DIFFERENTIAL GEOMETRY AND LIE GROUPS APPLICATION FOR "INITIATIVKOLLEG DER UNIVERSITÄT WIEN"

ROBERT BEIG
DIETRICH BURDE
ANDREAS ČAP
STEFAN HALLER
MICHAEL KUNZINGER
PETER W. MICHOR
ROLAND STEINBAUER

Abstract. The disciplines covered by the proposed Initiativkolleg include differential geometry, groups of symmetries, (non-linear) PDEs, singularities, and mathematical relativity. The local expertise together with the obvious synergies between these fields has the potential of creating a group of students who have close interaction both among themselves and with the existing research groups. The organizers' scientific connections allow international co-supervision of the PhD theses. The Initiativkolleg would also provide students with the opportunity to gain experience in scientific presentation, to visit international conferences and establish contacts with distinguished scientists.

1. Description of mathematical subfields and collaboration between them

We describe several mathematical subfields, list the scientists working in them or interested, beginning with the main investigators of these subfields. Citations solely made of numbers ([1], [2], etc.) refer to the references at the end of this proposal. If they are preceeded by a letter ([Bu1], [C1], etc.) they refer to the literature in the corrsponding individual curriculum vitae (CV) attached to this document.

Mathematical Relativity. (Robert Beig, Michael Kunzinger, Roland Steinbauer, Andreas Čap, Dietrich Burde, Peter Michor) Robert Beig is a leading expert in mathematical relativity, in particular for applying modern PDE theory to solve Einstein's equations. See the list of recent publications in his CV and his research statement for a succinct description of the field. Michael Kunzinger and Roland Steinbauer are interested in relativity and are actively working on applying their well developed multiplicative theory of generalized functions to solutions of Einstein's equations near singularities (like black holes). The interest of Andreas Čap stems from the importance of conformal geometry, which is one of the main examples of a parabolic geometry, in relativity. Dietrich Burde's interest is deformation and degenerations of groups and Lie algebras which happen during taking the non-relativistic limit of certain relativistic equations. Peter Michor's interest is directed towards the orbit space of Lorentzian metrics under the action of the diffeomorphism group, where Einstein's equations really live.

Nonlinear distributional geometry. (Michael Kunzinger, Roland Steinbauer, interested is Peter Michor) Linear distributional geometry, i.e., the extension of methods from differential geometry to the generalized functions setting was initiated by L. Schwartz and G. de Rham. Soon, however, the inherent limitations of distribution theory with respect to nonlinear operations, began to be felt increasingly (cf. [10] in case of general relativity). Based on J.F. Colombeau's construction of algebras of generalized functions canonically extending the space of distributions ([8]) on the one hand and A. Kriegl and P. Michor's calculus in infinite dimensional spaces ([19]) on the other, a suitable 'nonlinear distributional geometry' was developed by various authors over the past decade (cf. [15, 16, 17]). The distinguishing features of this construction are diffeomorphism invariance, canonical embedding of spaces of distributions and maximal compatibility with respect to classical analysis (in the smooth setting). The current main directions of research in nonlinear distributional geometry (within the focus of this proposal) are nonsmooth differential geometry, nonsmooth geometric analysis of partial differential operators, Lie group analysis of partial differential equations involving singularities, and nonsmooth general relativity.

The main guideline in the development of nonlinear distributional geometry is to keep a close connection to relevant fields of applications, in particular to Lie group analysis of PDEs and general relativity. Due to recent theoretical developments new and exciting applications have come into reach and research in that direction is currently pursued within the START project of the DIANA group (see below). In particular, recent progress on the solvability of the wave equation on singular static space times has been made in the course of the PhD of E. Mayerhofer (supervised by M. Kunzinger and R. Steinbauer; cf. [14]). For a detailed description of further active research topics see the statements of M. Kunzinger and R. Steinbauer attached to this proposal.

Geometric Theory of PDEs. (Peter Michor, Stefan Haller, Robert Beig, Andreas Čap, Michael Kunzinger, Roland Steinbauer) Here nearly all subfield have a common intersection.

The search for geodesics on shape manifolds leads to very interesting nonlinear PDEs and their conserved momenta. An elementary introduction to the methods used is [21].

Stefan Haller's study of (complex valued) analytic torsion and its relation to Morse, Morse–Bott, and Morse–Novikov theory theory belongs here. For more details on the planed research and possible thesis problems we refer to Haller's research statement below. The computation of the Ray–Singer torsion [26] in the presence of a compact group of symmetries with the help of an invariant Morse–Bott–Smale function would require the computation of the Ray–Singer torsion for certain symmetric spaces. Students involved in this problem would certainly benefit from Burde, Čap and Michor's expertise in the field of representation theory. There exists an intensive collaboration with

Dan Burghelea from the Ohio State University. This could also provide the possibility for the students to stay abroad.

The research in parabolic geometries led to a class of geometrically overdetermined systems of PDEs and methods to solve them involving connections and curvatures.

Structures on Manifolds, in particular parabolic geometry. (Andreas Čap, Peter Michor, Stefan Haller) The main contribution of the theory of parabolic geometries to the program of the Initiativkolleg (IK) is that it provides a systematic way to apply techniques of Lie theory, representation theory, and cohomology of Lie algebras to problems in differential geometry. The concept of Cartan connections, which is a basic ingredient of the theory builds a bridge between the geometry of homogeneous spaces (which can be described in terms of Lie theory) and differential geometry, see [27].

Among the main successes of parabolic geometries during the last years was a very general construction of differential operators which are intrinsic to such structures, see [C2] and [6]. For each of the geometries, this leads to a large number of geometric overdetermined systems of PDEs, together with an approach to attack questions about these systems, see [C7].

Peter Michor is interested in Riemannian and symplectic geometry, the latter is also of interest to Stefan Haller.

Cohomology, extensions, and degenerations of Lie groups and Lie algebras. (Dietrich Burde, Peter Michor, Andreas Čap) This quite specialized fields has connections to general relativity where the non-relativistic limit leads to degenerations of symmetry groups and Lie algebras, see Dietrich Burde's contribution below. The connection to Andreas Čap's interests is provided by the theory of Bernstein–Gelfand–Gelfand sequences. These are a central tool in the theory of parabolic geometries and involve cohomology of nilpotent Lie algebras as a crucial ingredient. Peter Michor has contributed to the extension theory of Lie super algebras.

Group actions, orbit spaces, and invariant theory. (Peter Michor, Andreas Čap) Group actions play an important role in all fields described here, in particular as symmetry groups of equations. Orbit spaces and invariant theory are investigated as a primary research object by Peter Michor: mainly the question of lifting smooth curves from the orbit space to the manifold was looked into in a series of papers, see [1, 18] and [M65, M91]. The question of choosing roots of polynomials smoothly, or of choosing eigenvalues of a smooth curve of (even unbounded) operators belongs here. A successful thesis was finished in this field 2 years ago.

For any parabolic geometry, there is a homogeneous model, which is endowed with a transitive action of a semisimple Lie group. Structures which are locally isomorphic to this homogeneous model are an interesting class which can be studied using Lie theory. Finding invariants for parabolic geometries is a very hard problem which has natural relations to classical invariant theory.

Geometry of the infinite dimensional shape manifolds. (Peter Michor) Here the main question is to find reasonable metrics on spaces of plane curves modulo reparametrizations, or spaces of surfaces in space modulo reparametrizations. This is important for automated pattern recognition. Here some very surprising results were proved recently: The L^2-metric induces 0 geodesic distance on each kind of shape space, and even on each diffeomorphism group, see [29] and [M98, M102, M107]. Here there are very tight connections to applications, via the 'NSF-Focused Research Group: The geometry, mechanics, and statistics of the infinite dimensional shape manifolds (based in Johns Hopkins and Brown University)' where Peter Michor is an associated investigator. This part has connections to the fields of orbit spaces, structures on manifolds, geometric theory of PDEs, and others.

2. Existing and planned collaborations in and outside of the project

For several decades Andreas Čap, Stefan Haller, Peter Michor, and others run two joint seminars: One is called Lie theory and is devoted to scan all available literature on Lie groups and Lie algebras together with students. The other one is devoted to jointly supervising diploma and doctoral students of all participating faculty and to discuss common research. This has establish a certain tradition of graduate studies in the research group differential geometry. In the project we want to extend this tradition in a flexible way to the whole Initiativkolleg (IK).

The newly founded *Eduard Čech Center for Algebra and Geometry* in Brno and Prague, `http://ecc.sci.muni.cz`, is devoted to graduate and post-graduate studies in algebra and geometry. There are strong research groups working on parabolic geometries at the Masaryk University in Brno and the Charles University in Prague associated to this center. There is a regular joint seminar between Andreas Čap and these two research groups which takes place in Brno three times per semester. Peter Michor is member of the scientific board of this center.

Peter Michor is also member (2004–2008) in the 'comite scientifique du *College Doctoral Europeen* des Universites de Strasbourg', see `http://edc.u-strasbg.fr/`. This opens the possibility of exchanging doctoral students for certain periods with this College.

Robert Beig is member of 'Fachbeirat des *MPI für Gravitationsphysik*' in Potsdam, and has close scientific ties with members of the mathematical relativity group there, in particular Bernd G. Schmidt. He also collaborates with Piotr T. Chruściel (Departement de Mathématique, University of Tour, France.)

Peter Michor is associated investigator of 'NSF-Focused Research Group: The geometry, mechanics, and statistics of the infinite dimensional shape manifolds' based in Johns Hopkins and Brown University.

Andreas Čap and Peter Michor have co-organized several scientific programs at the *Erwin Schrödinger Institute* of Mathematical Physics (ESI) in Vienna. Former Doctoral students have profited considerably from these programs.

For decades Andreas Čap and Peter Michor have participated and have organized yearly trips of groups of students to the traditional *Winter School on Geometry and Physics* which takes place for a full week each mid-January in a small Village (Srni) in the Bohemian woods in the Czech republic. Many former doctoral students have profited a lot from the winter school, and have used it to present their results for the time to an international audience and to gain experience in international scientific conferences.

Michael Kunzinger and Roland Steinbauer are members of the Vienna-branch of the research group DIANA (DIfferential Algebras and Nonlinear Analysis, `www.univie.ac.at/~diana`) formed by mathematicians from Innsbruck, Torino, Southampton and Vienna with their research interests centered around nonlinear theories of generalized functions and their applications. These include, in particular, M. Grosser (Univ. Vienna), working in the structural theory of nonlinear generalized functions, Günther Hörmann (Univ. Vienna), an expert in nonsmooth geometric analysis of PDEs, Michael Oberguggenberger (Univ. Innsbruck), one of the main architects of the theory, as well as James Vickers, a relativist working at the University of Southampton. The START-group also includes two Post-docs (James Grant, relativist and differential geometer, Shantanu Dave, PhD at Penn State with V. Nistor), as well as 4 current PhD students.

Further international cooperations relevant to the proposal include scientists working in the structural theory of nonlinear generalized functions, in particular, the groups of S. Pilipović (University of Novi Sad, Serbia and Montenegro) and J.A. Marti (Université des Antilles et de la Guyane, Guadeloupe, France) as well as specialist in the respective fields of applications such as J. Podolsky (relativist at the Charles University, Prague), J.M. Heinzle (relativist; AEI Golm, Germany, to return to the Faculty of Physics at Vienna University at the beginning of next year) and M. de Hoop (mathematical geophysics; MIT and Purdue University, USA). The START-project will allow frequent visits of these scientists which will certainly benefit the proposed PhD program.

3. Training aspects

The proposed Initiativkolleg (IK) aims at employing nine students for a period of three years (14000 Euro each per year.) One of the guiding principles of the program will be co-supervision of the PhD students. On the one hand, this means joint PhD projects of the research groups sustaining the program. Additionally, as can be seen from the individual research statements we will provide PhD topics co-supervised by distinguished outside scientists. This conforms with the second major goal of the IK, namely to involve the students as early as possible into up-to-date research on an international level and make contact with the scientific community.

Apart from a variety of advanced lecture courses given by the organizers and geared to the needs of the IK students, the main foundation of the proposed training program are regular joint seminars providing a platform for all members (students as well as supervisors) of the IK. These will allow the PhD candidates to profit in an optimal way from the local expertise.

The IK will provide the opportunity of inviting top class international scientists, both as guest lecturers and as co-supervisors. The budget for financing guests is estimated rather low (7000 Euro per year) in view of the fact that several of the organizers are principal investigators of substantial third party funded research projects, and thus have additional funds at their disposal to support this purpose. Another cornerstone of the proposed program is the student's participation in international conferences on a regular basis. The budget to cover the students' travel expenses is assessed so that every student can be supported by 1000 Euro per year.

An additional training aspect (which, nevertheless, we consider central to the program's success) is a careful introduction into techniques of scientific presentation and teaching. Based on these foundations, students will be expected to regularly give talks in the joint seminars and at the Dissertantenkolloquium of the University of Vienna. Participation in international conferences, eg. the Winter School on Geometry and Physics, will help them gain confidence in addressing international audiences.

Progress of the PhD theses will receive feedback on a regular basis by the team of supervisors, as well as by the IK-fellows in the weekly seminars and talks. Additionally, once per year the team of organizers will discuss with each student his/her individual progress. In this way it should be possible to closely keep track of each student's advances and to avoid most of the common problems which PhD students working alone usually have to face.

Selection of candidates for the IK will be based on standard web-based job markets (e.g., `mathjobs.com`), as well as personal international contacts. Selection criteria primarily are mathematical quality, but also additional skills like team-orientation and communicative skills will be taken into account. Wherever possible, personal interviews will provide additional input for the selection process.

4. Infrastructure

The IK will be situated at the Faculty of Mathematics (the largest Mathematics department in Austria offering a broad variety of mathematical disciplines, cf. `www.mat.univie.ac.at`), jointly with the department of theoretical physics of the University of Vienna (`www.thp.univie.ac.at`, accommodating the gravitational physics group co-organizing the program). The full infrastructure of both organizations will be made available to the students of the proposed IK. In particular, each student will obtain a fully equipped workplace at one of the departments, and full access to both libraries (also outside the opening hours). Complying with one of the main features of the program, the students will be given the opportunity of acquiring teaching experience in the course of the PhD program.

A significant enhancement of the opportunities presented to the PhD candidates of the IK is provided by a number of research projects of the applicants:

- Robert Beig: FWF Project P16745 *Elasticity as Matter Model in Relativity,* January 2004 – June 2006.
- Andreas Čap: FWF-Project P15747 *Parabolic Geometries,* October 2002 – September 2006.
- Michael Kunzinger: FWF-Project P16742 *Geometric Theory of Generalized Functions,* February 2004 – February 2007. START-Project Y237 *Nonlinear Distributional Geometry,* February 2005 – February 2011.
- Peter Michor: FWF Project P17108-N04: *Lie Theory and Applications II,* October 2004 – September 2007.

5. Research statement of Robert Beig

Mathematical Relativity. I am a theoretical physicist working on Mathematical Relativity (MR). In MR one studies problems arising in a theory of physics, namely Einstein's general relativity (GR), in a rigorous fashion. The nature of this study, in very broad terms and from the mathematical point of view, is as follows. The basic object is a symmetric 2-tensor of Lorentzian signature on a manifold, which models the gravitational field on the spacetime manifold. This spacetime metric is subject to Einstein's equations (EEs), a system of quasilinear partial differential equations (PDEs) of second order. In the presence of sources ("matter") these equations are coupled to a further system of equations, which depends on the nature of these sources, and is also quasilinear and of first or second order. Much of MR concerns solutions of EEs in vacuo or with different matter sources: their existence and uniqueness under appropriate boundary conditions and their properties, e.g. their asymptotic behavior, presence of symmetries, etc. There are two main difficulties in solving EEs: One is their intrinsic nonlinearity. The other one is the fact that their solution space carries a natural action of a huge gauge group, namely the group of diffeomorphisms of the underlying manifold. These difficulties turn the study of EEs into a subtle undertaking, in which the time-honored search for solutions in closed form offers little help, and for which up-to-date methods from differential geometry, Lie group theory and PDE theory have to be used.

Nature of Research. I now list specific topics which have interested me over the years, and which have left some loose ends, which are of current interest and suitable for a PhD project.

- Perfect fluids: It has been an open problem to show that static, self-gravitating perfect fluid bodies have to be spherically symmetric. Using a combination of elliptic PDE theory with methods inspired by 3-dimensional conformal geometry, this was established in [3] for a wide class of equations of state for the fluid. In its most general form the conjecture is still open and may require completely different methods. E.g. using as fluid variable, rather than pressure or density, the deformations of some reference state, the system becomes a gauge theory with the volume preserving diffeomorphisms as gauge group. This viewpoint has so far not been exploited.
- Asymptotics of time-independent solutions of EEs: It has been proved in [4] that static, asymptotically flat solutions of the vacuum EEs are in a suitable sense conformally analytic near infinity. With the recent developments in string theory there is now interest in versions of this result for spacetime dimension higher than $d = 3 + 1$. This should crucially involve the set of conformally invariant tensors recently constructed by Fefferman and Graham (see [13]) and hence is intimately related to the theory of parabolic geometries, see the statement of A. Čap below. The case $d = 5$ has been recently solved by P.T. Chruściel and R. Beig (unpublished). At present it is conjectured that there exists a generalization for $d = 1 + 2k, k = 2, 3 \dots$. Even if a proof can be

found soon, there exist wide possibilities for further generalizations — such as the inclusion of certain matter fields or replacing "static" by "stationary" — which furnish possible PhD projects.

- Causality in continuum mechanics: The characteristic polynomial associated with the leading-order symbol of the PDEs describing matter sources in GR gives rise to a geometry [2] which vastly generalizes Lorentzian geometry and which would furnish a topic worthy of further study. In this context Michor's expertise should be helpful, cf. [M91].

6. Research statement of Dietrich Burde

Affine structures on Lie groups. Affine manifolds and their fundamental groups play an important role within the theory of compact manifolds with geometric structure. Milnor (see [22]) has posed in connection with Auslander's conjecture on affine crystallographic groups, the following question:

Does every solvable Lie group admit a complete left-invariant affine structure?

Despite of the evidence for the existence of such structures, Yves Benoist constructed in 1993 a counterexample in dimension 11 consisting of a filiform nilpotent Lie group without any left-invariant affine structure. We have produced a whole family of counterexamples [Bu1], [Bu3], [Bu7] for the dimensions $10 \leq n \leq 13$, out of which Benoist's example emerges as just one in a series.

The goal here is to establish new criteria for the existence question of left-invariant affine structures on Lie groups. Among the open questions we will study the conjectures saying that any two-step solvable Lie group, respectively any four-step nilpotent Lie group admits a left-invariant affine structure. For recent results see [Bu12].

The question for reductive Lie groups asks for semisimple connected, simply connected algebraic groups S possessing a module V with an orbit of codimension 1 and $\dim V = \dim S + 1$. For results where $S = SL(n)$ see [Bu2].

Crystallographic groups. A classical crystallographic group is a uniform discrete subgroup of Isom($\mathbb{R}^n$). Such groups act properly, discontinuously and cocompactly on $\mathbb{R}^n$. Their structure is well known by the three Bieberbach theorems. In fact, all these groups are finitely generated virtually abelian. As a generalization of this concept, one also studies affine crystallographic groups. These are subgroups of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ acting crystallographically (by which we will mean properly discontinuously and cocompactly) on $\mathbb{R}^n$. The structure of these affine crystallographic groups is not at all as well known as in the case of the classical crystallographic groups. A prominent open question here is the conjecture of Auslander and its generalizations. For our recent results in this question see [Bu10].

Degenerations of Lie algebras and algebraic groups. Let $\mu \in \mathrm{Hom}(\Lambda^2 V, V)$ be a Lie algebra structure over a vector space V. The set of such μ forms an algebraic subset $\mathcal{L}_n(K)$ of the variety $\mathrm{Hom}(\Lambda^2 V, V)$. The group $GL_n(K)$ acts on $\mathcal{L}_n(K)$ by $(g * \mu)(x \wedge y) = g(\mu(g^{-1}(x) \wedge g^{-1}(y)))$. The orbits under this action are the isomorphism classes of n-dimensional Lie algebras. The closure of the orbits are called degenerations, i.e., λ degenerates to μ, if $\mu \in \overline{O(\lambda)}$. The classification of Lie algebras and their degenerations is rather difficult. We have obtained such a classification in dimension 4, see [Bu6], and a partial classification for nilpotent Lie algebras in dimension 7, see [Bu9], [Bu5]. The invariants and coinvariants given there can be generalized. An open question of Vergne is, whether every nilpotent Lie algebra is the degeneration of some other Lie algebra. A very interesting concept, due to Slodowy, is the study of orbit closures for algebraic groups.

Representation theory and cohomology of Lie algebras. We want to obtain new results on Betti numbers, Lie algebra cohomology and representation theory. To be more explicit, we will study the spaces $H^p(\mathfrak{g}, K)$, $H^p(\mathfrak{g}, \mathfrak{g}^*)$, the toral rank conjecture of Alperin and the invariant $\mu(\mathfrak{g})$, the minimal dimension of a faithful module of a finite-dimensional Lie algebra $\mathfrak{g}$. The toral rank conjecture of Alperin claims that $2^{\dim Z(\mathfrak{g})}$ does not exceed the sum of all Betti numbers $b_p(\mathfrak{g})$, where $\mathfrak{g}$ denotes a finite-dimensional nilpotent Lie algebra. This is only proved in special cases. For affine and symplectic structures on Lie algebras, the spaces $H^2(\mathfrak{g}, K)$ are of interest. In [Bu8] we have classified these spaces for filiform Lie algebras of dimension $n \leq 11$, and for important series of nilpotent Lie algebras of dimension $n \geq 12$. Another interesting problem which is connected with affine structures is a refinement of Ado's theorem, i.e., to determine $\mu(\mathfrak{g})$. If $\mathfrak{g}$ admits an affine structure, then $\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1$. However, it is rather difficult to verify this condition for a given Lie algebra $\mathfrak{g}$, see [Bu1], [Bu3]. The invariant $\mu(\mathfrak{g})$ may or may not grow polynomially with $\dim \mathfrak{g}$.

7. Research statement of Andreas Čap

My main field of research is the theory of parabolic geometries. The basis this theory is the general concept of Cartan geometries, which builds a bridge from geometry in the classical sense of F. Klein's Erlangen program to differential geometry. Starting from a Lie group G and a closed subgroup $H \subset G$ one obtains the concept of a Cartan geometry of type (G, H) whose instances can be thought of as "curved analogs" of the homogeneous space G/H. Parabolic geometries are the special case of this concept in which G is semisimple and $H \subset G$ is a parabolic subgroup, so G/H is a generalized flag variety. This gives rise to a strong connection to semisimple representation theory, which is a characteristic feature of the theory. The other main feature is that one may study a large variety of seemingly very diverse geometric structures in a uniform manner. Among these structures are important examples like conformal and quaternionic structure, hypersurface type CR structures, and quaternionic contact structures.

Some of the examples of parabolic geometries, particularly conformal structures and CR structures, have been studied independently for a long time. This leads to an interesting interplay between the general theory of parabolic geometries and results on specific examples of such structures. During the last years the concepts of parabolic geometries have established themselves not only as a unified approach to various structures but also as a source of new ideas and results for the classical examples, see e.g. [C1]–[C8], [6], [11], [12], [9].

The projected IK would provide further possibilities to exploit the interplay between the general theory of parabolic geometries and specific examples of such structures. On the one hand, conformal geometry plays an important role in mathematical relativity, since conformally invariant properties are particularly robust properties of Riemannian metrics. An example of an element of the general theory of parabolic geometries which should be very useful in relativity is provided by the multilinear invariant differential operators introduced in [6]. They are known to implement helicity raising and lowering in special cases, but in general they are not well understood, even in the special case of conformal structures. Likewise, some of the recent general results on infinitesimal automorphisms and deformations in [C8] could be relevant for mathematical relativity.

On the other hand, the theory of Bernstein–Gelfand–Gelfand sequences introduced in [C2] and [6] gives rise to a large number of geometric overdetermined systems of PDEs. It has been shown in [C7] that for a subclass of structures, one can use the methods developed there to explicitly prolong arbitrary semi-linear systems with the same principal symbols to a closed form. Only the simplest examples of this process have been worked out explicitly and there are evident possibilities of generalizations of this procedure. Therefore, this should provide an excellent basis for exchange and interaction with other approaches to PDE problems.

There is a wide variety of future problems related to parabolic geometries that could be worked on by participants of the projected IK. These can be studied with different emphasis depending on the prior education and the interests of students taking part in the IK. As a selection of such problems, let me mention the following.

- Study those examples of the multilinear natural operators introduced in [6] for conformal structures, which are related to linear operators that are of interest in relativity.
- The relation between the canonical Cartan connection and the Fefferman–Graham ambient metric for conformal structures has been described in [C4]. A similar description should be possible for the Poincaré metric associated to a conformal structure. This should lead to advances in the theory of conformally compact Einstein manifolds.
- On the homogeneous models for general parabolic geometries, the multilinear invariant operators of [6] can be studied using only Lie theory and representation theory. These operators form a rather complicated algebraic structure similar to an A_∞–algebra. Improving the understanding of this structure is a very fruitful problem.
- Generalizations of the prolongation procedure of [C7] should lead to results on overdetermined systems on manifolds endowed with certain types of bracket–generating distributions, the simplest examples being provided by contact structures. Results in that direction are also of interest for fields like sub–Riemannian geometry.

8. Research statement of Stefan Haller

The Ray–Singer analytic torsion [26] essentially is the super determinant of the deRham complex associated to a flat vector bundle over a closed manifold. It is defined with the help of zeta regularized determinants of Laplacians associated with a Riemannian metric on the base manifold and a Hermitian fiber metric on the vector bundle. As shown in [H8] the analytic torsion can be modified so that it provides an invariant (essentially a positive real number) depending only on the flat connection and a coEuler structure, i.e. an element in an affine version of $H^{n-1}(M)$. CoEuler structures are Poincaré dual to Euler structures, which have been introduced by Turaev [28] in order to remove the ambiguities involved in the definition of the combinatorial Reidemeister torsion (essentially a non-vanishing complex number.) A result of Cheeger [7], Müller [23] and Bismut–Zhang [5] tells that the analytic torsion is the absolute value of the combinatorial torsion. Replacing the Hermitian fiber metric by a fiber wise symmetric non-degenerate bilinear form one can define generalized Laplacians and use them to define a complex valued Ray–Singer torsion. One can show that this analytic torsion coincides with the combinatorial torsion. Particularly, it computes the phase of the combinatorial torsion in analytic terms. For trivial line bundles this is contained in [H11], the general case is in preparation. An obvious question and possible thesis problem(s) is how to extend this picture to the case of bordisms. What is the right concept of coEuler structures in this relative setting? How to define the relative complex valued analytic torsion? Show that it coincides with the relative combinatorial torsion.

In [H9] we extended the result of Cheeger, Müller and Bismut–Zhang to the Morse–Bott situation. This can be interpreted as a localization result for the Ray–Singer torsion. It immediately permits to recover (and slightly generalize) a theorem of Lück, Schick and Thielmann [20] about the analytic torsion of fiber bundles. Another application we have in mind is the computation of the Ray–Singer torsion in the presence of a compact group of symmetries. In such a situation one typically does not have an invariant Morse–Smale function. However, in many cases there is an invariant Morse–Bott–Smale function which can be used to compute the analytic torsion.

In [H3] and [H11] we studied the Morse–Novikov complex from an analytic point of view. Recall that the Novikov incidence numbers [24] replace the Morse incidence numbers when passing from the gradient of an exact one form to the gradient of a closed one form. Novikov incidence numbers are elements of a large formal completion of a group ring. We showed that if the vector field satisfies the so-called exponential growth condition then the Novikov incidence numbers actually are elements in a small subring and have a Laplace transform which is related to spectral geometry. The question whether this exponential growth condition is true or not is a very basic yet open problem. Precisely, consider the unstable manifold of a vector field corresponding to a closed one form via a Riemannian metric on a closed manifold. Let $V(r)$ denote the volume of the ball of radius r in this unstable manifold. Does one have an estimate of the form $V(r) \leq e^{Cr}$, C a constant? If this was always the case it would, together with [H11], prove a conjecture of Novikov. In view of [25] the latter conjecture is known to be true C^0-generically. A possible (tough) thesis problem would be to study this exponential growth condition. Prove that it is satisfied for all vector fields; or come up with a counter-example; or show that the vector fields with this property are at least C^1-generic.

9. Research statement of Michael Kunzinger

Nonsmooth differential geometry. The theory of nonlinear of generalized functions (in the sense of J.F. Colombeau) has seen a fundamental restructuring over the past decade. The main objective which was achieved in the course of this development was the creation of a fully diffeomorphism invariant version of an algebra of generalized functions canonically containing the space of distributions while at the same time being optimally consistent with the smooth setting. Based on pioneering work by Colombeau, Oberguggenberger, Jelinek, Pilipović, Vickers, Wilson and others, this aim was finally achieved in [K2] and [K13], see also [K1] for a comprehensive survey. Moreover, global analysis and differential geometry in the so-called 'special' (not allowing for a canonical embedding of spaces of distributions, as opposed to the 'full' version addressed above) version of Colombeau's construction was initiated, among others, in [K4, K6, K8, K9, K10, K12]. Based on these foundations, the following research topics will be pursued in our research group DIANA (Differential Algebras and Nonlinear Analysis) in the coming years, each providing possible topics of doctoral theses under my supervision:

- Development of a theory of vector valued sections in the full version of the construction, canonically extending the distributional setting (based on [K2] and [K13]).
- Transferring differential geometric concepts from the special setting of the theory of algebras of generalized functions ([K6], [K9], [K12]) to the full version as developed in the previous item.
- Relating the theory of generalized connections on principal fiber bundles introduced in [K4] to the theory of characteristic currents of singular connections of Harvey and Lawson.
- Studying sheaf and embedding properties of the space of manifold valued generalized functions ([K8], [K10]).
- Continuing the study of generalized pseudo-Riemannian geometry initiated in [K9].

Lie group analysis of singular differential equations. My research in this field over the past years has been focused upon systematically extending classical and distributional methods of symmetry group analysis in order to allow the treatment of nonlinear differential equations involving singularities ([K3, K6, K11, K14]). The basic methods of group analysis of PDEs have thereby been made available in the nonlinear setting of generalized functions and have been brought to a level where methods of nonlinear distributional geometry can successfully be employed. Building on these foundations, it is now possible to pursue the following main directions, each giving rise to possible topics for doctoral theses:

- Connecting to the global distributional theory of Lie group actions developed by Ziemian and Schmidt.

- Studying group invariant generalized solutions of linear and nonlinear PDEs, as well as applications to calculating group invariant fundamental solutions ([K11], [K1]).
- Developing a theory of variational symmetries in the framework of algebras of generalized functions.

Applications to general relativity. Nonlinear distributional geometry has already found a number of interesting applications in general relativity, initiated by R. Steinbauer, and followed by Balasin, Vickers, Wilson, and myself. I refer to Roland Steinbauer's list of research topics for a more detailed description of future work in this area. I am interested in co-supervising doctoral dissertations in this field (in collaboration with R. Steinbauer, J. Grant (University of Vienna), and J. Vickers (University of Southampton)).

10. Research statement of Peter W. Michor

I am active in several fields and I offer topics for doctoral and diploma theses in most of them, usually after thorough discussions with the student. Some of these fields are shortly described in the following.

Infinite dimensional differential geometry. Based on convenient calculus in infinite dimensions which was jointly developed by A.Frölicher, A.Kriegl, L.Nel, and myself (see [MF]), this is probably the center of my mathematical expertise. Manifolds of mappings, diffeomorphism groups, various orbit spaces and weak symplectic and Riemannian metrics on these are the center of my interest. In particular, *shape spaces* are orbit spaces of plane curves under reparametrization groups, or of surfaces in 3-space under reparametrizations, are very much in the center of my interest since my collaboration with David Mumford started in 2001, see the papers [M98], [M102], [M107], and also [M108]. The geodesic equations we found on shape space are very interesting nonlinear PDEs, closely related to Burgers' equation and the Camassa-Holm equation. Thus my strong new interest in *geometric theory of PDEs*, including their conserved momenta. The lecture notes [21] of my spring 2005 lecture course is devoted mainly to the questions of finding conserved momenta (like the reparametrization momentum on shape space) and using it for solving some of these equations, following methods of Ebin, Marsden, Skoller, Constantin, Kapeller, et.al.

After having determined the geodesic distance on shape space the question of the Cauchy completion of shape space in this distance arises. Here one quickly leaves the realm of smooth mappings and collaboration with Michael Kunzinger und Roland Steinbauer is envisaged to described the shape in the completion.

A surprising result that came out of our investigation of the geometry of shape spaces is that the geodesic distance for the L^2-Riemannian metric vanishes on any kind of orbit space under the action of the diffeomorphism group of a (compact) manifold M on the space of immersions of M into a Riemannian manifold (N, g). This also holds for the right invariant L^2-metric on each full diffeomorphism group. But a small change in the L^2 integrand like adding the product of the divergences of the vector fields immediately yields that the geodesic distance is positive. Thus for volume preserving diffeomorphisms the L^2-metric (which is the metric of incompressible fluid mechanics) geodesic distance separates distinct diffeomorphisms. See [M102] for these results.

Lie groups, algebraic groups, group actions, orbit spaces, and invariant theory. In [M65] we investigated the following problem: Let $P(t) = x^n - \sigma_1(t)x^{n-1} + \dots + (-1)^n\sigma_n(t)$ be a polynomial with all roots real (a hyperbolic polynomial), smoothly parameterized by t in $\mathbb{R}$. Can we find n smooth functions $x_1(t), \dots, x_n(t)$ of the parameter t defined near 0, which are the roots of $P(t)$ for each t? The answer is yes if everything is real analytic (Rellich 1940, different proof in [M65]), or no two roots meet of infinite order [M65]. The lift can always be chosen twice differentiable [M91] (using a result by Bronstein) but not better. This problem can be reformulated by asking to lift smooth curves from the orbit space of a representation over the invariants; our results are nearing those for polynomials, in [M73], [M99], [M105]. Some lifting results are only possible for algebraic actions of finite groups, [M88] and [M90].

In [M87] we consider the generalized Cayley transform from the group into the Lie algebra induced by a rational representation of an algebraic group. Many interesting results concerning the invariance under the Cayley transform of the Jordan decomposition of elements into the semisimple part (further into elliptic and hyperbolic parts) and the nilpotent part are derived. Since the Cayley transform is a regular algebraic mapping it also makes sense for finite groups of Lie type in terms of algebraic geometry over a finite field, where the exponential mapping does not make sense. This is completely unexplored up to now.

In [M97] we studied reflection groups on Riemannian manifolds which also allowed reconstruction of manifold from the Riemannian chamber. Here the problem is that not every reflection disects the manifold (for example for the Weyl group acting on the maximal torus of compact simple Lie groups of rank ≥ 2). This research is ongoing.

Actions of Lie algebras on manifolds. In the paper [M56] we started to investigate the differential geometry of an action of a Lie algebra on a manifold, i.e., only an infinitesimal Lie group action. We want to study how this action can be extended to a Lie group action on an enlarged manifold. There always exists a universal solution to this problem, as shown in the papers [M83], [M92], and [M96], in a way which differs from the original solution of Palais. In particular, in [M92] we have constructed the flow completion of Burgers' equation which we viewed as a dynamical system on an infinite dimensional space where the solutions run into singularities in finite time.

Cohomology of Lie algebras and groups. [M81] (unpublished) is a coherent exposition of the extension theory of Lie algebras which is not easily accessible in the literature, [M84] carries this over to super Lie algebras. [M101] and

[M106] study extensions of diffeomorphism groups and Lie algebras of vector fields which respect exact forms.

11. Research statement of Roland Steinbauer

General Relativity in a non-smooth setting. The significant advances of the theory of nonlinear generalized functions (in the sense of J.F. Colombeau) and especially the development of a nonlinear distributional geometry in recent years (see also M. Kunzinger's list of research topics and [S1] for an overview) were achieved in close connection with applications in a geometric context and, in particular, in general relativity, contributed among others by J. Vickers, J. Wilson, H. Balasin, M. Kunzinger and myself ([S12, S11, S9, S5]). Especially due to the intrinsic construction of a diffeomorphism invariant (full) algebra of generalized functions on differentiable manifolds ([S10]) as well as the development of a generalized Pseudo-Riemannian geometry ([S8]) in the special version of the theory—which, in particular, allows for a flexible modeling of singular, i.e., distributional data—new and even more exciting applications in relativity such as to the cosmic censorship hypothesis have come into reach. Some of these are currently investigated in the course of several third party funded projects within our research group DIANA (DIfferential Algebras and Nonlinear Analysis) by J. Vickers (Southampton), J. Grant (Vienna), M. Kunzinger and myself. Two of these which qualify perfectly as areas for a PhD project are in some more detail:

- The wave equation on singular space times: Local unique solvability of the Cauchy problem for the wave equation in (weakly) singular space times (i.e., with a metric of low differentiability as e.g. cosmic strings) was put forward by C. Clarke as a criterion to exclude these geometries as counterexamples to the cosmic censorship hypothesis. Based upon earlier work by J. Vickers and J. Wilson an existence and uniqueness result in nonlinear generalized functions has recently been proven for a large class of static generalized space times ([S13]). Ongoing research pursued together with a PhD student (co-supervised by M. Kunzinger and myself) and a diploma student (supervised by myself) will also be concerned with clarifying the relationship between second order strictly hyperbolic operators with non-smooth coefficients and generalized pseudo-Riemannian metrics.

- Generalized singularity theorems: We study geodesics and their (non-)extendibility in the presence of singularities, resp. in space-times of low differentiability using, in particular, the techniques of [S12, S8, S4]. We aim at formulating and proving analogues of the singularity theorems by S. Hawking and R. Penrose in generalized space-times.

Nonsmooth differential geometry. My contributions in this field over the last years were centered around providing a geometric nonlinear distributional setting suitable for applications in mathematical physics, in particular relativity [S10, S8] and classical mechanics [S4, S3]. For more details and especially for areas of ongoing research I refer to M. Kunzinger's statement included in this proposal. I have a strong interest in (co-)supervising PhD projects in this field with M. Kunzinger, J. Vickers (Southampton) and M. Grosser (Vienna).

Local existence results in algebras of generalized functions. The aim of this line of development is to explore the extent to which the group of (classically) equivalent local existence results: Implicit Function Theorem – Inverse Function Theorem – Existence of Solutions of ODEs – Frobenius Theorem can be recovered in the generalized setting. Up to date the main focus has been on ODEs: recently a theory of generalized flows on differentiable manifolds was developed extending the linear distributional setting of J. Marsden (see [S4], and [S7] for the underlying concept of manifold valued generalized functions). A second starting point for the analysis of these interconnected topics is the encouraging fact that in a certain concrete example from relativity it has been possible to single out "nonsmooth diffeomorphisms" and to provide a rigorous description in the framework of nonlinear distributional geometry ([S11]).

Currently a PhD thesis mainly concerned with generalizations of the inverse function theorem is carried out within our research group and I am supervising a diploma thesis on singular ODEs in the generalized functions setting.

Further interests. I have retained interest in several questions of classical general relativity since the time of my PhD (started at the Department of Theoretical Physics at Vienna University). The proposed project would provide me with the possibility of joint teaching and research perspectives in this direction especially with R. Beig.

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Abbreviations

CV … Curriculum Vitae
EE … Einstein's Equation
FWF … Fonds zur Förderung der Wissenschaftlichen Forschung, Austrian Science Fund
GR … General Relativity
IK … Initiativkolleg
MR … Mathematical Relativity
ODE … Ordinary Differential Equation
PDE … Partial Differential Equation