IN THE NAME OF HIGHEST

SUPER HILBERT SPACE

BY

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To My Family

SAYIN SOYUMA SUNURUM

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Chapter 1

PRELIMINARIES

1. PRELIMINARIES

This chapter contains some preliminary concepts which we will use during of this work. The concept of Hilbert C^*-modules are given completely from [46]. A good additional reference for this concept is [26]. Direct limit and inductive limit of spaces are given from [30] and [9] respectively.

Also we give preliminary knowledge about Riesz spaces, Banach lattices and Riesz algebras which will used in the structure of Riesz algebra of supernumbers.

1.1 Hilbert C^*-modules

Hilbert modules form a category in between Banach spaces and Hilbert spaces. A Hilbert module obeys the same axioms as an ordinary Hilbert space except that the inner product, takes values in a more general C^*-algebra A than $\mathbb{C}$. Fundamental and familiar Hilbert space properties like Pythagoras equality, self duality, and even decomposition into orthogonal complements must be given up.

The theory of operators on Hilbert modules, generalizing the well known theory of $\mathcal{B}(H)$ for an ordinary Hilbert space H, is a little tricky. In the Hilbert space case the existence of adjoint operators is automatic, mainly because Hilbert spaces are self-dual. Since in general a Hilbert module E need not be self-dual, not all maps in the Banach algebra of all bounded linear maps in E need have an adjoint. In the next of this subsection consider the letter A as a C^*-algebra.

Definition 1.1.1. A **pre Hilbert A-module** is a right A-module E (which is at the same time a complex vector space) equipped with an A valued

inner product $\langle .|. \rangle : E \times E \longrightarrow A$ that is sesquilinear, positive definite, and respects the module action. In other words:

1. $\langle x|y_1 + y_2\rangle = \langle x|y_1\rangle + \langle x|y_2\rangle$ for $x, y_1, y_2 \in E$;
2. $\langle x|ya\rangle = \langle x|y\rangle a$ for $x, y \in E,\ a \in A$;
3. $\langle x|cy\rangle = c\langle x|y\rangle$ for $x, y \in E,\ c \in \mathbb{C}$;
4. $\langle x|y\rangle = \langle y|x\rangle^*$ for $x, y \in E$;
5. $\langle x|x\rangle \geq 0$ for $x \in E$, and $\langle x|x\rangle = 0 \Leftrightarrow 0$.

Notice that the positivity condition 5 is a statement about positive elements in the C^*-algebra A. Now we notice the **Cauchy-Schwartz inequality**.

Lemma 1.1.1. *If E is a pre Hilbert module and $x, y \in E$, then*

$$\|\langle x|y\rangle\|^2 \leq \|\langle x|x\rangle\|\|\langle y|y\rangle\|.$$

Proof. See lemma 15.1.3 of [46]. □

Definition 1.1.2. The **norm** of an element $e \in E$ in defined as

$$\|x\| := \|\langle x|x\rangle\|^{\frac{1}{2}}. \tag{1.1.1}$$

If a pre Hilbert A-module is complete with respect to its norm, it is said to be a **Hilbert A-module**.

Remark 1.1.1. Notice that $\|xa\| \leq \|x\|\|a\|$ for $x \in E$ and $a \in A$, because

$$\|xa\|^2 = \|\langle xa|xa\rangle\| = \|a^*\langle x|x\rangle a\| \leq \|a^*a\|\|\langle x|x\rangle\| = \|a\|^2\|x\|^2.$$

Also, one of the most important characteristic of inner product of Hilbert A-module is that it is separately continuous in each variable since

$$\|\langle x_n|y\rangle - \langle x|y\rangle\| = \|\langle x_n - x|y\rangle\| \leq \|x - x_n\|\|y\|.$$

Definition 1.1.3. Let E be a Hilbert A-module. A map $T : E \longrightarrow E$ is said to be **adjointable** if there exists a map $T^* : E \longrightarrow E$ satisfying

$$\langle x|Ty\rangle = \langle T^*x|y\rangle \tag{1.1.2}$$

for all x, y in E. Such a map T^* is then called the **adjoint** of T.

By $\mathcal{B}(E)$ we denote the set of all adjointable maps in E, whereas $\mathcal{B}_b(E)$ is the set of all bounded module maps in E.

Lemma 1.1.2. *If T is adjointable, then its adjoint is unique and adjointable with $T^{**} = T$. If both T and S are adjointable, then so is ST with $(ST)^* = T^*S^*$.*

If T is adjointable, then T and T^* are module maps which are bounded with respect to the operator norm. In particular, $\mathcal{B}(E) \subset B_b(E)$.

Proof. See lemma 15.2.3 of [46]. □

Proposition 1.1.3. *When equipped with the operator norm*

$$\|T\| = \sup\{\|Tx\| \;\mid\; \|x\| \leq 1\}, \tag{1.1.3}$$

$B_b(E)$ is a Banach algebra and $\mathcal{B}(E)$ a C^*-algebra.

Proof. See Proposition 15.2.4 of [46]. □

Proposition 1.1.4. *Let $T \in \mathcal{B}(E)$ be an adjointable operator. Then T is self-adjoint with positive spectrum if and only if $\langle Tx|x\rangle \geq 0$ in A for all $x \in E$.*

Proof. See proposition 15.2.5 of [46]. □

1.2 Limits

1.2.1 Direct Limit

If $(G_n)_{n=1}^{\infty}$ is a sequence of groups, and if for each n we have a homomorphism $\tau_n : G_n \longrightarrow G_{n+1}$, then we call $(G_n)_{n=1}^{\infty}$ a **direct sequence of groups**.

Given a such sequence and positive integers $n \leq m$, we set $\tau_{nn} = I_{G_n}$ and we define $\tau_{nm} : G_n \longrightarrow G_m$ inductively by setting $\tau_{n,m+1} = \tau_m \tau_{nm}$. If $n \leq m \leq k$, we have $\tau_{nk} = \tau_{mk}\tau_{nm}$.

If G' is a group and we have homomorphisms $\rho^n : G_n \longrightarrow G'$ such that $\rho^n = \rho^{n+1}\tau_n$, then $\rho^n = \rho^m \tau_{nm}$ for all $m \geq n$.

The product $\prod_{k=1}^{\infty} G_k$ is a group with the pointwise defined operations and if we let G' be the set of all elements $(x_k)_k$ in $\prod_{k=1}^{\infty} G_k$ such that there is an integer N for which $x_{k+1} = \tau_k(x_k)$ for all $k \geq N$, then G' is a subgroup of $\prod_{k=1}^{\infty} G_k$. Let e_k be the unit of G_k. The set F of all $(x_k)_k \in \prod_{k=1}^{\infty} G_k$ such that there exists N for which $x_k = e_k$ for all $k \geq N$ is a normal subgroup of G', and we denote the quotient group G'/F by G. We call G the **direct limit** of the sequence $(G_n, \tau_n)_{n=1}^{\infty}$, and sometimes write $G = \lim\limits_{\rightarrow} G_n$.

1.2.2 Inductive Limit

Definition 1.2.1. Let X be a vector space and M be a subset of X. M is said to be *convex* if $tx + (1-t)y \in M$ for all $x, y \in M$ and $t \in [0, 1]$.

In the vector space X a *linear manifold* of X , is a linear subspace of X that is not necessarily closed.

A *directed set* is a partially ordered set $(I, \leq)$ such that if $i_1, i_2 \in I$ then there is an $i_3 \in I$ such that $i_3 \geq i_1$ and $i_3 \geq i_2$.

A set $A \subset X$ is said *balanced set* if rx is in A, whenever $x \in A$ and $|r| \leq 1$.

Definition 1.2.2. An **inductive system** is a pair $(X, \{X_i : i \in I\})$, where X is a vector space , X_i is a linear manifold in X which has a topology τ_i such that (X_i, τ_i) is a locally convex space(LCS), and moreover:

(i) I is a directed set and $X_i \subset X_j$ if $i \leq j$;

(ii) If $i \leq j$ and $U_j \in \tau_i$ then $U_j \cap X_i \in \tau_i$;

(iii) $X = \cup\{X_i : i \in I\}$.

Proposition 1.2.1. *If $(X, \{X_i, \tau_i\})$ is an inductive system, let $\mathcal{B}$ be the set of all convex balanced sets V such that $V \cap X_i \in \tau_i$ for all i and also let τ be the collection of all subsets U of X such that for every x_0 in U there is a V in $\mathcal{B}$ with $x_0 + V \subseteq U$. Then (X, τ) is a (not necessarily Hausdorff)LCS.*

Proof. See proposition IV.5.3 of [9] . ☐

Definition 1.2.3. If $(X, \{X_i\})$ is an inductive system and τ is the topology defined in (1.2.1), τ is called **inductive limit topology** and (X, τ) is said to be **inductive limit** of $\{X_i\}_{i \in I}$.

A **strict inductive system** is an inductive system $(X, \{X_n, \tau_n\}_{n=1}^{\infty})$ such that for every $n \geq 1, X_n \subseteq X_{n+1}, \tau_{n+1}|X_n = \tau_n$ and X_n is closed in X_{n+1}.

The inductive limit topology defined on X by such a system is called a **strict inductive limit topology** and X is said to be **strict inductive limit** of $\{X_n\}_{n=1}^{\infty}$.

Proposition 1.2.2. *Any strict inductive system is complete.*

Proof. See Theorem 13.1 of [45] ☐

1.3 Riesz spaces

In this section we give some definitions and properties of Riesz space theory to using them in the next chapters.

1.3.1 Real Riesz Spaces

Hear, all vector spaces are assumed on real numbers.

Definition 1.3.1. A partially ordered vector space $(V, \geq)$ is called a **lattice** if each pair of elements u, v of V has a supremum and infimum. We have the following notations:

$$u \vee v = \sup\{u, v\} \quad \text{and} \quad u \wedge u = \inf\{u, v\}.$$

An ordered vector space which is also a lattice is called **Riesz space** or a **vector lattice**. Every element v in a vector lattice has modulus $|v| = v \vee (-v)$, positive part $v^+ = v \vee 0$, and negative part $v^- = (-v) \vee 0$, and the usual identities $v = v^+ - v^-$, $|v| = v^+ + v^-$, and $v^+ \wedge v^- = 0$ are hold. We say that v and u are disjoint if $|v| \wedge |u| = 0$. It will be denoted by $v \perp u$.

Given the ordered vector space V, the subset $V^+ = \{v \in V : \ v \geq 0\}$ is called the **positive cone** of V which has the following properties:

(i) $u, v \in V^+$ implies $u + v \in V^+$;

(ii) $v \in V^+$ implies $rv \in V^+$ for any real number $r \geq 0$;

(iii) $v, -v \in V^+$ implies $v = 0$ $(V^+ \cap (-V^+) = \{0\})$.

Definition 1.3.2. A subset U of a Riesz space V is **order bounded from above(below)** if there is a vector v (called an **upper(lower) bound** of U) satisfying $u \leq v (u \geq v)$ for each $u \in U$. A subset U of a Riesz space V is **order bounded** if U is both order bounded from above and below.

A **box** or **order interval** is any set of the form

$$[a, d] = \{c \in V : \quad a \leq c \leq d\}. \tag{1.3.1}$$

Definition 1.3.3. A nonempty subset U of a Riesz space V has a **supremum** (or a **least upper bound**) if there is an upper bound u of U such that $a \leq v$ for all $a \in U$ implies $u < v$. Clearly the supremum, if it exist, is unique and is denoted by $\sup U$.

Definition 1.3.4. A net $\{v_\tau\}$ in a Riesz space V is **decreasing**, written $v_\tau \downarrow$ if $\tau \geq \mu$ implies $v_\tau \leq v_\mu$. The symbol $v_\tau \uparrow$ indicates an **increasing** net, while $v_\tau \uparrow \leq v$ (resp. $v_\tau \downarrow \geq v$) denotes an increasing (resp. decreasing) net that is order bounded from above (resp. below) by v.

The notation $v_\tau \downarrow v$ means that $v_\tau \downarrow$ and $\inf\limits_\tau v_\tau = v$. Also the notation $v_\tau \uparrow v$ means that $v_\tau \uparrow$ and $\sup\limits_\tau v_\tau = v$.

Definition 1.3.5. A net $\{v_\tau\}$ in a Riesz space V **converges in order** or is **order convergent** to some $v \in V$, written $v_\tau \xrightarrow{o} v$, if there is a net $\{u_\tau\}$ (with the same directed set) satisfying $u_\tau \downarrow o$ and $|v_\tau - v| \leq u_\tau$ for each τ. In this case v is called **order limit** of $\{v_\tau\}$.

A sequence $\{v_n\}_{n=1}^{\infty}$ in V is said to be an **order Cauchy sequence** if there is a sequence $u_n \downarrow 0$ such that $|v_n - v_m| \leq u_n$ for all $m \geq n \geq 1$. One Riesz space is **order complete** if every order Cauchy sequence has an order limit. Equivalently a Riesz space is order complete if every subset of it has a supremum.

The main properties of order convergence can be find in any book of Riesz space theory such as Theorem 10.2 of [54].

Definition 1.3.6. Let $u \geq 0$ be an element of a Riesz space V. We say that the sequence $\{v_n\}_{n=1}^{\infty}$ in V **converges u-uniformly** to an element

$v \in V$ whenever, for every $\varepsilon > 0$, there exists a natural number N_ε such that $|v_n - v| \leq \varepsilon u$ holds for all $n \geq N_\varepsilon$. In this case v is called **u-uniform limit** of $\{v_n\}$ and written as $v_n \longrightarrow v(u\text{-un})$.

It is said that the sequence $\{v_n\}_{n=1}^{\infty}$ in V **converges relatively uniformly** to $v \in V$ whenever v_n converges u-uniformly to v for some $u \in V^+$. This kind of convergence is denoted by $v_n \longrightarrow v(\text{un})$.

The element u is then called the **regulator** of the relatively uniform convergence.

Definition 1.3.7. A sequence $\{v_n\}$ in V is called a **u-uniform Cauchy sequence**, whenever for any $\varepsilon > 0$, there exists a positive integer $n_1 = n_1(\varepsilon)$ such that $|v_m - v_n| \leq \varepsilon u$ holds for all $m, n \geq n_1$. A Riesz space is called **u-uniformly complete** if every u-uniform Cauchy sequence has a u-uniform limit. A Riesz space is said to be **uniformly complete** whenever, for every positive element u, any u-uniformly Cauchy sequence has a u-uniform limit.

Definition 1.3.8. A Riesz space V is called **Archimedean** if $\frac{1}{n}v \downarrow 0$ for each $v \in V^+$. V is said to be **Dedekind complete** if every nonempty subset of V which is bounded from above has a supremum. Also a **Dedekind σ-complete** Riesz space is a space that every non-empty at most countable subset of it which is bounded from above has a supremum.

It is interesting to note that in any Riesz space, relatively uniform convergence is **stable**, i.e., it has the property that for any sequence $v_n \longrightarrow 0(\text{un})$ there exists a sequence of real numbers $\{r_n\}_{n=1}^{\infty}$ such that $0 \leq r_n \uparrow \infty$ and $r_n v_n \longrightarrow 0(\text{un})$. Indeed, given that $v_n \longrightarrow 0(\text{un})$ there exists a sequence of positive real numbers $(s_n : n = 1, 2, ...)$ and an element $u \in V^+$ such that $s_n \downarrow 0$ and $|v_n| \leq s_n u$ for all n, and so $r_n = s_n^{-\frac{1}{2}}$ satisfies the above mentioned condition. Order convergence is not necessarily stable.

Theorem 1.3.1. *In an Archimedian Riesz space order convergence is stable if and only if order convergence and relatively uniform convergence are equivalent.*

Proof. See theorem 16.3 of [28]. □

Theorem 1.3.2. *If V is Archimesean, then V is u-uniformly complete if and only if every monotone u-uniform Cauchy sequence has an u-uniform limit.*

Proof. See theorem 39.4 of [28]. □

Lemma 1.3.3. *In a Dedekind σ-complete space every monotone u-uniform Cauchy sequence is u-uniformly convergent.*

Proof. See the last part of lemma 39.2 of [28]. □

Definition 1.3.9. A Riesz space V is called **order separable** if every non-empty subset V possessing a supremum contains at most a countable subset possessing the same supremum as V. It is said to be **strong order separable** whenever every non-empty subset V which is bounded above contains an at most countable subset having the same upper bounds as V (this property is called property (*) in [28]).

Definition 1.3.10. A subset U of a Riesz space V is called **solid set** if $|u| \leq |v|$ and $v \in U$ implies $u \in U$.

Definition 1.3.11. A subset U of a Riesz space V is **order closed** if $\{u_\tau\} \subset U$ and $u_\tau \xrightarrow{o} u$ imply $u \in U$. In a similar way U is called **σ-order closed** if these statements are true for sequences.

A solid linear subspace of a Riesz space is called an **ideal**. An order closed ideal is called a **band**. An ideal J is a band if and only if $\{j_\tau\} \subset J$ and $0 \leq j_\tau \uparrow j$ imply $j \in J$.

The ideal J_U generated by the non-empty subset U, is

$$J_U = \bigcup\{n[-u, u] : \quad n \in \mathbb{N}; \quad u = |u_1| \vee \cdots \vee |u_r| \quad , u_1, \ldots, u_r \in U\}.$$

A **principal ideal** is an ideal generated by a singleton $\{u\}$ and is denoted by J_u. For any $u \in V^+$ the ideal J_u generated by u is

$$J_u = \bigcup\{n[-u, u] : \quad n \in \mathbb{N}\}.$$

For any $u \in V$ the principal ideal J_u generated by u is

$$J_u = \{v \in V \quad | \quad \exists \lambda > 0 \quad \text{with} \quad |v| \leq \lambda|u|\}. \tag{1.3.2}$$

The band generated by an ideal J of V is given by

$$B_J = \{v \in V \quad | \quad \exists \text{a net } \ \{v_\tau\} \subset J \quad \text{with} \quad 0 \leq v_\tau \uparrow |v|\}.$$

Also for $v \in V$ the band B_{J_v}(will be denoted by B_v) generated by ideal J_v is called **principal band** generated by v.

Let $v \in V^+$. Any element $u \in V^+$ satisfying $u \wedge (v - u) = 0$ is called a **component** of v. The set C_v of all components of v is a Boolean algebra.

$$C_v = \{u \in V^+ \quad | \quad u \wedge (v - u) = 0\}. \tag{1.3.3}$$

Definition 1.3.12. Any band B in the Riesz space V, having the property that $B \bigoplus B^d = V$, is called **projection band**. In this case, if $v = b_1 + b_2$ is decomposition of an arbitrary $v \in V$ as the sum of $b_1 \in B$ and $b_2 \in B^d$, then b_1 and b_2 are called the components of v in B and B^d respectively.

The Riesz space V is said to have the **projection property** if every band in V is a projection band, and V is said to have the **principal projection property** if every principal band in V is a projection band.

Theorem 1.3.4. *Every Dedekind complete Riesz space has the projection property.*

Proof. See theorem 24.9(i) of [28]. □

Theorem 1.3.5. *Every Dedekind σ-complete Riesz space has the principal projection property and therefore has sufficiently many projection.*

Proof. See the main inclusion theorem 25.1 and theorem 30.4 of [28]. □

Definition 1.3.13. The element $e \in V^+$ is called a **strong order unit** in V if the principle ideal generated by e is the whole space V, i.e, if for every $e' \in V$, there exists a positive number r, depending upon e', such that $|e'| \leq re$.

The element $e \in V^+$ is called a **weak unit** in V if $v \perp e$ implies that $v = 0$ for $v \in V$.

Theorem 1.3.6. *Let the Archimedean Riesz space V have a strong order unit, and let V is either Dedekind σ-complete or have a projection property. Then order convergence in V is stable if and only if V is of finite dimension.*

Proof. See theorem 70.3 of [28]. □

Definition 1.3.14. The Riesz space V is said to have the **diagonal property** whenever, given any double sequence $\{v_{n,k}\}_{n,k=1}^{\infty}$ in V, any sequence $\{v_n\}_{n=1}^{\infty}$ in V any $v_0 \in V$ such that $v_{n,k} \longrightarrow v_n$ for all n(”as $k \longrightarrow \infty$”) and $v_n \longrightarrow v_0$, there exists for any n an appropriate $k = k(n)$ such that $v_{n,k(n)} \longrightarrow v_0$.

The Riesz space V has **diagonal gap property** , if under the same hypothesis that $v_{n,k} \longrightarrow v_n$ for any n and $v_n \longrightarrow v_0$, there exists a sequence $v_{n_i,k(n_i)}$ with $n_1 < n_2 < ...$, an infinite sequence containing at most one member from each sequence $\{v_{n,k}\}_{k=1}^{\infty}$ such that $v_{n_i,k(n_i)} \longrightarrow v_0$ (as $i \longrightarrow \infty$).

Definition 1.3.15. If $\{v_{n,k}\}_{n,k=1}^{\infty}$ is a double sequence of elements of a Riesz space V and if the element $v' \in V$ has the property that for every n

there exists a positive integer $k(n)$ such that $v' \leq v_{n,k(n)}$, then we will write $v' \ll \{v_{n,k}\}$.

The element $v \in V$ is said to have the **Egoroff property**, if given any double sequence $\{v_{n,k}\}_{n,k=1}^{\infty}$ in V such that $0 \leq v_{n,k} \uparrow_k |v|$ for $n = 1, 2, ...$ there exists a sequence $0 \leq v'_m \uparrow v$ such that $v'_m \ll \{v_{n,k}\}$ holds for every m (i.e., for every m and n there exists a positive integer $k(m,n)$ such that $v'_m \leq v_{n,k(m,n)}$ holds). The space V is said to have **Egoroff property** if every element of V has the Egoroff property.

The space V is said to have the **strong Egoroff property**, if given any double sequence $\{v_{n,k}\}_{n,k=1}^{\infty}$ in V^+ such that $v_{n,k} \downarrow_k 0$ for $n = 1, 2, ...$, there exists a sequence $v'_m \downarrow 0$ in V_+ with the property that, for every pair (m, n) of positive integers, we have $v'_m \geq v_{n,k(m,n)}$ for an appropriate $k = k(m, n)$.

Definition 1.3.16. The Riesz space V is said to have the **d-property** whenever given the double sequence $\{v_{n,k}\}_{n,k=1}^{\infty}$ in V^+ such that $v_{n,k} \downarrow_k 0$ holds for $n = 1, 2, ...$, there is an element $v' \in V^+$ with the property that, for every n, we have $v' \geq v_{n,k(n)}$ for an appropriate $k = k(n)$.

The Riesz space V is said to have the **σ-property** whenever, given the sequence $\{v_n\}_{n=1}^{\infty}$ in V^+, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ of strictly positive numbers such that the sequence $\{r_n v_n\}_{n=1}^{\infty}$ is bounded from above.

Theorem 1.3.7. *In an Archimedean Riesz space V the following properties are equivalent:*

(i) V has the diagonal property for order convergence;

(ii) V has the d-property;

(iii) V has the σ-property and order convergence in V is stable.

Proof. See theorem 70.2 of [28]. □

Theorem 1.3.8. *The Riesz space V has the strong Egoroff property if and only if V has the Egoroff property and the d-property.*

Proof. See theorem 68.4 of [28]. □

Theorem 1.3.9. *The Archimedean Riesz space V has the strong Egoroff property if and only if V has d-property.*

Proof. See theorem 68.5 of [28]. □

Theorem 1.3.10. *The following conditions for Archimedean Riesz space V are equivalent:*

(i) V has the σ-property;

(ii) V has the diagonal property for relatively uniform convergence;

(iii) V has the diagonal gap property for relatively uniform convergence.

Proof. See theorem 72.2 of [28]. □

1.3.2 Complex Riesz Spaces

Let V be a (real) Riesz space and let $V+iV$ be its complexification. The space $V+iV$ can be partially ordered coordinatewise, i.e, $v_1+iu_1 \leq v_2+iu_2$ whenever $v_1 \leq u_1$ and $v_2 \leq u_2$. Then $V+iV$ is a Riesz space and, for $w = v+iu($ v and u in $V)$, the element $|w|$ is given by $|w| = |v| + i|u|$.

If V is an arbitrary Riesz space and $w = v + iu$ is any element of $V + iV$, we wish to define an absolute value $|w|$ of w such that $|w| \in V^+$ and such that if w itself is an element of V, then $|w| = w \vee (-w)$. We have the following theorem.

Theorem 1.3.11. *If the Riesz space V is Archimedian and uniformly* *complete, then*

$$|w| = \sup\{Re(we^{-i\theta}) \ : \quad 0 \leq \theta \leq 2\pi\} = \sup(v\cos(\theta) + u\sin(\theta) \ \ ; 0 \leq \theta \leq 2\pi) \tag{1.3.4}$$

exists in V for every $w = v + iu \in V + iV$.

Proof. See theorem 13.4 of [54]. □

Definition 1.3.17. A subset V of $V + iV$ is called an **ideal** if V is a complex linear subspace of $V + iV$ and if V is a solid subspace of V. The set of all real elements in the ideal V is denoted by V_r, i.e., $\mathsf{V}_r = \mathsf{V} \cap V$. The ideal V in $V + iV$ is called a **band** if the real part of V_r is a band in V. The band V in $V + iV$ is called a **projection band** if V_r is a projection band in V.

1.3.3 Banach Lattices

Definition 1.3.18. Let V be a real Riesz space, equipped with a norm. The norm in V is called a **Riesz norm** if $|u| \leq |v|$ in V implies $\|u\| \leq \|v\|$. Note that this implies that for any $v \in V$ the elements v amd $|v|$ have the same norm. Any Riesz space equipped with a Riesz norm, is called a **normed Riesz space**. If the normed Riesz space V is norm complete then V is called **Banach lattice**.

Definition 1.3.19. Let V be a uniformly complete normed Riesz space. Every elemnt $w = u + iv$ in $V + iV$, with $u, v \in V$, has an absolute value $|w|$ in V. For this element w we define the number $\|w\|$ by $\|w\| = \||w|\|$ as its norm. This norm is normaly a Riesz norm on $V + iV$. If V is Banach lattice then $V + iV$ will be Banach space which is called a **complex Banach lattice**.

Definition 1.3.20. A normed Riesz space E is said to have **order continuous norm** if, for any subset $D \downarrow 0$ we have $\inf(\|d\| \ , \ d \in D) = 0$. The norm is said to be σ-**order continuous** norm if, for any sequence $e_n \downarrow 0$ in E, we have $\|e_n\| \downarrow 0$.

Definition 1.3.21. The Dedekind σ-complete Riesz space is called **a space of countable type** if every bounded subset of pairwise disjoint elements of it which are different from 0, is at most countable.

The norm in a Dedekind complete Banach lattice is **Fatou** if $0 \leq x_\tau \uparrow x$ implies $\|x_\tau\| \uparrow \|x\|$. Equivalently if x_τ converges to x in order, then $\|x\| \leq \liminf_\tau \|x_\tau\|$. Also the norm is **weakly Fatou** if there is a constant $m \geq 1$ such that $0 \leq x_\tau \uparrow x$ implies $\|x\| \leq m \sup \|x_\tau\|$.

Definition 1.3.22. A Riesz norm on a Riesz space is an M-**norm** if $x, y \geq 0$ implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ and is an L-**norm** if $x, y \geq 0$ implies $\|x + y\| = \|x\| + \|y\|$. A normed Riesz space with an M-norm (resp. an L-norm) is called an M-**space** (resp. an L-**space**). A norm complete M-space is called an AM-**space**. Similarly a norm complete L-space is an AL-**space**.

Proposition 1.3.12. *Let V be an Archimedian vector lattic possessing an order unit e. The gauge function of $[-e, e]$, given as*

$$P_e(v) = \inf\{r \in \mathbb{R} \ : \quad -re \leq v \leq re\} \qquad (v \in V) \tag{1.3.5}$$

is an M-norm on V. (V, P_e) is an AM-space (with unit e) if and only if V is relatively uniformly complete.

Proof. See proposition 7.2 of chapter 2 of [41]. □

Definition 1.3.23. An ordered normed space $(V, C, \| \ \ \|)$ is called an *order-unit normed space* if there exists an order unit e such that $\| \ \ \|$ is the gauge of $[-e, e]$. Also $(V, C, \| \ \ \|)$ is an *approximate order-unit normed space* if there is an approximate order unit $\{e_\tau, \tau \in T, \geq\}$ in C such that the given norm $\| \ \ \|$ is the gauge of the circled convex set $S_\tau = \cup\{[-e_\tau, e_\tau] \ : \ \tau \in T\}$.

1.3.4 Functional Calculus on Riesz spaces

Let E be a Riesz space having the principal projection property. Let $0 < e \in E$ be strong order unit in E, hence, the principal ideal J_e generated by e is the whole space E, i. e., $J_e = E$. Given the element $f \in E = J_e$, there exist real numbers a, b with $a < b$ and a number $\delta > 0$ such that $ae \leq f \leq (b-\delta)e$. The interval $[a, b]$ is then sometimes called a **spectral interval** of f. Let $\mathsf{P} : a = k_0 < k_1 < \cdots < k_n = b$ be a partition of $[a, b]$. The elements

$$s = \sum_{i=1}^{i=n} k_{i-1}(P_{k_i} - P_{k_{i-1}})e \quad \text{and} \quad S = \sum_{i=1}^{i=n} k_i(P_{k_i} - P_{k_{i-1}})e$$

are called the **lower sum** and **upper sum** belonging to f and the partition where $s \leq f \leq S$. If $k_i - k_{i-1} \leq \epsilon$ for $i = 1, 2, \ldots, n$, then $\leq S - s \leq \epsilon e$, so $0 \leq f - s \leq \epsilon e$ and $0 \leq S - f \leq \epsilon e$.

Let F be a real continuous function on the spectral interval $[a, b]$ and let, for $i = 1, 2, \ldots, n$, the numbers

$$m_j = \min_{k_{j-1} \leq k \leq k_j} F(k) \qquad M_j = \max_{k_{j-1} \leq k \leq k_j} F(k) \tag{1.3.6}$$

be the minimum and maximum of F in the interval $[k_{i-1}, k_i]$ of the partition P of $[a, b]$ respectively. In rest, we sall sometimes write $s(\mathsf{P})$ and $S(\mathsf{P})$ to denote that s and S depend on P.

Theorem 1.3.13. *Let e be a strong order unit in the Dedekind σ-complete space E, and $f \in E$ be fixed. Let $[a, b]$ be a spectral interval for f and F be a real continuous function on $[a, b]$. For any partition P of $[a, b]$, let $s = s(\mathsf{P})$ and $S = S(\mathsf{P})$ be the corresponding lower and upper sums (for f and P). Then the set of all $s(\mathsf{P})$, for all possible partitions P of $[a, b]$, has a supremum in E which is at the same time the infimum of all possible $S(\mathsf{P})$. We shall denote this element by $F(f)$.*

Proof. See theorem 34.1 of [54] □

In a similar way of integration since we have upper sums and lower sums, this common value is denoted often with $F(f) = \int_a^b F(k)dp_k$.

This method of defining $F(f)$ for any $f \in E$ and any (real) continuous function F on $[a, b]$ is an example of what is known as a **functional calculus**.

If n is a natural number and $F(k) = k^n$ for all $k \in [a, b]$, then f^n can be define easily by method of functional calculus. Since for f and g in E the elements f^2, g^2 and $(f + g)^2$ are now defined, one might try to define the product fg by

$$fg = \frac{1}{2}\{(f + g)^2 - f^2 - g^2\}.$$

The properties of this new multiplication are given in section 35 of [54] which we list them without proof.

(i) The multiplication is associative, i.e., $f(g + h) = fg + fh$ and $(g + h)f = gf + hf$ for any $f, g, h \in E$;

(ii) The order unit e in E is a multiplicative unit element;

(iii)The multiplication is positive, i. e., if f and g are positive, then fg is positive;

(iv) The multiplication is commutative, i. e., $fg = gf$ for all $f, g \in E$;

(v) The multiplication is associative, i. e., $(fg)h = f(gh)$ for all $f, g \in E$;

(vi) $|fg| \leq |f||g|$ for all $f, g \in E$;

(vii) If $g \perp h$, then $fg \perp h$ for all $f \in E$;

(viii) For any component s of e we have $s^2 = s$ (and so $s^n = s$ for $n = 2, 3, \dots$);

(ix) If $f = \sum r_k s_k$ and $g = \sum r'_k s_k$ are e-step functions with all components s_k pairwise disjoint, then $fg = \sum r_k r'_k s_k$;

(x) $f \perp g$ if and only if $fg = 0$;

(xi) If $f^2 = 0$, then $f = 0$. Moreover, if $f^n = 0$ for some positive integer n, then $f = 0$;

(xii) For any $f \in E$ we have $f^+f^- = f^-f^+ = 0$ and hence

$$f^2 = (f^+ - f^-)^2 = (f^+)^2 + (f^-)^2 \geq 0;$$

(xiii) It follows from e-uniform convergence of f_n to f and g_n to g that $f_n g_n$ e-uniformly converge to fg;

(xiv) If $f \in E$ and there exists an element $g \in E$ such that $fg = gf = e$ (e is strong order unit), then g is called an **inverse** of f and g is then denoted by f^{-1}. If f^{-1} exist then it is unique;

(xv) If $f \geq 0$ and f^{-1} exists, then $f^{-1} \geq 0$. Also for $f \geq 0$ which f^{-1} exists,$f \geq re$ for some positive number r. Conversly, any $f \in E$ satisfying $f \geq re$ for some $r > 0$ has an inverse.

Theorem 1.3.14. *Let $e > 0$ be given in the Dedekind σ-complete Riesz space V and let the commutative multiplication in J_e with e as unit element be introduced as explained above. The multiplication is extended to the complexification $J_e + iJ_e$ in the natural manner, i.e.,*

$$(u + iv)(w + ix) = (uw - vx) + i(ux + vw).$$

Then, for any $w = u + iv \in J_e + iJ_e$ we have $|w|^2 = u^2 + v^2$, i.e., $|w|$ is the unique positive square root of $u^2 + v^2$.

Proof. See theorem 44.4 of [54]. □

Remark 1.3.1. The multiplication in $J_e + iJ_e$ is commutative with e as unit element. The further properties of it are given in corollary 44.5 of [54].

1.3.5 Riesz Algebra

In this subsection we give the basic definitions and results of Riesz algebra which are given in [19].

Definition 1.3.24. A(real) **Riesz algebra** X is an algebra and Riesz space with the additional property that the multiplication and ordering are compatible, i. e. $x, y \in X^+ \implies xy \in X^+$ where X^+ is the positive cone of X.

C.B Hujsmans [15] proved its validity in the following instance:

Theorem 1.3.15. *Let X be a real Archimedean relatively uniformly complete Riesz algebra. Then*

$$|z_1 z_2| \leq |z_1||z_2|,$$

for all $z_1, z_2 \in X_{\mathbb{C}}$.

Theorem 1.3.16. *Let X be a Riesz algebra with a strong order unit, then*

$$|z|^2 = x^2 + y^2$$

for all $x, y \in X$.

L.Venter [47] Showed that if X is a Banach lattice algebra, then the inequality in 1.3.15 implies that

$$\|z_1 z_2\| \leq \|z_1\|\|z_2\|.$$

Now we recall some definitions and results from [8].

Definition 1.3.25. A Riesz algebra F is called an f**-algebra** if F has the additional property that $f \wedge g = 0$ in F implies

$$(fh) \wedge g = (hf) \wedge g = 0$$

for all $0 \leq h \in F$.

Some preliminary properties of f-algebras are collected below:

(i) Multiplication by a positive element of $\mathcal{A}$ is a Riesz homomorphism;

(ii) $|fg| = |f|.|g|$ for all $f, g \in F$;

(iii) $f \perp g$ implies $fg = 0$;

(iv) $f^2 = (f^+)^2 + (f^-)^2 \geq 0$ for all $f \in F$;

(v) $ff^+ = (f^+)^2 \geq 0$ for all $f \in F$;

(vi) If F is semiprime (i.e., the only nilpotent element in F is 0 or, equivalently, f^2=0 in F implies $f = 0$), then $f^2 \leq g^2$ if and only if $|f| \leq |g|$;

(vii) If F is semiprime, then $f \perp g$ if and only if $fg = 0$;

(viii) Every unital f-algebra is semiprime.

Some of the above properties characterize the class of f-algebra amongst a certain class of Riesz algebra. In fact, every semiprime Riesz algebra satisfying one of the conditions (i), (ii) or (iii) is an f-algebra. Furthermore, every unital Riesz algebra satisfying (iv) or (v) is an f-algebra as well.

Definition 1.3.26. Let F be an f-algebra.

(a) F has property (*) if for all $0 \leq f, g \in F$ satisfying $0 \leq f \leq g^2$, there exists $0 \leq h \in F$ such that $f = hg$.

(b) F is said to have the multiplicative decomposition property if it follows from $0 \leq f \leq gh$ with $0 \leq g, h \in F$ that there exists $p, q \in F$ such that $f = pq$, $0 \leq p \leq g$ and $0 \leq q \leq h$.(M.D. property).

Theorem 1.3.17. *Every uniformly completely unital f-algebra has both Property (*) and the M.D. property.*

Proof. See theorems 3.11 and 3.16 of [15]. □

Theorem 1.3.18. *Let F be a uniformly complete semiprime f-algebra and $0 \leq f, g \in F$. Then $\sqrt{fg}$ exists in F. In particular, $\sqrt{f}$ exists for all positive f of a uniformly complete unital f-algebra.*

Proof. See theorem 4.2 of [8]. □

Corollary 1.3.19. *As above, F is a uniformly complete semiprime f-algebra. Then $\sqrt{f^2+g^2}$ exists for all $0 \leq f, g \in F$. Hence $\sqrt{f^2+g^2} = \sqrt{|f^2|+|g^2|}$ exists for all $f, g \in F$.*

Proof. See corolary 4.3 of [8]. □

Theorem 1.3.20. *Let F be a uniformly complete semiprime f-algebra. Then*

$$\sqrt{f^2+g^2} = \sup_{0 \leq \theta \leq 2\pi} (f \cos\theta + g \sin\theta)$$

for all $f, g \in F$.

Proof. See theorem 5.2 of [8].

Chapter 2

INTRODUCTION TO RIESZ SUPERNUMBERS

2. INTRODUCTION TO RIESZ SUPERNUMBERS

In introduction of this work we stated some historical remarks and applications of supermathematics. This new branch has many elements such as supermanifolds, superalgebra, supergroup, superfunctions, supernumbers and etc. Also it's subbranch "Superanalysis", becomes a high developed branch in recent years. As examples of source works on it we can consider the books of Berezin [7] and Khrennikov [22]. The concept of diffrentiation and integration of functions of anti-commuting variables are studies in this area and also it contains investigation of some vector spaces over the algebra of supernumbers (as a scalar field), operators on them and its related structures such as " super Hilbert space". The inner product of these spaces are considered to be supernumber-valued which we study deeply them in the next chapter. Therefore supernumbers (as ordinary numbers) play important role in the structure of supermathematics.

The elements of infinite dimensional Grassman algebra is taken as supernumbers by Dewitt [10] and in terminology of Kobayashi and Nagamachi [24], the elements of finite and infinite dimensional σ-commutative G-graded algebras are called supernumbers. These spaces are called, spaces of supernumbers. In the contexts of supermathematics, supernumbers are used instead of ordinary numbers. Note that, the body of any element of Graßmann algebra is real or complex number. It has another part (soul) which has not similarities in ordinary numbers. Hence, supernumbers are more general than numbers. Of course, unlike of his name, they has not behavior as ordinary numbers. They are not comparable and invertiblity of them is meaningless. If a is a supernumber, what is the meaning of $\sqrt{a}$ or $|a|$? Whether $\sqrt{a}$ exist always ?

In this chapter first we give some basic definitions and notions of super structures and then investigate the algebra of supernumbers according to terminology of Kabayashi and Nagamachi [24]. Also we introduce the Riesz space, Banach lattice and Riesz algebra structure on the algebra of supernumbers and finally we study the Graßmann algebra.

2.1 Some on Graded Structures

As stated above, supermathematics has some elements such as supervector space, superalgebra, supermanifolds and etc. In present section we consider to some of them which will be used in remainder of this work. Of course we give only elementry definitions and interested readers can refer to [23, 24, 25, 42, 43].

The remainder of this thesis can be reformulate for any commutative field which its characteristic is not equal to 2.

Definition 2.1.1. Let G be a finite additive abelian group and $\mathbb{F}$ be the real or complex field. A map $\sigma : G \times G \longrightarrow \mathbb{F}$ is called the **sign** or **commutation factor** of G if it satisfies

(i) $\sigma(\alpha + \beta, \gamma) = \sigma(\alpha, \gamma)\sigma(\beta, \gamma)$

(ii) $\sigma(\alpha, \beta)\sigma(\beta, \alpha) = 1$

for any $\alpha, \beta, \gamma \in G$. The pair (G, σ) is called **signed group**.

It is easy to verify that $\sigma(\alpha, \alpha) = \pm 1$ for any $\alpha \in G$. An element α of G is called **even** (resp. **odd**) if $\sigma(\alpha, \alpha) = 1$(resp. -1). The even part $\{\alpha \in G : \sigma(\alpha, \alpha) = 1\}$ and the odd part $\{\alpha \in G : \sigma(\alpha, \alpha) = -1\}$ of G are denoted by G_0 and G_1 respectively. G_0 is a subgroup of G of index at most 2 and we have $G = G_0 \cup G_1$(disjoint union).

Definition 2.1.2. A mapping $\phi : G \times G \longrightarrow \mathbb{F} - \{0\}$ is called a **factor system** on G, if it satisfies

(i) $\phi(\alpha, \beta + \gamma)\phi(\beta, \gamma) = \phi(\alpha, \beta)\phi(\alpha + \beta, \gamma)$;

(ii) $\phi(0,0) = 1$,

for any $\alpha, \beta, \gamma \in G$. It follows from (i) that

(i)$'$ $\phi(\alpha, 0) = \phi(0, \alpha) = 1$;

(i)$''$ $\phi(\alpha, -\alpha) = \phi(-\alpha, \alpha) = \phi(\alpha, \beta)\phi(-\alpha, \alpha + \beta)$;

for any $\alpha, \beta \in G$.

Proposition 2.1.1. *Let (G, σ) be an even signed group and assume that G is finitely generated. Then there is a factor system ϕ on G such that $\sigma(\alpha, \beta) = \phi(\alpha, \beta)/\phi(\beta, \alpha)$ for $\alpha, \beta \in G$. Moreover, if $|\sigma(\alpha, \beta)| = 1$ for all $\alpha, \beta \in G$, we can choose ϕ so that $|\phi(\alpha, \beta)| = 1$ for all $\alpha, \beta \in G$.*

Proof. See [43]. □

Remark 2.1.1. If we take $G = \mathbb{Z}_2$ and $\sigma(\alpha, \beta) = (-1)^{\alpha\beta}$ for any $\alpha, \beta \in \mathbb{Z}_2$, we obtain an important signed group which has spread usage in contexts of supermathematics and theoretical physics. In what follows, either $G = \mathbb{Z}_2$, with mentioned sign, or G is a signed group in general.

Definition 2.1.3. A vector space V is said to be G**-graded** if we are given a family $(V_\alpha)_{\alpha \in G}$ of subspaces of V such that V is their direct sum, $V = \bigoplus_{\alpha \in G} V_\alpha$.

An element of V is said to be **homogeneous of grade** $\alpha \in G$ if it is an element of V_α. Let V and W be two G-graded vector spaces. A linear mapping $T : V \longrightarrow W$ is said to be **homogeneous of grade** $\alpha \in G$ if $T(V_\beta) \subset W_{\alpha+\beta}$ for all $\beta \in G$.

Let $L(V, W)$ denote the vector space of all linear mappings of V into W and let $L_\alpha(V, W)$ denote the subspace of those linear mappings of V into W which are homogeneous of grade α. We define $L_{gr}(V, W)$ to be the sum of these subspaces, obviously this sum is directed:

$$L_{gr}(V,W) = \bigoplus_{\alpha\in G} L_\alpha(V,W).$$

Thus $L_{gr}(V,W)$ is a G-graded vector space. Note that $L_{gr}(V,W)$ is equal to $L(V,W)$ if (for example) $V_\alpha = \{0\}$ and $W_\alpha = \{0\}$ for all but a finite number of degrees [43]. In the case where $V = W$ and $V_\alpha = W_\alpha$ for all $\alpha \in G$, we shall simplify the notations and write $L(V)$ and $L_{gr}(V)$ instead of $L(V,V)$ and $L_{gr}(V,V)$, respectively.

Let U, V and W be three G-graded vector spaces and let $h : U \longrightarrow V$ and $k : V \longrightarrow W$ be two linear mappings. If h is homogeneous of grade α and k is homogeneous of grade β, then koh is homogeneous of degree $\alpha + \beta$.

Definition 2.1.4. An algebra $\mathcal{A}$ is called G**-graded algebra** if $\mathcal{A}$ has direct sum decomposition $\mathcal{A} = \bigoplus_{\alpha\in G} \mathcal{A}_\alpha$ where $\mathcal{A}_\alpha$ is a subalgebra of $\mathcal{A}$ of grade α for any $\alpha \in G$ with additional condition that $\mathcal{A}_\alpha\mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta}$ for all $\alpha, \beta \in G$.

A G-graded (associative) algebra $\mathcal{A} = \bigoplus_{\alpha\in G} \mathcal{A}_\alpha$ over $\mathbb{F}$ is called σ**-commutative** algebra if $ab = \sigma(\alpha,\beta)ba$ holds for any $a \in \mathcal{A}_\alpha,\ b \in \mathcal{A}_\beta$ and $\alpha, \beta \in G$.

It is important to note that in the case char $\mathbb{F}$=2 we must add the condition $a^2 = \sigma(\alpha,\alpha)a^2$ for any $a \in \mathcal{A}_\alpha$ [43].

Example 2.1.1. *As an important example of a G-graded algebra which is used frequently, is Graßmann algebra. The Graßmann algebra(or exterior algebra)Λ_n with n generators is the associative algebra (over $\mathbb{C}$) generated by a set of n anticommuting generators $\{\xi_i\}_{i=1}^n$ and by $1 \in \mathbb{C}$ with the property*

$$\xi_i\xi_j = -\xi_j\xi_i \qquad \textit{for all } i,j, \tag{2.1.1}$$

in particular $\xi_i^2 = 0$. This algebra is $\mathbb{Z}_2$-graded algebra which we will investigate it later.

Definition 2.1.5. Graded tensor product

For two G-graded algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{F}$, the G-graded vector space $\mathcal{A}\otimes\mathcal{B} = \bigoplus_{\alpha\in G}(\bigoplus_{\beta+\gamma=\alpha}(\mathcal{A}_\beta\otimes\mathcal{B}_\gamma))$ is a G-graded algebra if, we define the multiplication by $(a\otimes b).(c\otimes d) = \sigma(\beta,\gamma)(ac\otimes bd)$ for $\beta,\gamma\in G$ and $a\in\mathcal{A},\ b\in\mathcal{B}_\beta,\ c\in\mathcal{A}_\gamma$ and $d\in\mathcal{B}$. The algebra $\mathcal{A}\otimes\mathcal{B}$ is called the **graded tensor product** of $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{F}$. If $\mathcal{A}$ and $\mathcal{B}$ are σ-commutative, so is $\mathcal{A}\otimes\mathcal{B}$.

Now, let V be a G-graded vector space. Then the G-graded vector space $L_{gr}(V)$, equipped in addition with the usual multiplication (i. e., composition) of linear mappings, is a G-graded algebra.

Definition 2.1.6. A G-graded algebra $\mathcal{A} = \bigoplus_{\alpha\in G}\mathcal{A}_\alpha$ over $\mathbb{F}$ is called a **G-graded Banach algebra** if it satisfies

(i) $\mathcal{A}_\alpha$ is a complete normed space, for any $\alpha\in G$;

(ii) for any $\alpha,\beta\in G$ and $a\in\mathcal{A}_\alpha,\ b\in\mathcal{A}_\beta$ we have $\|ab\|\leq\|a\|\|b\|$.

Definition 2.1.7. Tensor Algebra

Let $V = \bigoplus_{\alpha\in G}V_\alpha$ be a G-graded vector space over $\mathbb{F}$ and $T(V)$ be tensor algebra of V over $\mathbb{F}$. As is well-known, $T(V)$ has a natural $\mathbb{Z}\times G$-gradation which is fixed by the condition that the grade of a tensor $v_1\otimes\cdots\otimes v_n$, with $v_i\in V_{\alpha_i},\ \ \alpha_i\in G$ for $1\leq i\leq n$, is equal to $(n,\alpha_1,\ldots,\alpha_n)$. The subspace of $T(V)$ consisting of the homogeneous tensors of order $n\in\mathbb{Z}$ will be denoted by $T_n(V)$; of course, $T_n(V)=\{0\}$ if $n\leq -1$. Let I be the ideal of $T(V)$ generated by the elements of the form $x\otimes y-\sigma(\alpha,\beta).y\otimes x$ where $\alpha,\beta\in G$ and $x\in V_\alpha,\ y\in V_\beta$. The quotient algebra $\tilde{U}(V) = T(V)/I$ is a σ-commutative algebra.

Definition 2.1.8. Crossed Product

If a factor system ϕ on G is given, we can construct a G-graded σ-commutative algebra $C = \bigoplus_{\alpha\in G}C_\alpha$ over $\mathbb{F}$, called the **crossed product** of $\mathbb{F}$ and G as follows:

$C_\alpha = \mathbb{F}.u_\alpha$ is the one dimensional vector space over $\mathbb{F}$ with a generator u_α of grade α. The multiplication in C is given by $u_\alpha.u_\beta = \phi(\alpha,\beta)u_{\alpha+\beta}$ for $\alpha,\beta \in G$. By 2.1.2 (i)$'$, u_0 is an identity element of C.

Let G_0 be the subgroup of even elements of the signed group (G,σ). Since $\sigma|_{G_0}$ is an even sign, there is a factor system $\phi : G_0 \times G_0 \longrightarrow \mathbb{F}-\{0\}$ associated with $\sigma|_{G_0}$, that is, ϕ satisfies

(i) $\phi(\alpha,\beta+\gamma)\phi(\beta,\gamma) = \phi(\alpha,\beta)\phi(\alpha+\beta,\gamma)$

(ii) $\phi(0,0) = 1$

(iii) $\phi(\alpha,\beta)/\phi(\beta,\alpha) = \sigma(\alpha,\beta)$

where $\alpha,\beta,\gamma \in G_0$. We can choose ϕ so that $|\phi(\alpha,\beta)| = 1$ (proposition 2.1.1).

Let $C = \bigoplus_{\alpha\in G_0} C_\alpha$ be the crossed product of $\mathbb{F}$ and G_0 by means of ϕ. Then C is a σ-commutative algebra over $\mathbb{F}$ by (iii) of above.

Definition 2.1.9. Involution of a graded algebra

Let $\mathcal{A} = \bigoplus_{\alpha\in G} \mathcal{A}_\alpha$ be a G-graded σ-commutative algebra over $\mathbb{C}$, which means that, for any $a \in \mathcal{A}_\alpha$ and $b \in \mathcal{A}_\beta$ we have $ab = \sigma(\alpha,\beta)ba$.

A mapping $* : \mathcal{A} \longrightarrow \mathcal{A}$ is an involution of $\mathcal{A}$, if it satisfies the following conditions:

(i) $a^{**} = a$;

(ii) $(a+b)^* = a^* + b^*$;

(iii) $(ca)^* = \bar{c}a^*$;

(iv) $(ab)^* = b^*a^*$;

for any $a,b \in \mathcal{A}$ and $c \in \mathbb{C}$.

If moreover it satisfies

(v)$a^* \in \mathcal{A}_\alpha$, for any $a \in \mathcal{A}_\alpha$ and $\alpha \in G$,

then $*$ is called a **conjugation** of $\mathcal{A}$. On the other hand if it satisfies

(vi)$a^* \in \mathcal{A}_{-\alpha}$ for any $a \in \mathcal{A}_\alpha$ and $\alpha \in G$,

then $*$ is called a **transposition** of $\mathcal{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be G-graded σ-commutative algebras with a conjugation (resp. transposition) $*$. On the graded tensor product $\mathcal{A} \otimes \mathcal{B}$ we define $*$ by $(a \otimes b)^* = \sigma(\beta, \alpha) a^* \otimes b^*$ for $a \in \mathcal{A}_\alpha, b \in \mathcal{B}_\beta, \alpha, \beta \in G$. Then $*$ is a conjugation (resp. transposition) on $\mathcal{A} \otimes \mathcal{B}$.

Definition 2.1.10. Graded modules

Let $\mathcal{A}$ be an associative G-graded algebra with identity. A **G-graded left $\mathcal{A}$-module** $\mathcal{M}$ is a G-graded vector space with a left action of $\mathcal{A}$ on $\mathcal{M}$, that is, a mapping $\mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M}$ satisfying the following conditions:

(i) $a(bm) = (ab)m$

(ii) $(a + b)m = am + bm$, $\quad a(m + m') = am + am'$

(iii) $1.m = m$

(iv) If $m \in \mathcal{M}_\alpha$, and $a \in \mathcal{A}_\beta$ then $am \in \mathcal{M}_{\alpha+\beta}$.

G-graded right $\mathcal{A}$-modules are defined in a similar way.

Definition 2.1.11. Let $\mathcal{A}$ be a G-graded algebra. A G-graded vector space V over $\mathcal{A}$ is called **generalized supervector space(GSVS)** if it has additional property that

$$av = \sigma(\alpha, \beta) va$$

for any $a \in \mathcal{A}_\alpha$ and $v \in V_\beta$.

A $\mathbb{Z}_2$-graded vector space over Graßmann algebra Λ_n with this property that

$$\lambda v = (-1)^{|\lambda||v|} v\lambda$$

for any $\lambda \in \Lambda_{n,|\lambda|}$ and $v \in V_{|v|}$ is called **supervector space(SVS)**. If V and W are GSVS(or SVS) then $L(V, W)$ is so.

Definition 2.1.12. A G-graded algebra $\mathcal{A}$ over a G-graded algebra $\mathcal{B}$ is called **generalized superalgebra(GSA)** if it satisfies in the following conditions:

(i) $ab = \sigma(\alpha, \beta)ba$ for any $a \in \mathcal{A}_\alpha$ and $b \in \mathcal{B}_\beta$;

(ii) $aa' = \sigma(\alpha, \beta)a'a$ for any $a \in \mathcal{A}_\alpha$ and $a' \in \mathcal{A}_\beta$.

A $\mathbb{Z}_2$-graded algebra over the Graßmann algebra is called **superalgebra(SA)**.

A GSA $\mathcal{A}$ is called **generalized Banach superalgebra(GBSA)** if it satisfies in the two condition of definition 2.1.6. We will see later that Graßmann algebra is an example of BSA.

2.2 Finite Dimensional Riesz Algebra of Supernumbers

In this section we will consider an important example of finite dimensional σ-commutative G-graded algebras which is introduced by Kobayashi and Nagamachi in [24]. Particular case of this algebra is Graßmann algebra which has wide usage in supermathematics and theoretical physics. We will investigate this algebra deeply in the end of this chapter. The elements of a particular σ-commutative G-graded algebra has been called **supernumbers** in [24]. Supernumbers, in spite of his name, do not behave as ordinary numbers. They are incomparable and positivity of them is meaningless. In current section, first we give the structure of the algebra of supernumbers according to [24] and define some new norms on it [5]. Then we introduce the concept of Riesz spaces on this algebra over the real and complex number fields. Next, with using of its norm, we prove that it is Banach lattice. The method of functional calculus on this Riesz space gives a new multiplication on it. Finally, we will see that it is commutative Banach algebra.

2.2.1 The Algebra of Supernumbers

Let (G, σ) be a finite additive abelian signed group with G_0 and G_1 as its even and odd parts, respectively and $\mathbb{F}$ be the real or complex number field. Also let $C = \bigoplus_{\alpha \in G_0} C_\alpha$ be the crossed product of $\mathbb{F}$ and G_0, where $C_\alpha = \mathbb{F}.u_\alpha$ is the one-dimensional vector space over $\mathbb{F}$ with generator u_α of grade α. A finite set L is called a G-**set** if L is linearly ordered and there is a map $g : L \longrightarrow G$ such that any $\ell \in L$ has a grade $g(\ell) \in G$. If $g(\ell) \in G_0$ then ℓ has an even grade and if $g(\ell) \in G_1$ then ℓ has odd grade. Let L be an odd G-set , that is, each element ℓ of L has an odd grade. Suppose that V is the G-graded vector space over $\mathbb{F}$ with basis $\{v_\ell |\ \ell \in L\}$, where the grade of v_ℓ is $g(\ell)$. Let B be the σ-commutative algebra over V defined by $B = T(V)/K$, where K is an ideal of the tensor algebra $T(V)$ over V generated by the elements $v_i v_j - \sigma(g(i), g(j)) v_j v_i$ with $i, j \in L$. A subset M of L is a G-set in a natural way. The ordered product $\prod\limits_{m \in M} v_m$ is written as v_M. Then B is a G-graded σ-commutative algebra with a linear basis $\{v_M \ : \ M \subseteq L\}$ over $\mathbb{F}$. If $\mathcal{A} = C \otimes B$ is the graded tensor product of crossed product C and B over $\mathbb{F}$, then $\mathcal{A}$ is a finite dimensional σ-commutative algebra and the elements of $\mathcal{A}$ are called **supernumbers**.

Any element a of $\mathcal{A}$ is expressed uniquely as

$$a = \sum_{\substack{M \subset L \\ \alpha \in G_0}} a_{\alpha,M} u_\alpha \otimes v_M, \tag{2.2.1}$$

where $a_{\alpha,M} \in \mathbb{F}$ and α ranges over the elements of G_0 and M the subsets of L.

For $a \in \mathcal{A}$ given as 2.2.1 we define the body and the soul of a by

$$b(a) = \sum_{\alpha \in G_0} a_{\alpha,\phi} u_\alpha \otimes 1 \quad \text{and} \quad S(a) = \sum_{\substack{\alpha \in G_0 \\ \phi \neq M \subset L}} a_{\alpha,M} u_\alpha \otimes v_M. \tag{2.2.2}$$

Any element of $\mathcal{A}$ is uniquely decomposed as a sum of its body and its soul. The body of a is invertible if it is nonzero and homogeneous while the soul of a is nilpotent. Let $b = \sum\limits_{\beta,N} b_{\beta,N} u_\beta \otimes v_N$ be another element of $\mathcal{A}$. Then the operations additivity, scalar multiplicativity and multiplicativity are defined as follows:

$$a + b = \sum_{\alpha,M} (a_{\alpha,M} + b_{\alpha,M}) u_\alpha \otimes v_M \ ; \tag{2.2.3}$$

$$ra = \sum_{\alpha,M} (r a_{\alpha,M}) u_\alpha \otimes v_M \ ; \qquad r \in \mathbb{F} \tag{2.2.4}$$

$$ab = \sum_{\gamma,K} \Big(\sum_{\substack{M \cup N = K \\ \alpha + \beta = \gamma}} a_{\alpha,M} b_{\beta,N} \varepsilon_{\alpha,\beta,M,N} \Big) u_\gamma \otimes v_K \ ; \tag{2.2.5}$$

where $M \cap N = \phi$ and $\varepsilon_{\alpha,\beta,M,N}$ are elements of $\mathbb{F}$ with absolute value 1.

For a subset A of $\mathcal{A}$ we set $b(\mathsf{A}) = \{b(a) \ : a \in \mathsf{A}\}$ and $s(\mathsf{A}) = \{s(a) \ : a \in \mathsf{A}\}$ which are called body and soul of A respectively.

With returning to the definition 2.1.4, $\mathcal{A}$ is a G-graded algebra whose homogeneous components $\mathcal{A}_\alpha$ of grade $\alpha \in G$ is the set of elements

$$a_\alpha = \sum_{\substack{M \subset L \\ \beta \in G_0}} a_{\beta,M} u_\beta \otimes v_{_M}$$

where $\beta + g(M) = \alpha$ with $g(M) = \sum_{\ell \in M} g(\ell)$.

Example 2.2.1. *An important example of a G-graded σ-commutative algebra is the Graßmann algebra over $\mathbb{F}$ which obtains with letting $G = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}, \sigma(\alpha, \beta) = (-1)^\alpha \beta$ for $\alpha, \beta \in \mathbb{Z}_2$ and $L = \{1, 2, ..., n\}$. Therefore we will have $G_0 = \{0\}$ and $C = \mathbb{F}$, the crossed product. This algebra is denoted by Λ_n and has a base $\{\xi_1, ..., \xi_n\}$ such that $\xi_i \xi_j = -\xi_j \xi_i$ for $i, j = 1, 2, ..., n$ and $i \neq j$. Every element λ of Λ_n is represented as*

$$\begin{aligned} \lambda &= \lambda_0 + \sum_{k=1}^{n} \lambda_k \xi_k + \sum_{\substack{k_1, k_2 \in L \\ k_1 < k_2}} \lambda_{k_1 k_2} \xi_{k_1} \xi_{k_2} + \cdots \\ &+ \sum_{\substack{k_1, \ldots, k_i \\ k_1 < \ldots < k_i}} \lambda_{k_1 k_2 \ldots k_i} \xi_{k_1} \xi_{k_2} \ldots \xi_{k_i} + \cdots + \lambda_{12 \ldots n} \xi_1 \xi_2 \ldots \xi_n \end{aligned}$$

where $\lambda_{k_1 k_2 \ldots k_i} \in \mathbb{F}$ for any $1 \leq i \leq n$. Note that the body of λ is λ_0, i.e, $b(\lambda) = \lambda_0 \in \mathbb{F}$.

For $a \in \mathcal{A}$ and $1 \leq k < \infty$ let

$$\|a\|_k = \{\sum_{\alpha, M} |a_{\alpha,M}|^k\}^{\frac{1}{k}} \tag{2.2.6}$$

and for $k = \infty$ define

$$\|a\|_\infty = \sup_{\alpha, M} |a_{\alpha,M}|. \tag{2.2.7}$$

This norms make $\mathcal{A}$ to be a Banach space which is proved in the following proposition.

Proposition 2.2.1. *The algebra $\mathcal{A}$ with the norm $\| \quad \|_k$, for $1 \leq k \leq \infty$, is Banach space.*

Proof. Let $1 \leq k < \infty$ and $a, b \in \mathcal{A}$ which have representations

$$a = \sum_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M \qquad \text{and} \qquad b = \sum_{\alpha,M} b_{\alpha,M} u_\alpha \otimes v_M.$$

First we prove that $\|.\|_k$ defines a norm on $\mathcal{A}$. It is clear that $\|a\|_k \geq 0$ for any a in $\mathcal{A}$. Let $a = 0$, this means that $a_{\alpha,M} = 0$ for any α, M which implies that $\|a\|_k = 0$. Conversely if $\|a\|_k = 0$ then $\sum_{\alpha,M} |a_{\alpha,M}|^k = 0$. Since L and G are finite sets then $|a_{\alpha,M}|^k = 0$ for any α, M and so $a = 0$. Now for $a, b \in \mathcal{A}$, with using the Minkowski inequality we have

$$\begin{aligned} \|a+b\|_k &= \{\sum_{\alpha,M} |a_{\alpha,M} + b_{\alpha,M}|^k\}^{\frac{1}{k}} \\ &\leq \{\sum_{\alpha,M} |a_{\alpha,M}|^k\}^{\frac{1}{k}} + \{\sum_{\alpha,M} |b_{\alpha,M}|^k\}^{\frac{1}{k}} \\ &= \|a\|_k + \|b\|_k. \end{aligned}$$

Also for any $a \in \mathcal{A}$ and $r \in \mathbb{F}$ it is clear that $\|ra\|_k = |r|\|a\|_k$. Therefore $\|.\|_k$ defines a norm on $\mathcal{A}$ that changes it to be a normed space. Now we prove that this normed space is complete. For this we must prove that every Cauchy sequence of elements of $\mathcal{A}$, as $\{a_n\}_{n=1}^{\infty}$, converges to an element a of $\mathcal{A}$.

For any $n \geq 1$, a_n has representation as $a_n = \sum_{\alpha,M} a^{(n)}_{\alpha,M} u_\alpha \otimes v_M$ and we have

$$\forall \varepsilon \geq 0 \quad \exists N \geq 0 \quad \forall m, n \in \mathbb{N} \quad \text{which } \ m, n > N; \ \ \|a_n - a_m\|_k \leq \varepsilon.$$

and so$\|a_n - a_m\|_k^k = \sum_{\alpha,M} |a^{(n)}_{\alpha,M} - a^{(n)}_{\alpha,M}|^k \leq \varepsilon^k$. Therefore $|a^{(n)}_{\alpha,M} - a^{(m)}_{\alpha,M}|^k \leq \varepsilon^k$ for any α, M. Since $\mathbb{F}$ is complete, so there is $a_{\alpha,M} \in \mathbb{F}$ such that $|a^{(n)}_{\alpha,M} - a_{\alpha,M}| \leq \varepsilon$ for any α, M.

Now let $a = \sum_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$. Therefore we have

$$\|a_n - a\|_k^k = \sum_{\alpha,M} |a^n_{\alpha,M} - a_{\alpha,M}|^k \leq \sum_{\alpha,M} \varepsilon^k.$$

Since L and G_0 are finite sets, there exists $p \geq 0$ such that $\sum_{\alpha,M} \varepsilon^k = p\varepsilon^k$. By taking convenient ε we obtain $a_n \longrightarrow a \in \mathcal{A}$. Therefore $\mathcal{A}^k$ is complete and so is Banach space for $1 \leq k < \infty$.

For the case $k = \infty$, first we prove that $\|.\|_\infty$ defines a norm on $\mathcal{A}$. Positivity is clear and also it is obvious that if $a = 0$ then $\|a\|_\infty = 0$.

Let $a \in \mathcal{A}$ and $\|a\|_\infty = 0$, this means that the supremum of a set of positive real numbers is zero and it is possible only when all of them are zero , that is, for any α, M we have $a_{\alpha,M} = 0$ which implies $a = 0$.

Now if $a, b \in \mathcal{A}$ we can write

$$\|a + b\|_\infty = \sup_{\alpha,M} |a_{\alpha,M} + b_{\alpha,M}| \leq \sup_{\alpha,M} |a_{\alpha,M}| + \sup_{\alpha,M} |b_{\alpha,M}| = \|a\|_\infty + \|b\|_\infty.$$

Also for $r \in \mathbb{F}$ and $a \in \mathcal{A}$ we have

$$\|ra\|_\infty = \sup_{\alpha,M} |ra_{\alpha,M}| = |r| \sup_{\alpha,M} |a_{\alpha,M}| = |r|\|a\|_\infty.$$

Therefore the algebra $\mathcal{A}$ with $\|.\|_\infty$ is also normed space.

For completeness, let $\{a_n\}_{n=1}^\infty$ be a Cauchy sequence of elements of $\mathcal{A}$, that is

$$\forall \varepsilon \geq 0 \quad \exists N \geq 0 \ ; \ \forall m, n \in \mathbb{N} \text{ which } m, n > N \ ; \quad \|a_n - a_m\|_\infty \leq \varepsilon,$$

where $a_n = \sum_{\alpha,M} a^{(n)}_{\alpha,M} u_\alpha \otimes v_M$. Now

$$\|a_n - a_m\|_\infty \leq \varepsilon \implies \sup_{\alpha,M} |a^{(n)}_{\alpha,M} - a^{(m)}_{\alpha,M}| \leq \varepsilon \implies \forall \alpha, M; \quad |a^{(n)}_{\alpha,M} - a^{(m)}_{\alpha,M}| \leq \varepsilon.$$

This implies that the sequence $\{a^{(n)}_{\alpha,M}\}_{n=1}^\infty$ is Cauchy in $\mathbb{F}$, then by completeness of $\mathbb{F}$, there is $a_{\alpha,M} \in \mathbb{F}$ such that $a^{(n)}_{\alpha,M} \longrightarrow a_{\alpha,M}$ for any α, M. So

$|a^{(n)}_{\alpha,M} - a_{\alpha,M}| < \varepsilon$. Now by letting $a = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$, we will have $\|a_n - a_m\|_\infty = \sup\limits_{\alpha,M} |a^{(n)}_{\alpha,M} - a_{\alpha,M}| \leq \varepsilon$.

This obtain completeness of $\mathcal{A}$ with this norm. Therefore $\mathcal{A}$ is Banach space.

□

In 2.2.6 by taking $k = 1$ we obtain easily that $\mathcal{A}$ with this norm is Banach algebra. To see this let $a = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$ and $b = \sum\limits_{\beta,N} b_{\beta,N} u_\beta \otimes v_N$ be any elements of $\mathcal{A}$. We know that $ab = \sum_{\gamma,K} (\sum_{\substack{M \cup N = K \\ \alpha+\beta=\gamma}} a_{\alpha,M} b_{\beta,N} \varepsilon_{\alpha,\beta,M,N}) u_\gamma \otimes v_K$ and we have

$$\begin{aligned}
\|ab\|_1 &= \sum_{\gamma,K} |\sum_{\substack{M \cup N = K \\ \alpha+\beta=\gamma}} a_{\alpha,M} b_{\beta,N} \varepsilon_{\alpha,\beta,M,N}| \\
&\leq \sum_{\gamma,K} \sum_{\substack{M \cup N = K \\ \alpha+\beta=\gamma}} |a_{\alpha,M}||b_{\alpha,M}||\varepsilon_{\alpha,\beta,M,N}| \\
&\leq \{\sum_{\alpha,M} |a_{\alpha,M}|\}\{\sum_{\beta,N} |b_{\beta,N}|\} \\
&= \|a\|_1 \|b\|_1.
\end{aligned}$$

By considering to the structure of $\mathcal{A}$ it is observes that the norms $\| \ \|_k$ for $1 \leq k \leq \infty$, defined as in 2.2.6 and 2.2.7, are depend to the choice of the basis u_α for C_α and $\{v_l \ : \ l \in L\}$ for G-graded vector space V and we can obtain another norms by taking different basis for C and V. Now by changing the structure of $\mathcal{A}$, we want to define another norm on $\mathcal{A}$ which be independent from the choice of basis. For this purpose by assuming the first hypothesis, let

$$[M] = \{M' \subseteq L \ : \ \text{card}M' = \text{card}M\}$$

which card M means cardinal number of M. For any $M \subseteq L$, the set $[M]$ is equivalence class of M and the collection of these equivalence classes make a partition for L. Let $V_M = \{ \sum\limits_{M' \in [M]} a_{M'} v_{M'} \ : \ a_{M'} \in \mathbb{F}\}$. Also for $\alpha \in G_0$ set

$$\begin{aligned} V_{\alpha,M} &= C_\alpha \otimes V_M \\ &= \{v_{\alpha,M} = \sum_{M' \in [M]} a_{\alpha,M'} u_\alpha \otimes v_{M'} \ : \ a_{\alpha.M'} = c_\alpha a_{M'} \in \mathbb{F}\}. \end{aligned}$$

Note that $V_{\alpha,M}$ is a vector space and since $v_{M'}$'s for $M' \in [M]$ are linearly independent, $v_{\alpha,M} = 0$ implies that $a_{\alpha,M} = 0$ for any $M' \in [M]$. Now by letting $\mathcal{A} = \bigoplus\limits_{\alpha,M} V_{\alpha,M}$, where M ranges over all elements of partition, we will have the same finite dimensional σ-commutative G-graded algebra of supernumbers. Every element of $\mathcal{A}$ can be written as $a = \sum\limits_{\alpha,M} v_{\alpha,M}$. For defining a new norm on $\mathcal{A}$ first consider $\| \ \ \|_{\alpha,M}$ on $V_{\alpha,M}$ as follows:

$$\|v_{\alpha,M}\|_{\alpha,M} = \inf\{ \sum_{M' \in [M]} |a_{\alpha,M'}| : \ v_{\alpha,M} = \sum_{\alpha,M'} a_{\alpha,M'} u_\alpha \otimes v_{M'} \ ; a_{\alpha,M'} \in \mathbb{F}\} \tag{2.2.8}$$

where infimum is taken over all possible choices of the set of generators of the G-graded vector space V and all possible choices of the generators u_α of grade α for C_α. Now for $a \in \mathcal{A}$, which has the form $a = \sum\limits_{\alpha,M} v_{\alpha,M}$, define

$$m(a) = \sum_{\alpha,M} \|v_{\alpha,M}\|_{\alpha,M}. \tag{2.2.9}$$

Proposition 2.2.2. *The equation 2.2.9 defines a norm on $\mathcal{A}$ and $\mathcal{A}$ with this norm is a Banach space.*

Proof. For proving the first part of proposition, it suffices to prove that the relation 2.2.8 is a norm on $V_{\alpha,M}$. Let $u_{\alpha,M}$ and $v_{\alpha,M}$ be two elements of $V_{\alpha,M}$. They has representation as $u_{\alpha,M} = \sum\limits_{M' \in [M]} a_{\alpha,M'} u_\alpha \otimes v_{M'}$ and

$v_{\alpha,M} = \sum\limits_{M'\in[M]} b_{\alpha,M'} u_\alpha \otimes v_{M'}$. Therefore

$$\begin{aligned}
\|u_{\alpha,M} + v_{\alpha,M}\|_{\alpha,M} &= \inf\{\sum_{M'\in[M]} |a_{\alpha,M'} + b_{\alpha,M'}|\} \\
&\leq \inf\{\sum_{M'\in[M]} |a_{\alpha,M'}| + \sum_{M'\in[M]} |b_{\alpha,M'}|\} \\
&\leq \inf\{\sum_{M'\in[M]} |a_{\alpha,M'}|\} + \inf\{\sum_{M'\in[M]} |b_{\alpha,M'}|\} \\
&= \|u_{\alpha,M}\|_{\alpha,M} + \|v_{\alpha,M}\|_{\alpha,M}.
\end{aligned}$$

We can easily see that if $r \in \mathbb{R}$ then $\|ru_{\alpha,M}\|_{\alpha,M} = |r|\|u_{\alpha,M}\|_{\alpha,M}$ and also if $u_{\alpha,M} = 0$ then $\|u_{\alpha,M}\|_{\alpha,M} = 0$. Conversly if $\|u_{\alpha,M}\|_{\alpha,M} = 0$ then for any $\varepsilon > 0$ there is a representation $u_{\alpha,M} = \sum\limits_{M'\in[M]} a_{\alpha,M'} u_\alpha \otimes v_{M'}$ such that $\sum\limits_{M'\in[M]} |a_{\alpha,M'}| < \varepsilon$. We can choose ε to be sufficiently small and hence $|a_{\alpha,M'}| < \varepsilon$ for any $M' \in [M]$. This implies that $a_{\alpha,M'} = 0$ for any $M' \in [M]$ and so $u_{\alpha,M} = 0$. Therefore $\| \quad \|_{\alpha,M}$ defines a norm on $V_{\alpha,M}$. Now we prove that $\| \quad \|_{\alpha,M}$ is complete norm on $V_{\alpha,M}$. For this let $\{v^{(n)}_{\alpha,M}\}_{n=1}^{\infty}$ be any Cauchy sequence in $V_{\alpha,M}$. Hence for each $\varepsilon \geq 0$, there is a positive integer N such that for each $m, n \in \mathbb{N}$ with $m, n > N$ we have $\|v^{(n)}_{\alpha,M} - v^{(m)}_{\alpha,M}\|_{\alpha,M} \leq \varepsilon$. This implies that there is $v^{(n)}_{\alpha,M} = \sum\limits_{M'\in[M]} b^{(n)}_{\alpha,M'} u_\alpha \otimes v_{M'}$ which for any M', $\{b^{(n)}_{\alpha,M'}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$. The completeness of $\mathbb{F}$ implies that

$$\forall M' \in [M] \quad \exists b_{\alpha,M'} \in \mathbb{F} \quad \text{which } |b^{(n)}_{\alpha,M'} - b_{\alpha,M'}| \leq \varepsilon.$$

By choosing a suitable ε we can have $\sum\limits_{M'\in[M]} |b^{(n)}_{\alpha,M'} - b^{(m)}_{\alpha,M'}| \leq \varepsilon$. Now let $v_{\alpha,M} = \sum\limits_{M'\in[M]} b_{\alpha,M'} u_\alpha \otimes v_{M'}$. Since at first ε was arbitrary, we have $\|v^{(n)}_{\alpha,M} - v^{(m)}_{\alpha,M}\|_{\alpha,M} \leq \varepsilon$, that is, $\{v^{(n)}_{\alpha,M}\}_{n=1}^{\infty}$ converges to $v_{\alpha,M}$ and so $\| \quad \|_{\alpha,M}$ is a complete norm on $V_{\alpha,M}$.

Now m, defined in 2.2.9, is a finite sum of complete norms. Therefore it is complete norm on $\mathcal{A}$ and $\mathcal{A}$ with this norm is Banach space. □

This norm is called **mass norm** on $\mathcal{A}$ and $\mathcal{A}$ with this norm will be denoted by$\mathcal{A}_m$.

Remark 2.2.1. By taking $G = \mathbb{Z}_2$ and $\sigma(\alpha, \beta) = (-1)^{\alpha\beta}$ for $\alpha, \beta \in G$ we will obtain the finite dimensional Graßmann algebra Λ_n with $L = \{1, \ldots, n\}$ which is the particular case of the algebra of supernumbers. In [38], Rudolph has defined mass norm on Λ_n and we generalize it for $\mathcal{A}$.

According to [25], let $*$ be a transposition of $\mathcal{A}$, then we know that $u_\alpha^* = \pm u_{-\alpha}$ for any $\alpha \in G_0$. If $u_\alpha^* = u_{-\alpha}$ holds for every $\alpha \in G_0$, $*$ is called **standard transposition**.

Proposition 2.2.3. *If $*$ is a standard transposition of $\mathcal{A}$, then the additive mapping $-: \mathcal{A} \longrightarrow \mathcal{A}$ defined by $\overline{a} = u_{2\alpha} a^*$ for $\alpha \in G$ and $a \in \mathcal{A}_\alpha$ is conjugation of $\mathcal{A}$.*

Proof. See proposition 2.4 of [25]. □

This conjugation is said to be associated with the standard transposition $*$ of $\mathcal{A}$. In [25] one transposition $*$ on $\mathcal{A}$ and its associated conjugation is defined as follows:

For $c \in \mathbb{C}$ and $\alpha \in G_0$ define $(cu_\alpha)^* = \overline{c} u_{-\alpha}$ and extend $*$ additively to C. To define a transposition of B, suppose the G_1-set L, used to define B, has a transposition $*$, that is, $*$ is a mapping from L to L such that $g(l^*) = -g(l)$ and $l^{**} = l$ for all $l \in L$. Then the graded vector space V, defined by L, has a transposition $*$ induced from the transposition $*$ of L; $(cv_l)^* = \overline{c} v_{l^*}$, for $c \in \mathbb{C}$ and $l \in L$. Now the σ-commutative algebra B of V over $\mathbb{C}$ has the transposition $*$ given by $(v_1 \ldots v_n)^* = \overline{c} v_n^* \ldots v_1^*$ and $\mathcal{A} = C \otimes B$ also has the transposition $*$ given as

$$(c \otimes b)^* = \sigma(\beta, \alpha) c^* \otimes b^* \ ,$$

for $c \in C_\alpha, b \in B_\beta, \alpha, \beta \in G$.

Let $-$ be the conjugation of $\mathcal{A}$ associated with $*$ (that is, $\overline{a} = u_{2\alpha}a^*$ for $\alpha \in G$). Then we have $\overline{u_\alpha} = u_\alpha$ for $\alpha \in G_0$ and $\overline{v_M} = u_{2g(M)} \otimes v_{M^*}$ where $M = \{l_1, \ldots, l_m\} \subseteq L, v_M = v_{l_1} \ldots v_{l_m}, g(M) = g(l_1) + \cdots + g(l_m)$, and $v_{M^*} = v_{l_m^*} \ldots v_{l_1^*}$.

For $a \in \mathcal{A}_{\mathbb{C}}$, which has the form $a = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$, with $a_{\alpha,M} \in \mathbb{C}$ for any α, M, transposition $*$ and conjugation $-$ are defined by

$$a^* = \sum_{\alpha,M} \overline{a}_{\alpha,M} \sigma(g(M), \alpha) u_{-\alpha} \otimes v_{M^*} \tag{2.2.10}$$

and

$$\overline{a} = \sum_{\alpha,M} \overline{a}_{\alpha,M} \sigma(g(M), \alpha) u_\alpha \otimes (u_{2g(M)} \otimes v_{M^*}) \tag{2.2.11}$$

respectively.

2.2.2 Real Riesz Space of Supernumbers

In this subsection we use the letter $\mathcal{A}$ to indicate the finite-dimensional σ-commutative G-graded algebra over $\mathbb{R}$ which its elements are called *supernumbers* according to the previous subsection. Here we equip $\mathcal{A}$ with Riesz space structure. First define relation $\leq$ in $\mathcal{A}$ as follows:
for any $a, b \in \mathcal{A}$, which have the form $a = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$ and $b = \sum\limits_{\alpha,M} b_{\alpha,M} u_\alpha \otimes v_M$, define $a \leq b$ if and only if $a_{\alpha,M} \leq b_{\alpha,M}$ for any α, M. Also $a < b$ means that $a \leq b$ and there exists $\alpha \in G_0$ or $M \subseteq L$ such that $a_{\alpha,M} \neq b_{\alpha,M}$ (i.e, $a \neq b$). Obviously, this relation is transitive, reflexive and antisymmetric. Therefore, the relation $\leq$ is **partial order** relation and $(\mathcal{A}, \leq)$ is **partially ordered vector space**. It is evident to prove that this partial order satisfies the following conditions:

(i) for $a, b, c \in \mathcal{A}$, $a \leq b$ implies that $a + c \leq b + c$;

(ii) for $a, b \in \mathcal{A}$ and $r \in \mathbb{R}^+$, $a \leq b$ implies that $ra \leq rb$;

which change $\mathcal{A}$ to be an **ordered vector space**.

An element $a \in \mathcal{A}$ is called **positive** if $a_{\alpha,M} \geq 0$ for any α, M. Also it is called **totally positive** if $a_{\alpha,M} > 0$ for any α, M.

The set

$$\begin{aligned}\mathcal{A}^+ &= \{a \in \mathcal{A} : \quad a \geq 0\} \\ &= \{a \in \mathcal{A} : \quad a = \sum_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M, \quad a_{\alpha,M} \geq 0 \text{ for any } \alpha, M\}\end{aligned}$$

is **positive cone** of $\mathcal{A}$ and any element of $\mathcal{A}^+$ is called **positive supernumber**. The set of all totally positive supernumbers is denoted by $\mathcal{A}_t^+$.

The supremum and infimum of two elements a, b of $\mathcal{A}$, which are denoted by $a \vee b$ and $a \wedge b$ respectively, are defined as below:

$$\begin{aligned}a \vee b &= \sup\{a, b\} = \sum_{\alpha,M} \max\{a_{\alpha,M}, b_{\alpha,M}\} u_\alpha \otimes v_M \qquad (2.2.12)\\ a \wedge b &= \inf\{a, b\} = \sum_{\alpha,M} \min\{a_{\alpha,M}, b_{\alpha,M}\} u_\alpha \otimes v_M\end{aligned}$$

This supremum and infimum are always exist. Therefore $\mathcal{A}$ is a **lattice**. So the ordered vector space $\mathcal{A}$ which is also lattice will be a Riesz space and we call it **Riesz space of supernumbers**.

Remark 2.2.2. Now since $\mathcal{A}$ is a G-graded algebra then $\mathcal{A} = \bigoplus_{\alpha \in G} \mathcal{A}_\alpha$ and we can define order relation in $\mathcal{A}_\alpha$ for any $\alpha \in G$ same as $\mathcal{A}$:

if $a_\alpha = \sum_{\beta,M} a_{\beta,M} u_\beta \otimes v_M$ and $b_\alpha = \sum_{\beta,M} a_{\beta,M} u_\beta \otimes v_M$ are elements of $\mathcal{A}_\alpha$, then $a_\alpha \leq b_\alpha$ if and only if $a_{\beta,M} \leq b_{\beta,M}$ for any β, M with condition $\beta + g(M) = \alpha$.

For an element a of a Riesz space $\mathcal{A}$, the **positive part** a^+, the **negative part** a^- and the **absolute value** $|a|$ are defined by:

$$a^+ = a \vee 0 = \sum_{\alpha,M} \max\{a_{\alpha,M}, 0\} u_\alpha \otimes v_M,$$

$$a^- = (-a) \vee 0 = \sum_{\alpha,M} \max\{-a_{\alpha,M}, 0\} u_\alpha \otimes v_M,$$

$$|a| = a \vee (-a) = \sum_{\alpha,M} \max\{a_{\alpha,M}, -a_{\alpha,M}\} u_\alpha \otimes v_M.$$

With these definitions, we have

$$(a \vee b)^+ = a^+ \vee b^+ \qquad (a \vee b)^- = a^- \vee b^- \tag{2.2.13}$$
$$(a \wedge b)^+ = a^+ \wedge b^+ \qquad (a \wedge b)^- = a^- \wedge b^-.$$

Remark 2.2.3. The following identities indicate some properties of elements of $\mathcal{A}$ that are true in any Riesz spaces. For more details refer to [28] and [54].

(i) $a^+, a^- \in \mathcal{A}^+$, $a^+ = (-a)^-$ and $a^- = (-a)^+$, $|a| = |-a|$;

(ii) $a = a^+ - a^-$ and $|a| = a^+ + a^-$;

(iii) $-a^- \leq a \leq a^+$;

(iv) $a \leq b$ if and only if $a^+ \leq b^+$ and $b^- \leq a^-$;

(v) $|a| \leq b$ if and only $-b \leq a \leq b$;

(vi) $(a+b)^+ \leq a^+ + b^+$, $(a+b)^- \leq a^- + b^-$, $||a| - |b|| \leq |a+b| \leq |a| + |b|$ for any $a, b \in \mathcal{A}$.

Definition 2.2.1. The linear subspace A of $\mathcal{A}$ is called a **Riesz subspace** of $\mathcal{A}$ if, for every pair of elements a, b in A, the elements $\sup(a, b)$ and $\inf(a, b)$ are also in A.

Lemma 2.2.4. *The body of Riesz space of supernumbers $\mathcal{A}$, $b(\mathcal{A})$, is a Riesz subspace of $\mathcal{A}$.*

In the consideration $\mathcal{A} = \bigoplus_{\alpha \in G} \mathcal{A}_\alpha$ of $\mathcal{A}$, for any $\alpha \in G, \mathcal{A}_\alpha$ is a Riesz subspace of $\mathcal{A}$.

Now consider the following particular subsets of $\mathcal{A}$ with its usual order,

$$\underline{\mathbb{N}} = \{\sum_{\alpha,M} n u_\alpha \otimes v_M : n \in \mathbb{N}\} \qquad \underline{\mathbb{Z}} = \{\sum_{\alpha,M} z u_\alpha \otimes v_M : z \in \mathbb{Z}\}$$

$$\underline{\mathbb{Q}} = \{\sum_{\alpha,M} q u_\alpha \otimes v_M : q \in \mathbb{Q}\} \qquad \underline{\mathbb{R}} = \{\sum_{\alpha,M} r u_\alpha \otimes v_M : r \in \mathbb{R}\}$$

Evidently $\underline{\mathbb{N}} \subset \underline{\mathbb{Z}} \subset \underline{\mathbb{Q}} \subset \underline{\mathbb{R}}$ and these sets are totally ordered , i.e, any two element of them are comparable. Only $\underline{\mathbb{R}}$ is a Riesz subspace of $\mathcal{A}$. Note that in the case of Graßmann algebra we have $b(\underline{\mathbb{N}}) = \mathbb{N}$, $b(\underline{\mathbb{Z}}) = \mathbb{Z}$, $b(\underline{\mathbb{Q}}) = \mathbb{Q}$ and $b(\underline{\mathbb{R}}) = \mathbb{R}$, the usual number sets.

It is evident to see that $\mathcal{A}$ hasn't minimal or maximal elements. Therefore according to terminology of [54] $\mathcal{A}$ hasn't zero element and unit element. But since we work with numbers, we say zero element to $\underline{0} = \sum\limits_{\alpha,M} 0_{\alpha,M} u_\alpha \otimes vM$ and unit to $\underline{1} = \sum\limits_{\alpha,M} 1_{\alpha,M} u_\alpha \otimes v_M$, where $1_{\alpha,M} = 1$ and $0_{\alpha,M} = 0$ for any α, M.

Now we restate some definitions of Riesz space theory in language of supernumbers. Reformulation of these definitions for supernumbers will be used in later structures.

Definition 2.2.2. Two elements a and d of $\mathcal{A}$ are said to be **weakly disjoint**, written $a \perp_w d$, if $|b(a)| \wedge |b(d)| = 0$, that is,

$$\inf\{\sup\{a_{\alpha,\phi}, -a_{\alpha,\phi}\}, \sup\{b_{\alpha,\phi}, -b_{\alpha,\phi}\}\} = 0$$

for any $\alpha \in G_0$.

Two elements a and b of $\mathcal{A}$ are said to be **disjoint** , written $a \perp b$, if $|a| \wedge |b| = 0$ that is

$$\inf\{\sup\{a_{\alpha,M}, -a_{\alpha,M}\}, \sup\{b_{\alpha,M}, -b_{\alpha,M}\}\} = 0$$

for any α, M. For $a \in \mathcal{A}$, let a^d be the set of all elements of $\mathcal{A}$ which are disjoint with a, that is, $a^d = \{b \in \mathcal{A} : a \perp b\}$. In fact, for given $a, b \in \mathcal{A}$, we have: $a \perp b$ if and only if $a_{\alpha,M} = 0$ if $b_{\alpha,M} \neq 0$ and $a_{\alpha,M} \neq 0$ if $b_{\alpha,M} = 0$ for any α, M. We can see evidently that for any $0 \neq a \in \underline{\mathbb{R}}$, $a^d = \{0\}$.

For any subset A of $\mathcal{A}$, the sets

$$A^d = \{b \in \mathcal{A} : \; b \perp a \; \text{ for any } \; a \; \text{ in } \; A\}$$

and

$$b(A^d) = \{b(d) \in b(\mathcal{A}) : \; b(d) \perp b(a) \; \text{ for any } \; a \; \text{ in } \; b(A)\}$$

are called **disjoint complement** and **weakly disjoint complement** of A respectively. Evidently we have $A^d \subseteq b(A^d)$.

Note that $\underline{\mathbb{Z}}^d = \underline{\mathbb{Q}}^d = \underline{\mathbb{R}}^d = \mathcal{A}^d = \{0\}$. Also for zero element we have $\{0\}^d = \mathcal{A}$.

Definition 2.2.3. A subset A of a Riesz space $\mathcal{A}$ is **(weakly) order bounded from above** if there is a vector u (called an **(weak) upper bound** of A) satisfying $(b(a) \leq b(u))$ $a \leq u$ for each $a \in \mathsf{A}$. In other words u is (weak) upper bound of A if $(a_{\alpha,\phi} \leq u_{\alpha,\phi})$ $a_{\alpha,M} \leq u_{\alpha,M}$ for any α, M. The sets **(weakly) order bounded from below** are defined similarly. For example the set $\underline{\mathbb{N}}$ is bounded from below with zero element.

A subset A of a Riesz space $\mathcal{A}$ is **(weakly) order bounded** if A is both (weakly) order bounded from above and below.

A **weak box** or **weakly order interval** is any set of the form

$$\begin{aligned} b([a,d]) &= \{b(c) \in b(\mathcal{A}): \ b(a) \leq b(c) \leq b(d)\} \\ &= \{b(c) \in b(\mathcal{A}): \ a_{\alpha,\phi} \leq c_{\alpha,\phi} \leq d_{\alpha,\phi} \text{ for any } \alpha \in G_0\}. \end{aligned}$$

A **box** or **order interval** is any set of the form

$$\begin{aligned} [a,d] &= \{c \in \mathcal{A}: \ a \leq c \leq d\} \\ &= \{c \in \mathcal{A}: \ a_{\alpha,M} \leq c_{\alpha,M} \leq d_{\alpha,M} \text{ for any } \alpha \in G_0 \text{ and } M \subset L\}. \end{aligned}$$

Obviously $[a,d] \subseteq b([a,d])$ and for incomparable elements a, d, we have $b([a,d]) = \phi$.

Definition 2.2.4. A nonempty subset A of a Riesz space $\mathcal{A}$ has a **(weak) supremum** (or a **(weak) least upper bound**) if there is (a)an (weak) upper bound $(b(u))u$ of $(b(\mathsf{A}))\mathsf{A}$ such that $(b(a) \leq b(v))$ $a \leq v$ for all $(b(a) \in b(\mathsf{A}))$ $a \in \mathsf{A}$ implies $(b(u) < b(v))$ $u < v$. Clearly the supremum, if it exist, is unique and is denoted by $\sup \mathsf{A}$. If $b(\mathcal{A})$, the body of $\mathcal{A}$, is considered as a subset of $\mathcal{A}$ then weak supremum is not unique. But if we consider $b(\mathcal{A})$ as a Riesz space independently, then weak supremum is unique. The**(weak) infimum** (or **(weak) greatest lower bound**) of a nonempty subset A is defined similarly and is denoted by $(\inf b(\mathsf{A}))\inf \mathsf{A}$.

Definition 2.2.5. A net $\{a_\tau\}$ in a Riesz space $\mathcal{A}$ is **(weakly)decreasing**, written $(a_\tau \downarrow_w)a_\tau \downarrow$ if $\tau \geq \mu$ implies $(b(a_\tau) \leq b(a_\mu))a_\tau \leq a_\mu$. In other words $\tau \geq \mu$ implies $(a^{(\tau)}_{\alpha,\phi} \leq a^{(\mu)}_{\alpha,\phi}$ for any $\alpha)$ $a^{(\tau)}_{\alpha,M} \leq a^{(\mu)}_{\alpha,M}$ for any α, M. The symbol $(a_\tau \uparrow_w)a_\tau \uparrow$ indicates an **(weakly) increasing** net, while $(a_\tau \uparrow_w \leq a)$ $a_\tau \uparrow \leq a$ (resp. $(a_\tau \downarrow_w \geq a)$ $a_\tau \downarrow \geq a$) denotes an (weakly) increasing (resp. (weakly) decreasing) net that is (weakly) order bounded from above (resp. below) by a.

The notation $(a_\tau \downarrow_w a)$ $a_\tau \downarrow a$ means that $(a_\tau \downarrow_w)a_\tau \downarrow$ and $(\inf_\tau\{a^{(\tau)}_{\alpha,\phi}\} = a_{\alpha,\phi}$ for any $\alpha)$ $\inf_\tau\{a^{(\tau)}_{\alpha,M}\} = a_{\alpha,M}$ for any α, M. Also the notation $(a_\tau \uparrow_w a)a_\tau \uparrow a$ means that $(a_\tau \uparrow_w)a_\tau \uparrow$ and $(\sup_\tau\{a^{(\tau)}_{\alpha,\phi}\} = a_{\alpha,\phi}$ for any $\alpha)$ $\sup_\tau\{a^{(\tau)}_{\alpha,M}\} = a_{\alpha,M}$ for any α, M.

Definition 2.2.6. A net $\{a_\tau\}$ in a Riesz space $\mathcal{A}$ **(weakly) converges in order** or is **(weakly) order convergent** to some $a \in \mathcal{A}$, written $(a_\tau \xrightarrow{w.o} a)a_\tau \xrightarrow{o} a$, if there is a net $\{d_\tau\}$ (with the same directed set) satisfying $(d_\tau \downarrow_w o)d_\tau \downarrow o$ and $(|b(a_\tau) - b(a)| \leq b(d_\tau))|a_\tau - a| \leq d_\tau$ for each τ (equivalently for any τ we have $(|a^{(\tau)}_{\alpha,\phi} - a_{\alpha,\phi}| \leq d^{(\tau)}_{\alpha,\phi}$ for any $\alpha)$ $|a^{(\tau)}_{\alpha,M} - a_{\alpha,M}| \leq d^{(\tau)}_{\alpha,M}$ for any $\alpha, M)$. In this case a is called **(weak) order limit** of $\{a_\tau\}$.

A sequence $\{a_n\}_{n=1}^\infty$ in $\mathcal{A}$ is said to be an **(weak) order Cauchy sequence** if there is a sequence $d_n \downarrow 0$ such that $(|b(a_n) - b(a_m)| \leq b(d_n))|a_n - a_m| \leq d_n$ for all $m \geq n \geq 1$. One Riesz space is **order complete** if every order Cauchy sequence has an order limit. Equivalently a Riesz space is order complete if every subset of it has a supremum. It is easy to see from this equivalence definition that $\mathcal{A}$ is not order complete.

The main properties of order convergence are listed in Theorem 10.2 of [54].

Definition 2.2.7. Let $u \geq 0$ be an element of a Riesz space of supernumbers $\mathcal{A}$. We say that the sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ is **(weakly) converges u-uniformly** to an element $a \in \mathcal{A}$ whenever, for every $\varepsilon > 0$, there exists a natural number N_ε such that ($|a^n_{\alpha,\emptyset} - a_{\alpha,\emptyset}| \leq \varepsilon u_{\alpha,\emptyset}$ for any $\alpha \in G_0$) $|a^n_{\alpha,M} - a_{\alpha,M}| \leq \varepsilon u_{\alpha,M}$ holds for all $n \geq N_\varepsilon$, where $a_n = \sum\limits_{\alpha,M} a^{(n)}_{\alpha,M} u_\alpha \otimes v_M$ and $u = \sum\limits_{\alpha,M} u_{\alpha,M} u_\alpha \otimes v_M$. In this case a is called **(weakly) u-uniform limit** of $\{a_n\}$ and written as ($a_n \xrightarrow{w} a(u\text{-un})$) $a_n \longrightarrow a(u\text{-un})$.

It is said that the sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ is **(weakly) converges relatively uniformly** to $a \in \mathcal{A}$ whenever a_n (weakly) converges u-uniformly to a for some $u \in \mathcal{A}^+$. This kind of convergence is denoted by ($a_n \xrightarrow{w} a(\text{un})$) $a_n \longrightarrow a(\text{un})$.

In next proposition we will prove that $\mathcal{A}$ is Archimedean and therefore the uniqueness of uniform limit and some of its properties follows from Theorem 10.3 of [54]. The Riesz space of supernumbers has some characteristics which we prove them now. For definitions of some concepts refer to section 1.3.

Proposition 2.2.5. *The Riesz space of supernumbers $\mathcal{A}$ is Archimedean.*

Proof. We know that every element a of $\mathcal{A}$ has the form $a = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$ and $a \in \mathcal{A}^+$ if and only if $a_{\alpha,M} \geq 0$ for any α, M. So

$$\frac{1}{n}a = \sum_{\alpha,M} \frac{1}{n} a_{\alpha,M} u_\alpha \otimes v_M \geq \sum_{\alpha,M} \frac{1}{n+1} a_{\alpha,M} u_\alpha \otimes v_M = \frac{1}{n+1} a$$

holds for any positive integer n, which means $\frac{1}{n}a \downarrow$. Also it is clear that $\inf\limits_n\{\frac{1}{n}a\} = 0$ or $\inf\limits_n\{\frac{1}{n}a_{\alpha,M}\} = 0$ for any α, M. Therefore $\frac{1}{n}a \downarrow 0$ which implies $\mathcal{A}$ is Archimedean. $\square$

Proposition 2.2.6. *The Riesz space of supernumbers $\mathcal{A}$ is Dedekind complete.*

Proof. Let A be a non-empty subset of $\mathcal{A}$ which is order bounded from above by b. Hence for any a in A we have $a_{\alpha,M} \leq b_{\alpha,M}$ for any α, M. Set $A_{\alpha,M} = \{a_{\alpha,M} : a \in A\}$ for any α, M. So $A_{\alpha,M}$ is a nonempty subset of real line $\mathbb{R}$ that has an upper bound $b_{\alpha,M}$. Therefore by completeness axiom of $\mathbb{R}$, $A_{\alpha,M}$ has a supremum, as $S_{\alpha,M}$, which $S_{\alpha,M} \leq b_{\alpha,M}$. Now take $S = \sum_{\alpha,M} S_{\alpha,M} u_\alpha \otimes v_M$. Obviously $S \leq b$ and S is the supremum of A. □

Proposition 2.2.7. *The Riesz space of supernumbers $\mathcal{A}$ is Dedekind σ-complete.*

Here we give the concepts of *limit superior(limsup)* and *limit inferior(liminf)* in the Riesz space of supernumbers. Let $\{a_\tau\}$ be a net of supernumbers in $\mathcal{A}$ with the property that $b_\tau = \sup_{\mu \geq \tau}\{a_\mu\}$ exists in $\mathcal{A}$ for any τ . Evidently $\{b_\tau\}$ is decreasing net. If there exists an element $b \in \mathcal{A}$ such that $b_\tau \downarrow b$, we write $b = \limsup a_\tau$. Similarly we write $c = \liminf a_\tau$ if $c_\tau = \inf_{\mu \geq \tau}\{a_\mu\}$ exists for all τ and $c_\tau \uparrow c$. In other words

$$\limsup_\tau a_\tau = \inf_\tau \sup_{\mu \geq \tau} a_\mu = \sum_{\alpha,M} (\inf_\tau \sup_{\mu \geq \tau} a^{(k)}_{\alpha,M}) u_\alpha \otimes v_M$$

$$\liminf_\tau a_\tau = \sup_\tau \inf_{\mu \geq \tau} a_\mu = \sum_{\alpha,M} (\sup_\tau \inf_{\mu \geq \tau} a^{(k)}_{\alpha,M}) u_\alpha \otimes v_M$$

The limit superior and limit inferior characterize order convergence in Dedekind complete Riesz space, i.e., an order bounded net $\{a_\tau\}$ in $\mathcal{A}$ satisfies $a_\tau \xrightarrow{o} a$ if and only if $a = \limsup_\tau a_\tau = \liminf_\tau a_\tau$.

Proposition 2.2.8. *The Riesz space of supernumbers $\mathcal{A}$ is u-uniformly complete space($u \in \mathcal{A}^+$).*

Proof. Since $\mathcal{A}$ is Archimedian, by theorem 39.4 of [28], it suffices to prove that every monotone u-uniformly Cauchy sequence has an u-uniform limits. This is easily obtainable by considering to the Dedekind σ- completeness of $\mathcal{A}$ (previous proposition)and last part of lemma 39.2 of [28]. □

Now by definition of uniformly complete we have the following.

Proposition 2.2.9. *The Riesz space of supernumbers is uniformly complete space.*

Proposition 2.2.10. *The Riesz space of supernumbers $\mathcal{A}$ is order separable.*

Proof. Let A be a non-empty subset of $\mathcal{A}$ possessing a supremum $\sup \mathsf{A} = S$. Hence for any a of A, $a \leq S$ and so $a_{\alpha,M} \leq S_{\alpha,M}$ for any α, M. Let $\mathsf{A}_{\alpha,M} = \{a_{\alpha,M} : \ a \in \mathsf{A}\}$. Therefore $\mathsf{A}_{\alpha,M} \subseteq \mathbb{R}$ and $\sup \mathsf{A}_{\alpha,M} = S_{\alpha,M}$. Since $\mathbb{R}$ is separable, $\mathsf{A}_{\alpha,M} \cap \mathbb{Q}$ is a countable dense subset of $\mathsf{A}_{\alpha,M}$ which $\sup(\mathsf{A}_{\alpha,M} \cap \mathbb{Q}) = S_{\alpha,M}$. Now let A' be the set of all $a \in \mathsf{A}$ such that its component $a_{\alpha,M}$ belongs in $\mathsf{A}_{\alpha,M} \cap \mathbb{Q}$, that is,

$$\mathsf{A}' = \{a \in \mathsf{A} : \ a = \sum_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M \ \text{ and } \ a_{\alpha,M} \in \mathsf{A}_{\alpha,M} \cap \mathbb{Q} \ \text{ for any } \ \alpha, M\}$$

A' is at most countable subset of A. Since G_0 and L are finite sets then the number of the sets $\mathsf{A}_{\alpha,M} \cap \mathbb{Q}$ is finite. Therefore A has at most countable subset. □

Now since $\mathcal{A}$ is Archimedean, the recent result and theorem 23.5 of [28]imply the next assertion.

Proposition 2.2.11. *The Riesz space of supernumbers $\mathcal{A}$ is strong order separable.*

Definition 2.2.8. A subset A of a Riesz space $\mathcal{A}$ is called **weak solid** if $|b(d)| \leq |b(a)|$ and $b(a) \in b(\mathsf{A})$ implies $b(d) \in b(\mathsf{A})$. A is said to be **solid set** if $|d| \leq |a|$ and $a \in \mathsf{A}$ implies $d \in \mathsf{A}$.

For instance the positive cone of $\mathcal{A}$, $\mathcal{A}^+$, is not solid but the sets $\underline{\mathbb{N}}, \underline{\mathbb{Z}}, \underline{\mathbb{Q}}$ and $\underline{\mathbb{R}}$ are solid sets. Also for any $\alpha \in G$ the set $\mathcal{A}_\alpha$ is solid.

Definition 2.2.9. A subset F of a Riesz space E is **order closed** if $\{x_\tau\} \subset F$ and $x_\tau \xrightarrow{o} x$ imply $x \in F$. It is called **weakly order closed** if $\{x_\tau\} \subset F$ and $x_\tau \xrightarrow{w.o} x$ imply $x \in F$. In a similar way F is called **(weakly)** σ**-order closed** if these statements are true for sequences.

A solid linear subspace of a Riesz space is called an **ideal**. An order closed ideal is called a **band**. An ideal F is a band if and only if $\{x_\tau\} \subset F$ and $0 \leq x_\tau \uparrow x$ imply $x \in F$.

The ideal J_A generated by the non-empty subset A of $\mathcal{A}$, is

$$J_\mathsf{A} = \bigcup \{n[-a, a] : \quad n \in \mathbb{N}; \quad a = |a_1| \vee \cdots \vee |a_r| \quad , a_1, \ldots, a_r \in \mathsf{A}\}.$$

A **principal ideal** is an ideal generated by a singleton $\{a\}$ and is denoted by J_a. For any $a \in \mathcal{A}^+$ the ideal J_a generated by a is

$$J_a = \bigcup \{n[-a, a] : \quad n \in \mathbb{N}\}.$$

For any $a \in \mathcal{A}$ the principal ideal J_a generated by A is

$$\begin{aligned} J_a &= \{b \in \mathcal{A} \quad | \quad \exists \lambda > 0 \quad \text{with} \quad |b| \leq \lambda |a|\} \\ &= \{b \in \mathcal{A} \quad | \quad \exists \lambda > 0 \quad \text{with} \quad |b_{\alpha,M}| \leq \lambda |a_{\alpha,M}| \quad \text{for any} \quad \alpha, M\}. \end{aligned}$$

The band generated by an ideal J of $\mathcal{A}$ is given by

$$B_J = \{a \in \mathcal{A} \quad | \quad \exists \text{a net } \{a_\tau\} \subset J \quad \text{with} \quad 0 \leq a_\tau \uparrow |a|\}.$$

Also for $a \in \mathcal{A}$ the band B_{J_a}(will be denoted by B_a) generated by ideal J_a is called **principal band** generated by a.

Let $a \in \mathcal{A}^+$. Any element $b \in \mathcal{A}^+$ satisfying $b \wedge (a - b) = 0$ is called a **component** of a. The set C_a of all components of a is a Boolean algebra.

$$\begin{aligned} C_a &= \{b \in \mathcal{A}^+ \quad | \quad b \wedge (a-b) = 0\} \\ &= \{b \in \mathcal{A}^+ \quad | \quad b_{\alpha,M} = 0 \ \text{ or } \ b_{\alpha,M} = a_{\alpha,M} \quad \text{for any} \ \ \alpha, M\} \end{aligned}$$

Proposition 2.2.12. *The Riesz space of supernumbers $\mathcal{A}$ has projection property.*

Proof. This is an immediate consequence of proposition 2.2.6 and theorem 24.9(i) of[28]. □

Proposition 2.2.13. *The Riesz space of supernumbers has principal projection property.*

Proof. This is obtainable from proposition 2.2.7 and Theorem 25.1 of [28]. □

Proposition 2.2.14. *The Riesz space of supernumbers $\mathcal{A}$ has sufficiently many projection.*

Proof. Recall that a Riesz space has sufficiently many projection if every nonzero band contains a nonzero projection band. The assertion can be obtain from preceding proposition and Theorem 30.4(i) of[28]. □

According to principal projection property of $\mathcal{A}$, any principal band generated by $a \in \mathcal{A}$ will be projection band ($B_a + B_a^d = \mathcal{A}$) and if $a \in \mathcal{A}^+$ then for any $b \in \mathcal{A}^+$ the element

$$c = \sup(b \wedge na \quad n = 1, 2, \dots) = \sum_{\alpha,M} \min\{b_{\alpha,M}, \sup_n(na_{\alpha,M})\} u_\alpha \otimes v_M$$

exists always(by theorem 11.5 of [54]). In this case c is the component of b in B_a and is denoted by $P_{B_a}(b)$. In fact P_{B_a} is a projection mapping from $\mathcal{A}$ into $\mathcal{A}$ itself which is linear and idempotent for any $b \in \mathcal{A}^+$. Therefore $P_{B_a} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$P_{B_a}(b) = \sum_{\alpha,M} \min\{b_{\alpha,M}, \sup_n(na_{\alpha,M})\}u_\alpha \otimes v_M$$

for any $b \in \mathcal{A}^+$.

Proposition 2.2.15. *In the Riesz space of supernumbers, totally positive elements are order unit.*

Proof. Let $e \in \mathcal{A}_t^+$. We must prove that for any $a \in \mathcal{A}$, there is a positive real number r, depending upon a, such that $|a| \leq re$.

$$e = \sum_{\substack{M \subseteq L \\ \alpha \in G_0}} e_{\alpha,M} u_\alpha \otimes v_M \qquad e_{\alpha,M} > 0$$

According to Archimedean property of $\mathbb{R}$, there is a positive real numbers $r_{\alpha,M}$ such that $|a_{\alpha,M}| \leq (r_{\alpha,M})e_{\alpha,M}$ for any α, M. Therefore $|a| \leq (\sum_{\alpha,M} r_{\alpha,M})e$ and so e is strong unit. □

Remark 2.2.4. Since $\mathcal{A}$ is finite dimentional Archimedean Dedekind σ-complete Riesz space then stablity of order convergence obtains from Theorem 70.3 of [28]. Also according to Theorem 16.3 of [28], stability of order convergence implies that order convergence and relatively uniform convergence are equivalent.

Definition 2.2.10. The element $e \in \mathcal{A}^+$ is called a **weak unit** in $\mathcal{A}$ if $a \perp e$ implies that $a = 0$ for $a \in \mathcal{A}$. It is an immediate consequence of definition 2.2.2 that $e \in \mathcal{A}^+$ is weak unit of $\mathcal{A}$ if and only if $e_{\alpha,M} \neq 0$ for any α, M. In fact $e^d = \{0\}$ if and only if e is weak unit.

Proposition 2.2.16. *The Riesz space of supernumbers has σ-property.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{A}^+$. For any n, a_n has the form

$$a_n = \sum_{\substack{M \subseteq L \\ \alpha \in G_0}} a_{\alpha,M}^{(n)} u_\alpha \otimes v_M$$

where $a_{\alpha,M}^{(n)} \in \mathbb{R}$. If $a_n = 0$ then take r_n be the positive integer 1, else take $r_n = 1/\sum\limits_{\alpha,M} a_{\alpha,M}^{(n)}$. It is clear that $r_n > 0$ for any n and $\{r_n a_n\}_{n=1}^{\infty}$ is bounded from above by the element $\sum\limits_{\alpha,M} u_\alpha \otimes v_M$. □

Proposition 2.2.17. *The Riesz space of supernumbers $\mathcal{A}$ has strong diagonal property.*

Proof. According to remark 2.2.4, the order convergence in $\mathcal{A}$ is stable. By proposition 2.2.16 and Theorem 70.2 of [28], $\mathcal{A}$ has d-property. Also $\mathcal{A}$ is Archimedean and the Proof is complete by the Theorem 68.5 of [28]. □

Proposition 2.2.18. *The Riesz space of supernumbers has diagonal property for order convergence and relatively uniform convergence.*

Proof. Since $\mathcal{A}$ is Archimedean, it is an immediate consequence of theorems 70.2 and 72.2 0f [28], proposition 2.2.16 and remark 2.2.4. □

Proposition 2.2.19. *The Riesz space of supernumbers has strong Egoroff property.*

Proof. Since $\mathcal{A}$ is Archimedean, according to previous proposition and Theorem 70.2($i \longrightarrow ii$) of [28] $\mathcal{A}$ has the *d-property* and so by Theorem 68.5 of [28], the assertion is obtaining. □

2.2.3 Complex Riesz space of supernumbers

In this subsection we investigate complexification of Riesz space of supernumbers, $\mathcal{A}$, by considering complex supernumbers.

Definition 2.2.11. Let $\mathcal{A}_{\mathbb{C}}$ be the space of finite-dimensional σ-commutative G-graded algebra over the complex number field $\mathbb{C}$ which it's elements are called **complex supernumber**. Every element of $\mathcal{A}_{\mathbb{C}}$ has the form $c = \sum\limits_{\alpha,M} c_{\alpha,M} u_\alpha \otimes v_M$ where $c_{\alpha,M} \in \mathbb{C}$ for any α, M. Therefore $c_{\alpha,M} = a_{\alpha,M} + ib_{\alpha,M}$ where $a_{\alpha,M}$,$b_{\alpha,M}$ are elements of $\mathbb{R}$ for any α, M. Also it is clear that we can write

$$\begin{aligned} c &= \sum_{\alpha,M} c_{\alpha,M} u_\alpha \otimes v_M \\ &= \sum_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M + i \sum_{\alpha,M} b_{\alpha,M} u_\alpha \otimes v_M. \end{aligned}$$

By letting $a = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$ and $b = \sum\limits_{\alpha,M} b_{\alpha,M} u_\alpha \otimes v_M$ we have $c = a + ib$ where $a, b \in \mathcal{A}$. So we can write $\mathcal{A}_{\mathbb{C}} = \mathcal{A}$+i$\mathcal{A}$. The operations additivity and scalar multiplicativity are defined evidently as follows:
for any $c_1, c_2 \in \mathcal{A}_{\mathbb{C}}$ where $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ and any $r, s \in \mathbb{R}$:

$$c_1 + c_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$(r + is)c = (r + is)(a + ib) = ra - sb + i(sa + rb)$$

The set $\mathcal{A}_{\mathbb{C}}$ is called complexification of a real vector space $\mathcal{A}$.

Now, the space $\mathcal{A}_{\mathbb{C}}$ can be partially ordered coordinatewise, i.e, $c_1 \leq c_2$ whenever $a_1 \leq a_2$ and $b_1 \leq b_2$ in $\mathcal{A}$. Also the supremum and infimum of two elements of $\mathcal{A}_{\mathbb{C}}$ can be defined evidently as:

$$\sup(c_1, c_2) = \sup(a_1, a_2) + i \sup(b_1, b_2)$$

and

$$\inf(c_1, c_2) = \inf(a_1, a_2) + i \inf(b_1, b_2)$$

It is easy to prove that $\mathcal{A}_{\mathbb{C}}$ is a Riesz space, and for $c = a + ib \in \mathcal{A}_{\mathbb{C}}$ the element $|c|$ is given by $|c| = |a| + i|b|$.

Remark 2.2.5. For a moment we restrict our attention to the complex number field $\mathbb{C}$. Let z be an element of $\mathbb{C}$. Therefore $z = a + ib$ where $a, b \in \mathbb{R}$. We have the familiar absolute value $|z| = (a^2 + b^2)^{\frac{1}{2}}$ which is a non-negative real number. But since $\mathbb{C}$ is a Riesz space the absolute value of z would be $|z| = |a| + i|b|$ instead of $(a^2 + b^2)^{\frac{1}{2}}$. Also for any z(real or complex) it is hold that $|z| = z \vee (-z)$ in accordance with a definition of Riesz space. It is straightforward to see that

$$|z| = \sup\{Re(ze^{-i\theta}) : 0 \leq \theta \leq 2\pi\} = \sup_{0\leq\theta\leq 2\pi} (a\cos(\theta) + b\sin(\theta))$$

Now for $c \in \mathcal{A}_{\mathbb{C}}$ with above notations and considerations we obtain two statements for absolute value of c. We wish to define an absolute value $|c|$ of c such that $|c| \in \mathcal{A}^+$ and such that if c itself is an element of $\mathcal{A}$, then $|c| = c \vee (-c)$. For this purpose with letting $c \in \mathcal{A}_{\mathbb{C}}$ we will have:

$$|c| = c \vee (-c) = \sum_{\alpha,M} (a^2_{\alpha,M} + b^2_{\alpha,M})^{\frac{1}{2}} u_\alpha \otimes v_M.$$

It is clear that in this case $|c| \in \mathcal{A}^+$ and if $c \in \mathcal{A}$ then $|c| = c \vee (-c)$. Also for any two elements $c_1, c_2 \in \mathcal{A}_{\mathbb{C}}$ we have

$$\begin{aligned} c_1 \vee c_2 &= \sup\{c_1, c_2\} \\ &= \sup\{\sum_{\alpha,M} c^{(1)}_{\alpha,M} u_\alpha \otimes v_M \quad , \sum_{\alpha,M} c^{(2)}_{\alpha,M} u_\alpha \otimes v_M\} \\ &= \sum_{\alpha,M} \max\{a^{(1)}_{\alpha,M}, a^{(2)}_{\alpha,M}\} u_\alpha \otimes v_M + i \sum_{\alpha,M} \max\{b^{(1)}_{\alpha,M}, b^{(2)}_{\alpha,M}\} u_\alpha \otimes v_M \end{aligned}$$

where $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$. If we take $c = c_1 = c_2$ in above equalities it is straightforward to see that $|c| = |a| + i|b|$. On the other hand $\mathcal{A}$ is Archimedean and uniformly complete (propositions 2.2.5 and 2.2.9), hence according to Theorem 13.4 of [54]

$$|c| = \sup\{Re(ce^{-i\theta}) : \quad 0 \leq \theta \leq 2\pi\} = \sup(a\cos(\theta) + b\sin(\theta) : 0 \leq \theta \leq 2\pi)$$

exists in $\mathcal{A}$ for any $c = a + ib \in \mathcal{A}_{\mathbb{C}}$ and we have

$$|c| = \sum_{\alpha,M} \sup_{0 \leq \theta \leq 2\pi} (a_{\alpha,M}\cos(\theta) + b_{\alpha,M}\sin(\theta))u_\alpha \otimes v_M$$

where $c = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M + i \sum\limits_{\alpha,M} b_{\alpha,M} u_\alpha \otimes v_M$.

Remark 2.2.6. Note that the absolute value in $\mathcal{A}+i\mathcal{A}$ has some properties as the familiar absolute value for complex numbers. Some of them are listed in below which are proved in [54], Theorems 13.5 and 13.6:

(i) $|a| \leq |c|, |b| \leq |c|$ and $|c| \leq |a| + |b|$, for which $c = a + ib$;

(ii) $|c| = 0$ if and only if $c = 0$,$|rc| = |r||c|$ for any complex number r;

(iii) $|c_1 - c_2| = |c_1 + c_2| = |c_1| + |c_2| = ||c_1 - |c_2|| = |c_1| \vee |c_2|$.

Definition 2.2.12. A subset A of $\mathcal{A} + i\mathcal{A}$ is called an **ideal** if A is a complex linear subspace of $\mathcal{A} + i\mathcal{A}$ and if A is a solid subspace of $\mathcal{A}$. The set of all real elements in the ideal A is denoted by A_r, i.e., $\mathsf{A}_r = \mathsf{A} \cap \mathcal{A}$. The ideal A in $\mathcal{A} + i\mathcal{A}$ is called a **band** if the real part of A_r is a band in $\mathcal{A}$. The band A in $\mathcal{A} + i\mathcal{A}$ is called a **projection band** if A_r is a projection band in $\mathcal{A}$.

It is clear that $b(\mathcal{A}_{\mathbb{C}}) = b(\mathcal{A}) + ib(\mathcal{A})$ and $b(\mathcal{A}_{\mathbb{C}})$ is a Riesz subspace of $\mathcal{A} + i\mathcal{A}$. By letting $\underline{\mathbb{C}} = \{\sum\limits_{\alpha,M} cu_\alpha \otimes v_M \quad : \quad c \in \mathbb{C}\}$ we find out that in the case of Graßmann algebra $b(\underline{\mathbb{C}}) = \mathbb{C}$.

Theorem 2.2.20. *The complex Riesz space of supernumbers has all properties of $\mathcal{A}$ such as Archimedian, Dedekind completeness, projection property, principal projection property and so on.*

According to [25], let $*$ be the transposition on $\mathcal{A}_{\mathbb{C}}$ and $-$ be its associated conjugation. In fact for $a \in \mathcal{A}_{\mathbb{C}}$, which has the form $a = \sum\limits_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$, with $a_{\alpha,M} \in \mathbb{C}$ for any α, M, transposition $*$ and conjugation $-$ are defined by

$$a^* = \sum_{\alpha,M} \overline{a}_{\alpha,M} \sigma(g(M),\alpha) u_{-\alpha} \otimes v_{M^*}$$

and

$$\overline{a} = \sum_{\alpha,M} \overline{a}_{\alpha,M} \sigma(g(M),\alpha) u_\alpha \otimes (u_{2g(M)} \otimes v_{M^*})$$

respectively.

For $\alpha \in G$, the real part $\mathrm{Re}(\mathcal{A}_{\mathbb{C}})$ and the imaginary part $\mathrm{Im}(\mathcal{A}_{\mathbb{C}})$ of $\mathcal{A}_{\mathbb{C}}$ are defined by

$$\mathrm{Re}(\mathcal{A}_{\mathbb{C}}) = \{a \in \mathcal{A}_{\mathbb{C}} \mid \quad \overline{a} = a \ \} \qquad \mathrm{Im}(\mathcal{A}_{\mathbb{C}}) = \{a \in \mathcal{A}_{\mathbb{C}} \mid \quad \overline{a} = -a \ \}.$$

Also we have $\mathcal{A}_{\mathbb{C}} = \mathrm{Re}(\mathcal{A}_{\mathbb{C}}) \oplus \mathrm{Im}(\mathcal{A}_{\mathbb{C}})$, $\mathrm{Im}(\mathcal{A}_{\mathbb{C}}) = i\mathrm{Re}(\mathcal{A}_{\mathbb{C}})$ where $i = \sqrt{-1}$. Therefore an element $c \in \mathcal{A}_{\mathbb{C}}$ is written as $c = \mathrm{Re}(c) + \mathrm{Im}(c) = a + ib$ with $a, b \in \mathrm{Re}(\mathcal{A}_{\mathbb{C}})$. Since $\mathcal{A}_{\mathbb{C}} = \bigoplus\limits_{\alpha \in G} \mathcal{A}_\alpha$ then

$$\begin{aligned}
\mathcal{A}_{\mathbb{C}} &= \bigoplus_{\alpha \in G} [\mathrm{Re}(\mathcal{A}_\alpha) \oplus \mathrm{Im}(\mathcal{A}_\alpha)] \\
&= [\bigoplus_{\alpha \in G} \mathrm{Re}(\mathcal{A}_\alpha)] \oplus [\bigoplus_{\alpha \in G} \mathrm{Im}(\mathcal{A}_\alpha)] \\
&= \mathrm{Re}(\mathcal{A}_{\mathbb{C}}) \oplus \mathrm{Im}(\mathcal{A}_{\mathbb{C}}).
\end{aligned}$$

Now it is clear that if $a \in \mathcal{A}_{\mathbb{C}}$ and $\overline{a} = a$ then $a \in \mathrm{Re}(\mathcal{A}_{\mathbb{C}})$ and in this case we will say that a is **superreal**.

2.2.4 Banach Lattice of Supernumbers

We know that $\mathcal{A}$ is a normed space with some different norms. In this subsection we prove that $\mathcal{A}$ is a Banach lattice with some of them. First we consider $\mathcal{A}$ as a real Riesz space and in the last, we consider the norms on the complexification of $\mathcal{A}$.

We saw that, $\mathcal{A}$ is an Archimedean Dedekind complete Riesz space which has the projection and principal projection properties with totally positive elements as its order units. Now it is straightforward to check that $\| \ \|_k$ is a *Riesz norm* for any $1 \leq k \leq \infty$, that is, if $|a| \leq |b|$ in $\mathcal{A}$ then $\|a\|_k \leq \|b\|_k$. To see this, let $|a| \leq |b|$ for $a, b \in \mathcal{A}$. This implies that $\sum\limits_{\alpha,M} |a_{\alpha,M}| u_\alpha \otimes v_M \leq \sum\limits_{\alpha,M} |b_{\alpha,M}| u_\alpha \otimes v_M$, i.e., $|a_{\alpha,M}| \leq |b_{\alpha,M}|$ for any α, M. So for $1 \leq k < \infty$ we have $|a_{\alpha,M}|^k \leq |b_{\alpha,M}|^k$ for any α, M and hence $\sum\limits_{\alpha,M} |a_{\alpha,M}|^k \leq \sum\limits_{\alpha,M} |b_{\alpha,M}|^k$. Therefore $\|a\|_k \leq \|b\|_k$. On the other hand $\mathcal{A}$ is norm complete with $\| \ \|_k$, and so it will be *a Banach lattice*. But the mass norm on $\mathcal{A}$ is not Riesz norm and therefore $\mathcal{A}$ with this norm is not normed Riesz space.

Proposition 2.2.21. *The norm $\| \ \|_k$ for $1 \leq k \leq \infty$ in Banach lattice $\mathcal{A}$ is σ-order continuous.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{A}$ which $a_n \downarrow 0$ in $\mathcal{A}$. Every a_n has the form $a_n = \sum\limits_{\alpha,M} a^{(n)}_{\alpha,M} u_\alpha \otimes V_M$ where $a_n \downarrow 0$ implies that $a^{(n)}_{\alpha,M} \geq a^{(n+1)}_{\alpha,M}$, $a^{(n)}_{\alpha,M} \geq 0$, and $\inf\limits_n \{a^{(n)}_{\alpha,M}\} = 0$ for any n and for any α, M. Now let $1 \leq k < \infty$. It is easily seen that $\|a_n\|_k \geq \|a_{n+1}\|_k$. $\inf\limits_n \{a^{(n)}_{\alpha,M}\} = 0$ for any α, M implies that $\inf\limits_n \{|a^{(n)}_{\alpha,M}|^k\} = 0$ for any $\alpha, M$and so $\inf\limits_n \{\sum\limits_{\alpha,M} |a^{(n)}_{\alpha,M}|^k\} = 0$. Therefore $\inf\limits_n \{\sum\limits_{\alpha,M} |a^{(n)}_{\alpha,M}|^k\}^{\frac{1}{k}} = 0$ and hence $\inf\limits_n \|a_n\|_k = 0$ For $k = \infty$ it is straightforward. ☐

Remark 2.2.7. Theorem 17.9 of [54], Dedekind σ-completeness of $\mathcal{A}$ and previous proposition imply that $\| \ \|_k$ is an order continuous norm on $\mathcal{A}$. Also with using theorem 17.8 of [54] we obtain that $\mathcal{A}$ is super Dedekind complete. Since $\mathcal{A}$ is Banach lattice, theorem 16.2 of [54] implies the Riesz-Fischer property for $\mathcal{A}$, that is, the property that convergence of $\sum\limits_{1}^{\infty} \|a_n\|$ implies the existence of $\sum_1^\infty a_n$, or equivalently for any sequence $\{a_n\}_{n=1}^\infty$ of positive elements in $\mathcal{A}$ for which $\sum \|a_n\|$ converges, the order limit $\sum_1^\infty a_n$ exists. Lemma 16.1 of [54] indicate that $\|\sum a_n\| \leq \sum \|a_n\|$ for any such sequences.

The Riesz space of supernumbers is Dedekind complete Banach lattice with order continuous norm $\| \ \|_k$. Hence $\mathcal{A}$ has countable type property and also its norm is Fatou(definition 1.3.21). It is straightforward to see that for any $1 \leq k \leq \infty$, the norm $\| \ \|_k$ is not M-norm or L-norm. Since $\mathcal{A}$ is Archimedean relatively uniformly complete Riesz space with totally positive supernumbers as its order units, proposition 7.2 of [41] indicate that for any totally positive e, the gauge function of $[-e, e]$, given as

$$P_e(a) = \inf\{r \in \mathbb{R} \ : \quad -re \leq a \leq re\} \tag{2.2.14}$$

is an M-norm and $\mathcal{A}$ with this norm is AM-space. Now since $\mathcal{A}$ is Banach lattice with the norms $\| \ \|_k$ and P_e, for any $e \in \mathcal{A}_t^+$, they are equivalent.

We know that every totally positive supernumber of $\mathcal{A}$ is its order unit and P_e, defined as above, is the gauge function of $[-e, e]$, therefore $(\mathcal{A}, \mathcal{A}^+, P_e)$ is order unit normed space. Now by theorem 9.11 of [52], there is a compact convex set K in a locally convex space such that $(\mathcal{A}, \mathcal{A}^+, P_e)$ is isometrically order-isomorphic to $A(K)$ where $A(K)$ is the closed subspace of the set of all real valued continuous functions on K, consisting of all affine functions.

The Riesz space of supernumbers has some topologies which are equivalent. First of them is *order topology*, defined by order closed subsets of $\mathcal{A}$. The second is *relatively uniform topology* which is defined by relatively uniformly closed subsets. Since order convergence in $\mathcal{A}$ is stable, this two kind of topologies are equivalent. They satisfies in T_1-separation axiom. We know that $\mathcal{A}$ is a Banach lattice with the norms $\| \ \|_k$ for $1 \leq k \leq \infty$ and P_e for $e \in \mathcal{A}_t^+$, therefore they are equivalent and so induced topology by them are same. On the other hand, since $\mathcal{A}$ is finite dimensional, all topologies on $\mathcal{A}$ are equivalent.

Every strong order unit e of ordered topological vector space $\mathcal{A}$ is an interior point of positive cone $\mathcal{A}^+$ and the order box $[-e, e]$ is a neighborhood of 0. The lattice operations

$$(a,b) \mapsto a \vee b \ , \ (a,b) \mapsto a \wedge b \ , \ a \mapsto a^+ \ , \ a \mapsto a^- \ , \ a \mapsto |a|$$

are continuous in order topology. Uniformly continuity of them follows by proposition 5.2 of [41].

Now consider to $\mathcal{A}_{\mathbb{C}}$ as a complexification of real Riesz space $\mathcal{A}$. We know that $\mathcal{A}$ is uniformly complete Riesz space and every element $a = b + ic$ of $\mathcal{A}_{\mathbb{C}}$ has an absolute value $|a|$ in $\mathcal{A}$. For this element a define the number $\|a\|_{\mathbb{C}}$ by $\|a\|_{\mathbb{C}} = \||a|\|$. This defined a Riesz norm on $\mathcal{A}_{\mathbb{C}}$. Since $\mathcal{A}$ is Banach lattice, thus $\mathcal{A}_{\mathbb{C}}$ will be Banach lattice so which is called **complex Banach lattice of supernumbers**. Exercise 15.12 of [54] states some properties of complex Banach lattices.

Proposition 2.2.22. *The norm of $\mathcal{A}_{\mathbb{C}}$ is a σ-order continuous norm.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{A}_{\mathbb{C}}$ such that $a_n \downarrow 0$. For any n, $a_n = a_1^{(n)} + ia_2^{(n)}$ and $a_n \downarrow 0$ implies that $a_1^{(n)} \downarrow 0$ and $a_2^{(n)} \downarrow 0$ in $\mathcal{A}$. Therefore $|a_n| \downarrow 0$. Since the norm of $\mathcal{A}$ is Riesz norm, we will have $\||a_n|\| \downarrow 0$ which means that $\|a_n\|_{\mathbb{C}} \downarrow 0$. □

2.2.5 Riesz Algebra of Supernumbers

We know that the multiplication in the algebra of supernumbers has not multiplicative unit and so the inverse of supernumbers is not defined. But, as we stated in the first of current chapter, we wish these supernumbers behave as ordinary numbers. Therefore in this subsection by using the method of functional calculus in Riesz spaces we will obtain a new multiplication on $\mathcal{A}$ to gain our goal. This new multiplication make a new algebric structure to $\mathcal{A}$ which is called Riesz algebra. In the subsection 1.3.4 we saw that the method of functional calculus arises for such Riesz spaces which they have strong order unit and principal projection property. In this Riesz spaces, the principal ideal generated by an order unit element is the whole space. Also the strong units of a Riesz spaces is not unique and we can use different strong order units to obtain different results.

Remmember that the Riesz space of supernumbers, has principal projection property and every totally positive element is its strong order unit. We are familiar with the sets $\underline{\mathbb{N}}, \underline{\mathbb{Z}}, \underline{\mathbb{Q}}, \underline{\mathbb{R}}$ and $\underline{\mathbb{C}}$ as supernatural, superinteger, superrational, superreal and supercomplex supernumbers respectively. Also we know that the supernumber $\underline{1}$ is strong order unit. Then we can use it for reformulating of functional calculus to Riesz space of supernumbers. We do it now and for this first we consider the Riesz space of supernumbers $\mathcal{A}$ over real number field $\mathbb{R}$.

The set $C_{\underline{1}}$ of all components of $\underline{1}$ is

$$C_{\underline{1}} = \{a \in \mathcal{A}^+ \quad | \quad a_{\alpha,M} = 0 \ \text{ or } \ a_{\alpha,M} = 1 \quad \text{for any } \ \alpha, M\}$$

which is Boolean algebra. Evidently $\underline{0}$, $\underline{1} \in C_{\underline{1}}$. The principal ideal $J_{\underline{1}}$ generated by $\underline{1}$ is whole space $\mathcal{A}$, i. e., $J_{\underline{1}} = \mathcal{A}$.

$$J_{\underline{1}} = \{a \in \mathcal{A} \quad | \quad \exists \lambda > 0 \quad \text{with} \quad |a_{\alpha,M}| \leq \lambda \quad \text{for any} \alpha, M\}.$$

Given an element $a \in J_{\underline{1}} = \mathcal{A}$, there exist real numbers r, t with $r < t$ and a number $\delta > 0$ such that $r\underline{1} \leq a \leq (t - \delta)\underline{1}$. For example we can take

$$r = \min_{\alpha,M}\{a_{\alpha,M}\} - 1 \quad ; \quad t = \max_{\alpha,M}\{a_{\alpha,M}\} + 1 \quad ; \quad \delta = 1.$$

The interval $[r,t] \subset \mathbb{R}$ is called **spectral interval** of a. Let $\mathsf{P} : r = k_0 < k_1 < \cdots < k_m = t$ be a partition of $[r,t]$ and P_k be the band projection onto the band generated by $(k\underline{1} - a)^+$ for any $k \in [r,t]$. In fact, for any $b \in \mathcal{A}^+$,

$$\begin{aligned} P_k(b) &= \sup_n (b \wedge [n(k\underline{1} - a)^+]) = b \wedge \sup_n [n(k\underline{1} - a)^+] \\ &= \sum_{\alpha,M} \min\{b_{\alpha,M}, \sup_n [n(k\underline{1} - a)^+_{\alpha,M}]\} u_\alpha \otimes v_M \\ &= \sum_{\alpha,M} \min\{b_{\alpha,M}, \sup_n [n \max\{k - a_{\alpha,M}, 0\}]\} u_\alpha \otimes v_M. \end{aligned}$$

The elements $s = \sum_{j=1}^{m} k_{j-1}(P_{k_j} - P_{k_{j-1}})\underline{1}$ and $S = \sum_{j=1}^{m} k_j(P_{k_j} - P_{k_{j-1}})\underline{1}$ are called **lower sum** and **upper sum** belonging to a and partition P where $s \le a \le S$. If $k_j - k_{j-1} < \varepsilon$ for $j = 1, 2, \ldots, m$ then $0 \le a - s \le \varepsilon\underline{1}$ and $0 \le S - a \le \varepsilon\underline{1}$.

Let F be a continuous real valued function on the spectral interval $[r,t]$ and consider the numbers

$$m_j = \min_{k_{j-1} \le k \le k_j} F(k) \qquad M_j = \max_{k_{j-1} \le k \le k_j} F(k) \tag{2.2.15}$$

for $j = 1, 2, \ldots, m$. Also let

$$s = \sum_{j=1}^{m} m_j(P_{k_j} - P_{k_{j-1}})\underline{1} \qquad \text{and} \qquad S = \sum_{j=1}^{m} M_j(P_{k_j} - P_{k_{j-1}})\underline{1} \tag{2.2.16}$$

be the corresponding lower and upper sums.

Let $s(\mathsf{P})$ and $S(\mathsf{P})$ denote the lower and upper sums depending on P, respectively. According to theorem 34.1 of [54], the set of all $s(\mathsf{P})$, for all partitions P of $[r,t]$, has a supremum in $\mathcal{A}$ which is at the same time infimum of all possible $S(P)$ for all partitions. This common value is denoted by $F(a)$ and expressed with

$$F(a) = \int_r^t F(k)\, dP_k.$$

This method of defining $F(a)$ for any $a \in \mathcal{A}$ and any real continuous function F on $[r,t]$ is known as a **functional calculus**. By taking $F(k) = k^n$ for all $k \in [r,t]$ and natural number n, we can define a^n for any $a \in \mathcal{A}$.

Proposition 2.2.23. *For any $a \in \mathcal{A}^+$, there exists $b \in \mathcal{A}^+$ such that $a = b^2$.*

Proof. For $a \in \mathcal{A}^+$ we can take $[0, t]$ as its spectral interval. The function $F(k) = k^{\frac{1}{2}}$ is a continuous real valued function on $[0, t]$. Hence by letting $b = F(a)$ the assertion will be obtain. □

For $a, b \in \mathcal{A}$, the elements a^2 , b^2 and $(a + b)^2$ are defined as above and the product ab can be define by $ab = \frac{1}{2}\{(a + b)^2 - a^2 - b^2\}$. Distributivity, associativity and commutativity are hold for this multiplication. The strong order unit $\underline{1}$ is as its multiplicative unit. Also we have the following statements:

(1) $|ab| \leq |a||b|$ for all $a, b \in \mathcal{A}$;

(2) for positive elements $a, b \in \mathcal{A}^+$ the element ab is in $\mathcal{A}^+$;

(3) $a \perp b$ if and only if $ab = 0$;

(4) If $a^n = 0$ for some $n \in \mathbb{N}$, then $a = 0$;

(5) $a^+a^- = a^-a^+ = 0$ for any $a \in \mathcal{A}$ and hence $a^2 \geq 0$.

If $a \in \mathcal{A}$ and there exists an element $b \in \mathcal{A}$ such that $ab = ba = \underline{1}$, then b is called an **inverse** of a and is denoted by a^{-1}. This element, if it exists, is unique. If $a \geq 0$ and a^{-1} exists then $a \geq r\underline{1}$ for some positive number r. Conversely, any $a \in \mathcal{A}$ satisfying $a \geq r\underline{1}$ for some $r > 0$ has an inverse. Therefore $a \in \mathcal{A}$ is invertible if and only if there is an element $0 < r \in \mathbb{R}$ such that $a_{\alpha,M} \geq r$ for any α, M.

According to above argument and definition 1.3.24 $\mathcal{A}$ will be a Riesz algebra which we call it **Riesz algebra of supernumbers**. Also (3) and (4) of above imply that $\mathcal{A}$ is a f-algebra. We know that $\mathcal{A}$ is unital with unit element $\underline{1}$ and since it is uniformly complete, it has property (*) and M.D. (According to terminology of [8]). Theorem 4.2 and corollary 4.3 of [8] indicate that for every positive supernumber a, the elements $\sqrt{a}$ and $\sqrt{a^2 + b^2}$ always exist.

Proposition 2.2.24. *The set $\mathcal{A}$ is commutative Banach algebra.*

Proof. $\mathcal{A}$ is Archimedean uniformly complete Riesz algebra (with product induced by functional calculus) and also is Banach lattice. L. Venter [47], showed that in any Banach lattice algebra X, $\|z_1 z_2\| \leq \|z_1\|\|z_2\|$ holds for any $z_1, z_2 \in X_{\mathbb{C}}$. On the other hand we know that the new product in $\mathcal{A}$ is associative and commutative. Therefore we have our aim. □

Now let $\mathcal{A}$ be the Riesz space of supernumbers over the complex number field. The complexification of $\mathcal{A}$ as a complex Riesz space is considered in 2.2.3. Since $\mathcal{A}$ is a Dedekind σ-complete Riesz space, as previous argument, let $\underline{1} > 0$ be the strong order unit in $\mathcal{A}$ and let the commutative multiplication in $J_{\underline{1}}$ with $\underline{1}$ as unit element be introduced as explained before. Therefore according to theorem 44.4 of [54] the multiplication can be extended to the complexification $J_{\underline{1}} + iJ_{\underline{1}}$ in a natural way, i. e.,

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Then, for any $h = a + ib \in J_{\underline{1}} + iJ_{\underline{1}}$ we have

$$|h|^2 = a^2 + b^2,$$

i. e., $|h|$ is(unique) positive square root of $a^2 + b^2$.

In corollary 44.5 of [54], some properties of this multiplication are mentioned.

(i) $|h_1 h_2| = |h_1||h_2|$ for all $h_1, h_2 \in J_{\underline{1}} + iJ_{\underline{1}}$;

(ii) $h_1 h_2 = 0$ if and only if $|h_1| \perp |h_2|$;

(iii) $h^k = 0$ for some natural number k implies that $h = 0$.

This last argument indicate that $\mathcal{A}_{\mathbb{C}}$ is also Riesz algebra. Theorem 5.2 of [8] indicates that

$$\sqrt{a^2 + b^2} = \sup_{0 \leq \theta \leq 2\pi} (a \cos(\theta) + b \sin(\theta))$$

for all $a, b \in \mathcal{A}$ and this means that the absolute value of any complex supernumber $z = a + ib$ can be define $|z| = \sqrt{a^2 + b^2}$ in a natural way. Also it is straightforward to see that $\mathcal{A}_{\mathbb{C}}$ is also uniformly complete semiprime unital f-algebra (see [8]).

We can easily reformulate the concept of functional calculus to $\mathcal{A}_{\mathbb{C}}$. For $a, \underline{1}_{\mathbb{C}} \in \mathcal{A}_{\mathbb{C}}$ we have $a = a_1 + ia_2$ and $\underline{1}_{\mathbb{C}} = \underline{1} + i\underline{1}$ where $a_1, a_2 \in \mathcal{A}$ and $\underline{1} \in \underline{\mathbb{R}}$. Let F be a complex valued continuous function on the spectral interval $[r, t]$ of a. On the One hand for any $k \in [r, t]$, $F(k) = f_1(k) + iF_2(k) \in \mathbb{C}$ which F_1 and F_2 are continuous real valued functions on $[r, t]$. On the other hand $r\underline{1}_{\mathbb{C}} \leq a \leq (t - \delta)\underline{1}_{\mathbb{C}}$ implies that $r\underline{1} \leq a_1 \leq (t - \delta)\underline{1}$ and $r\underline{1} \leq a_2 \leq (t - \delta)\underline{1}$. Indeed $[r, t]$ is also spectral interval of both a_1 and a_2. We want to define $F(a)$ for $a \in \mathcal{A}_{\mathbb{C}}$. For this by using the method of functional calculus of $\mathcal{A}$ we can associate $F_1(a_1)$ to a_1, $F_2(a_2)$ to a_2 and finally $F(a) = F_1(a_1) + iF_2(a_2)$ to a. About invertiblity of any $a \in \mathcal{A}_{\mathbb{C}}$ we have the following assertion: $a = a_1 + ia_2 \in \mathcal{A}_{\mathbb{C}}$ is invertible if and only if a_1 and a_2 are invertible.

2.3 Infinite Dimensional Riesz Algebra of Supernumbers

The elements of infinite dimensional σ-commutative G-graded algebra is also called supernumbers in [31]. Therefore investigation of properties of these supernumbers seems is natural. In current section we do this and prove the similar propositions for infinite dimensional supernumbers. Indeed, the structure of Riesz spaces, Banach lattices, Riesz algebras and functional calculus on this algebra will be obtain. As an important result we will see that this algebra is commutative Banach algebra.

2.3.1 Infinite Dimensional Algebra of Supernumbers

Let us consider a strict increasing sequence of finite G_1-sets $L_n, n = 0, 1, \ldots$ where $L_0 = \phi$. Let $L = \cup L_n$. For a finite subsets M of L we define the height $h(M)$ of M by $h(M) = \min\{n \; : \; M \subseteq L_n\}$. Let $B_n = B_{L_n}$ and $\mathcal{A}_n = C \otimes B_n$ (according to construction for any n). Since there is a natural inclusion mapping $i_n : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$, we have an increasing sequence of finite dimensional σ-commutative Banach algebras $\mathcal{A}_n$. We consider the countable strict inductive limit $\mathcal{A}$ of this sequence $\{\mathcal{A}_n\}$(see subsection 1.2.2). $\mathcal{A}$ is the union of $\mathcal{A}_n$'s and the topology of $\mathcal{A}$ is defined as follows: a subset A of $\mathcal{A}$, assumed to be convex, is a neighbourhood of zero if and only if, for every n=1,2,..., the intersection $\mathsf{A} \cap \mathcal{A}_n$ is a neighbourhood of zero in the Banach space $\mathcal{A}_n$. Actually, this topology is defined by the following system of norms on $\mathcal{A}$.

Let $\omega = \{\omega_n\}$ be an increasing sequence of positive integers, then we define a norm $P_\omega(a)$ on $\mathcal{A}$ by

$$P_\omega(a) = \sum_{\alpha,M} \omega_{h(M)} |a_{\alpha,M}|, \tag{2.3.1}$$

where $a = \sum_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M \in \mathcal{A}$. The space I_n spanned by the elements of the form $u_\alpha \otimes v_M$ with $h(M) = n$ is an ideal of $\mathcal{A}_n$. Every element a of $\mathcal{A}$ is written uniquely as $a = \sum_n a_n, \ \ a_n \in I_n$ and we have

$$P_\omega(a) = \sum_n \omega_n \|a_n\| \tag{2.3.2}$$

where $\|a_n\|$ is the norm of a_n.

Now ranging over all the increasing sequences ω of positive integer, $\{U_\omega\}$ is a fundamental system of neighborhoods of zero in $\mathcal{A}$, where $U_\omega = \{a \in \mathcal{A} \ : \ \ P_\omega(a) < 1\}$.

Proposition 2.3.1. $P_\omega(ab) \leq P_\omega(a)P_\omega(b)$.

Proof. Let $a = \sum_{\alpha,M} a_{\alpha,M} u_\alpha \otimes v_M$ and $b = \sum_{\beta,N} b_{\beta,N} u_\beta \otimes v_M$ be arbitrary elements of $\mathcal{A}$. Then

$$ab = \sum_{\substack{K \subset L \\ \gamma \in G_0}} \Big(\sum_{\substack{M \cup N = K \\ \alpha + \beta = \gamma}} a_{\alpha,M} b_{\beta,N} \varepsilon_{\alpha,\beta,M,N} \Big) u_\gamma \otimes v_K,$$

where $\alpha, \beta \in G_0, \ \ M, N \subseteq L, \ \ M \cap N = \emptyset$ and $\varepsilon_{\alpha,\beta,N,M}$ are elements of $\mathbb{F}$ with absolute value 1. Thus we have

$$\begin{aligned}
P_\omega(a) p_\omega(b) &= \Big(\sum_{\alpha,M} \omega_{h(M)} |a_{\alpha,M}|\Big)\Big(\sum_{\beta,N} \omega_{h(N)} |b_{\beta,N}|\Big) \\
&= \sum_{\alpha,\beta,M,N} \omega_{h(M)} \omega_{h(N)} |a_{\alpha,M} b_{\beta,N}| \\
&\geq \sum_{\alpha,\beta,M,N} \omega_{h(M \cup N)} |a_{\alpha,M} b_{\beta,N}| \\
&\geq \sum_{\substack{K \subset L \\ \gamma \in G_0}} \Big(\sum_{\substack{M \cup N = K \\ \alpha + \beta = \gamma}} |a_{\alpha,M} b_{\beta,N}| \Big) \\
&\geq P_\omega(ab).
\end{aligned}$$

□

The multiplication in $\mathcal{A}$ is induced from those of $\mathcal{A}_n$, and it is continuous in $\mathcal{A}$ as easily seen from above proposition. $\mathcal{A}$ is a G-graded algebra as follows: $\mathcal{A} = \bigoplus_{\alpha \in G} \mathcal{A}_\alpha$ which $\mathcal{A}_\alpha = \bigcup_{n=1}^{\infty} (\mathcal{A}_n)_\alpha$. Clearly $\mathcal{A}$ is σ-commutative. Thus, $\mathcal{A}$ is a σ-commutative locally convex topological algebra.

Proposition 2.3.2. *$\mathcal{A}$ is complete.*

Proof. Since $\mathcal{A}$ is a countable strict inductive limit of Banach spaces, it is complete by proposition 1.2.2. □

This topological algebra $\mathcal{A}$ is not Banach algebra but it is complete and its topology is defined by a system of Banach norms [31]. Again with refereing to [5] we can have the following norms on $\mathcal{A}$. Evidently with using $\| \quad \|_k$ for $1 \leq k \leq \infty$, we obtain that

$$P_\omega^{(k)}(a) = \sum_n \omega_n \|a_n\|_k \tag{2.3.3}$$

defines a norm on $\mathcal{A}$.

In a similar way of finite dimensional case, we can define the mass norm on infinite dimensional $\mathcal{A}$, which be independent from the choice of basis $\{v_l^{(n)} \ : \ l \in L_n\}$ for G-graded vector spaces V_n, over $\mathbb{F}$ for any n, and base u_α of grade α for C_α. For this purpose , by considering the first hypothesis, we can follow our investigation.

For $M \subseteq L_n$ let $[M]_n = \{M' \subseteq L_n \ : \ \text{card}M' = \text{card}M\}$ which is an equivalence class for L_n. The set of all such equivalence classes makes a partition for L_n. Now for $M \subseteq L_n$ let

$$V_M^{(n)} = \{ \sum_{M' \in [M]_n} a_{M'}^{(n)} v_{M'}^{(n)} \ : \ a_{M'}^{(n)} \in \mathbb{F}\}.$$

Also for $\alpha \in G_0$ and $M \subseteq L_n$ set

$$V_{\alpha,M}^{(n)} = \{v_{\alpha,M}^{(n)} = \sum_{M' \in [M]_n} a_{\alpha,M'}^{(n)} u_\alpha \otimes v_{M'}^{(n)} \ : a_{\alpha,M'}^{(n)} = c_\alpha a_{M'}^{(n)} \in \mathbb{F}\}.$$

This set, $V^{(n)}_{\alpha,M}$, is a vector space for any n. We know that , for any n, the set $\mathcal{A}_n = \bigoplus_{\alpha,M} V^{(n)}_{\alpha,M}$ is a finite dimensional σ-commutative G-graded algebra where M ranges over all elements of partition of L_n. Any $a_n \in \mathcal{A}_n$ has the form $a_n = \sum_{\alpha,M} v^{(n)}_{\alpha,M}$. For defining the mass norm on $\mathcal{A}$ first we define the norm $\| \ \|_{\alpha,M}$ on $V^{(n)}_{\alpha,M}$ in the same as

$$\|v^{(n)}_{\alpha,M}\|_{\alpha,M} = \inf\{ \sum_{M' \in [M]_n} |a^{(n)}_{\alpha,M'}| \ : \ V^{(n)}_{\alpha,M} = \sum_{M' \in [M]_n} a^{(n)}_{\alpha,M'} u_\alpha \otimes v^{(n)}_{M'} \ ; \ a^{(n)}_{\alpha,M'} \in \mathbb{F}\} \tag{2.3.4}$$

where infimum is taken over all possible choices of basis for V_n and C_α. We have the mass norm

$$m_n(a_n) = \sum_{\alpha,M} \|v^{(n)}_{\alpha,M}\|_{\alpha,M} \tag{2.3.5}$$

on $\mathcal{A}_n$, and the norm

$$m_\omega(a) = \sum_n \omega_n m_n(a_n) \tag{2.3.6}$$

on $\mathcal{A}$ for $a = \sum_n a_n \in \mathcal{A}$. This last norm is called **mass norm** on infinite dimensional $\mathcal{A}$.

2.3.2 Infinite Dimensional Riesz Space and Banach Lattice of Supernumbers

Now since $\mathcal{A}$ is the strict inductive limit of a sequence $\{\mathcal{A}_n\}$, by proposition 11.9 of [52] and its preceding argument $\mathcal{A}$ is a locally convex Riesz space. Of course we can define a relation $\leq$ in $\mathcal{A}$ as follows:

for any $a, b \in \mathcal{A}, a \leq b$ if and only if $a_n \leq b_n$ in I_n for any n. $\mathcal{A}$ with this relation is an ordered vector space. An element $a \in \mathcal{A}$ is called **positive** if a_n is positive for any n. Also it called **totally positive** if a_n is totally positive in I_n for any n. The set

$$\mathcal{A}^+ = \{a \in \mathcal{A} \ : \ a_n \in I_n^+ \ \text{ for any } \ n\}$$

is the positive cone of $\mathcal{A}$. Also $\mathcal{A}_t^+$ denotes the set of totally positive supernumbers. The supremum and infimum of two elements a, b of $\mathcal{A}$, are defined as follows:

$$a \vee b = \sup\{a, b\} = \sum_n \sup\{a_n, b_n\} \quad ; \quad a \wedge b = \inf\{a, b\} = \sum_n \inf\{a_n, b_n\} \tag{2.3.7}$$

These supremum and infimum are exist always. Therefore $\mathcal{A}$ is a lattice and so it will be Riesz space which we call it **infinite-dimensional Riesz space of supernumbers**.

For an element a of $\mathcal{A}$, the positive part a^+, the negative part a^- and the absolute value $|a|$ are defined by

$$a^+ = \sum_n a_n^+ \quad , \quad a^- = \sum_n a_n^- \quad , \quad |a| = \sum_n |a_n|$$

where $a = \sum_n a_n$.

In a similar way of finite dimensional case we can see easily the following identities in the same as 2.3.2 for $a, b \in \mathcal{A}$: $(a \vee b)^+ = a^+ \vee b^+$ $\qquad (a \vee b)^- = a^- \vee b^-$

$(a \wedge b)^+ = a^+ \wedge b^+$ $\qquad (a \wedge b)^- = a^- \wedge b^-$.It is straightforward to see that any norms P_ω and $P_\omega^{(k)}$ for $1 \leq k \leq \infty$ are Riesz norms. Also we can see evidently that these norms are not M-norm or L-norm. On the other hand $\mathcal{A}$ is complete with these norms and so $\mathcal{A}$ will be Banach lattice. This last statement implies that the norms P_ω and $P_\omega^{(k)}$ for $1 \leq k \leq \infty$ are equivalent.

In proposition 2.2.15 we proved that every totally positive supernumber is strong order unit of finite dimensional Riesz space of supernumbers. This is true for infinite-dimensional case. To see this let e be a totally positive element of $\mathcal{A}$. Then by definition e_n is a totally positive element of I_n for any n and so P_{e_n}, defined as in 2.2.14, is an M-norm on $\mathcal{A}_n$. Now since $\mathcal{A}$ is a strict inductive limit of $\mathcal{A}_n$,

$$P_\omega^{(e)}(a) = \sum_n \omega_n P_{e_n}(a_n) \tag{2.3.7}$$

defines a norm on $\mathcal{A}$. It is straightforward to see that if e is a totally positive element of $\mathcal{A}$ and a is an arbitrary element of $\mathcal{A}$, then $|a| \leq P_\omega^{(e)}(a)e$.

For weak units of infinite-dimensional $\mathcal{A}$, we have simpler condition. Let $e \in \mathcal{A}^+$. Then e has the form $e = \sum_n e_n$ where $e_n \in I_n^+$. Also let $a \in \mathcal{A}$ and $a \perp e$ then $a_n \perp e_n$ for any n. Therefore $e \in \mathcal{A}^+$ is weak unit of $\mathcal{A}$ if and only if e_n is a weak unit in $\mathcal{A}_n$ for any n.

Beforehand we saw some particular subsets of finite dimensional Riesz space of supernumbers which were the similarities of ordinary real(or complex) numbers. Here we consider to the similar subsets for infinite dimensional $\mathcal{A}$. First recall that

$$\underline{\mathbb{N}} = \{\sum_{\alpha,M} n u_\alpha \otimes v_M : n \in \mathbb{N}\} \qquad \underline{\mathbb{Z}} = \{\sum_{\alpha,M} z u_\alpha \otimes v_M : z \in \mathbb{Z}\}$$

$$\underline{\mathbb{Q}} = \{\sum_{\alpha,M} q u_\alpha \otimes v_M : q \in \mathbb{Q}\} \qquad \underline{\mathbb{R}} = \{\sum_{\alpha,M} r u_\alpha \otimes v_M : r \in \mathbb{R}\}$$

$$\underline{\mathbb{C}} = \{\sum_{\alpha,M} c u_\alpha \otimes v_M : c \in \mathbb{C}\}.$$

Evidently $\underline{\mathbb{N}} \subset \underline{\mathbb{Z}} \subset \underline{\mathbb{Q}} \subset \underline{\mathbb{R}}$.

Remember the structure of $\mathcal{A}$. For any n, the set $\mathcal{A}_n$ is a finite dimensional σ-commutative G-graded algebra and we can have the usual subsets $\underline{\mathbb{N}}^{(n)}, \underline{\mathbb{Z}}^{(n)}, \underline{\mathbb{Q}}^{(n)}$ and $\underline{\mathbb{R}}^{(n)}$ of $\mathcal{A}_n$ as above. It is easily seen that the following inclusions are hold for any n.

$$\underline{\mathbb{N}}^{(n)} \subset \underline{\mathbb{N}}^{(n+1)}, \;\; \underline{\mathbb{Z}}^{(n)} \subset \underline{\mathbb{Z}}^{(n+1)}, \;\; \underline{\mathbb{Q}}^{(n)} \subset \underline{\mathbb{Q}}^{(n+1)}, \;\; \underline{\mathbb{R}}^{(n)} \subset \underline{\mathbb{R}}^{(n+1)}.$$

We introduce the following subsets of $\mathcal{A}$:

$$\underline{\mathbb{N}}^\infty = \bigcup_n \underline{\mathbb{N}}^{(n)}, \;\; \underline{\mathbb{Z}}^\infty = \bigcup_n \underline{\mathbb{Z}}^{(n)}, \;\; \underline{\mathbb{Q}}^\infty = \bigcup_n \underline{\mathbb{Q}}^{(n)}, \;\; \underline{\mathbb{R}}^\infty = \bigcup_n \underline{\mathbb{R}}^{(n)}.$$

The set $\underline{\mathbb{R}}^{(n)}$ is a Riesz subspace of $\mathcal{A}_n$ for any n, and hence $\underline{\mathbb{R}}^{\infty}$ will be Riesz subspace of $\mathcal{A}$.

If we choose the complex field as our background, then with a similar argument, we have $\underline{\mathbb{C}}^{(n)} \subset \mathcal{A}_{\mathbb{C}}^{(n)}$ and $\underline{\mathbb{C}}^{(n)} \subset \underline{\mathbb{C}}^{(n+1)}$ for any n. Also the set $\underline{\mathbb{C}}^{\infty} = \bigcup\limits_{n} \underline{\mathbb{C}}^{(n)}$ is a Riesz subspace of $\mathcal{A}_{\mathbb{C}}$.

We saw that $\underline{1} = \sum\limits_{\alpha,M} 1u_{\alpha} \otimes v_M \in \{\sum_{\alpha,M} nu_{\alpha} \otimes v_M : n \in \mathbb{N}\}$. Therefore in particular we can have $\underline{1} = \sum\limits_{n} \underline{1}_n \in \underline{\mathbb{N}}^{\infty} \subset \mathcal{A}$ where $\underline{1}_n \in \underline{\mathbb{N}}^{(n)}$ for any n. This element is strong order unit of $\mathcal{A}$ and will be used next.

Definition 2.3.1. Two elements a and d of $\mathcal{A}$ are **weakly disjoint**, if and only if $|b(a)| \wedge |b(d)| = 0$, that is, $b(a_n) \perp_w b(d_n)$ in I_n for any n.
They called **disjoint**, written $a \perp d$, if and only if $|a| \wedge |d| = 0$, that is, $a_n \perp d_n$in I_n for any n. For $a \in \mathcal{A}$, let a^d be the set of all elements of $\mathcal{A}$ which are disjoint with a, that is, $a^d = \{b \in \mathcal{A} : a \perp b\}$.

Definition 2.3.2. A subset A of a Riesz space $\mathcal{A}$ is **(weakly) order bounded from above** if there is a vector u (called an **(weak) upper bound** of A) satisfying $(b(a) \leq b(u))$ $a \leq u$ for each $a \in \mathsf{A}$. In other words u is (weak) upper bound of A if $(b(a_n) \leq b(u_n))a_n \leq u_n$ in I_n for any n. The **(weakly) order bounded from below** are defined similarly. A subset A of a Riesz space $\mathcal{A}$ is **(weakly) order bounded** if A is both (weakly) order bounded from above and below.

A **weak box** or **weak order interval** is any set of the form

$$\begin{aligned} b([a,d]) &= \{b(c) \in b(\mathcal{A}) : \ b(a) \leq b(c) \leq b(d)\} \\ &= \{b(c) \in b(\mathcal{A}) : \ b(a_n) \leq b(c_n) \leq b(d_n) \text{ in } I_n; \text{ for any } n \ \}. \end{aligned}$$

A **box** or **order interval** is any set of the form

$$\begin{aligned}[a,d] &= \{c\in\mathcal{A}:\ \ a\le c\le d\}\\ &= \{c\in\mathcal{A}:\ \ a_n\le c_n\le d_n \text{ in } I_n \text{ for any } \ n\}.\end{aligned}$$

Obviously $[a,d]\subseteq b([a,d])$ and for incomparable elements a,d we have $[a,d]=b([a,d])=\phi$.

Definition 2.3.3. A nonempty subset A of a Riesz space $\mathcal{A}$ has a **(weak) supremum** (or a **(weak) least upper bound**) if there is (a)an (weak) upper bound $(b(u))u$ of $(b(\mathsf{A}))\mathsf{A}$ such that $(b(a)\le b(v))$ $a\le v$ for all $(b(a)\in b(\mathsf{A}))$ $a\in\mathsf{A}$ implies $(b(u)<b(v))$ $u<v$. Clearly the supremum, if it exist, is unique and is denoted by $\sup\mathsf{A}$. If $b(\mathcal{A})$, the body of $\mathcal{A}$, is considered as a subset of $\mathcal{A}$ then the weak supremum is not unique. But if we consider $b(\mathcal{A})$ as a Riesz space independently, then weak supremum is unique. The**(weak) infinum** (or **(weak) greatest lower bound**) of a nonempty subset A is defined similarly and is denoted by $(\inf b(\mathsf{A}))\inf\mathsf{A}$.

Definition 2.3.4. A net $\{a_\tau\}$ in a Riesz space $\mathcal{A}$ is **(weakly)decreasing**, written $(a_\tau\downarrow_w)a_\tau\downarrow$ if $\tau\ge\mu$ implies $(b(a_\tau)\le b(a_\mu))a_\tau\le a_\mu$. In other words $\tau\ge\mu$ implies $(b(a_n^{(\tau)})\le b(a_n^{(\mu)})$ in I_n for any n where $a_\tau=\sum\limits_n a_n^{(\tau)}$. The symbol $(a_\tau\uparrow_w)a_\tau\uparrow$ indicates an **(weakly) increasing** net, while $(a_\tau\uparrow_w\le a)$ $a_\tau\uparrow\le a$ (resp. $(a_\tau\downarrow_w\ge a)$ $a_\tau\downarrow\ge a$) denotes an (weakly) increasing (resp. (weakly) decreasing) net that is (weakly) order bounded from above (resp. below) by a.

The notation $(a_\tau\ \downarrow_w\ a)$ $a_\tau\downarrow a$ means that $(a_\tau\downarrow_w)a_\tau\downarrow$ and $(\inf\limits_\tau\{b(a_n^{(\tau)})\}=b(a_n))\inf\limits_\tau\{a_n^{(\tau)}\}=a_n$ in I_n for any n. Also the notation $(a_\tau\uparrow_w a)a_\tau\uparrow a$ means that $(a_\tau\uparrow_w)a_\tau\uparrow$ and $(\sup\limits_\tau\{b(a_n^{(\tau)})\}=a_n)$ $\sup\limits_\tau\{a_n^{(\tau)}\}=a_n$ in I_n for any n.

Definition 2.3.5. A net $\{a_\tau\}$ in a Riesz space $\mathcal{A}$ **(weakly)converges in**

order or is **(weakly) order convergent** to some $a \in \mathcal{A}$, written $(a_\tau \xrightarrow{w.o} a) a_\tau \xrightarrow{o} a$, if there is a net $\{d_\tau\}$ (with the same directed set) satisfying $(d_\tau \downarrow_w o) d_\tau \downarrow o$ and $(|b(a_\tau) - b(a)| \leq b(d_\tau))|a_\tau - a| \leq d_\tau$ for each τ [equivalently for any τ we have $(|b(a_n^{(\tau)}) - b(a_n)| \leq b(d_n^{(\tau)}))$ $|a_n^{(\tau)} - a_n| \leq d_n^{(\tau)}$ in I_n for any n.] In this case a is called **(weak) order limit** of $\{a_\tau\}$.
A sequence $\{a_k\}_{k=1}^\infty$ in $\mathcal{A}$ is said to be an **(weak)order Cauchy sequence** if there is a sequence $d_k \downarrow 0$ such that $(|b(a_l) - b(a_k)| \leq b(d_k))|a_l - a_k| \leq d_k$ for all $k \geq l \geq 1$. Equivalently $(|b(a_n^{(l)}) - b(a_n^{(k)})| \leq b(d_n^{(k)}))|a_n^{(l)} - a_n^{(k)}| \leq d_n^{(k)}$ for all $k \geq l \geq 1$ and for all n. One Riesz space is **order complete** if every order Cauchy sequence has an order limit. Equivalently a Riesz space is order complete if every subset of it has a supremum. It is easy to see from this equivalence definition that $\mathcal{A}$ is not order complete.

Definition 2.3.6. Let $u \geq 0$ be an element of a Riesz space of supernumbers $\mathcal{A}$. We say that the sequence $\{a_k\}_{k=1}^\infty$ in $\mathcal{A}$ is **(weakly) converges** u**-uniformly** to an element $a \in \mathcal{A}$ whenever, for every $\varepsilon > 0$, there exists a natural number K_ε such that $(|b(a_k) - b(a)| \leq \varepsilon b(u))$ $|a_k - a| \leq \varepsilon u$ holds for all $n \geq K_\varepsilon$. In this case a is called **(weakly)** u**-uniform limit** of $\{a_n\}$ and written as $(a_k \xrightarrow{w} a(u\text{-un}))$ $a_k \longrightarrow a(u\text{-un})$. It is said that the sequence $\{a_k\}_{k=1}^\infty$ in $\mathcal{A}$ is **(weakly) converges relatively uniformly** to $a \in \mathcal{A}$ whenever a_k (weakly) converges u-uniformly to a for some $u \in \mathcal{A}^+$. This kind of convergence is denoted by $(a_k \xrightarrow{w} a(\text{un}))$ $a_k \longrightarrow a(\text{un})$.

Now we find out some characteristics of $\mathcal{A}$ by means of propositions.

Proposition 2.3.3. *The infinite dimensional Riesz space of supernumbers is Archimedean.*

Proof. Every element a of $\mathcal{A}$ has the form $a = \sum\limits_n a_n$ where $a_n \in I_n \subseteq \mathcal{A}_n$.

Also $a \in \mathcal{A}^+$ if and only if $a_n \geq 0$ in I_n for any n. So

$$\frac{1}{m}a = \frac{1}{m}\sum_n a_n = \sum_n \frac{1}{m}a_n \geq \sum_n \frac{1}{m+1}a_n = \frac{1}{m+1}\sum_n a_n = \frac{1}{m+1}a$$

holds for any positive integer m, which means $\frac{1}{m}a \downarrow$. Also we have

$$\inf_m \frac{1}{m}a = \inf_m \frac{1}{m}\sum_n a_n = \inf_m \sum_n \frac{1}{m}a_n = \sum_n \inf_m \frac{1}{m}a_n = 0.$$

For any n, I_n is Archimedean and hence $\mathcal{A}$ is so. □

Proposition 2.3.4. *$\mathcal{A}$ is Dedekind complete Riesz space.*

Proof. Let A be a nonempty subset of $\mathcal{A}$ which is order bounded from above by b. Hence for any a in $\mathcal{A}$ we have $a \leq b$. Therefore $a_n \leq b_n$ in I_n for any n. Let $\mathsf{A}_n = \{a_n \in I_n \ : \ a \in \mathsf{A}\}$. Evidently A_n is nonempty set which is bounded from above by b_n. Dedekind completeness of I_n implies that A_n has a supremum as $s_n \in I_n$ such that $s_n \leq b_n$. It is clear that $s = \sum_n s_n$ is an element of $\mathcal{A}$ and $s \leq b$. So supremum of A is s and hence $\mathcal{A}$ is Dedekind complete. □

Proposition 2.3.5. *$\mathcal{A}$ is Dedekind σ-complete.*

Remark 2.3.1. Archimedean property and Dedekind σ-completeness of $\mathcal{A}$ with theorem 39.4 and lemma 39.2 of [28] imply the u-uniform completeness of $\mathcal{A}$, for any $u \in \mathcal{A}^+$. Uniformly completeness of $\mathcal{A}$ follows from definition.

Proposition 2.3.6. *$\mathcal{A}$ is order separable.*

Proof. Let A be a non-empty subset of $\mathcal{A}$ possessing a supremum $\sup \mathsf{A} = s$. Hence for any $a \in \mathsf{A}$, $a \leq s$ and so $a_n \leq s_n$ for any $n = 1, 2, ...$ Let $\mathsf{A}_n = \{a_n \ : \ a \in \mathsf{A}\}$. Therefore $\mathsf{A}_n \subset I_n$ and has a supremum s_n. Order separability of I_n implies that A_n has at most countable subset A'_n possessing the same supremum as A_n. Now take $\mathsf{A}' = \{a \in \mathcal{A} \ : \ a_n \in \mathsf{A}'_n \text{ for any } n\}$. This set is almost countable subset of A. □

Remark 2.3.2. Since $\mathcal{A}$ is Dedekind complete, Theorem 24.9(i) of [28] implies that $\mathcal{A}$ has projection property. Also Dedekind σ-completeness of $\mathcal{A}$ and Theorem 25.1 of [28] imply the principal projection property. Theorem 30.4 of [28] implies that $\mathcal{A}$ has sufficiently many projection.

Proposition 2.3.7. *The norms P_ω and $P_\omega^{(k)}$ in Banach lattice $\mathcal{A}$ are σ-order continuous.*

Proof. Let $\{a_m\}_{m=1}^{\infty}$ be a sequence in $\mathcal{A}$. For any m, $a_m = \sum\limits_n a_n^{(m)}$. Let $a_m \downarrow 0$ in $\mathcal{A}$. Then by definition $a_m \leq a_{m+1}$ for any m and $\inf\limits_m a_m = 0$. In other words $a_n^{(m)} \leq a_n^{(m+1)}$ for any m, n and $\inf\limits_m a_n^{(m)} = 0$ for any n. This says that $a_n^{(m)} \downarrow 0$ for any n. If $\omega = \{\omega_n\}$ be an increasing sequence of positive numbers then $\omega_n \|a_n^{(m)}\|_k \downarrow 0$ for any n and therefore $P_\omega^{(k)}(a_m) \downarrow 0$. □

By using remark 2.2.7 for $\mathcal{A}$ we can obtain similarly that the norms on $\mathcal{A}$ are order continuous and also $\mathcal{A}$ has Riesz-Fischer property. Now because of $\mathcal{A}$ is Dedekind complete Banach lattice with order continuous norms P_ω and $P_\omega^{(k)}$, then $\mathcal{A}$ has countable type property(Exercise 4(e), CHA. II of [41]). Also order continuity of norm of Banach lattice implies Fauto Property for norm. Since $\mathcal{A}$ is infinite dimensional Archimedean Dedekind σ-complete Riesz space which has order unit, order convergence in $\mathcal{A}$ is not stable(theorem 70.3 of [28]). For investigation of Egoroff and Diagonal Properties for $\mathcal{A}$, we have the following proposition and remark.

Proposition 2.3.8. *The infinite dimensional Riesz space of supernumbers has σ-property.*

Remark 2.3.3. Since the order convergence in $\mathcal{A}$ is not stable then it hasn't the d-property by theorem 70.2 of [28] which this and Archimedean property of $\mathcal{A}$ imply that $\mathcal{A}$ hasn't strong Egoroff property. Theorem 68.8 of [28] implies that the order convergence hasn't diagonal and diagonal gap properties. On the other hand since $\mathcal{A}$ has σ-property therefore $\mathcal{A}$ has diagonal and diagonal gap property for relatively uniform convergence.

2.3.3 Riesz Algebra Structure

Now we want to reformulate the concept of functional calculus for infinite dimensional Riesz space of Supernumbers $\mathcal{A}$. Recall that any element $a \in \mathcal{A}$ has the unique form $a = \sum\limits_{n} a_n$ where $a_n \in I_n$ for any n. By considering the stated definitions in subsection 1.3.5 we can work in a similar way.

For any $a \in \mathcal{A}$ the principal ideal J_a generated by a is

$$\begin{aligned} J_a &= \{b \in \mathcal{A} \quad | \quad \exists \lambda > 0 \quad \text{with} \quad |b| \le \lambda |a|\} \\ &= \{b \in \mathcal{A} \quad | \quad \exists \lambda > 0 \quad \text{with} \quad |b_n| \le \lambda |a_n| \quad \text{for any} \quad n\} \\ &\subseteq \{b \in \mathcal{A} \quad | \quad b_n \in J_{a_n} \quad \text{for any} \quad n\}. \end{aligned}$$

If a is strong order unit in $\mathcal{A}$ then $J_a = \mathcal{A}$. Now for $a \in \mathcal{A}^+$, the set C_a of all components of a will be as follows:

$$\begin{aligned} C_a &= \{b \in \mathcal{A}^+ \quad | \quad b \wedge (a - b) = 0\} \\ &= \{b \in \mathcal{A}^+ \quad | \quad b_n \in C_{a_n} \quad \text{for any} \ \ n \ \}. \end{aligned}$$

The band generated by an ideal J of $\mathcal{A}$ is given by

$$\begin{aligned} B_J &= \{a \in \mathcal{A} \quad | \quad \exists \text{a net} \ \{a_\tau\} \subset J \quad \text{with} \quad 0 \le a_\tau \uparrow |a|\} \\ &= \{a \in \mathcal{A} \quad | \quad \exists \text{a net} \ \{a_\tau\} \subset J \quad \text{with} \quad 0 \le a_\tau \uparrow \text{and} \\ & \qquad \sup_\tau \{a_n^\tau\} = |a_n| \text{in } I_n \quad \text{for any} \quad n\}. \end{aligned}$$

According to remark 2.2.13 $\mathcal{A}$ has principal projection property, therefore any principal band generated by an element a will be projection band and if $a \in \mathcal{A}^+$

then for any $b \in \mathcal{A}^+$ the element

$$\begin{aligned} P_{B_a}(b) &= \sup(b \wedge ma \;\; m = 1, 2, \dots) \\ &= \sum_n P_{B_{a_n}}(b_n) \end{aligned}$$

exists always.

We know that every totally positive supernumbers is strong order unit of $\mathcal{A}$. We will use the element $\underline{1} = \sum_n \underline{1}_n \in \mathcal{A}$ in the same way as finite dimensional case. The set $C_{\underline{1}}$ of all components of $\underline{1}$ is

$$\begin{aligned} C_{\underline{1}} &= \{a \in \mathcal{A}^+ \;\; | \;\; a \wedge (\underline{1} - a) = 0\} \\ &= \{a \in \mathcal{A}^+ \;\; | \;\; a_n \in C_{\underline{1}_n} \;\; \text{for any} \;\; n\}. \end{aligned}$$

Evidently $\underline{0}, \underline{1} \in C_{\underline{1}}$. The principal ideal $J_{\underline{1}}$ generated by $\underline{1}$ is the same $\mathcal{A}$, i. e., $J_{\underline{1}} = \mathcal{A}$ and we have

$$\begin{aligned} J_{\underline{1}} &= \{a \in \mathcal{A} \;\; | \;\; \exists \lambda > 0 \;\; \text{with} \;\; |a| \leq \lambda \underline{1}\} \\ &= \{a \in \mathcal{A} \;\; | \;\; \exists \lambda > 0 \;\; \text{with} \;\; |a_n| \leq \lambda \underline{1}_n \;\; \text{for any} \;\; n\} \\ &\subseteq \{a \in \mathcal{A} \;\; | \;\; a_n \in J_{\underline{1}_n} \;\; \text{for any} \;\; n\}. \end{aligned}$$

Given an element $a \in J_{\underline{1}} = \mathcal{A}$, there exist a spectral interval $[r, t]$ for a and partition $\mathsf{P} : r = k_0 < k_1 < \cdots < k_m = t$ for $[r, t]$. Let P_k be the band projection onto the band generated by $(k\underline{1} - a)^+$ for any $k \in [r, t]$. In fact, for any $b \in \mathcal{A}^+$

$$P_k(b) = \sup_m(b \wedge [m(k\underline{1} - a)^+]) = \sum_n (b_n \wedge \sup_m(m(k\underline{1}_n - a_n))^+ = \sum_n P_k(b_n).$$

If F be a continuous real valued function on $[r, t]$, in a similar way of finite dimensional case we can have upper and lower sums as 2.2.16 for given P. Therefore since $\mathcal{A}$ is Dedekind σ-complete, according to theorem 34.1 of [54], the sets of all lower and upper sums, induced by different partitions of $[r, t]$, have supremum and infimum in $\mathcal{A}$ respectively which are equal. This common value is denoted by $F(a)$. Similar results as section 2.2.5 are hold and $\mathcal{A}$ will be a Riesz algebra too. Also it has similar proof as finite dimensional case that $\mathcal{A}$ is a commutative Banach algebra with this new multiplication.

2.4 Graßmann Algebra

In this section we consider to an important algebra which has wide usage in theoretical physics and Mathematics. Of course physicists and mathematicians use this algebra instead of number fields. So it's elements is also called **supernumbers**. In example 2.2.1 we saw that its structure is in accordance of σ-commutative G-graded algebras. Here, we consider to structure of this algebra in more details.

2.4.1 Finite dimensional Graßmann algebra

The **Graßmann algebra**(or **exterior algebra**)Λ_n with n generators is the associative algebra (over $\mathbb{C}$) generated by a set of n anticommuting generators $\{\xi_i\}_{i=1}^n$ and by $1 \in \mathbb{C}$ with the property

$$\xi_i \xi_j = -\xi_j \xi_i \qquad \text{for all } i, j, \tag{2.4.1}$$

in particular $\xi_i^2 = 0$.

It follows from 2.4.1 that any element of Λ_n is linear combination of monomials $\xi_{m_1}\xi_{m_2}...\xi_{m_k}$ with $1 \leq m_1 < m_2 < \cdots < m_k \leq n$ and the unit such that $1 \leq k \leq n$. Since all monomials among ξ_i follow from 2.4.1, these monomials are linearly independent. Consequently, together with the unit, they form a basis of Λ_n as a linear space. Since their number is equal to the number of subsets of n elements, we have dim $\Lambda_n = 2^n$.

Any element $q \in \Lambda_n$ may be written as

$$q = q(\xi) = \sum_{k \geq 0} \sum_{m_1,...,m_k} q_{m_1,...,m_k} \xi_{m_1}\xi_{m_2}...\xi_{m_k}. \tag{2.4.2}$$

The term corresponding to $k = 0$ is proportional to the unit. The relation $q = q(\xi)$ shows the fact that q is expressed in the form of a polynomial in ξ_m. In what follows we see that polynomials $q = q(\xi)$ has many formal properties of the usual functions. We shall call them **functions of anticommuting variables**. The expression in elements of Λ_n in the above form is not unique in general. This becomes unique if supplementary conditions are imposed on coefficients $q_{m_1,\ldots,m_k}$. For instance, we may require that $q_{m_1,\ldots,m_k} = 0$ whenever the relation $m_1 < m_2 < \cdots < m_k$ fails or that $q_{m_1,\ldots,m_k}$ are skew-symmetric with respect to indices $m_1, \ldots, m_k$ (i. e., $q_{m_1,\ldots,m_k}$ change the sign under permutation of any two indices). In what follows we always assume first condition ($q_{m_1,\ldots,m_k} = 0$ if $m_1 < m_2 < \cdots < m_k$ fails). Let

$$M_n = \{(m_1, \ldots, m_k)| \ ;1 \leq k \leq n \ ;1 \leq m_1 < \cdots < m_k \leq n \ n, m_i \in \mathbb{N}\}. \tag{2.4.3}$$

Therefore with this condition any element $q \in \Lambda_n$ can be written uniquely as

$$q = \sum_{k \geq 0} \sum_{\substack{m_1,\ldots,m_k \\ m_1 < m_2 < \cdots < m_k}} q_{m_1,\ldots,m_k} \xi_{m_1} \xi_{m_2} \cdots \xi_{m_k} \tag{2.4.4}$$

$$= q_0 + \sum_{(m_1,\ldots,m_k) \in M_n} q_{m_1,\ldots,m_k} \xi_{m_1} \xi_{m_2} \cdots \xi_{m_k} \tag{2.4.5}$$

such that $q_0 \in \mathbb{C}$ is the number for $k = 0$. A. Rogers in [36] defined a norm on Λ_n as follows:

$$\|q\| = \sum_{(m_1,\ldots,m_k) \in M_n} |q_{m_1,\ldots,m_k}| + |q_0| \quad \text{for} \quad q \in \Lambda_n.$$

This norm makes Λ_n to be Banach algebra which sometimes called **Rogers norm** and $(\Lambda_n, \| \ \|)$, the **Rogers algebra**.

Now we return to example 2.2.1 and consider to the σ-commutative G-graded algebra structure of Λ_n. Indeed we shall show that these two kind of representing of Graßmann algebra are coincide.

Let $G = \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ and $\sigma(\alpha, \beta) = (-1)^{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$. Note that $(-1)^{\overline{0}} = 1$ and $(-1)^{\overline{1}} = -1$. Therefore according to definition of finite dimensional σ-commutative G-graded algebra (introduced in section 2.2) we will have $G_0 = \{0\}$ and the factor system ϕ is trivial ($\phi(\alpha, \beta) = 1$ for all $\alpha, \beta \in \mathbb{Z}_2$). Also the crossed product C is equal to the base field $\mathbb{C}$.

Now let L be a finite linearly ordered odd $\mathbb{Z}_2$-set, that is, each element l of L has odd grade:

$$g : L \longrightarrow \mathbb{Z}_2 \qquad \forall l \in L \quad ; \quad g(l) = 1.$$

Let V be the $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$ with basis $\{\xi_l \ : \ l \in L\}$ where the grade of v_l is $g(l)$. Let B be quotient algebra over V defined by $B = T(V)/I$, where I is the ideal of the tensor algebra $T(V)$ over V generated by the elements $\xi_l\xi_{l'} - (-1)^{g(l)g(l')}\xi_{l'}\xi_l$, which is equal with $\xi_l\xi_{l'} + \xi_{l'}\xi_l$. Let M be a subset of L. The ordered product $\prod_{\ell \in M} v_\ell$ is written as v_M, that is, if $M = \{l_1, l_2, \dots, l_k\}$ then $v_M = v_{l_1}v_{l_2}\dots v_{l_k}$ such that $l_1 < l_2 < \cdots < l_k$. Hence B is a $\mathbb{Z}_2$-graded $(-1)^{\alpha\beta}$-commutative algebra with a linear basis $\{v_M \mid M \subseteq L\}$ over $\mathbb{C}$. Remember that with this assumptions, the crossed product C is the same $\mathbb{C}$. Let $\mathcal{A} = C \otimes_{\mathbb{C}} B = B$. Since B is the σ-Graßmann algebra over V, if L has n element then B is the same Graßmann algebra Λ_n. As in 2.2.1 any element of $\mathcal{A} = B = \Lambda_n$ is expressed uniquely as

$$q = \sum_{\alpha, M} q_{\alpha,M} u_\alpha \otimes v_M = \sum_M q_M v_M$$

where $\alpha \in G_0 = \{0\}$,$q_M = q_{0,M}u_0$ with $u_0 = 1$ and summation is taken over all subsets M of L and $q_M \in \mathbb{C}$.

According to equation 2.2.2 the body and the soul of q are as follows:

$$b(q) = \sum_{\alpha \in G_0} q_{\alpha,\varnothing} u_\alpha \otimes 1 = q_{0,\varnothing}u_0 1 = q_B 1 \qquad q_B \in \mathbb{C}, \quad u_0 \in \mathbb{C}$$

and

$$s(q) = \sum_{\substack{M = \varnothing \\ \alpha \in G_0}} q_{\alpha,M} u_\alpha \otimes v_M = \sum_{M \neq \varnothing} q_{0,M} u_0 \otimes v_M = \sum_{M \neq \varnothing} q_M v_M.$$

2.4.2 Infinite dimensional Graßmann algebra

The concept of infinite dimensional Graßmann algebra is defined in two ways. A. Rogers in [37] took it to be the direct limit of finite dimensional Graßmann algebras and Nagamachi and Kobayashi [32] took it to be the inductive limit of them. Here we give the Rogers approach and in the next chapter we will see the Nagamachi and Kobayashi's terminology.

Let m and n be two positive integers, with $n > m$. The Graßmann algebras Λ_n and Λ_m generated by the sets $\{\xi_i^{(n)} : i = 1, 2, ..., n\} \cup \{1^{(n)}\}$ and $\{\xi_i^{(m)} : i = 1, 2, ..., m\} \cup \{1^{(m)}\}$ respectively, are assumed as previous subsection. Then there is a natural injection $j_{m,n} : \Lambda_m \longrightarrow \Lambda_n$, which is the unique algebra homomorphism satisfying

$$j_{m,n}(\xi_i^{(m)}) = \xi_i^{(n)} \qquad i = 1, \ldots, n \quad ; \quad j_{m,n}(1^{(m)}) = 1^{(n)}.$$

Λ_n naturally has a Λ_m-module structure with, given $\lambda^{(n)} \in \Lambda_n$ and $\lambda^{(m)} \in \Lambda_m$;

$$\lambda^{(m)}\lambda^{(n)} = j_{m,n}(\lambda^{(m)})\lambda^{(n)}.$$

This means that every Graßmann algebra with n generators can be embedded in a Graßmann algebra with $n+1$ generators. Therefore the set $\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots\}$ make a direct system and according to subsection 1.2.1, we can consider the direct limit of this direct system. We will denote by Λ_∞ the direct limit of this sequence, i.e., $\Lambda_\infty = \lim\limits_{\rightarrow} \Lambda_n$.

Chapter 3

SUPER HILBERT SPACE

3. SUPER HILBERT SPACE

During of suberization of elements of ordinary mathematics, some authors consider to Hilbert space. Suberization of Hilbert space is done by different authors. The first notion of "Super Hilbert Space" has been considered by Dewitt [10]. He defines a super Hilbert space $\mathcal{H}$ basically as a $\mathbb{Z}_2$-graded Λ_n-module, where n is possibly infinite, with a Λ_n-valued inner product. Others are gave different definitions. Khrennikov, in [20] and [21] defined a super Hilbert space to be a Banach(commutative) Λ-module which is isometric to the space $\ell_2(\Lambda)$ of square-summable sequences in Λ with the inner product $\langle x, y\rangle = \sum x_n y_n^*$ and norm $\|x\|^2 = \langle x, x\rangle$. According to Schmitt definition, [44], a super Hilbert space is just a complex $\mathbb{Z}_2$-graded ordinary Hilbert space. Nagamachi and Kobayashi formulated and refined Dewitt's definition by taking also into account the topological and norm structure on super Hilbert spaces [32]. El Gradechi and Nieto [11] defined a super Hilbert space to be a direct sum $\mathcal{H} = H_0 \oplus H_1$ of two complex Hilbert spaces $(H_0, \langle ., .\rangle_0)$ and $(H_1, \langle ., .\rangle_1)$ equipped with the super hermitian form $\langle\langle ., .\rangle\rangle = \langle ., .\rangle_0 + i\langle ., .\rangle_1$. Samsonov considered the super Hilbert space as a $\mathbb{Z}_2$-graded infinite dimensional linear space equipped with a super- hermitian form (superscalar product) and in some sense complete [40]. The last definition of super Hilbert space is given by Rudolph, [38], which is a module over a Grammann algebra endowed with a Graßmann number-valued inner product.

The aim of present chapter is investigation of these definitions in a mathematical framework. Of course super Hilbert spaces, as ordinary Hilbert spaces, have many applications in theoretical physics and Quantum physics. But we do not work in this area. Also some of definitions has more implicit examples which require many other information about physics. Therefore we argue only the basic structure of definitions. The last definition of super Hilbert space, which is given by Rudolph, has a mathematical framework and is more general than others such that many of examples of other definitions can be considered as an example of it. In current chapter we give different definitions of super Hilbert space with more focus on Rudolph's. The last section include comparison between different definitions of super Hilbert space.

3.1 Dewitt

In this section we consider the definition of super Hilbert space according to Bryce Dewitt. As stated above, his definition is the first which is given. We need some terminology to understand it. Therefore we state some notions about supernumbers, which are elements of infinite dimensional Graßmann algebra, and supervector spaces, which are $\mathbb{Z}_2$-graded vector spaces over Graßmann algebra.

3.1.1 Supernumbers

Let $\xi_i, i = 1, \ldots, n$ be a set of generators for an algebra, which anticommute:

$$\xi_i \xi_j = -\xi_j \xi_i \qquad (\xi_i)^2 = 0 \quad \text{for all } i, j. \tag{3.1.1}$$

The algebra is called a **Graßmann algebra** and will be denoted by Λ_n. We shall usually, though not always, deal with the formal limit $n \longrightarrow \infty$. The corresponding algebra will be denoted by Λ_∞. The elements $1, \xi_i, \xi_{ij}, \ldots,$ where the indices in each product are all different, form an infinite basis for Λ_∞. The elements of Λ_∞ are called **supernumbers**. Every supernumber can be expressed in the form $z = z_B + z_s$, where z_B is an ordinary complex number and

$$z_S = \sum_{n=1}^{\infty} \frac{1}{n!} q_{a_1,\ldots,a_n} \xi_{a_n} \ldots \xi_{a_1}, \tag{3.1.2}$$

the q's also being complex numbers. The number z_B is called the **body** and remainder z_S will be called the **soul** of the supernumber z. For n finite the soul of supernumber is always nilpotent: $z_S^{n+1} = 0$.

Any supernumber may be split into its even and odd parts; $z = u + v$ where

$$u = z_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} q_{a_2,\ldots,a_{2n}} \xi_{a_{2n}} \ldots \xi_{a_2} \tag{3.1.3}$$

and

$$v = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} q_{a_1,\ldots,a_{2n+1}} \xi_{a_{2n+1}} \ldots \xi_{a_1}. \tag{3.1.4}$$

Odd supernumbers anticommute among themeselves and will be called a-**numbers**. Even supernumbers commute with everything and will be called c-**numbers**. The set of all c-numbers is commutative subalgebra of Λ_∞, which will be denoted by C_c. The set of all a-numbers will be denoted by C_a; it is not subalgebra. The product of two c-number, or two a-number is a c-number. The product of an a-number and a c-number is an a-number.

For any $z_1, z_2 \in \Lambda_\infty$, the rules complex conjugation is valid:

$$(z_1 + z_2)^* = z_1^* + z_2^* \qquad (z_1 z_2)^* = z_2^* z_1^*. \tag{3.1.5}$$

The complex conjugate of the body of a supernumber will be taken to be as its ordinary complex conjugate, and the generators of Λ_∞ will be assumed to be real;

$$\xi_i^* = \xi_i \quad \text{for all } i. \tag{3.1.6}$$

Evidently; $(\xi_i \ldots \xi_j)^* = \xi_j \ldots \xi_i$ and from this, together with the anticommutation law ($\xi_i \xi_j = -\xi_j \xi_i$), one may infer that the basis element $\xi_{a_1} \ldots \xi_{a_n}$ is real when $\frac{1}{2}n(n-1)$ is even and imaginary when $\frac{1}{2}n(n-1)$ is odd. (As for ordinary numbers, a supernumber z is said to be real if $z^* = z$ and imaginary if $z^* = -z$). A general element of Λ_∞ is real if and only if both of its body and soul are real. The soul will be real if and only if the coefficients $q_{a_1,\ldots,a_n}$ in the expansion 3.1.2 are real when $\frac{1}{2}n(n-1)$ is even and imaginary when $\frac{1}{2}n(n-1)$ is odd.

We shall denote by $\mathbb{R}_c$ the subset of all real elements of C_c and by $\mathbb{R}_a$ the subset of all real elements of C_a. The set $\mathbb{R}_c$ is a subalgebra of C_c.

3.1.2 Supervector space

Definition 3.1.1. A **supervector space** is a set V of elements called **supervectors**, together with a collection of mappings, having properties as follows:

(a) There exists a binary operation mapping $+ : V \times V \longrightarrow V$, called addition, which V is a commutative group.

(b) For every supernumber $\lambda \in \Lambda_\infty$ there exist two mappings, $\lambda_L : V \longrightarrow V$ and $\lambda_R : V \longrightarrow V$, called *left multiplication* and *right multiplication* respectively and conventionally expressed by the notation

$$\lambda_L(v) = \lambda v \qquad \lambda_R(v) = v\lambda \quad \text{for all } v \in V \tag{3.1.7}$$

These mappings satisfy the linear laws:

$$(\alpha+\beta)v = \alpha v + \beta v \qquad v(\alpha+\beta) = v\alpha + v\beta \tag{3.1.8}$$

$$\alpha(u+v) = \alpha u + \alpha v \qquad (u+v)\alpha = u\alpha + v\alpha \tag{3.1.9}$$

$$(\alpha\beta)v = \alpha(\beta v) = \alpha\beta v \qquad v(\alpha\beta) = (v\alpha)\beta = v\alpha\beta \tag{3.1.10}$$

$$1v = v \qquad v1 = v \tag{3.1.11}$$

for all $\alpha, \beta \in \Lambda_\infty$ and for all $u, v \in V$. In 3.1.11, 1 is the ordinary number one.

(c) Left and right multiplication are related. Firstly

$$(\alpha v)\beta = \alpha(v\beta) = \alpha v\beta \qquad \text{for all } \alpha, \beta \in \Lambda_\infty \text{ and for all } v \in V. \tag{3.1.12}$$

Secondly, if α is a c-number then it commutes with all supervectors. That is

$$\alpha v = v\alpha \qquad \text{for all } \alpha \in C_c \quad \text{and all } v \in V. \tag{3.1.13}$$

Thirdly, for every w in V, there exist unique supervectors v and u in V such that

$$w = u + v, \qquad \alpha u = u\alpha \qquad \text{and } \alpha v = -v\alpha, \quad \text{for } \alpha \in C_c. \tag{3.1.14}$$

The supervectors u and v are called, respectively, the **even** and **odd** parts of w. If the odd part of a supervector vanishes(i.e., equals to the zero supervector) the supervector is said to be of **type** c. If its even part vanishes it is said to be of **type** a. The zero supervector is the only supervector which is simultaneously c-type and a-type.

A supervector that has a definite type will be called **pure**. Similarly, a supervector that is either a c-number or an a-number will be called pure. For pure supernumbers and supervectors equations 3.1.13 and 3.1.14 may be summarized into formula

$$\alpha v = (-1)^{\alpha v} v\alpha. \tag{3.1.15}$$

(d) There exists a mapping $* : V \longrightarrow V$ called **complex conjugation** conventionally written in the form $*(v) = v^*$ for all $v \in V$ which satisfies

$$v^{**} = v; \quad (v+u)^* = v^* + u^*; \quad (\alpha v)^* = v^*\alpha^*; \quad (v\alpha)^* = \alpha^* v^*$$

for all $u, v \in V$ and all $\alpha \in \Lambda_\infty$. It is easy to verify that the complex conjugate of a pure supernumber is pure, the type remaining unaffected by the complex conjugation mapping. A supervector z will be said to be real if $z^* = z$, imaginary if $z^* = -z$, and complex otherwise. A complex supervector can always be decomposed into its real and imaginary parts. Note that the product of a real c-number and a real supervector is a real supervector. The product of a real a-number and a real c-type supervector is a real a-type supervector, but the product of a real a-number and a real a-type supervector is an imaginary c-type supervector.

3.1.3 Dual supervector space

Let V be a supervector space. The supervector space dual to V, denoted by V^*, is defined to be the set of all mappings $\omega : V \longrightarrow \Lambda_\infty$, conventionally expressed by the notion

$$\omega(v) = v.\omega \qquad \text{for all } v \in V, \tag{3.1.16}$$

which satisfy the linear laws

$$(\alpha v).\omega = \alpha(v.\omega) \qquad (v_1 + v_2).w = v_1.\omega + v_2.\omega$$

for all α in Λ_∞ and all $v_1, v_2 \in V$. $v.\omega$ is called the inner product of v and ω.

The set V^* has the structure of supervector space. The unique even and odd parts of ω are defined by

$$\omega = \omega_0 + \omega_1 \quad w\omega_0 = (u.\omega)_0 + (v.\omega)_0 \quad w\omega - 1 = (u.\omega)_1 + (v.\omega)_1$$

for all $w \in V$, where $w = u + v$, u and v are even and odd parts of w, respectively. The c-type elements of V^* map c-type elements of V into c-numbers and a-type elements of V into a-numbers. With a-type elements of V^* the association is reversed. From these facts we have

$$\alpha\omega = (-1)^{\alpha\omega}\omega\alpha.$$

Also $v\omega^* = (-1)^{v\omega}(v^*.\omega)^*$ for all $v \in V$ and all $\omega \in V^*$.

3.1.4 DeWitt's super Hilbert spaces

As we stated beforehand, the first definition of super Hilbert space is given by Bryce Dewitt [10]. He used the Dirac notation for his work and many mathematics students are not familiar with it. Rudolph [38] brings his definition in a usual notation and we will use it.

Definition 3.1.2. A **super Hilbert space** $\mathcal{H}$ is a supervector space for which the notion of a real (or imaginary) supervector is undefined and for which the complex conjugation mapping is replaced by an inner product, i.e., by a one-to-one mapping $* : \mathcal{H} \longrightarrow \mathcal{H}^*$, from $\mathcal{H}$ to its dual $\mathcal{H}^*$, which satisfies the following axioms:

(1) $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$, for $x, y_1, y_2 \in \mathcal{H}$;

(2) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle = \langle \alpha^* x, y \rangle$, for $x, y \in \mathcal{H}, \alpha \in \mathbb{C}$;

(3) $\langle x, yq \rangle = \langle x, y \rangle q$ for all $x, y \in \mathcal{H}, q \in \Lambda_n$.

(4) $\langle x, y \rangle = \langle y, x \rangle^*$, for $x, y \in \mathcal{H}$;

(5) $\langle x, x \rangle_B \geq 0$ for $x \in \mathcal{H}$; $x \in \mathcal{H}$ has nonvanishing body if and only if $\langle x, x \rangle_B > 0$;

(6) $\langle x_s, y_r \rangle q_t = (-1)^{t(s+r)} q_t \langle x_s, y_r \rangle$ for all pure $x_s \in \mathcal{H}_s, y_r \in \mathcal{H}_r$ and $q \in \Lambda_n$, $\deg(q_t) = t$.

DeWitt moreover requires that the body of $\mathcal{H}$ is an ordinary complex Hilbert space. As has already remarked, super Hilbert spaces are generalization of ordinary Hilbert spaces. Every element $x \in \mathcal{H}$ is called *physical* if it merely has non vanishing body. Physical elements of $\mathcal{H}$ are also called *state vectors*. It is easy to show that if x is physical, it can be normalized(i.e., multiplied by an appropriate supernumber with nonvanishing body) so that $\langle x, x \rangle = 1$.

Also it is easy to see that sesqui-Λ-linearity implies $\langle x_B, y_B \rangle = \langle x, y \rangle_B$ for all $x, y \in \mathcal{H}$.

3.1.5 Linear operators

A mapping $T : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be a linear operator if and only if, for all $x, y \in \mathcal{H}$ and all $\alpha \in \Lambda_\infty$,

$$T(x\alpha) = T(x)\alpha \qquad T(x+y) = T(x) + T(y), \tag{3.1.17}$$

where Tx is shortand for $T(x)$. A linear operator acting on $\mathcal{H}$ may equally well be regarded as acting on $\mathcal{H}^*$ through the rule

$$(\omega T x) = \omega(T(x)) = \omega T x,$$

for all $x \in \mathcal{H}$ and all $\omega \in \mathcal{H}^*$. Evidently T is also a linear operator when acting on $\mathcal{H}^*$. Linear operators may be combined with each other and with supernumbers. Hence the set of all linear operators costitutes what may be called **superalgebra**.

The adjoin T^* of a linear operator T is defined by $T^*x = (Tx)^*$ for all $x \in \mathcal{H}$. T^* is also a linear operator. The operator T is said to be *self-adjoint* if and only if $T^* = T$.

3.1.6 Physical observables

A linear operator T, whenever c-type or a-type will be called **physical observable** if and only if

(i) it is self-adjoint;

(ii) all its eigenvalues are c-numbers;

(iii) for every eigenvalue, there is at least one corresponding physical eigenvector;

(iv) the set of physical eigenvectors that correspond to soulless eignvalues, contains a complete basis.

The soulless eigenvalues will be called *physical eignvalues*.

3.2 Khrennikov's Hilbert Super space

In this section we review the definition of Hilbert super space which is given by Andrev Khrennikov in [20] and [21]. He considered superspaces over commutative superalgebras, commutative supermodules and commutative Hilbert supermodules. By using of them, he obtained three versions for Hilbert superspace. Here we bring them.

Let Λ be a vector space over a field $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$ with a given decomposition into the direct sum of subspaces Λ_0 and $\Lambda_1 : \Lambda = \Lambda_0 \oplus \Lambda_1$(a $\mathbb{Z}_2$-graded space). Any element a in Λ_0 is said to be of *even* pairity $|a| = 0$ and any element a in Λ_1 is said to be of *odd* parity $|a| = 1$. The elements in Λ_0 and Λ_1 are said to be *homogeneous.*

Definition 3.2.1. Commutative Banach Superalgebra

A **superalgebra** is a space $\Lambda = \Lambda_0 \oplus \Lambda_1$ endowed with the structure of an associative algebra with identity and an even multiplication operator (i.e., $|ab| = |a||b| \bmod 2$ for any homogeneous elements a, b). The **supercommutator** $[a, b]$ of homogeneous elements a and b in the superalgebra Λ is defined as

$$[a, b] = ab - (-1)^{|a||b|}ba. \tag{3.2.1}$$

A superalgebra Λ is *commutative* if $[a, b] = 0$ for arbitrary homogeneous elements $a, b \in \Lambda$. In the remainder of this section it is assumed that $\Lambda = \Lambda_0 \oplus \Lambda_1$ is a Banach CSA.

Example 3.2.1. Infinite dimensional Graßmann algebra

Let $(q_j)_{j=1}^{\infty}$ be a system of anticommuting generators. We denote by Λ the linear space consisting of the series $q = \sum_r c_r q^r$ with $r = (r_1, \ldots, r_n, \ldots)$, $r_j = 0, 1,\ |r| < \infty,\ c_r \in \mathbb{F},\ q^r = q_1^{r_1} \ldots q_n^{r_n} \ldots$ and $\|q\| = \sum_r |c_r| < \infty$. The subspaces Λ_0 and Λ_1 consists of the series with even and odd $|r|$ respectively.

The topology in $\Lambda = \Lambda_0 \oplus \Lambda_1$ defined by the norm $\|\ \ \|$ agrees with the algebric structure and the grading.

Definition 3.2.2. The space $\Lambda_0^n \times \Lambda_1^m$ is called a **superspace** over the CSA Λ and is denoted by $\mathbb{F}_\Lambda^{n,m}$.

Definition 3.2.3. commutative supermodule(CSM)

A **supermodule** is a two-sided unitary $\mathbb{Z}_2$-graded module $m = m_0 \oplus m_1$ over a superalgebra $A = A_0 \oplus A_1$ in which the multiplication by elements of A is even. A supermodule $m = m_0 \oplus m_1$ over a CSA $\Lambda = \Lambda_0 \oplus \Lambda_1$ is said to be *commutative* if $[a, \lambda] = 0$ for any homogeneous elements $a \in m$ and $\lambda \in \Lambda$, where the supercommutator $[\ \ ]$ is defined as in 3.2.1.

In what follows we shall assume that all CSM's are endowed with locally convex topologies which agree with the algebric structure and grading.

Definition 3.2.4. A space $\mathbb{F}^{m_0,n_1} = m_0 \times n_1$, where $m = m_0 \oplus m_1$ and $n = n_0 \oplus n_1$ are CSM, is called a **superspace** over the CSM's m and n.

Definition 3.2.5. The CSM m and n are said to be **dual** to one another if there exists a bilinear form $(.,.) : m \times n \longrightarrow \Lambda$ continuous on compact sets which separates points of the CSM and satisfies the condition

$$(\lambda u \mu, v \gamma) = \lambda (u, \mu v) \gamma \qquad (3.2.2)$$

for any $\lambda, \mu, \gamma \in \Lambda$ and $u \in m, v \in n$.

The form $(.,.)$ is called a **duality form**. We can also introduce a duality form $(.,.) : n \times m \longrightarrow \Lambda$ by setting $(u, v) = (-1)^{|u||v|}(v, u)$ for homogeneous $v \in m$ and $u \in n$. Condition 3.2.2 is also true for this duality form. Superspaces over dual CSM are called **dual superspace**.

On superspaces $V = \mathbb{F}^{m_0,n_1}$ and $W = \mathbb{F}^{p_0,q_1}$, where m, n and p, q are dual CSM's, the duality form is defined as

$$(v, w) = (\pi_0 v, \pi_0 w) + (\pi_1 v, \pi_1 w)$$

where π_0 and π_1 are projections onto the even and odd parts of a $\mathbb{Z}_2$-graded space.

Definition 3.2.6. Hilbert Superspace

A superspace Z is said to be *Hilbert* if it is complete and the duality form 3.2.2 is defined on $Z \times Z$.

Let us suppose that compatible involutions $*$ that converse pairity are defined on a Banach CSA algebra Λ and on a Hilbert superspace Z. The form $\langle z_1, z_1 \rangle = (z_1, z_2^*)$ is called the Λ_0**-product**. This is a natural generalization of the inner product on a Hilbert space.

Example 3.2.2. *We denote by* $L_2(\mathbb{R}^n, \Lambda)$ *the space of functions* $\phi : \mathbb{R}^n \longrightarrow \Lambda$ *such that*

$$\|\phi\|_2^2 = \int \|\phi(x)\|^2 d^n x < \infty.$$

The space $L_2(\mathbb{R}^n, \Lambda)$ *is a Hilbert superspace with the* Λ_0*-product*

$$\langle \phi_1, \phi_2 \rangle = \int (\pi_0 \phi_1(x) \pi_0 \phi_2^*(x) + \pi_1 \phi_1(x) \pi_1 \phi_2^*(x)) d^n x.$$

Definition 3.2.7. The Banach algebra B is called a $\sum$**-algebra** if for every elements $b_1, \ldots, b_m \in B$,

$$\sum_{j=1}^{m} \|b_j\| = \max_{\|b_j\| \leq 1} \| \sum_{j=1}^{m} \lambda_j b_j \| \quad \lambda_j \in \Lambda.$$

Let Λ be a CSA. We introduce the Banach spaces of sequences of elements of Λ, which are the superanaloges of the standard Banach spaces of the numerical sequences c_0, m, ℓ_p:

$$c_0(\Lambda) = \{x = (x_1, \ldots, x_n, \ldots) \in \Lambda^\infty : \quad x_n \longrightarrow 0\};$$

$$m(\Lambda) = \{x = (x_1, \dots, x_n, \dots) \in \Lambda^\infty : \quad \|x\|_\infty = \sup \|x_n\| < \infty\};$$

$$\ell_p(\Lambda) = \{x = (x_1, \dots, x_n, \dots) \in \Lambda^\infty : \quad \|x\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{\frac{1}{p}} < \infty\} \quad p \geq 1.$$

Proposition 3.2.1. *The spaces of Λ-sequences $c_0(\Lambda), m(\Lambda), \ell_p(\Lambda), p \leq 1$ are commutative Banach supermodules over the CSA Λ.*

Definition 3.2.8. Coordinate Hilbert Module

The Banach commutative supermodule $\ell_2(\Lambda)$ (with the norm $\|x\|_2^2 = \sum\limits_{n=1}^{\infty} \|x\|^2$) is called the **coordinate Hilbert supermodule**. A scalar product (Λ-valued) in $\ell_2(\Lambda)$ is defined by the duality

$$\langle x.y \rangle = \sum_{n=1}^{\infty} x_n y_n. \tag{3.2.3}$$

This scalar product has the following properties:

$$\langle \alpha x \beta, y\gamma \rangle = \alpha \langle x, \beta y \rangle \gamma \quad \text{for any } \alpha, \beta, \gamma \in \Lambda, x, y \in \ell_2(\Lambda); \tag{3.2.4}$$

$$\langle x, y \rangle = (-1)^{|x||y|} \langle y, x \rangle \quad \text{for any } x, y \in \ell_2(\Lambda); \tag{3.2.5}$$

The canonical basis $e_n = (0, \dots, 1, \dots)$ in $\ell_2(\Lambda)$ is orthonormal with respect to this inner product. The above properties are called **Λ-linearity** and **supersymmetry** respectively.

Definition 3.2.9. Hilbert supermodules

Let $M = M_0 \oplus M_1$ be a commutative supermodule; then a bilinear form $\langle x, y \rangle : M \times M \longrightarrow \Lambda$, which has the above properties, is called the **scalar product** on M. The triplet $(M, \langle ., . \rangle, \|.\|)$, where M is a scalar product on M, is called the **Hilbert commutative supermodule** if there exist the Λ-isomorphism $\gamma : M \longrightarrow \ell_2(\Lambda)$, such that $\langle \gamma m_1, \gamma m_2 \rangle = \langle m_1, m_2 \rangle, \|\gamma m\| = \|m\|$ (i.e, the operator γ is unitary with respect to scalar products and isomorphic with respect to the norm).

It follows from definition of the Hilbert supermodules that in M exists the orthonormal basis $a_n = \gamma^{(-1)}e_n$. Now we obtain the superanalog of Riesz theorem.

Theorem 3.2.2. *Let Λ be a $\sum$-algebra. Then, for any Λ-linear continuous functional f on the commutative Hilbert supermodule $M = M_0 \oplus M_1$ there exist a unique element $u \in M$ such that*

$$f(x) = \langle x, u\rangle, \quad x \in M \tag{3.2.6}$$

and, moreover, $\|f\| = \sup\limits_{\|x\|\leq 1} \|\langle x, u\rangle\| = \|u\|$. Conversely, if $u \in M$, then the above formula defines the Λ-linear continuous functional f such that $\|f\| = \|u\|$. Therefore, the equality 3.2.6 defines the isomorphism $f \longrightarrow u$ between modules $M' = (M')_0 \oplus (M')_1$ and $M = M_0 \oplus M_1$, where M' denotes the set of all Λ-linear continuous functionals on M.

Proposition 3.2.3. *In any commutative Hilbert supermodule the Cauchy Bonyakovski inequality*

$$\|\langle x, y\rangle\| \leq \|x\|\|y\|$$

holds.

Example 3.2.3. *The space*

$$L_2^\Lambda(\mathbb{R}^n, dx) = \{f(x) = \sum_\alpha f_\alpha \Phi_\alpha(x) : f_\alpha \in \Lambda\},$$

where $\Phi_\alpha(x)$ are Hermite functions on $\mathbb{R}^n$, with the norm $\|f\|^2 = \sum\limits_\alpha \|f_\alpha\|^2$, and the scalar product $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)dx$ is a commutative Hilbert supermodule.

Let Λ be a complex Banach CSA with involution $*$, compatible with the $\mathbb{Z}_2$-gradation. Involution in Λ induces involution in $\ell_2(\Lambda)$. We introduce a scalar product, which is compatible with involution $*$ (an analog of the scalar product on a complex Hilbert space), under the assumption that

$$(x, y) = \sum_{n=1}^{\infty} x_n y_n^*.$$

This scalar product possess a normal **symmetry** property

$$(x, y)^* = (y, x)$$

and a property of **Λ-antilinearity**

$$(\lambda x, \mu y \alpha) = \lambda (x \alpha^*, y) \mu^*.$$

Using the space $\ell_2(\Lambda)$ with the scalar product $(.,.)$, one defines the commutative Hilbert supermodules with involution.

Definition 3.2.10. Hilbert supermodule with involution

Let $M = M_0 \oplus M_1$ be a commutative supermodule, then a bilinear form $(x, y) : M \times M \longrightarrow \Lambda$, which has the above properties, is called the **scalar product** on M. The triplet $(M, (.,.), \| \ \|)$, where $(.,.)$ is a scalar product on M, is called **commutative Hilbert supermodule with involution** if there exists a Λ-isomorphism $\gamma : M \longrightarrow \ell_2(\Lambda)$, such that

$$(\gamma m_1, \gamma m_2) = (m_1, m_2) \qquad , \|\gamma m\| = \|m\|.$$

Definition 3.2.11. Self adjoint operators in Hilbert supermodules with involution

Let a be a continuous left Λ-linear operator in the commutative Hilbert supermodule M with involution $*$. The adjoint operator a^* is defined by the equation $(ax, y) = (x, a^* y)$. This operator is continuous and left Λ-linear in M, if the CSA Λ is a $\sum$-algebra. As usual, the operator a is called self adjoint if $a = a^*$.

Following of first definitions of current section, one can define the Hilbert superspace over a pair of commutative Hilbert supermodules.

Definition 3.2.12. A space $\Lambda^{M_0,N_1} = M_0 \times N_1$, where $M = M_0 \oplus M_1$ and $N = N_0 \oplus N_1$ are commutative Hilbert supermodules, is called a **superspace over the CHSM M and N.**

Definition 3.2.13. Two CHSM M and N are said to be **dual** to one another if there exists a bilinear form $((.,.)) : M \times N \longrightarrow \Lambda$ continuous on compact sets which seperates points of the CHSM and satisfies the condition

$$(\lambda u \mu, v\gamma) = \lambda(u, \mu v)\gamma \tag{3.2.7}$$

for any $\lambda, \mu, \gamma \in \Lambda$ and $u \in M,\ v \in N$.

A**Hilbert superspace** over a pair of commutative Hilbert supermodules is a superspace Z over the CHSM such that it is complete and the duality form 3.2.7 is defined on $Z \times Z$.

3.3 Schmitt

In this section we consider to the definition of super Hilbert space given by Thomas Schmitt. He considered to Hermitian structures on his work [44] which did not used in definition of super Hilbert space. His definition need not to have any preknowledge and so we give it directly.

Definition 3.3.1. A *super Hilbert space* is a complex $\mathbb{Z}_2$-graded vector space $H = H_0 \oplus H_1$ together with a $\mathbb{C}$-linear pairing

$$\overline{H} \times H \longrightarrow \mathbb{C}, \qquad (\overline{h}, k) \mapsto \langle \overline{h}|k\rangle$$

such that

(i) $\overline{\langle \overline{h}|k\rangle} = \langle \overline{k}|h\rangle \quad$ for $h, k \in H$;

(ii) $\langle \overline{h}|h\rangle > 0 \quad$ for $h \in H, h \neq 0$;

(iii) H is complete with respect to the topology defined by the norm $h \mapsto ||h|| = \langle \overline{h}|h\rangle^{\frac{1}{2}}$.

3.4 Nagamachi and Kobayashi

In this section we consider the definition of Hilbert superspace given by Shigeaki Nagamachi and Yuji Kobayashi's in them paper [32]. It consists more topological aspects of Hilbert superspace than algebric. One of remarkable notes in this paper is that they consider infinite dimensional Graßmann algebra as inductive limit topology of finite dimensional Graßmann algebras which was introduced in them paper [31] which we investigate it in section 2.3 of previous chapter. Of course as we seen beforehand, the concept of infinite dimensional Graßmann algebra was considered to be direct limit of finite dimensional Graßmann algebras by Rogers in [37] and we saw it in section 2.4 of previous chapter.

3.4.1 Preliminaries

The algebra $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ is a $\mathbb{Z}_2$-graded and a direct sum of the even part $\Lambda_{\bar{0}}$ and the odd part $\Lambda_{\bar{1}}$. Every $\lambda \in \Lambda$ is a sum of the body $b(\lambda)$ which is in $\mathbb{C}$ and the soul $s(\lambda)$ which is nilpotent. The mapping $b : \Lambda \longrightarrow \mathbb{C}$ is a surjective algebra homomorphism. For the infinite-dimensional Λ, we need a suitable topology on it. We employ the inductive limit topology of finite dimensional Graßmann algebras, which introduced in subsection 1.2.2.

Let $\Lambda^{(n)}$ be the subalgebra of Λ generated by $\xi_1, \ldots, \xi_n$ and I_n be the ideal of $\Lambda^{(n)}$ generated by ξ_n. Then we have $\Lambda^{(n)} = \Lambda^{(n-1)} \oplus I_n$ and every element of Λ is uniquely written as

$$\lambda = \sum_{n \geq 0} \lambda_n \qquad ; \lambda_n \in I_n. \tag{3.4.1}$$

Each $\Lambda^{(n)}$ is a Banach algebra with the Rogers norm defined by

$$\|\lambda\| = \sum_M |q_M| \quad ; \text{for} \quad \lambda = \sum_M q_M \xi_M, \tag{3.4.2}$$

where $\xi_M = \xi_{l_1} \ldots \xi_{l_m}, q_M \in \mathbb{C}$ and $M = \{l_1, \ldots, l_m\}, \quad l_1 < \cdots < l_m \leq n$.

We consider the inductive limit topology of the sequence $\{\Lambda^{(n)}\}$ on Λ, that is, the finest locally convex topology on Λ such that every injection $\phi_n : \Lambda^{(n)} \longrightarrow \Lambda$ is continuous, which is equivalent to the finest locally convex topology on Λ such that every injection $\psi_n : I_n \longrightarrow \Lambda = \sum_n \oplus I_n$ is continuous (the direct sum topology on $\Lambda = \sum_n \oplus I_n$). This topology is defined by the following system $\{p_\omega\}$ of norms on Λ. Let $\omega = \{\omega_n\}$ ba an arbitrary increasing sequence of positive integers and define a norm $P_\omega(\lambda)$ on Λ by

$$P_\omega(\lambda) = \sum_n \omega_n \|\lambda_n\|, \tag{3.4.3}$$

where λ is expressed as 3.4.1 and $\|\lambda_n\|$ is the Rogers norm of Λ_n. Since each norm satisfies $P_\omega(\lambda\mu) \leq P_\omega(\lambda)P_\omega(\mu)$ for $\lambda, \mu \in \Lambda$ and Λ is a topological algebra.

Note that the body map $b : \Lambda \longrightarrow \mathbb{C}$ is continuous. The algebra Λ has the following fundamental properties which are essential in our discussions:

(i) Λ is a complete and nuclear space;

(ii) The soul of every element of Λ is nilpotent;

(iii) any bounded set of Λ is contained in $\Lambda^{(n)}$ for some n.

In the following we assume that Λ has a continuous involution $*$ satisfying

$$\lambda^{**} = \lambda \; ; \; (\lambda + \mu)^* = \lambda^* + \mu^* \; ; \; (c\lambda)^* = \overline{c}\lambda^* \; ; \; (\lambda\mu)^* = \mu^*\lambda^* \; ; \; P_\omega(\lambda^*) = P_\omega(\lambda)$$

for $\lambda, \mu \in \Lambda, \quad \lambda^* \in \Lambda_\alpha \quad$ for any $\lambda \in \Lambda_\alpha \quad$ and $\alpha \in \mathbb{Z}$.

If we define an involution $*$ for generators ξ_n by

$$\xi^*_{2n-1} = \xi_{2n}, \quad \xi^*_{2n} = \xi_{2n-1}, \quad (c\xi_{l_1} \ldots \xi_{l_m})^* = \overline{c}\xi^*_{l_m} \ldots \xi^*_{l_1},$$

and extend it additively to Λ, then Λ is a Graßmann algebra with a continuous involution $*$ having the above properties.

3.4.2 Hilbert superspace

In this subsection we give the definition of Hilbert superspace and study some of its properties.

Definition 3.4.1. A $\mathbb{Z}_2$-graded Λ-module $\mathcal{H} = \bigoplus_{\alpha \in \mathbb{Z}_2} \mathcal{H}_\alpha$ is called an **inner product superspace** if it has a $\mathbb{Z}_2$-graded inner product $(.,.) : \mathcal{H}_\alpha \times \mathcal{H}_\beta \longrightarrow \Lambda_{\beta-\alpha}$ satisfying the following conditions:

(i) $(g, f) = (f, g)^*$ (symmetric);

(ii) $(f, g + h) = (f, g) + (f, h)$ (biadditive);

(iii) $(f, g\lambda) = (f, g)\lambda$ (sesquilinear);

(iv) $b(f, f) \geq 0$ (positive definite);

for$f, g, h \in \mathcal{H}$ and $\lambda \in \Lambda$, where $b(f, f)$ is an abbreviation of $b((f, f))$.

One of the most important notions that we interested to it, is the positive definiteness of, only, body of inner product (not inner product itself). As we seen earlier, Dewitt's definition has similar restriction. We know that Nagamachi and Kobayashi introduced a general algebra, the σ-commutative G-graded algebra, which Graßmann algebra is particularly a $\mathbb{Z}_2$-garaded $(-1)^{\alpha\beta}$-commutative algebra with $\alpha, \beta \in \mathbb{Z}_2$. They restrict themselves to Graßmann algebra. Why?. Maybe one of them reason was that the positivity of supernumbers, as an element of σ-commutative G-graded algebra, is meaningless even for its body. While the body of any Graßmann number is a complex number which it can be positive. In the next chapter we will try to correct this. For an inner product superspace $\mathcal{H}$ we define the soul $s(\mathcal{H})$ and the body $b(\mathcal{H})$ as follows:

$$s(\mathcal{H}) = \{h \in \mathcal{H} \mid h\lambda = 0 \text{ for some } 0 \neq \lambda \in \Lambda \},$$

$$b(\mathcal{H}) = \mathcal{H}/s(\mathcal{H}).$$

An element h of $\mathcal{H}$ is called a **supervector** and if $h \in s(\mathcal{H})$, it is called a **soul vector**. Since $s(\mathcal{H})$ is a $\mathbb{Z}_2$-graded subspace of $\mathcal{H}$ over $\mathbb{C}$, $b(\mathcal{H})$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$.

Definition 3.4.2. Let $\{P_\omega\}$ be the system of norms which defines the topology of Λ. The system of open neighborhoods of 0 is generated by the following family of sets:

$$U(h, \omega, \epsilon) = \{g \in \mathcal{H} \mid \quad P_\omega((g, h)) < \epsilon\},$$

where $\epsilon > 0$ and $h \in \mathcal{H}$. The **σ-topology** is the weakest among those topologies under which the inner product $(.,.)$ is separately continuous.

Definition 3.4.3. A $\mathbb{Z}_2$-graded subspace H of $\mathcal{H}$ over $\mathbb{C}$ is a base **pre-Hilbert space (base Hilbert space)** of $\mathcal{H}$ if the following conditions are satisfied:

(i) H is a pre-Hilbert (Hilbert) space with the inner product $(.,.)$ of $\mathcal{H}$, this means that $(f, g) \in \mathbb{C}$ for all $f, g \in H$, (and H is complete).

(ii) The norm topology of the pre-Hilbert (Hilbert) space H is stronger than the induced topology of $\mathcal{H}$.

(iii) $(H, h) = 0$ implies $h = 0$ for $h \in \mathcal{H}$.

A complete orthonormal basis of H is called a **complete orthonormal basis** of $\mathcal{H}$.

Definition 3.4.4. An inner product superspace $\mathcal{H}$ is called a **(pre)-Hilbert superspace** if it has a base (pre-)Hilbert space H.

Since the algebra Λ of supernumbers is not a field, a Λ submodule of a Hilbert superspace is not always a free Λ-module. This causes some difficulties in treating subsuperspaces.

Definition 3.4.5. A $\mathbb{Z}_2$-graded submodule Φ of $\mathcal{H}$ is called a **Hilbert subsuperspace** of $\mathcal{H}$, if the following conditions are satisfied:

(i) Φ is a Hilbert superspace with the inner product $(.,.)$ of $\mathcal{H}$;

(ii) The topology of Φ coincides with the induced topology from $\mathcal{H}$.

3.5 El Gradechi and Nieto

This section is devoted to definition of super Hilbert space according to A. M. Elgradechi and M. Nieto's terminology. They considered this space as a direct sum of two complex Hilbert spaces equipped with a super Hermitian form. In [11], they gave some examples of this space which are used in physics.

Here we give some basic definitions to approach to them definition. Of course for this we use also [17] and [35].

A complex superalgebra is a complex vector superspace (i.e. a $\mathbb{Z}_2$-graded linear space) $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ equipped with a $\mathbb{Z}_2$-compatible product, namely, $\mathcal{B}_k \cdot \mathcal{B}_l \subset \mathcal{B}_{k+l}$; $\mathcal{B}$ is considered associative and possesses a unity. Note that, $\mathcal{B}_0$ (resp. $\mathcal{B}_1$) is called the *even* (resp. *odd*) part of $\mathcal{B}$. Accordingly, elements of $\mathcal{B}_0$ (resp. $\mathcal{B}_1$) are called *even* (resp. *odd*) elements of $\mathcal{B}$. A homogeneous element of $\mathcal{B}$ is either even or odd. The *parity* (or *degree* of such an element $u \in \mathcal{B}_k$, denoted $\epsilon(u)$, is defined by $\epsilon(u) = k$. The superalgebra $\mathcal{B}$ is supercommutative if

$$uv = (-1)^{\epsilon(u)\epsilon(v)} vu, \tag{3.5.1}$$

for u and v two homogeneous elements of $\mathcal{B}$.

The *complex supercommutative superalgebra with unit* $\mathcal{B}$ considered in the present section is the complex Graßmann algebra [7] generated by (θ, χ) and their complex conjugates (θ, χ). These are anticommuting, and hence nilpotent variables. In other words $\mathcal{B}$ is the complex exterior algebra over $\mathbb{C}^4 = \mathbb{C}^2 \oplus \overline{\mathbb{C}}^2$. Its even (resp. odd) part is spanned by the products of an even (resp. odd) number of generators, and the dimension of $\mathcal{B}$ is 16. The decomposition of any element $\theta \in \mathcal{B}$ in a given basis of $\mathcal{B}$, assumes the following form

$$\theta = \widetilde{\theta} \cdot \mathrm{I} + \theta_{nil}, \tag{3.5.2}$$

where, the purely nilpotent component θ_{nil} is called the *soul* of θ, while the component $\widetilde{\theta}$ along the identity of $\mathcal{B}$ is called the *body* of θ.

Finally, all vector superspaces appearing in this work are considered as left $\mathcal{B}$-modules. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be such a $\mathcal{B}$-module. Then, for v and Θ homogeneous elements in V and $\mathcal{B}$ respectively, we have,

$$\Theta\, v = (-1)^{\epsilon(\Theta)\epsilon(v)} v\, \Theta. \tag{3.5.3}$$

There is a natural way of defining a bracket $[,]$ in a superalgebra $\mathcal{U}$, i.e., be equality

$$[a, b] = ab - (-1)^{(\deg a)(\deg b)} ba. \tag{3.5.4}$$

A superalgebra is called **commutative** if $[a, b] = 0$ for all $a, b \in \mathcal{U}$. Associativity of a superalgebra is defined as for an algebra. For an associative superalgebra $\mathcal{U}$ we have the following important identity:

$$[a, bc] = [a, b]c + (-1)^{(\deg a)(\deg b)} b[a, c].$$

Example 3.5.1. *Let $\Lambda_n = \Lambda(n)$ be the Graßmann algebra in n variables $\xi_1, \ldots, \xi_n$. Then $\Lambda(n)$ becomes $\mathbb{Z}_2$-graded if we set $deg\xi_i = \bar{1}, i = 1, \ldots, n$. The result is called* **Graßmann superalgebra**. *It is commutative and associative. Evidently $\Lambda(m) \otimes \Lambda(n) = \Lambda(m + n)$.*

Definition 3.5.1. Lie superalgebra

A **Lie superalgebra** is a superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with an operation $[.,.]$ satisfying the following axioms:

$[a, b] = -(-1)^{(\deg a)(\deg b)}[b, a]$ (anticommutativity)

$[a, [b, c]] = [[a, b], c] - (-1)^{(\deg a)(\deg b)}[b, [a, c]]$ (Jacobi identity)

Example 3.5.2. *If $\mathcal{U}$ is an associative superalgebra, then the bracket 3.5.4 turns $\mathcal{U}$ into a Lie superalgebra. We denote the resulting Lie superalgebra by $\mathcal{U}_L$.*

Example 3.5.3. *Let G be a Lie superalgebra and $\Lambda(n)$ a Graßmann superalgebra. The $G \otimes \Lambda(n)$ is also a Lie superalgebra.*

Definition 3.5.2. Let $G = G_0 \oplus G_1$ be a Lie superalgebra over $\mathbb{R}$ and $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded linear space. A **linear representation** ρ of a Lie superalgebra G in V is a homomorphism $\rho : G \longrightarrow \ell(V)$. For brevity we often say in this case that V is a G**-module**, and instead of $\rho(g)(v)$ we write $g(v)$, for $g \in G, v \in V$. Note that by definition, $G_i(V_j) \subseteq V_{i+j}$ for $i, j \in \mathbb{Z}_2$ and

$$[g_1, g_2](v) = g_1(g_2(v)) - (-1)^{(\deg g_1)(\deg g_2)} g_2(g_1(v)).$$

Note also that the map ad : $G \longrightarrow \ell(G)$ for which $(\text{ad } g)(a) = [g, a]$ is a linear representation of G. It is called the **adjoint representation**.

Definition 3.5.3. We call a representation (ρ, V) of G **superunitary** if $V = V_0 \oplus V_1$ admits a non-degenerate bilinear form $(.,.)$ which satisfies:

1) super Hermitian condition, i. e., $(u, v) = -(-1)^{(\deg u)(\deg v)}(v, u)$;

2) The form is homogeneous of degree zero, i. e., $(u, v) \neq 0$ only if $\deg u + \deg v = 0$ for $u, v \in V$ and homogeneous;

3) The form is positive definite on V_0. There is a constant $\delta = \pm 1$ depending only on (ρ, V) such that $\delta\sqrt{-1}(.,.)$ is positive definite on V_1;

4) The operators $\{\rho(g) \mid g \in G\}$ leave $(.,.)$ invariant:

$$(\rho(g)u, v) + (-1)^{(\deg g)(\deg u)}(u, \rho(g)v) = 0, \tag{3.5.5}$$

for homogeneous $g \in G$ and $u, v \in V$.

Here we call $(.,.)$ a **super Hermitian form** when

$$(u, v) = (-1)^{(\deg u)(\deg v)}\overline{(v, u)}$$

holds. Let $V = V_0 \oplus V_1$ be a superspace over $\mathbb{R}$ (or $\mathbb{C}$). A bilinear form b on V is called **super skew symmetric** if b satisfies

$$b(u, v) = -(-1)^{(\deg u)(\deg v)} b(v, u),$$

where $u, v \in V$ are homogeneous elements.

Definition 3.5.4. A **super Hilbert space** is a pair $(\mathcal{H}, \langle\langle ., . \rangle\rangle)$, where $\mathcal{H} \equiv \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ is a superspace equipped with a super Hermitian form $\langle\langle ., . \rangle\rangle = \langle\langle ., . \rangle\rangle_{\bar{0}} + i\langle\langle ., . \rangle\rangle_{\bar{1}}$, such that $(\mathcal{H}_{\bar{0}}, (\mathcal{H}, \langle\langle ., . \rangle\rangle)_{\bar{0}}$ and $(\mathcal{H}_{\bar{1}}, (\mathcal{H}, \langle\langle ., . \rangle\rangle)_{\bar{1}})$ are both Hilbert spaces.

3.6 Samsonov

This section contains the definition of super Hilbert space, which is given by Boris F. Samsonov in his paper [40].

3.6.1 Basic definitions

Let Λ be a $\mathbb{Z}_2$-graded linear space $\Lambda = \Lambda_1 \oplus \Lambda_2$. When an element $\lambda \in \Lambda_0$, it is called *even*[parity $p(\lambda) = 0$] and when $\lambda \in \Lambda_1$ it is called *odd*[parity $p(\lambda) = 1$]. The elements from Λ_0 and Λ_1 are called *homogeneous*. When the structure of associative algebra with unit $e \in \Lambda_0$ and even multiplication operation [i.e., $p(\lambda\mu) = p(\lambda)p(\mu)$, mod 2 for homogeneous λ and μ] is introduced in Λ, it is called *superalgebra*. Superalgebra Λ is called *commutative* if *supercommutator* $[\lambda, \mu] = \lambda\mu - (-1)^{p(\lambda)p(\mu)}\mu\lambda = 0$ for homogeneous $\lambda, \mu \in \Lambda$. Further, the commutative superalgebra $\Lambda = \Lambda_1 \oplus \Lambda_2$ is supposed to be a Banach space with the norm $\|\lambda\mu\| \leq \|\lambda\|.\|\mu\|, \lambda, \mu \in \Lambda, \|e\| = 1$. The components Λ_0 and Λ_1 are closed subspaces in Λ. When Λ is defined over the real number field $\mathbb{R}$ we obtain the real superalgebra $\Lambda(\mathbb{R})$, and for the case of complex number field $\mathbb{C}$ we obtain $\Lambda(\mathbb{C})$.

Given a real superalgebra $\Lambda(\mathbb{R})$, **real superspace** $\mathbb{R}_\Lambda^{m,n}$ of dimension (m, n) over $\Lambda(\mathbb{R})$ is defined as follows:

$$\mathbb{R}_\Lambda^{m,n} = \Lambda_0 \otimes \cdots \otimes \Lambda_0 \otimes \Lambda_1 \cdots \otimes \Lambda_1 = \Lambda_0^m \otimes \Lambda_1^n. \tag{3.6.1}$$

A **complex superspace** $\mathbb{C}_\Lambda^{m,n}$ over $\Lambda(\mathbb{C})$ is defined in the same way but with the help of the complex superalgebra $\Lambda(\mathbb{C})$. If for every point $X = (x, \xi) = (x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) \in \mathbb{R}_\Lambda^{m,n}$ we introduce the norm

$$\|X\|^2 = \|x\|^2 + \|\xi\|^2 = \sum_{k=1}^{m} \|x_k\|^2 + \sum_{j=1}^{n} \|\xi\|^2, \tag{3.6.2}$$

then $\mathbb{R}_\Lambda^{m,n}$ becomes a Banach space. Every connected open set $\mathbf{O} \subset \mathbb{R}_\Lambda^{m,n}$ is called *domain* in $\mathbb{R}_\Lambda^{m,n}$.

Let us have two superspaces $\mathbb{R}_\Lambda^{m,n}$ and $\mathbb{R}_{\Lambda'}^{m',n'}$ with the norms $\|.\|$ and $\|.\|'$, $\Lambda \subseteq \Lambda'$, and a domain $\mathbf{O}$ in $\mathbb{R}_\Lambda^{m,n}$. Function $f(X) : \mathbf{O} \longrightarrow \mathbb{R}_{\Lambda'}^{m',n'}$ is called *continuous* in the point $X \in \mathbf{O}$ if $\|f(X+H) - f(X)\|' \longrightarrow 0$ when $\|H\| \longrightarrow 0$. The same function is called *superdifferentiable from the left* in the point $X \in \mathbf{O}$ if elements $F_k(X) \in \mathbb{R}_{\Lambda'}^{m',n'}$, $k = 1, \ldots, m+n$, such that

$$f(X+H) = f(X) + \sum_{k=1}^{m+n} H_k F_k(X) + \tau(X,H), \tag{3.6.3}$$

where $\|\tau(X,H)\|'/\|H\| \longrightarrow 0$ when $\|H\| \longrightarrow 0$ exist. The functions $F_k(x)$ are called *left partial superderivatives* of f with respect to X_k in the point $X \in \mathbf{O}$:

$$F_k(x) = \frac{\partial f(X)}{\partial X_k}, \quad F_{m+j}(x) = \frac{\partial f(X)}{\partial X_{m+j}}, \quad k = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{3.6.4}$$

The expression $\sum_{k=1}^{m+n} H_k \partial f(X)/\partial X_k$ is called *left superdifferential* of the function $f(X)$ in rhe point X.

One can find more details about superanalysis in [48], [49] and [50].

3.6.2 Hilbert superspace

Consider the real superspace $\mathbb{R}_\Lambda^{1,1}$ defined over $\Lambda(\mathbb{R}) = \Lambda_0(\mathbb{R}) \otimes \Lambda_1(\mathbb{R})$ where $\Lambda_0(\mathbb{R}) = \mathbb{R}$ and $\Lambda_1(\mathbb{R})$ has two generators ξ and $\overline{\xi}$ with the properties $\xi^2 = \overline{\xi^2} = \xi\overline{\xi} + \overline{\xi}\xi = 0, \overline{\overline{\xi}} = \xi$. The complex superspace $\mathbb{C}_\Lambda^{1,1}$ is defined over $\Lambda(\mathbb{C}) = \Lambda_0(\mathbb{C}) \otimes \Lambda_1(\mathbb{C})$ where $\Lambda_0(\mathbb{C}) = \mathbb{C}$ and $\Lambda_1(\mathbb{C})$ has the same generators ξ and $\overline{\xi}$.

Consider now functions from $\mathbb{R}_\Lambda^{1,1}$ to $\mathbb{C}_\Lambda^{1,1}$ of the following form:
$\Psi^0(t,x,\xi,\overline{\xi}) = \psi(x,t), \psi(x,t) \in H^0$ and $\Psi^1(t,x,\xi,\overline{\xi}) = \xi\varphi(x,t), \varphi(x,t) \in H^1$. We shall designate the collection of the functions $\Psi^0(t,x,\xi,\overline{\xi})$ and $\Psi^1(t,x,\xi,\overline{\xi})$ as $H_{\overline{0}}$ and $H_{\overline{1}}$, respectively. It follows from these constructions that $H_{\overline{0}}$ and

$H_{\overline{1}}$ are linear spaces (over the field $\mathbb{C}$), and $H_s = H_{\overline{0}} \oplus H_{\overline{1}}$ is a $\mathbb{Z}_2$-graded linear space of functions. The elements from $H_{\overline{0}}$ and $H_{\overline{1}}$ are called *homogeneous* with the parity $p(\Phi) = 0$ when $\Phi \in H_{\overline{0}}$ and $p(\Phi) = 1$ when $\Phi \in H_{\overline{1}}$.

Define in the space H_s *scalar product (super-Hermitian form)* as follows:

$$(\Phi_1|\Phi_2) = \int \overline{\Phi_1}(t, x, \xi, \overline{\xi})\Phi_2(t, x, \xi, \overline{\xi}) i e^{-i\overline{\xi}\xi} dx d\xi d\overline{\xi} \in \mathbb{C}. \tag{3.6.5}$$

Since the integration in superspaces is developed in references [48], [49] and [50] for sufficiently smooth functions (it is a supergeneralization of various integral constructions based on Rimann integral and not on Lebesgue integral) we should make more precise the sence of integral in 3.6.5. If functions Φ_1 and Φ_2 are defined by their homogeneous components

$$\Phi_l(x, \xi, \overline{\xi}) = \Phi_l^0(x, \xi, \overline{\xi}) + \Phi_l^1(x, \xi, \overline{\xi}),$$

where $\Phi_l^0(x, \xi, \overline{\xi}) = \chi_l^0(x) \in H_{\overline{0}}$, and $\Phi_l^1(x, \xi, \overline{\xi}) = \chi_l^1(x) \in H_{\overline{1}}, l = 1, 2$, and functions $\chi_l^j(x), j = 0, 1$ are sufficiently smoth, then we may interpret the integral 3.6.5 in the sence defined in [48], [49] and [50]. In our case this integral becomes equal to a product of two integrals. The first one is a conventional integral with respect to the variables ξ and $\overline{\xi}$. The only integral with respect to the Graßmann variables different from zero is $\int \overline{\xi}\xi d\xi d\overline{\xi} = 1$. Thus, for the integral 3.6.5 we obtain the expression

$$(\Phi_1|\Phi_2) = (\Phi_1^0|\Phi_2^0)_0 + (\Phi_1^1|\Phi_2^1)_1, \tag{3.6.6}$$

$$(\Phi_1^i|\Phi_2^i)_j = i^j \langle \chi_l^i | \chi_2^i \rangle_j, \quad \chi_l^i \in H^i, \quad l = 1, 2, \quad i, j = 0, 1.$$

We note that the spaces $H_{\overline{0}}$ and $H_{\overline{1}}$ are mutually orthogonal with respect to the scalar product 3.6.5 and are complete in the sence we shall make more precise so that $(.|.)_j, j = 0, 1$, are the restrictions of the scalar product 3.6.5 on the spaces $H_{\overline{j}}$.

In the case when functions $\chi_l^j \in L_2(\mathbb{R})$ are not sufficiently smooth for applying the definition of the integral given in [48], [49] and [50], we directly apply the formula 3.6.6 for calculating the integral 3.6.5. We remind the reader that the scalar product $\langle .|.\rangle$ in $L_2(\mathbb{R})$ is defined with the help of the Lebesgue integral. We will notice that the formula 3.6.6 is in accord with the definition of the super-Hermitian form in the abstract Hilbert superspace given in [11]. The super-Hermitian form in 3.6.6 is positive definite in the sense that the Hermitian forms $\langle .|.\rangle_j, j = 0, 1$, from which it is expressed are positive definite.

The super-Hermitian form generates a norm in H_s. For every $\Phi = \Phi^0 + \Phi^1 \in H_s$, $\Phi^0 = \chi^0(x,t)$, and $\Phi^1 = \xi\chi^1(x,t)$ we put by definition

$$\|\Phi\|^2 = |(\Phi|\Phi)| = \|\chi^0\|_0^2 + \|\chi^1\|_1^2, \tag{3.6.7}$$

where $\|.\|_j$ are the norms in $H^j, j = 0, 1$, generated by the appropriate scalar products. It is not difficult to see that the properties of the norm so defined correspond to the axioms of the conventional norm:

(i) $\|\Phi\| \geq 0$;

(ii) $\|\Phi\| = 0$ if and only if $\Phi = 0$;

(iii) $\|c\Phi\| = |c|.\|\Phi\|, \quad \forall c \in \mathbb{C}$;

(iv) $\|\Phi_1 + \Phi_2\| \leq \|\Phi_1\| + \|\Phi_2\|$.

It follows that H_s is a normed space in the usual sense. Conditions (i), (iii) and (iv) mean that the norm is a convex functional in H_s. Condition (ii) means that the set $\{\|.\|\}$ formed from a single convex topological space. Just in this sense we shall understand the completeness of the space H_s which we shall call the **Hilbert superspace**. This signifies that the space H_s contains only linear functionals of the variable ξ with the coefficients from H. Since the functions $\Psi_n^0(t,x,\xi,\overline{\xi}) = \psi_n(x,t)$ and $\Psi_n^1(t,x,\xi,\overline{\xi}) = \xi\varphi_n(x,t)$ form bases in the spaces $H_{\overline{0}}$ and $H_{\overline{1}}$, respectively, we have obtained a **separable** Hilbert superspace.

3.7 Rudolph

In this section we consider the last definition of super Hilbert space which is given by Oliver Rudolph in [38]. Unlike of previous definitions, Rudolph's terminology has mathematical framework.[1]

A super Hilbert space in sense of Rudolph, is a module over a Graßmann algebra endowed with a Graßmann number-valued inner product. Of course it is almost analog of Hilbert C^*-modules with some differences. As it is well-known, in Hilbert C^*-modules, positive definiteness of inner product has meaning and hence the Cauchy Schwartz inequality is valid which implies the continuity of inner product with respect to the both arguments. By using of this inequality it is proved that the set of all adjointable operators on a Hilbert C^*-module forms a C^*-algebra. But in the Rudolph's definition, the positive definiteness of Graßmann number-valued inner product is weaker (it holds only for its body) and hence the Cauchy Schwarz inequality is valid only for body of inner product which this implies that the inner product is not always continuous. Also the set of all adjointable operators makes only an involutive Banach algebra with continuous involution.

Rudolph has mentioned some interesting examples for super Hilbert space which are slightly complex. In fact he makes them for applying in physics.

3.7.1 Exterior Algebra with mass norm

In the present subsection we give the structure of exterior algebra endowed with mass norm according to [12].

For any vector space V, we can construct a particular graded algebra

[1] Consider to this note that the Rudolph's paper is expressive alone and then we will use in some where it without any explanation. Also he used the concept of super Hilbert space to the Schrdinger representation of spinor quantum field theory in section 6 of his paper, which we do not investigate it, because of we are not work on physics.

$\otimes_* V = \bigoplus_{n=0}^{\infty} \otimes_n V$ called the **Tensor algebra** of V where $\otimes_n V$ is the n fold tensor product with all n factors equal to V. We define multiplication in $\otimes_* V$ so that its restriction to $\otimes_m V \times \otimes_n V$ is simply the (bilinear) composition

$$\otimes_m V \times \otimes_n V \longrightarrow \otimes_m V \otimes \otimes_n V = \otimes_{m+n} V.$$

The set $\otimes_0 V = \mathbb{R}$ is as unit element of the ring $\otimes_* V$(associative algebra).

In the associative tensor algebra $\otimes_* V$ we consider the **two sided ideal** $\mathcal{U}V$ generated by all the elements $x \otimes x$ in $\otimes_2 V$ corresponding to $x \in V$. The quotient algebra

$$\wedge_* V = \otimes_* V / \mathcal{U}V$$

is called the **exterior algebra** of the vector space V. Clearly $\mathcal{U}V$ is a homogeneous ideal, in fact

$$\mathcal{U}V = \oplus_{m=2}^{\infty} (\otimes_m V \cap \mathcal{U}V)$$

and therefore

$$\wedge_* V = \oplus_{m=0}^{\infty} \wedge_m V$$

where

$$\wedge_m V = \otimes_m V / (\otimes_m V \cap \mathcal{U}V);$$

in particular $\wedge_0 V = \mathbb{R}$ and $\wedge_1 V = V$. The elements of $\wedge_m V$ are called m**-vectors** of V. The multiplication in $\wedge_* V$ is called **exterior multiplication** and denoted by the wedge symbol $\wedge$. It follows that if $v_1, \ldots, v_m \in V$, then the canonical homomorphism maps the product $v_1 \otimes \cdots \otimes v_m \in \otimes_m V$ on to the product $v_1 \wedge \cdots \wedge v_m \in \wedge_m V$. Clearly $\wedge_m V$ is the vector space generated by all such products.

If u and v belong to V, then $u \otimes v + v \otimes u = (u+v) \otimes (u+v) - u \otimes u - v \otimes v \in \mathcal{U}V$, hence $u \wedge v = -v \wedge u$. Therefore

$$(v_{p+1} \wedge \cdots \wedge v_{p+q}) \wedge (v_1 \wedge \cdots \wedge v_p) = (-1)^{pq} (v_1 \wedge \cdots \wedge v_p) \wedge (v_{p+1} \wedge \cdots \wedge v_{p+q})$$

whenever $v_1, \dots v_{p+q} \in V$, which implies that the anticommutative law holds for exterior multiplication. Among all anticommutative associative algebras with a unit, whose direct summand of index 1 is linearly isomorphic to V, the exterior algebra $\wedge_* V$ is characterized (up to isomorphism) by the following property:

For every anticommutative associative algebra $\mathcal{A}$ with a unit element, each linear map of V into $\mathcal{A}_1$ can be uniquely extended to a unit preserving algebra homomorphism of $\wedge_* V$ into $\mathcal{A}$, carrying $\wedge_m V$ into $\mathcal{A}_m$ for each m. Such an extension is unique because the algebra $\wedge_* V$ is generated by $V \cup \{1\}$.

If $e_1, e_2, \dots$ form a basis of V, then the products

$$e_\lambda = e_{\lambda(1)} \vee e_{\lambda(2)} \vee \cdots \vee e_{\lambda(m)}$$

corresponding to all increasing m termed sequences λ form a basis of $\wedge_m V$. In fact $\wedge_m V$ has a basis equipotent with the set $I(n, m)$ of all increasing maps of $\{1, \dots, m\}$ into $\{1, \dots, n\}$.

An element of $\wedge_m V$ is called **simple** (or **decomposable** if and only if it equals the exterior product of m elements of V. With each $\xi \in \wedge_m V$ we associate the vector subspace

$$T = V \cap \{v : \xi \vee v = 0\}.$$

A nonzero m-vector ξ is simple if and only if its associated subspace T has dimension m; in this case ξ equals the exterior product of m suitable base vectors of T. The associated subspaces of two nonzero simple m-vectors ξ and η are equal if and only if $\xi = c\eta$ with $0 \neq \in \mathbb{R}$.

If ξ is a nonzero simple m-vector and η is a nonzero simple n-vector, then $\xi \vee \eta \neq 0$ if and only if the subspace associated with $\xi \wedge \eta$ is the direct sum of the two subspaces associated with ξ and η. The subspace associated with a nonzero simple m-vector ξ is contained in the subspace associated with a nonzero simple n-vector η if and only if $\eta = \xi \vee \zeta$ for some $\zeta \in \wedge_{n-m} V$.

A nonzero simple m-vector $\xi \in \wedge_m V$ is called **complex** if and only if the $\mathbb{R}$ vector subspace of V with ξ is a $\mathbb{C}$ vector subspace of V. It follows that ξ is complex if and only if m is even, say $m = 2p$, and $\xi = rv_\vee iv_1 \vee \cdots \vee v_p \vee iv_p$ for some $r \in \mathbb{R}$ and $v_1, \ldots, v_p \in V$. Moreover sign(r) is uniquely determined by ξ. We term ξ positive in case $r > 0$.

Now for defining the mass norm on $\wedge_m V$, we need to know some notions.

An m-linear function f which maps the m fold cartesian product V^m of a vector space V into some other vector space W, is called *alternating* if and only if $f(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m \in V$ and $v_i = v_j$ for some $i \neq j$. Let $\wedge^m(V, W)$ be the vector space of all m-linear alternating functions mapping V^m into W and define

$$\wedge^*(V, W) = \oplus_{m=0}^{\infty} \wedge^m (V, W).$$

Note that $\wedge^0(V, W) = W$. Most frequently we use $w = \mathbb{R}$; hence we abbreviate

$$\wedge^m(V, \mathbb{R}) = \wedge^m V \quad \text{and} \quad \wedge^* (V, \mathbb{R}) = \wedge^* V.$$

The elements of $\wedge^m V$ are called m-covectors of V. According to 1.4.1 of [12] we have $\wedge^m(V, W) \backsimeq \mathrm{Hom}(\wedge_m V, W)$ and so in an extension of the usual notation $\langle \zeta, h \rangle = h(\zeta)$ for $\zeta \in \wedge_m V,\ h \in \mathrm{Hom}(\wedge_m V, W)$. Next we discuss the manner in which inner products for the spaces $\wedge_m V$ are induced by the given inner product for V. The polarity $\beta : V \longrightarrow \wedge^1 V$ can be uniquely extended to a unit preserving algebra homomorphism $\gamma : \wedge_* V \longrightarrow \wedge^* V$, which is the direct sum of linear maps $\gamma_m : \wedge_m V \longrightarrow \wedge^m V$. Composing γ_m with $\wedge^m V \backsimeq \wedge^1(\wedge_m V)$ which satisfy the condition

$$\langle \xi, \beta_m(\eta) \rangle = \langle \eta, \beta_m(\xi)$$

for $\xi, \eta \in \wedge_m V$. Thus β_m is a polarity, and we define a symmetric bilinear function $\bullet$ on $\wedge_m V \times \wedge_m V$ by the formula

$$\xi \bullet \eta = \langle \xi, \beta_m(\eta) \rangle \tag{3.7.1}$$

for $\xi, \eta \in \wedge_m V$. Now if $e_1, \ldots, e_n$ form an orthonormal base for V, then the base vectorse_λ of $\wedge_m V$, corresponding to $\lambda \in I(n, m)$, are likewise orthonormal. For any m-vectors ξ and η the representations

$$\xi = \sum_{\lambda \in I(n,m)} \xi_\lambda e_\lambda, \qquad \eta = \sum_{\lambda \in I(n,m)} \eta_\lambda e_\lambda$$

are the bilinearity of $\bullet$ lead to the formula

$$\xi \bullet \eta = \sum_{\lambda \in I(n,m)} \xi_\lambda \eta_\lambda.$$

In case $\xi = \eta \neq 0$, we obtain $\xi \bullet \xi = \sum\limits_{\lambda} (\xi_\lambda)^2 > 0$. So $\bullet$ is in fact an inner product for $\wedge_m V$, and we can define

$$|\xi| = (\xi \bullet \xi)^{\frac{1}{2}} = (\sum_{\lambda} (\xi_\lambda)^2)^{\frac{1}{2}}.$$

Now let ξ be a p-vector and η a q-vector. In case ξ or η is simple, then $|\xi \wedge \eta| \leq |\xi|.|\eta|$. In case ξ and η are simple and nonzero, equality holds if and only if the subspaces associated with ξ and η are orthogonal. Always $|\xi \wedge \eta| \leq ()^{\frac{1}{2}} |\xi|.|\eta|$. For proof and more details refer to [12]. The set $C = \wedge_m V \cap \{\xi : \ \|\xi\| \leq 1\}$ is the convex hull of the compact connected set $S = \wedge_m V \cap \{\xi : \ \xi$ is simple and $|\xi| \leq 1\}$, hence C consists of all finite sums

$$\sum_{i=1}^{N} c_i \xi_i \quad \text{with } \xi_i \in S, \quad c_i > 0, \quad \sum_{i=1}^{N} c_i = 1$$

and $N \leq \dim \wedge_m V = (?)$. It follows that for each $\xi \in \wedge_m V$ there exist simple m-vectors $\xi_1, \ldots, \xi_N$ with

$$\xi = \sum_{i=1}^{N} \xi_i, \quad \|\xi\| = \sum_{i=1}^{N} |\xi_i|.$$

Consequently

$$\|\xi\| == \inf\{\sum_{i=1}^{N} |\xi_i| : \ \xi_i \text{ are simple and } \xi = \sum_{i=1}^{N} \xi\}. \tag{3.7.2}$$

If $\xi \in \wedge_p V$ and $\eta \in \wedge_q V$, then

$$\|\xi \wedge \eta\| \|\xi\|.\|\eta\|. \tag{3.7.3}$$

This norm is called **Mass norm** on exterior algebra.

3.7.2 Graßmann algebra

Recall that the Graßmann algebra (or exterior algebra) Λ_n with n generators is the algebra (over $\mathbb{C}$) generated by a set of n anticommuting generators $\{\xi_i\}_{i=1}^n$ and by $1 \in \mathbb{C}$

$$\xi_i \xi_j = -\xi_j \xi_i, \text{ for all } i, j.$$

Also for countably infinite set of generators, which will be denoted by Λ_∞, it is defined by the direct limit of finite dimensional Graßmann algebras. In the rest of this section we shall write Λ_n where $n \in \mathbb{N} \cup \{\infty\}$ is possibly infinite unless indicated otherwise. We saw in section 2.4 of previous chapter that the Graßmann algebra carries a natural $\mathbb{Z}_2$-grading: $\Lambda_n = \Lambda_{n,0} \oplus \Lambda_{n,1}$, where $\Lambda_{n,0}$ consists of the *even* (commuting) elements in Λ_n and $\Lambda_{n,1}$ consists of the *odd* (anticommuting) elements of Λ_n, i.e., for $a_r \in \Lambda_{n,r}$ and $a_s \in \Lambda_{n,s}$ we have $a_r a_s = (-1)^{rs} a_s a_r \in \Lambda_{n,r+s(\text{mod} 2)}$. We also write $\deg(a_r) = r$ if $a_r \in \Lambda_{n,r}$ and call $\deg(a_r)$ the *degree* of a_r (this degree is called somewhere, the parity of a_r). Let

$$M_n^0 := \{(m_1, \ldots, m_k) \,|\, 0 \leq k \leq n, m_i \in \mathbb{N}, 1 \leq m_1 < \cdots < m_k \leq n\}. \tag{3.7.4}$$

Every element $q \in \Lambda_n$ can be uniquely written as

$$q = \sum_{(m_1, \cdots, m_k) \in M_n^0} q_{m_1, \cdots, m_k} \xi_{m_1} \cdots \xi_{m_k},$$

where $q_{m_1,\cdots,m_k} \in \mathbb{C}$ with $q_\phi = q_B($ the body of $q)$. If p is another element of Λ_n with representation $p = \sum_{(m_1,\cdots,m_k)\in M_n^0} p_{m_1,\cdots,m_k}\xi_{m_1}\cdots\xi_{m_k}$, the addition, scalar multiplication and multiplication on Λ_n are defined as follows:

$$p + q = \sum_{(m_1,\cdots,m_k)\in M_n^0} (p_{m_1,\cdots,m_k} + q_{m_1,\cdots,m_k})\xi_{m_1}\cdots\xi_{m_k},$$

$$cq = \sum_{(m_1,\cdots,m_k)\in M_n^0} (cq_{m_1,\cdots,m_k})\xi_{m_1}\cdots\xi_{m_k},$$

$$pq = \sum_{(m_1,\cdots,m_k)\in M_n^0} \sum_{k=0}^{r} \sideset{}{'}\sum_{\sigma} (-1)^{\mathrm{sgn}(\sigma)} \quad p_{\sigma(m_1),\cdots,\sigma(m_k)} q_{\sigma(m_{k+1}),\cdots,\sigma(m_r)} \xi_{m_1}\cdots\xi_{m_k}\xi_{m_{k+1}}\cdots\xi_{m_r}, \tag{3.7.5}$$

where the sum $\sideset{}{'}\sum_{\sigma}$ runs over all permutations σ of $(m_1,\ldots,m_k)$ such that $(\sigma(m_1),\ldots,\sigma(m_r)) \in M_n^0$. Recall that $\mathrm{sgn}(\sigma)$ is the number of pairs (i,j) such that $i < j$ and $\sigma(i) > \sigma(j)$ for $i,j \in \{m_1,\ldots,m_r\}$. Note that if we take

$$M_n := \{(m_1,\cdots,m_k)\,|\,1 \le k \le n, m_i \in \mathbb{N}, 1 \le m_1 < \cdots < m_k \le n\}, \tag{3.7.6}$$

then every $q \in \Lambda_n$ can be written as

$$q = q_B 1 + q_S = q_B 1 + \sum_{(m_1,\cdots,m_k)\in M_n} q_{m_1,\cdots,m_k}\xi_{m_1}\cdots\xi_{m_k}, \tag{3.7.7}$$

where $q_B, q_{m_1,\cdots,m_k} \in \mathbb{C}$. Now for each $1 \le \kappa < \infty$ we can define

$$|q|_\kappa := \left(|q_B|^\kappa + \sum_{(m_1,\cdots,m_k)\in M_n} |q_{m_1,\cdots,m_k}|^\kappa\right)^{1/\kappa} \tag{3.7.8}$$

and for $\kappa = \infty$,

$$|q|_\infty = \sup_{(m_1,\ldots,m_k)\in M_n} |q_{m_1,\ldots,m_k}|. \tag{3.7.9}$$

If n is finite, it is straightforward to verify that each $|\cdot|_\kappa$ defines a norm on Λ_n and that Λ_n becomes a complex Banach space with each of the norms $|\cdot|_\kappa$, $1 \le \kappa \le \infty$, which we denote by $\Lambda_n(\kappa)$ respectively. In fact we proved this assertion in proposition 2.2.1 of previous chapter in general case for the σ-commutative G-graded algebra $\mathcal{A}$. In the case of Λ_∞, $|\cdot|_\kappa$ defines a seminorm on Λ_∞ and we denote the set of all $q \in \Lambda_\infty$ for which the above expression for $|q|_\kappa$ satisfies $|q|_\kappa < \infty$ by $\Lambda_\infty(\kappa)$. Again it is easy to see that $\Lambda_\infty(\kappa)$ with the norm $|\cdot|_\kappa$ is a Banach space for all $1 \leq \kappa \leq \infty$.

Define an involution on Λ_n as a map $* : \Lambda_n \longrightarrow \Lambda_n$ which satisfies in following conditions:

(i) $(q^*)^* = q$;

(ii) $(qp)^* = p^* q^*$;

(iii) $(\alpha q)^* = \bar{\alpha} q^*$;

(iv) $1^* = 1$;

(v) $\xi_i^* = \xi_i$, for all i;

(vi) $*$ extends linearly to all of Λ_n,

where $p, q \in \Lambda_n$ and $\alpha \in \mathbb{C}$.

The norms $|\cdot|_\kappa$ in (3.7.8) depend implicitly on the choice of the set of generators of the Graßmann algebra and are not invariant under a change of the set of generators of Λ_n. Since for fix $q \in \Lambda_n$, its coefficient may be vary by varying of generators of Λ_n and hence its norm is also may be not invariant.

For n finite, Λ_n is finite dimensional and so not only all the norms in 3.7.8 are equivalent and therefore generate the same topology on Λ_n for all $1 \leq \kappa \leq \infty$, but the resulting topology is in fact independent of the choice of generators of the Graßmann algebra.

For n finite there is an isomorphism $\star : \Lambda_n \longrightarrow \Lambda_n$ known as the **Hodge star operator**. Consider the ordered sequence $\{\xi_1, \cdots, \xi_n\}$ of all generators of Λ_n, then $\star$ is defined on the element $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_d}$ by

$$\star[\xi_{i_1}\xi_{i_2}\cdots\xi_{i_d}] := \xi_{j_1}\xi_{j_2}\cdots\xi_{j_{n-d}},$$

where $(j_1, \cdots, j_{n-d})$ is chosen such that $(i_1, \cdots, i_d, j_1, \cdots, j_{n-d})$ is an even permutation of $(1, \cdots, n)$. We extend $\star$ to all of Λ_n by conjugate linearity, i.e., we require that $\star[\alpha q] := \bar{\alpha}\star[q]$ for $\alpha \in \mathbb{C}$ and $q \in \Lambda_n$ and that $\star$ is a real linear transformation. Let $q \in \Lambda_n$ be expressed as in 3.7.7. Then the action of $\star$ on q is as follows:

$$\star[q] = \overline{q}_B 1 + \sum_{(m_1,\cdots,m_k)\in M_n} \overline{q}_{m_1,\cdots,m_k}\xi_{m'_1}\cdots\xi_{m'_{n-k}}.$$

such that $(m_1, \ldots, m_k, m'_1, \ldots, m'_{n-k})$ is an even permutation of $(1, \ldots, n)$. It is well-known that the Hodge star operator is independent of the basis used to define it. Also this operator is continuous in the unique topology induced by the norms 3.7.8. To see this let $\{q_n\}_{n=1}^{\infty}$ be a sequence of elements of Λ. It can be represented as

$$q_n = q_B^{(n)} 1 + \sum_{(m_1,\cdots,m_k)\in M_n} q^{(n)}_{m_1,\cdots,m_k}\xi_{m_1}\cdots\xi_{m_k}.$$

Let $q_n \longrightarrow q \in \Lambda$ in the norm 3.7.8. So for any $\varepsilon > 0$ there exists positive N such that for any $n \geq N$, $|q_n - q|_k < \infty$. This implies that $q_B^{(n)} \longrightarrow q_B$ and $q^{(n)}_{m_1,\cdots,m_k} \longrightarrow q_{m_1,\cdots,m_k}$ in $\mathbb{C}$ for any $(m_1, \cdots, m_k) \in M_n$. Since the complex conjugation is a continuous function, we obtain that $\overline{q}_B^{(n)} \longrightarrow \overline{q}_B$ and $\overline{q}^{(n)}_{m_1,\cdots,m_k} \longrightarrow \overline{q}_{m_1,\cdots,m_k}$ in $\mathbb{C}$ for any $(m_1, \cdots, m_k) \in M_n$. On the other hand, the action of Hodge star operator on $\{q_n\}_{n=1}^{\infty}$ and q implies that

$$\star[q_n] = \overline{q}_B^{(n)} 1 + \sum_{(m_1,\cdots,m_k)\in M_n} \overline{q}^{(n)}_{m_1,\cdots,m_k}\xi_{m'_1}\cdots\xi_{m'_{n-k}}.$$

and

$$\star[q] = \overline{q}_B 1 + \sum_{(m_1,\cdots,m_k)\in M_n} \overline{q}_{m_1,\cdots,m_k}\xi_{m'_1}\cdots\xi_{m'_{n-k}}.$$

Above discussion, simply imply $|\star[q_n] - \star[q]|_k \longrightarrow 0$ and hence the Hodge star operator is continuous. For $k = \infty$, we can easily obtain this continuity.

Now we want to define a new norm on Λ_n which be invariant under change of generators. For this purpose, note that, the Graßmann algebra can also be written as a direct sum

$$\Lambda_n = \oplus_{r=0}^{n} \mathsf{V}_r,$$

where V_r is the complex vector space spanned by the elements of the form $\xi_{m_1} \cdots \xi_{m_r}$, r fixed. Therefore any $q \in \Lambda_n$ can be uniquely decomposed as $q = \sum_{r=0}^{n} q_r$ with $q_r \in \mathsf{V}_r$. Any choice of a basis of V_1 may serve as a possible choice of (possibly complex) generators of Λ_n. For defining the required norm, firstly, it is known that there is a norm $\|\cdot\|_r$ on V_r given by, [12, 51],

$$\|q_r\|_r = \inf \left\{ \sum_{(m_1,\cdots,m_r)\in M_n} |q_{m_1,\cdots,m_r}| \right\}, \tag{3.7.10}$$

for $q_r \in \mathsf{V}_r$ where the infimum is taken over all possible choices of the set of generators of the Graßmann algebra. To see that this defines a norm on V_r, it is easily seen that it is a seminorm, i. e., subadditive positive or zero and satisfies $\|\alpha q_r\|_r = |\alpha| \|q_r\|_r$. To show that it is a norm, it has be shown that $\|q_r\|_r = 0$ implies $q_r = 0$. Assume $\|q_r\|_r = 0$. Then for every $\epsilon > 0$ there is representation of q_r analogous to 3.7.7 such that the sum on the right hand side of 3.7.10 (without the infimum) is smaller than (or possibly equal to)ϵ. AS ϵ can be arbitrarily small and as the terms in the expansion in 3.7.7 are linearly independent and as for all complex numbers a, b we have $\|a + b\| \leq \|a\| + \|b\|$, this is only possible if $q_r = o$. Thus equation 3.7.10 defines a norm on V_r. The norm $\|\cdot\|_r$ satisfies

$$\|q_r p_s\|_{r+s} \leq \|q_r\|_r \|p_s\|_s,$$

for all $q_r \in \mathsf{V}_r$ and $p_s \in \mathsf{V}_s$, see [12, 51]. Now define a seminorm on Λ_n by

$$\|q\| := \sum_{r=0}^{n} \|q_r\|_r. \tag{3.7.11}$$

For n finite it is obvious that $\|\cdot\|$ is a norm on Λ_n. This norm $\|\cdot\|$ is called the **mass (norm)** on Λ_n (n finite). By construction the mass norm is independent of the choice of the set of generators of Λ_n.

If $n = \infty$, then every finite subset $\{\xi_{i_1}, \cdots \xi_{i_m}\} \cup \{1\}$ of the set of all generators $\{\xi_i\}_i$ of Λ_∞ generates an m-dimensional Graßmann subalgebra of Λ_∞ denoted by $\Lambda_{i_1,\cdots,i_m}$. The collection of all such Graßmann subalgebras of Λ_∞ forms a directed set and the canonical imbedding morphisms obviously preserve the mass norm. We consider the algebraic direct limit Δ_∞ of this directed set. Indeed, $\Lambda_{i_1,\cdots,i_m} \subset \Lambda_{i_1,\cdots,i_{m+1}}$ and $\imath : \Lambda_{i_1,\cdots,i_m} \longrightarrow \Lambda_{i_1,\cdots,i_{m+1}}$ is canonical imbedding morphism which satisfies $\imath(\|q\|) = \|\imath(q)\|$ for any $q \in \Lambda_{i_1,\cdots,i_m}$. The mass norm on the finite dimensional Graßmann subalgebras induces a *mass norm* $\|\cdot\|$ on Δ_∞. We denote the completion of Δ_∞ with respect to the mass norm by Λ_∞^m. Obviously, Λ_∞^m consists of all $q \in \Lambda_\infty$ with $\|q\| = \sum_{r=0}^\infty \|q_r\|_r < \infty$. The norm on Λ_∞^m is again called the **mass norm**.

We can see that the mass norm is submultiplicative

$$\begin{aligned}\|pq\| &= \sum_r \|(pq)_r\|_r \leq \sum_r \sum_{k\leq r} \|p_{r-k}q_k\|_r \\ &\leq \sum_r \sum_{k\leq r} \|p_{r-k}\|_{r-k}\|q_k\|_k \leq \sum_r \sum_k \|p_r\|_r\|q_k\|_k = \|p\|\|q\|.\end{aligned}$$

3.7.3 Hilbert Λ modules

Definition 3.7.1. A **pre-Hilbert Λ module** is a $\mathbb{Z}_2$-graded right Λ module $E = E_0 \oplus E_1$ equipped with a Λ-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to \Lambda$ that is sesquilinear, definite, and whose body is Hermitian and positive. In other words:

i) $\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$, and $\langle y_1 + y_2, x\rangle = \langle y_1, x\rangle + \langle y_2, x\rangle$ for $x, y_1, y_2 \in E$;

ii) $\langle x, \alpha y\rangle = \alpha\langle x, y\rangle = \langle \alpha^* x, y\rangle$, for $x, y \in E, \alpha \in \mathbb{C}$

iii) $\langle x, y\rangle_B = \langle y, x\rangle_B^*$, for $x, y \in E$;

iv) $\langle x, x\rangle_B \geq 0$ for $x \in E$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

An element x of a pre-Hilbert Λ module $E = E_0 \oplus E_1$ is called *even* if $x \in E_0$ and *odd* if $x \in E_1$, respectively.

Immediate consequences of Definition 3.7.1 are that every pre-Hilbert Λ module is a complex vector space and that every element x of a pre-Hilbert Λ module E can be uniquely written as a sum of an even and an odd element of E, i.e., $x = x_0 + x_1$, where $x_0 \in E_0$ and $x_1 \in E_1$.

We may now use a norm $\|\cdot\|_\Lambda$ defined on Λ to define a function P_E on a pre-Hilbert Λ module E by

$$P_E^2(x) = \|\langle x, x\rangle\|_\Lambda. \tag{3.7.12}$$

For instance, if Λ equals Λ_n or$\Lambda_\infty(\kappa)$ endowed with the norm $|\cdot|_\kappa$, then this function on E is given by

$$P_\kappa^2(x) := |\langle x, x\rangle|_\kappa, \tag{3.7.13}$$

for $x \in E$ and $1 \leq \kappa \leq \infty$. The function

$$P^2(x) := \|\langle x, x\rangle\| \tag{3.7.14}$$

corresponding to the mass norm on $\Lambda = \Lambda_n$ or $\Lambda = \Lambda_\infty^m$ in Equation 3.7.11 is called the **mass function** on the Hilbert Λ module E.

Remark 3.7.1. It is important to note that, unlike the theory of Hilbert C^*-modules, the equation 3.7.12 is not a norm on E. Because, as we see as soon as, the Cauchy-Schwartz inequality is not hold in general and hence the function P is not subadditive. But it is easy to see that P has other properties of a norm.

Lemma 3.7.1 (Cauchy-Schwartz inequality). *If E is a pre-Hilbert Λ module and $x, y \in E$, then*

$$|\langle x, y\rangle_B|^2 \leq \langle x, x\rangle_B \langle y, y\rangle_B.$$

Proof. Let $p_x := \langle x, x\rangle_B, p_y := \langle y, y\rangle_B, q := \langle x, y\rangle_B$ and $\lambda \in \mathbb{R}$, then

$$0 \leq \langle x - y\lambda q^*, x - y\lambda q^*\rangle_B = p_x - 2\lambda q q^* + \lambda^2 q p_y q^*.$$

Adding $2\lambda q q^*$ on both sides and taking norms yield

$$2\lambda|q|^2 \leq 2|\lambda||q|^2 \leq |p_x + \lambda^2 q p_y q| \leq |p_x| + \lambda^2 |q|^2 |p_y|. \tag{3.7.15}$$

Now since $|p_y|$ is positive, we multiply both sides of 3.7.15 to it and add $(|q|)^2$ to obtain

$$(\lambda|q||p_y| - |q|)^2 \geq (|q|)^2 - |p_x||p_y|.$$

If $|p_y| \neq 0$, then setting $\lambda := \frac{1}{|p_y|}$ yields the required inequality. Moreover, we find that $|p_x| = 0$ and $|p_y| \neq 0$ implies $|q| = 0$ (let $\lambda = 1$). From symmetry considerations (or from Equation 3.7.15) we also get that $|p_y| = 0$ and $|p_x| \neq 0$ implies $|q| = 0$. In the case that $|p_x| = |p_y| = 0$ we infer from Equation 3.7.15 by taking λ to be positive that $|q| = 0$. □

On any pre-Hilbert Λ module E there is a *body operation*, i.e., a linear map $B : E \to E_0, x \mapsto x_B$ such that $(x\lambda)_B = x_B \lambda_B$ for all $\lambda \in \Lambda$ [32]. First define the *soul* $s(E)$ and the *body* $b(E)$ of E by

$$\begin{aligned} s(E) &:= \{x \in E | x\lambda = 0 \text{ for some } \lambda \in \Lambda, \lambda \neq 0\}, \\ b(E) &:= E/s(E). \end{aligned}$$

The body operation $B : E \to E_0$ is the canonical surjection from E to $b(E)$.

If the inner product of E satisfies $\langle x_B, y_B\rangle = \langle x, y\rangle_B$, then the body of E endowed with the induced inner product is a pre-Hilbert space whose completion is a Hilbert space (by virtue of the Cauchy-Schwartz inequality). To see this, let the inner product has the mentioned property. We know that the body operation $B : E \longrightarrow E_0$ has the following properties:

$$B(x+y) = (x+y)_B = B(x) + B(y) \qquad B(x\lambda) = B(x)B(y)$$

where $x, y \in E$ and $\Lambda \in \Lambda$.

We can see easily that, if $x_B, y_B, z_B \in E$ then

$$\begin{aligned} \langle x_B + y_B, z_B \rangle = \langle (x+y)_B, z_B \rangle &= \langle x+y, z \rangle_B = B(\langle x, z \rangle + \langle y, z \rangle) \\ &= \langle x, z \rangle_B + \langle y, z \rangle_B = \langle x_B, z_B \rangle + \langle y_B, z_B \rangle. \end{aligned}$$

The other conditions of inner product is straightforward.

But even if the inner product does not respect the body operation, we can prove

Proposition 3.7.2. *Let E be a pre-Hilbert Λ module. Then there exists a map $x \to [x]$ from E into a dense subspace of a Hilbert space H such that*

$$\langle [x], [y] \rangle_H = \langle x, y \rangle_B,$$

for all $x, y \in E$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on H.

Proof. Let $\mathcal{N} := \{x \in E | \langle x, x \rangle_B = 0\}$. Let $[x] := x + \mathcal{N}$. Then $\langle \cdot, \cdot \rangle_B$ induces a well-defined inner product on $E/\mathcal{N}$ by virtue of Lemma 3.7.1. Therefore $E/\mathcal{N}$ with this inner product is a pre-Hilbert space. □

Definition 3.7.2. Let E be a pre-Hilbert Λ module and $\| \cdot \|$ a norm on E, then E is said to be a **Hilbert Λ module** if E is complete with respect to its norm. A **Hilbert submodule** of a Hilbert module E is a closed submodule of E.

Definition 3.7.3. Let E and F be Hilbert Λ modules. A $\mathbb{C}$-linear map $O : E \to E$ is called an **operator** on E. We denote the set of all bounded operators on E by $\mathcal{L}(E)$. An operator $T : E \to E$ is called **unitary** if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in E$. An operator S is called **weakly unitary** if $\langle S(x), S(y)\rangle_B = \langle x, y\rangle_B$ for all $x, y \in E$. A **(Hilbert) module map** is a linear map $T : E \to F$ which respects the module action: $T(xq) = T(x)q$, for $x \in E, q \in \Lambda$.

Definition 3.7.4. A Hilbert Λ module E is said to satisfy the **strong definiteness condition** if $\langle x, x\rangle_B = 0$ implies $x = 0$ for all $x \in E$.

Every Hilbert Λ module E satisfying the strong definiteness condition becomes a pre-Hilbert space with respect to the norm $\|\cdot\|_B^2 := \langle\cdot,\cdot\rangle_B$.

Every Hilbert Λ module E is endowed with a $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$. This induces a $\mathbb{Z}_2$-grading on $\mathcal{L}(E)$: every operator $T : E \to E$ can be written as sum of an *even* map $T_0 : E_i \to E_i$ and an *odd* map $T_1 : E_i \to E_{i+1(\text{mod}2)}$, i.e. $T = T_0 + T_1$ where T_0 and T_1 are defined by $T_0u := (Tu_0)_0 + (Tu_1)_1$ and $T_1u := (Tu_0)_1 + (Tu_1)_0$ respectively where $u = u_0 + u_1$.

Definition 3.7.5. Let E be a Hilbert Λ module. An operator $T : E \to E$ is said to be **adjointable** if there exists an operator $T^* : E \to E$ satisfying $\langle x, Ty\rangle = \langle T^*x, y\rangle$ for all $x, y \in E$. Such an operator T^* is called an **adjoint** of T. We denote the set of all adjointable operators on E by $\mathcal{B}(E)$. An adjointable operator $T \in \mathcal{B}(E)$ is called **self-adjoint** if $T^* = T$.
An operator $T : E \to E$ is said to be **weakly adjointable** if there exists an operator $T^\dagger : E \to E$ satisfying $\langle x, Ty\rangle_B = \langle T^\dagger x, y\rangle_B$ for all $x, y \in E$. Such an operator $T^\dagger$ is called a **weak adjoint** of T. We denote the set of all weakly adjointable operators on E by $\mathcal{B}_w(E)$. A weakly adjointable operator

$T \in \mathcal{B}_w(E)$ is called **weakly self-adjoint** if $T^\dagger = T$.

Remark 3.7.2. Obviously, any adjointable operator is also weakly adjointable. Thus, $\mathcal{B}(E) \subset \mathcal{B}_w(E)$. Accordingly we also expect that the set $\mathcal{B}_w(E)$ plays a distinguished role and that the operators representing physical observables or physical operations will be elements of $\mathcal{B}_w(E)$.

The following Lemma can be proven in analogy to the corresponding result for Hilbert C^*-modules, see [46].

Lemma 3.7.3. *(a) Let E be a Hilbert Λ module and $T : E \to E$ be an adjointable operator. The adjoint T^* of T is unique. If both $T : E \to E$ and $S : E \to E$ are adjointable operators, then ST is adjointable and $(ST)^* = T^*S^*$.*
(b) Let E be a Hilbert Λ module satisfying the strong definiteness condition and $T_w : E \to E$ be a weakly adjointable operator. Then the weak adjoint $T_w^\dagger$ of T_w is unique. If both $T_w : E \to E$ and $S_w : E \to E$ are adjointable operators, then S_wT_w is adjointable and $(S_wT_w)^\dagger = T_w^\dagger S_w^\dagger$.

Proof. (a) Assume that $\overline{T}$ and T^* are adjoints of T, then

$$0 = \langle \overline{T}x, y\rangle - \langle T^*x, y\rangle = \langle (\overline{T} - T^*)x, y\rangle,$$

for all $x, y \in E$. Let $y = (\overline{T} - T^*)x$. This implies $\overline{T} = T^*$. Also $\langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle$ which implies that $(ST)^* = T^*S^*$. A similar argument proves (b). □

3.7.4 Super Hilbert spaces

The Definitions 3.7.3 and 3.7.5 are analogous to parallel definitions in the theory of Hilbert C^*-modules [46] and [26]. However, the positivity requirement in the definition of a Hilbert Λ module is weaker than the positivity requirement for Hilbert C^*-modules and all results for Hilbert C^*-modules depending on the positivity of the inner product may in general not be valid for a Hilbert Λ module. The Cauchy-Schwartz inequality in Lemma 3.7.1 is a first example. As a consequence of the failure of the general Cauchy-Schwartz inequality the inner product on a pre-Hilbert Λ module may in general not be continuous in each argument and therefore in general an inner product on a pre-Hilbert Λ module does not extend to an inner product on its completion. In the sequel we shall be mainly interested in inner products which are continuous.[2]

Definition 3.7.6. We shall call a (pre-) Hilbert Λ module $\mathcal{H}$ a **super (pre-) Hilbert space** if the inner product on $\mathcal{H}$ is continuous, i.e., if there exists a constant $C > 0$ such that $\|\langle x, y\rangle\| \leq C\|x\|\|y\|$.

Remark 3.7.3. It is important to note that if $\mathcal{H}$ is a super (pre-) Hilbert space, then the function $\|.\|$H defined on $\mathcal{H}$, as in 3.7.12, is a norm on $\mathcal{H}$. It suffices to show its subadditivity which it can be obtain from continuity of inner product. The completion of a super pre-Hilbert space is a Hilbert space.

We have already noticed above that the physical transition amplitudes are given by the body of the inner product of a Hilbert Λ module. This gives rise to the following definition.

Definition 3.7.7. Let $\mathcal{H}$ be a super Hilbert space. An element $x \in \mathcal{H}$ is called **physical** if $\langle x, x\rangle_B \neq 0$. An element $g \in \mathcal{H}$ with $g \neq 0$ and $\langle g, g\rangle_B = 0$ is called a **ghost**.

[2]In the next chapter we will see that by using the theory of Riesz spaces on Graßmann algebra, in particular, Hilbert Λ-modules behave exactly as Hilbert C^*-modules, i.e, the positivity of inner product holds. Also we shall prove the Cauchy-Schwartz inequality in general which implies the continuity of inner product on a pre-Hilbert Λ-module.

Example 3.7.1. *Let n be finite. The Graßmann algebra Λ_n endowed with the mass norm $\|\cdot\|$ becomes a super Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ given by*

$$\langle p, q\rangle := \star\,[p \star [q]] \tag{3.7.16}$$

for all $p, q \in \Lambda_n$, where $\star$ denotes the Hodge star operator. First we want see explicit version of this inner product. Let $p, q \in \Lambda_n$ with following representations:

$$p = p_B 1 + \sum_{(m_1,\cdots,m_k)\in M_n} p_{m_1,\cdots,m_k}\xi_{m_1}\cdots\xi_{m_k},$$

and

$$q = q_B 1 + \sum_{(m_1,\cdots,m_k)\in M_n} q_{m_1,\cdots,m_k}\xi_{m_1}\cdots\xi_{m_k},.$$

Then according to action of Hodge star operator we will have

$$\star[q] = \overline{q}_B 1 + \sum_{(m_1,\cdots,m_k)\in M_n} \overline{q}_{m_1,\cdots,m_k}\xi_{m'_1}\cdots\xi_{m'_{n-k}}.$$

such that $(m_1,\ldots,m_k,m'_1,\ldots,m'_{n-k})$ is an even permutation of $(1,\ldots,n)$. On the other hand

$$p\star[q] = \sum_{(m_1,\cdots,m_k)\in M_n^0}\sum_{k=0}^{r}\sum_{\sigma}{}'(-1)^{sgn(\sigma)} \quad p_{\sigma(m_1),\cdots,\sigma(m_k)}\overline{q}_{\sigma(m_{k+1}),\cdots,\sigma(m_r)}$$
$$\xi_{m_1}\cdots\xi_{m_k}\xi_{m_{k+1}}\cdots\xi_{m_r},$$

where the sum $\sum_{\sigma}'$ runs over all permutations σ of $(m_1,\ldots,m_k)$ such that $(\sigma(m_1),\ldots,\sigma(m_r)) \in M_n^0$ where M_n^0 is as in 3.7.4. The action of Hodge star operator on this element can be shown as

$$\star[p\star[q]] = \sum_{(m_1,\cdots,m_k)\in M_n^0}\sum_{k=0}^{r}\sum_{\sigma}{}'(-1)^{sgn(\sigma)} \quad q_{\sigma(m_{k+1}),\cdots,\sigma(m_r)}\overline{p}_{\sigma(m_1),\cdots,\sigma(m_k)}$$
$$\xi_{m_1}\cdots\xi_{m_k}\xi_{m_{k+1}}\cdots\xi_{m_r}.$$

The submultiplicativity of the mass norm implies

$$\|\langle p, q\rangle\| \le \|p\|\|q\|$$

for all $p, q \in \Lambda_n$. By considering to the inner product 3.7.16 we see that

$$\langle q, q\rangle_B = \star[q \star [q]]_B = |q_B|^2 + \sum_{(m_1,\cdots,m_k)\in M_n} |q_{m_1,\ldots,m_k}|^2.$$

It is easily seen that if $\langle q, q\rangle_B = 0$ then $q = 0$ which this implies that Λ_n satisfies the strong definiteness condition. Also we can check all conditions of definition 3.7.1, for the inner product 3.7.16. Now it remains to check that the inner product is continuous and this obtain easily from the continuity of Hodge star operator.

More general super Hilbert spaces can be constructed by building the tensor product $\Lambda_n \otimes \mathfrak{H}$ of Λ_n with a complex Hilbert space $\mathfrak{H}$. The inner product of $\Lambda_n \otimes \mathfrak{H}$ is given on simple tensors by $\langle p\otimes\varphi, q\otimes\psi\rangle = \langle p, q\rangle\langle\varphi, \psi\rangle$, for $p, q \in \Lambda_n$ and $\varphi, \psi \in \mathfrak{H}$, and extended to arbitrary elements of $\Lambda_n \otimes \mathfrak{H}$ by linearity and continuity. We omit the details of the construction as a more general example will be given below in Example 3.7.5.

Example 3.7.2. *Consider a measure space (X, Ω), where X is a set and Ω a σ-algebra of subsets of X, endowed with a σ-finite measure μ. Every function $f : X \to \Lambda_n$ can be expanded as*

$$f(x) = f_B(x) + \sum_{(m_1,\cdots,m_k)\in M_n} f_{m_1,\cdots,m_k}(x)\xi_{m_1}\cdots\xi_{m_k},$$

with complex-valued functions $f_B : X \to \mathbb{C}$ and $f_{m_1,\cdots,m_k} : X \to \mathbb{C}$. We restrict ourselves here to the case that n is finite. Now consider the set E of all functions $f : X \to \Lambda_n$ such that f_B and all $f_{m_1,\cdots,m_k}$ are square integrable with respect to μ. This requirement is independent of the basis chosen. We define a Λ_n-valued inner product on E by

$$\langle f, g\rangle = \int f(x)^* g(x) d\mu(x), \tag{3.7.17}$$

for all $f, g \in E$. This inner product can be explicitly rewritten as follows. Let $f, g \in E$ which have representations as:

$$f(x) = f_B(x)1 + \sum_{(m_1,\cdots,m_k)\in M_n} f_{m_1,\cdots,m_k}(x)\xi_{m_1}\cdots\xi_{m_k},$$

$$f^*(x) = \bar{f}_B(x)1 + \sum_{(m_1,\cdots,m_k)\in M_n} \bar{f}_{m_1,\cdots,m_k}(x)\xi_{m_k}\cdots\xi_{m_1},$$

$$f^*(x)g(x) = \sum_{(m_1,\cdots,m_k)\in M_n^0} \sum_{k=0}^{r} \sum_{\sigma}{}'(-1)^{sgn(\sigma)} \quad \bar{f}_{\sigma(m_1),\cdots,\sigma(m_k)}(x) g_{\sigma(m_{k+1}),\cdots,\sigma(m_r)}(x)$$
$$\xi_{m_1}\cdots\xi_{m_k}\xi_{m_{k+1}}\cdots\xi_{m_r},$$

where M_n^0 is as 3.7.4. Now we have

$$\langle f, g\rangle = \int f^*(x)g(x)d\mu(x) =$$
$$\sum_{(m_1,\cdots,m_k)\in M_n^0} \sum_{k=0}^{r} \sum_{\sigma}{}'(-1)^{sgn(\sigma)}[\int \quad \bar{f}_{\sigma(m_1),\cdots,\sigma(m_k)}(x) g_{\sigma(m_{k+1}),\cdots,\sigma(m_r)}(x)d\mu]$$
$$\xi_{m_1}\cdots\xi_{m_k}\xi_{m_{k+1}}\cdots\xi_{m_r}$$

and we can see easily that

$$\langle f, f\rangle = \sum_{(m_1,\cdots,m_k)\in M_n^0} \sum_{k=0}^{r} \sum_{\sigma}{}'(-1)^{sgn(\sigma)}[\int \quad |f_{\sigma(m_1),\cdots,\sigma(m_r)}(x)|^2 d\mu]$$
$$\xi_{m_1}\cdots\xi_{m_k}\xi_{m_{k+1}}\cdots\xi_{m_r}.$$

If Λ_n is furnished with the Rogers norm $|\cdot|_1$, then define

$$\|f\| := \sum_{(m_1,\cdots,m_r)\in M_n^0} \sqrt{\int |f_{m_1,\cdots,m_r}(x)|^2 d\mu(x)}. \tag{3.7.18}$$

Further let $\mathcal{N} := \{f \in E \mid \|f\| = 0\}$. It is easy to see that Equation 3.7.18 defines a norm on $E/\mathcal{N}$ and that $E/\mathcal{N}$ equipped with the norm (3.7.18) becomes *a super Hilbert space*[3]. *Indeed, let $f, g \in E$, then*

$$\begin{aligned}
|\langle f,g\rangle|_1 &= \sum_{(m_1,\cdots,m_r)\in M_n^0} \left| \sum_{k=0}^{r} \sum_{\sigma}{}' (-1)^{\mathrm{sgn}(\sigma)} \int f^*_{\sigma(m_1),\cdots,\sigma(m_k)} g_{\sigma(m_{k+1}),\cdots,\sigma(m_r)} d\mu \right| \\
&\leq \sum_{(m_1,\cdots,m_r)\in M_n^0} \sum_{k=0}^{r} \sum_{\sigma}{}' \int |f_{\sigma(m_1),\cdots,\sigma(m_k)}|\,|g_{\sigma(m_{k+1}),\cdots,\sigma(m_r)}| d\mu \\
&\leq \sum_{(m_1,\cdots,m_r)\in M_n^0} \sum_{k=0}^{r} \sum_{\sigma}{}' \left[\int |f_{\sigma(m_1),\cdots,\sigma(m_k)}|^2 d\mu \int |g_{\sigma(m_{k+1}),\cdots,\sigma(m_r)}|^2 d\mu \right]^{\frac{1}{2}} \\
&\leq \|f\|\,\|g\|,
\end{aligned}$$

where the sum $\sum_\sigma'$ in the first three lines runs over all permutations σ of $(m_1, \cdots, m_r)$ such that $(\sigma(m_1), \cdots, \sigma(m_k)) \in M_n^0$ and $(\sigma(m_{k+1}), \cdots, \sigma(m_r)) \in M_n^0$. If we replace (3.7.17) by $\langle f, g\rangle = \int \star[f(x)]g(x)d\mu(x)$, a similar argument holds.

Example 3.7.3. *For n infinite we also can make Λ_∞^m a super Hilbert space by defining an appropriate inner product. For simplicity we assume that the set of all generators is countable $\{\xi_i\}_{i\in\mathbb{N}}$. The generalization of the following to the situation where the set of generators is uncountable is obvious. First of all we observe that the inner product (3.7.16) is not well-defined as the Hodge star operator is not defined on Λ_∞^m. This difficulty can be overcome by suitably imbedding Λ_∞^m into the direct sum $\Lambda_\infty^m \oplus \Lambda_\infty^m$ of two copies of Λ_∞^m. The basic idea is to introduce the formal infinite product of all generators $\xi_\infty \equiv \prod_i \xi_i$. We do not make any attempt to give a precise meaning to this infinite product of Graßmann numbers and just introduce ξ_∞ as an auxiliary object which has certain properties we would expect from the product of all generators of the Graßmann algebra. Namely, we require that $q\xi_\infty = q_B\xi_\infty$ for all $q \in \Lambda_\infty^m$.*

[3] *It is important to note that the norm 3.7.18 is not induced from the inner product 3.7.17. Although it may be preferable to have a norm induced by the inner product, but Rudolph's intention with his paper was to provide a general framework for super Hilbert spaces that may arise in theoretical physics. For that reason he wanted to be quite general and not too restrictive. Therefore he also allow super Hilbert spaces where the norm is not induced by the inner product and current example is as an example for such a situation.*

Analogously we define cofinite products of the generators of the Graßmann algebra, i.e., infinite products obtained from ξ_∞ by removing at most finitely many terms in the product. E.g., the infinite product $\prod_{i\neq 1}\xi_i$ of all generators except ξ_1 is denoted by $\hat{\xi}_1 \equiv \frac{\partial}{\partial\xi_1}\xi_\infty$. We require

$$\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j} = -\frac{\partial}{\partial\xi_j}\frac{\partial}{\partial\xi_i}$$

and $\xi_i\hat{\xi}_i = \xi_\infty$ and $\xi_i\frac{\partial}{\partial\xi_j} = -\frac{\partial}{\partial\xi_j}\xi_i$, for all $i \neq j$. Moreover we require ξ_∞ to be even. Therefore the algebra $\star[\Lambda^m_\infty]$ generated by the $\frac{\partial}{\partial\xi_i}$ and 1 is isomorphic to Λ^m_∞.

Now we are able to define the action of the Hodge star operator *on Λ^m_∞ by setting*

$$\star[q] \equiv q^*_B\xi_\infty + \sum_{(m_1,\cdots,m_k)\in M_\infty} q^*_{m_1,\cdots,m_k}\frac{\partial}{\partial\xi_{m_k}}\cdots\frac{\partial}{\partial\xi_{m_1}}\xi_\infty, \tag{3.7.19}$$

for all $q \in \Lambda^m_\infty$. Moreover, we require $\star[\star[q]] = q$, for all q. The algebra generated by the $\frac{\partial}{\partial\xi_i}$ is isomorphic to Λ^m_∞ with the isomorphism given by the Hodge star operator (3.7.19).

The inner product $\langle p, q\rangle = \star[p\star[q]]$, for all $p, q \in \Lambda^m_\infty$ is now well-defined. Notice that although $\star[q] \notin \Lambda^m_\infty$ for all $q \in \Lambda^m_\infty$, the inner product satisfies $\langle p, q\rangle \in \Lambda^m_\infty$ if $p, q \in \Lambda^m_\infty$. Since, by virtue of the properties of the mass norm, we also have $\|\langle p, q\rangle\| \leq \|p\|\|q\|$ for all $p, q \in \Lambda^m_\infty$ and since

$$\langle q, q\rangle_B = |q_B|^2 + \sum_{(m_1,\cdots,m_r)\in M_n} |q_{m_1,\cdots,m_r}|^2$$

we see that Λ^m_∞ with the inner product (3.7.16) is a super Hilbert space satisfying the strong definiteness condition.

Example 3.7.4. *$\star[\Lambda^m_\infty]$ can be made a super Hilbert space (over Λ^m_∞) by setting*

$$\langle p, q\rangle = \star[p]q,$$

for all $p, q \in \star[\Lambda_\infty^m]$ (when we identify ξ_∞ formally with $1 \in \mathbb{C}$). Obviously $\star[\Lambda_\infty^m]$ satisfies the strong definiteness condition.

Recall the following definition.

Definition 3.7.8. Given normed spaces U, V, a norm c on $U \otimes V$ is said to be a **cross-norm** if $\mathsf{c}(u \otimes v) = \|u\|\|v\|(u \in U, v \in V)$.

Example 3.7.5. *We are now going to construct the tensor product of two super Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We denote the inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively, and the norms on $\mathcal{H}_1$ and $\mathcal{H}_2$ are denoted by $\| \cdot \|_1$ and $\| \cdot \|_2$ respectively.*

The algebraic tensor product $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ is defined as usual as the set of all finite sums of the form $\sum_i p_i \otimes q_i$ with $p_i \in \mathcal{H}_1$ and $q_i \in \mathcal{H}_2$. We define a function μ on $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$ by

$$\mu(t) := \inf \left\{ \sum_i \|p_i\|_1 \|q_i\|_2 \,\middle|\, t = \sum_i p_i \otimes q_i \right\}. \tag{3.7.20}$$

μ is a cross norm on $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$ and the completion of $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$ with respect to μ is a Banach algebra which we denote by $\mathcal{H}_1 \otimes_\mu \mathcal{H}_2$ (for a proof, see, e.g., Proposition T.3.6 in [46]). The inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$ induce an inner product on $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$ given by

$$\langle a, b \rangle = \sum_{i,j} \langle p_i, t_j \rangle_1 \otimes \langle q_i, s_j \rangle_2$$

if $a = \sum_i p_i \otimes q_i$ *and* $b = \sum_j t_j \otimes s_j$. *As*

$$\begin{aligned}
\mu(\langle a, b\rangle) &= \inf \left\{ \sum_l \|c_l\|_1 \|d_l\|_2 \,\middle|\, \langle a, b\rangle = \sum_l c_l \otimes d_l \right\} \\
&\leq \text{restr.}\inf \sum_{i,j} \|\langle p_i, t_j\rangle_1\|_1 \|\langle q_i, s_j\rangle_2\|_2 \\
&\leq \text{restr.}\inf \sum_{i,j} \|p_i\|_1 \|q_i\|_2 \|t_j\|_1 \|s_j\|_2 \\
&= \text{restr.}\inf \left(\sum_i \|p_i\|_1 \|q_i\|_2\right)\left(\sum_j \|t_j\|_1 \|s_j\|_2\right) \\
&= \mu(a)\mu(b),
\end{aligned}$$

where the infimum in the first line runs over all possible decompositions of $\langle a, b\rangle$ *as sums over elementary tensors, whereas the 'restricted infima' in the following three lines run over all decompositions of* a *and* b *into sums of elementary tensors. Consequently the inner product* μ *on* $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$ *is continuous and can be extended to the completion* $\mathcal{H}_1 \otimes_\mu \mathcal{H}_2$ *of* $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$. *We denote this extension also by* μ. *Therefore* $\mathcal{H}_1 \otimes_\mu \mathcal{H}_2$ *is a super Hilbert space when endowed with the norm* μ.

When both $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *satisfy the strong definiteness condition, both* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are pre-Hilbert spaces with respect to the body of their inner products. Therefore also the body* μ_B *of* μ *is a complex-valued scalar product on* $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$ *and, by virtue of the Cauchy-Schwartz inequality,* μ_B *can be extended to a complex-valued scalar product* $\widetilde{\mu}_B$ *on* $\mathcal{H}_1 \otimes_\mu \mathcal{H}_2$. $\widetilde{\mu}_B$ *obviously coincides with the body of the extension of* μ *to* $\mathcal{H}_1 \otimes_\mu \mathcal{H}_2$. *Therefore we conclude that* $\mathcal{H}_1 \otimes_\mu \mathcal{H}_2$ *is a super Hilbert space satisfying the strong definiteness condition.*

In Section 6 we shall be interested in the case $\mathcal{H}_1 = \Lambda^m_\infty$ *and* $\mathcal{H}_2 = \star[\Lambda^m_\infty]$. *The norm* μ_m *arising from the mass norms on* Λ^m_∞ *and* $\star[\Lambda^m_\infty]$ *via Equation 3.7.20 is called the* ***mass norm*** *on* $\Lambda^m_\infty \otimes_{\mu_m} \star[\Lambda^m_\infty]$. *It follows from our discussion above that* $\Lambda^m_\infty \otimes_{\mu_m} \star[\Lambda^m_\infty]$ *is a super Hilbert space satisfying the strong definiteness condition. We shall see in Section 6 that in the functional*

Schrödinger representation of spinor quantum field theory the super Hilbert space $\Lambda_\infty^m \otimes_{\mu_m} \star[\Lambda_\infty^m]$ arises naturally as the quantum theoretical state space.

Proposition 3.7.4. *Let $\mathcal{H}$ be a super Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be an adjointable operator. Then T and T^* are bounded with respect to the operator norm*

$$\|T\| := \sup\{\|Tx\| \mid \|x\| \leq 1\}. \tag{3.7.21}$$

If Equation 3.7.12 holds, then $\|T\| = \|T^*\|$.

Proof. Let $x_\lambda, x, y \in \mathcal{H}$, such that $x_\lambda \to x$ and $Tx_\lambda \to y$. The inner product of a super Hilbert space is separately continuous in each variable. Thus

$$0 = \langle T^*e, x_\lambda\rangle - \langle T^*e, x_\lambda\rangle = \langle e, Tx_\lambda\rangle - \langle T^*e, x_\lambda\rangle \to \langle e, y\rangle - \langle T^*e, x\rangle = \langle e, y - Tx\rangle,$$

for all $e \in \mathcal{H}$. Putting $e = y - Tx$ implies $y = Tx$. The boundedness of T and T^* follows now from the closed graph theorem. As $\|Tx\|^2 = \|\langle T^*Tx, x\rangle\| \leq \|T^*Tx\|\|x\| \leq \|T^*\|\|T\|\|x\|^2$, we find $\|T\| \leq \|T^*\|$. But then also $\|T^*\| \leq \|T^{**}\| = \|T\|$. □

A similar argument proves

Proposition 3.7.5. *Let $\mathcal{H}$ be a super Hilbert space satisfying the strong definiteness condition and $T : \mathcal{H} \to \mathcal{H}$ be a weakly adjointable operator. Then T and $T^\dagger$ are bounded with respect to the operator norm in Equation 3.7.21 and with respect to the norm*

$$\|T\|_w := \sup\{|\langle Tx, Tx\rangle_B|^{1/2} \mid \|x\| \leq 1\} \tag{3.7.22}$$

and $\|T\|_w = \|T^\dagger\|_w$.

Proof. The boundedness of T and $T^\dagger$ with respect to the norm in Equation 3.7.21 follows as in the proof of Proposition 3.7.4. The boundedness with respect to $\|\cdot\|_w$ follows from $\|q\|_B \leq \|q\|$ for all $q \in \Lambda$. □

Proposition 3.7.6. *Let $\mathcal{H}$ be a super Hilbert space. When equipped with the operator norm (3.7.21) $\mathcal{B}(\mathcal{H})$ is an involutive Banach algebra with continuous involution.*

[4]

Proof. It is easy to see that (3.7.21) defines a norm on $\mathcal{B}(\mathcal{H})$. The operator norm is clearly submultiplicative. It remains to show that $\mathcal{B}(\mathcal{H})$ is norm complete. If $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of adjointable operators, then $(T_n x)_{n\in\mathbb{N}}$ and $(T_n^* x)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{H}$ for every $x \in \mathcal{H}$. We call the limits Tx and $\overline{T}x$ respectively. Since $\langle y, Tx\rangle = \lim\langle y, T_n x\rangle = \lim\langle T_n^* y, x\rangle = \langle \overline{T}y, x\rangle$, we see that T is adjointable and $T^* = \overline{T}$. This shows that $\mathcal{B}(\mathcal{H})$ is norm complete. From $\|T_n - T\| = \|T_n^* - T^*\|$ it is easy to see that the involution is continuous. □

3.7.5 Physical observables

Definition 3.7.9. Let E be a Hilbert Λ module and $T \in \mathcal{B}_w(E)$. Then we say that a Graßmann number λ is a **spectral value** for T when $T - \lambda I$ does not have a two-sided inverse in $\mathcal{B}_w(E)$. The set of spectral values for T is called the **spectrum** of T and is denoted by $\mathrm{sp}(T)$. The subset $\mathrm{sp}_{\mathbb{C}}(T) := \mathrm{sp}(T) \cap \mathbb{C}$ is called the **complex spectrum** of T.

It is well-known that a Graßmann number $q \in \Lambda_n, n$ finite, has an inverse if and only if its body q_B is nonvanishing [10]. Therefore the following proposition that the spectrum of a bounded module map T on a Hilbert Λ_n module, n finite, is fully determined by the complex spectrum of T is not surprising.

[4] As we saw in 1.1.3, the set of all adjoitable operators on Hilbert C^*-modules forms a C^*-algebra and we know that proving the C^*-condition needs the Cauchy-Schwartz inequality.

Proposition 3.7.7. *Let E be a Hilbert Λ_n module, n finite, and $T \in \mathcal{B}_w(E)$ be a Hilbert module map. Then $\lambda \in \mathrm{sp}(T)$ if and only if $\lambda_B \in \mathrm{sp}_{\mathbb{C}}(T)$.*

Proof. : Let $\lambda \notin \mathrm{sp}(T)$. Then $T - \lambda I$ has a two-sided inverse in $\mathcal{B}_w(E)$, denoted by T_λ^{-1}. Evidently T_λ^{-1} is a module map. Now let s be a Graßmann number with vanishing body. Then $T_{\lambda-s,L}^{-1} := \left(\sum_{n=0}^{\infty}(-T_\lambda^{-1}s)^n\right) T_\lambda^{-1}$ is a left inverse for $T-(\lambda-s)I$ and $T_{\lambda-s,R}^{-1} := T_\lambda^{-1}\left(\sum_{n=0}^{\infty}(-sT_\lambda^{-1})^n\right)$ is a right inverse for $T-(\lambda-s)I$. Both sums are actually finite. This follows from the bodylessness of s and from the fact that T_λ^{-1} is decomposable into an even and an odd part: $T_\lambda^{-1} = T_{\lambda,0}^{-1} + T_{\lambda,1}^{-1}$. Therefore the left and right inverse exist for all $s \in \Lambda_n$ with $s_B = 0$. As $T_{\lambda-s,L}^{-1}(T-(\lambda-s)I)\,T_{\lambda-s,R}^{-1} = T_{\lambda-s,R}^{-1} = T_{\lambda-s,L}^{-1}$ the left and right inverse coincide. This proves that $\lambda \notin \mathrm{sp}(T)$ implies $\lambda - s \notin \mathrm{sp}(T)$ for all $s \in \Lambda_n$ with $s_B = 0$. □

Example 3.7.6. *Consider Λ_n endowed with the inner product (3.7.16). Let $\xi_1, \cdots, \xi_n$ denote the set of generators of Λ_n. Consider the module map $\hat{\xi}_1 : \Lambda_n \to \Lambda_n, \hat{\xi}_1 q := \xi_1 q$. Obviously 0 is the only complex spectral value of $\hat{\xi}_1$ and, as $\hat{\xi}_1 - sI$ does not have an inverse for all bodyless $s \in \Lambda_n$, all Graßmann numbers with vanishing body are spectral values for $\hat{\xi}_1$. The element $\xi_1 \cdots \xi_n \in \Lambda_n$ is an "Eigenstate" for $\hat{\xi}_1$ for any bodyless spectral value: $\hat{\xi}_1 \xi_1 \cdots \xi_n = s\xi_1 \cdots \xi_n = 0$, for all $s \in \Lambda_n$ with $s_B = 0$.*

Definition 3.7.10. Let $\mathcal{H}$ be a super Hilbert space. A **physical observable** on $\mathcal{H}$ is a weakly self-adjoint operator $\mathcal{O} : \mathcal{H} \to \mathcal{H}$.

Proposition 3.7.8. *Let $\mathcal{H}$ be a super Hilbert space and let H be the Hilbert space from Proposition 3.7.2. Then there exists a * homomorphism φ from $\mathcal{B}_w(\mathcal{H}) \cap \mathcal{L}(\mathcal{H})$ (equipped with the norm $\|\cdot\|_w$) into the C^*-algebra $\mathcal{B}(H)$ of bounded operators on H.*

Proof. : Let $\mathcal{N} := \{x \in \mathcal{H} | \langle x, x \rangle_B = 0\}$ and let $T \in \mathcal{B}_w(\mathcal{H})$. For $n \in \mathcal{N}$ we have by virtue of Lemma 3.7.1: $|\langle Tn, Tn \rangle_B|^2 \leq \langle T^\dagger Tn, T^\dagger Tn \rangle_B \langle n, n \rangle_B = 0$. Thus $T(\mathcal{N}) \subset \mathcal{N}$. This shows that every $T \in \mathcal{B}_w(\mathcal{H})$ induces a bounded linear operator on $\mathcal{H}/\mathcal{N}$ which we denote by $\phi(T)$ via $\phi(T)(x+\mathcal{N}) := T(x)+\mathcal{N}$. The operator $\phi(T)$ can be uniquely extended to a bounded linear operator $\varphi(T)$ on H (compare, e.g., Theorem 1.5.7 in [18]). Obviously, the correspondence φ is linear, multiplicative and satisfies $\varphi(T^\dagger) = \varphi(T)^*$ and $\varphi(I) = I_{|H}$, i.e., φ is a * homomorphism. □

Proposition 3.7.9. *Let $\mathcal{H}$ be a super Hilbert space satisfying the strong definiteness condition. Then the * homomorphism φ from Proposition 3.7.8 is an isometric isomorphism from $\mathcal{B}_w(\mathcal{H})$ to the C^*-algebra $\mathcal{B}(H)$. Hence $\mathcal{B}_w(\mathcal{H})$ is a C^*-algebra with norm $\|T\|_w := \sup\{|\langle Tx, Tx \rangle_B|^{1/2} | \|x\| \leq 1\}$.*

Proof. : This follows, e.g., from Theorem 1.5.7 in [18]. □

Chapter 4

GENERALIZED SUPER HILBERT SPACE

4. GENERALIZED SUPER HILBERT SPACE

4.1 Introduction

As we saw in previous chapter, in defining of super Hilbert space, some authors restrict themselves to positivity of body of Graßmann number valued inner product. The induced spaces have many applications which more fully discussed in related papers. In fact we can say that, most of ideas in defining of super Hilbert space are revealed by necessity and so some mathematical structures are ignored. But we wish to have not any restriction. Also we wish our work be a generalization of ordinary mathematics.

The aim of present chapter is devoted to two purpose. First we want to work with the algebra of supernumbers, in general, instead of the Graßman algebra. The second purpose is extending the positivity of body of inner product to positivity of inner product in ordinary meaning.

4.2 Some definitions

We saw in chapter 2 that the finite and infinite dimensional σ-commutative G-graded algebras are Riesz spaces and Banach lattices with some order continuous norms. Also by using the method of functional calculus on them we obtained a new multiplication on $\mathcal{A}$ which makes it to be a Riesz algebra. Also we proved that the Riesz algebra of supernumbers is a commutative Banach algebra with respect to this new multiplication. Therefore we can use the supernumbers freely. They behave as ordinary numbers and have unit element, inverse and other characteristics of ordinary numbers.

During of present chapter we assume the letter $\mathcal{A}$ denotes the Riesz algebra and Banach algebra of supernumbers. For giving some examples we need the following definitions which can be fined in [24].

Let $I = \{1, \ldots, p, p+1, \ldots, p+q\}$ be a G-set such that $g(i)$are even for $i = 1, \ldots, p$ and odd for $i = p+1, \ldots, p+q$. Let $X = \mathcal{A}_I = \bigoplus_{i \in I} \mathcal{A}_{g(i)}$ be the direct sum of $\mathcal{A}_{g(i)}$. The X is a Banach space by the product topology induced from $\mathcal{A}$ and is called **superspace** over $\mathcal{A}$.

For a point $x = (x^{(i)} | i \in I)$ in a superspace X, $b(x) = (b(x^{(i)}) | i \in I)$ and $s(x) = (s(x^{(i)}) | i \in I)$ are called the body and the soul of x, respectively. Let U be a (connected open) domain of X. Here $b(U) = \{b(x) | x \in U\}$ is called the body of U. Then $b(U)$ is contained in the even part

$$U_0 = \{(x^{(1)}, \ldots, x^{(p)}, 0, \ldots, 0) | \quad (x^{(1)}, \ldots, x^{(p)}, x^{(p+1)}, \ldots, x^{(p+q)}) \in U\}$$

of U.

Let $(x^{(1)}, \ldots, x^{(p)}, 0, \ldots, 0)$ be in $b(X)$. Then $x = \tilde{x}^{(i)} u_{g(i)}$ for some $\tilde{x}^{(i)} \in \mathbb{F}$. The mapping $\sim$ which sends x to the point $\tilde{x} = (\tilde{x}^{(1)}, \ldots, \tilde{x}^{(p)})$ is a homeomorphism of $b(X)$ onto the (real or complex) p-dimensional space $\mathbb{F}^p$. For a domain V of $b(X)$, $\tilde{V} = \{\tilde{x} | x \in V\}$ is a domain of $\mathbb{F}^p$. We sometimes write the odd coordinates $x^{(p+j)}$ as $\xi^{(j)}$ and express a point of X as $(x, \xi) = (x^{(1)}, \ldots, x^{(p)}, \xi^{(1)}, \ldots, \xi^{(q)})$ in order to distinguish between the even and odd coordinates.

Let U be a domain of X. Here $\mathcal{A}^U$ denotes the set of functions(superfields) on U which take their values in $\mathcal{A}$. A function $f \in \mathcal{A}^U$ is said to be *homogeneous* of grade $\alpha \in G$, if $f(x) \in \mathcal{A}_\alpha$, for all $x \in U$. Thus $\mathcal{A}^U$ is naturally a σ-commutative G-graded algebra over $\mathbb{F}$.

For a domain V of $b(X)$, $\mathcal{A}^U$ also denotes the set of $\mathcal{A}$-valued functions defined on V. A function $f \in \mathcal{A}^U$ is written uniquely as

$$f(x) = \sum_{\alpha,M} f_{\alpha,M}(x) u_\alpha \otimes V_M \qquad (x \in V) \tag{4.2.1}$$

where $\alpha \in G_0, M \subseteq L$, and $f_{\alpha,M}(x) \in \mathbb{F}$. The functions $\tilde{f}_{\alpha,M}$, which are defined by $\tilde{f}_{\alpha,M} = f_{\alpha,M}(x)$ for $x \in V$, are $\mathbb{F}$-valued functions on the domain $\tilde{V}$.

4.3 Generalized super Hilbert space

In present section we consider the structure of Riesz algebra of supernumbers with defined Riesz norms on it. It is obvious that the $\mathcal{A}$ is considered as an algebra over real or complex field.

Definition 4.3.1. A **generalized Super pre-Hilbert space(GSpHS)** is a GSM $E = \bigoplus_{\alpha \in G} E_\alpha$ equipped with an $\mathcal{A}$-valued mapping $\langle , \rangle : E \times E \longrightarrow \mathcal{A}$ which is $\mathbb{C}$-linear positive definite and Hermitian form. In fact it has the following conditions:

(i) $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$ and $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$for all $x, x_1, x_2, y, y_1, y_2 \in E$;

(ii) $\langle x, cy \rangle = \langle x, y \rangle c$ and $\langle cx, y \rangle = \overline{c} \langle x, y \rangle$ for any $x, y \in E$ and $c \in \mathbb{C}$;

(iii) $\overline{\langle x, y \rangle} = \langle y, x \rangle$ for any $x, y \in E$;

(iv) $\langle x, x \rangle \geq 0$ in $\mathcal{A}$ for any $x \in E$ and if $\langle x, x \rangle = 0$ then $x = 0$.

An element x of GSpHS $E = \bigoplus_{\alpha \in G} E_\alpha$ is called **homogeneous of grade** $\alpha \in G$ if $x \in E_\alpha$. Also any $x \in E$ can be written uniquely as $x = \sum_{\alpha \in G} x_\alpha$ where $x_\alpha \in E_\alpha$. Now we prove the following famous lemma.

Lemma 4.3.1. Cauchy-Schwartz Inequality
If E is a GSpHS and $x, y \in E$, then

$$\|\langle x, y \rangle\|^2 \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\|.$$

Proof. If $\langle x, y\rangle = 0$, this is trivial. If $\langle x, y\rangle \neq 0$ the inequality follows from the positivity of $\mathcal{A}_{\mathbb{C}}$-valued inner product. We have the following inequality for any $a \in \mathcal{A}$.

$$0 \leq \langle x + ay, x + ay\rangle = \langle x, x\rangle + a\langle x, y\rangle + \bar{a}\langle y, x\rangle + a\bar{a}\langle y, y\rangle$$

Now by letting $a = \frac{-\langle x,y\rangle}{\|\langle x,y\rangle\|}$ and taking norm we will have

$$0 \leq \langle x, x\rangle - \frac{\langle x, y\rangle\langle x, y\rangle}{\|\langle y, y\rangle\|} - \frac{\overline{\langle x, y\rangle}\langle x, y\rangle}{\|\langle y, y\rangle\|} + \frac{\overline{\langle x, y\rangle}\langle x, y\rangle\langle y, y\rangle}{\|\langle y, y\rangle\|}$$

and so

$$\|\langle x, y\rangle\|^2 \leq \|\langle x, x\rangle\|\|\langle y, y\rangle\|.$$

□

We can define some real valued norms on E by using the different norms of $\mathcal{A}$. If $\| \ \|_{\mathcal{A}}$ denotes the norm of $\mathcal{A}$, then for any $x \in E$,

$$\|x\|_E^2 = \|\langle x, x\rangle\|_{\mathcal{A}} \tag{4.3.1}$$

defines a norm on E. Unlike equation 3.7.12, this is naturally like as the theory of Hilbert C^*-modules and we can see easily the subadditivity property by using the Cauchy-Schwartz inequality. It is important to note that Cauchy-Schwartz inequality yields automatically the continuity of inner product with respect to its components. As some examples for equation 4.3.1, for any $1 \leq k \leq \infty$ and $x \in E$ define

$$\|x\|_k = \|\langle x, x\rangle\|_k^{\frac{1}{2}} \tag{4.3.2}$$

If m denote the mass norm on $\mathcal{A}$ then for any $x \in E$ we can have

$$m(x) = m(\langle x, x\rangle)^{\frac{1}{2}} \tag{4.3.3}$$

We know from 2.2.14 that for strong order unit $\underline{1} \in \mathcal{A}$ the equations

$$P_{\underline{1}}(a) = \inf\{r \in \mathbb{R} \ : \quad -r\underline{1} \leq a \leq r\underline{1}\} \tag{4.3.4}$$

and

$$P^{(\underline{1})}_{\omega}(a) = \sum_n \omega_n P_{\underline{1}_n}(a_n) \tag{4.3.5}$$

define an M-norm on $\mathcal{A}$ in the finite and infinite dimensional cases respectively. Therefore if use them for $\mathcal{A}$, we will obtain new norm on E.

Definition 4.3.2. Let E be a GSpHS and $\|\quad\|$ a norm on it. Then E is said to be **generalized super Hilbert space(GSHS)** if E is complete with respect to its norm.

As a trivial example, the finite dimensional σ-commutative G-graded algebra $\mathcal{A}$ over $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$ is a GSHS with inner product defined as

$$\langle a, b\rangle = \sum_{\alpha,M} \overline{a}_{\alpha,M} b_{\alpha,M} u_\alpha \otimes v_M \tag{4.3.6}$$

Example 4.3.1. *Let X be a superspace over $\mathcal{A}$ and U be a domain of $b(X)$, the body of X. Any $f \in \mathcal{A}^U$ has the form of 4.2.1. Define an inner product on $\mathcal{A}^U$ as follows*

$$\langle f, g\rangle = \sum_{\alpha,M} \overline{f}_{\alpha,M}(x) g_{\alpha,M}(x) u_\alpha \otimes v_M \tag{4.3.7}$$

where $f, g \in \mathcal{A}^U$ as we saw above. It is easily seen that $\mathcal{A}^U$ is GSHS.

Example 4.3.2. *Let W be a subset of $U \subset b(X)$ such that $\tilde{W}$ is a measurable subset of $\mathbb{R}^p$. Let $f \in \mathcal{A}^U$ be integrable on W, that is, all $\tilde{f}_{\alpha,M}$ are integrable on $\tilde{W}$. According to definition 4.5 of [24] we have*

$$\begin{aligned}\int_W f(x)dx &= \int_W f(x^{(1)}, \ldots, x^{(p)})dx^{(1)} \ldots dx^{(p)} \\ &= \sum_{\alpha,M} [\int_{\tilde{W}} \tilde{f}_{\alpha,M}(\tilde{x}^{(1)} \ldots \tilde{x}^{(p)})d\tilde{x}^{(1)} \ldots d\tilde{x}^{(p)}](u_\alpha \otimes v_M).\end{aligned} \tag{4.3.8}$$

Denote by $\mathcal{A}^U_W$ the set of all integrable superfields $f \in \mathcal{A}^U$ over W. This set is subspace of $\mathcal{A}^U$ and also is GSM. We equip it with following inner product *which make it to be GSHS. For this let $f, g \in \mathcal{A}_W^U$ and define*

$$\langle f, g\rangle = \sum_{\alpha,M}[\int_{\tilde{W}} \overline{\tilde{f}}_{\alpha,M}(\tilde{x}^{(1)}\dots\tilde{x}^{(p)})\tilde{g}_{\alpha,M}(\tilde{x}^{(1)}\dots\tilde{x}^{(p)})d\tilde{x}^{(1)}\dots d\tilde{x}^{(p)}](u_\alpha\otimes v_M). \tag{4.3.9}$$

Example 4.3.3. *Here we consider graded tensor product of two GSHS.*

Let H_1 and H_2 be two GSHS equipped with inner products $\langle,\rangle_1$ and $\langle,\rangle_2$ respectively. H_1 and H_2 are G-graded $\mathcal{A}$-modules. Therefore $H_1 = \bigoplus_{\beta\in G} H_\beta^1$ and $H_2 = \bigoplus_{\gamma\in G} H_\gamma^2$ which H_β^1 and H_γ^2 are subspaces of H_1 and H_2 respectively for any $\beta \in G$ and $\gamma \in G$. We can restrict inner products $\langle,\rangle_1$ and $\langle,\rangle_2$ to H_β^1 and H_γ^2 respectively which make them to be ordinary Hilbert spaces. Therefore the algebraic tensor product $H_\beta^1 \otimes H_\gamma^2$ can be define in a natural way for any $\beta, \gamma \in G$. The graded tensor product of H_1 and H_2 is defined by

$$H_1 \otimes_{gr} H_2 = \bigoplus_{\alpha\in G}(\bigoplus_{\beta+\gamma=\alpha}(H_\beta^1\otimes H_\gamma^2))$$

which is also GSM over $\mathcal{A}$. Now let $h_1, k_1 \in H_1$ and $h_2, k_2 \in H_2$. We have

$$h_1 = \sum_{\beta\in G} h_\beta^1\ , \quad k_1 = \sum_{\beta\in G} k_\beta^1\ , \quad h_2 = \sum_{\gamma\in G} h_\gamma^2\ , \quad k_2 = \sum_{\gamma\in G} k_\gamma^2$$

where $h_\beta^1, k_\beta^1 \in H_\beta^1$ for any $\beta \in G$ and $h_\gamma^2, k_\gamma^2 \in H_\gamma^2$ for any $\gamma \in G$. The inner product

$$\langle h_\beta^1 \otimes h_\gamma^2, k_\beta^1 \otimes k_\gamma^2\rangle = \langle h_\beta^1, k_\beta^1\rangle\langle h_\gamma^2, k_\gamma^2\rangle$$

makes $H_\beta^1 \otimes H_\gamma^2$ to be Hilbert space for any $\beta, \gamma \in G$. Now define inner product on $H_1 \otimes_{gr} H_2$ as follows:

$$\begin{aligned}\langle h_1\otimes h_2, k_1\otimes k_2\rangle &= \sum_{\alpha\in G}\sum_{\beta+\gamma=\alpha}\langle h_\beta^1\otimes h_\gamma^2, k_\beta^1\otimes k_\gamma^2\rangle \\ &= \sum_{\alpha\in G}\sum_{\beta+\gamma=\alpha}\langle h_\beta^1, k_\beta^1\rangle\langle h_\gamma^2, k_\gamma^2\rangle.\end{aligned}$$

By this inner product the graded tensor product of two GSHS will be GSHS.

4.4 Linear operators

Definition 4.4.1. Let E and F be two GSHS. An $\mathcal{A}$-linear map $T : E \longrightarrow F$ is called an *operator* on E into F. $\mathcal{A}$-linear means that

(1) $T(x+y) = Tx + Ty$ for any $x, y \in E$;

(2) $T(ax) = aTx$ for $x \in E$ and $a \in \mathcal{A}$;

(3) $T(xa) = \sigma(\alpha, \beta)T(x)a$ for any $x \in E_\alpha$ and $a \in \mathcal{A}_\beta$ and $\alpha, \beta \in G$.

Moreover T is called **grade preserving** if $TE_\alpha \subset F_\alpha$ for any $\alpha \in G$. For brevity we shall usually say operator instead of $\mathcal{A}$-linear operator and otherwise we shall say explicitly. Evidently the set $L(E, F)$ of all operators from E into F is a vector space and the set of all grade preserving operators of E into F is a subspace of it which we denote by $L_{gp}(E, F)$.

Now $T : E \longrightarrow F$ is said to be **homogeneous of grade** $\alpha \in G$ if $T(E_\beta) \subset F_{\alpha+\beta}$ for all $\beta \in G$. Let $L_\alpha(E, F)$ denote the subspace of $L(E, F)$ consisting of all homogeneous operators of grade α. By defining $L_{gr}(E, F) = \bigoplus_{\alpha \in G} L_\alpha(E, F)$, we obtain a G-graded vector space(GSV). Note that $L_{gr}(E, F)$ is equal to $L(E, F)$ if $E_\alpha = \{0\}$ and $F_\alpha = \{0\}$ for all but a finite number of degrees [43]. Since at first we assume that G is finite group, then we will have the equality. Therefore every operator $T \in L(E, F)$ can be written uniquely as $T = \sum_{\alpha \in G} T_\alpha$ which $T_\alpha \in L_\alpha(E, F)$ for any α. If $E = F$ then the G-graded vector space $L(E, F)$, will be denoted by $L(E)$, equipped in addition with the usual multiplication(i. e., composition) is generalized superalgebra(GSA).

Definition 4.4.2. An operator $T : E \longrightarrow F$ is called **unitary** if

$$\langle Tx, Ty \rangle = \langle x, y \rangle \tag{4.4.1}$$

for all $x, y \in E$. Also an operator $S : E \longrightarrow F$ is called **weakly unitary** if

$$b(\langle Tx, Ty \rangle) = b(\langle x, y \rangle) \tag{4.4.2}$$

for all $x, y \in E$. An operator $T : E \longrightarrow F$ is called **graded unitary** if

$$\langle Tx, Ty\rangle = \sigma(\alpha, \beta)\langle x, y\rangle \tag{4.4.3}$$

for all $x \in E_\alpha, y \in E_\beta$ and $\alpha, \beta \in G$. It is called **weakly graded unitary** if

$$b(\langle Tx, Ty\rangle) = \sigma(\alpha, \beta)b(\langle x, y\rangle) \tag{4.4.4}$$

for all $x \in E_\alpha, y \in E_\beta$ and $\alpha, \beta \in G$.

Definition 4.4.3. Let E be a GSHS. An operator $T \in L(E)$ is said to be **adjointable** if there exists an operator $T^* : E \longrightarrow E$ satisfying

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \tag{4.4.5}$$

for all $x, y \in E$. Such an operator T^* is called an **adjoint** of T and T is called **self adjoint** if $T = T^*$. We have the following properties:

(i) $(aT_1 + bT_2)^* = \overline{a}T_1^* + \overline{b}T_2^*$;

(ii) $\langle T_1x, T_2y\rangle^* = \langle T_2^*x, T_1^*y\rangle$;

(iii) $(T^*)^* = T$,

for any T, T_1 and T_2 in $L(E)$. The set of all adjointable operators will be denoted by $\mathcal{B}(E)$.

An operator $S : E \longrightarrow E$ is said to be **weakly adjointable** if there is an operator $S_w^* : E \longrightarrow E$ satisfying

$$b(\langle x, Sy\rangle) = b(\langle S_w^*x, y\rangle) \tag{4.4.6}$$

for all $x, y \in E$. Such operator is called **weak adjoint** of S. If $S_w^* = S$ then it called **weakly self adjoint**. The set of all weakly adjointable operators is denoted by $\mathcal{B}^w(E)$.

Definition 4.4.4. An operator T on E is called **generalized super adjointable(GSa)** if there is an operator $\tilde{T} : E \longrightarrow E$ satisfying

$$\langle x, Ty\rangle = \sigma(\alpha, \beta)\langle \tilde{T}x, y\rangle \tag{4.4.7}$$

for all $x \in E_\alpha, y \in E_\beta$ and $\alpha, \beta \in G$.

A homogeneous operator T of grade α is called **self GSa** if $\tilde{T} = \sigma(\alpha, \alpha)T$. We have the following properties:

(i) $(aT_1 \tilde{+} bT_2) = \overline{a}\tilde{T}_1 + \overline{b}\tilde{T}_2$ for any $T_1, T_2 \in \mathrm{L}(E)$ and $a, b \in \mathcal{A}$;

(ii) $\langle T_1 x \tilde{,} T_2 y\rangle = \sigma(\alpha, \beta)\langle \tilde{T}_2 x, \tilde{T}_1 y\rangle$ for all homogeneous operators T_1, T_2 of grade α, β respectively;

(iii) $\tilde{\tilde{T}} = \sigma(\alpha, \alpha)T$ for homogeneous operator T of grade $\alpha \in G$.

The set of all generalized super-adjointable operators on E will be denoted by $\mathcal{B}_{GSa}(E)$. Also an operator T is **weakly GSa** if there exists an operator $\tilde{T}_w : E \longrightarrow E$ satisfying

$$b(\langle x, Ty\rangle) = \sigma(\alpha, \beta)b(\langle \tilde{T}_w x, y\rangle) \tag{4.4.8}$$

for all $x \in E_\alpha, y \in E_\beta$ and $\alpha, \beta \in G$. The set of all weakly GSa operators is denoted by $\mathcal{B}^w_{GSa}(E)$. A homogeneous operator T of grade α is called **weakly self-GSa** if $\tilde{T}_w = \sigma(\alpha, \alpha)T$. Evidently we have the following inclusions:

$$\mathcal{B}(E) \subset \mathcal{B}^w(E) \qquad \& \qquad \mathcal{B}_{GSa}(E) \subset \mathcal{B}^w_{GSa}(E)$$

Definition 4.4.5. An adjointable or GSa operator T is called **positive operator** if $\langle Tx, x\rangle \geq 0$ for all $x \in E$. It is called **weakly positive** if $b(\langle Tx, x\rangle) \geq 0$ for all $x \in E$, and **strictly positive** if $\langle Tx, x\rangle\rangle 0$ for some $x \in E$.

Proposition 4.4.1. *Let H be a GSpHS and $T : H \longrightarrow H$ be an adjointable operator. Then T and T^* are bounded with respect to the operator norm*

$$||T|| = \sup\{||Tx|| \quad | \quad ||x|| \leq 1\} \tag{4.4.9}$$

and $\mathcal{B}(H)$ is a C^*-algebra.

Proof. Let $x_\tau, x, y \in H$, such that $x_\tau \longrightarrow x$ and $Tx_\tau \longrightarrow y$. The inner product of a GSpHS is separately continuous in each variables. Thus

$$0 = \langle T^*e, x_\tau\rangle - \langle T^*e, x_\tau\rangle = \langle e, Tx_\tau\rangle - \langle T^*e, x_\tau\rangle.$$

Therefore

$$\langle h, y\rangle - \langle T^*h, x\rangle = \langle h, y - Tx\rangle \qquad \text{for all } \; h \in H.$$

Let $h = y - Tx$, then $\langle h, y - x\rangle = 0$ implies that $\langle y - Tx, y - x\rangle = 0$. So $y = Tx$. The boundedness of T and T^* follows from the closed graph theorem. Submultiplicativity of operator norm is easy. According to Cauchy-schwarz inequality we have $\|T\| = \|T^*\|$. Since

$$\|Tx\|^2 = \|\langle Tx, Tx\rangle\| = \|\langle T^*Tx, x\rangle\| \leq \|\langle T^*Tx, T^*Tx\rangle\|\|\langle x, x\rangle\| \leq \tag{4.4.10}$$

$$\|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2 \leq \|T^*\|\|T\|\|x\|^2.$$

Then $\|T\| \leq \|T^*\|$. Also we can have $\|T^*\| \leq \|T^{**}\| = \|T\|$.

To establish the C^*-equation, use submultiplicativity of operator norm to get $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. On the other hand 4.4.10 implies that also $\|T\|^2 \leq \|T^*T\|$. It remains to show that $\mathcal{B}(H)$ is norm complete. If $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of adjointable operators, then $(T_nx)_{n\in\mathbb{N}}$ and $(T_n^*x)_{n\in\mathbb{N}}$ are Cauchy sequences in H for $x \in H$. We call the limits Tx and $\overline{T}x$ respectively. Since $\langle y, Tx\rangle = \lim\langle y, T_nx\rangle = \lim\langle T_n^*y, x\rangle = \langle \overline{T}y, x\rangle$, we see that T is adjointable and $T^* = \overline{T}$. This shows that $\mathcal{B}(H)$ is norm complete. From $\|T_n - t\| = \|T_n^* - T^*\|$ it is easy to see that the involution is continuous. □

Proposition 4.4.2. *If E be a GSpHS and T be an adjointable operator, then T is self-adjoint if and only if $\langle Tx, x\rangle \in Re(\mathcal{A}_\mathbb{C})$ for all $x \in E$.*

Proof. If T is self-adjoint then $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ hence $\langle Tx, x\rangle \in$ $\mathrm{Re}(\mathcal{A}_\mathbb{C})$.

For converse, assume that $\langle Tx, x\rangle \in \mathrm{Re}(\mathcal{A}_{\mathbb{C}})$ for any $x \in E$. If $c \in \mathbb{C}$ and $x, y \in E$, then

$$\langle T(x+cy), x+cy\rangle = \langle Tx, x\rangle + c\langle Tx, y\rangle + \overline{c}\langle Ty, x\rangle + c\overline{c}\langle Ty, y\rangle \in \mathrm{Re}(\mathcal{A}_{\mathbb{C}}).$$

So this expression equals with its conjugation in $\mathcal{A}_{\mathbb{C}}$. Using the facts that $\langle Tx, x\rangle$ and $\langle Ty, y\rangle$ are in $\mathcal{A}_{\mathbb{C}}$ yield

$$\begin{aligned} c\langle Tx, y\rangle + \overline{c}\langle Ty, x\rangle &= \overline{c}\overline{\langle Tx, y\rangle} + c\overline{\langle Ty, x\rangle} \\ &= \overline{c}\langle y, Tx\rangle + c\langle x, Ty\rangle \\ &= \overline{c}\langle T^*y, x\rangle + c\langle T^*x, y\rangle \end{aligned}$$

By first taking $c = 1$ and $c = i$, we obtain two equations:

$$\langle Tx, y\rangle + \langle Ty, x\rangle = \langle T^*y, x\rangle + \langle T^*x, y\rangle$$

$$i\langle Tx, y\rangle - i\langle Ty, x\rangle = -i\langle T^*y, x\rangle + i\langle T^*x, y\rangle$$

A little arithmetic implies $\langle Tx, y\rangle = \langle T^*x, y\rangle$, and so $T = T^*$. $\square$

ABSTRACT AND TITLE PAGE IN PERSIAN

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