GENERALIZED NONLINEAR SUPERPOSITION PRINCIPLES *

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We study the possible existence of first integrals of the form $I(x, y) = (y - g_1(x))^{\alpha_1}(y-g_2(x))^{\alpha_2} \cdots (y-g_\ell(x))^{\alpha_\ell} h(x)$, where $g_1(x), \ldots, g_\ell(x)$ are unknown particular solutions of $dy/dx = Q(x, y)/P(x, y)$, α_i are unknown constants and $h(x)$ is an unknown function. For certain systems some of the particular solutions remain arbitrary and the other ones are explicitly determined or are functionally related to the arbitrary particular solutions. We obtain in this way a nonlinear superposition principle that generalizes the classical nonlinear superposition principle of the Lie theory. In general, the first integral contains some arbitrary solutions of the system but also quadratures of these solutions and an explicit dependence on the independent variable, see [1].

1. Introduction

We consider in this paper two–dimensional systems

$$\frac{dx}{dt} = \dot{x} = P(x, y) \ , \frac{dy}{dt} = \dot{y} = Q(x, y) \ , \tag{1}$$

in which $P, Q \in \mathbb{R}[x, y]$ are polynomials in the real variables x and y and the independent variable (the time) t is real. Throughout this paper we

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will denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1). Obviously, we can also express system (1) as the differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \ . \tag{2}$$

We associate to system (1) the vector field $\mathcal{X}$ defined by $\mathcal{X} = P\partial/\partial x + Q\partial/\partial y$. H is a first integral of the system (1) on U if and only if $\mathcal{X}H \equiv 0$ on U. We are lead to the problem of characterizing the systems of differential equations for which a superposition function, allowing to express the general solution in terms of a certain finite number of particular solutions, does exist. As it is well known, this problem has been studied by Lie [2]. Let $\Sigma = \{g_1(x), \ldots, g_n(x)\}$ be a set of particular solutions of equation (2). Then $F(y, g_1(x), \ldots, g_n(x))$ is defined as a *connecting function* of (2) if $F = 0$ is also an implicitly defined particular solution. Formally, a nonlinear superposition principle is an operation $F : \mathbb{R} \times \mathcal{F}^n \to \mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ are function spaces such that the former properties hold.

Moreover, we will say that Σ is a *fundamental set of solutions* of (2) if a connecting function F exists, such that F is a first integral or equivalently $F = c$ is the general solution of equation (2), where c is an arbitrary constant. The standard example of nonlinear first order differential equation with a fundamental set of solutions is the Riccati equation $dy/dx = A_0(x) + A_1(x)y + A_2(x)y^2$ for which the general solution is given by the cross ratio

$$F(y, g_1(x), g_2(x), g_3(x)) = \frac{(y - g_1(x))(g_3(x) - g_2(x))}{(y - g_2(x))(g_3(x) - g_1(x))} = c,$$

where c is an arbitrary constant. It follows from the work of Lie and Scheffers [2] that the real equation (2) with n arbitrary particular solutions defining a fundamental set of solutions is associated with finite dimensional Lie algebras of vector fields on $\mathbb{R}$. In fact, Lie showed that there is a fundamental set of n arbitrary solutions for the differential equation (2) if and only if it can be written in the form $\mathrm{d}y/\mathrm{d}x = \sum_{i=0}^{s} A_i(x)B_i(y)$, where the vector fields $\mathcal{X}_i = B_i(y)\partial/\partial y$ with $i = 0, 1, \ldots, s$, generate an r-dimensional Lie algebra with $s + 1 \leq r \leq n$. Moreover, the notion of a fundamental set of solutions developed by Lie is extremely restrictive as can be seen from the following theorem proved by Lie [2].

Theorem 1.1. *The only ordinary differential equations of the form $dy/dx = f(x,y)$, with $f \in \mathcal{C}^1$, allowing a fundamental set of arbitrary* *solutions are the Riccati equation* $dy/dx = A_0(x) + A_1(x)y + A_2(x)y^2$ *and any equation obtained from it by a change of dependent and independent variables of the form* $\psi = \psi(y)$, $\tau = \tau(x)$.

In the expression of the first integral given by the Lie theory, the particular solutions $g_i(x)$ of the fundamental set are arbitrary. The natural question is: are there intermediate cases of nonlinear superposition principles for which some of the particular solutions remain arbitrary and the others are explicitly determined ? To answer this question we introduce the following ansatz for the first integral $I(x,y) = (y - g_1(x))^{\alpha_1}(y - g_2(x))^{\alpha_2} \ldots (y - g_\ell(x))^{\alpha_\ell} h(x)$, where $g_j(x)$ are unknown particular solutions of (2), $h(x)$ is an unknown function of x and the α_i are unknown constants such that $\prod_{i=1}^{\ell} \alpha_i \neq 0$, in order to detect first integrals of (1).

Example. Let us consider the following polynomial differential system $\dot{x} = -y + x^4$, $\dot{y} = x$. The equation for the orbits is

$$\frac{dx}{dy} = \frac{-y + x^4}{x} \ . \tag{3}$$

We propose the following first integral $I(x,y) = (x - g_1(y))^{\alpha_1}(x - g_2(y))^{\alpha_2}(x - g_3(y))^{\alpha_3}(x - g_4(y))^{\alpha_4} h(y)$, where the functions $g_i(y)$ for $i = 1, \ldots, 4$ are particular solutions for equation (3). Applying the method we obtain that $g_2(y) = -g_1(y)$, $g_4(y) = -g_3(y)$, $\alpha_2 = \alpha_1$, $\alpha_3 = \alpha_4 = -\alpha_1$ and $h(y) = e^{2\alpha_1 \int (g_3^2(y) - g_1^2(y))dy}$. Therefore, a generalized nonlinear superposition principle is given by the first integral

$$I(x,y) = \frac{(x^2 - g_1^2(y))}{(x^2 - g_3^2(y))} e^{2\int (g_3^2(y) - g_1^2(y))dy} \ ,$$

where $g_1(y)$ and $g_3(y)$ are arbitrary particular solutions of equation (3). This nonlinear superposition principle is not a particular case of the Lie theory, because the expression of the first integral contains quadratures of the particular solutions $g_1(y)$ and $g_3(y)$.

References

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