# New symmetry solution techniques for first order non-linear PDEs *

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**Abstract**

Using recent extensions of work of S. Lie and É. Cartan for integrating Frobenius integrable vector field distributions via symmetry, we examine some symmetry techniques for finding local solutions of first order non-linear partial differential equations. In the language of exterior differential systems, we develop a technique for solving first order quasilinear partial differential equations that is then applied to general, first order non-linear partial differential equations. Our results are significant inasmuch as we give algorithms for solving first order partial differential equations in the presence of symmetry that are essentially mechanical in nature and done in the original coordinate system.

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*Short running title: Solutions of PDEs using symmetry

# 1 Introduction

In recent times, authors such as Basarab-Horwath [2], Duzhin and Lychagin [7], Hartl and Athorne [11], and Sherring and Prince [15] have studied ideas of Lie [14] and Cartan [4, 5] for integrating Frobenius integrable vector field distributions, largely with an intention to solving ordinary differential equations (ODEs). This paper examines how that same symmetry approach may be used to find local solutions of partial differential equations (PDEs) and in particular those differential equations which are first order non-linear. Our work is motivated by the fact that any such partial differential equation can be solved using ordinary differential equation techniques (e.g. by Cauchy characteristics), and we look to transform the problem of finding a solution into one of integrating a Frobenius integrable vector field distribution, which we then do using symmetry. Our techniques are significant in the sense that they remain the whole time in the original coordinate systems, as opposed to well-known symmetry reduction methods by such authors as Bluman and Kumei [3] who introduce canonical coordinates.

The fundamental assumption we make is the existence of certain very general symmetries known as *solvable structures* which are not necessarily of point (Lie) type. These may be generated most easily from existing symbolic symmetry finders. Our work uses the symmetry determination software package `Dimsym` [16] operating as a `REDUCE` overlay to find such symmetries, and while there are many other symbolic programs for Lie symmetry analysis such as `Symmgrp.max` [13], `LIE` [12], `Crackstar` [23], and `Liesymm` for Maple, we choose to work with `Dimsym` since it allows us to easily insert Lie bracket symmetry conditions. It then generates and attempts to solve the resulting linear determining equations. In the Appendix, we give the required input code for `Dimsym`.

Throughout this paper, we work in a smooth category, however there exist systems of PDEs in this category that are without solution, e.g. Lewy's famous example. Alternatively, we could elect to work in an analytic category with 'involutive' systems of PDEs, for which we are guaranteed local solutions by the Cauchy-Kowalevski theorem. Instead, since our main tool for this paper is a smooth results in Theorem 2.1, we choose to remain in a smooth category and simply assume for all our PDEs that there exist smooth local solutions. The reason we do this is also because we are not so much concerned with the existence and uniqueness of solutions of PDEs, but rather with the formulation of algorithmic approaches based on symmetry for extracting smooth local solutions.

The plan of this paper is to first provide a simple technique using symmetry for generating local solutions of first order quasilinear PDEs of one dependent variable and an arbitrary number of independent variables, based on work also found in Edelen [8, 9] on *fundamental ideals*. It is well-known that such PDEs can be solved by the method of characteristics using ordinary differential equations (see for example Duff [6]) and our aim here is to replace this approach with an algorithm solvable symmetry structures. We then give two symmetry algorithms for finding local solutions of general non-linear first order PDEs of one dependent variable and two independent variables, one which applies work by Vessiot [20, 21, 22], and another, simpler approach that deals with a special class of such PDEs that do not involve the dependent variable.

# 2 Preliminary results

It is assumed throughout this paper that for any integers $n$ and $m$, $U^n$ and $V^m$ are, respectively, some open, convex neighbourhoods of $\mathbb{R}^n$ and $\mathbb{R}^m$, with coordinates $x^1, \ldots, x^n$ and $z^1, \ldots, z^m$. On the $\kappa$-th jet bundle $J^\kappa(U^n, V^m)$, we say that the set of exterior differential $p$-forms $\Lambda^p(J^\kappa(U^n, V^m))$ is a section of the bundle of all homogeneous differential forms $\Lambda(J^\kappa(U^n, V^m))$. We define $\mathfrak{X}(J^\kappa(U^n, V^m))$ to be the module of all smooth vector fields over $C^\infty(J^\kappa(U^n, V^m))$. We also assume throughout this paper that, on their domains of definition, all vector field distributions are of constant dimension, and all mappings and differential one-forms are of constant rank.

The *Cauchy characteristic* space of a differential ideal $I$ generated by some finite collection of differential forms is denoted $A(I)$, and contains all vector fields $X \in \mathfrak{X}(U)$ such that $X \lrcorner I \subset I$. A vector field $X \in \mathfrak{X}(J^\kappa(U^n, V^m))$ is said to be a *symmetry* (isovector) of $I$ if it satisfies the Lie derivative condition $\mathcal{L}_X I \subset I$. A vector field $X \in \mathfrak{X}(J^\kappa(U^n, V^m))$ is a symmetry of a vector field distribution $D \subset \mathfrak{X}(J^\kappa(U^n, V^m))$ if $\mathcal{L}_X D \subset D$. We say that a vector field is a *non-trivial* symmetry if, in terms of a differential ideal, it is not Cauchy characteristic, or in terms of a vector field distribution, it is not in the distribution. In this paper, unless otherwise stated, we assume all symmetries are non-trivial. For any vector field distribution $D$, we say that a collection of $q$ linearly independent vector fields $X_1, \ldots, X_q \in \mathfrak{X}(J^\kappa(U^n, V^m))$ forms a *solvable symmetry structure* for $D$ if

$$\begin{gathered}
\mathcal{L}_{X_1}\left(sp\{X_2, \ldots, X_q\} \oplus D\right) \subset sp\{X_2, \ldots, X_q\} \oplus D, \\
\vdots \\
\mathcal{L}_{X_{q-1}}\left(sp\{X_q\} \oplus D\right) \subset sp\{X_q\} \oplus D, \\
\mathcal{L}_{X_q} D \subset D.
\end{gathered}$$

Such a solvable structure may found using `Dimsym`. This is done in stages by first finding a symmetry $X_q$ of $D$, then finding a symmetry $X_{q-1}$ of $D$ spanned with $X_q$, and so on until $X_1$ is found. For each stage, we input the necessary Lie bracket relations, and let `Dimsym` solve the linear determining equations. See the Appendix for further details. Note that $X_1, \ldots, X_q$ are quite general symmetries, and need not be of Lie point type. Moreover, while $X_q$ is a genuine symmetry of $D$, $X_1$ is a much weaker symmetry. In fact, $X_1$ is only a symmetry of $D$, modulo $X_2, \ldots, X_q$.

Finally, our main tool for this paper is Theorem 3.14 in [1] (or Proposition 4.7 in [15]), which is adapted from a result for integrating a Frobenius integrable vector field distribution based on a solvable structure of symmetries:

**Theorem 2.1.** *Let $\Omega \in \Lambda^p(J^\kappa(U^n, V^m))$ such that $\Omega$ is decomposable and $d\Omega \equiv 0 \bmod \Omega$. Suppose there exists a solvable structure of linearly independent symmetries $X_1, \dots, X_p \in \mathfrak{X}(J^\kappa(U^n, V^m))$ such that $X_p$ is a non-trivial symmetry of the Cauchy characteristic space $A(\langle\Omega\rangle)$, and that for all $i < p$, $X_i$ is a non-trivial symmetry of the distribution spanned by the generators of $A(\langle\Omega\rangle)$, and $X_{i+1}, \dots, X_p$. For $1 \le i \le p$, define $\omega^i$ by*

$$\omega^i := \frac{X_1 \lrcorner \dots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \dots \lrcorner X_p \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \dots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \dots \lrcorner X_p \lrcorner \Omega}.$$

*Then $\{\omega^1, \dots, \omega^p\}$ is dual to $\{X_1, \dots, X_p\}$, and for all $\omega^i$ up to $i = p$,*

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2) d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3) d\gamma^1, \\
&\vdots \\
\omega^p &\equiv d\gamma^p \mod d\gamma^1, \dots, d\gamma^{p-1},
\end{aligned}$$

*for some functionally independent $\gamma^1, \dots, \gamma^p \in C^\infty(J^\kappa(U^n, V^m))$. Finally, define $\gamma^0 := \Omega(X_1, \dots, X_p)$. Then $\Omega = \gamma^0 d\gamma^1 \wedge \dots \wedge d\gamma^p$.*

Using the decomposability of $\Omega$, it has been shown in [1] that the symmetry conditions in Theorem 2.1 are equivalent to having that

$$\begin{aligned}
\mathcal{L}_{X_1} (X_2 \lrcorner \dots \lrcorner X_p \lrcorner \Omega) &= \lambda_1 (X_2 \lrcorner \dots \lrcorner X_p \lrcorner \Omega), \\
&\vdots \\
\mathcal{L}_{X_{p-1}} (X_p \lrcorner \Omega) &= \lambda_{p-1} (X_p \lrcorner \Omega), \\
\mathcal{L}_{X_p} \Omega &= \lambda_p \Omega,
\end{aligned}$$

for some $\lambda_1, \dots \lambda_p \in C^\infty(J^\kappa(U^n, V^m))$.

# 3 Quasilinear PDEs

Suppose we have a single quasilinear PDE of the form

$$\left( f_{11} \frac{\partial u^1}{\partial x^1} + \dots + f_{n1} \frac{\partial u^1}{\partial x^n} \right) + \dots + \left( f_{1m} \frac{\partial u^m}{\partial x^1} + \dots + f_{nm} \frac{\partial u^m}{\partial x^n} \right) = k, \quad (1)$$

where $x^1, \dots, x^n$ are the independent variables, $u^1, \dots, u^m$ are the dependent variables of the PDE, and $f_{ij}$, where $1 \le i \le n$, $1 \le j \le m$, and $k$ are smooth functions of $x^1, \dots, x^n, u^1, \dots, u^m$.

On the first jet bundle $J^1(U^n, V^m)$ with coordinates $x^1, \ldots, x^n, z^1, \ldots, z^m, z_1^1, \ldots, z_1^m$, we have the first-order contact (Pfaffian) system $\Sigma$ (i.e. a submodule over $C^\infty(J^1(U^n, V^m))$) generated by

$$C^j := dz^j - z_1^j dx^1 - \cdots - z_n^j dx^n,$$

where $1 \leq j \leq m$. Now define

$$F := \left(f_{11}z_1^1 + \cdots + f_{n1}z_n^1\right) + \cdots + \left(f_{1m}z_1^m + \cdots + f_{nm}z_n^m\right) - k.$$

A solution of the PDE in (1) can be thought of as an immersion or rank $n$ mapping into the locus of $J^1(U^n, V^m)$ described by the equation $F = 0$, that also satisfies the $nm$ partial derivative relations

$$z_i^j = \frac{\partial z^j}{\partial x^i},$$

where $1 \leq i \leq n$ and $1 \leq j \leq m$.

It is well-known (e.g. Theorem 2.3.1 in Stormark [17]) that an integral manifold (immersion) in some $\kappa$-jet bundle that annihilates all the contact forms generating the $\kappa$th-order contact system is a $\kappa$-jet if and only if on the integral manifold we have $dx^1 \wedge \cdots \wedge dx^n \neq 0$. Using this result and the discussion in the previous paragraph we can then say the following:

**Theorem 3.1.** *Let there exist a (rank $n$) immersion*

$$\Phi : U^n \longrightarrow J^1(U^n, V^m),$$

*satisfying the following $(m+2)$-conditions:*

1. $\Phi^* C^j = 0$ *for all* $1 \leq j \leq m$,
2. $\Phi^* F = 0$,
3. $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$.

*Then $\Phi(U^n) = j^1 f(U^n)$ for some smooth solution map $f : U^n \longrightarrow V^m$ of the PDE in (1).*

The inequality condition of Theorem 3.1 is a transverse (or independence) requirement which allows us to express the dependent variables as functions of the independent variables.

# 4 Ideals of quasilinear PDEs

Following [8, 9], the *fundamental ideal* of the PDE in (1) is defined as:

$$I_F := \langle C^1, \ldots, C^m, dC^1, \ldots, dC^m, F dx^1 \wedge \cdots \wedge dx^n \rangle.$$

From Lemma 1.1 in [8],

**Lemma 4.1.**

$$d\left(F dx^1 \wedge \cdots \wedge dx^n\right) \equiv 0 \mod C^1, \ldots, C^m, dC^1, \ldots, dC^m.$$

This means that $I_F$ is in fact a differential ideal. Our aim is to look for an $n$-dimensional integral manifold of $I_F$, i.e. an immersion

$$\Phi : U^n \longrightarrow J^1(U^n, V^m),$$

such that $\Phi^* I_F = 0$ and $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$. Such an immersion obviously satisfies items 1 and 3 in Theorem 3.1. Item 2 in the theorem is seen to be satisfied if we recall that

$$0 = \Phi^*(F dx^1 \wedge \cdots \wedge dx^n) = (\Phi^* F)\Phi^*(dx^1 \wedge \cdots \wedge dx^n)$$

implies that $\Phi^* F = 0$, using item 3. Therefore, from Theorem 3.1:

**Theorem 4.2.** *With $I_F$ defined as above corresponding to the PDE in* (1)*, suppose the immersion*

$$\Phi : U^n \longrightarrow J^1(U^n, V^m),$$

*is an $n$-dimensional integral manifold of $I_F$ such that $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$. Then $\Phi(U^n) = j^1 f(U^n)$ for some smooth solution map $f : U^n \longrightarrow V^m$ of the PDE in* (1)*.*

We will now show in the following theorem that the first order quasilinear nature of our PDE means the $n$-form $F dx^1 \wedge \cdots \wedge dx^n$ can be simplified somewhat so that, modulo $C^1, \ldots, C^m$, it does not depend on any of the first derivative coordinates $z^1_1, \ldots, z^1_n, \ldots, z^m_1, \ldots, z^m_n$.

**Theorem 4.3.**

$$I_F = \langle C^1, \ldots, C^m, dC^1, \ldots, dC^m, K \rangle,$$

*where*

$$\begin{aligned} K := &\left(f_{11} dz^1 + \cdots + f_{1m} dz^m\right) \wedge dx^2 \wedge \cdots \wedge dx^n + \ldots \\ &+ dx^1 \wedge \cdots \wedge dx^{n-1} \wedge \left(f_{n1} dz^1 + \cdots + f_{nm} dz^m\right) \\ &- k dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

*Proof.* We have that

$$\begin{aligned} F dx^1 \wedge \cdots \wedge dx^n = &\{\left(f_{11} z^1_1 + \cdots + f_{n1} z^1_n\right) + \ldots \\ &+ \left(f_{1m} z^m_1 + \cdots + f_{nm} z^m_n\right) - k\} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Now for any given $1 \leq i \leq n$ and $1 \leq j \leq m$ (no sum on $i$ or $j$),

$$\begin{aligned} f_{ij} z^j_i dx^1 \wedge \cdots \wedge dx^n &= f_{ij} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge z^j_i dx^i \wedge dx^{i+1} \wedge \cdots \wedge dx^n \\ &= f_{ij} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \left(du^j - C^j\right) \wedge dx^{i+1} \wedge \cdots \wedge dx^n \\ &\equiv f_{ij} dx^1 \wedge \cdots \wedge dx^{i-1} \wedge du^j \wedge dx^{i+1} \wedge \cdots \wedge dx^n \mod C^j, \end{aligned}$$

where in the second line we have used that $dz^j - C^j = z^j_1 dx^1 + \cdots + z^j_i dx^i + \cdots + z^j_n dx^n$. Therefore,

$$\begin{aligned} F dx^1 \wedge \cdots \wedge dx^n \equiv \{ & \left( f_{11} dz^1 \wedge dx^2 \wedge \cdots \wedge dx^n \right) + \dots \\ & + \left( f_{n1} dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^1 \right) \} + \dots \\ & + \{ \left( f_{1m} dz^m \wedge dx^2 \wedge \cdots \wedge dx^n \right) + \dots \\ & + \left( f_{nm} dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^m \right) \} \\ & - k dx^1 \wedge \cdots \wedge dx^n \mod C^1, \dots, C^m. \end{aligned}$$

We can collect terms so that

$$\begin{aligned} F dx^1 \wedge \cdots \wedge dx^n &\equiv \left( f_{11} dz^1 + \cdots + f_{1m} dz^m \right) \wedge dx^2 \wedge \cdots \wedge dx^n + \dots \\ &\quad + dx^1 \wedge \cdots \wedge dx^{n-1} \wedge \left( f_{n1} dz^1 + \cdots + f_{nm} dz^m \right) \\ &\quad - k dx^1 \wedge \cdots \wedge dx^n \mod C^1, \dots, C^m, \\ &\equiv K \mod C^1, \dots, C^m. \end{aligned}$$

To complete the proof, since

$$K \equiv F dx^1 \wedge \cdots \wedge dx^n \mod C^1, \dots, C^m,$$

using Lemma 4.1 we obtain

$$\begin{aligned} dK &\equiv d \left( F dx^1 \wedge \cdots \wedge dx^n \right) \mod C^1, \dots, C^m, \\ &\equiv 0 \mod C^1, \dots, C^m, dC^1, \dots, dC^m. \end{aligned}$$

Hence the result. □

We define

$$I_{\overline{F}} := \langle C^1, \dots, C^m, dC^1, \dots, dC^m, K \rangle.$$

Technically speaking, $I_{\overline{F}} := I_F$ (by Theorem 4.3), and the notation $I_{\overline{F}}$ might appear redundant. However we will use $I_{\overline{F}}$ as a brief way of referring to the particular choice of generators $C^1, \dots, C^m, dC^1, \dots, dC^m, K$.

Theorem 4.3 now means that the task of determining local solutions of (1) becomes that of finding $n$-dimensional integral manifolds of $I_{\overline{F}}$. Note that the $n$-form $K$ in the ideal contains no first order derivative coordinates.

We end this section with an obvious result that uses Theorems 3.1 and 4.3:

**Theorem 4.4.** *Let*

$$\Phi : U^n \longrightarrow U^n \times V^m,$$

*be an immersion such that* $\Phi^* K = 0$ *and* $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$. *Then* $\Phi(U^n) = j^0 f(U^n)$ *for some smooth solution map* $f : U^n \longrightarrow V^m$ *of the PDE in (1).*

Theorem 4.4 means that if the pull-back of a rank $n$ immersion mapping into the graph space satisfies the transverse condition and annihilates $K$, then this is enough to guarantee in this case that the 1-jet $j^1 f$ is an $n$-dimensional integral manifold of the differential ideal $I_{\overline{F}}$ (and hence a local solution of PDE in (2)). This is because $K$ contains no first order derivative coordinates and all the contact forms are automatically annihilated.

# 5 Quasilinear PDEs of one dependent variable

This section addresses single quasilinear PDEs of one dependent variable of the form

$$f_1 \frac{\partial u}{\partial x^1} + \cdots + f_n \frac{\partial u}{\partial x^n} = k, \tag{2}$$

where $f_1, \ldots, f_n, k \in C^\infty(U^n \times V^1)$. For this PDE, the corresponding $K$ in $I_{\overline{F}}$ is

$$K = \left(f_{11} dz^1 \wedge dx^2 \wedge \cdots \wedge dx^n\right) + \cdots + \left(f_{n1} dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^1\right) \\ - k dx^1 \wedge \cdots \wedge dx^n.$$

Now $K$ is an $n$-form in the $(n+1)$-dimensional space $U^n \times V^1$. From Corollary 4.2 and Theorem 4.3 in [1], respectively, it follows that $K$ is decomposable and $dK \equiv 0 \bmod K$. Suppose we are given $n$ non-trivial symmetries $X_1, \ldots, X_n \in \mathfrak{X}(U^n \times V^1)$ such that

$$\begin{aligned} \mathcal{L}_{X_n} K &= \lambda_n K, \\ \mathcal{L}_{X_{n-1}} (X_n \lrcorner K) &= \lambda_{n-1} (X_n \lrcorner K), \\ &\vdots \\ \mathcal{L}_{X_1} (X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K) &= \lambda_1 (X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K), \end{aligned}$$

for some $\lambda_1, \ldots, \lambda_n \in C^\infty(U^n \times V^1)$. Applying Theorem 2.1, we can explicitly compute some $\gamma^0, \ldots, \gamma^n \in C^\infty(U^n \times V^1)$ so that

$$K = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^n.$$

Now consider the $n$-dimensional regular submanifold of $U^n \times V^1$ described by

$$H(\gamma^1, \ldots, \gamma^n) = 0, \tag{3}$$

where $H$ is any non-constant smooth function of $\gamma^1, \ldots, \gamma^n$. It is assumed $H$ is constant rank one on $U^n \times V^1$. Then

$$dH = \frac{\partial H}{\partial \gamma^1} d\gamma^1 + \cdots + \frac{\partial H}{\partial \gamma^n} d\gamma^n \stackrel{*}{=} 0, \tag{4}$$

where we use $\stackrel{*}{=}$ to mean equality on the tangent space of the submanifold of $U^n \times V^1$ described by (3). We must have that at each point of this submanifold there exists some $1 \leq p \leq n$ such that

$$\frac{\partial H}{\partial \gamma^p} \neq 0.$$

Otherwise $H$ is independent of all $\gamma^1, \ldots, \gamma^n$ at some point, but it is assumed $H$ is constant rank one. Now from inserting (4),

$$0 \stackrel{*}{=} dH \wedge d\gamma^1 \wedge \cdots \wedge d\gamma^{p-1} \wedge d\gamma^{p+1} \wedge \cdots \wedge d\gamma^n$$
$$= \frac{\partial H}{\partial \gamma^p} d\gamma^p \wedge d\gamma^1 \wedge \cdots \wedge d\gamma^{p-1} \wedge d\gamma^{p+1} \wedge \cdots \wedge d\gamma^n.$$

This implies that $K = 0$ on the submanifold described by equation (3).

If on some neighbourhood of $U^n \times V^1$ we have

$$\frac{\partial H}{\partial z^1} \neq 0,$$

then by the implicit function theorem,

$$z^1 = \overline{H}(x^1, \ldots, x^n), \tag{5}$$

for some smooth $\overline{H}$. Therefore $j^0\overline{H}^* K = 0$ (and hence $j^1\overline{H}^* I_{\overline{F}} = 0$). Since $j^0\overline{H}^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$, Theorem 4.4 means that equation (5) is then a local solution of the quasilinear PDE in (2). We summarise the above result in the following theorem:

**Theorem 5.1.** *Suppose we have a first order quasilinear PDE of the form*

$$f_1 \frac{\partial u}{\partial x^1} + \cdots + f_n \frac{\partial u}{\partial x^n} = k, \tag{6}$$

*for some* $f_1, \ldots, f_n, k \in C^\infty(U^n \times V^1)$, *with the corresponding* $K$ *in* $I_{\overline{F}}$ *as*

$$K := \left(f_1 dz^1 \wedge dx^2 \wedge \cdots \wedge dx^n\right) + \cdots + \left(f_n dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dz^1\right)$$
$$- k dx^1 \wedge \cdots \wedge dx^n.$$

*If there exist* $n$ *non-trivial symmetries* $X_1, \ldots, X_n \in \mathfrak{X}(U^n \times V^1)$ *such that*

$$\mathcal{L}_{X_n} K = \lambda_n K,$$
$$\mathcal{L}_{X_{n-1}}(X_n \lrcorner K) = \lambda_{n-1}(X_n \lrcorner K),$$
$$\vdots$$
$$\mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K) = \lambda_1(X_2 \lrcorner \ldots \lrcorner X_n \lrcorner K),$$

*for some* $\lambda_1, \ldots, \lambda_n \in C^\infty(U^n \times V^1)$, *then there exist some functionally independent* $\gamma^1, \ldots, \gamma^n \in C^\infty(U^n \times V^1)$ *so that*

$$K = K(X_1, \ldots, X_n) d\gamma^1 \wedge \cdots \wedge d\gamma^n.$$

*Furthermore, any regular submanifold of* $U^n \times V^1$ *given by*

$$H(\gamma^1, \ldots, \gamma^n) = 0,$$

*for any smooth* $H$ *such that* $\frac{\partial H}{\partial z^1} \neq 0$ *on some neighbourhood is then the graph space coordinate representation of a local solution of the PDE in (6).*

Using the fact that $K$ is decomposable, any symmetry of the vector field distribution $\{X \in \mathfrak{X}(U^n \times V^1) : X \lrcorner K = 0\}$ is also a symmetry of $K$ (and vice versa). This fact also holds true for any differential form that is an interior product of $K$ by some vector field. See [1] for further details. Therefore we may use `Dimsym` to generate the required symmetries in Theorem 5.1.

Next, we give an example:

**Example 5.2.** Consider the following first order quasilinear PDE of two independent variables and one dependent variable:

$$x^1 \frac{\partial u}{\partial x^1} - x^2 \frac{\partial u}{\partial x^2} = x^2 \exp(u). \tag{7}$$

Our corresponding two-form $K$ on $U^2 \times V^1$ is

$$K = x^1 dz^1 \wedge dx^2 + x^2 dz^1 \wedge dx^1 - x^2 \exp(z^1) dx^1 \wedge dx^2.$$

Using `Dimsym`, we find that $X_2 := \frac{1}{x^2}\frac{\partial}{\partial x^1}$ is a non-trivial symmetry of $K$. Then with $X_2 \lrcorner K = -dz^1 - \exp(z^1)dx^2$, it is easy to see that $X_1 := \frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $X_2 \lrcorner K$. Hence from Theorem 2.1,

$$\frac{X_2 \lrcorner K}{X_1 \lrcorner X_2 \lrcorner K} = d\left(x^2 - \exp(-z^1)\right),$$

and

$$\frac{X_1 \lrcorner K}{X_2 \lrcorner X_1 \lrcorner K} = d(x^1 x^2) + x^1 d\left(x^2 - \exp(-z^1)\right). \tag{8}$$

Equation (8) can easily be derived by performing a coordinate transformation with the introduction of $x^2 - \exp(-z^1)$ as a new coordinate.

We now obtain

$$K = K(X_2, X_1) d(x^1 x^2) \wedge d\left(x^2 - \exp(-z^1)\right).$$

One can then say that

$$H\left(x^1 x^2, x^2 - \exp(-z^1)\right) = 0,$$

is, in implicit form in terms of the graph space coordinates, a local solution of (7) for any suitable smooth $H$. Thus

$$u = -\ln\left|x^2 - l(x^1 x^2)\right|,$$

gives local solutions for arbitrary choice of smooth $l$ that is a function of $x^1 x^2$.

While the symmetries used in Theorem 5.1 do not have to be point symmetries, there exists a relationship between Lie point symmetries and symmetries of $K$ that we explore below.

First, we introduce the following definition:

**Definition 5.3.** A vector field $X \in \mathfrak{X}(U^n \times V^1)$ is said to be a *Lie point symmetry* of the first order quasilinear PDE in (2) if

$$X^{(1)}(F) = 0,$$

whenever $F = 0$, where $X^{(1)}$ is the first prolongation of $X$ (i.e. $X^{(1)}$ projects to $X$ and preserves contact structure).

Using this, we obtain the following:

**Theorem 5.4.** *Given a first order quasilinear PDE of the form in (2), a vector field $X \in \mathfrak{X}(U^n \times V^1)$ is a symmetry of its corresponding $n$-form $K$ if and only if $X$ is a Lie point symmetry of the PDE.*

*Proof.* First suppose $X \in \mathfrak{X}(U^n \times V^1)$ is a symmetry of $K$ corresponding to the quasilinear PDE in (2), i.e. $\mathcal{L}_X K = \lambda K$ for some $\lambda \in C^\infty(U^n \times V^1)$. Since $K$ does not contain any first derivative coordinates, we can write

$$\mathcal{L}_{X^{(1)}} K = \lambda K, \tag{9}$$

with (9) defined on the first jet bundle $J^1(U^n, V^1)$. Now $K \equiv F dx^1 \wedge \cdots \wedge dx^n$ mod $C^1$, so

$$\mathcal{L}_{X^{(1)}} \left( F dx^1 \wedge \cdots \wedge dx^n \wedge C^1 - K \wedge C^1 \right) = 0. \tag{10}$$

It is well-known (and not hard to show) that for any point symmetry, the Lie derivative of any first order contact form with respect to the first prolongation of the symmetry is a contact form. So putting $\mathcal{L}_{X^{(1)}} C^1 = \rho C^1$ for some $\rho \in C^\infty(J^1(U^n, V^1))$, we have from (10),

$$\begin{aligned} X^{(1)}(F) dx^1 \wedge \cdots \wedge dx^n \wedge C^1 + F \mathcal{L}_{X^{(1)}} \left( dx^1 \wedge \cdots \wedge dx^n \wedge C^1 \right) \\ = (\lambda + \rho) K \wedge C^1. \end{aligned} \tag{11}$$

Now $F dx^1 \wedge \cdots \wedge dx^n \wedge C^1 = K \wedge C^1$ and $dx^1 \wedge \cdots \wedge dx^n \wedge C^1 = dx^1 \wedge \cdots \wedge dx^n \wedge dz^1$. Hence

$$X^{(1)}(F) dx^1 \wedge \cdots \wedge dx^n \wedge dz^1 = 0,$$

whenever $F = 0$. This implies that $X^{(1)}(F) = 0$ whenever $F = 0$.

Conversely, suppose that $X$ is a Lie point symmetry of the quasilinear PDE in (2). Hence

$$X^{(1)}(F) = 0, \tag{12}$$

whenever $F = 0$. Now

$$\begin{aligned} \mathcal{L}_{X^{(1)}} \left( K \wedge C^1 \right) &= \mathcal{L}_{X^{(1)}} \left( F dx^1 \wedge \cdots \wedge dx^n \wedge C^1 \right), \\ &= X^{(1)}(F) dx^1 \wedge \cdots \wedge dx^n \wedge C^1 \\ &\quad + F \mathcal{L}_{X^{(1)}} \left( dx^1 \wedge \cdots \wedge dx^n \wedge C^1 \right). \end{aligned}$$

Therefore from (12),

$$\mathcal{L}_{X^{(1)}}\left(K \wedge C^1\right) = 0,$$

whenever $F = 0$. Expanding, and using the fact that $\mathcal{L}_{X^{(1)}} C^1 = \rho C^1$ for some $\rho \in C^\infty(J^1(U^n, V^1))$, we obtain

$$(\mathcal{L}_{X^{(1)}} K) \wedge C^1 + \rho K \wedge C^1 = 0,$$

whenever $F = 0$. Then using $K \wedge C^1 = F dx^1 \wedge \cdots \wedge dx^n \wedge C^1$, we find

$$(\mathcal{L}_{X^{(1)}} K) \wedge C^1 = 0,$$

whenever $F = 0$. We also have

$$(\mathcal{L}_{X^{(1)}} K) \wedge C^1 = L dx^1 \wedge \cdots \wedge dx^n \wedge dz^1, \tag{13}$$

for some $L \in C^\infty(J^1(U^n, V^1))$ because (9) implies $\mathcal{L}_{X^{(1)}} K$ is an $n$-form expressed entirely in terms of the $(n+1)$ coordinates of the graph space. Furthermore, from the definition of $C^1$ we obtain that $L$ is linear in $z^1_1$ and $z^1_2$. Since (13) is zero whenever $F = 0$, and $F$ is also linear in $z^1_1$ and $z^1_2$, we can therefore say that $L = hF$ for some $h \in C^\infty(U^n \times V^1))$. Since $hF dx^1 \wedge \ldots dx^n \wedge dz^1 = hK \wedge C^1$, we may write

$$\mathcal{L}_{X^{(1)}} K \equiv hK \mod C^1.$$

Hence

$$\mathcal{L}_X K \equiv hK \mod C^1.$$

As $h$ is expressed only in terms of coordinate of the graph space, we therefore have $\mathcal{L}_X K = hK$. □

Theorem 5.4 has the following corollary:

**Corollary 5.5.** *A vector field $X \in \mathfrak{X}(U^n \times V^1)$ is a Lie point symmetry of the first order quasilinear PDE in (2) if and only if $X^{(1)}$ is a symmetry of its corresponding $I_{\overline{F}}$.*

## 6 First order non-linear PDEs

In this section, we examine two approaches to solving single first order non-linear PDEs of one dependent variable and two independent variables. The first involves using Vessiot theory while the second employs a simpler technique for the special case when the PDE does not explicitly involve the dependent variable. We begin with the former.

## 6.1 Vessiot theory

This section summarises the main points of Vessiot's theory [20, 21, 22] of differential equations, as reviewed by Fackerell [10], Stormark [17], and Vassiliou [18, 19].

Consider the system of $\rho$ PDEs of $n$ independent and $m$ dependent variables

$$F_\nu(x^i, u^j, u^j_{i_1}, u^j_{i_1 i_2}, \ldots, u^j_{i_1\cdots i_\kappa}) = 0, \qquad \nu = 1, \ldots, \rho, \tag{14}$$

where the $n$ $x^i$ and $m$ $u^j$ are, respectively, the independent and dependent variables. The subscripts $1 \leq i_1 \leq \cdots \leq i_\kappa \leq n$ are used to specify partial derivatives of $u^j$, where $\kappa$ is the maximum order of the system.

In the $\kappa$th-jet bundle $J^\kappa(U^n, V^m)$ with coordinates $x^i, z^j, z^j_{i_1}, z^j_{i_1 i_2}, \ldots, z^j_{i_1\cdots i_\kappa}$, we may express a solution of the system of PDEs above as a regular $n$-dimensional submanifold that

1. Satisfies the relations $F_\nu(x^i, z^j, z^j_{i_1}, z^j_{i_1 i_2}, \ldots, z^j_{i_1\cdots i_\kappa}) = 0$, for all $\nu = 1, \ldots, \rho$,

2. Satisfies the transverse requirement,

3. Has a tangent space that annihilates the $\kappa$th-order contact system generated by (with sum):

$$\begin{aligned}
C^j &:= dz^j - z^j_{i_1} dx^{i_1}, \\
C^j_{i_1} &:= dz^j_{i_1} - z^j_{i_1 i_2} dx^{i_2}, \\
C^j_{i_1 i_2} &:= dz^j_{i_1 i_2} - z^j_{i_1 i_2 i_3} dx^{i_3}, \\
&\vdots \\
C^j_{i_1\cdots i_{\kappa-1}} &:= dz^j_{i_1\cdots i_{\kappa-1}} - z^j_{i_1\cdots i_\kappa} dx^{i_\kappa}.
\end{aligned}$$

We denote the span of the contact system by $\Omega^\kappa(U^n, V^m)$, that includes all $1 \leq j \leq m$.

Using the independence forms $dx^1, \ldots, dx^n$, we know from before that if $\Phi : U^n \longrightarrow J^\kappa(U^n, V^m)$ is any immersion whose pull-back annihilates the contact system $\Omega^\kappa(U^n, V^m)$, and satisfies the transverse condition $\Phi^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0$, then $\Phi(U^n)$ is the image of some $\kappa$-jet. To incorporate the system of PDEs into the contact system, we introduce a inclusion map $\Phi_F$ mapping onto the regular submanifold of the $\kappa$-th jet bundle described by the PDEs $\{F_\nu = 0 : \nu = 1, ..., \rho\}$ (here we put the subscript $F$ on $\Phi$ to indicate the single or system of PDEs of the form in (14)). We then pull-back the contact forms in $\Omega^\kappa(U^n, V^m)$ by $\Phi_F$. The *Vessiot distribution* is then defined as the vector field dual of the pulled-back contact system, i.e. $\Phi_F{}^*\Omega^\kappa(U^n, V^m))^\perp$. Using the Vessiot distribution, our task is to look for some immersion $\Phi$ of rank $n$ that maps into the image of $\Phi_F$ and annihilates the contact system, while at the same time being transverse.

We illustrate with a simple example:

**Example 6.1.** Suppose we have a single PDE of two dependent variables $u^1, u^2$ and two independent variables $x^1, x^2$ given by

$$u^1_{22} = F(x^1, x^2, u^1, u^2, u^1_1, u^2_1, u^1_2, u^2_2, u^1_{11}, u^1_{12}, u^2_{11}, u^2_{12}, u^2_{22}). \tag{15}$$

Then a local solution of the PDE is a two-dimensional regular submanifold of the thirteen-dimensional locus of $J^2(U^2, V^2)$ described by the map $\Phi_F : \Sigma \longrightarrow J^2(U^2, V^2)$, where

$$\begin{aligned}\Phi_F : &(x^1, x^2, z^1, z^2, z^1_1, z^2_1, z^1_2, z^2_2, z^1_{11}, z^1_{12}, z^2_{11}, z^2_{12}, z^2_{22}) \\ &\longmapsto (x^1, x^2, z^1, z^2, z^1_1, z^2_1, z^1_2, z^2_2, z^1_{11}, z^1_{12}, F, z^2_{11}, z^2_{12}, z^2_{22}).\end{aligned}$$

Thus from the discussion immediately before this example, the image of a rank two immersion mapping into $\Phi_F(\Sigma)$ is a 2-jet image of a local solution of the PDE if it annihilates the contact system and the transverse condition is satisfied. Explicitly,

$$\begin{aligned}\Omega^2(U^2, V^2) = sp\{&dz^1 - z^1_1 dx^1 - z^1_2 dx^2, dz^2 - z^2_1 dx^1 - z^2_2 dx^2, \\ &dz^1_1 - z^1_{11} dx^1 - z^1_{12} dx^2, dz^2_1 - z^2_{11} dx^1 - z^2_{12} dx^2, \\ &dz^1_2 - z^1_{12} dx^1 - z^1_{22} dx^2, dz^2_2 - z^2_{12} dx^1 - z^2_{22} dx^2\}.\end{aligned}$$

Pulling this back onto the regular submanifold of $J^2(U^2, V^2)$ described by (15), we get

$$\begin{aligned}\Phi_F^*\Omega^2(U^2, V^2) = sp\{&dz^1 - z^1_1 dx^1 - z^1_2 dx^2, dz^2 - z^2_1 dx^1 - z^2_2 dx^2, \\ &dz^1_1 - z^1_{11} dx^1 - z^1_{12} dx^2, dz^2_1 - z^2_{11} dx^1 - z^2_{12} dx^2, \\ &dz^1_2 - z^1_{12} dx^1 - F dx^2, dz^2_2 - z^2_{12} dx^1 - z^2_{22} dx^2\}.\end{aligned}$$

Therefore the Vessiot distribution is

$$\begin{aligned}\left(\Phi_F^*\Omega^2(U^2, V^2)\right)^\perp = sp\Big\{&\frac{\partial}{\partial x^1} + z^1_1 \frac{\partial}{\partial z^1} + z^2_1 \frac{\partial}{\partial z^2} + z^1_{11} \frac{\partial}{\partial z^1_1} + z^2_{11} \frac{\partial}{\partial z^2_1} + z^1_{12} \frac{\partial}{\partial z^1_2} \\ &+ z^2_{12} \frac{\partial}{\partial z^2_2}, \frac{\partial}{\partial x^2} + z^1_2 \frac{\partial}{\partial z^1} + z^2_2 \frac{\partial}{\partial z^2} + z^1_{12} \frac{\partial}{\partial z^1_1} + z^2_{12} \frac{\partial}{\partial z^2_1} \\ &+ F \frac{\partial}{\partial z^1_2} + z^2_{22} \frac{\partial}{\partial z^2_2}, \frac{\partial}{\partial z^1_{11}}, \frac{\partial}{\partial z^1_{12}}, \frac{\partial}{\partial z^2_{11}}, \frac{\partial}{\partial z^2_{12}}, \frac{\partial}{\partial z^2_{22}}\Big\}.\end{aligned}$$

Given a Vessiot distribution for some arbitrary system of $\rho$ PDEs of $n$ independent and $m$ dependent variables, we look for an $n$-dimensional Frobenius integrable subdistribution that satisfies the transverse condition. This is done in stages by generating a finite sequence of higher dimensional subdistributions, each containing the previous, beginning with dimensional one and ending at dimension $n$. We describe this below:

**Definition 6.2.** For a vector field distribution $D$ on some smooth manifold $M$, the submodule $E$ of $D$ is said to be an *involution* if $[X, Y] \equiv 0 \bmod D$ for all $X, Y \in E$.

If $E$ is Frobenius integrable, then it is an involution. Moreover, if $E$ is spanned by a single vector field, say for example representing the solution curve an ordinary differential equation field, then it is trivially an involution.

Given some $b$-dimensional Vessiot distribution $D_F = sp\{X_1, \ldots, X_b\}$ in $\mathfrak{X}(J^\kappa(U^n, V^m))$ corresponding to some system of PDEs in (14), the process of generating an $n$-dimensional submanifold involves first setting up a chain of lower dimensional involutions up to dimension $n$, where in each step, the next involution is contained in the previous. Beginning with one-dimensional involutions, since every vector field in $D_F$ generates a one-dimensional involution, we let the distribution spanned by $Y_1 := a_1^k X_k$ generate our involution, where the $a_1^k$ are any smooth functions defined on the $(\dim(J^\kappa(U^n, V^m)) - \rho)$-dimensional regular submanifold of $J^\kappa(U^n, V^m)$ described by the PDEs. Given any $Y_1$, we typically distinguish between two types of involutions, namely those *regular* and those *singular*. In determining which of the two our one-dimensional involution may be, a two-dimensional involution containing it is constructed. We do this by first defining $Y_2 := a_2^k X_k$ for some smooth $a_2^k$. Then the requirement that $[Y_1, Y_2] \equiv 0$ mod $D_F$ generates a system linear algebraic equations, which in matrix form is

$$\mathbf{M}(Y_1) \cdot \mathbf{a_2} = \mathbf{0}, \tag{16}$$

where

$$\mathbf{a_2} := \begin{pmatrix} a_2^1 \\ \vdots \\ a_2^b \end{pmatrix}.$$

Define $s := \text{rank}(\mathbf{M}(Y_1))$. In general, over all involutions of dimension one, $\mathbf{M}(Y_1)$ will have a maximal rank $s_1$. If $\text{rank}(\mathbf{M}(Y_1)) = s_1$, then the one-dimensional involution is said to be *regular*. If, however, $\text{rank}(\mathbf{M}(Y_1)) < s_1$, then we say that the involution is *singular*. If $\text{rank}(\mathbf{M}(Y_1)) = 0$, then $[Y_1, Y_2] \in D_F$ for all $Y_2$, and here we can say further that the singular one-dimensional involution is *characteristic*. Once we have a $Y_1$ that generates some one-dimensional involution, we then look for all possible two-dimensional involutions of $D_F$ containing $Y_1$ by solving (16) for some $Y_2$.

The process continues until we have an $n$-dimensional involution that may be regular or singular. To illustrate further, suppose we are give some $j$-dimensional involution and wish to find a $(j+1)$-dimensional involution containing it. First define $Y_{(j+1)} := a_{(j+1)}^k X_k$. Then the requirement that $[Y_i, Y_{(j+1)}] \equiv 0$ mod $D_F$ for all $i = 1, \ldots, j$ generates a system of linear algebraic equations, which in matrix form is

$$\mathbf{M}(Y_1, \ldots, Y_j) \cdot \mathbf{a_{(j+1)}} = \mathbf{0},$$

where

$$\mathbf{a_{(j+1)}} := \begin{pmatrix} a_{(j+1)}^1 \\ \vdots \\ a_{(j+1)}^b \end{pmatrix}.$$

Once again define $s := \mathrm{rank}(\mathbf{M}(Y_1, \ldots, Y_j))$. Over all involutions of dimension $j$, let $s_j$ be the maximal rank of the matrix. If $s = s_j$ the $j$-dimensional involution is regular. If $s < s_j$ then the involution is singular. If $s = 0$, then $[Y_i, Y_{(j+1)}] \in D_F$ for all $i = 1, \ldots, j$ and the singular involution is characteristic.

A $j$-dimensional involution is regular if the rank of the matrix used to determine all $(j+1)$-dimensional involutions containing it is *maximised.* In the subset of the Grassmann bundle of $j$-planes consisting of all $j$-dimensional involutions of $D_F$, those which are regular form a dense open subset of this space. Therefore all $j$-dimensional involutions of $D_F$ in some neighbourhood of a regular $j$-dimensional involution are also regular. For a characteristic $j$-dimensional involution, any choice of vector field in $D_F$ that is linearly independent of any vector field in the involution will generate a singular (and not necessarily characteristic) $(j+1)$-dimensional involution containing the characteristic involution. If at some stage during the process of building up a chain of higher dimensional involutions we have a singular subinvolution, then our $n$-dimensional involution at the end of the process will also be singular.

For any Vessiot distribution, the maximal dimension of the regular involutions in the system is defined to be the genus $g$. In many situations, $g$ will be greater than or equal to the dimension of the desired involutions for the particular PDE problem at hand, which will be $n$, the number of independent variables. Problems arise when we are looking for $n$-dimensional involutions when $g < n$. One way around this is to first find a *singular* $g$-dimensional involution. The rank of $\mathbf{M}(Y_1, \ldots, Y_g)$ is then not at a maximum, so it will be possible to find a singular $(g+1)$-dimensional involution containing the $g$-dimensional involution.

Once we have an $n$-dimensional regular or singular involution (that also satisfies the transverse condition), the final requirement that the distribution be Frobenius integrable will then give us a system of first order quasilinear PDEs where the arbitrary functions are the dependent variables. We take up this issue in the next section.

## 6.2 Application of Vessiot theory to single first order non-linear PDEs

In this section, we use Vessiot theory to examine symmetry solutions of single first order non-linear PDEs of one dependent variable and two independent variables. Suppose then that we are given a first order non-linear PDE of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}\right),$$

for some smooth function $F$. On $J^1(U^2, V^1)$ with coordinates $x^1, x^2, z^1, z^1_1, z^1_2$, our first order contact system is generated by the element $C^1 = dz^1 - z^1_1 dx^1 -$

$z^1_2 dx^2$. Restricted to regular submanifold $M^4 \subset J^1(U^2, V^1)$ described by $z^1_2 = F(x^1, x^2, z^1, z^1_1)$, the contact system on $M^4$ with coordinates $x^1, x^2, z^1, z^1_1$ is generated by $C^1 := dz^1 - z^1_1 dx^1 - F dx^2$. The Vessiot distribution $D_F$ is generated by

$$X_1 := \frac{\partial}{\partial x^1} + z^1_1 \frac{\partial}{\partial z^1}, \qquad X_2 := \frac{\partial}{\partial x^2} + F \frac{\partial}{\partial z^1}, \qquad X_3 := \frac{\partial}{\partial z^1_1}.$$

In looking for a one-dimensional involution of $D_F$ which is regular, let

$$Y_1 := a^k_1 X_k, \qquad Y_2 := a^k_2 X_k.$$

We have the commutator relations

$$[X_1, X_2] = X_1(F) \frac{\partial}{\partial z^1}, \qquad [X_1, X_3] = -\frac{\partial}{\partial z^1}, \qquad [X_2, X_3] = -X_3(F) \frac{\partial}{\partial z^1},$$

with all others zero. Demanding that $[Y_1, Y_2] \equiv 0 \bmod D_F$ means

$$\left(a^1_1 a^2_2 - a^2_1 a^1_2\right) X_1(F) + \left(a^3_1 a^2_2 - a^2_1 a^3_2\right) X_3(F) - a^1_1 a^3_2 + a^3_1 a^1_2 = 0.$$

In matrix form,

$$\begin{pmatrix} -a^2_1 X_1(F) + a^3_1 & a^1_1 X_1(F) + a^3_1 X_3(F) & -a^2_1 X_3(F) - a^1_1 \end{pmatrix} \cdot \begin{pmatrix} a^1_2 \\ a^2_2 \\ a^3_2 \end{pmatrix} = 0. \quad (17)$$

We choose a one-dimensional involution spanned by $Y_1$ by letting $a^1_1 = 1$ and $a^2_1 = 0$. Then

$$\begin{pmatrix} a^3_1 & X_1(F) + a^3_1 X_3(F) & -1 \end{pmatrix}.$$

is rank one, and hence in a neighbourhood of one-dimensional involutions about $Sp\{Y_1\}$, the matrix on the left in (17) remains rank one. Therefore $Sp\{Y_1\}$ is a regular involution.

In looking for a two-dimensional involution satisfying the transverse condition, we let $a^1_2 = 0$ and $a^2_2 = 1$ so that (17) holds with

$$Y_1 = X_1 + a^3_1 X_3, \qquad Y_2 = X_2 + (X_1(F) + a^3_1 X_3(F)) X_3,$$

thus generating a two-dimensional involution for arbitrary $a^3_1$. To see that the involution is regular, let $Y_3 = a^k_3 X_k$. Requiring that $[Y_1, Y_3] \equiv 0 \bmod D_F$ and $[Y_2, Y_3] \equiv 0 \bmod D_F$ means that

$$\begin{pmatrix} a^3_1 & X_1(F) + a^3_1 X_3(F) & -1 \\ a^3_1 X_3(F) & \left(X_1(F) + a^3_1 X_3(F)\right) X_3(F) & -X_3(F) \end{pmatrix} \cdot \begin{pmatrix} a^1_3 \\ a^2_3 \\ a^3_3 \end{pmatrix} = \mathbf{0},$$

where the matrix on the left is of rank one. The space of all possible $Y_3$ must contain $Y_1$ and $Y_2$, so it follows that in a neighbourhood of the two-dimensional involution $Sp\{Y_1, Y_2\}$, this rank one condition must be maintained by dimension. Therefore $Sp\{Y_1, Y_2\}$ is a regular involution.

Given a two-dimensional involution spanned by $Y_1$ and $Y_2$, we finally require that it be Frobenius integrable. We introduce the condition $[Y_1, Y_2] = 0$ which forces $a_1^3$ to satisfy the following first order quasilinear PDE:

$$\begin{aligned} & -X_3(F)\frac{\partial a_1^3}{\partial x^1} + \frac{\partial a_1^3}{\partial x^2} + \left(F - z_1^1 X_3(F)\right)\frac{\partial a_1^3}{\partial z^1} + X_1(F)\frac{\partial a_1^3}{\partial z_1^1} \\ & \qquad = X_1\left(X_1(F)\right) + a_1^3 X_1\left(X_3(F)\right) + a_1^3 X_3\left(X_1(F)\right) + (a_1^3)^2 X_3\left(X_3(F)\right), \end{aligned}$$

where $a_1^3$ is some function of $x_1, x_2, z^1, z_1^1$. The problem is now reduced to that of finding a solution of a first order quasilinear PDE.

We can summarise the above in the following theorem:

**Theorem 6.3.** *Consider the first order PDE*

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}\right), \tag{18}$$

*for smooth $F$. On the regular submanifold of $J^1(U^2, V^1)$ described by $z_2^1 = F(x^1, x^2, z^1, z_1^1)$, let*

$$X_1 := \frac{\partial}{\partial x^1} + z_1^1\frac{\partial}{\partial z^1}, \qquad X_2 := \frac{\partial}{\partial x^2} + F\frac{\partial}{\partial z^1}, \qquad X_3 := \frac{\partial}{\partial z_1^1}.$$

*Define the vector fields*

$$Y_1 := X_1 + a_1^3 X_3, \qquad Y_2 := X_2 + (X_1(F) + a_1^3 X_3(F))X_3,$$

*with $a_1^3$ satisfying the first order quasilinear PDE*

$$\begin{aligned} & -X_3(F)\frac{\partial a_1^3}{\partial x^1} + \frac{\partial a_1^3}{\partial x^2} + \left(F - z_1^1 X_3(F)\right)\frac{\partial a_1^3}{\partial z^1} + X_1(F)\frac{\partial a_1^3}{\partial z_1^1} \\ & \qquad = X_1\left(X_1(F)\right) + a_1^3 X_1\left(X_3(F)\right) + a_1^3 X_3\left(X_1(F)\right) + (a_1^3)^2 X_3\left(X_3(F)\right), \end{aligned} \tag{19}$$

*where $a_1^3$ is some smooth function of $x_1, x_2, z^1, z_1^1$. Then $Sp\{Y_1, Y_2\}$ generates a two-dimensional regular submanifold of $J^1(U^2, V^1)$ that is the image of the 1-jet of some local solution of the PDE in (18).*

*Remark 1.* Of course, solving (19) using Theorem 5.1 will generally yield $a_1^3$ in terms of an arbitrary function which typically cannot be left arbitrary when integrating $sp\{Y_1, Y_2\}$.

*Remark 2.* In normal applications, Theorem 6.3 would be used if (18) is non-linear. However it is obvious that the theorem still holds if the PDE is linear or quasilinear. For such situations, Theorem 5.1 clearly provides a simpler alternative.

In spite of the fact that our resulting first order quasilinear PDE appears much more complicated than the original (typically non-linear) PDE, the situation is somewhat simpler because it may be solved using the symmetry technique outlined earlier in Theorem 5.1 to generate local solutions of (19) depending on an arbitrary function. Once we have chosen a suitable $a^3_1$, Theorem 2.1 (or Proposition 4.7 in Sherring and Prince [15]) for integrating Frobenius integrable distributions may be applied to the vector field distribution spanned by $Y_1$ and $Y_2$.

We close this section with the following example:

**Example 6.4.** Consider the following non-linear PDE:

$$u = \frac{\partial u}{\partial x^1}\frac{\partial u}{\partial x^2}. \tag{20}$$

On the regular submanifold of some suitably chosen $J^1(U^2, V^1)$ specified by

$$z^1_2 = \frac{z^1}{z^1_1},$$

with coordinates $x^1, x^2, z^1, z^1_1$ (where $z^1_1 \neq 0$), the Vessiot distribution is generated by

$$X_1 = \frac{\partial}{\partial x^1} + z^1_1\frac{\partial}{\partial z^1}, \qquad X_2 = \frac{\partial}{\partial x^2} + \frac{z^1}{z^1_1}\frac{\partial}{\partial z}, \qquad X_3 = \frac{\partial}{\partial z^1_1}.$$

A two-dimensional involution satisfying the transverse condition is generated by

$$Y_1 = \frac{\partial}{\partial x^1} + z^1_1\frac{\partial}{\partial z^1} + f\frac{\partial}{\partial z^1_1}, \qquad Y_2 = \frac{\partial}{\partial x^2} + \frac{z^1}{z^1_1}\frac{\partial}{\partial z^1} + \left(1 - \frac{z^1 f}{(z^1_1)^2}\right)\frac{\partial}{\partial z^1_1},$$

where $f$ is some arbitrary smooth function of $x^1, x^2, z^1, z^1_1$. The integrability condition means that $f$ must satisfy

$$\frac{z^1}{(z^1_1)^2}\frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2} + \frac{2z^1}{z^1_1}\frac{\partial f}{\partial z^1} + \frac{\partial f}{\partial z^1_1} = \frac{f}{z^1_1}\left(\frac{2z^1 f}{(z^1_1)^2} - 1\right). \tag{21}$$

At this point we would use the ideas in Section 2 and `Dimsym` to find suitable $f$, then integrate the distribution using Theorem 2.1. Quite often however, a simple observation may yield a trivial solution for $f$ that gives a non-trivial solution to the original non-linear PDE. For example, let $f = 0$. Then integrating the resulting distribution results in the rather obvious solution to (20),

$$u = (c_1 + x^1)(c_2 + x^2),$$

where $c_1, c_2$ are arbitrary constants. We leave it to the reader to generate local solutions of (21) using Theorem 5.1. For now though, by observing from (21) that there exists a solution of $f$ that is only a function of $z^1$ and $z_1^1$, we have found another suitable $f$ to be

$$f = \frac{(z_1^1)^2}{z^1(\sqrt{z^1}+2)}.$$

This gives

$$Y_1 = \frac{\partial}{\partial x^1} + z_1^1 \frac{\partial}{\partial z^1} + \frac{(z_1^1)^2}{z^1(\sqrt{z^1}+2)} \frac{\partial}{\partial z_1^1},$$
$$Y_2 = \frac{\partial}{\partial x^2} + \frac{z^1}{z_1^1} \frac{\partial}{\partial z^1} + \left(1 - \frac{1}{\sqrt{z^1}+2}\right) \frac{\partial}{\partial z_1^1},$$

as generators for our Frobenius integrable distribution. It has two obvious commuting symmetries which are $Z_1 := \frac{\partial}{\partial x^1}$, and $Z_2 := \frac{\partial}{\partial x^2}$. They make it easier to integrate our distribution, as shown in Theorem 3.16 in [1] or Corollary 3.3 in [15] (which are simple extensions of Theorem 2.1). We can then integrate the distribution to give

$$\frac{z^1 + 2\sqrt{z^1}}{z_1^1} - x^1 = c_1, \qquad \frac{(\sqrt{z^1}+2)z_1^1}{\sqrt{z^1}} - x^2 = c_2,$$

where $c_1$ and $c_2$ are two arbitrary constant functions. Finally, eliminating $z_1^1$ and replacing $z^1$ with $u$ yields the following local solution to the original non-linear PDE in (20):

$$u = \left(\pm\sqrt{(x^1+c_1)(x^2+c_2)} - 2\right)^2.$$

# 7 First order non-linear PDEs not involving the dependent variable

In the previous section it was shown that solution of a given first order non-linear PDE of one dependent variable and two independent variables could be found by generating a corresponding Vessiot distribution whose integrability condition was in the form of a first order quasilinear PDE that could be solved using Theorem 5.1. The major disadvantage of generating local solutions of such non-linear PDEs in this way is that even for basic examples, the resulting first order quasilinear PDE is usually quite complicated and of four independent variables, that requires a solvable structure of four symmetries to solve. In addition, a further solvable structure of two symmetries is then required to integrate the resulting Frobenius integrable distribution.

In this section, we present a simpler alternative to the Vessiot integration scheme for solving single first order non-linear PDEs that also generates a corresponding first order quasilinear PDE, but which is of only two independent variables and requires a single solvable structure of just two symmetries.

Unfortunately, the disadvantage here is that this technique can only be applied to first order non-linear PDEs of two independent variables and one dependent variable that do not depend on the dependent variable.

Suppose then, that our PDE is of the form

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, \frac{\partial u}{\partial x^1}\right), \tag{22}$$

for smooth $F$, where $x^1, x^2$ are the independent variables, and $u$ is the dependent variable. This gives the corresponding fundamental ideal

$$I_F = \langle dz^1 - z_1^1 dx^1 - z_2^1 dx^2, dz_1^1 \wedge dx^1 + dz_2^1 \wedge dx^2, \left(z_2^1 - F\right) dx^1 \wedge dx^2\rangle,$$

where $F$ is now a function of $x^1, x^2, z_1^1$.

The main result of this section the following theorem:

**Theorem 7.1.** *Consider the first order PDE*

$$\frac{\partial u}{\partial x^2} = F\left(x^1, x^2, \frac{\partial u}{\partial x^1}\right), \tag{23}$$

*for smooth $F$. In terms of coordinates of $J^1(U^2, V^1)$, set $z_1^1 = f(x^1, x^2)$ and $z_2^1 = F(x^1, x^2, f)$. Then any smooth solution $f(x^1, x^2)$ of the quasilinear PDE*

$$\frac{\partial F}{\partial f}\frac{\partial f}{\partial x^1} - \frac{\partial f}{\partial x^2} = -\frac{\partial F}{\partial x^1}, \tag{24}$$

*has the property that $z_1^1 dx^1 + z_2^1 dx^2 = dg$ for some $g \in C^\infty(U^2)$. Moreover, the expression $u = g$ is a local solution of the PDE in (23).*

*Proof.* Let $f \in C^\infty(U^2)$ be any function. Using (23), set the following:

$$z_1^1 = f(x^1, x^2), \qquad z_2^1 = F(x^1, x^2, f). \tag{25}$$

We have

$$I_F := \langle C^1, dC^1, (z_2^1 - F) dx^1 \wedge dx^2\rangle,$$

where $C^1 := dz^1 - z_1^1 dx^1 - z_2^1 dx^2$, and wish to look for conditions on $f$ such that

$$-dC^1 = dz_1^1 \wedge dx^1 + dz_2^1 \wedge dx^2 = 0. \tag{26}$$

Supposing this, we obtain by inserting (25) into (26),

$$\frac{\partial f}{\partial x^2} dx^2 \wedge dx^1 + \left(\frac{\partial F}{\partial x^1} + \frac{\partial F}{\partial f}\frac{\partial f}{\partial x^1}\right) dx^1 \wedge dx^2 = 0. \tag{27}$$

Now if our $f$ satisfies the PDE in (24), then from (27),

$$dz_1^1 \wedge dx^1 + dz_2^1 \wedge dx^2 = 0,$$

so $z^1_1 dx^1 + z^1_2 dx^2$ is closed. Therefore,

$$z^1_1 dx^1 + z^1_2 dx^2 = dg,$$

for some $g \in C^\infty(U^2)$. If we now set $z^1 = g$, then

$$C^1 := dz^1 - z^1_1 dx^1 - z^1_2 dx^2 = 0.$$

Therefore the immersion

$$j^1 g : U^2 \longrightarrow J^1(U^2, V^1),$$

maps onto the two-dimensional regular submanifold of $J^1(U^2, V^1)$ defined by the equations $z^1 = g$, $z^1_1 = f$, and $z^1_2 = F$, and has the property that $j^1 g^* I_F = 0$. Hence the expression $u = g$ is a local solution of (22). □

*Remark.* The second remark for Theorem 6.3 is valid here. In addition, since the PDE in (23) is independent of $u$, it is obvious that $\frac{\partial}{\partial u}$ is a symmetry of (23), and so all local solutions may have the addition of an arbitrary constant.

Finally, we apply Theorem 7.1 to an example:

**Example 7.2.** Consider the following first order non-linear PDE:

$$\frac{\partial u}{\partial x^2} = \left( \frac{\partial u}{\partial x^1} \right)^{-1}. \tag{28}$$

Applying Theorem 7.1, let $f \in C^\infty(U^2)$ be non-zero on $U^2$, and set

$$z^1_1 = f, \qquad z^1_2 = \frac{1}{f},$$

so that

$$\begin{aligned} dz^1_1 \wedge dx^1 + dz^1_2 \wedge dx^2 &= d\left( f dx^1 + \frac{1}{f} dx^2 \right), \\ &= df \wedge dx^1 - \frac{1}{f^2} df \wedge dx^2, \\ &= -\left( \frac{1}{f^2} \frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2} \right) dx^1 \wedge dx^2. \end{aligned}$$

In order to solve for $f$ in the first order quasilinear PDE

$$\frac{1}{f^2} \frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2} = 0, \tag{29}$$

we will use Theorem 5.1. The corresponding two-form $K$ is

$$K = df \wedge dx^1 - \frac{1}{f^2} df \wedge dx^2.$$

The vector field $X_2 := \frac{\partial}{\partial x^1}$ is a non-trivial symmetry of $K$, and $X_1 := \frac{\partial}{\partial f}$ is a non-trivial symmetry of $X_2 \lrcorner K = -df$. Then following Theorem 5.1, we obtain that

$$K = df \wedge d\left(x^1 - \frac{x^2}{f^2}\right).$$

Hence in implicit form,

$$G\left(f, x^1 - \frac{x^2}{f^2}\right) = 0,$$

is a local solution of (29) for any suitably defined smooth function $G$. Suppose we choose $G$ so that

$$G\left(f, x^1 - \frac{x^2}{f^2}\right) = \frac{1}{f^2} + x^1 - \frac{x^2}{f^2} - c_1,$$

for any constant function $c_1$. Then

$$f = \sqrt{\frac{1 - x^2}{c_1 - x^1}}$$

is a local solution of (29), assuming that we are in some neighbourhood where $(1 - x^2)/(c_1 - x^1) > 0$. Therefore

$$z_1^1 = \sqrt{\frac{1 - x^2}{c_1 - x^1}}, \qquad z_2^1 = \sqrt{\frac{c_1 - x^1}{1 - x^2}},$$

From Theorem 7.1, these expressions for $z_1^1$ and $z_2^1$ mean that

$$d\left(z_1^1 dx^1 + z_2^1 dx^2\right) = 0.$$

So a simple integration yields

$$z_1^1 dx^1 + z_2^1 dx^2 = d\left(-2\sqrt{(c_1 - x^1)(1 - x^2)}\right).$$

Putting

$$u = -2\sqrt{(c_1 - x^1)(1 - x^2)},$$

then gives a local solution of the original non-linear PDE in (28). In fact,

$$u = -2\sqrt{(c_1 - x^1)(c_2 - x^2)},$$

is a local solution of the PDE for any appropriate choice of constant functions $c_1$ and $c_2$.

Finally, if we suppose that

$$G\left(f, x^1 - \frac{x^2}{f^2}\right) = f\left(x^1 - \frac{x^2}{f^2}\right) - c_3,$$

for some constant $c_3$, then we may solve the quadratic equation $x^1 f^2 - c_3 f - x^2 = 0$ to give

$$f = \frac{c_3 \pm \sqrt{c_3^2 + 4x^1x^2}}{2x^1}.$$

If we choose the positive option for $f$, and put

$$z_1^1 = \frac{c_3 + \sqrt{c_3^2 + 4x^1x^2}}{2x^1}, \qquad z_2^1 = \frac{2x^1}{c_3 + \sqrt{c_3^2 + 4x^1x^2}},$$

then one obtains

$$z_1^1 dx^1 + z_2^1 dx^2 = d\left(\sqrt{c_3^2 + 4x^1x^2} + \frac{c_3}{2}\ln\left|\frac{x^1\left(\sqrt{c_3^2 + 4x^1x^2} - c_3\right)}{x^2\left(\sqrt{c_3^2 + 4x^1x^2} + c_3\right)}\right|\right),$$

so

$$u = \sqrt{c_3^2 + 4x^1x^2} + \frac{c_3}{2}\ln\left|\frac{x^1\left(\sqrt{c_3^2 + 4x^1x^2} - c_3\right)}{x^2\left(\sqrt{c_3^2 + 4x^1x^2} + c_3\right)}\right|$$

is another local solution of the original non-linear PDE in (28).

# 8 Summary

Using the Lie symmetry analysis software package `Dimsym` for generating solvable structures of symmetries, a single first order quasilinear PDE of one dependent variable and $n$ independent variables may be solved in terms of an arbitrary function of $n$ so-called ‘invariants’ using a solvable structure of $n$ symmetries. For a single first order non-linear PDE of one dependent variable and two independent variables we implement a two stage process. Vessiot theory is first introduced to generate a two dimensional involution whose integrability condition is in the form of a first order quasilinear PDE that requires a solvable structure of four symmetries. Once this is solved, another solvable structure of two symmetries is required to integrate the resulting Frobenius integrable distribution and finally yield a solution to the PDE. Finally, for situations where the first order non-linear PDE is not a function of the dependent variable, but only depends on its derivatives and the independent variables, we may avoid the two stage process above, and generate solutions by using simply one solvable structure of two symmetries.

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# Appendix: Solvable structures using Dimsym

Here we illustrate using Example 5.2 how to use `Dimsym` to find a solvable structure of an arbitrary vector field distribution. In the example, we are given a one dimensional vector field distribution corresponding to the Cauchy characteristic space of $\langle K \rangle$, and we wish to find solvable symmetry structure of two non-trivial symmetries . Then working in `REDUCE` (with `x(1),x(2),x(3)` representing, respectively, coordinates $x^1, x^2, z^1$), we load the `Dimsym` package and enter the Cauchy characteristic vector field like so:

```
Y1 := x(1)*@x(1) - x(2)*@x(2) + x(2)*exp(x(3))*@x(3);
```

We then define some arbitrary functions:

```
for i := 1:3 do
xi(i) := newarb(x(1),x(2),x(3));
```

and also a symmetry vector:

```
symvec := xi(1)*@x(1) + xi(2)*@x(2) + xi(3)*@x(3);
```

Now defining the following arbitrary function:

```
a(1) := newarb(x(1),x(2),x(3));
```

we set up the Lie bracket symmetry relation:

```
zvec1 := comm(symvec,Y1) - a(1)*Y1;
```

Next, we introduce determining equations:

```
for i := 1:3 do
deteqn(i) := vecder(zvec1,x(i));
```

where `vecder` is a `Dimsym` command which in this case simply extracts from the vector field in the first argument, the coefficient of the basis vector in the second argument. We then read the determining equations into `Dimsym` using:

```
readdets();
```

Next, we ask `Dimsym` to solve the determining equations using the standard algorithm:

```
solvedets(std);
```

Any unsolved determining equations that were too difficult for Dimsym to solve will be shown using:

```
showdets();
```

Finally, the infinitesimal generators of all the trivial as well as nontrivial symmetries that Dimsym has been able to find are given using the following:

```
mkgens();
end;
```

Although Example 5.2 uses simple observation to obtain a symmetry of $X_2 \lrcorner K$ (or equivalently, a symmetry of the span of symvec and Y1), for completeness, we give a brief indication below of how the code above may be extended to find this additional symmetry.

First we include the symmetry of the distribution spanned by Y1 and call this X2:

```
X2 := 1/x(2)*@x(1);
```

While the definition for symvec stays the same, we make the following changes:

```
for i := 1:4 do
a(i) := newarb(x(1),x(2),x(3));

zvec1 := comm(symvec,Y1) - a(1)*Y1 - a(2)*X2;
zvec2 := comm(symvec,X2) - a(3)*Y1 - a(4)*X2;

for i := 1:3 do
begin
deteqn(i) := vecder(zvec1,x(i));
deteqn(i+3) := vecder(zvec2,x(i));
end;
```

Then remainder of the code is the same.

In performing the above routines, we would ideally like no unsolved determining equations to exist when we execute showdets() each time, however in most cases, there will be unsolved determining equations if the symvec coefficients xi(1),xi(2),xi(3),xi(4),xi(5) are left purely arbitrary as they are above. Typically, for at least one of the coefficients, it is required to reduce the number of arguments of the corresponding newarb, or even define the coefficient to be zero.

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