(3) + (x(4)-x(5))*@x(4);
X5 := @x(3);
X4 := @x(2);
X3 := @x(1);

for i := 1:6 do
xi(i) := newarb(x(1),x(2),x(3),x(4),x(5),x(6));

symvec := xi(1)*@x(1) + xi(2)*@x(2) + xi(3)*@x(3)
+ xi(4)*@x(4) + xi(5)*@x(5) + xi(6)*@x(6);

for i := 1:16 do
a(i) := newarb(x(1),x(2),x(3),x(4),x(5),x(6));

zvec1 := comm(symvec,Y)
- a(1)*Y - a(2)*X5 - a(3)*X4 - a(4)*X3;
zvec2 := comm(symvec,X5)
- a(5)*Y - a(6)*X5 - a(7)*X4 - a(8)*X3;
zvec3 := comm(symvec,X4)
- a(9)*Y - a(10)*X5 - a(11)*X4 - a(12)*X3;
zvec4 := comm(symvec,X3)
```

```
- a(13)*Y - a(14)*X5 - a(15)*X4 - a(16)*X3;

for i := 1:6 do
begin
deteqn(i) := vecder(zvec1,x(i));
deteqn(i+6) := vecder(zvec2,x(i));
deteqn(i+12) := vecder(zvec3,x(i));
deteqn(i+18) := vecder(zvec4,x(i));
end;

readdets();
solvedets(std);
showdets();
mkgens();
end;
```

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