Solvable symmetry structures in differential form applications

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Abstract

We investigate symmetry techniques for expressing various exterior differential forms in terms of simplified coordinate systems. In particular, we give extensions of the Lie symmetry approach to integrating Frobenius integrable distributions based on a solvable structure of symmetries and show how a solvable structure of symmetries may be used to find local coordinates for the Pfaffian problem and Darboux's theorem.

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1 Introduction

In this paper we present several methods based on symmetry techniques for expressing various differential forms in simplified coordinate systems. We use work by Lie [10] and Cartan [3] to explore how symmetries may be used to integrate Frobenius integrable distributions. In recent times, Barasab-Horwath [1], Duzhin and Lychagin [6], Hartl and Athorne [9], and Sherring and Prince [13] have extended Lie and Cartan's work from the perspective of constructing first integrals of a completely integrable distribution by quadratures. Our work uses such results to examine conditions under which a given differential form can be expressed in a simpler coordinate system.

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The plan of this paper is to first review Lie's solvable symmetry structure approach to integrating Frobenius integrable vector field distributions. For a given Frobenius integrable distribution an exterior product of one-forms is used to generate invariants of the distribution. We then apply the approach to Pfaffian and Darboux systems, and in both cases present an example.

It is assumed throughout this paper that our expressions apply locally on some n-dimensional, open, simply connected differentiable submanifold, U, of $\mathbb{R}^n$, with coordinates $x^1, \ldots, x^n \in C^\infty(U)$. One further assumption that we make on U is that it be convex. This allows us to use the converse of the Poincaré Lemma on the whole of U, i.e. if $\Delta \in \Lambda^k(U)$ is closed ($d\Delta = 0$), then $\Omega = d\Theta$ for some $\Theta \in \Lambda^{k-1}(U)$ [12, 15].

Consider the differentiable manifold U of dimension n. TU is the tangent bundle of vector fields with $T_p(U)$, $p \in U$ as its fibres. Let $\mathfrak{X}(U)$ be be the module of all smooth vector fields over $C^\infty(U)$. T^*U is the cotangent bundle with $T^*_p(U)$, $p \in U$ as its fibres. The set of exterior differential m-forms is a section of the bundle of all homogeneous differential forms, $\Lambda(U)$. For any $\Omega \in \Lambda^m(U)$ define its kernel by $\ker(\Omega) := \{X \in \mathfrak{X}(U) : X \lrcorner \Omega = 0\}$.

For the remainder of this paper we will also assume all vector field distributions non-singular in the sense that their dimension is constant on U, and also that all one-forms have constant rank on U.

2 Ideals, Cauchy characteristics and symmetries

Following Bryant *et al.* [2], for any $\alpha^1, \ldots, \alpha^p \in \Lambda(U)$ up to some $p \in \mathbb{N}$, we write $I := \langle \alpha^1, \ldots, \alpha^p \rangle$ to mean that I is the (homogeneous) algebraic ideal generated by the elements $\alpha^1, \ldots, \alpha^p$. An ideal I is a *differential* ideal if the exterior derivative of every member of I is also in I. A vector field Y is called a *Cauchy characteristic* vector field of an ideal I if $Y \lrcorner I \subset I$. Define $A(I)$ to be the set of all Cauchy characteristic vector fields of I. It is not hard to show that $A(I)$ is Frobenius integrable.

A vector field $X \in \mathfrak{X}(U)$ is said to be a *symmetry* (or *isovector*) of an ideal, I, if $\mathcal{L}_X I \subset I$. It is easy to see that in order to show that X is a symmetry of I, it is enough to show that the Lie derivative with respect to X of merely the generators of I, is also in I. A vector field $X \in \mathfrak{X}(U)$ is a *symmetry* of a vector field distribution $D \subset \mathfrak{X}(U)$ if $\mathcal{L}_X D \subset D$. Once again, it is enough to look at simply the generators of D when determining whether a vector field is a symmetry of the distribution.

We now present some results connecting symmetries, ideals, and Cauchy characteristic spaces.

Proposition 2.1. *Let I be an ideal. Suppose $A(I)$ is not zero-dimensional. If a vector field X is a symmetry of I then X is a symmetry of $A(I)$.*

Proof. Let X be a symmetry of the ideal I. Let $Y \in A(I)$ and $\beta \in I$. Then, from rearranging the identity $\mathcal{L}_X(Y \lrcorner \beta) = [X,Y] \lrcorner \beta + Y \lrcorner (\mathcal{L}_X \beta)$, we obtain

$$[X,Y] \lrcorner \beta = \mathcal{L}_X(Y \lrcorner \beta) - Y \lrcorner (\mathcal{L}_X \beta).$$

Now the first term on the right hand side is in I since $Y \lrcorner \beta \in I$ and X is a symmetry of I. The second term is also in I since $\mathcal{L}_X \beta \in I$ and $Y \in A(I)$. Hence $[X,Y] \lrcorner \beta \in I$. Therefore $[X,Y] \in A(I)$. □

Proposition 2.2. *Let $\{\alpha^1, \ldots, \alpha^p\}$ be some finite set of linearly independent one-forms in $\Lambda^1(U)$, and define $I := \langle \alpha^1, \ldots, \alpha^p, d\alpha^1, \ldots, d\alpha^p \rangle$. With the dual space of the Pfaffian system generated by $\alpha^1, \ldots, \alpha^p$ defined by $D := \{X \in \mathfrak{X}(U) : X \lrcorner \alpha^i = 0, \ \ for\ all\ \ 1 \leq i \leq p\}$, then a vector field, $Y \in D$ is a Cauchy characteristic of I if and only if $[X,Y] \in D$ for all $X \in D$.*

Proof. Let Y be a Cauchy characteristic vector field of I, i.e. $Y \lrcorner \alpha^i = 0$ and $Y \lrcorner d\alpha^i \in I$ for all $1 \leq i \leq p$. This implies that for all i,

$$\mathcal{L}_Y \alpha^i = Y \lrcorner d\alpha^i \in I.$$

Hence Y is a symmetry of I. Let $X \in D$, where D is defined in the theorem. Using the property

$$\mathcal{L}_Y \left(X \lrcorner \alpha^i \right) = [Y,X] \lrcorner \alpha^i + X \lrcorner \left(\mathcal{L}_Y \alpha^i \right),$$

we know that the term on the left is zero and the second term on the right is also zero. Hence for all i, $[X,Y] \lrcorner \alpha^i = 0$, so that $[X,Y] \in D$.

Conversely, let $Y \in D$ and $[X,Y] \in D$ for all $X \in D$. We therefore have that for all i,

$$Y \lrcorner \alpha^i = 0 = [X,Y] \lrcorner \alpha^i.$$

Now once again using the property

$$\mathcal{L}_Y \left(X \lrcorner \alpha^i \right) = [Y,X] \lrcorner \alpha^i + X \lrcorner \left(\mathcal{L}_Y \alpha^i \right),$$

we have that for each i,

$$X \lrcorner \left(\mathcal{L}_Y \alpha^i \right) = 0. \tag{1}$$

Since (1) must hold for all $X \in D$, we must have that $\mathcal{L}_Y \alpha^i \in I$. Since $Y \lrcorner \alpha^i = 0$,

$$\mathcal{L}_Y \alpha^i = Y \lrcorner d\alpha^i \in I,$$

so Y is a Cauchy characteristic vector field of I. □

At this point we will introduce the idea of a *trivial symmetry*. Given a *differential* ideal I, we call all Cauchy characteristics of I trivial symmetries of I. The reason for this is contained in the next proposition:

Proposition 2.3. *Let I be a differential ideal, and let Y be a Cauchy characteristic vector field of I. Then Y is a symmetry of I.*

Proof. let $\beta \in I$ and $Y \in A(I)$.

$$\mathcal{L}_Y \beta = d\,(Y \lrcorner\, \beta) + Y \lrcorner\, d\beta.$$

The first term on the right is in I because $Y \lrcorner\, \beta \in I$, and consequently $d\,(Y \lrcorner\, \beta) \in I$, since I is a differential ideal. The second term is obviously in I. □

Similarly, given a vector field distribution D, a *trivial symmetry* of D is a symmetry of D that is also in D.

A fundamental distinction between trivial and non-trivial symmetries is as follows: Given a trivial symmetry, multiplying it by any non-constant function will yield a trivial symmetry, however doing the same to a non-trivial symmetry will in general not produce a non-trivial symmetry.

For a differential ideal generated by a Pfaffian system we have the following extension of Proposition 2.3:

Proposition 2.4. *Let I be a differential ideal generated by some finite collection of linearly independent one-forms $\alpha^1, \ldots, \alpha^p \in \Lambda^1(U)$. A vector field $X \in \mathfrak{X}(U)$ is a symmetry of I in the annihilating space $D := \{X \in \mathfrak{X}(U) : X \lrcorner\, \alpha^i = 0, \quad for\ all \quad 1 \le i \le p\}$ if and only if X is a trivial symmetry (Cauchy characteristic vector field) of I.*

Proof. With X as a symmetry of I, if $X \lrcorner\, \alpha^i = 0$ for all $1 \le i \le p$, then for each i

$$I \ni \mathcal{L}_X \alpha^i = X \lrcorner\, d\alpha^i.$$

The converse is also obvious using Proposition 2.3. □

Definition 2.5. A differential p-form said to be *decomposable* (or *simple*) if it can be written as the wedge product of p one-forms.

Decomposability is a local property, and a p-form is decomposable if and only if the dimension of the kernel is of dimension $n - p$.

Consider the following two simple propositions, the first of which is proved in Sherring and Prince [13]:

Proposition 2.6. *A vector field $X \in \mathfrak{X}(U)$ is a symmetry of a decomposable m-form $\Omega \in \Lambda^m(U)$ if and only if X is a symmetry of* $\ker(\Omega)$.

Proposition 2.7. *Let $\Omega \in \Lambda^m(U)$ and $I := \langle \Omega, d\Omega \rangle$. If $d\Omega = 0$ mod Ω, then* $\ker(\Omega) = A(I)$.

Proof. First suppose $\ker(\Omega)$ is not zero-dimensional, so that there exists a non-zero vector field $W \in \mathfrak{X}(U)$ such that $W \lrcorner\, \Omega = 0$. Now since $d\Omega = 0$ mod Ω, $W \lrcorner\, d\Omega = W \lrcorner\, (\alpha \wedge \Omega) = (W \lrcorner\, \alpha) \wedge \Omega$ for some $\alpha \in \Lambda^1(U)$. Therefore $W \in A(I)$.

Now suppose $A(I)$ is not zero-dimensional. This means there exists a non-zero vector field, $X \in \mathfrak{X}(U)$ such that $X \lrcorner \Omega = 0$ and $X \lrcorner d\Omega = f\Omega$ for some smooth $f \in C^\infty(U)$. Hence from the first part, $X \in \ker(\Omega)$.

If $\ker(\Omega)$ is zero-dimensional, then $Y \lrcorner \Omega \neq 0$ for all non-zero $Y \in \mathfrak{X}(U)$. This means $Y \lrcorner \Omega \notin I$, and hence $Y \notin A(I)$. Therefore $A(I)$ is zero-dimensional.

Finally, if $A(I)$ is zero-dimensional, then $Z \lrcorner \Omega \neq 0$ for all $Z \in \mathfrak{X}(U)$. Hence $\ker(\Omega)$ is zero-dimensional. □

Using the above two results, we obtain the following extension to differential ideals thus giving us a condition under which the converse of Proposition 2.1 holds true:

Proposition 2.8. *Let I be a differential ideal generated by some $\Omega \in \Lambda^m(U)$ with $d\Omega = 0$ mod Ω. Furthermore, let Ω be decomposable on U and $A(I)$ not zero-dimensional. Then X is a symmetry of I if and only if X is a symmetry of $A(I)$.*

Proof. From Proposition 2.7, $d\Omega = 0 \text{ mod } \Omega$ implies that $\ker(\Omega) = A(I)$. Hence the result follows from Proposition 2.6. □

Remark. If $\Omega \in \Lambda^m(U)$ with $m = n$, and I is the differential ideal generated by Ω (note $d\Omega = 0$), then *any* non-zero vector field in $\mathfrak{X}(U)$ is a symmetry of I. Moreover, $A(\langle\Omega\rangle)$ is zero-dimensional, and therefore *any* non-zero vector field in $\mathfrak{X}(U)$ is also a symmetry of a zero-dimensional $A(\langle\Omega\rangle)$.

3 The Frobenius theorem and integration via symmetry

First, we present a basic result:

Lemma 3.1. [5] *Let $\Omega \in \Lambda^{n-m}(U)$ for some $m \leq n-1$. Then* $\ker(\Omega)$ *can be at most m-dimensional. Moreover,* $\ker(\Omega)$ *is precisely m-dimensional if and only if Ω is decomposable.*

Lemma 3.1 has the following corollary:

Corollary 3.2. *Let $D := sp\{Y_1, \ldots, Y_m\}$ be some m-dimensional distribution in $\mathfrak{X}(U)$, where $m < n-1$. If $\Omega := Y_1 \lrcorner \ldots \lrcorner Y_m \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \in \Lambda^{n-m}(U)$, then Ω is decomposable and equal to the wedge product of some $n-m$ linearly independent generators of $D^\perp$.*

Proof. With D and Ω defined as in the corollary, let $X \in \mathfrak{X}(U)$ be any non-zero vector field in D. Then from the definition of Ω, $X \lrcorner \Omega = 0$. Hence $\ker(\Omega)$ is at least m-dimensional. But from Lemma 3.1, since Ω is an $(n-m)$-form, its kernel can not be greater than m-dimensional, and therefore Ω is decomposable.

Now we can write $\Omega = \theta^1 \wedge \cdots \wedge \theta^{n-m}$ for some linearly independent $\theta^1, \ldots, \theta^{n-m} \in \Lambda^1(U)$. Since for each $1 \leq i \leq m$, $Y_i \lrcorner \Omega = 0$, we then have that for each $1 \leq j \leq (n-m)$, $Y_i \lrcorner \theta^j = 0$. Hence $\theta^1, \ldots, \theta^{n-m}$ generate $D^\perp$. □

Theorem 3.3. (Frobenius) *Let D be an m-dimensional distribution generated by the vector fields $Y_1, \ldots, Y_m \in \mathfrak{X}(U)$, where $m \leq n-1$. Define $D^\perp$ to be the submodule of all one-forms that annihilate D. Let $\Omega := Y_1 \lrcorner \ldots \lrcorner Y_m \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \in \Lambda^{n-m}(U)$. Then D has m-dimensional integral submanifolds on U if and only if either of the following two equivalent conditions are true:*

1. *For all $X, Y \in D$, $[X,Y] \in D$,*
2. *For all $\theta \in D^\perp$, $d\theta \wedge \Omega = 0$.*

We say that a distribution D is *Frobenius integrable* (or generates a *foliation* of U) if the first condition in the Frobenius theorem holds. The Frobenius theorem means that D generates an m-dimensional foliation of U whose leaves are described by some set of $n-m$ functions $\gamma^1 = c_1, \ldots, \gamma^{n-m} = c_{n-m}$ of rank $n-m$, where $\gamma^1, \ldots, \gamma^{n-m} \in C^\infty(U)$ and $c_1, \ldots, c_{n-m}$ are some appropriate constant functions.

Using Corollary 3.2, we have the following corollary to the Frobenius theorem:

Corollary 3.4. *Let D be an m-dimensional distribution generated by the vector fields $Y_1, \ldots, Y_m \in \mathfrak{X}(U)$, where $m \leq n-1$. Let $\Omega := Y_1 \lrcorner \ldots \lrcorner Y_m \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \in \Lambda^{n-m}(U)$. For all $\theta \in D^\perp$, $d\theta \wedge \Omega = 0$ (i.e. D is Frobenius integrable) if and only if $d\Omega = 0$ mod Ω.*

Proof. With Ω defined as in the corollary, Corollary 3.2 implies $\Omega = \theta^1 \wedge \cdots \wedge \theta^{n-m}$ for some linearly independent $\theta^1, \ldots, \theta^{n-m} \in \Lambda^1(U)$ that generate $D^\perp$. Now for each $1 \leq i \leq (n-m)$, the Frobenius condition $d\theta^i \wedge \Omega = 0$ is equivalent to the condition that $d\theta^i = 0 \bmod \theta^1, \ldots, \theta^{n-m}$. Hence

$$\begin{aligned} d\Omega &= d\left(\theta^1 \wedge \cdots \wedge \theta^{n-m}\right), \\ &= 0 \mod \Omega. \end{aligned}$$

To prove the converse, suppose $d\Omega = 0 \bmod \Omega$. Now for all i,

$$d\theta^i \wedge \Omega = d\left(\theta^i \wedge \Omega\right) + \theta^i \wedge d\Omega. \tag{2}$$

Since $\theta^i \wedge \Omega = 0$, and Ω is closed modulo itself, we find from (2) that $d\theta^i \wedge \Omega = 0$. □

From Sherring and Prince [13] we have the following definition:

Definition 3.5. A differential m-form $\Omega \in \Lambda^m(U)$ is *Frobenius integrable* if $\ker(\Omega)$ is Frobenius integrable and of dimension $n-m$.

From this definition we have the following lemma:

Lemma 3.6. *A differential m-form $\Omega \in \Lambda^m(U)$ is Frobenius integrable if and only if Ω is decomposable and $d\Omega = 0$ mod Ω.*

Proof. First suppose $\Omega \in \Lambda^m(U)$ is Frobenius integrable. By definition, $\ker(\Omega)$ is maximal dimension, and hence Ω is decomposable. We can write $\Omega = \theta^1 \wedge \dots \theta^{n-m}$ for some $\theta^1, \dots \theta^{n-m} \in \Lambda^1(U)$. Since $\ker(\Omega)$ is Frobenius integrable, it follows that for each $1 \leq i \leq n-m$, $d\theta^i = \text{mod } \theta^1, \dots, \theta^{n-m}$. Hence $d\Omega = 0 \text{ mod } \Omega$.

Conversely, let Ω be decomposable and $d\Omega = 0 \text{ mod } \Omega$. It is clear that $\ker(\Omega)$ is maximal rank. Further, $\ker(\Omega) = A(\langle\Omega\rangle)$ is Frobenius integrable from Proposition 2.7. □

Theorem 3.7. *Let $\Omega \in \Lambda^m(U)$ for some $m > 1$ be decomposable, and let $X \in \mathfrak{X}(U)$ with the property $X \lrcorner \Omega \neq 0$. Then there exists $\theta \in \Lambda^1(U)$ such that $\Omega = \theta \wedge (X \lrcorner \Omega)$.*

Proof. Let $\Omega \in \Lambda^m(U)$ be decomposable, and let $X \in \mathfrak{X}(U)$ with $X \lrcorner \Omega \neq 0$. Let $Y_{m+1}, \dots, Y_n \in \mathfrak{X}(U)$ be a basis for $\ker(\Omega)$. Since $X \lrcorner \Omega \neq 0$, the vector fields $X, Y_{m+1}, \dots, Y_n$ are linearly independent. We can extend these vector fields to a basis by including some $Y_2, \dots, Y_m \in \mathfrak{X}$. Let $\{\theta^1, \dots, \theta^m\}$ be a dual basis of one forms for $\{X, Y_2, \dots, Y_m\}$. Then $\Omega = f\theta^1 \wedge \dots \wedge \theta^m$, and moreover, $X \lrcorner \Omega = f\theta^2 \wedge \dots \wedge \theta^m$. Hence the result follows. □

By an obvious iteration, we have the following corollary to Theorem 3.7:

Corollary 3.8. *Let $\Omega \in \Lambda^m(U)$ be decomposable. Let $X_1, \dots, X_p \in \mathfrak{X}(U)$ up to some $p < m$ such that $X_1 \lrcorner \dots \lrcorner X_p \lrcorner \Omega \neq 0$. Then there exist $\theta^1, \dots, \theta^p \in \Lambda^1(U)$ such that*

$$\begin{aligned} \Omega &= \theta^p \wedge \dots \wedge \theta^1 \wedge (X_1 \lrcorner \dots \lrcorner X_p \lrcorner \Omega), \\ X_p \lrcorner \Omega &= \theta^{p-1} \wedge \dots \wedge \theta^1 \wedge (X_1 \lrcorner \dots \lrcorner X_p \lrcorner \Omega), \\ &\vdots \\ X_2 \lrcorner \dots \lrcorner X_p \lrcorner \Omega &= \theta^1 \wedge (X_1 \lrcorner \dots \lrcorner X_p \lrcorner \Omega). \end{aligned}$$

Proposition 3.9. *Let $\Omega \in \Lambda^m(U)$ be decomposable, and let $X \in \mathfrak{X}(U)$ such that $X \lrcorner \Omega \neq 0$. Then $\ker(X \lrcorner \Omega) = \ker(\Omega) \oplus sp\{X\}$.*

Proof. It is clear that $\ker(X \lrcorner \Omega) \supset \ker(\Omega)$. Since $X \in \ker(X \lrcorner \Omega)$, we therefore have $\ker(X \lrcorner \Omega) \supset \ker(\Omega) \oplus sp\{X\}$. By assumption Ω is decomposable, so Lemma 3.1 implies $\ker(\Omega)$ has maximal dimension $n - m$. Since $X \notin \ker(\Omega)$, it follows that $\ker(\Omega) \oplus sp\{X\}$ has dimension $n - m + 1$. Hence Lemma 3.1 implies $X \lrcorner \Omega$ is decomposable. □

We have the following corollary to Proposition 3.9, which can also be found in Sherring and Prince [13]:

Corollary 3.10. *Let $\Omega \in \Lambda^m(U)$ for some $m > 1$ be decomposable, and let $X \in \mathfrak{X}(U)$ such that $X \lrcorner \Omega \neq 0$. Then $X \lrcorner \Omega$ is decomposable.*

Before we present the next result, we require the following central definition:

Definition 3.11. Let D be a distribution in $\mathfrak{X}(U)$. Then a set of p linearly independent vector fields, $X_1, \ldots, X_p \in \mathfrak{X}(U)$, form a *solvable symmetry structure* for D if

$$\begin{gathered} \mathcal{L}_{X_1}\left(sp\{X_2, \ldots, X_p\} \oplus D\right) \subset sp\{X_2, \ldots, X_p\} \oplus D, \\ \vdots \\ \mathcal{L}_{X_{p-1}}\left(sp\{X_p\} \oplus D\right) \subset sp\{X_p\} \oplus D, \\ \mathcal{L}_{X_p} D \subset D. \end{gathered}$$

Theorem 3.12. *Let $\Omega \in \Lambda^m(U)$ be Frobenius integrable. Further, let $X \in \mathfrak{X}(U)$ such that $A(\langle\Omega\rangle) \oplus sp\{X\}$ is Frobenius integrable and $X \lrcorner \Omega \neq 0$. Then $X \lrcorner \Omega$ is Frobenius integrable.*

Proof. This theorem is obvious from Definition 3.5, Propositions 2.7 and 3.7, and Corollary 3.10. □

We have the following corollary to Theorem 3.12:

Corollary 3.13. *Let $\Omega \in \Lambda^m(U)$ be Frobenius integrable, and suppose there exist $X_1, \ldots, X_p \in \mathfrak{X}(U)$ up to some $p < m$ such that $X_1 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega \neq 0$. If $A(\langle\Omega\rangle) \oplus sp\{X_p\}$ is a Frobenius integrable distribution, and for all $1 \leq i < p$, $A(\langle\Omega\rangle) \oplus sp\{X_i, \ldots, X_p\}$ is also Frobenius integrable, then $X_p \lrcorner \Omega$, $\ldots$, $X_1 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega$ are Frobenius integrable. Moreover, $\{X_1, \ldots, X_p\}$ form a solvable symmetry structure for $A(\langle\Omega\rangle)$ if and only if*

$$\begin{gathered} \mathcal{L}_{X_p}\Omega = \lambda_p \Omega, \\ \mathcal{L}_{X_{p-1}}(X_p \lrcorner \Omega) = \lambda_{p-1}(X_p \lrcorner \Omega), \\ \vdots \\ \mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega) = \lambda_1(X_2 \lrcorner \ldots \lrcorner X_p \lrcorner \Omega), \end{gathered} \tag{3}$$

for some $\lambda_1, \ldots, \lambda_p \in C^\infty(U)$.

Corollary 3.13 provides a direct connection between a solvable symmetry structure for $\ker(\Omega) = A(\langle\Omega\rangle)$ and one for Ω (the equations in (3) will be frequently referred to as a solvable symmetry structure for Ω).

The papers by Sherring and Prince [13] and Basarab-Horwath [1] extend Lie's approach to integrating a Frobenius integrable distribution via a solvable structure of symmetries. In those papers, a Frobenius integrable distribution is given first. The one-form annihilating space is then generated and all generators wedged to give a decomposable form with a Frobenius integrable kernel. The result is reproduced below:

Theorem 3.14. [13] *Let $D := sp\{Y_1, \dots, Y_q\} \subset \mathfrak{X}(U)$ be a q-dimensional Frobenius integrable vector field distribution. Define $\Omega := Y_1 \lrcorner \dots \lrcorner Y_q \lrcorner (dx^1 \wedge \dots \wedge dx^n) \in \Lambda^{n-q}(U)$, and suppose there exists a solvable structure of linearly independent symmetries $X_1, \dots, X_{n-q} \in \mathfrak{X}(U)$ such that X_{n-q} is a non-trivial symmetry of D, and that for all $1 \le i < n-q$, X_i is a non-trivial symmetry of $D \oplus sp\{X_{i+1}, \dots, X_{n-q}\}$. For all $1 \le i \le n-q$, define ω^i by*

$$\omega^i := \frac{X_1 \lrcorner \dots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \dots \lrcorner X_{n-q} \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \dots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \dots \lrcorner X_{n-q} \lrcorner \Omega}.$$

Then $\{\omega^1, \dots, \omega^{n-q}\}$ is dual to $\{X_1, \dots, X_{n-q}\}$, and for all ω^i up to $i = n-q$,

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\
&\vdots \\
\omega^{n-q} &= d\gamma^{n-q} \mod d\gamma^1, \dots, d\gamma^{n-q-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \dots, \gamma^{n-q} \in C^\infty(U)$. Moreover, on U, the submanifolds described by D generate a q-dimensional foliation of U whose leaves have $\gamma^1, \dots, \gamma^{n-q}$ constant.

In our work, we will start with a decomposable m-form Ω with a Frobenius integrable kernel. This is achieved by also demanding that $d\Omega = 0 \mod \Omega$. Hence by Proposition 2.7, the Cauchy characteristic space of the differential ideal generated by Ω is Frobenius integrable and equal to $\ker(\Omega)$. Using these facts, we show below in Theorem 3.15 how a solvable structure of symmetries for Ω (as in Corollary 3.13) can assist in generating a simplified expression for Ω. Theorem 3.15 is the key result of this paper.

Theorem 3.15. *Let $\Omega \in \Lambda^m(U)$ be Frobenius integrable. Suppose there exists a solvable structure of linearly independent symmetries $X_1, \dots, X_m \in \mathfrak{X}(U)$ such that X_m is a non-trivial symmetry of $A(\langle\Omega\rangle)$, and that for all $1 \le i < m$, X_i is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus sp\{X_{i+1}, \dots, X_m\}$. For all $1 \le i \le m$, define ω^i by*

$$\omega^i := \frac{X_1 \lrcorner \dots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \dots \lrcorner X_m \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \dots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \dots \lrcorner X_m \lrcorner \Omega}. \tag{4}$$

Then $\{\omega^1, \dots, \omega^m\}$ is dual to $\{X_1, \dots, X_m\}$, and for all ω^i up to $i = m$,

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\
&\vdots \\
\omega^m &= d\gamma^m \mod d\gamma^1, \dots, d\gamma^{m-1},
\end{aligned} \tag{5}$$

for some functionally independent $\gamma^1, \dots, \gamma^m \in C^\infty(U)$*. Finally, define* $\gamma^0 := \Omega(X_1, \dots, X_m)$*. Then* $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$*.*

Proof. Since from Lemma 3.6, Ω is decomposable, we may write $\Omega = \theta^1 \wedge \cdots \wedge \theta^m$ for some linearly independent $\theta^1, \dots, \theta^m \in \Lambda^1(U)$. Now $\ker(\Omega) = sp\{Y_1, \dots, Y_{n-m}\}$ for some $Y_1, \dots, Y_{n-m} \in \mathfrak{X}(U)$. From Lemma 3.6 and Proposition 2.7, we have that $A(\langle\Omega\rangle) = \ker(\Omega)$ is Frobenius integrable. Applying Theorem 3.14 with the linearly independent symmetries $X_1, \dots, X_m \in \mathfrak{X}(U)$ for $A(\langle\Omega\rangle)$ given in Theorem 3.15, we obtain that

$$\{Y_1, \dots, Y_{n-m}, X_1, \dots, X_m\}$$

spans $\mathfrak{X}(U)$ and is dual to

$$\left\{\phi^1, \dots, \phi^{n-m}, \omega^1, \dots, \omega^m\right\},$$

for some linearly independent $\phi^1, \dots, \phi^{n-m} \in \Lambda^1(U)$ with $\omega^1, \dots, \omega^m$ defined as in (4). Since $Y_j \lrcorner \Omega = 0$ for all $1 \leq j \leq n-m$, it follows that

$$\Omega = \Omega(X_1, \dots, X_m)\omega^1 \wedge \cdots \wedge \omega^m. \tag{6}$$

Now Theorem 3.14 implies the equations in (5), so (6) simplifies to give

$$\Omega = \Omega(X_1, \dots, X_m) d\gamma^1 \wedge \cdots \wedge d\gamma^m.$$

□

Remark 1. The fact that the symmetries in Theorem 3.15 are non-trivial means that the denominator is non-zero in each of the definitions for ω^i.

Remark 2. The expression for γ^0 is easily derived since $\Omega = \Omega(X_1, \dots, X_m)$ $\omega^1 \wedge \cdots \wedge \omega^m$ as $Y_j \lrcorner \Omega = 0$ for all $1 \leq j \leq (n-m)$ linearly independent vector fields $Y_1, \dots, Y_{n-m}$ in $A(\langle\Omega\rangle)$ that are used with $X_1, \dots, X_m$ to span $\mathfrak{X}(U)$.

Theorem 3.15, for a given Ω and solvable symmetry structure of vector fields, gives us explicit expressions for the relations described in Proposition 3 in [9].

In later sections, we will illustrate Theorem 3.15 with some applications. For now though, we have the following consequence of Theorem 3.15 regarding the its second remark:

Theorem 3.16. *Given some Frobenius integrable* $\Omega \in \Lambda^m(U)$ *and a solvable structure* $X_1, \dots, X_m \in \mathfrak{X}(U)$ *for* $A(\langle\Omega\rangle)$ *as in Theorem 3.15, then*

$$\begin{aligned}
\mathcal{L}_{X_m}\Omega &= \{X_m \lrcorner\, d(\ln|\Omega(X_1, \dots, X_m)|)\}\Omega, \\
\mathcal{L}_{X_{m-1}}(X_m \lrcorner\, \Omega) &= \{X_{m-1} \lrcorner\, d(\ln|\Omega(X_1, \dots, X_m)|)\}(X_m \lrcorner\, \Omega), \\
&\ \ \vdots \\
\mathcal{L}_{X_1}(X_2 \lrcorner \dots \lrcorner\, X_m \lrcorner\, \Omega) &= \{X_1 \lrcorner\, d(\ln|\Omega(X_1, \dots, X_m)|)\}(X_2 \lrcorner \dots \lrcorner\, X_m \lrcorner\, \Omega).
\end{aligned}$$

Proof. First we will show that for all $1 \leq i \leq m$, $d(\omega^1 \wedge \cdots \wedge \omega^i) = 0$. From Theorem 3.15 it is obvious that $d\omega^1 = 0$ and for each $1 < i \leq m$ that $d\omega^i = 0$ mod $\omega^1, \ldots, \omega^{i-1}$. Thus for all $i > 1$,

$$d(\omega^1 \wedge \cdots \wedge \omega^i) = 0. \tag{7}$$

From Theorem 3.15 it is clear that

$$\Omega = \Omega(X_1, \ldots, X_m)\omega^1 \wedge \cdots \wedge \omega^m. \tag{8}$$

Hence

$$d\left(\frac{\Omega}{\Omega(X_1, \ldots, X_m)}\right) = 0. \tag{9}$$

Using that $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$ and contracting (8) with X_m, we obtain

$$\omega^1 \wedge \cdots \wedge \omega^{m-1} = \frac{X_m \lrcorner \, \Omega}{(-1)^{m-1}\Omega(X_1, \ldots, X_m)}.$$

From repeating this contraction with X_{m-1} and so on down to X_1, we obtain for all $1 \leq i \leq m-1$,

$$\omega^1 \wedge \cdots \wedge \omega^i = \frac{X_{i+1} \lrcorner \, \ldots \lrcorner \, X_m \lrcorner \, \Omega}{(-1)^{((m-1)+\cdots+i)}\Omega(X_1, \ldots, X_m)}.$$

Hence from (7),

$$d\left(\frac{X_{i+1} \lrcorner \, \ldots \lrcorner \, X_m \lrcorner \, \Omega}{(-1)^{((m-1)+\cdots+i)}\Omega(X_1, \ldots, X_m)}\right) = 0. \tag{10}$$

Equation (9) implies

$$d\Omega = d\left(\ln |\Omega(X_1, \ldots, X_m)|\right) \wedge \Omega, \tag{11}$$

while equation (10) means

$$d\left(X_{i+1} \lrcorner \, \ldots \lrcorner \, X_m \lrcorner \, \Omega\right) = d\left(\ln |\Omega(X_1, \ldots, X_m)|\right) \wedge \left(X_{i+1} \lrcorner \, \ldots \lrcorner \, X_m \lrcorner \, \Omega\right), \tag{12}$$

for all $1 \leq i \leq (m-1)$. Now

$$\begin{aligned}
\mathcal{L}_{X_m}\Omega &= X_m \lrcorner \, d\Omega + d\left(X_m \lrcorner \, \Omega\right), \\
&= X_m \lrcorner \left\{d\left(\ln |\Omega(X_1, \ldots, X_m)|\right) \wedge \Omega\right\} + d\left(\ln |\Omega(X_1, \ldots, X_m)|\right) \wedge \left(X_m \lrcorner \, \Omega\right), \\
&= \{X_m \lrcorner \, d(\ln |\Omega(X_1, \ldots, X_m)|)\}\Omega,
\end{aligned}$$

where in the second line we have inserted equations (11) and (12). To obtain the third line we used the identity $X \lrcorner \, (\omega \wedge \sigma) = (X \lrcorner \, \omega) \wedge \sigma + (-1)^{deg(\omega)}\omega \wedge (X \lrcorner \, \sigma)$ for differential forms σ, ω.

Finally, let $1 \leq i \leq (m-1)$. Then in a similar fashion to before, we get

$$\begin{aligned}\mathcal{L}_{X_i}(X_{i+1}\lrcorner \ldots \lrcorner X_m\lrcorner \Omega) = X_i\lrcorner \{d(\ln|\Omega(X_1,\ldots,X_m)|) \wedge (X_{i+1}\lrcorner \ldots \lrcorner X_m\lrcorner \Omega)\} \\ + d(\ln|\Omega(X_1,\ldots,X_m)|) \wedge (X_i\lrcorner \ldots \lrcorner X_m\lrcorner \Omega),\end{aligned}$$

which simplifies to

$$\mathcal{L}_{X_i}(X_{i+1}\lrcorner \ldots \lrcorner X_m\lrcorner \Omega) = \{X_i\lrcorner d(\ln|\Omega(X_1,\ldots,X_m)|)\}(X_{i+1}\lrcorner \ldots \lrcorner X_m\lrcorner \Omega).$$

□

In general, each $\omega^2,\ldots,\omega^m$ in Theorem 3.15 is not exact. Our final results for this section examine some conditions on the symmetries $X_1,\ldots,X_m$ in Theorem 3.15 that force at least one of $\omega^2,\ldots,\omega^m$ to be exact.

Theorem 3.17. *Let $\Omega \in \Lambda^m(U)$ for some $m \geq 3$ such that Ω is Frobenius integrable. Let there exist a solvable structure of linearly independent symmetries $X_3,\ldots,X_m \in \mathfrak{X}(U)$ such that X_m is a non-trivial symmetry of $A(\langle\Omega\rangle)$, and that for all $3 \leq i < m$, X_i is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus sp\{X_{i+1},\ldots,X_m\}$. Also, let there exist two linearly independent vector fields $X_1, X_2 \in \mathfrak{X}(U)$ that are non-trivial symmetries of $A(\langle\Omega\rangle) \oplus sp\{X_3,\ldots,X_m\}$ such that*

$$[X_1,X_2] = 0 \mod A(\langle\Omega\rangle) \oplus sp\{X_3,\ldots,X_m\}. \tag{13}$$

For all $1 \leq i \leq m$, define ω^i by

$$\omega^i := \frac{X_1\lrcorner \ldots \lrcorner X_{i-1}\lrcorner X_{i+1}\lrcorner \ldots \lrcorner X_m\lrcorner \Omega}{X_i\lrcorner X_1\lrcorner \ldots \lrcorner X_{i-1}\lrcorner X_{i+1}\lrcorner \ldots \lrcorner X_m\lrcorner \Omega}.$$

Then $\{\omega^1,\ldots,\omega^m\}$ is dual to $\{X_1,\ldots,X_m\}$ and for all ω^i up to $i=m$,

$$\begin{aligned}
\omega^1 &= d\gamma^1,\\
\omega^2 &= d\gamma^2,\\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)d\gamma^2 - X_1(\gamma^3)d\gamma^1,\\
\omega^4 &= d\gamma^4 - X_3(\gamma^4)(d\gamma^3 - X_2(\gamma^3)d\gamma^2 - X_1(\gamma^3)d\gamma^1) - X_2(\gamma^4)d\gamma^2 - X_1(\gamma^4)d\gamma^1,\\
&\vdots\\
\omega^m &= d\gamma^m \mod d\gamma^1,\ldots,d\gamma^{m-1},
\end{aligned}$$

for some functionally independent $\gamma^1,\ldots,\gamma^m \in C^\infty(U)$. Finally, define $\gamma^0 := \Omega(X_1,\ldots,X_m)$. Then $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.

Proof. We begin by showing that X_1 is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus sp\{X_2,\ldots,X_m\}$. Since X_1 is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus sp\{X_3,\ldots,X_m\}$, we have from Corollary 3.13 that

$$\mathcal{L}_{X_1}(X_3\lrcorner \ldots \lrcorner X_m\lrcorner \Omega) = \lambda(X_3\lrcorner \ldots \lrcorner X_m\lrcorner \Omega),$$

for some $\lambda \in C^\infty(U)$. Using this fact and equation (13) then gives

$$\begin{aligned}\mathcal{L}_{X_1}\left(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right) &= [X_1, X_2] \lrcorner X_3 \lrcorner \ldots X_m \lrcorner \Omega + X_2 \lrcorner \mathcal{L}_{X_1}\left(X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right), \\ &= \lambda\left(X_2 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right).\end{aligned}$$

From Theorem 3.13, our symmetries at this point satisfy Theorem 3.15. Therefore

$$\begin{aligned}\omega^1 &= d\gamma^1, \\ \omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\ \omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\ &\vdots \\ \omega^m &= d\gamma^m \mod d\gamma^1, \ldots, d\gamma^{m-1},\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^m \in C^\infty(U)$. To show that $X_1(\gamma^2) = 0$, we must show that

$$d\omega^2 = d\left(\frac{X_1 \lrcorner X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}\right) = 0. \tag{14}$$

This can be proved by observing that since $\ker(X_1 \lrcorner X_3 \lrcorner \ldots X_m \lrcorner \Omega) = A(\langle\Omega\rangle) \oplus sp\{X_1, X_3, \ldots, X_m\}$ is a Frobenius integral distribution, we therefore have that

$$d(X_1 \lrcorner X_3 \lrcorner \ldots X_m \lrcorner \Omega) = 0 \mod X_1 \lrcorner X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega.$$

Then to show that X_2 is a non-trivial symmetry of $A(\langle\Omega\rangle) \oplus sp\{X_1, X_3, \ldots, X_m\}$ we use the formula

$$\begin{aligned}\mathcal{L}_{X_2}\left(X_1 \lrcorner X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right) = [X_2, X_1] \lrcorner X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega \\ + X_1 \lrcorner \mathcal{L}_{X_2}\left(X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega\right).\end{aligned}$$

Now using equation (13) and that X_2 is a non-trivial symmetry of $X_3 \lrcorner \ldots \lrcorner X_m \lrcorner \Omega$, we get the desired result. Equation (14) can then be deduced from simple algebraic manipulation, or by applying Theorem 3.15. ☐

Remark. While Theorem 3.17 assumes $m \geq 3$, it is clear that is still holds when $m = 2$. In this situation, there is no need for symmetries other than X_1, X_2, with (13) reducing to $[X_1, X_2] = 0 \mod A(\langle\Omega\rangle)$. Further, the expressions for ω^i in the conclusion of the theorem vanish for $i > 2$.

We can generalise Theorem 3.17 in the following way:

Theorem 3.18. *Let $\Omega \in \Lambda^m(U)$ for some $m \geq 3$, and suppose Ω is Frobenius integrable. For some $1 \leq l < m$, let there exist a solvable structure of $m - l$ linearly independent symmetries $X_{l+1}, \ldots, X_m \in \mathfrak{X}(U)$ such that X_m is a non-trivial symmetry of $A(\langle\Omega\rangle)$, and that for all $l + 1 \leq i < m$, X_i is a* *non-trivial symmetry of* $A(\langle\Omega\rangle) \oplus sp\{X_{i+1}, \ldots, X_m\}$. *Also, let there exist* l *linearly independent vector fields* $X_1, \ldots, X_l \in \mathfrak{X}(U)$ *that are non-trivial symmetries of* $A(\langle\Omega\rangle) \oplus sp\{X_{l+1}, \ldots, X_m\}$ *such that*

$$[X_u, X_v] = 0 \mod A(\langle\Omega\rangle) \oplus sp\{X_{l+1}, \ldots, X_m\}, \tag{15}$$

for all $1 \leq u < v \leq l$. *For all* $1 \leq i \leq m$, *define* ω^i *by*

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega}.$$

Then $\{\omega^1, \ldots, \omega^m\}$ *is dual to* $\{X_1, \ldots, X_m\}$ *and for all* ω^i *up to* $i = l$,

$$\begin{aligned} \omega^1 &= d\gamma^1, \\ &\vdots \\ \omega^l &= d\gamma^l, \end{aligned}$$

with for each i *greater than* l *up to* $i = m$,

$$\begin{aligned} \omega^{l+1} &= d\gamma^{l+1} - X_l(\gamma^{l+1})d\gamma^l - X_{l-1}(\gamma^{l+1})d\gamma^{l-1} - \cdots - X_1(\gamma^{l+1})d\gamma^1, \\ \omega^{l+2} &= d\gamma^{l+2} - X_{l+1}(\gamma^{l+2})\left(d\gamma^{l+1} - X_l(\gamma^{l+1})d\gamma^l - X_{l-1}(\gamma^{l+1})d\gamma^{l-1} - \ldots\right. \\ &\qquad \left. -X_1(\gamma^{l+2})d\gamma^1\right) - X_l(\gamma^{l+2})d\gamma^l - \cdots - X_1(\gamma^{l+2})d\gamma^1, \\ &\vdots \\ \omega^m &= d\gamma^m \mod d\gamma^1, \ldots, d\gamma^{m-1}, \end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^m \in C^\infty(U)$. *Finally, define* $\gamma^0 := \Omega(X_1, \ldots, X_m)$. *Then* $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.

Proof. (Outline) The proof is similar to that of Theorem 3.17, and essentially involves repeating the proof of Theorem 3.17 $l-1$ more times. To do this, from the fact that Ω is decomposable and $d\Omega = 0 \mod \Omega$, we can then apply Corollary 3.13 to obtain

$$\begin{aligned} \mathcal{L}_{X_m} &= \lambda_m \Omega, \\ \mathcal{L}_{X_{m-1}}(X_m \lrcorner \Omega) &= \lambda_{m-1}(X_m \lrcorner \Omega), \\ &\vdots \\ \mathcal{L}_{X_{l+1}}(X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) &= \lambda_{l+1}(X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega), \end{aligned}$$

and also that

$$\begin{aligned} \mathcal{L}_{X_l}(X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) &= \lambda_l (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega), \\ \mathcal{L}_{X_{l-1}}(X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) &= \lambda_{l-1} (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega), \\ &\vdots \\ \mathcal{L}_{X_1}(X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega) &= \lambda_1 (X_{l+1} \lrcorner X_{l+2} \lrcorner \ldots \lrcorner X_m \lrcorner \Omega), \end{aligned}$$

for some $\lambda_1, \ldots, \lambda_m \in C^\infty(U)$. Next, using (15), it is easy to show that

$$\begin{aligned}\mathcal{L}_{X_m}\Omega &= \lambda_m\Omega,\\ \mathcal{L}_{X_{m-1}}(X_m \lrcorner\, \Omega) &= \lambda_{m-1}(X_m \lrcorner\, \Omega),\\ &\vdots\\ \mathcal{L}_{X_1}(X_2 \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega) &= \lambda_1(X_2 \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega).\end{aligned}$$

Then we may apply Theorem 3.15 to give us that $\{\omega^1, \ldots, \omega^m\}$ is dual to $\{X_1, \ldots, X_m\}$, and that for all ω^i up to $i = m$,

$$\begin{aligned}\omega^1 &= d\gamma^1,\\ \omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1,\\ \omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,\\ &\vdots\\ \omega^m &= d\gamma^m \mod d\gamma^1, \ldots, d\gamma^{m-1},\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^m \in C^\infty(U)$. Now since we know already that $d\omega^1 = 0$, we only have to show that for each $1 < j \leq l$,

$$d\omega^j = d\left(\frac{X_1 \lrcorner \ldots \lrcorner\, X_{j-1} \lrcorner\, X_{j+1} \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega}{X_j \lrcorner\, X_1 \lrcorner \ldots \lrcorner\, X_{j-1} \lrcorner\, X_{j+1} \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega}\right) = 0. \qquad (16)$$

The original symmetry relations for $X_1, \ldots, X_m$ tell us that for each j, $A(\langle\Omega\rangle) \oplus sp\{X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_m\}$ is Frobenius integrable, so

$$\begin{aligned}&d\,(X_1 \lrcorner \ldots \lrcorner\, X_{j-1} \lrcorner\, X_{j+1} \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega)\\ &\qquad\qquad = 0 \mod X_1 \lrcorner \ldots \lrcorner\, X_{j-1} \lrcorner\, X_{j+1} \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega.\end{aligned}$$

Finally, using (15), and in similar fashion to the end of the proof of Theorem 3.17, we get that for each j, X_j is a non-trivial symmetry of $X_1 \lrcorner \ldots \lrcorner\, X_{j-1} \lrcorner\, X_{j+1} \lrcorner \ldots \lrcorner\, X_m \lrcorner\, \Omega$. Simple algebraic manipulation then yields (16). □

Remark. As in Theorem 3.17, it is easy to see that Theorem 3.18 holds for all $m \geq 2$. However, here we can also say that the theorem holds if $l = m$, so (15) becomes $[X_u, X_v] = 0 \mod A(\langle\Omega\rangle)$ for all $1 \leq u < v \leq l$. In this situation, all ω^i become exact, which is in accordance with the corollary to Proposition 2 given in [1].

The next section gives a simple application of some of the ideas presented above.

4 Differential forms in $\Lambda^m(\mathbb{R}^{m+1})$

In this section we show that, provided we have enough symmetries, any differential form in $\Lambda^m(\mathbb{R}^{m+1})$ can be expressed locally in terms of m functionally independent functions as in the conclusion of Theorem 3.15. Further details will be given in Theorem 4.3 below, but first, consider the following result:

Lemma 4.1. *Let $\Omega \in \Lambda^m(U)$ for some $m < n$ be non-zero, where U is defined as in previous sections (though the requirement that U be convex is not necessary here). Suppose Ω is of the form*

$$\Omega := \gamma_1\theta^2 \wedge \theta^3 \wedge \cdots \wedge \theta^{m+1} + \gamma_2\theta^1 \wedge \theta^3 \wedge \cdots \wedge \theta^{m+1} + \dots \\ + \gamma_{m+1}\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^m,$$

for some linearly independent $\theta^1, \dots, \theta^{m+1} \in \Lambda^1(U)$ and $\gamma_1, \dots, \gamma_{m+1} \in C^\infty(U)$. Then Ω is decomposable.

Proof. Let $\Omega \in \Lambda^m(U)$ be as in the theorem. We can write

$$\Omega = X \lrcorner \left(\theta^1 \wedge \cdots \wedge \theta^{m+1}\right),$$

where

$$X := \sum_{i=1}^{m+1} (-1)^{i-1}\gamma_i X_i,$$

for some $X_1, \dots, X_{m+1} \in \mathfrak{X}(U)$ dual to $\theta^1, \dots, \theta^{m+1}$. Hence from Corollary 3.10 the result follows. □

From Lemma 4.1 we obtain the following useful result for m-forms in $(m+1)$-dimensional spaces also found in [8] by Godbillon. Define W to be some open neighbourhood of $\mathbb{R}^{m+1}$.

Proposition 4.2. *Let $\Omega \in \Lambda^m(W)$. Then Ω is Frobenius integrable.*

Proof. Let $\Omega \in \Lambda^m(W)$. Lemma 4.1 implies

$$\Omega = \theta^1 \wedge \cdots \wedge \theta^m,$$

for some linearly independent $\theta^1, \dots, \theta^m \in \Lambda^1(W)$. Now $d\Omega$ is an $(m+1)$-form in $\Lambda^{m+1}(W)$, so we may complete $\theta^1, \dots, \theta^m$ to a basis by including some linearly independent $\phi \in \Lambda^1(W)$ with the property that

$$d\Omega = \theta^1 \wedge \cdots \wedge \theta^m \wedge \phi.$$

□

Theorem 4.3. *Let $\Omega \in \Lambda^m(W)$, where W to be some open, convex neighbourhood of $\mathbb{R}^{m+1}$. If there exists a solvable structure of m symmetries for $A(\langle\Omega\rangle)$ as in Theorem 3.15, then we can compute functions $\gamma^0, \dots, \gamma^m \in C^\infty(W)$ so that $\Omega = \gamma^0 d\gamma^1 \wedge \cdots \wedge d\gamma^m$.*

Proof. We know from Proposition 4.2 and Proposition 3.6 respectively that Ω is decomposable and that $d\Omega = 0 \bmod \Omega$, so Theorem 3.15 gives us a direct algorithm for finding $\gamma^0, \dots, \gamma^m$. □

5 Some necessary conditions

For an arbitrary form differential form $\Delta \in \Lambda^m(U)$, we use ideas in the previous section to examine some necessary conditions for Δ to be decomposable and $d\Delta = 0 \bmod \Delta$, so that we can apply Theorem 3.15. Of course if $m = n$, these two conditions trivially hold, and Proposition 4.2 and means they still hold if $m = n-1$. In this section we examine the situation when $m < n-1$. In what follows, we assume U is some open, convex neighbourhood of $\mathbb{R}^n$.

Theorem 5.1. *Let $\Delta \in \Lambda^m(U)$ for some $m < n-1$. If there exist $n-m-1$ linearly independent vector fields $\Gamma_1, \dots \Gamma_{n-m-1} \in \mathfrak{X}(U)$ in $\ker(\Delta)$, then Δ is decomposable. Moreover, if for each $1 \leq i \leq n-m-1$,*

$$\mathcal{L}_{\Gamma_i}\Delta = \lambda_i \Delta, \tag{17}$$

for some $\lambda_i \in C^\infty(U)$, then $d\Delta = 0$ mod Δ.

Proof. Let $\Delta \in \Lambda^m(U)$ with $m < n-1$, and let there exist linearly independent $\Gamma_1, \dots \Gamma_{n-m-1} \in \mathfrak{X}(U)$ such that for all $1 \leq i \leq n-m-1$,

$$\Gamma_i \lrcorner \Delta = 0. \tag{18}$$

Now

$$(sp\{\Gamma_1, \dots, \Gamma_{n-m-1}\})^\perp = sp\left\{\theta^1, \dots, \theta^{m+1}\right\},$$

for some $\theta^1, \dots, \theta^{m+1} \in \Lambda^1(U)$. Hence from (18), we must have

$$\Delta = \Delta_{j_1 \dots j_m} \theta^{j_1} \wedge \dots \wedge \theta^{j_m},$$

for some $\Delta_{j_1 \dots j_m} \in C^\infty(U)$, with summation over $1 \leq j_1 < \dots < j_m \leq m+1$. Therefore by Lemma 4.1, Δ is decomposable.

For the second part of the proof, we choose without loss,

$$\Delta = \theta^1 \wedge \dots \wedge \theta^m.$$

We can complete $\theta^1, \dots, \theta^{m+1}$ to a basis for $\Lambda^1(U)$ by adding linearly independent $\phi^1, \dots, \phi^{n-m-1} \in \Lambda^1(U)$ such that

$$\left\{\phi^1, \dots, \phi^{n-m-1}, \theta^1, \dots, \theta^{m+1}\right\} \tag{19}$$

is dual to

$$\{\Gamma_1, \dots \Gamma_{n-m-1}, Y_1, \dots, Y_{m+1}\}, \tag{20}$$

for some linearly independent $Y_1, \dots, Y_{m+1} \in \mathfrak{X}(U)$. Now with summation on k over $1 \leq k \leq m$, we can write

$$d\Delta = \sigma_k \wedge \theta^1 \wedge \dots \wedge \theta^{k-1} \wedge \theta^{k+1} \wedge \dots \wedge \theta^m + \beta \wedge \Delta, \tag{21}$$

for some $\sigma_1, \ldots, \sigma_m \in \Lambda^2(U)$ and $\beta \in \Lambda^1(U)$ with the property that each σ_k only depends on the basis vectors $\phi^1, \ldots, \phi^{n-m-1}, \theta^{m+1}$. Hence from the dual basis property in (19) and (20), we have for each k,

$$Y_j \lrcorner\, \sigma_k = 0, \tag{22}$$

for all $1 \leq j \leq m$. By combining the assumptions in (17) and (18), we have for all i,

$$\Gamma_i \lrcorner\, d\Delta = \lambda_i \Delta. \tag{23}$$

Using the dual basis property once more, we get that for each i and $1 \leq l \leq m+1$, $\Gamma_i \lrcorner\, \theta^l = 0$. Hence substituting (21) into (23) gives (with sum),

$$(\Gamma_i \lrcorner\, \sigma_k) \wedge \theta^1 \wedge \cdots \wedge \theta^{k-1} \wedge \theta^{k+1} \wedge \cdots \wedge \theta^m + (\Gamma_i \lrcorner\, \beta) \wedge \Delta = \lambda_i \Delta, \tag{24}$$

for each i. Since each $\Gamma_i \lrcorner\, \sigma_k$ only depends on the basis vectors $\phi^1, \ldots, \phi^{n-m-1}$, θ^{m+1}, for (24) to hold we must have

$$\Gamma_i \lrcorner\, \sigma_k = 0, \tag{25}$$

for each i and k. Hence from (22) and (25), $\ker(\sigma_k)$ is at least $(n-1)$-dimensional. This means $\sigma_k(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(U)$. Thus $\sigma_k = 0$ for each k. Therefore $d\Delta = \beta \wedge \Delta$. □

Theorem 5.1 has the following two corollaries:

Corollary 5.2. *Let $\Delta \in \Lambda^m(U)$ such that $m < n-1$. If there exist $n-m-1$ linearly independent Cauchy characteristic vector fields of the differential ideal $\langle \Delta, d\Delta \rangle$, then Δ is decomposable and $d\Delta = 0$ mod Δ.*

Proof. Since the Cauchy characteristic vector fields are in the kernel of Δ, Theorem 5.1 implies Δ is decomposable. Now it is clear that (17) in Theorem 5.1 still holds for some $\lambda_1, \ldots, \lambda_{n-m-1} \in C^\infty(U)$. Hence from the theorem, $d\Delta = 0 \bmod \Delta$ □

Corollary 5.3. *Let $\Delta \in \Lambda^m(U)$ such that $m < n-1$. If there exist $n-m-1$ linearly independent Cauchy characteristic vector fields of the differential ideal $\langle \Delta, d\Delta \rangle$, then the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is $(n-m)$-dimensional*

Proof. From Corollary 5.2, Δ is decomposable, so $\ker(\Delta)$ is $(n-m)$-dimensional. The corollary also means Δ is closed modulo itself which implies $\langle \Delta \rangle = \langle \Delta, d\Delta \rangle$, and hence their Cauchy characteristic spaces are equal. From Proposition 2.7 the result follows. □

Now the dimension of the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is always less than or equal to that of $\ker(\Delta)$, and the maximum dimensional of $\ker(\Delta)$ is $n-m$, which occurs when Δ is decomposable. Theorem 5.1 therefore means that if $\ker(\Delta)$ is at least $(n-m-1)$-dimensional, then it is $(n-m)$-dimensional. Similarly, Corollary 5.3 means that if the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is at least $(n-m-1)$-dimensional, then it is $(n-m)$-dimensional.

Next, we illustrate Corollary 5.2 with the following example:

Example 5.4. Suppose U^4 is some suitably chosen open, convex neighbourhood of $\mathbb{R}^4$ with coordinates x^1, x^2, x^3, x^4, and

$$\Delta := \frac{2x^2x^4}{x^3} dx^3 \wedge dx^2 - \left(\frac{x^4}{x^3}\right) dx^3 \wedge dx^1 - 2dx^4 \wedge dx^1 + \frac{1}{x^3x^4} dx^1 \wedge dx^2 + 4x^2 dx^4 \wedge dx^2.$$

Now the vector field

$$\Gamma := 4x^2 \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{1}{x^3x^4}\frac{\partial}{\partial x^2},$$

is a Cauchy characteristic of $\langle \Delta, d\Delta \rangle$. Hence from Corollary 5.2, Δ is decomposable and $d\Delta = 0 \mod \Delta$. Note from Corollary 5.3 that the Cauchy characteristic space of $\langle \Delta, d\Delta \rangle$ is two-dimensional.

We will now proceed to apply Theorem 3.15 to Δ. It is easy to see that $\frac{\partial}{\partial x^1}$ is a non-trivial symmetry of Δ. With

$$\frac{\partial}{\partial x^1} \lrcorner \Delta = \frac{1}{x^3x^4} dx^2 + \frac{x^4}{x^3} dx^3 + 2dx^4,$$

it is also easy to see that $\frac{\partial}{\partial x^2}$ is a non-trivial symmetry of $\frac{\partial}{\partial x^1} \lrcorner \Delta$. Now from Theorem 3.15 and Corollary 3.13,

$$\omega^1 := \frac{\frac{\partial}{\partial x^1} \lrcorner \Delta}{\frac{\partial}{\partial x^2} \lrcorner \frac{\partial}{\partial x^1} \lrcorner \Delta} = dx^2 + (x^4)^2 dx^3 + 2x^3x^4 dx^4 = d\left(x^2 + x^3(x^4)^2\right).$$

Also, it is not hard to show that

$$\omega^2 := \frac{\frac{\partial}{\partial x^2} \lrcorner \Delta}{\frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial x^2} \lrcorner \Delta} = dx^1 + 2x^2(x^4)^2 dx^3 + 4x^2x^3x^4 dx^4, = d\left(x^1 - (x^2)^2\right) + 2x^2 d\left(x^2 + x^3(x^4)^2\right).$$

Hence

$$\Delta = \frac{1}{x^3x^4} d\left(x^1 - (x^2)^2\right) \wedge d\left(x^2 + x^3(x^4)^2\right).$$

6 Pfaffian equations

In this section we examine how symmetries may be used to express a differential one-form 'normal form' given in (26). We begin with the following definition and theorem:

Definition 6.1. Let $\alpha \in \Lambda^1(U)$. The *rank*of the Pfaffian equation $\alpha = 0$ at the point $p \in U$ is the non-negative integer r such that $(d\alpha)^r \wedge \alpha \neq 0$ and $(d\alpha)^{r+1} \wedge \alpha = 0$ at p.

If a one-form α is exact, i.e. $\alpha = df$ for some $f \in C^\infty(U)$, then it (and any linearly dependent one-form) has rank zero.

Theorem 6.2. *Let $\alpha \in \Lambda^1(U)$ and suppose the equation $\alpha = 0$ is of constant rank r on U. Then there exists a coordinate system $\gamma^1, \dots, \gamma^n \in C^\infty(U)$, where $2r + 1 \leq n$, so that the equation becomes*

$$d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1} = 0.$$

Theorem 6.2 is known as the Pfaff problem. A proof of this theorem may be found in [2].

It is easy to see that multiplying any one-form of constant rank on U by a nowhere zero smooth function f leaves the rank unchanged, using the fact that for any $m \in \mathbb{N}$, we have $(d(f\alpha))^m \wedge (f\alpha) = f^{m+1}(d\alpha)^m \wedge \alpha$. This allows us to express any $\alpha \in \Lambda^1(U)$ of constant rank r on U as

$$\alpha = \gamma^0(d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1}), \tag{26}$$

for some $\gamma^0, \dots, \gamma^{2r+1} \in C^\infty(U)$.

Theorem 6.3. *Let $\alpha \in \Lambda^1(U)$. Suppose α is of constant rank r on U. Define $\Omega := (d\alpha)^r \wedge \alpha$. Then Ω is decomposable and $d\Omega = 0 \mod \Omega$.*

Proof. Let $\alpha \in \Lambda^1(U)$ with α of constant rank r on U. Hence

$$\alpha = \gamma^0(d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1}),$$

for some $\gamma^0, \dots, \gamma^{2r+1} \in C^\infty(U)$. Define

$$\overline{\alpha} := d\gamma^1 + \gamma^2 d\gamma^3 + \cdots + \gamma^{2r} d\gamma^{2r+1}.$$

Further, let $\overline{\Omega} := (d\overline{\alpha})^r \wedge \overline{\alpha}$. We will first show that $d\overline{\Omega} = 0$. Simple computation yields

$$(d\overline{\alpha})^r = r! d\gamma^2 \wedge \cdots \wedge d\gamma^{2r+1}.$$

Hence

$$\begin{aligned}\overline{\Omega} &= \overline{\alpha} \wedge (d\overline{\alpha})^r, \\ &= r! d\gamma^1 \wedge d\gamma^2 \wedge \cdots \wedge d\gamma^{2r+1}.\end{aligned}$$

We then have $d\overline{\Omega} = 0$. Now

$$\Omega = (\gamma^0)^{r+1}(d\overline{\alpha})^r \wedge \overline{\alpha}.$$

Since $d\overline{\Omega} = 0$, we get

$$d\Omega = d((\gamma^0)^{r+1}) \wedge (d\overline{\alpha})^r \wedge \overline{\alpha}.$$

But, $(d(\gamma^0\overline{\alpha}))^r \wedge ((\gamma^0)\overline{\alpha}) = (\gamma^0)^{r+1}(d\overline{\alpha})^r \wedge \overline{\alpha}$. Hence $d\Omega = 0 \mod \Omega$ as γ^0 is nowhere zero on U. Finally, since $\overline{\Omega}$ is decomposable and $\Omega = (\gamma^0)^{r+1}\overline{\Omega}$, Ω is therefore decomposable. □

Our aim is to use Theorem 6.3 with Theorem 3.15 to ultimately find some coordinates for the Pfaff problem in Theorem 6.2. The next theorem illustrates how this may be done for one-forms that are of constant rank one on U, which will be later extended to one-forms of any constant rank $r \geq 1$. The case $r = 0$ involves a trivial application of Theorem 3.15, and will therefore be ignored.

To assist in finding coordinates for the Pfaff problem, the following lemma will be needed:

Lemma 6.4. *Let $\alpha \in \Lambda^1(U)$ and suppose α is of constant non-zero rank r on U. Let $\Omega := (d\alpha)^r \wedge \alpha$ and $X \in \mathfrak{X}(U)$ such that $X \lrcorner \Omega = 0$. Then $X \lrcorner \alpha = 0$.*

Proof. Let $\alpha \in \Lambda^1(U)$. Suppose α is of constant non-zero rank r on U, and define Ω as in the lemma. Let $X \in \mathfrak{X}(U)$ with $X \lrcorner \Omega = 0$. Now

$$0 = X \lrcorner \Omega = (X \lrcorner (d\alpha)^r) \wedge \alpha + (X \lrcorner \alpha)(d\alpha)^r.$$

By taking the exterior product with α, we obtain

$$(X \lrcorner \alpha)(d\alpha)^r \wedge \alpha = 0.$$

Since α is of rank r, $(d\alpha)^r \wedge \alpha \neq 0$, and hence $X \lrcorner \alpha = 0$. □

Theorem 6.5. *Let $\alpha \in \Lambda^1(U)$ such that α is of constant rank one on U. Let $\Omega := d\alpha \wedge \alpha$ and $\langle \Omega \rangle$ be the differential ideal generated by Ω. Suppose $X_1, X_2, X_3 \in \mathfrak{X}(U)$ is a solvable structure of linearly independent symmetries such that X_3 is a non-trivial symmetry of $A(\langle \Omega \rangle)$ with the extra condition that $X_3 \lrcorner \alpha = 0$, X_2 is a non-trivial symmetry of $A(\langle \Omega \rangle) \oplus sp\{X_3\}$, and X_1 is a non-trivial symmetry of $A(\langle \Omega \rangle) \oplus sp\{X_2, X_3\}$. Then with $\omega^1, \omega^2, \omega^3 \in \Lambda^1(U)$ defined by*

$$\omega^1 := \frac{X_2 \lrcorner X_3 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner \Omega},$$
$$\omega^2 := \frac{X_1 \lrcorner X_3 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner \Omega},$$
$$\omega^3 := \frac{X_1 \lrcorner X_2 \lrcorner \Omega}{X_3 \lrcorner X_1 \lrcorner X_1 \lrcorner \Omega},$$

we have

$$\omega^1 = d\gamma^1,$$
$$\omega^2 = d\gamma^2 - X_1(\gamma^2) d\gamma^1,$$
$$\omega^3 = d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1,$$

for some functionally independent $\gamma^1, \gamma^2, \gamma^3 \in C^\infty(U)$, and

$$\alpha = (X_2 \lrcorner \alpha) \left(d\gamma^2 + \frac{(X_1 \lrcorner \alpha) - (X_2 \lrcorner \alpha) X_1(\gamma^2)}{(X_2 \lrcorner \alpha)} d\gamma^1 \right). \tag{27}$$

Proof. With $\Omega := d\alpha \wedge \alpha$, Theorem 6.3 means that Ω is decomposable and $d\Omega = 0 \mod \Omega$. Theorem 3.15 can be used to obtain $\{\omega^1, \omega^2, \omega^3\}$ dual to $\{X_1, X_2, X_3\}$, where

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2) d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1,
\end{aligned}$$

for some functionally independent $\gamma^1, \gamma^2, \gamma^3 \in C^\infty(U)$. Now from Lemma 6.4, $X \lrcorner\, \alpha = 0$ for all $X \in A(\langle\Omega\rangle)$. And since $X_3 \lrcorner\, \alpha = 0$, we are left with

$$\alpha = (X_1 \lrcorner\, \alpha)\omega^1 + (X_2 \lrcorner\, \alpha)\omega^2.$$

Now $X_2 \lrcorner\, \alpha \neq 0$ in the neighbourhood, since α is nowhere rank zero by assumption. Hence

$$\alpha = (X_2 \lrcorner\, \alpha) \left(d\gamma^2 + \frac{(X_1 \lrcorner\, \alpha) - (X_2 \lrcorner\, \alpha) X_1(\gamma^2)}{(X_2 \lrcorner\, \alpha)} d\gamma^1 \right).$$

□

Remark 1. The extra condition in Theorem 6.5 that the *non-trivial* symmetry X_3 satisfies $X_3 \lrcorner\, \alpha = 0$ implies from Proposition 2.6 that the symmetry is not a Cauchy characteristic vector field of $\langle \alpha, d\alpha \rangle$. Therefore $X_3 \lrcorner\, d\alpha$ is not some multiple of α (as α is of rank one, it is impossible that $d\alpha = 0 \mod \alpha$). Such a symmetry exists since if $\gamma^1, \ldots, \gamma^n$ are coordinates for U and $\alpha := \gamma^0 \left(d\gamma^1 + \gamma^2 d\gamma^3 \right)$ is already in normal form for some $\gamma^0 \in C^\infty(U)$, then it is easy to show that Theorem 6.5 can be applied to such α with X_3 chosen as $\frac{\partial}{\partial \gamma^2}$ or $\frac{\partial}{\partial \gamma^3} - \gamma^2 \frac{\partial}{\partial \gamma^1}$.

Remark. 2 In deriving our expression for α in (27), we do not need to calculate γ^3. This significantly reduces the number of algebraic manipulations required.

We illustrate Theorem 6.5 with an example:

Example 6.6. Suppose we are in some open, convex neighbourhood of $\mathbb{R}^3$, denoted by U^3, with coordinates x^1, x^2, x^3. Define on some suitably chosen U^3,

$$\alpha := -\frac{x^2 x^3}{(x^1)^2} dx^1 + \left(\frac{x^1}{x^2} + \frac{x^3}{x^1} \right) dx^2 + \frac{x^1}{x^3} dx^3.$$

By dimension, $(d\alpha)^2 \wedge \alpha = 0$, and it is easy to show that $d\alpha \wedge \alpha \neq 0$ on some region of U^3. Suppose U^3 is chosen such that $d\alpha \wedge \alpha \neq 0$ everywhere. Since *any* non-zero vector field is a non-trivial symmetry of $d\alpha \wedge \alpha \in \Lambda^3(U^3)$, we may choose any X_3 such that $X_3 \lrcorner\, \alpha = 0$. So let

$$X_3 := \frac{x^2 x^3}{(x^1)^2} \frac{\partial}{\partial x^3} + \frac{x^1}{x^3} \frac{\partial}{\partial x^1}$$

be the symmetry. Now

$$X_2 := (x^3)^2 \frac{\partial}{\partial x^3}$$

is a non-trivial symmetry of $sp\{X_3\}$ ($A(\langle d\alpha \wedge \alpha \rangle)$ is zero-dimensional), and by inspection that

$$X_1 := \frac{\partial}{\partial x^2}$$

is a non-trivial symmetry of $sp\{X_2, X_3\}$. These yield

$$\omega^1 := \frac{X_2 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)}{X_1 \lrcorner X_2 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)} = dx^2,$$

and

$$\begin{aligned}\omega^2 := \frac{X_1 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)}{X_2 \lrcorner X_1 \lrcorner X_3 \lrcorner (d\alpha \wedge \alpha)} &= -\frac{x^2}{(x^1)^3} dx^1 + \frac{dx^3}{(x^3)^2}, \\ &= d\left(\frac{x^2}{2(x^1)^2} - \frac{1}{x^3}\right) - \frac{1}{2(x^1)^2} dx^2.\end{aligned}$$

Hence a simple calculation gives

$$\alpha = x^1 x^3 \left(d\left(\frac{x^2}{2(x^1)^2} - \frac{1}{x^3}\right) + \left(\frac{1}{x^2 x^3} + \frac{1}{2(x^1)^2}\right) dx^2 \right).$$

Such expressions for α are in general not unique, and may be found by choosing different symmetries. For example, we have also obtained

$$\alpha = x^3 \left(d\left(\frac{x^2}{x^1}\right) + \frac{x^1}{x^3} d\left(\ln |x^2 x^3|\right) \right).$$

We now present a generalisation of Theorem 6.5:

Theorem 6.7. *Let $\alpha \in \Lambda^1(U)$ have constant rank r on U, and define $\Omega := (d\alpha)^r \wedge \alpha$. Let $X_1, \ldots, X_{2r+1} \in \mathfrak{X}(U)$ be a solvable structure of linearly independent symmetries such that X_{2r+1} is a non-trivial symmetry of $A(\langle \Omega \rangle)$, and for each $1 < i < 2r+1$, X_i is a non-trivial symmetry of $A(\langle \Omega \rangle) \oplus \{X_{i+1}, \ldots, X_{2r+1}\}$. Suppose, in addition, that for the r vector fields $X_{r+2}, \ldots, X_{2r+1}$, we have $X_{r+2} \lrcorner \alpha = 0, \ldots, X_{2r+1} \lrcorner \alpha = 0$. For all $1 \leq i \leq 2r+1$, define ω^i by*

$$\omega^i := \frac{X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_{2r+1} \lrcorner \Omega}{X_i \lrcorner X_1 \lrcorner \ldots \lrcorner X_{i-1} \lrcorner X_{i+1} \lrcorner \ldots \lrcorner X_{2r+1} \lrcorner \Omega}.$$

Then for all ω^i up to $i = 2r+1$,

$$\begin{aligned}\omega^1 &= d\gamma^1, \\ \omega^2 &= d\gamma^2 - X_1(\gamma^2) d\gamma^1, \\ \omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1, \\ &\vdots \\ \omega^{2r+1} &= d\gamma^{2r+1} \mod d\gamma^1, \ldots, d\gamma^{2r},\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^{2r+1} \in C^\infty(U)$*, and*

$$
\begin{aligned}
\alpha = (X_1 \lrcorner\, \alpha) d\gamma^1 &+ (X_2 \lrcorner\, \alpha)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) \\
&+ (X_3 \lrcorner\, \alpha)(d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2) d\gamma^1) - X_1(\gamma^3) d\gamma^1) + \ldots \\
&\quad + (X_{r+1} \lrcorner\, \alpha)(d\gamma^{r+1} - X_r(\gamma^{r+1})(d\gamma^r - \cdots - X_1(\gamma^r) d\gamma^1) - \ldots \\
&\qquad - X_1(\gamma^{r+1}) d\gamma^1),
\end{aligned}
$$

which when rearranged give α *in the form of (26).*

Proof. The proof follows in a similar fashion to Theorem 6.5. The conditions $X_{r+2} \lrcorner\, \alpha = 0, \ldots, X_{2r+1} \lrcorner\, \alpha = 0$ and Lemma 6.4 ensure that α is a linear combination of $d\gamma^1, \ldots, d\gamma^{r+1}$. Further, since α is of constant rank r, $X_{r+1} \lrcorner\, \alpha \neq 0$, so we are permitted to divide by it, and hence express α in the form of (26). □

Remark. Both remarks for Theorem 6.5 may be extended to Theorem 6.7 as follows: Firstly, from the proof of Theorem 6.3 it is clear that there exist r non-trivial symmetries $X_{r+2}, \ldots, X_{2r+1}$ of $(d\alpha)^r \wedge \alpha$ in $\ker(\alpha)$, and secondly, in deriving our expression for α, we do not need to calculate any $\gamma^{r+2}, \ldots, \gamma^{2r+1}$.

7 Darboux systems

This section gives an algorithm based on vector fields for generating a set of coordinates in Darboux's theorem given below in Theorem 7.4. To begin with, we present some preliminary material. In Bryant *et al.* [2] there is the following fundamental theorem:

Theorem 7.1. *Let* $\Omega \in \Lambda^2(U)$ *and let* r *be the natural number such that* $\Omega^r \neq 0$ *and* $\Omega^{r+1} = 0$*. Then there exist* $2r$ *linearly independent elements* $\omega^1, \ldots, \omega^{2r} \in \Lambda^1(U)$ *such that*

$$\Omega = \omega^1 \wedge \omega^2 + \cdots + \omega^{2r-1} \wedge \omega^{2r}.$$

In what follows, we will also make use of the following lemma:

Lemma 7.2. *Let* $\Omega \in \Lambda^2(U)$ *and* $r \in \mathbb{N}$ *such that* $\Omega^r \neq 0$ *and* $\Omega^{r+1} = 0$*. Also let* $X \in \mathfrak{X}(U)$*. Then* $X \lrcorner\, \Omega^r = 0$ *if and only if* $X \lrcorner\, \Omega = 0$*.*

Proof. Let $\Omega \in \Lambda^2(U)$ with $X \lrcorner\, \Omega^r = 0$ for some vector field $X \in \mathfrak{X}(U)$. Then from Theorem 7.1 we have

$$\Omega = \omega^1 \wedge \omega^2 + \cdots + \omega^{2r-1} \wedge \omega^{2r}, \tag{28}$$

for some linearly independent $\omega^1, \ldots, \omega^{2r} \in \Lambda^1(U)$. This implies

$$\Omega^r = r! \omega^1 \wedge \cdots \wedge \omega^{2r}.$$

Now $X \lrcorner\, \Omega^r = 0$ implies that $X \lrcorner\, \omega^i = 0$ for all $1 \leq i \leq 2r$. Hence using the expression for Ω in (28) gives $X \lrcorner\, \Omega = 0$. Proving the converse is obvious since if Y is any vector field in $\mathfrak{X}(U)$, then $Y \lrcorner\, \Omega^r = r(Y \lrcorner\, \Omega) \wedge \Omega^{r-1}$. □

Theorem 7.3. *Let $\Omega \in \Lambda^2(U)$ be closed. Suppose r is the natural number such that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$. Further suppose we have a solvable structure of $2r$ linearly independent symmetries $X_1, \ldots, X_{2r} \in \mathfrak{X}(U)$ such that X_{2r} is a non-trivial symmetry of $A(\langle\Omega^r\rangle)$, and for all $1 \leq i < 2r$, X_i is a non-trivial symmetry of $A(\langle\Omega^r\rangle) \oplus sp\{X_{i+1}, \ldots, X_{2r}\}$. Then Theorem 3.15 gives us an algorithm for expressing Ω solely in terms of the $2r$ functionally independent functions $\gamma^1, \ldots, \gamma^{2r} \in C^\infty(U)$ and their exterior derivatives*

Proof. Let $\Omega \in \Lambda^2(U)$ be closed with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. Since $d\Omega = 0$ implies that $d(\Omega^r) = 0$, from Proposition 2.7, $\ker(\Omega^r) = A(\langle\Omega^r\rangle)$ is therefore Frobenius integrable. The fact that Ω^r is decomposable of degree $2r$ means that $A(\langle\Omega^r\rangle)$ is generated by $n - 2r$ linearly independent vector fields. Suppose we have a set of linearly independent symmetries $X_1, \ldots, X_{2r} \in \mathfrak{X}(U)$ such that X_{2r} is a non-trivial symmetry of $A(\langle\Omega^r\rangle)$, and for all $1 \leq i < 2r$, X_i is a non-trivial symmetry of $A(\langle\Omega^r\rangle) \oplus sp\{X_{i+1}, \ldots, X_{2r}\}$. Then by Theorem 3.15 we have on U, $\{\omega^1, \ldots, \omega^{2r}\}$ dual to $\{X_1, \ldots, X_{2r}\}$, where for all $1 \leq j \leq 2r$,

$$\omega^j := \frac{X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_{2r} \lrcorner \Omega^r}{X_j \lrcorner X_1 \lrcorner \ldots \lrcorner X_{j-1} \lrcorner X_{j+1} \lrcorner \ldots \lrcorner X_{2r} \lrcorner \Omega^r},$$

and

$$\begin{aligned}
\omega^1 &= d\gamma^1, \\
\omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\
\omega^3 &= d\gamma^3 - X_2(\gamma^3)(d\gamma^2 - X_1(\gamma^2)d\gamma^1) - X_1(\gamma^3)d\gamma^1, \\
&\vdots \\
\omega^{2r} &= d\gamma^{2r} \mod d\gamma^1, \ldots, d\gamma^{2r-1},
\end{aligned}$$

for some functionally independent $\gamma^1, \ldots, \gamma^{2r} \in C^\infty(U)$. Then by Lemma 7.2, and using the fact that $\{X_1, \ldots, X_{2r}\}$ plus any set of generators of $A(\langle\Omega^r\rangle)$ spans $\mathfrak{X}(U)$, we can therefore write

$$\Omega = \Omega(X_k, X_l)\omega^k \wedge \omega^l, \qquad 1 \leq k < l \leq 2r,$$

where we are implying a double summation. This means that

$$\Omega = \Omega_{kl} d\gamma^k \wedge d\gamma^l, \qquad 1 \leq k < l \leq 2r, \tag{29}$$

for some functions $\Omega_{kl} \in C^\infty(U)$. But since Ω is closed, we must have for all $\Gamma \in A(\langle\Omega^r\rangle)$,

$$\mathcal{L}_\Gamma \Omega = d(\Gamma \lrcorner \Omega) = 0,$$

also using Lemma 7.2. Since $\Gamma(\gamma^i) = 0$ for all i, it follows that (with sum)

$$0 = \mathcal{L}_\Gamma \Omega = \Gamma(\Omega_{kl}) d\gamma^k \wedge d\gamma^l.$$

Therefore $\Gamma(\Omega_{kl}) = 0$ for each k and l. Hence Ω only depends on the $2r$ functions $\gamma^1, \ldots, \gamma^{2r}$ and their exterior derivatives. □

Remark. In applying Theorem 7.3, there will exist situations when it may be difficult to express each Ω_{kl} in terms of the known $\gamma^1, \ldots, \gamma^{2r}$.

Next, consider Darboux's theorem proved in [2, 5]:

Theorem 7.4. (Darboux) *Let $\Omega \in \Lambda^2(U)$ be closed so that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. Then there exist coordinates $\gamma^1, \ldots, \gamma^n$ such that*

$$\Omega = d\gamma^1 \wedge d\gamma^2 + \cdots + d\gamma^{2r-1} \wedge d\gamma^{2r}.$$

Theorem 7.3 may be applied to Darboux's theorem; however, the difficulty is that Theorem 7.3 expresses Ω in terms of a sum of a maximum of $\binom{2r}{2}$ two-form components, which must then be simplified to r components with unit one coefficients if we wish to find a set of coordinates in Darboux's theorem.

As an alternative approach extending work in [5] by Crampin and Pirani in their proof of Darboux's theorem (though similar proofs can be found in the literature), we now look to formulate an extraction process for generating a set of coordinates in the theorem using solvable symmetry structures. The next three theorems will be useful in establishing this.

Theorem 7.5. *Let $\Omega \in \Lambda^2(U)$ with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \geq 2$. Suppose there exist $X_1, X_2 \in \mathfrak{X}(U)$ such that $\Omega(X_1, X_2) = 1$ and $(X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega) \neq 0$. If $\overline{\Omega}$ is defined by $\overline{\Omega} := \Omega + (X_2 \lrcorner\, \Omega) \wedge (X_1 \lrcorner\, \Omega)$, then $\overline{\Omega}^{r-1} \neq 0$ and $\overline{\Omega}^r = 0$.*

Proof. Let $\Omega \in \Lambda^2(U)$ such that $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \geq 2$. Using the definition for $\overline{\Omega}$ in the theorem gives

$$\overline{\Omega}^r = \Omega^r + r\Omega^{r-1} \wedge (X_2 \lrcorner\, \Omega) \wedge (X_1 \lrcorner\, \Omega). \tag{30}$$

Now from $\Omega(X_1, X_2) = 1$ we have

$$\begin{aligned} \Omega^r &= \Omega^r (X_2 \lrcorner\, X_1 \lrcorner\, \Omega), \\ &= X_2 \lrcorner\, (\Omega^r \wedge (X_1 \lrcorner\, \Omega)) - (X_2 \lrcorner\, \Omega^r) \wedge (X_1 \lrcorner\, \Omega), \\ &= X_2 \lrcorner\, (\Omega^r \wedge (X_1 \lrcorner\, \Omega)) - (r(X_2 \lrcorner\, \Omega) \wedge \Omega^{r-1}) \wedge (X_1 \lrcorner\, \Omega). \end{aligned} \tag{31}$$

In the second line we have used the property $X_2 \lrcorner\, (\Omega^r \wedge (X_1 \lrcorner\, \Omega)) = (X_2 \lrcorner\, \Omega^r) \wedge (X_1 \lrcorner\, \Omega) + (X_2 \lrcorner\, X_1 \lrcorner\, \Omega)\Omega^r$, and in the third, we have expanded $X_2 \lrcorner\, \Omega^r$. If we substitute the end result in (31) into the expression for $\overline{\Omega}^r$ in (30), we obtain

$$\overline{\Omega}^r = X_2 \lrcorner\, (\Omega^r \wedge (X_1 \lrcorner\, \Omega)). \tag{32}$$

By Theorem 7.1, there exist linearly independent one-forms $\omega^1, \ldots, \omega^{2r} \in \Lambda^1(U)$ such that

$$\Omega = \omega^1 \wedge \omega^2 + \cdots + \omega^{2r-1} \wedge \omega^{2r}.$$

Hence $X_1 \lrcorner\, \Omega = a_1\omega^1 + \cdots + a_{2r}\omega^{2r}$ for some $a_1, \ldots, a_{2r} \in C^\infty(U)$. Since

$$\Omega^r = r!\omega^1 \wedge \cdots \wedge \omega^{2r},$$

it follows that $\Omega^r \wedge (X_1 \lrcorner\, \Omega) = 0$. Thus from (32) we get $\overline{\Omega}^r = 0$.

Now suppose $\overline{\Omega}^{r-1} = 0$. Then

$$0 = \overline{\Omega}^{r-1} = \Omega^{r-1} + (r-1)\Omega^{r-2} \wedge (X_2 \lrcorner\, \Omega) \wedge (X_1 \lrcorner\, \Omega).$$

This implies

$$\Omega^{r-1} = (r-1)\Omega^{r-2} \wedge (X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega). \tag{33}$$

Taking the exterior product with Ω gives

$$\Omega^r = (r-1)\Omega^{r-1} \wedge (X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega) = 0, \tag{34}$$

where the second equality comes from substituting Ω^{r-1} in (34) with its expression in (33). The calculations still holds for $r = 2$, and hence we reach a contradiction for all $r \geq 2$. □

Remark. Although Theorem 7.5 demands that X_1, X_2 be such that $\Omega(X_1, X_2) = 1$, we can relax this condition by saying that all we need is to find two vector fields $Y_1, Y_2 \in \mathfrak{X}(U)$ such that $\Omega(Y_1, Y_2) \neq 0$. Then we can choose X_1, X_2 as, respectively, scaled Y_1, Y_2 such that $\Omega(X_1, X_2) = 1$.

The second theorem we require concerns the foliated exterior derivative, as explained by Vaisman [15]:

Theorem 7.6. *Let $\omega \in \Lambda^1(U)$ and $\alpha^1, \ldots, \alpha^s \in \Lambda^1(U)$ be s linearly independent one-forms such that for all $1 \leq i \leq s$,*

$$d\alpha^i = 0 \mod \alpha^1, \ldots, \alpha^s,$$

(i.e. the Frobenius condition holds so that $\ker(\alpha^1 \wedge \cdots \wedge \alpha^s)$ is Frobenius integrable).

Then if

$$d\omega = 0 \mod \alpha^1, \ldots, \alpha^s,$$

then

$$\omega = df \mod \alpha^1, \ldots, \alpha^s,$$

for some $f \in C^\infty(U)$.

Using the foliated exterior derivative, we prove the following theorem:

Theorem 7.7. *Let $\Omega \in \Lambda^2(U)$ be closed. If there exists a pair of vector field $X_1, X_2 \in \mathfrak{X}(U)$ such that*

1. $\mathcal{L}_{X_1}\Omega = 0$,
2. $\mathcal{L}_{X_2}\Omega = 0 \mod X_1 \lrcorner\, \Omega$,
3. $(X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega) \neq 0$,

then on U,

$$(X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega) = df \wedge dg,$$

for some functionally independent smooth f and g.

Proof. Let $\Omega \in \Lambda^2(U)$ be closed and let there exist vector fields $X_1, X_2 \in \mathfrak{X}(U)$ that satisfy the three conditions in the theorem. Now $\mathcal{L}_{X_1}\Omega = 0$ implies $d(X_1 \lrcorner\, \Omega) = 0$, using the property $\mathcal{L}_{X_1}\Omega = X_1 \lrcorner\, d\Omega + d(X_1 \lrcorner\, \Omega)$ and that Ω is closed. Hence $X_1 \lrcorner\, \Omega = df$ for some smooth f.

Now suppose $\mathcal{L}_{X_2}\Omega = 0$. Then by the same argument to above, $X_2 \lrcorner\, \Omega = dg_1$ for some smooth g_1. If, however, $\mathcal{L}_{X_2}\Omega \neq 0$, then by assumption,

$$0 \neq \mathcal{L}_{X_2}\Omega = \alpha \wedge (X_1 \lrcorner\, \Omega),$$

for some $\alpha \in \Lambda^1(U)$. Therefore

$$(\mathcal{L}_{X_2}\Omega) \wedge (X_1 \lrcorner\, \Omega) = 0.$$

Using $\mathcal{L}_{X_2}\Omega = X_2 \lrcorner\, d\Omega + d(X_2 \lrcorner\, \Omega)$ and the fact that Ω is closed gives

$$d(X_2 \lrcorner\, \Omega) \wedge (X_1 \lrcorner\, \Omega) = 0.$$

Hence

$$d(X_2 \lrcorner\, \Omega) = 0 \mod (X_1 \lrcorner\, \Omega).$$

Using Theorem 7.6, we then get

$$X_2 \lrcorner\, \Omega = dg_2 \mod df,$$

for some smooth g_2. Hence in both cases the result is proved. □

We now present the main result of this section:

Theorem 7.8. *Let $\Omega \in \Lambda^2(U)$ be closed with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. Then the following algorithm explicitly computes a set of $2r$ functions for Ω described in Darboux's theorem:*

1. *Find vector fields $X_1, X_2 \in \mathfrak{X}(U)$ such that:*
   - *(a)* $\mathcal{L}_{X_1}\Omega = 0$,
   - *(b)* $\mathcal{L}_{X_2}\Omega = 0 \mod X_1 \lrcorner\, \Omega$,
   - *(c)* $(X_1 \lrcorner\, \Omega) \wedge (X_2 \lrcorner\, \Omega) \neq 0$,
   - *(d)* $\Omega(X_1, X_2) = 1$,
2. *Let $\Omega + (X_2 \lrcorner\, \Omega) \wedge (X_1 \lrcorner\, \Omega)$ be our new Ω,*
3. *Repeat steps 1 and 2 a further $r - 2$ more times until $\Omega^2 = 0$,*
4. *Apply Theorem 3.15 with a solvable structure of two symmetries $X_3, X_4 \in \mathfrak{X}(U)$ for Ω, such that X_3 is a non-trivial symmetry of Ω and X_4 is a non-trivial symmetry of $X_3 \lrcorner\, \Omega$ with the property that $\Omega(X_3, X_4) = 1$.*

Proof. Let $\Omega \in \Lambda^2(U)$ be closed with $\Omega^r \neq 0$ and $\Omega^{r+1} = 0$ for some $r \in \mathbb{N}$. From Theorem 7.7 and then Theorem 7.5, we can compute $\Omega_1 \in \Lambda^2(U)$, where

$$\Omega_1 = \Omega + dg_1 \wedge df_1,$$

for some smooth f_1 and g_1, with $\Omega_1^{r-1} \neq 0$ and $\Omega_1^r = 0$. Then once again from Theorem 7.7 followed by Theorem 7.5, $\Omega_2 \in \Lambda^2(U)$ can be computed so that

$$\Omega_2 = \Omega + dg_1 \wedge df_1 + dg_2 \wedge df_2,$$

for some smooth f_2 and g_2, with $\Omega_2^{r-2} \neq 0$ and $\Omega_2^{r-1} = 0$. Continuing in this way, we reach a stage when Ω_{r-1} is of the form

$$\Omega_{r-1} = \Omega + dg_1 \wedge df_1 + dg_2 \wedge df_2 + \cdots + dg_{r-1} \wedge df_{r-1},$$

such that $\Omega_{r-1} \neq 0$ and $\Omega_{r-1}^2 = 0$. Applying step 4, Ω_{r-1} is closed, and from Theorem 7.1, Ω_{r-1} is also decomposable. From Theorem 3.15 and Corollary 3.13, with X_3 as a non-trivial symmetry of Ω_{r-1} and X_4 as a non-trivial symmetry of $X_3 \lrcorner\, \Omega_{r-1}$ such that $\Omega_{r-1}(X_3, X_4) = 1$, then

$$\frac{X_3 \lrcorner\, \Omega_{r-1}}{X_4 \lrcorner\, X_3 \lrcorner\, \Omega_{r-1}} = dg_r,$$
$$\frac{X_4 \lrcorner\, \Omega_{r-1}}{X_3 \lrcorner\, X_4 \lrcorner\, \Omega_{r-1}} = df_r + \lambda dg_r,$$

for some smooth f_r, g_r and λ, with

$$\Omega_{r-1} = \Omega_{r-1}(X_3, X_4) df_r \wedge dg_r = df_r \wedge dg_r.$$

Therefore

$$\Omega = df_1 \wedge dg_1 + df_2 \wedge dg_2 + \cdots + df_{r-1} \wedge dg_{r-1} + df_r \wedge dg_r.$$

□

Remark 1. In looking for two symmetries that satisfy the four conditions in Theorem 7.8, condition (d) can be relaxed a little by only requiring that $X_2 \lrcorner\, X_1 \lrcorner\, \Omega = const$. Then X_1 or X_2 may be scaled appropriately by constants while still satisfying the other three conditions. The same holds true for the two symmetries in step 4.

Remark 2. Conditions (a) and (b) are strong requirements, and may be difficult in practice to satisfy. Since Ω is closed, they imply X_1, X_2 must be chosen such that $X_1 \lrcorner\, \Omega$ is closed and $X_2 \lrcorner\, \Omega$ is closed, modulo $X_1 \lrcorner\, \Omega$. Hence the result in Theorem 7.8 is of more theoretical significance than practical use.

We can provide an alternative to the requirement in step 4 in Theorem 7.8 as follows:

Lemma 7.9. *Let $\Omega \in \Lambda^2(U)$ be some arbitrary closed two-form. Suppose there exists some $X_3 \in \mathfrak{X}(U)$ not in* $\ker(\Omega)$ *such that such that*

$$\mathcal{L}_{X_3}\Omega = 0, \tag{35}$$

and $X_4 \in \mathfrak{X}(U)$ satisfies $\Omega(X_3, X_4) = 1$. Then

$$\mathcal{L}_{X_4}(X_3 \lrcorner\, \Omega) = 0.$$

Proof.

$$\mathcal{L}_{X_4}(X_3 \lrcorner\, \Omega) = d(X_4 \lrcorner\, X_3 \lrcorner\, \Omega) + X_4 \lrcorner\, d(X_3 \lrcorner\, \Omega) = X_4 \lrcorner\, (\mathcal{L}_{X_3}\Omega) = 0,$$

using that $X_4 \lrcorner\, X_3 \lrcorner\, \Omega = 1$, equation (35), and that Ω is closed. □

We now apply the algorithm in Theorem 7.8 and the modification of Step 4 in Lemma 7.9 to an example. It is important to realise that the difficult part in applying Theorem 7.8 is in finding the first $r-1$ pairs of symmetries X_1, X_2. Nevertheless, the main purposes of this example are to illustrate: i) the crucial role Theorem 7.5 plays in reducing the number of terms in a two-form by one; and ii) the flexibility in choosing X_4 in Lemma 7.9.

Example 7.10. Consider the following two-form $\Omega \in \Lambda^2(U^4)$, where U^4 is some suitably chosen four-dimensional, open, convex neighbourhood of $\mathbb{R}^4$ with coordinates x^1, x^2, x^3, x^4:

$$\begin{aligned}\Omega := &\left(\frac{x^1}{x^2}\right)\left(\frac{x^3}{x^2} - 2\right) dx^1 \wedge dx^2 + \frac{x^1}{x^2} dx^1 \wedge dx^3 - \frac{2x^1}{x^4} dx^1 \wedge dx^4 \\ &- \left(\frac{x^1}{x^2}\right)^2 dx^2 \wedge dx^3.\end{aligned}$$

Now it is easy to show that $d\Omega = 0$, $\Omega^2 \neq 0$ and $\Omega^3 = 0$. We may then proceed to apply Theorem 7.8. Let

$$X_1 := -\frac{1}{x^3}\left(\frac{x^2}{x^1}\right)^2 \frac{\partial}{\partial x^2} + \frac{x^2 x^4}{(x^1)^2 x^3}\frac{\partial}{\partial x^4}.$$

Now

$$\begin{aligned}\mathcal{L}_{X_1}\Omega &= d\left(X_1 \lrcorner\, \Omega\right), \\ &= d\left(\frac{1}{x^3} dx^3 + \frac{2x^2}{x^1 x^3} dx^1 + \frac{1}{x^3}\left(\frac{x^2}{x^1}\right)\left(\frac{x^3}{x^2} - 2\right) dx^1\right), \\ &= d\left(\frac{1}{x^3} dx^3 + \frac{1}{x^1} dx^1\right) = 0,\end{aligned}$$

so condition (a) of step 1 in Theorem 7.8 is met. Hence

$$X_1 \lrcorner\, \Omega = d\left(\ln|x^1 x^3|\right).$$

Let

$$X_2 := x^3 \frac{\partial}{\partial x^3}.$$

We have $X_2 \lrcorner X_1 \lrcorner \Omega = 1$, so condition (d) is satisfied. Then using that

$$X_2 \lrcorner \Omega = \frac{x^1 x^3}{x^2} dx^1 + x^3 \left(\frac{x^1}{x^2}\right)^2 dx^2,$$

it is not hard to show that $(X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) \neq 0$, so condition (c) is satisfied. Also,

$$\begin{aligned}
(\mathcal{L}_{X_2}\Omega) \wedge (X_1 \lrcorner \Omega) &= d\left(X_2 \lrcorner \Omega\right) \wedge \left(X_1 \lrcorner \Omega\right), \\
&= \left(\frac{x^1}{x^2} dx^1 \wedge dx^3 - \frac{x^1 x^3}{(x^2)^2} dx^1 \wedge dx^2 - \left(\frac{x^1}{x^2}\right)^2 dx^2 \wedge dx^3 \right. \\
&\qquad \left. + \frac{2x^1 x^3}{(x^2)^2} dx^1 \wedge dx^2 \right) \wedge \left(\frac{1}{x^1} dx^1 + \frac{1}{x^3} dx^3\right), \\
&= 0,
\end{aligned}$$

so condition (b) is met. Now

$$d\left(X_2 \lrcorner \Omega\right) = 0 \mod X_1 \lrcorner \Omega.$$

Using the foliated derivative, this implies

$$X_2 \lrcorner \Omega = dg_1 + \lambda_1 d\left(\ln |x^1 x^3|\right), \tag{36}$$

for some $g_1, \lambda_1 \in C^\infty(U^4)$. Performing a coordinate substitution gives

$$X_2 \lrcorner \Omega = -d\left(\frac{(x^1)^2 x^3}{x^2}\right) + \frac{(x^1)^2 x^3}{x^2} d\left(\ln |x^1 x^3|\right).$$

Therefore

$$(X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega) = -d\left(\frac{(x^1)^2 x^3}{x^2}\right) \wedge d\left(\ln |x^1 x^3|\right) = -d\left(\frac{x^1}{x^2}\right) \wedge d(x^1 x^3).$$

For other choice of X_1, X_2, we may obtain an expression for the other two-form component of Ω.

Now define $\Omega_1 := \Omega + (X_2 \lrcorner \Omega) \wedge (X_1 \lrcorner \Omega)$ as in step 2. We then get

$$\Omega_1 = -\frac{2x^1}{x^2} dx^1 \wedge dx^2 - \frac{2x^1}{x^4} dx^1 \wedge dx^4.$$

It is clear that $d\Omega_1 = 0$ and $\Omega_1^2 = 0$ as expected, so we may proceed to apply the final step in Theorem 7.8 on Ω_1. Defining

$$X_3 := x^1 x^4 \frac{\partial}{\partial x^4},$$

we have

$$\mathcal{L}_{X_3}\Omega_1 = d\left(X_3 \lrcorner\, \Omega_1\right) = d\left(2(x^1)^2 dx^1\right) = 0.$$

This implies

$$X_3 \lrcorner\, \Omega_1 = d\left(\frac{2(x^1)^3}{3}\right). \tag{37}$$

Now choose

$$X_4 := \frac{1}{2(x^1)^2}\frac{\partial}{\partial x^1},$$

so that $X_4 \lrcorner\, X_3 \lrcorner\, \Omega_1 = 1$. From Lemma 7.9, $\mathcal{L}_{X_4}\left(X_3 \lrcorner\, \Omega_1\right) = 0$, and hence from Theorem 3.15,

$$X_4 \lrcorner\, \Omega_1 = df_2 + \lambda_2 d\left(\frac{2(x^1)^3}{3}\right), \tag{38}$$

for some $f_2, \lambda_2 \in C^\infty(U^4)$. To find f_2, it is easy to show that

$$X_4 \lrcorner\, \Omega_1 = -d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \mod dx^1,$$

and hence

$$\Omega_1 = d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \wedge d\left(\frac{2(x^1)^3}{3}\right).$$

Once again we may simplify this:

$$\begin{aligned} d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \wedge d\left(\frac{2(x^1)^3}{3}\right) &= 2(x^1)^2 d\left(\frac{1}{x^1}\ln|x^2x^4|\right) \wedge dx^1, \\ &= 2x^1 d\left(\ln|x^2x^4|\right) \wedge dx^1, \\ &= d\left(\ln|x^2x^4|\right) \wedge d\left((x^1)^2\right). \end{aligned}$$

Thus

$$\Omega = d\left(\frac{x^1}{x^2}\right) \wedge d\left(x^1x^3\right) + d\left(\ln|x^2x^4|\right) \wedge d\left((x^1)^2\right).$$

8 Summary

Using the idea of a solvable symmetry structure we presented various algorithms for expressing certain classes of differential forms in terms of simplified coordinate systems. We began by reviewing Lie's symmetry approach and then showed that it may applied to simplify differential forms which are decomposable and closed modulo themselves. We then gave a result showing that certain types of symmetry structures in Theorem 3.15 forced more than one of the ω^i to become closed, and looked at under what conditions a given differential form was decomposable and closed modulo itself.

Next, we examined the problem of finding simplifying coordinates for the Pfaffian problem. This was treated by imposing a special condition on the solvable symmetry structure applied to the Cauchy characteristic space of the differential ideal generated by the differential form $(d\alpha)^r \wedge \alpha$, where α was the Pfaffian form, and r was its rank.

Finally, we looked at differential two-forms where the main result there was an algorithm for finding the coordinates in Darboux's theorem, derived from the well-known iterative scheme, where a pair of new coordinates is extracted each time.

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