LIE SYMMETRIES, QUALITATIVE ANALYSIS AND EXACT SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS WITH INHOMOGENEOUS NONLINEARITIES

Juan Belmonte-Beitia, Víctor M. Pérez-García, Vadym Vekslerchik

Departamento de Matemáticas, E.T.S.I Industriales, and
Instituto de Matemática Aplicada a la Ciencia y la Ingeniería (IMACI)
Avda. de Camilo José Cela, 3
Universidad de Castilla-La Mancha
13071 Ciudad Real, Spain

Pedro J. Torres

Departamento de Matemática Aplicada. Facultad de Ciencias.
Universidad de Granada Campus de Fuentenueva s/n
Campus de Fuentenueva s/n, 18071 Granada, Spain.

(Communicated by ???)

Abstract. Using Lie group theory and canonical transformations, we construct explicit solutions of nonlinear Schrödinger equations with spatially inhomogeneous nonlinearities. We present the general theory, use it to study different examples and use the qualitative theory of dynamical systems to obtain some properties of these solutions.

1. **Introduction.** The Nonlinear Schrödinger Equation (NLSE) in its many versions is one of the most important models of mathematical physics, with applications to different fields [33] as for example in semiconductor electronics [6, 21], nonlinear optics [18], photonics [17], plasma physics [10], fundamentation of quantum mechanics [28], dynamics of accelerators [13], mean-field theory of Bose-Einstein condensates [8, 35] or biomolecule dynamics [9] to cite only a few examples. In some of these fields and many others, the NLSE appears as an asymptotic limit for a slowly varying dispersive wave envelope propagating in a nonlinear medium [30]. Moreover, the range of applicability of this equation is large because of the well-known universality of this equation [2].

The study of these equations has served as a catalyzer of the development of new ideas or even mathematical concepts such as solitons [36] or singularities in partial differential equations [31, 15].

In the last years there has been an increased interest in a one-dimensional nonlinear Schrödinger equation with inhomogeneous nonlinearity (INLSE):

$$i\psi_t = -\psi_{xx} + V(x)\psi + g(x) |\psi|^2 \psi, \tag{1}$$

with $x \in \mathbb{R}$, $V(x)$ is an external potential and $g(x)$ describes the spatial modulation of the nonlinearity. This equation arises in different physical contexts such as nonlinear optics and dynamics of Bose-Einstein condensates. But it is in the later

field where the possibility of using the Feschbach resonance management techniques to modify spatially the collisional interactions between atoms [27, 34, 1, 32, 16, 26, 25, 11] the one which has motivated a lot of theoretical research in the last few years focusing on questions of direct applicability to experiments. Different aspects of the dynamics of solitons in these contexts have been studied such as the emission of solitons [27, 34] and the propagation of solitons when the space modulation of the nonlinearity is a random [1], periodic [29, 25], linear [32] or localized function [26]. Although many exact solutions of the NLSE with spatially homogeneous nonlinearities and without potentials ($V = 0$) have been known for a long time, the problem of finding exact solutions even of the NLS with homogeneous nonlinearities and general potentials is a very difficult one.

In this paper, using Lie symmetries we find general classes of potentials $V(x)$ and nonlinearity functions $g(x)$ for which exact solutions can be constructed by combining solutions of the integrable NLS and solvable potentials $V(x)$. The basic idea of the Lie symmetries method is to study the invariance properties of given differential equations under continuous groups of transformations. This method has been applied successfully to different equations, such as, for example, differential equations that model anharmonic oscillators [19, 20] and Madelung fluid equations [3]. In Ref. [4] we have presented some examples of this methodology of specific physical interest. Here we complement that analysis by presenting the general theory, provide more examples, study the case of asymmetric solutions and use qualitative theory of dynamical systems to provide a much more complete analysis of the method and its applications to equations of physical relevance.

The paper is organized as follows. In Section 2, we introduce the general theory of the Lie symmetry analysis for ordinary differential equations and particularize it for our model problem: the nonlinear Schrödinger equation with an inhomogeneous nonlinearity. In Section 3, we study the canonical transformations of the INLSE. In Section 4, we present the connection between the NLSE and the INLSE. In Section 5, we use the method to construct explicit solutions of the stationary nonlinear Schrödinger equation with an inhomogeneous nonlinearity and study the qualitative behaviour of the NLSE and its qualitative connection with the INLSE in different examples. Finally, in Section 6, we present asymmetric solutions of the INLSE. To our knowledge, this is the first time that such solutions are calculated.

2. **General Theory of Lie symmetries.** In this paper we will look for localized stationary solutions of Eq. (1), which are of the form

$$\psi(x, t) = \phi(x)e^{-i\lambda t}, \tag{2}$$

which satisfy the following nonlinear eigenvalue problem

$$-\phi_{xx} + V(x)\phi + g(x)\phi^3 = \lambda\phi, \tag{3a}$$
$$\lim_{|x|\to\infty} \phi(x) = 0. \tag{3b}$$

By definition [5, 23], a second-order differential equation $A(x, \phi, \phi', \phi'') = 0$ possesses a Lie group of point transformations or Lie point symmetry of the form

$$M = \xi(x, \phi)\partial/\partial x + \eta(x, \phi)\partial/\partial\phi, \tag{4}$$

if the action of the second extension of M, i.e. $M^{(2)}$ on A is equal to zero, i.e.

$$M^{(2)}A(x, \phi, \phi', \phi'') = \left[\xi(x, \phi)\frac{\partial}{\partial x} + \eta(x, \phi)\frac{\partial}{\partial \phi} + \right.$$
$$\left. \eta^{(1)}(x, \phi)\frac{\partial}{\partial \phi'} + \eta^{(2)}(x, \phi)\frac{\partial}{\partial \phi''} \right] A(x, \phi, \phi', \phi'') = 0, \quad (5)$$

with $\eta^{(k)}$ given by

$$\eta^{(k)}(x, \phi, \phi', \phi'', ..., \phi^k) = \frac{D\eta^{(k-1)}}{Dx} - \phi^{(k)}\frac{D\xi(x, \phi)}{Dx}, \quad k = 1, 2, ... \quad (6)$$

where $\eta^{(0)} = \eta(x, \phi)$ and D/Dx is the total derivative, i.e.

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + \phi'\frac{\partial}{\partial \phi} + \phi''\frac{\partial}{\partial \phi'} + ... + \phi^{(n+1)}\frac{\partial}{\partial \phi^{(n)}} + ... \quad (7)$$

For example, $\eta^{(1)}$ is equal to

$$\eta^{(1)} = \eta_x + [\eta_\phi - \xi_x]\phi_x - \xi_\phi(\phi_x)^2. \quad (8)$$

In our case, $A(x, \phi, \phi_x, \phi_{xx})$ is given by

$$A(x, \phi, \phi_x, \phi_{xx}) = -\phi_{xx} + f(x, \phi), \quad (9)$$

where $f(x, \phi) = V(x)\phi + g(x)\phi^3 - \lambda\phi$, and the action of the operator $M^{(2)}$ on $A(x, \phi, \phi_x, \phi_{xx})$ leads to a polynomial equation in ϕ_x. By equating coefficients of powers of ϕ_x, one obtains

$$\xi_{\phi\phi} = 0, \quad (10a)$$
$$\eta_{\phi\phi} - 2\xi_{\phi x} = 0, \quad (10b)$$
$$2\eta_{x\phi} - \xi_{xx} - 3f\xi_\phi = 0, \quad (10c)$$
$$\eta_{xx} - \xi f_x - \eta f_\phi + \eta_\phi f - 2\xi_x f = 0. \quad (10d)$$

Integrating Eqs. (10a) and (10b), we get

$$\xi(x, \phi) = a(x)\phi + b(x), \qquad \eta(x, \phi) = a'(x)\phi^2(x) + c(x)\phi + d(x). \quad (11)$$

Substituting these expressions into Eq. (10c) we obtain

$$2c'(x) = b''(x), \qquad a(x) = 0. \quad (12)$$

Finally, substituting Eqs. (11) and (12) in Eq. (10d), we get

$$\xi(x, \phi) = b(x), \quad (13a)$$
$$\eta(x, \phi) = c(x)\phi, \quad (13b)$$
$$c''(x) - b(x)V'(x) - 2b'(x)(V(x) - \lambda) = 0, \quad (13c)$$
$$2c(x)g(x) + b(x)g'(x) + 2b'(x)g(x) = 0. \quad (13d)$$

Also the substitution of Eq. (12) in Eq. (13d) gives

$$g(x) = g_0 b^{-3}(x)e^{-2C\int_0^x 1/b(s)ds}. \quad (14)$$

where g_0 and C are arbitrary constants. Summarizing the previous calculations, the Lie point symmetry is of the form

$$M = b(x)\frac{\partial}{\partial x} + c(x)\phi\frac{\partial}{\partial \phi}, \quad (15)$$

where

$$g(x) = g_0 b^{-3} e^{-2C \int_0^x 1/b(s) ds}, \quad (16a)$$

$$c(x) = \tfrac{1}{2} b'(x) + C, \quad (16b)$$

$$c''(x) - b(x) V'(x) - 2b'(x) \left(V(x) - \lambda \right) = 0. \quad (16c)$$

Eqs. (16) allow us to construct pairs $\{V(x), g(x)\}$ for which a Lie point symmetry exists. Thus given either $g(x)$ or $V(x)$, in principle we can choose the other in order to satisfy Eqs. (16). In what follows, we will study the implications of the existence of this Lie symmetry.

3. **Canonical transformations and invariants.** It is known [19], that the invariance of the energy is associated to the translational invariance. The generator of such a transformation is of the form $M = \partial/\partial X$. To use this fact, we define the transformation from variables (x, ψ) to new variables (X, U)

$$X = h(x), \qquad U = n(x)\phi, \quad (17)$$

where $h(x)$ and $n(x)$ will be determined by requiring that a conservation law of energy type $M = \partial/\partial X$ exists in the canonical variables. In fact, using Eq. (17), we get

$$\frac{\partial}{\partial \phi} = n(x) \frac{\partial}{\partial U}, \quad (18)$$

$$\frac{\partial}{\partial x} = n'(x)\phi \frac{\partial}{\partial U} + h'(x) \frac{\partial}{\partial X}. \quad (19)$$

Inserting the expressions (18) and (19) in Eq. (15) and assuming the condition $M = \partial/\partial X$, one finds

$$h'(x)b(x) = 1, \quad (20)$$

$$b(x)n'(x) + c(x)n(x) = 0. \quad (21)$$

By inserting Eq. (16b) into (21), and integrating, we obtain

$$h(x) = \int_0^x \frac{1}{b(s)} ds, \quad (22a)$$

$$n(x) = b(x)^{-1/2} e^{-C \int_0^x 1/b(s) ds}. \quad (22b)$$

We can now write Eq. (3a) in terms of the canonical coordinates U and X,

$$-\frac{d^2 U}{dX^2} - 2C \frac{dU}{dX} + g_0 U^3 - EU = 0, \quad (23)$$

with

$$E = (\lambda - V(x)) b(x)^2 - \tfrac{1}{4} b'(x)^2 + \tfrac{1}{2} b(x) b''(x) + C^2. \quad (24)$$

Equation (23) is the so-called Duffing equation which arises as a model of damped nonlinear oscillations [14]. It follows from Eqs. (16b)-(16c) that the quantity E given by Eq. (24) is a constant of motion.

When $C = 0$ the previous transformations preserve the Hamiltonian structure, because the canonical transformation is symplectic. In that case, Eq. (23) becomes

$$-\frac{d^2 U}{dX^2} + g_0 U^3 = EU. \quad (25)$$

As E is constant, this means that in the *new variables we obtain the nonlinear Schrödinger equation (NLSE) without external potential and with an homogeneous nonlinearity.*

Of course not all choices of $V(x)$ and $g(x)$ lead to the existence of a Lie symmetry or an appropriate canonical transformation, since they are linked by Eqs. (16). This fact imposes some obvious restrictions, for instance $b(x)$ must be smooth and positive.

Note that Eq. (25) is a stationary homogeneous NLSE without external potential, which can be reduced to quadratures and for which many solutions are known. So, we obtain

$$X - X_0 = \int_{U_0}^{U} \frac{dG}{\sqrt{2(N + \frac{1}{2}EG^2 + \frac{1}{4}g_0 G^4)}}, \tag{26}$$

with N a constant of integration. Moreover, the energy of the system is given by

$$H = \frac{1}{2}\left(\frac{dU}{dX}\right)^2 + \frac{1}{2}EU^2 - \frac{1}{4}g_0 U^4. \tag{27}$$

Many solutions of Eq. (25) are known. In this paper we will use the following ones

$$U_1(X) = \eta\frac{1}{\cosh(\mu X)}, \quad \left(E = -\mu^2, g_0 = -\frac{2\mu^2}{\eta^2}\right), \tag{28a}$$

$$U_2(X) = \eta\tanh(\mu X), \quad \left(E = 2\mu^2, g_0 = \frac{2\mu^2}{\eta^2}\right), \tag{28b}$$

$$U_3(X) = \eta\frac{\mathrm{sn}(\mu X, k)}{\mathrm{dn}(\mu X, k)}, \quad \left(E = \mu^2(1 - 2k^2), g_0 = -\frac{2\mu^2 k^2(1 - k^2)}{\eta^2}\right), \tag{28c}$$

$$U_4(X) = \eta\,\mathrm{dn}(\mu X, k), \quad \left(E = \mu^2(k^2 - 2), g_0 = -\frac{2\mu^2}{\eta^2}\right), \tag{28d}$$

with $0 \leq k \leq 1$. Table 1 summarizes the parameter values required for the existence of the solutions listed in Eqs. (28).

U	E	g_0	H
$U_1(X)$	negative	negative	0
$U_2(X)$	positive	positive	positive
$U_3(X)$	both	negative	positive
$U_4(X)$	negative	negative	negative

TABLE 1. Conditions on the parameters E, g_0 and energy H for the existence of the solutions $U_i, i = 1..4$ of Eq. (25) listed in Eqs. (28).

4. **Connection between the NLSE and INLSE via the LSE.** Setting $C = 0$ and eliminating $c(x)$ in Eqs. (16) we get

$$g(x) = g_0/b(x)^3, \tag{29}$$

plus an equation relating $b(x)$ and $V(x)$

$$b'''(x) - 2b(x)V'(x) + 4b'(x)\lambda - 4b'(x)V(x) = 0. \tag{30}$$

We notice that the simplest form to generate solutions for our problem, which involves constructing solutions pairs $(b(x), V(x))$ of Eq. (30) is to fix $b(x)$ and then, to calculate $V(x)$, since then we must solve a linear first order equation. Although we can eliminate $b(x)$ and obtain a nonlinear equation for the pairs $g(x)$
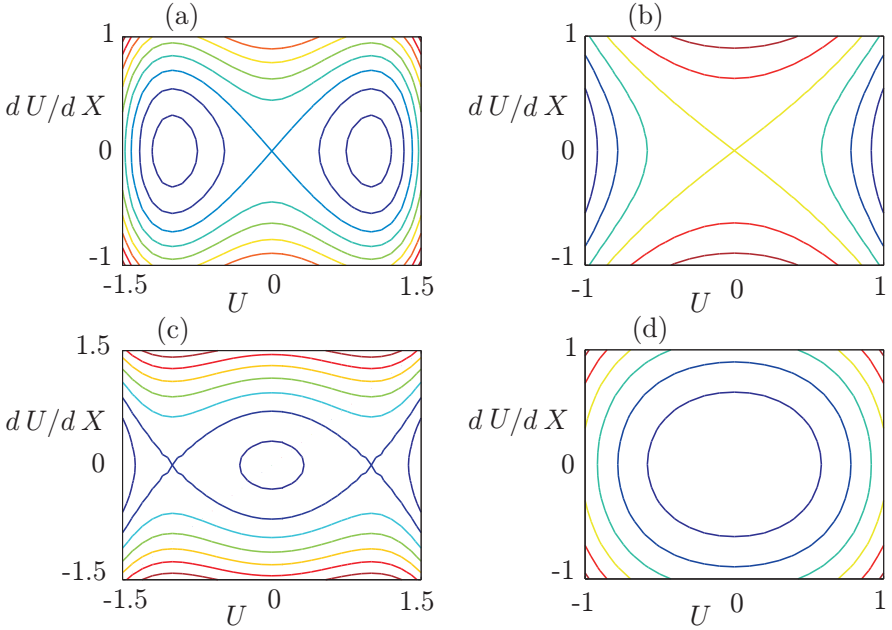
FIGURE 1. [Color online] Phase portrait of the real solutions of Eq. (25) for (a) $E < 0$, $g_0 < 0$, (b) $E < 0$, $g_0 > 0$ (c) $E > 0$, $g_0 > 0$ and (d) $E > 0$, $g_0 < 0$

and $V(x)$ for which there is a Lie symmetry, it is more convenient to work with (30), which is a linear equation. Alternatively, we can define $\rho(x) = b^{1/2}(x)$ and get an Ermakov-Pinney equation [12, 24]

$$\rho_{xx} + (\lambda - V(x))\,\rho = E/\rho^3, \tag{31}$$

whose solutions can be constructed as

$$\rho = \left(\alpha\varphi_1^2 + 2\beta\varphi_1\varphi_2 + \gamma\varphi_2^2\right)^{1/2}, \tag{32}$$

with α, β, γ constant and $\varphi_j(x)$ being two linearly independent solutions of the Schrödinger equation

$$\varphi_{xx} + (\lambda - V(x))\,\varphi = 0. \tag{33}$$

This choice leads to $E = \Delta W^2$ with $\Delta = \alpha\gamma - \beta^2$ and W being the (constant) Wronskian $W = \varphi_1'\varphi_2 - \varphi_1\varphi_2'$. Thus, given any arbitrary solution of the *linear Schrödinger equation* (33) *we can construct solutions of the nonlinear spatially inhomogeneous problem Eq. (3a) from the known solutions of Eq. (25).* Thus, using the huge amount of knowledge on the linear Schrödinger equation we can get potentials $V(x)$ for which φ_1 and φ_2 are known and construct $b(x)$, the canonical transformations $f(x), n(x)$, the nonlinearity $g(x)$ and the explicit solutions $\phi(x)$.

5. **Qualitative Analysis and Exact Solutions.** In this section, we will calculate exact solutions of Eq. (3a) for different specific choices of the nonlinear coefficient $g(x)$ and the external potential $V(x)$, using the method described in the previous sections, for $C = 0$. Moreover, using qualitative analysis, we will describe properties of the solutions of Eq. (3a), on the basis of the qualitative behaviour of Eq. (25). We begin by calculating the equilibrium points of this equation. One easily finds
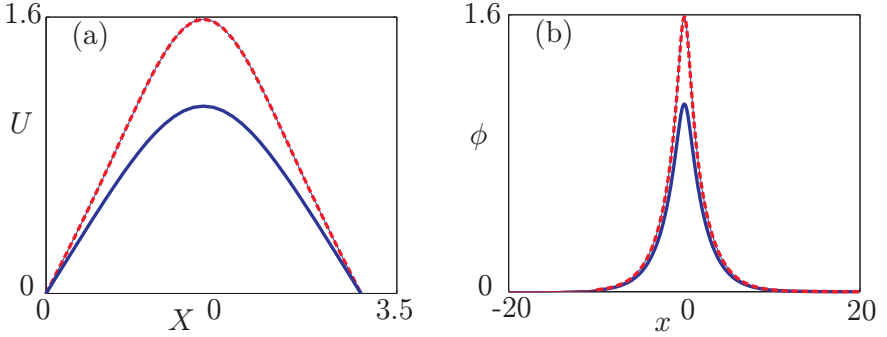
FIGURE 2. [Color online] Solutions of (a) Eq. (25) and (b) Eq. (3a), for $E = 0.15$, $g_0 = -1$ (solid blue line) and $E = -0.75$, $g_0 = -1$ (dashed red line) in both cases. We apply the transformation (39) to the solutions of Eq. (25) shown in Fig. 2(a) to obtain the solutions of Eq. (3a) shown in Fig 2(b).

that the Eq. (25) has three possible equilibrium points, depending of the signs of E and g_0:

$$U_\pm = \pm\sqrt{E/g_0}, \tag{34a}$$
$$U = 0. \tag{34b}$$

Then, we distinguish four cases:

1. For $E < 0, g_0 < 0$, we get three equilibrium points. $U = 0$ is a saddle point and $U_\pm$ are centers, Fig. 1(a).
2. When $E < 0, g_0 > 0$, we obtain that $U = 0$ is the only equilibrium point, which is a saddle point, Fig. 1(b).
3. When $E > 0, g_0 > 0$, we get three equilibrium points. $U_\pm$ are saddle points and $U = 0$ is a center, Fig. 1(c).
4. The last case corresponds to $E > 0, g_0 < 0$. For this case, the only equilibrium point is the trivial solution $U = 0$, which is a global center, Fig. 1(d).

Using Eq. (27), we can draw the phase portrait of Eq. (25), as we can see in Fig. 1.

In what follows we will present three examples as applications of our theory:

Example 1. Let us take $b(x) = \cosh(x)$. By using Eqs. (30) and (24), for $C = 0$, we obtain

$$V(x) = \lambda + \frac{1}{4} + \left(\frac{1}{4} - E\right)\frac{1}{\cosh^2(x)}. \tag{35}$$

Moreover, using Eq. (29), $g(x)$ is given by

$$g(x) = \frac{g_0}{\cosh^3(x)}, \tag{36}$$

with $X(x)$ being

$$\cos X(x) = -\tanh x, \tag{37}$$

where $0 \leq X \leq \pi$, subject to the Dirichlet boundary conditions $U(0) = U(\pi) = 0$. Any solution U of Eq. (25) gives a solution
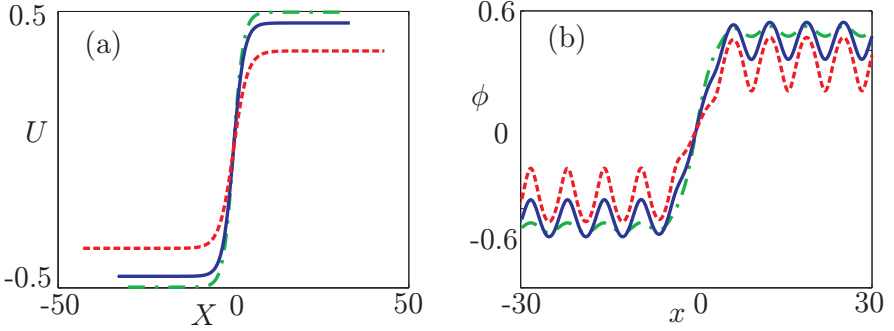
$$\phi(x) = b^{1/2}(x)U(X(x)), \tag{38}$$

FIGURE 3. [Color online] Example black solitons solutions of (a) Eq. (25) and (b) Eq. (3a) with $g_0 = 1$, $\lambda = 1/4$ and (i) $\alpha = 0.1$ (dashed-dot green line), (ii) $\alpha = 0.4$ (solid blue line) and (iii) $\alpha = 0.7$ (dashed red line).The solutions shown in Fig. 3(b) are obtained from those shown in Fig. 3(a) through the transformations (17).

of the original equation (3a). If $E < 0$ and $g_0 < 0$, we are in the first case, Fig. 1(a). The periodic solution (28c) of Eq. (25) is a closed orbit of the phase portrait $(U, dU/dX)$, corresponding to one of the external closed orbits to the homoclinic orbits, shown in Fig. 1(a). By an elementary application of L'Hopital rule, it is easy to verify that the solution

$$\phi(x) = b^{1/2}(x)U_3(X(x)), \tag{39}$$

is a homoclinic orbit to zero (bright soliton) of the original equation (3a).

If $E > 0$ and $g_0 < 0$, we are in the four case, where $U = 0$ is a center, Fig. 1(d). Again, $\phi(x) = b^{1/2}(x)U_3(X(x))$ is a homoclinic orbit (bright soliton). The solutions of Eq. (3a) for $E = 0.15$, $g_0 = -1$ and $E = -0.75$, $g_0 = -1$ are drawn in Fig. 2 (b). The case $E = 1/4$, where $V(x)$ is a constant, was studied in [4].

Example 2. Let us take $V(x) = 0$. Then Eq. (30) becomes $b'''(x) + 4\lambda b'(x) = 0$. For $\lambda > 0$, the solution can be written as

$$b(x) = 1 + \alpha \cos(2\sqrt{\lambda}x). \tag{40}$$

Using Eq. (29), we obtain a periodic nonlinearity

$$g(x) = g_0(1 + \alpha \cos(2\sqrt{\lambda}x))^{-3}. \tag{41}$$

For small α, this nonlinearity is approximately harmonic

$$g(x) \simeq g_0(1 - 3\alpha \cos(2\sqrt{\lambda}x)), \quad \alpha \ll 1. \tag{42}$$

We can construct our canonical transformation by using Eqs. (17) and obtain

$$X(x) = \frac{1}{\sqrt{\lambda(1 - \alpha^2)}} \arctan\left(\sqrt{\frac{1 - \alpha}{1 + \alpha}} \tan(\sqrt{\lambda}x)\right). \tag{43}$$

Using any solution of Eq. (25) with $E = \lambda\left(1 - \alpha^2\right)$ this transformation provides solutions of Eq. (3a) with $g(x)$ given by (41). For example, when $g_0 > 0$ we can use U_2 as defined by (28b), which in the phase portrait is the heteroclinic orbit shown

in Fig. 1(c). So, the solution of Eq. (3a) is of the form

$$\phi(x) = \sqrt{\frac{\lambda(1-\alpha^2)}{g_0}\left(1 + \alpha\cos(2\sqrt{\lambda})x\right)} \times \tanh\left[\sqrt{\frac{\lambda(1-\alpha^2)}{2}}\,X(x)\right]. \qquad (44)$$

As we can see in Fig. 3(b), $\phi(x)$ is also a heteroclinic orbit for Eq. (3a). It is important to note that in the asymptotic regions the profile of $\phi(x)$ is close to $b^{1/2}(x)$ multiplied by a constant. Therefore, the canonical transformation (17), in this case, transforms a heteroclinic orbit in the phase portrait $(U, dU/dX)$ into other heteroclinic orbit, Eq. (44). On the other hand, any closed orbit U inside the heteroclinic loop of the phase portrait $(U, dU/dX)$ provides a new heteroclinic solution of the original equation (3a).

Example 3. The last example is the so-called quasi-harmonic confinement $V(x) \sim x^2$.

If we choose $b(x) = \alpha/\sqrt{1 + \beta x^2}$, with $\alpha, \beta > 0$, then, we obtain

$$g(x) = \frac{g_0}{\alpha^3}(1 + \beta x^2)^{3/2}, \qquad (45)$$

and

$$V(x) = M(1 + \beta x^2) + \frac{1}{4}\frac{3\beta x^2 - 2\beta + 4\lambda + 8\lambda\beta x^2 + 4\lambda\beta^2 x^4}{(1 + \beta x^2)^2}, \qquad (46)$$

with M a positive constant. Although the expression of $V(x)$ is complicated this potential $V(x)$ is a quasi-harmonic potential and satisfies $V(x) \sim x^2$ for large x. Moreover $V(x)$ is a harmonic potential with a bounded perturbative term (see Fig. 4(a)). As to the nonlinear term, it satisfies, $g(x) \sim x^2$ for $|x| \ll 1$, and $g(x) \sim x^3$ for $|x| \gg 1$. Using Eq. (24) we get $E = -\alpha^2 M$. Taking $g_0 < 0$, we obtain the nonlinear Schrödinger equation with nonlinear attractive term, Eq. (25). As $E < 0$ and $g_0 < 0$, all the solutions of Eq. (25) are bounded, (see Fig. 1(a)). If $U(X)$ is one of these solutions, it is clear that

$$\phi(x) = b(x)^{1/2}U(X(x)), \qquad (47)$$

is a homoclinic orbit (bright soliton) of the original equation. In particular, the solution given by Eq. (28a) is

$$U_1(X) = \sqrt{\frac{2E}{g_0}}\frac{1}{\cosh(\sqrt{|E|}X)}. \qquad (48)$$

As $X(x) = x\sqrt{1 + \beta x^2}/(2\alpha) + \sinh^{-1}(\sqrt{\beta}x)/(2\alpha\sqrt{\beta})$, we get

$$\phi(x) = b(x)^{1/2}U_1(X(x)). \qquad (49)$$

In Fig. 4(b), we draw the solutions of Eq. (3a) for different values of the parameter β.

In this example, we have used the solution $U_1(X)$ of Eq. (25). Another possibility is to choose the closed periodic orbits inside the homoclinic loop (see Fig. 1(a)) given by $U_4(X)$ in Eq. (28d). We note that, for this case, Eq. (27) satisfies $H \leq 0$, as one can see in Table 1.

The analytical expression of the closed periodic orbits outside the homoclinic loop is given by $U_3(X)$, with $k > 1/\sqrt{2}$. Such orbits satisfy $H > 0$ (see Table 1).

As X is a bijective map on the real line, $b(x)$ is a positive function and $U_3(X)$ is a periodic function with infinite nodes on the real line, the function $\phi(x) = b(x)^{1/2}U_3(X)$ also has infinite nodes on the real line, as it is shown in Fig. 5. It is
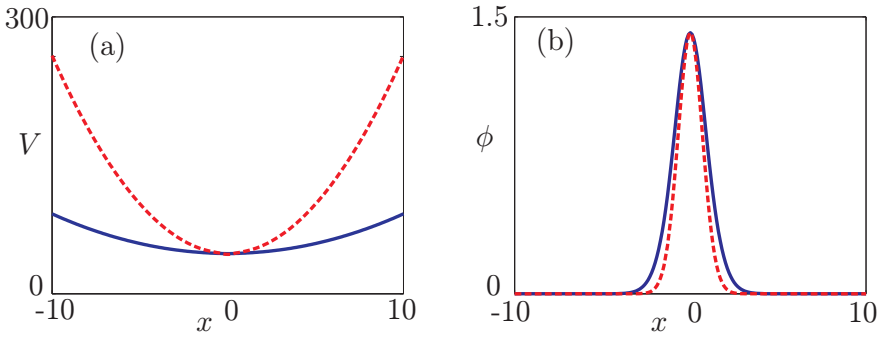
FIGURE 4. [Color online] (a) Quasi-harmonic potential for $M = 1$, $\lambda = 1$ and (i) $\beta = 0.5$ (solid blue line) and (ii) $\beta = 2.5$ (dashed red line). (b) Solutions of Eq. (3a) for $\alpha = 1$, $g_0 = -1$, $M = 1$ and (i) $\beta = 0.5$ (solid blue line) and (ii) $\beta = 2.5$ (dashed red line)
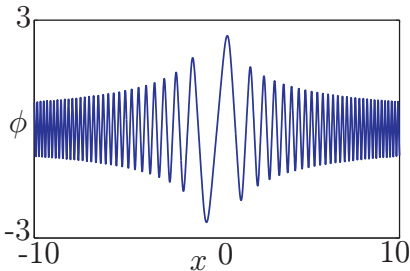

FIGURE 5. Solution of Eq. (3a) with infinite nodes for $\alpha = 1$, $\beta = 2.5$, $g_0 = -1$ and $k = 3/4$

immediate to check that the solution $\phi \to 0$ when $x \to \pm\infty$. Moreover, the zeroes of ϕ accumulate for large values of x, since the distance between two consecutive zeroes is given by $x_{n+1} - x_n \sim \sqrt{n+1} - \sqrt{n}$. Other localized solutions with infinite nodes have been studied in a different context in Ref. [7].

6. **Asymmetric modes of the INLSE.** In the previous section, we have explored the case $C = 0$ in the canonical transformation (17). In this case, the original equation (3a) is reduced to a Nonlinear Schrödinger Equation. In this section, we will study the case $C > 0$. If we take $g_0 < 0$ and $E > 0$, the resulting equation is

$$\frac{d^2U}{dX^2} + 2C\frac{dU}{dX} + |g_0|U^3 + EU = 0. \tag{50}$$

In general, this equation is not integrable and the energy is not a conserved quantity. However, exact solutions of Eq. (50) can be constructed analytically in particular cases. An exact integrability condition was given in [22]
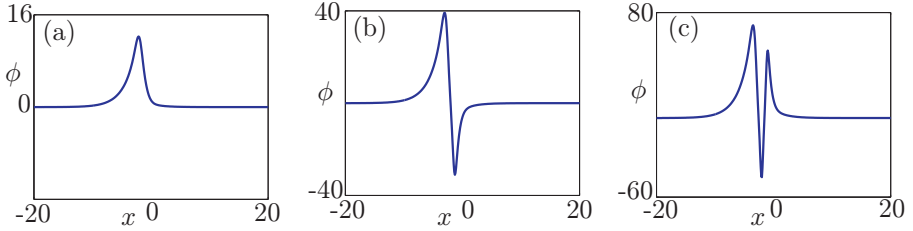
$$E = \frac{8}{9}C^2. \tag{51}$$

FIGURE 6. Asymmetric solutions of Eq. (3a) with $C = 4$ for (a) $n = 1$, (b) $n = 2$ and (c) $n = 3$.

In that case a family of exact analytical solutions of Eq. (50) is given by the expression

$$U_n(X) = \frac{\mu_n}{\sqrt{2|g_0|}} e^{-BX} \frac{\operatorname{sn}\left(\frac{\mu_n}{B}(1 - e^{-BX}), \sqrt{2}/2\right)}{\operatorname{dn}\left(\frac{\mu_n}{B}(1 - e^{-BX}), \sqrt{2}/2\right)}, \quad n = 1, 2, 3, \ldots \quad (52)$$

where μ_n and B are related to the boundary conditions of the problem .

We are going to solve Eq. (3a), using the solutions of Eq. (50). Choosing $b(x) = \cosh(x)$ and using Eqs. (30) and (24), we can calculate the potential $V(x)$:

$$V(x) = \lambda + 1/4 + \left(\frac{1}{4} + \frac{C^2}{9}\right) \frac{1}{\cosh^2(x)}. \quad (53)$$

The nonlinear term is

$$g(x) = g_0 \cosh^{-3}(x) e^{-2CX(x)}. \quad (54)$$

So, we can calculate the solutions of Eq. (3a) for the case $C \neq 0$ and to compare them with the solution obtained for the case $C = 0$, example 1. In this way, we can construct our canonical transformation by using Eqs. (17) and obtain

$$\cos X(x) = -\tanh x. \quad (55)$$

Then, $0 \leq X \leq \pi$ and using the boundary conditions for ϕ, $\phi(\pm\infty) = 0$, one has to impose $U(0) = U(\pi) = 0$.

Using these boundary conditions, we obtain the value of the amplitude μ_n as a function of the value of the integer n

$$\mu_n = \frac{4CK(\sqrt{2}/2)}{3(1 - e^{-2C\pi/3})} n, \quad (56)$$

and $B = 2C/3$. Thus, the solutions of Eq. (50) are

$$U_n(X) = \frac{\mu_n}{\sqrt{2|g_0|}} e^{-2CX/3} \frac{\operatorname{sn}\left(2nK(\sqrt{2}/2)(1 - e^{-2CX/3})/(1 - e^{-2C\pi/3}), \sqrt{2}/2\right)}{\operatorname{dn}\left(2nK(\sqrt{2}/2)(1 - e^{-2CX/3})/(1 - e^{-2C\pi/3}), \sqrt{2}/2\right)}, \quad (57)$$

$$n = 1, 2, 3, \ldots$$

where $K(k)$ is the elliptic integral of the first kind,

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}. \quad (58)$$

Then, the solutions of Eq. (3a) are

$$\phi_n(x) = b^{1/2}(x) e^{CX(x)} U_n(X(x)), \quad n = 1, 2, \ldots \quad (59)$$

By using L'Hopital's rule, $\phi_n(x) \to 0$ when $|x| \to \infty$ in (59). So, the solutions (59) are localized solutions of our problem as it can be seen in Fig. 6. These solutions are asymmetric solutions of Eq. (3a). Moreover, each of those solutions has exactly $n - 1$ zeroes. In Fig. 6, we plot some of them corresponding to $n = 1, 2, 3$. The picture in Fig. 6(a) shows clearly the difference between the positive solution given by (59), for $C \neq 0$, and the positive solution plotted in Fig. 2(b) and given by Eq. (39), for $C = 0$.

7. **Conclusions.** In this paper, we have used the method of Lie symmetries to find exact solutions of the INLSE. We have introduced the general framework of the Lie's theory and presented different examples as application to the theory. By using the qualitative theory of the dynamical systems, we can show the properties of the solutions of the INLSE and to classify such solutions. Finally, we have calculated asymetric solitons of the inhomogeneous nonlinear Schrödinger equation.

REFERENCES

[1] F. K. ABDULLAEV AND J. GARNIER, *Propagation of matter-wave solitons in periodic and random nonlinear potentials*, Phys. Rev. A, **72**, 061605, (2005).

[2] D. BAMBUSI, A. CARATI AND A. PONNO, *The nonlinear Schrödinger equation as a resonant normal form*, Discrete and Continuous Dynamical Systems - Series B, **2**, (2002) 109-128.

[3] G. BAUMANN AND T.F. NONNENMACHER, *Lie transformations, similarity reduction, and solutions for the nonlinear Madelung fluid equations with external potential*, J. Math. Phys., **28**, 6, (1987).

[4] J. BELMONTE-BEITIA, V. M. PÉREZ-GARCÍA, V. VEKSLERCHIK AND P. J. TORRES, *Lie symmetries and solitons in nonlinear systems with spatially inhomogeneous nonlinearities*, Phys. Rev. Lett., **98**, 064102, (2007).

[5] G. W. BLUMAN AND S. KUMEI, *Symmetries and Differential Equations*, Springer-Verlag, New York, (1989).

[6] F. BREZZI, P. A. MARKOWICH, *The three-dimensional Wigner-Poisson problem: existence, uniqueness and approximation*, Math. Meth. in Appl. Sci., **14**, 35-61, (1991).

[7] A. V. BURYAK AND N. N. AKHMEDIEV, *Stability criterion for stationary bound states of solitons with radiationless osciolating tails*, Phys. Rev. E, **51**, 4, (1995).

[8] F. DALFOVO, S. GIORGINI, L. P. PITAEVSKII, S. STRINGARI, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys., **71**, 463-512, (1999).

[9] A.S. DAVYDOV, *Solitons in Molecular Systems*, Reidel, Dordrecht, (1985).

[10] R.K. DODD, J.C. EILBECK, J.D. GIBBON, H.C. MORRIS, *Solitons and nonlinear wave equations*, Academic Press, New York, (1982).

[11] G. DONG, AND B. HU, *Management of Bose-Einstein condensates by a spatially periodic modulation of the atomic s-wave scattering length*, Phys. Rev. A. **75**, 013625 (2007).

[12] V. P. ERMAKOV, Univ. Izv. Kiev., **20**, 1-19, (1880).

[13] R. FEDELE, G. MIELE, L. PALUMBO, V. G. VACCARO, *Thermal wave model for nonlinear longitudinal dynamics in particle accelerators*, Phys. Lett. A, **173**, 407-413, (1993).

[14] Z. FENG, G. CHEN AND S-B. HSU, *A qualitative study of the damped Duffing equation and applications*, Discrete and Continuous Dynamical Systems - Series B, **6**, (2006) 1097-1112.

[15] G. FIBICH, G. PAPANICOLAU, *Self-focusing in the perturbed and unperturbed nonlinear Schrödinger equation in critical dimension*, SIAM J. Appl. Math., **60**, 183-240, (1999).

[16] J. GARNIER AND F. K. ABDULLAEV, *Transmission of matter-wave solitons through nonlinear traps and barriers*, Phys. Rev. A, **74**, 013604, (2006).

[17] A. HASEGAWA, *Optical Solitons in Fibers*, Springer-Verlag, Berlin, (1989).

[18] Y. KIVSHAR, G. P. AGRAWAL, *Optical Solitons: From fibers to Photonic crystals*, Academic Press, (2003).

[19] P. G. L. Leach, *An exact invariant for a class of time-dependent anharmonic oscillators with cubic anharmonicity*, J. Math. Phys., **22**, 3, (1981).

[20] P. G. L. Leach and S. D. Maharaj, *A first integral for a class of time-dependent anharmonic oscillators with multiple anharmonicities*, J. Math. Phys., **33**, 6, (1992).

[21] J. L. López, J. Soler, *Asymptotic behaviour to the 3D Schrödinger/Hartree-Poisson and Wigner-Poisson systems*, Math. Meth. in App. Sci., **10**, 923-943, (2000).

[22] D. Meimukhin, O.V. Gendelman, *Response regimes of integrable damped strongly nonlinear oscillator under impact periodic forcing*, Chaos, Solitons and Fractals, **32**, 405-414, (2007).

[23] P. J. Olver, *Applications of Lie groups to differential equations*, Springer, (1993).

[24] E. Pinney, *The nonlinear differential equation $y'' + p(x)y + cy^{-3} = 0$,* Proc. Amer. Math. Soc., **1**, (5), 681-681, (1950).

[25] M. A. Porter, P. G. Kevrekidis, B. A. Malomed, D. J. Frantzeskakis, *Modulated amplitude waves in collisionally inhomogenous Bose-Einstein condensates*, preprint `nlin.PS/0607009`.

[26] M. T. Primatarowa, K. T. Stoychev, R. S. Kamburova, *Interactions of solitons with extended nonlinear defects*, Phys. Rev. E, **72**, 036608, (2005).

[27] M.I. Rodas-Verde, H. Michinel and V.M. Pérez-García, *Controllable soliton emission from a Bose-Einstein condensate,* Phys. Rev. Lett., **95**, 15, 153903, (2005).

[28] J. L. Rosales, J. L. Sánchez-Gómez, *Nonlinear Schödinger equation coming from the action of the particles gravitational field on the quantum potential*, Phys. Lett. A, **66**, 111-115, (1992).

[29] H. Sakaguchi and B. Malomed, *Two-dimensional solitons in the Gross-Pitaevskii equation with spatially modulated nonlinearity*, Phys. Rev. E, **73**, 026601, (2006).

[30] A. Scott, *Nonlinear Science: Emergence and dynamics of coherent structures*, Oxford Appl. and Eng. Mathematics, Vol. 1, Oxford, (1999).

[31] C. Sulem, P. Sulem, *The nonlinear Schrödinger equation: Self-focusing and wave collapse*, Springer, Berlin, (2000).

[32] G. Teocharis, P. Schmelcher, P.G. Kevrekidis, and D.J. Frantzeskakis, *Matter-wave solitons of collisionally inhomogeneous condensates*, Phys. Rev. A, **72**, 033614, (2005).

[33] L. Vázquez, L. Streit, V. M. Pérez-García, Eds., *Nonlinear Klein-Gordon and Schrödinger systems: Theory and Applications*, World Scientific, Singapur, (1997).

[34] A. Vázquez-Carpentier, H. Michinel, M.I. Rodas-Verde and V.M. Pérez-García, *Analysis of an atom soliton laser based on the spatial control of the scattering length,* Phys. Rev. A, **74**, (1), 053610, (2006).

[35] V. Vekslerchik and V.M. Pérez-García, *Exact solution of the two-mode model of multicomponent Bose-Einstein condensates*, Discrete and Continuous Dynamical Systems - Series B, **3**, (2003) 179-192.

[36] V. E. Zaharov, V. S. L'vov, S. S. Starobinets, *Spin-wave turbulence beyond the parametric excitation threshold*, Sov. Phys. Usp., **17**, 6, 896-919, (1975).

E-mail address: `juan.belmonte@uclm.es`

E-mail address: `victor.perezgarcia@uclm.es`

E-mail address: `vadym@ind-cr.uclm.es`

E-mail address: `ptorres@ugr.es`