Notes on Topology

by

John L. Bell

Contents

0

Set-Theoretical Background

SINCE THE NOTION OF A *SET* is essential to everything we shall do here, we begin with some basic set-theoretic facts. Intuitively, a *set* is a well-defined collection of distinct objects: for example, the sets $\mathbb{N}$ of all natural numbers, Q of all rational numbers and $\mathbb{R}$ of all real numbers. If A is a set and a an object we write

$$a \in A, a \notin A$$

for "a is a *member* or *element* of A, or *belongs to* A" and "a is not a member of A", respectively. We shall regard a set as being uniquely determined by its members: thus, if two sets A and B have the same members, they are identical. This assertion is known as the *axiom of extensionality* and may be expressed formally by

$$A = B \Leftrightarrow \forall x[x \in A \Leftrightarrow x \in B].$$

A collection of sets is often called a *family* of sets.

We shall use the brace notation to form sets from properties: thus, if Px is the assertion that the object x has the property P,

$$\{x: Px\}$$

will denote the set of all objects x such that Px. For example, if N is the property of being a natural number, then

$$\mathbb{N} = \{x: Nx\}.$$

Evidently, for any object a we have

$$a \in \{x: Px\} \Leftrightarrow Pa.$$

Given a set A, we write $\{x \in A: Px\}$ for $\{x: x \in A \ \& \ Px\}$. For objects $a_1,...,a_n$, we define

$$\{a_1,...,a_n\} = \{x: x = a_1 \text{ or } x = a_2 \text{ or } ... \ x = a_n\};$$

thus $\{a_1,...,a_n\}$ is the set whose members are precisely the objects $a_1,...,a_n$. In particular, $\{a\}$ is the set —the *singleton* of a —whose sole member is a.

We extend the brace device for set formation to the following situation. Suppose that for each object x such that Px there is assigned a unique object x^*. Then $\{x^*: Px\}$ will denote the set of all x^* such that Px. That is,

$$\{x^*: Px\} = \{y: \exists x[y = x^* \ \& \ Px]\}.$$

The set A is said to be a *subset* of the set B, written $A \subseteq B$, if every member of A is also a member of B. If A is a subset of B we shall also say that A is *included* or *contained* in B. A subset of a family of sets will be called a *subfamily.* Clearly

$$A = B \Leftrightarrow A \subseteq B \ \& \ B \subseteq A.$$

We define the *power set* $\mathbf{P}A$ of a set A to be the family of all subsets of A: $\mathbf{P}A = \{X: X \subseteq A\}$.

The *union* of the sets A and B, written $A \cup B$, is the collection of objects which belong to A or B or both, that is:

$$A \cup B = \{x\!: x \text{ 0 } A \text{ or } x \text{ 0 } B\}.$$

The *intersection* of A and B, written $A \cap B$, is the collection of objects which belong both to A and to B:

$$A \cap B = \{x\!: x \in A \;\&\; x \in B\}.$$

The *empty set*—the set with no members—is denoted by $\varnothing$, and may be defined by

$$\varnothing = \{x\!: x \neq x\}.$$

The empty set may be regarded as a subset of *any* set: for, if A is an arbitrary set, then we have, since $x \notin \varnothing$ for every x,

$$\text{œ}x[x \notin A \Rightarrow x \notin \varnothing],$$

whence

$$\text{œ}x[x \in \varnothing \Rightarrow x \in A],$$

so that $\varnothing \subseteq A$.

The sets A and B are said to be *disjoint* if their intersection is empty, i.e., if $A \cap B = \varnothing$. In the contrary case, A is said to *meet* or *intersect* B.

The (relative) *complement* $B - A$ of a set A in a set B is the set of objects which belong to B but not to A: that is,

$$B - A = \{x\!: x \in B \;\&\; x \notin A\}.$$

The notions of union and intersection, so far defined just for pairs of sets, is easily extended to arbitrary finite (nonempty) families by defining, for sets $A_1, \ldots, A_n$,

$$A_1 \cup \ldots \cup A_n = \{x\!: x \in A_i \text{ for some } i = 1, \ldots, n\}$$

$$A_1 \cap \ldots \cap A_n = \{x\!: x \in A_i \text{ for all } i = 1, \ldots, n\}.$$

We shall also need to consider unions and intersections of families of sets which are *not necessarily finite*. Suppose that for each member i of a fixed set I (which we shall call an *index set*) we are given a set X_i. Then the *union* $\bigcup_{i \in I} X_i$ and the *intersection* $\bigcap_{i \in I} X_i$ of the *indexed family* of sets $\{X_i\!: i \in I\}$ are defined by

$$\bigcup_{i \in I} X_i = \{x : \exists i \in I.\ x \in X_i\} \qquad \bigcap_{i \in I} X_i = \{x : \forall i \in I.\ x \in X_i\}$$

An important special case of this notion arises when the index set is itself a family $\mathcal{A}$ of sets and X_A is the set A for each A 0 $\mathcal{A}$.. In this situation the union of the family $\{X_A\!: A \in \mathcal{A}\}$ is written $\bigcup\mathcal{A}$ and the intersection $\bigcap\mathcal{A}$:

$$\bigcup\mathcal{A} = \{x\!: \exists A \in \mathcal{A}(x \text{ 0 } A)\} \quad \bigcap\mathcal{A} = \{x\!: \forall A \in \mathcal{A}(x \in A)\}.$$

A *cover* of a set X is a family $\mathcal{A}$ of sets such that $X \subseteq \bigcup\mathcal{A}$; in this situation $\mathcal{A}$ is said to *cover* X.

We tabulate some of the basic facts about set-theoretical operations in the following

0.1. Proposition. Let A, B, C, X be sets, $\{X_i\!: i \in I\}$, $\{Y_j\!: j \in J\}$ indexed families of sets. Then

(i) $X - (X - A) = A \cap X;$

(ii) $A \cup B = B \cup A,\ \ A \cap B = B \cup A;$

(iii) $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C\,) = (A \cap B) \cap C,$

(iv) $X - (A \cup B) = (X - A) \cap (X - B),\ X - (A \cap B) = (X - A) \cup (X - B)$;

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C),\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

(vi) $X - \bigcup_{j \in J} Y_i = \bigcap_{j \in J} (X - Y_j) \qquad X - \bigcap_{j \in J} Y_j = \bigcup_{j \in J} (X - Y_j)$

(vii) $\bigcup_{i \in I} X_i \cap \bigcup_{j \in J} Y_j = \bigcup_{i \in I} \bigcup_{j \in J} X_i \cap Y_j \qquad \bigcap_{i \in I} X_i \cup \bigcap_{j \in J} Y_j = \bigcap_{i \in I} \bigcap_{j \in J} X_i \cup Y_j$.

Given two objects a, b, we assume that we can form a new object (a, b) called the *ordered pair* of a and b, which has the following characteristic property: for any objects u,v,

$$(a, b) = <u, v> \Leftrightarrow a = u \ \& \ b = v.$$

a and b are called the *first* and *second coordinates*, respectively, of the ordered pair (a, b). It is now easy to define ordered triples, quadruples, etc., by recursion. Thus $(a, b, c) = ((a, b), c)$ and in general

$$(a_1, \dots, a_n) = ((a_1, \dots, a_{n-1}), a_n).$$

Clearly

$$(a_1, \dots, a_n) = (b_1, \dots, b_n) \Leftrightarrow a_1 = b_1 \ \& \ \dots \ \& \ a_n = b_n.$$

Let $\{A_1, \dots, A_n\}$ be a finite family of sets. We define the (*Cartesian*) *product* $A_1 \times \dots \times A_n$ of the family to be the set of all n-tuples $(a_1, \dots, a_n)$ with $a_i \in A_i$ for each $i = 1, \dots, n$. If it should happen that each A_i is identical with some fixed set A, we write A^n for $A \times \dots \times A$. (Thus, for instance, $\mathbb{R}^n$ is Euclidean n-space.)

A set of ordered pairs is called a (binary) relation. If R is a relation, we often write xRy for $(x, y) \in R$. We define the *domain* dom(R) of R to be the set of first coordinates of members of R and the *range* ran(R) of R to be the set of second coordinates of members of R. That is,

$$\mathrm{dom}(R) = \{x: \exists y.xRy\}, \quad \mathrm{ran}(R) = \{y: \exists x.xRy\}.$$

The *composition* $R \circ S$ of two relations R and S is defined by

$$R \circ S = \{(x, z): \exists y.xSy \ \& \ yRz\}.$$

The *inverse* $R^{/1}$ of a relation R is defined by

$$R^{/1} = \{(x, y): yRx\}.$$

It is easy to see that, for any relations R, S,

$$(R^{/1})^{!1} = R, \quad (R \circ S)^{!1} = S^{!1} \circ R^{!1}.$$

A *function, map, or mapping* is a relation f with the property that for each $x \in \mathrm{dom}(f)$ there is a *unique* $y \in \mathrm{ran}(f)$ such that $(x, y) \in f$. This unique y is called the *value* of f at x, or the *image* of x under f, and is denoted by $f(x)$. We sometimes write $x \mapsto f(x)$ for f. Suppose now that A and B are sets and f is a function such that dom(f) = A and ran(f) $\subseteq B$. In this event we say that f is a function *from A to B*, or that *f maps A into B*, and write

$$f: A \to B.$$

In this event A is the *domain* of f and B is sometimes called the *codomain* of f. If, further, ran(f) = B, we say that f maps *A onto B*, or is a *surjection,* or *surjective.* The function f is said to be *one-one, injective,* or an *injection*, if distinct members of A have distinct images under f, i.e. if

$$x, y \in A \;\&\; f(x) \neq f(y) \Rightarrow x \neq y,$$

or, equivalently, if

$$x, y \in A \;\&\; f(x) = f(y) \Rightarrow x = y.$$

It is not hard to see that the inverse f^{-1} of f is a function from ran(f) onto A if and only f is injective. A function that is both injective and surjective is said to be *bijective.* It is readily shown that a function f: $A \to B$ is bijective iff its inverse f^{-1} is a function from B to A.

Given sets $A_1, \ldots, A_n$, for each $i = 1, \ldots, n$ we define the i^{th} *projection* π_i: $A_1 \times .. \times A_n \to A_i$ to be the map $(a_1, \ldots, a_n) \mapsto a_i$.

A set X is said to be *finite* if, for some $n \in \mathbb{N}$, there exists a bijection from X to the set $\{0, 1, 2, \ldots, n\}$. In the contrary case X is said to be *infinite*. X is said to be *countable* if it is finite or there is a bijection from X to the set $\mathbb{N}$ of natural numbers. In that case X can be presented as a set $\{x_n: n \in \mathbb{N}\}$ indexed by $\mathbb{N}$. If X is not countable, it is said to be *uncountable.* It can be shown that $\mathbb{Q}$ is countable and $\mathbb{R}$ uncountable. In the sequel we shall occasionally use the fact that the union of a countable family of countable sets is countable.

The *identity map* on a set A is the function $1_A = \{(x, x): x \in A\}$. Thus $1_A(x) = x$ for all $x \in A$.

If $f: A \to B$ and $X \subseteq A$, we define the *restriction* $f|X$ of f to X by $f|X = f \cap (X \times B)$. It is evident that $f|X$ is a function from X to B and that $f|X(x) = f(x)$ for each $x \in X$. Let $f: A \to B$; for each subset X of A we define the *image* $f[X]$ of X under f by

$$f[X] = \{f(x): x \in X\}.$$

Clearly, for each $x \in A$ we have

$$f[\{x\}] = \{f(x)\}.$$

Furthermore, for each $y \in B$ we define the *preimage* $f^{-1}(y)$ of y under f by

$$f^{-1}(y) = \{x \in A: f(x) = y\}.$$

Evidently

$$f^{-1}(y) \neq \varnothing \Leftrightarrow y \in \text{ran}(f).$$

If $Y \subseteq B$ we define the *preimage* $f^{-1}[Y]$ of Y under f by

$$f^{-1}[Y] = \{x \in A: f(x) \in Y\}.$$

Suppose now that $f: A \to B$ and $g: B \to C$. Then the composition $g \circ f$ is a function from A to C, and, for any $x \in A$, $(g \circ f)(x) = g(f(x))$. Moreover, for any $Z \subseteq C$,

$$(g \circ f)^{-1}[Z] = f^{-1}[g^{-1}[Z]].$$

We compile our final remarks on functions in the form of a proposition whose proof can be safely entrusted to readers (if any).

0.2. Proposition. Let A, B be sets, let $\{X_i: I \in I\}$, $\{Y_j: j \in J\}$ be indexed families of subsets of A and B, respectively, let $X \subseteq A$, $Y \subseteq B$, and finally let f be a function from A to B. Then we have

(i) $f[\bigcup_{i\in I} X_i] = \bigcup_{i\in I} f[X_i]$ **(ii)** $f[\bigcap_{i\in I} X_i] \subseteq \bigcap_{i\in I} f[X_i]$ **(iii)** $f^{-1}[\bigcup_{i\in I} Y_i] = \bigcup_{i\in I} f^{-1}[Y_i]$

(iv) $f^{-1}[\bigcap_{i\in I} Y_i] = \bigcap_{i\in I} f^{-1}[Y_i]$ **(v)** $f^{-1}[B - Y] = A - f^{-1}[Y]$; **(vi)** $f[f^{-1}[Y]] \subseteq Y$;

(vii) $X \subseteq f^{-1}[f[X]]$. ■

We conclude this chapter by defining two special types of relation. A relation R is said to be (*defined*) *on* a set A if $R \subseteq A \times A$. A relation R on A is said to be *reflexive* if aRa for all $a \in A$, *symmetric* if $aRb \Rightarrow bRa$ for all $a, b \in A$, *antisymmetric* if aRb & $bRa \Rightarrow a = b$ for all $a, b \in A$, and *transitive* if aRb & $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

A reflexive, transitive, and symmetric relation R on a set A is called an *equivalence relation* on A . If $a \in A$, the set $R[a] = \{x: aRx\}$ is called the *R-equivalence class* of a. Clearly distinct equivalence classes are disjoint, and, for all $a, b \in A$, $aRb \Leftrightarrow a$ and b belong to the same R-equivalence class.

A reflexive, transitive, and antisymmetric relation R on a set A is called a *partial ordering* on A: in this situation we shall usually write $\leq$ for R and $a \leq b$ for aRb. If $\leq$ is a partial ordering on A, the pair $(A, \leq)$ (or when confusion is unlikely, simply A) is said to be a *partially ordered set.* If for any elements $a, b \in A$ we have either $a \leq b$ or $b \leq a$, then $\leq$ is said to be a *total ordering* and $(A, \leq)$ a *totally ordered set.*

Examples. **1.** The standard "equal to or less than" relations on the sets ℕ of natural numbers, Q of rational numbers, and ℝ of real numbers are each total orderings.

2. The relation "m is a divisor of n" is a partial, but not total, ordering on ℕ.

3. The inclusion relation $\subseteq$ is a partial ordering on any family of sets.

Let $(A, \leq)$ be a partially ordered set and $X \subseteq A$. An element $a \in A$ is said to be an *upper* (*lower*) *bound* for X if $x \leq a$ ($a \leq x$) for all $x \in X$. An upper (lower) bound a for X such that $a \leq b$ ($b \leq a$) for every upper (lower) bound b for X is called a *supremum* (*infimum*) for X: it is clearly unique if it exists. We write sup X (inf X) for the supremum (infimum) of X if it exists. $(A, \leq)$ is said to be *complete* if every nonempty subset with an upper (lower) bound has a supremum (infimum). For example, ℝ with its standard ordering is complete.

1

Topological Structure of the Real Line

ROUGHLY SPEAKING, TOPOLOGY may be regarded as the analysis of concepts and entities which involve the notion of *proximity* or *continuity*: for example, points in n-dimensional space, plane or space curves, continuous real-valued functions, limits of sequences, etc. So our task is to specify a basic *structure* possessing sufficient breadth to enable the characteristic properties of these notions to be expressed. The resulting theory will then be applicable in any context in which these notions figure.

Suppose we are given a set X of elements, which may be such mathematical entities as mentioned above, although in general we make no assumptions as to their exact nature. The basic structure we single out for study will be a family $\mathfrak{T}$ of subsets of X, called *open sets*, which satisfy certain conditions, to be specified presently. A family satisfying these conditions will be called a *topology* on X, and X —or, to be precise, the ordered pair $(X, \mathfrak{T})$ —is called a *topological space*. Members of X will often be called *points.*

In order to motivate the general definition of a topology, we first discuss a very special topological space which lies at the heart of the subject, namely, the *real line*.We shall need to make some preliminary definitions.

1.1. Definition. For $a, b \in \mathbb{R}$ we define:

$$
\begin{aligned}
(a, b) &= \{x \in \mathbb{R} : a < x < b\},\\
[a, b] &= \{x \in \mathbb{R} : a \le x \le b\},\\
[a, b) &= \{x \in \mathbb{R} : a \le x < b\},\\
(a, b] &= \{x \in \mathbb{R} : a < x \le b\},\\
(a, \rightarrow) &= \{x \in \mathbb{R} : a < x\},\\
[a, \rightarrow) &= \{x \in \mathbb{R} : a \le x\},\\
(\leftarrow, a) &= \{x \in \mathbb{R} : x < a\},\\
(\leftarrow, a] &= \{x \in \mathbb{R} : x \le a\}.
\end{aligned}
$$

Subsets of $\mathbb{R}$ of any of the above forms, together with $\varnothing$ and $\mathbb{R}$ itself, are called *intervals*. Subsets of $\mathbb{R}$ of the form (a, b) or $[a, b]$ are called *open* or *closed intervals,* respectively. Intervals of the form $[a, b)$, $(a, b]$, $[a,\rightarrow)$ or $(\leftarrow, a]$ are called *half-open intervals.* It is easily shown that the intervals in $\mathbb{R}$ are precisely the subsets X of $\mathbb{R}$ possessing the *betweenness property*: if $a < b < c$ and $a, c \in X$, then $b \in X$.

1.2. Definition. A subset A of $\mathbb{R}$ is said to be *open* (in $\mathbb{R}$) if, for each $x \in A$, there is an open interval containing x and included in A.

In other words, an open set is a *union* of open intervals. Notice that in particular the empty set is open.

1.3. Proposition. In $\mathbb{R}$,

(i) the union of any family of open sets is open;

(ii) the intersection of any *finite* (nonempty) family of open sets is open;

(iii) $\mathbb{R}$ and $\varnothing$ are both open sets.

Proof. **(i)** and **(iii)** are obvious; **(ii)** follows from the fact that the intersection of finitely many open intervals is easily seen to be an open interval. ■

Despite its straightforward character, this proposition should be taken very seriously, for properties **(i)** – **(iii)** will later be taken as *characteristic* of open sets.

Observe that all open intervals and all subsets of $\mathbb{R}$ of the form $(a, \rightarrow)$ or $(\leftarrow, a]$ are open sets.

Although *finite* intersections of open sets are open, the intersection of an *infinite* family of open sets is not necessarily so. For instance, each interval of the form $(-1/n, 1/n)$ is open but $\bigcap_{n\in\mathbb{N}} (-1/n, 1/n) = \{0\}$ which is evidently not open.

1.4. Definition. A subset A of $\mathbb{R}$ is said to be *closed* if its complement $\mathbb{R} - A$ is open.

1.5. Proposition. In $\mathbb{R}$,

(i) the intersection of any (nonempty) family of closed sets is closed;

(ii) the union of any *finite* family of closed sets is closed;

(iii) $\mathbb{R}$ and $\varnothing$ are both closed sets.

Proof. Immediate from **1.3** and **1.4**. ■

From **1.3 (iii)** and **1.5 (iii)** we see that $\mathbb{R}$ and $\varnothing$ are both open *and* closed —they are *clopen* sets. (This is *not* a logical contradiction, merely a linguistic oddity!) Notice also that every closed interval $[a, b]$ is a closed set since its complement is the union of the two open sets $(\leftarrow, a)$ and $(b, \rightarrow)$ and is accordingly open. For similar reasons $[a, \rightarrow)$ and $(\leftarrow, a]$ are closed as well.

1.6. Definition. A subset V of $\mathbb{R}$ is said to be a *neighbourhood* of $a \in \mathbb{R}$ if there is an open set U such that $a \in U \subseteq V$.

In other words, V is a neighbourhood of a if it contains an open interval around a. It follows easily from this that *a set is open if and only if it is a neighbourhood of each of its points.*

Is every subset of $\mathbb{R}$ open or closed? No, the set Q of rational numbers is neither. For it is well known that every nonempty open interval contains both rationals *and* irrationals. Therefore neither Q nor its complement $\mathbb{R}$ –

Q is a neighbourhood of any of its points, so certainly neither is open. Accordingly, Q is neither open nor closed.

1.7. Definition. A point a of $\mathbb{R}$ is said to be a *limit point* of a subset A of $\mathbb{R}$ if every neighbourhood of a contains points of A different from a.

Clearly each neighbourhood of a limit point of of a set A must contain an infinity of points of A: it follows that only *infinite* subsets of $\mathbb{R}$ possess limit points. However, the mere fact that a set is infinite does not ensure that it possesses limit points: the set $\mathbb{N}$, for example, is infinite but does not have any limit points.

The concept of limit point yields a characterization of closed sets which provides some indication as to why such sets were called "closed" in the first place.

1.8. Proposition. A subset of $\mathbb{R}$ is closed iff it contains all its limit points.

Proof. Let A be a closed subset of $\mathbb{R}$. If $x \in \mathbb{R} - A$, the open set $\mathbb{R} - A$ is a neighbourhood of x which fails to intersect A, so x cannot be a limit point of A. It follows that A contains all its limit points.

Conversely, suppose that A is a subset of $\mathbb{R}$ which contains all its limit points, so that no point of $\mathbb{R} - A$ is a limit point of A. Then for each $x \in \mathbb{R} - A$ there is a neighbourhood of x which does not meet A, and is accordingly included in $\mathbb{R} - A$. So $\mathbb{R} - A$ is a neighbourhood of each of its points and is therefore open. This means that A is closed. ■

1.9. Definition. Let A be a subset of $\mathbb{R}$. A family $\mathcal{U}$ of open subsets of $\mathbb{R}$ which covers A is called an *open cover* of A. A subfamily of $\mathcal{U}$ which is also a cover of A is called a *subcover* of $\mathcal{U}$.

Our next result is one of the central theorems of classical analysis:

1.10. Heine-Borel Theorem. Each open cover of a closed interval in $\mathbb{R}$ has a finite subcover.

Proof. Let $\mathcal{U}$ be an open cover of a closed interval $[a, b]$ (where $a < b$, the result being trivial when $b \leq a$). Define X to be the set of all $x \in [a, b]$ such that there is a finite subfamily of $\mathcal{U}$ covering $[a, x]$. Proving the theorem amounts to showing that $b \in X$. Now $X \neq \varnothing$ since $a \in X$, and it is bounded above by b; since $(\mathbb{R}, \leq)$ is complete, X has a supremum c, say. Clearly $c \in [a, b]$ and $a < c$ since any member of $\mathcal{U}$ which contains a also includes some nonempty open interval around a.

Now choose $U \in \mathcal{U}$ so that $c \in U$, and choose a member d of the open interval (a, c) so that $[d, c] \subseteq U$. Since $d < c$, it follows from the definition of c that there is a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ which covers $[a, d]$, so that the finite family $\mathcal{G} = \mathcal{F} \cup \{U\}$ covers $[a, c]$. Accordingly $c \in X$. If $c < b$, then there is $e \in (c, b)$ such that $[c, e] \subseteq U$, so that $\mathcal{G}$ covers $[a, e]$, contradicting the definition of c. It follows that $b = c \in X$, and the theorem is proved. ■

Notice that this result fails for intervals which are not closed. For example, the family of open sets $(-1/n, 1)$ with $n \in \mathbb{N} - \{0\}$ covers the open interval (0, 1), but clearly no finite subfamily does.

As a corollary to this theorem we prove the classical result of *Bolzano-Weierstrass*.

1.11. Corollary. Any infinite subset of a closed interval has a limit point in that interval.

Proof. Let X be an infinite subset of the closed interval $[a, b]$. If no point of $[a, b]$ is a limit point of X, then each $x \in [a, b]$ has an open neighbourhood $V(x)$ such that $V(x) \cap X \subseteq \{x\}$. The family $\{V(x): x \in [a, b]\}$ is an open cover of $[a, b]$; by **1.10** it has a finite subcover $\{V(x_i): i = 1, ..., n\}$. But then

$$X = X \cap [a, b] \subseteq X \cap [V(x_1) \cup \ldots \cup V(x_n)] \subseteq \{x_1, ..., x_n\},$$

contradicting the assumption that X is infinite. ■

2

Topological Spaces

WE NOW INTRODUCE the definition of a topology on an arbitrary set.

2.1. Definition. A *topology* on a set X is a family $\mathfrak{I}$ of subsets of X satisfying the following conditions:

(i) the union of any subfamily of $\mathfrak{I}$ belongs to $\mathfrak{I}$;

(ii) the intersection of any finite nonempty subfamily of $\mathfrak{I}$ belongs to $\mathfrak{I}$;

(iii) $X \in \mathfrak{I}$ and $\varnothing \in \mathfrak{I}$.

The pair $(X, \mathfrak{I})$ (or simply X when confusion is unlikely) is called a *topological space*, or simply a *space*; the members of $\mathfrak{I}$ are called $\mathfrak{I}$ *-open*, *open in X,* or simply *open*, *sets*. The set X is called the *underlying set* of the topological space $(X, \mathfrak{I})$.

Examples. 1. The family of all open sets in $\mathbb{R}$ as laid down in Def.**1.2** is a topology on $\mathbb{R}$; it is known as the *usual topology* on $\mathbb{R}$.

2. For any set X, the family $\{\varnothing, X\}$ is a topology on X called the *indiscrete* or *trivial topology* on X. A set with the indiscrete topology is called an *indiscrete* or *trivial space*.

3. For any set X, the family of *all* subsets of X is evidently a topology on X: it is known as the *discrete topology* on X, and with this topology X is called a *discrete space.* Clearly, X is a discrete space iff $\{x\}$ is open for each $x \in X$.

4. For any set X, the family consisting of the empty set together with all subsets of X whose complement with X is finite is a topology on X called the *cofinite topology* on X.

5. Consider Euclidean n-space $\mathbb{R}^n$. For each pair of points $a = (a_1, \ldots, a_n)$ and $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ we define the *distance* $\mathbf{d}(a, x)$ between a and x to be the real number $\mathbf{d}(a, x) = \sqrt{\sum_{i=1}^{n}(x_i - a_i)^2}$. If ε is a positive real number, the *open sphere of radius* ε about a is the set

$$\{x \in \mathbb{R}^n \colon \mathbf{d}(a, x) < \varepsilon\}.$$

The family of unions of open spheres about points of $\mathbb{R}^n$ is a topology called the *usual topology* on $\mathbb{R}^n$. Thus a subset of $\mathbb{R}^n$ is open in the usual topology iff it contains an open sphere about each of its points.

As already indicated, to emphasize the geometrical content of topology, elements of topological spaces will often be called *points.*

2.2. Definition. A subset U of a topological space X is said to be a *neighbourhood* of a point $x \in X$ if U includes an open set to which x belongs. The family of all neighbourhoods of a point is called the *neighbourhood* *system* at that point. The neighbourhood system at a point reflects the properties of the ambient space in the vicinity of that point.

2.3. Proposition. Let X be a topological space. Then the neighbourhood system $\mathfrak{N}_x$ at any point x 0 X is a *filter* on X, that is, satisfies:

(i) $U, V \in \mathfrak{N}_x \Rightarrow U \cap V \in \mathfrak{N}_x$;

(ii) $U \in \mathfrak{N}_x$ & $U \subseteq V \subseteq X \Rightarrow V \in \mathfrak{N}_x$. ■

A concept which should be expressible in topological terms is that of *convergent sequence*. We shall see that a credible definition can indeed be framed within an arbitrary topological space.

2.4. Definition. Let X be a topological space and let $x_0, x_1, \ldots$ be a sequence of points of X. We shall say that this sequence *converges* to the point $x \in X$ if for each neighbourhood U of x there is an $n_0 \in \mathbb{N}$ such that $x_n \in U$ whenever $n \geq n_0$.

In other words, a sequence converges to a point provided that the sequence is eventually in every neighbourhood of that point. Notice that in an indiscrete space *every* sequence converges to *every* point, while—at the opposite extreme—in a discrete space the only convergent sequences are those which are *eventually constant*. (Two picturesque examples of *topological pathology*.)

2.5. Definition. Let $(X, \mathfrak{T})$ be a topological space. A subset A of X is said to be ($\mathfrak{T}$-) *closed* if its complement $X - A$ is ($\mathfrak{T}$-) open.

Thus a subset of a topological space is open iff its complement is closed. Our next proposition enumerates the characteristic properties of closed sets.

2.6. Proposition. In a topological space X,

(i) the intersection of an arbitrary nonempty family of closed sets is closed;

(ii) the union of a finite family of closed sets is closed;

(iii) X and $\varnothing$ are closed. ■

The next proposition, whose proof we entrust to the reader, shows that topological spaces could equally well have been defined in terms of the concept of closed set.

2.7. Proposition. Let $\mathfrak{F}$ be a family of subsets of a set X such that

(i) the intersection of any nonempty subfamily of $\mathfrak{F}$ belongs to $\mathfrak{F}$;

(ii) the union of any finite subfamily of $\mathfrak{F}$ belongs to $\mathfrak{F}$;

(iii) X and $\varnothing$ are both members of $\mathfrak{F}$.

Then the family $\mathfrak{I} = \{X - F: F \in \mathfrak{F}\}$ is a topology on X and $\mathfrak{F}$ is precisely the family of $\mathfrak{I}$-closed subsets of X. ■

2.8. Definition. Let $(X, \mathfrak{I})$ be a topological space. The ($\mathfrak{I}$-) *closure* $\overline{A}$ of a subset A of X is the intersection of all ($\mathfrak{I}$-) closed subsets of X which include A. A is said to be *dense* in X if $\overline{A} = X$.

Clearly $\overline{A}$ is closed and is in fact the *least* (under inclusion) closed set containing A. It follows from this that a subset of a topological space is closed iff it coincides with its closure.

2.9. Proposition. Let A be a subset of a topological space X. Then, for any point x of X, $x \in \overline{A}$ iff every (open) neighbourhood of x meets A.

Proof. If $x \notin \overline{A}$, then by definition there is a closed set F containing A but not x. So $X - F$ is an (open) neighbourhood of x which does not meet A. Conversely, if x has an open neighbourhood U disjoint from A, then $X - U$ is a closed set containing A but not x, so that $x \notin \overline{A}$. ■

It follows from this proposition that a subset of a topological space is dense if and only if it meets every nonempty open set. Thus, in particular, both the sets Q of rationals and ∅ of irrationals are dense in $\mathbb{R}$ (with its usual topology). We also have

2.10. Corollary. If X is a bounded subset of $\mathbb{R}$, then both sup X and inf X are members of $\overline{X}$, and so, if X is closed, of X.

Proof. Let $a = \sup X$ and $\varepsilon > 0$. Then since a is the least upper bound for X, there must be $x \in X$ for which $a - \varepsilon < x$. It follows that every open interval about a meets x, so that, by **2.9**, $a \in \overline{X}$. The argument for inf X is similar. ■

2.11. Theorem. In any topological space, we have

(i) $\overline{\varnothing} = \varnothing$; **(ii)** $A \subseteq \overline{A}$; **(iii)** $\overline{\overline{A}} = \overline{A}$; **(iv)** $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (i). Since $\varnothing$ is closed this is obvious. **(ii)** is evident from the definition of closure. **(iii)**. Since $\overline{A}$ is closed it coincides with its closure $\overline{\overline{A}}$. **(iv)**. Since $\overline{A \cup B}$ is a closed set containing both A and B, it must contain both $\overline{A}$ and $\overline{B}$ and so also their union. On the other hand, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$ and hence also $\overline{A \cup B}$. **(iv)** follows. ■

Our next result shows that the properties enumerated in **2.11** are actually characteristic of topological closure. In order to give this fact precise expression we need the following definition.

2.12. Definition. A *closure operation* on a set X is a function which assigns to each subset A of X a subset A^c of X in such a way that the following four statements, the *Kuratowski closure axioms*, are satisfied for all subsets A, B of X:

(i) $\varnothing^c = \varnothing$; **(ii)** $A \subseteq A^c$; **(iii)** $A^c = A^{cc}$; **(iv)** $(A \cup B)^c = A^c \cup B^c$.

Now we can prove

2.13. Theorem. Let ^{c} be a closure operation on X, let $\mathcal{F} = \{A\colon A^c = A\}$ and $\mathcal{T} = \{X - A\colon A \in \mathcal{F}\}$. Then $\mathcal{T}$ is a topology on X (called the topology *induced* by ^{c}) and, for any subset A of X, A^c is the $\mathcal{T}$-closure of A.

Proof. It follows immediately from **2.12 (i)** and **(ii)** that both $\varnothing$ and X are members of $\mathcal{F}$, and **(iv)** shows that the union of any pair of (and hence any finite collection of) members of $\mathcal{F}$ belongs to $\mathcal{F}$. We claim that the intersection of any nonempty subfamily of $\mathcal{F}$ is a member of $\mathcal{F}$. To prove this, observe that, if $B \subseteq A$, then $B^c \subseteq A^c$ because $A^c = (A \cup B)^c = A^c \cup B^c$. Now suppose that $\mathfrak{A}$ is a nonempty subfamily of $\mathcal{F}$, and let $B = \bigcap\mathfrak{A}$. Then $B \subseteq A$, and so $B^c \subseteq A^c$, for all $A \in \mathfrak{A}$. Therefore

$$B^c \subseteq \bigcap\{A^c\colon A \in \mathfrak{A}\} = \bigcap\{A\colon A \in \mathfrak{A}\} = B,$$

so that $B = B^c$ and $B \in \mathcal{F}$. This proves the claim. By **2.7**, $\mathcal{T}$ is a topology on X, and $\mathcal{F}$ is the family of $\mathcal{T}$-closed subsets of X.

We must finally show that $A^c = \overline{A}$, the $\mathcal{T}$-closure of A. Now it follows from **2.11(iii)** that $A^c \in \mathcal{F}$; since $A \subseteq A^c$ and $\overline{A}$ is, by definition, the intersection of all members of $\mathcal{F}$ containing A, we infer that $\overline{A} \subseteq A^c$. But $\overline{A}$, as the intersection of a subfamily of $\mathcal{F}$, itself belongs to $\mathcal{F}$, so that $\overline{A} = \overline{A}^c$; since $A \subseteq \overline{A}$, it follows that $A^c \subseteq \overline{A}^c = \overline{A}$, whence $\overline{A} = A^c$. ■

2.14. Definition. The *interior* $\overset{0}{A}$ of a subset A of a topological space is the union of all open sets contained in A. Points of $\overset{0}{A}$ are called *interior points* of A.

Evidently $\overset{0}{A}$ is open, and is in fact the *largest* open set contained in A; it follows that a set is open iff it coincides with its interior. It is easy to see that $\overset{0}{A}$ consists of all those points $a \in A$ such that A is a neighbourhood of a. As the next proposition shows, interior and closure are intimately related.

2.15. Proposition. Let A be a subset of a topological space X. Then $\overset{0}{A} = X - \overline{X - A}$.

Proof. We have $x \notin \overset{0}{A} \Leftrightarrow A$ contains no neighbourhood of $x \Leftrightarrow$ every neighbourhood of x meets $X - A$ $\Leftrightarrow x \in \overline{X - A}$. ■

From **2.11** and **2.15** we derive the corresponding dual properties of the interior operator, viz., **(i)** $\overset{0}{\varnothing} = \varnothing$; **(ii)** $\overset{o}{A} \subseteq A$; **(iii)** $\overset{0}{\overset{0}{A}} = \overset{0}{A}$; **(iv)** $\overset{0}{\widehat{A \cap B}} = \overset{0}{A} \cap \overset{0}{B}$.

3

Bases, Subbases, and Relative Topologies.

IN DEFINING THE USUAL topology on the reals we began with the family of all open intervals and from this constructed the topology. We now consider a useful extension of this procedure to arbitrary topological spaces.

3.1. Definition. Let $(X, \mathfrak{T})$ be a topological space. A subfamily $\mathfrak{B}$ of $\mathfrak{T}$ is said to be a *base* for $\mathfrak{T}$ if for each $x \in X$ and each neighbourhood U of x there is a member B of $\mathfrak{B}$ such that $x \in B \subseteq U$.

For example, the family of open intervals is a base for the usual topology on $\mathbb{R}$, and the family of open n-spheres is a base for the usual topology on $\mathbb{R}^n$. More interestingly, the countable family of open intervals with rational endpoints is also a base for the usual topology on $\mathbb{R}$, and the countable family of open n-spheres with radii of rational length and centres with rational coordinates is a base for the usual topology on $\mathbb{R}^n$.

We entrust the proof of the next proposition to the reader.

3.2. Proposition. A subfamily $\mathfrak{B}$ of a topology $\mathfrak{T}$ is a base for $\mathfrak{T}$ iff every member of $\mathfrak{T}$ is a union of members of $\mathfrak{B}$. ■

Now not every family of sets is a base for some topology. For instance, let $X = \{0, 1, 2\}$, $A = \{0, 1\}$, $B = \{1, 2\}$. If $\mathfrak{B} = \{X, A, B, \varnothing\}$ then by direct computation the union of members of $\mathfrak{B}$ is always in $\mathfrak{B}$, so were $\mathfrak{B}$ a base for some topology, that topology would have to none other than $\mathfrak{B}$ itself. But $\mathfrak{B}$ isn't a topology because $A \cap B = \{1\} \notin \mathfrak{B}$. Our next result shows exactly what conditions must be satisfied if a given family of sets is to be a base for some topology.

3.3. Proposition. A family $\mathfrak{B}$ of sets is a base for some topology on the set $X = \bigcup\mathfrak{B}$ iff for every pair of members U, V of $\mathfrak{B}$ and every $x \in U \cap V$ there is $W \in \mathfrak{B}$ such that $x \in W \subseteq U \cap V$.

Proof. Necessity is almost trivial. Conversely, if $\mathfrak{B}$ is a family of sets with the prescribed property it is a routine exercise to show that the family $\mathfrak{T}$ of unions of subfamilies of $\mathfrak{B}$ is a topology on X; $\mathfrak{B}$ is a base for $\mathfrak{T}$ by **3.2**. ■

Since not every family of sets is a base for a topology, it is natural to ask whether any family *determines* a topology in some natural way. Our next proposition shows that this question has an affirmative answer.

3.4. Proposition. If $\mathfrak{S}$ is a nonempty family of sets, the family $\mathfrak{B}$ of all finite intersections of members of $\mathfrak{S}$

is a base for a topology on the set $X = \bigcup \mathfrak{S}$.

Proof. The intersection of any pair of members of $\mathfrak{B}$ is easily seen to be a member of $\mathfrak{B}$. So by **3.3** $\mathfrak{B}$ is a base for a topology on X. ■

3.5. Definition. A family $\mathfrak{S}$ is a *subbase* for a given topology $\mathfrak{T}$ if the family of finite intersections of members of $\mathfrak{S}$ is a base for $\mathfrak{T}$. In this case $\mathfrak{S}$ is said to *generate* $\mathfrak{T}$.

That is, $\mathfrak{S}$ generates $\mathfrak{T}$ iff each member of $\mathfrak{T}$ is a union of finite intersections of members of $\mathfrak{S}$. In view of **3.4,** every nonempty family of sets generates a unique topology: it is easily seen that this topology is the *least* topology (under inclusion) containing the given family.

In general, there will be many different bases and subbases for a given topology and the most felicitous choice will depend on the problem at hand. For example, a natural subbase for the usual topology on $\mathbb{R}$ is the family of sets of the form $(a, \rightarrow)$ or $(\leftarrow, a)$. The family of all such sets with a rational is a subtler and more interesting subbase.

We next formulate a local version of the concept of base.

3.6. Definition. Let x a point of a topological space X. A *local base* at x is a family $\mathfrak{U}$ of neighbourhoods of x such that every neighbourhood of x contains a member of $\mathfrak{U}$.

3.7. Definition. A topological space is said to satisfy the *first axiom of countability*, or to be *first countable*, if every point has a *countable* local base.

For example, any Euclidean space with its usual topology is first countable.

In a first countable space one can formulate a very simple intuitive description of the closure of a set.

3.8. Theorem. Let A be a subset of a first countable space. Then for any point x, $x \in \overline{A}$ iff there is a countable sequence $a_0, a_1, \ldots$ of points of A converging to x.

Proof. If $a_0, a_1, \ldots.$ from A converges to x, then every neighbourhood of x contains members of this sequence, so *a fortiori* meets A. Hence $x \in \overline{A}$. Conversely, suppose that $x \in \overline{A}$, and let $\{U_n: n \in \mathbb{N}\}$ be a countable local base at x. For each $n \in \mathbb{N}$ define $V_n = U_1 \cap \ldots \cap U_n$; clearly $\{V_n: n \in \mathbb{N}\}$ is also a local base at x. Since $x \in \overline{A}$, each V_n meets A and so for each $n \in \mathbb{N}$ we can choose $a_n \in V_n \cap A$. But then the sequence $a_0, a_1, \ldots.$ converges to x. For if U is any neighbourhood of x, there is some $V_n \subseteq U$; if $m \geq n$ then $a_m \in V_m \subseteq V_n \subseteq U$, which proves the contention. ■

We next turn to a different problem. Suppose we are given a topological space $(X, \mathfrak{T})$ and a subset A of X. Is there a natural topology which is, so to speak, "inherited" by A? The affirmative answer to this question is provided by the next definition.

3.9. Definition. If (X, $\Im$) is a topological space and A is a subset of X, define $\Im_A$ to be the family of intersections of A with the members of $\Im$. Then $\Im_A$ is a topology on A called the *relative topology* on A induced by $\Im$. Each member U of $\Im_A$ is said to be *open in A*, and its relative complement $A - U$ *closed in A*. The space (A, $\Im_A$) (or simply A) is called a *subspace* of (X, $\Im$) (or simply of X). Formally, an arbitrary topological space (Y, $\mathfrak{U}$) is a *subspace* of a space (X, $\Im$) if $Y \subseteq X$ and $\mathfrak{U} = \Im_Y$.

If (X, $\Im$) is a topological space and $A \subseteq X$, then unless explicitly stated otherwise we shall assume that A has been assigned the relative topology induced by $\Im$. Thus if we assert that A has a certain topological property —first countability for instance —we shall mean that the space (A, $\Im_A$) has this property.

3.10. Proposition. Let (Y, $\mathfrak{U}$) be a subspace of a topological space (X, $\Im$), and let $A \subseteq Y$. Then

(i) A is $\mathfrak{U}$-closed iff it is the intersection with Y of a $\Im$-closed set;

(ii) the $\mathfrak{U}$-closure of A is the intersection with Y of the $\Im$-closure of A.

Proof. (i). A is $\mathfrak{U}$-closed $\Leftrightarrow$ $Y - A =$ V 1 Y for some V 0 $\Im$ $\Leftrightarrow$ $A = (X - V) \cap Y$ for some $V \in \Im$. This proves **(i)**.

(ii). The $\mathfrak{U}$-closure of $A = \bigcap\{B: B$ is $\mathfrak{U}$-closed and $A \subseteq B\}$

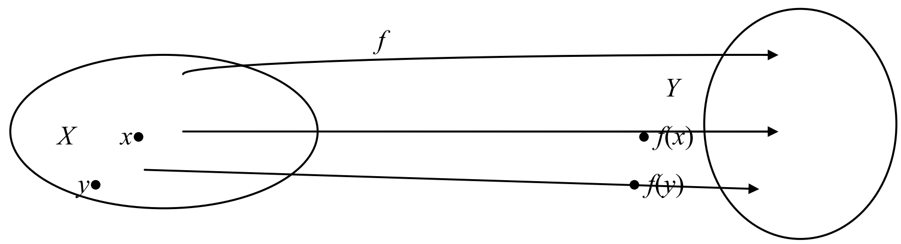
$= \bigcap\{C \cap Y: C$ is $\Im$-closed & $A \subseteq C\}$

$= Y \cap \bigcap\{C: C$ is $\Im$-closed & $A \subseteq C\}$

$= Y \cap \Im$-closure of A. ■

4

Continuous Functions.



CONSIDER TWO REGIONS X and Y in the plane, and a function $f\colon X \to Y$. Intuitively, we say that f is *continuous* if it carries neighbouring points of X to neighbouring points of Y, that is, if x and y are points of X, and x is close to y, then $f(x)$ is close to $f(y)$.

Now this notion of "closeness" is vague, and in classical analysis the definition of continuity is often given the following precise formulation in terms of the limit concept. Namely, f is continuous if whenever $x_0, x_1,...$ is a sequence of points of X converging to a point x of X, then the sequence $f(x_0), f(x_1),...$ converges to the point $f(x)$ of Y. But it is easy to see that this condition is equivalent —for plane regions at any rate —to the following: for any subset A of X, if x is in the closure of A, then $f(x)$ is in the closure of $f[A]$. That is, $f[\overline{A}] \subseteq \overline{f[A]}$ for all $A \subseteq X$. Obviously this condition makes sense even when X and Y are arbitrary topological spaces. We thus arrive at the following definition.

4.1. Definition. Let X and Y be topological spaces, and let $f\colon X \to Y$. Then f is said to be *continuous* (with respect to (the topologies on) X and Y) if, for all $A \subseteq X$, we have $f[\overline{A}] \subseteq \overline{f[A]}$. (That is, "for any subset of X, the image under f of its closure is a subset of the closure of its image under f".)

In other words, if we say that the points in the closure of a subset of a topological space are *close* to that subset, then a continuous function between topological spaces is one that preserves the closeness relation between points and subsets.

We next formulate a bundle of useful conditions each of which is equivalent to continuity.

4.2. Proposition. Let f be a function between topological spaces X and Y. The following conditions are equivalent:

(i) f is continuous;

(ii) the preimage under f of any closed subset of Y is a closed subset of X;

(iii) the preimage under f of any open subset of Y is an open subset of X;

(iv) for any $x \in X$, the preimage under f of any neighbourhood of $f(x)$ is a neighbourhood of x;

(v) for any $x \in X$ and any neighbourhood U of $f(x)$, there is a neighbourhood V of x such that $f[V] \subseteq U$.

Proof. (i) $\Rightarrow$ (ii). Assume (i), and let A be a closed subset of Y. By (i) we have

$$f[\overline{f^{-1}[A]}] \subseteq \overline{f[f^{-1}[A]]} \subseteq \overline{A} = A,$$

so that, applying f^{-1} to both sides,

$$\overline{f^{-1}[A]} \subseteq f^{-1}[f[\overline{f^{-1}[A]}]],$$

whence $\overline{f^{-1}[A]} = f^{-1}[A]$. Therefore $f^{-1}[A]$ is closed.

(ii) $\Rightarrow$ (iii). This is a simple consequence of the fact that preimages preserve complements (**0.2 (v)**).

(iii) $\Rightarrow$ (iv). Assuming (iii), if V is a neighbourhood of $f(x)$, then there is an open set U such that $f(x) \in U \subseteq V$. It follows that $x \in f^{-1}[U] \subseteq f^{-1}[V]$. Since $f^{-1}[U]$ is open by (iii), $f^{-1}[V]$ is a neighbourhood of x.

(iv) $\Rightarrow$ (v). Assuming (iv), if $x \in X$ and U is a neighbourhood of $f(x)$, then $f^{-1}[U]$ is a neighbourhood of x and $f[f^{-1}[U]] \subseteq U$.

(v) $\Rightarrow$ (i). Assuming (v), if $x \in \overline{A} \subseteq X$ and U is any neighbourhood of $f(x)$, then there is a neighbourhood V of x such that $f[V] \subseteq U$. But since $x \in \overline{A}$, we have $A \cap V \neq \varnothing$ so that

$$\varnothing \neq f[V \cap A] \subseteq f[V] \cap f[A] \subseteq U \subseteq f[A].$$

It follows that $x \in \overline{f[A]}$, so that $f[\overline{A}] \subseteq \overline{f[A]}$ and f is continuous. ■

Examples. 1. Every function from a discrete space, or to an indiscrete space, is continuous.

2. Every *constant* function, i.e. a function assuming just one value, is continuous.

3. Let $A \subseteq \mathbb{R}$ and $f\colon A \to \mathbb{R}$. Then f is continuous iff for every $a \in A$ and each $\delta > 0$ in $\mathbb{R}$ there is $\varepsilon > 0$ in $\mathbb{R}$ such that, for all $x \in A$, $|x - a| < \varepsilon \Rightarrow |f(x) - f(a)| < \delta$. (This is the classical "ε, δ" condition for continuity for real-valued functions.)

We shall frequently make use of condition **4.2** (iii): to illustrate, we prove

4.3. Corollary. If X, Y, Z are topological spaces and $f\colon X \to Y$, $g\colon Y \to Z$ are continuous, then so is $g \circ f\colon X \to Z$.

Proof. If U is open in Z, then $g^{-1}[U]$ is open in Y and $f^{-1}[g^{-1}[U]] = (g \circ f)^{-1}[U]$ is open in X. ■

4.4. Corollary. If $f\colon X \to Y$ is continuous and $A \subseteq X$, then the restriction $f|A\colon A \to Y$ of f to A is also continuous (with respect to the relative topology on A).

Proof. If U is open in Y, then $(f|A)^{-1}[U] = A \cap f^{-1}[U]$ is open in A. ■

The following local version of continuity is also useful.

4.5. Definition. A function f from a space X to a space Y is said to be *continuous at the point* $x \in X$ if for each neighbourhood U of $f(x)$ there is a neighbourhood V of x such that $f[V] \subseteq U$.

Clearly $f\colon X \to Y$ is continuous iff it is continuous at every point of X.

Additional important concepts are contained in the following

4.6. Definition. A mapping from a space X to a space Y is said to be *open* if it carries open sets to open sets, i.e. if $f[U]$ is open in Y whenever U is open in X. A bijective continuous open mapping from X to Y is called a *homeomorphism* of X and Y. Under these conditions X and Y are said to be *homeomorphic,* or *homeomorphs of* one another.

Evidently a bijective map f between topological spaces is a homeomorphism iff f and f^{-1} are both continuous. This means that a topological space is *topologically indistinguishable* from any of its homeomorphs. In other words, any purely topological property possessed by a space is shared by all of its homeomorphs. This may be illustrated as follows. Suppose we imagine a topological space to be some sort of geometric configuration, for instance, a diagram drawn on a sheet of rubber. Then a homeomorphism may be thought of as any deformation of the diagram (by stretching, bending, shrinking, etc.) which does not tear the sheet. Under deformations of this sort a circle, for example, can be changed into an ellipse, a triangle, or a square, but not into a figure eight, a horseshoe, or a single point. A topological property is any property which is preserved by such deformations. Notice that *metric properties* such as distances and angles are *not* topological because they can be altered by suitable homeomorphisms. In the case of the circle examples of topological properties include the fact that it has one "inside" and one "outside" and the fact that, if two points are removed from it, it falls into two pieces, whereas, if only one point is removed, just one piece remains. These remarks explain why topology is sometimes called "rubber-sheet geometry".

5

Basic Topological Properties

WE HAVE SEEN in the previous chapter that a topological property may be defined as one which is preserved under arbitrary homeomorphisms. We now introduce various properties of this sort, most of which are natural generalizations of properties possessed by subsets of $\mathbb{R}$ with its usual topology. We begin with the

5.1. Definition. A topological space is said to be a *Hausdorff* ("housed-off") space if each pair of distinct points have disjoint neighbourhoods.

For example, any Euclidean space with its usual topology is Hausdorff, as is any discrete space. Clearly, also, any subspace of a Hausdorff space is Hausdorff. It is also easily shown that in a Hausdorff space any subset consisting of a single point is closed.

Sequences behave particularly well in Hausdorff spaces. We have remarked that in certain spaces, in indiscrete spaces, for instance, it is possible for a sequence to converge to more than one point. But this pathology cannot arise in Hausdorff spaces:

5.2. Proposition. In a Hausdorff space a sequence can converge to at most one point.

Proof. Suppose X is a Hausdorff space and $x_0, x_1, \ldots$ is a sequence of points of X converging to the two distinct points a and b. Since X is Hausdorff, a and b have disjoint neighbourhoods U and V. But, for some $m \in \mathbb{N}$, U and V would both have to contain all x_n with $n \geq m$. Contradiction. ■

5.3. Definition. A topological space is said to be *connected* if it is not the union of a pair of disjoint nonempty open sets.

Intuitively, a space is connected if it *coheres* in the sense that it cannot be decomposed into a pair of disjoint nonempty open fragments. The straightforward proof of the next proposition is entrusted to the reader.

5.4. Proposition. The following conditions on a topological space are equivalent:

(i) X is connected;

(ii) X is not the union of a pair of disjoint nonempty closed sets;

(iii) the only subsets of X which are simultaneously open-and-closed ("clopen") are $\varnothing$ and X itself;

(iv) any continuous function from X to a 2-element discrete space is constant. ■

5.5. Definition. A subset of a topological space is said to be *connected* if it is connected in the relative topology.

It is easy to see that a subset A of a topological space X is connected iff, whenever $A \subseteq U \cup V$, where U and V are disjoint and open in X, then $A \subseteq U$ or $A \subseteq V$.

5.6. Theorem. The closure of a connected set is connected.

Proof. Let Y be a connected subset of a topological space X and suppose that $\overline{Y} = A \cup B$, where A and B are disjoint and open in $\overline{Y}$. Then Y is the union of the disjoint sets $A \cap Y$ and $B \cap Y$; since both of these are open in the connected subspace Y, at least one of them —$B \cap Y$, say —must be empty. In that case it follows that $Y \subseteq A$, so that $\overline{Y} \subseteq \overline{A}$. But A is closed in $\overline{Y}$, so that $A = \overline{A} \cap \overline{Y}$. Therefore $\overline{Y} \subseteq \overline{A} \cap \overline{Y} = A$, whence $B = \varnothing$. So $\overline{Y}$ is connected. ■

5.7. Corollary. A space with a connected dense subset is itself connected. ■

5.8. Theorem. The union of a family of pairwise nondisjoint connected sets is connected.

Proof. Let $\mathcal{A}$ be a (nonempty) family of pairwise nondisjoint connected subsets of a topological space X and suppose that $\bigcup\mathcal{A} \subseteq U \cup V$, where U and V are disjoint and open in X. Choose $A \in \mathcal{A}$; then $A \subseteq U \cup V$ so that, since A is connected, either $A \subseteq U$ or $A \subseteq V$; suppose the former. For any $B \in \mathcal{A}$, the same argument shows that $B \subseteq U$ or $B \subseteq V$. But the latter cannot hold since, if it did, A and B would be disjoint. Hence $B \subseteq U$ for all $B \in \mathcal{A}$, so that $\bigcup\mathcal{A} \subseteq U$. In the case $A \subseteq V$ we similarly conclude $\bigcup\mathcal{A} \subseteq V$. It follows that $\bigcup\mathcal{A}$ is connected. ■

The proof of the next proposition is left to the reader.

5.9. Proposition. If X is a connected space and f is a continuous mapping of X into a space Y, then $f[X]$ is a connected subset of Y. (That is, "the continuous image of a connected space is connected".) ■

What subsets of $\mathbb{R}$ are connected? Here we have the intuitively satisfying

5.10. Theorem. A subset of $\mathbb{R}$ is connected iff it is an interval. In particular, $\mathbb{R}$ itself is connected.

Proof. Let X be a nonempty subset of $\mathbb{R}$. If X is not an interval, then clearly there exist real numbers $a < b < c$ such that $a, c \in X$ and $b \notin X$. We can then express X in the form $X = (X \cap (\leftarrow, b))$ c $(X \cap (b, \rightarrow))$, which shows that X is disconnected.

Conversely, suppose that X is a nonempty interval, but that X is disconnected. Then $X = A \cup B$, where A and B are disjoint nonempty sets both closed in X. Choose points $a \in A$, $b \in B$. We may assume without loss of generality that $a < b$. Since X is an interval, $[a, b] \subseteq X$, and so each point in $[a, b]$ is either in A or in B. Define $c =$

sup($[a,b] \cap A$). Evidently $a \le c \le b$, so $c \in X$. Now $c \in \overline{A}$ by **2.10**, so that, since A is closed in X, $c \in A$. Therefore $c < b$. But now it follows from this and the definition of c that $c + \varepsilon \in B$ for every $\varepsilon > 0$ such that $c + \varepsilon \le b$, so that c is also in the closure of B. Since B is closed in X, $c \in B$. But this contradicts the disjointness of A and B, completing the proof. ■

The next result is an immediate consequence of **5.9** and **5.10.**

5.11. Corollary. The image of a connected space under a continuous function to $\mathbb{R}$ is an interval. ■

5.12. Corollary: *Weierstrass's intermediate value theorem*. Any continuous function from $\mathbb{R}$ to itself assuming two values $a < b$ also assumes any value c such that $a < c < b$.

Proof. By **5.11,** $f[\mathbb{R}]$ is an interval, and the result follows immediately. ■

Our next definition extends **1.9.**

5.13. Definition. An *open cover* of a subset A of a topological space X is a cover of A by sets open in X. A subfamily of an open cover $\mathcal{U}$ of A which also covers A is called a *subcover* of $\mathcal{U}$.

The following definition is motivated by the Heine-Borel theorem **1.10.**

5.14. Definition. A topological space X is said to be *compact* if every open cover of X has a finite subcover.

As examples of compact spaces we have: indiscrete spaces, any set with the cofinite topology, and — as we shall see — any closed bounded subspace of a Euclidean space. Clearly a *discrete* space is compact iff it is finite. It is also easy to see that a subset A of a topological space X is compact in the relative topology iff every open cover of A by sets open in X has a finite subcover.

5.15. Definition. A nonempty family of sets is said to have the *finite intersection property* (*fip* for short) if the intersection of each nonempty finite subfamily is nonempty.

5.16. Proposition. A topological space is compact iff every family of closed sets with the *fip* has nonempty intersection.

Proof. X is compact

$\Leftrightarrow$ every open cover of X has a finite subcover

$\Leftrightarrow$ if $\mathcal{U}$ is a family of open sets such that each finite subfamily fails to cover X, then $\mathcal{U}$ fails to cover X

$\Leftrightarrow$ if $\mathcal{U}$ is a family of open sets such that $\{X - U \colon U \in \mathcal{U}\}$ has the *fip*, then $\bigcap\mathcal{U} \neq \varnothing$

$\Leftrightarrow$ every family of closed sets with the *fip* has nonempty intersection. ■

5.17. Corollary. If $\mathcal{F}$ is a nonempty family of nonempty closed sets which is totally ordered by inclusion, then $\bigcap\mathcal{F} \neq \varnothing$. ■

5.18. Definition. Let X be a topological space and $A \subseteq V \subseteq X$. V is said to be a *neighbourhood* of A if there is an open set U such that $A \subseteq U \subseteq V$.

5.19. Theorem. If A is a compact subset of a Hausdorff space X and $x \in X - A$, then there are disjoint neighbourhoods of x and of A.

Proof. Since X is Hausdorff, for each y 0 A there are disjoint open neighborhoods $U(y)$ and $V(y)$ of x and y respectively. The family $\{V(y)\colon y \in A\}$ is an open cover of A which, since A is compact, has a finite subcover $\{V(y_i)\colon 1 \leq I \leq n\}$. Then $U(y_1) \cap \ldots \cap U(y_n)$ and $V(y_1) \cup \ldots \cup V(y_n)$ are the required disjoint neighbourhoods of x and A. ■

5.20. Corollary. A compact subset of a Hausdorff space is closed.

Proof. By **5.19** the complement of a compact subset A of a Hausdorff space contains a neighbourhood of each of its points and is therefore open. It follows that A is closed.

Notice, however, that a compact subset of an *arbitrary* topological space need not be closed: for instance, every subset of an indiscrete space is compact but only the empty set and the whole space are closed.

5.21. Proposition. A closed subset of a compact space is compact.

Proof. If $\mathcal{U}$ is an open cover of a closed subset A of a compact space X, then $\mathcal{U} \cup \{X - A\}$ is an open cover of X. Since X is compact, $\mathcal{U} \cup \{X - A\}$ has a finite subcover $\mathcal{V}$, and $\mathcal{V} - \{X - A\}$ is a finite subcover of $\mathcal{U}$. ■

These results enable us to characterize the compact subsets of $\mathbb{R}$.

5.22. Proposition. The compact subsets of $\mathbb{R}$ are precisely the closed bounded subsets.

Proof. Since $\mathbb{R}$ is Hausdorff, compact subsets must be closed by **5.20.** If $A \subseteq \mathbb{R}$ is not bounded, then A has a cover by open intervals of length 1 with no finite subcover, and so is not compact. Conversely, if A is closed and bounded, then it is included in some closed interval, which is compact by the Heine-Borel theorem **1.10**. As a closed subset A must be compact by **5.21.** ■

We next discuss the action of continuous functions on compact spaces.

5.23. Proposition. Let X be a compact space and $f\colon X \to Y$ a continuous surjection. Then Y is compact, and if Y is Hausdorff and f bijective, then f is a homeomorphism.

Proof. If $\mathcal{U}$ is an open cover of Y, then $\{f^{-1}[U]\colon u \in \mathcal{U}\}$ is an open cover of X which has a finite subcover since X has been assumed compact. The family of images under f of members of this subcover is a finite subfamily of $\mathcal{U}$ which covers Y; consequently, Y is compact. Suppose now that Y is Hausdorff and f bijective. If A is a closed subset of X, then A is compact by **5.21**, so its image $f[A]$ is compact by the first part of the proof and accordingly closed by **5.20.** Thus $(f^{/1})^{!1}[A] = f[A]$ is closed for each closed A, so that (by **4.2**(ii)) $f^{/1}$ is continuous, and f is a homeomorphism. ■

5.24. Corollary. Any continuous real-valued function on a compact space is bounded and attains its bounds.

Proof. Suppose X is compact, and $f\colon X \to \mathbb{R}$ continuous. Then $f[X]$ is compact —hence a closed bounded — subset of $\mathbb{R}$. Thus, by **2.10**, both $\inf(f[X])$ and $\sup(f[X])$ are members of $f[X]$, proving the corollary. ■

If in this corollary we take the compact space to be a closed interval $[a, b]$ in $\mathbb{R}$ we get a familiar result of classical analysis.

The final property we consider in this chapter is a kind of "poor man's" version of compactness.

5.25. Definition. A space is said to be *locally compact* if each point has at least one compact neighbourhood.

For example, any compact space is locally compact, while infinite discrete spaces and $\mathbb{R}$ are locally compact spaces which are not compact. Notice that the set Q of rational numbers (with the relative topology induced by the usual topology on $\mathbb{R}$) is *not* locally compact. In fact none of its points has a compact neighbourhood. For suppose, e.g. 0 had a compact neighbourhood U in Q. Then U would contain a subneighbourhood V of the form $\mathrm{Q} \cap [-a, a]$; since this is closed in Q, and hence in U, V would have to be compact. But this cannot be the case since, for each *irrational* $x \in [-a,a]$, the decreasing family of nonempty sets $V \cap [x - 1/n, x + 1/n]$ $(n \in \mathbb{N})$, each of which is closed in V, has nonempty intersection.

From this example we see that a subspace of a locally compact space need not be locally compact. However, providentially enough, we have the two following propositions.

5.26. Proposition. Any closed subspace of a locally compact space is locally compact.

Proof. Let A be a closed subset of a locally compact space. Then each $a \in A$ has a compact nighbourhood U. Since A is closed, $U \cap A$ is closed in U; since U is compact, so is $U \cap A$. Therefore $U \cap A$ is a compact neighbourhood of a in A, and so A is locally compact. ■

5.27. Proposition. An open subspace of a compact Hausdorff space is locally compact.

Proof. Let A be an open subset of a compact Hausdorff space X. Then $X - A$ is closed and hence, by **5.21**, compact. So, by **5.19**, for each $a \in A$ there are disjoint open neighbourhoods U of a and V of $X - A$. Then $U \subseteq X - V \subseteq A$ so that

$$\overline{U} \subseteq \overline{X - V} = X - V \subseteq A.$$

Since X is compact, so, by **5.21**, is the closed set $\overline{U}$. Therefore $\overline{U}$ is a compact neighbourhood of a in A, and accordingly A is locally compact. ■

In contrast with compact spaces, the continuous image of a locally compact space is not always locally compact. For any nonlocally compact space X (Q, for instance) is the continuous image —under the identity map on X —of the locally compact space X with its discrete topology.

6

Metric Spaces.

AS WE HAVE SEEN, the usual topology on $\mathbb{R}^n$ can be derived from the distance function on $\mathbb{R}^n$. We now examine this procedure in a general setting.

6.1. Definition. A *metric* on a set X is a function $\mathbf{d}$ from the product $X \times X$ to the set $\mathbb{R}^+$ of nonnegative real numbers such that, for all $x, y, z \in X$,

(i) $\mathbf{d}(x, y) = \mathbf{d}(y, x)$;
(ii) $\mathbf{d}(x, y) + \mathbf{d}(y, z) \geq \mathbf{d}(x, z)$;
(iii) $\mathbf{d}(x,y) = 0 \Leftrightarrow x = y$.

For obvious reasons, **(ii)** is known as the *triangle inequality*.

6.2. Definition. A *metric space* is a pair $(X, \mathbf{d})$ where X is a set and $\mathbf{d}$ is a metric on X. If ε is a positive real number and $x \in X$, the *open ε-sphere* about x is the set

$$S(x,\varepsilon) = \{y \in X\colon \mathbf{d}(x, y) < g\}$$

and the *closed ε-sphere* about x is the set

$$\{y \in X\colon \mathbf{d}(x, y) \leq \varepsilon\}.$$

We now show that any metric space can be assigned a natural topology.

6.3. Proposition. The family of open spheres in a metric space $(X, \mathbf{d})$ is a base for a topology on X.

Proof. We must show that the intersection of any pair of open spheres contains an open sphere about each of its points. Suppose that $x \in S(y, \varepsilon) \cap S(z, \eta)$, and write ζ for the lesser of the numbers $\varepsilon - \mathbf{d}(x, y)$, $\eta - \mathbf{d}(x, z)$. Then, for any $w \in S(x, \zeta)$, we have

$$\mathbf{d}(w, y) \leq \mathbf{d}(w, x) + \mathbf{d}(x, y) \leq \varepsilon - \mathbf{d}(x, y) + \mathbf{d}(x, y) = \varepsilon$$

and

$$\mathbf{d}(w, z) \leq \mathbf{d}(w, x) + \mathbf{d}(x, z) \leq \eta - \mathbf{d}(x, z) + \mathbf{d}(x, z) = \eta.$$

Thus w is contained in the intersection of the open spheres $S(y, \varepsilon)$ and $S(z, \eta)$; it follows that the open sphere $S(x, \zeta)$ about x is also contained in this intersection. This proves the contention, and the result follows. ■

The topology generated by the family of open spheres in a metric space $(X, \mathbf{d})$ is called the *metric topology* induced by $\mathbf{d}$. Unless otherwise specified, we shall assume that any metric space has been assigned the induced metric topology.

Examples. 1. Given a set X, define the metric $\mathbf{d}$ on X by setting $\mathbf{d}(x, y) = 1$ if $x \neq y$ and $\mathbf{d}(x, x) = 0$. Then $S(x, 1) = \{x\}$ for each $x \in X$; so $\{x\}$ is open in the metric topology induced by $\mathbf{d}$ and the metric space $(X, \mathbf{d})$ is discrete.

2. The *standard metric* $\mathbf{d}$ on $\mathbb{R}^n$ is given by

$$\mathbf{d}(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\,.$$

If $(X, \mathbf{d})$ is a metric space and Y is a subset of X, then $\mathbf{d}$ induces a metric on Y in the obvious way. It is easy to verify that the topology on Y induced by this metric coincides with the relative topology induced on Y by the metric topology on X.

6.4. Definition. Let $\mathbf{d}$ be a metric on a set X and let A and B be subsets of X. The *distance* $\mathbf{d}(A, B)$ from A to B is defined by

$$\mathbf{d}(A, B) = \inf\{\mathbf{d}(x, y) \colon x \in A \;\&\; y \in B\}.$$

In particular, the distance $\mathbf{d}(a, B)$ from a point $a \in X$ to B is

$$\mathbf{d}(a, B) = \mathbf{d}(\{a\}, B) = \inf\{\mathbf{d}(a, y) \colon y \in B\}.$$

6.5. Proposition. Suppose that $\mathbf{d}$ is a metric on X (and X is assigned the metric topology induced by $\mathbf{d}$). If A is a fixed subset of X, then the function $x \mapsto \mathbf{d}(x, A) \colon X \to \mathbb{R}$ is continuous.

Proof. Since $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$, it follows, by taking infima over $z \in A$, that $\mathbf{d}(x,A) \leq \mathbf{d}(x,y) + \mathbf{d}(y,A)$. The same inequality holds with x and y interchanged, and accordingly $|\mathbf{d}(x,A) - \mathbf{d}(y,A)| \leq \mathbf{d}(x, y)$. Consequently, if $y \in S(x,\varepsilon)$, then $|\mathbf{d}(x,A) - \mathbf{d}(y,A)| < \varepsilon$, and continuity follows. ■

6.6. Corollary. The closure of a subset of a metric space is the set of points at zero distance from it.

Proof. Let $\mathbf{d}$ be a metric on X and let $A \subseteq X$. Since (**6.5**) the function $x \mapsto \mathbf{d}(x, A)$ is continuous, as the preimage of $\{0\}$ under this function the set $\{x \in X \colon \mathbf{d}(x, A) = 0\}$ is closed. Therefore, since it contains A, it must also contain $\overline{A}$. If $y \notin \overline{A}$, then there is a neighbourhood of y, which may be taken to be an open ε-sphere about y, failing to meet A. But then $\mathbf{d}(y,A) \geq g > 0$, so that $\{x\colon \mathbf{d}(x,A) = 0\}$ is included in $\overline{A}$ and hence the two coincide. ■

6.7. Proposition. Any metric space is first countable.

Proof. For any point x of a metric space, the family of open spheres $S(x,\varepsilon)$ with rational ε is a countable neighbourhood base at x. ■

It follows from this and **3.8** that the closure of a subset A of a metric space coincides with the set of points x for which there is a sequence from A converging to x.

6.8. Definition. A topological space is said to be *separable* if it contains a countable dense subset.

For example, any Euclidean space $\mathbb{R}^n$ is separable, since the countable subset consisting of all points with rational coordinates is dense in it.

6.9. Proposition. Any topological space with a countable base is separable.

Proof. Choose a point out of each member of the base, thus obtaining a countable set A. The complement of the closure of A is an open set which, being disjoint from A, contains no nonempty member of the base and is hence empty. So A is dense. ∎

But observe that the converse of **6.9** fails. For let X be an uncountable set with the cofinite topology. Then every infinite subset of X is dense because it intersects every nonempty open set, so that, *a fortiori*, X is separable. On the other hand, suppose that X has a countable base **B** and let a be a fixed point of X. The intersection of the family of all open sets to which a belongs must be $\{a\}$ itself because the complement of every singleton is open. It follows that the intersection of all members of **B** that contain a is $\{a\}$. But the complement of this intersection is the union of countably many finite sets, which must itself be countable. Since this complement coincides with $X - \{a\}$, this set must also be countable. In that case, X itself would have to be countable, a contradiction.

However, this situation cannot arise for metric spaces:

6.10. Theorem. Any separable metric space has a countable base.

Proof. Suppose that $(X, \mathbf{d})$ is a separable metric space, let Y be a countable dense subset of X and let $\mathcal{U}$ be the family of all open g-spheres with rational g about points of Y. Then $\mathcal{U}$, as the union of a countable family of countable sets, is countable. If U is a neighbourhood of a point x of X, then there is a rational g such that $S(x,\varepsilon) \subseteq U$. Since Y is dense, $Y \cap S(x, \varepsilon/3)$ is nonempty; choose a point y in this intersection. Then $x \in S(y, 2\varepsilon/3) \subseteq S(x, \varepsilon) \subseteq U$, and $S(y, 2\varepsilon/3) \in \mathcal{U}$. So $\mathcal{U}$ is a base for X. ∎

6.11. Theorem. Any compact metric space has a countable base, and is therefore separable.

Proof. Let $(X, \mathbf{d})$ be a compact metric space. Then for each $n > 0$ the family $\{S(x, 1/n): x \in X\}$ is an open cover of X; since X is compact this cover has a finite subcover $\mathcal{B}_n$. We claim that the family $\mathcal{B} = \bigcup\{\mathcal{B}_n: n > 0\}$ is a base for X. If U is open in X, then for each $x \in U$ there is $n > 0$ for which $S(x, 1/n) \subseteq U$. Since the family $\mathcal{B}_{2n}$ covers X, there is $y \in X$ such that $x \in S(y, 1/2n) \in \mathcal{B}_{2n}$. Clearly $S(y, 1/2n) \subseteq S(x, 1/n) \subseteq U$. Therefore $\mathcal{B}$ is a base for X as claimed; since $\mathcal{B}$, as the union of a countable family of countable sets, is countable, it follows that X is separable.
∎

Call a topological space $(X, \mathfrak{I})$ *metrizable* if there is a metric **d** on X such that the metric topology induced by **d** is identical with $\mathfrak{T}$. We shall see later that a compact space is metrizable if and only if it possesses a countable base.

We conclude this chapter with a discussion of connected metric spaces.

6.12. Definition. Let $(X, \mathbf{d})$ be a metric space. For $\varepsilon > 0$, x, y 0 X, an ε-*chain* in X *linking* x and y is a finite sequence $x_1, \ldots, x_n$ of points of X such that $x_1 = x$, $x_n = y$ and $\mathbf{d}(x_i, x_{i+1}) \leq \varepsilon$ for $i = 1, \ldots, n-1$. $(X, \mathbf{d})$ is said to be *fine-grained* if for all $\varepsilon > 0$, any pair of points of X can be linked by an ε-chain.

6.13. Theorem. Any connected metric space is fine-grained.

Proof. Let $(X, \mathbf{d})$ be a connected metric space and for $x \in X$, $\varepsilon > 0$, write $X(x, \varepsilon)$ for the set of points of X which can be linked to x by an ε-chain. Then $X(x, \varepsilon)$ is nonempty since it contains x; it is open, since if y is a point of $X(x, \varepsilon)$, the same is true of every point z such that $\mathbf{d}(y, z) < \varepsilon$. It is closed, because, if $y \in \overline{X(x,\varepsilon)}$, there is $z \in X(x,$ g$)$ such that $\mathbf{d}(x, y) < \varepsilon$. Since X is connected, $X(x, \varepsilon)$ must then coincide with X. This holds for arbitrary $x \in X$, so X is fine-grained. ■

The converse of this theorem is false in general: for example, the metric space Q is easily seen to be fine-grained, but not connected. However, the converse does hold for *compact* metric spaces. To prove this assertion, we require the following

6.14. Lemma. If A and B are disjoint closed subsets of a metric space $(X, \mathbf{d})$ and A is compact, then $\mathbf{d}(A, B) > 0$.

Proof. If $x \in A$ then $x \notin B = \overline{B}$, and it follows from **6.6** that $\mathbf{d}(x, B) > 0$. The set $D = \{\mathbf{d}(x, B): x \in A\}$ is the image of the compact set A under the continuous map **(6.5)** $x \mapsto \mathbf{d}(x, B)$. Therefore, by **5.23** and **5.20**, D is a compact closed subset of $\mathbb{R}$ which does not contain 0. Then $\mathbf{d}(A, B) = \inf(D) \in \overline{D} = D$, and it follows, in particular, that $\mathbf{d}(A, B) \neq 0$, whence $\mathbf{d}(A, B) > 0$. ■

6.15. Theorem. Any compact fine-grained metric space is connected.

Proof. Let $(X, \mathbf{d})$ be a compact metric space. If X is not connected, then $X = A \cup B$ with A, B disjoint and closed, hence compact, subsets of X. By the previous lemma, $\mathbf{d}(A, B) = \delta > 0$. If $0 < \varepsilon < \delta$, then no point of A can be linked to a point of B by an ε-chain. For if $x_1, \ldots, x_n$ were such a chain, let $i > 1$ be the smallest index such that $x_i \in B$; then $x_{i-1} \in A$ and $\mathbf{d}(x_{i-1}, x_i) \leq$ g, contradicting $\mathbf{d}(A, B) = \delta$. So X is not fine-grained. ■

This last result can be employed to give another proof of **5.10**. For each closed interval in $\mathbb{R}$ is compact and obviously fine-grained, and so, by **6.15**, connected. Furthermore, it is easily seen that any nonempty interval in $\mathbb{R}$ is the union of closed —hence connected intervals with a common point, and so, by **5.8,** connected.

7

Product Spaces

WE SHALL NEED the following

7.1. Definition. Let X be a set, $\{X_i: i \in I\}$ an indexed family of topological spaces, and for each $i \in I$ let f_i be a function from X to X_i. The topology generated by the family of all subsets of the form $f_i^{-1}[U]$, with U open in X_i, for arbitrary $i \in I$ is called the *weak topology* induced by the set of functions $\{f_i: i \in I\}$ (or, for short, by the f_i). Sets of the form $f_i^{-1}[U]$ are called *subbasic open sets* in the weak topology.

Observe that, under the conditions of **7.1**, the family of subsets of X of the form $f_{i1}^{-1}[U_1] \cap \ldots \cap f_{in}^{-1}[U_n]$ with each U_j open in X_i , and $\{i_1,...,i_n\} \subseteq I$ is a base for the weak topology induced by the f_i. Sets of this form will be called the *basic open sets* determined by the f_i.

7.2. Proposition. Assume the conditions of **7.1**. Then:

(i) the weak topology induced by the f_i is the least topology (under inclusion) on X with respect to which each f_i is continuous;

(ii) a map g from a topological space Y to X is continuous iff each composition $f_i \circ g$ is continuous.

Proof. (i). Suppose that X has been assigned the weak topology. Let $x \in X$ and let U be an open neighbourhood of $f_i(x)$. Then by definition $f_i^{-1}[U]$ is an open neighbourhood of x in X. Since x was arbitrary, f_i is continuous. If $\mathfrak{I}$ is a topology on X with respect to which every f_i is continuous, then, for any $i \in I$ and open U in X_i, $f_i^{-1}[U]$ must be a member of $\mathfrak{I}$. Therefore $\mathfrak{I}$ contains every member of the family of sets generating the weak topology. Since the weak topology is the least topology containing this family, it follows that $\mathfrak{I}$ contains the weak topology.

(ii). One direction follows immediately from **(i)** and the fact that a composition of continuous functions is continuous. Conversely, suppose $f_i \circ g$ continuous for all $i \in I$. If $y \in Y$ and V is a neighbourhood of $g(y)$, then there is a subset $\{i_1,...,i_n\} \subseteq I$ and sets $U_1, ..., U_n$ open in $X_1 ,..., X_i$, respectively, such that

$$g(y) \in f_{i1}^{-1}[U_1] \cap \ldots \cap f_{in}^{-1}[U_n] \subseteq V.$$

Applying g^{-1} to this gives

$$(*) \qquad y \in (f_{i1} \circ g)^{-1}[U_1] \cap \ldots \cap (f_{in} \circ g)^{-1}[U_n] \subseteq g^{-1}[V].$$

But in view of the continuity of each $f_i \circ g$, each intersectand in (*) is open in Y so that $g^{-1}[V]$ is a neighbourhood of y. Since y was arbitrary, g is continuous. ■

We now make our central

7.3. Definition. Let $X_1, ..., X_n$ be topological spaces, and write X for their product $X_1 \times ... \times X_n$. Recall that for each $i = 1, ..., n$ there is a projection map $\pi_i: X \to X_i$. The weak topology induced by the set of projections $\{\pi_1, ... \pi_n\}$ is called the *product topology* on X. The resulting topological space is called the *product* of the spaces $X_1, ..., X_n$; these latter are called *coordinate spaces*.

It is readily shown that each basic open set in $X_1 \times ... \times X_n$ determined by the π_i is of the form $U_1 \times ... \times U_n$ with U_i open in X_i for $i = 1, ..., n$.

7.4. Example. *Products of metric spaces.* Let $(X_1, \mathbf{d}_1), ..., (X_n, \mathbf{d}_n)$ be a finite sequence of metric spaces, and let $X = X_1 \times ... \times X_n$. Define the map $\mathbf{d}: X \times X \to \mathbb{R}$ by setting, for each $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in X,

$$\mathbf{d}(x, y) = \sqrt{\sum_{i=1}^{n} \mathbf{d}(x_i, y_i)^2}$$

Using standard inequalities, it is not hard to show that $\mathbf{d}$ is a metric on X: it is called the *product* of the metrics $\mathbf{d}_1, ..., \mathbf{d}_n$. The resulting metric space $(X, \mathbf{d})$ is called the *product* of the metric spaces $(X_1, \mathbf{d}_1), ..., (X_n, \mathbf{d}_n)$. Each of these, equipped with the metric topology, is a topological space, so we may assign the resulting product topology to X. Happily, this product topology turns out to be identical with the metric topology induced by $\mathbf{d}$. To verify this, we need only show that each basic open neighbourhood $U = U_1 \times ... \times U_n$ of a point $x \in X$ there is a neighbourhood of the form $V(\varepsilon) = \{y \in X: \mathbf{d}(x, y) < \varepsilon\}$ contained in U and conversely. Now each U_i is of the form $\{u \in X_i: \mathbf{d}_i(u, x_i) < \varepsilon_i\}$, so that if we take $\varepsilon = \inf \{\varepsilon_i: 1 \leq i \leq n\}$ we see that $V(\varepsilon) \subseteq U$. Conversely, given a neighbourhood of the form $V(\varepsilon)$, if we define $U_i = \{u \in X_i: \mathbf{d}_i(u, x_i) < \varepsilon/\sqrt{n}\}$, it is easily shown that $U = U_1 \times ... \times U_n \subseteq V(\varepsilon)$.

We now turn our attention to the following problem. Call a property $\mathscr{P}$ of topological spaces *productive* if, whenever $X_1, ..., X_n$ is a finite sequence of spaces each of which has $\mathscr{P}$, the product space $A_1 \times ... \times A_n$ also has $\mathscr{P}$. Which properties are productive? It is obvious that, in order to establish the productivity of a given property $\mathscr{P}$, we need only prove that if a pair of spaces X, Y have $\mathscr{P}$, then so does $X \times Y$: the general case then follows by induction. Armed with this knowledge, we now set about showing that many of the topological properties we have introduced are productive.

7.5. Proposition. Metrizability is a productive property.

Proof. By example **7.4.** ■

7.6. Proposition. "Hausdorffness" is a productive property.

Proof. Let X, Y be Hausdorff spaces and let (x, y), (u, v) be distinct points of $X \times Y$. Then either $x \neq u$ or $y \neq v$; suppose the former obtains. Since X is Hausdorff, there are disjoint neighbourhoods U, V of x, u respectively, and $U \times Y$, $V \times Y$ are then the requisite disjoint neighbourhoods of (x, y), (u, v), respectively, in $X \times Y$. The argument in the case $y \neq v$ is similar. ■

7.7. Theorem. Compactness is a productive property.

Before proceeding to the proof of this result we require the following

7.8. Lemma. For any topological spaces X, Y and any point x of X, Y is homeomorphic to the subspace $\{x\} \times Y$ of $X \times Y$.

Proof. Let $f\colon Y \to \{x\} \times Y$ be the function $y \mapsto (x, y)$: we claim that f is a homeomorphism. It is obviously bijective; also $\pi_1 \circ f$ is the constant function with value x, and $\pi_2 \circ f$ is the identity function on Y. Since both are continuous, so, by **7.2(ii)**, is f. Finally, if U is open in Y, then $f[U] = \{x\} \times U = (X \times U) \cap (\{x\} \times Y)$ which is open in $\{x\} \times Y$ since $X \times U$ is open in $X \times Y$. Therefore f is a homeomorphism as claimed. ■

Proof of 7.7. Let X and Y be compact topological spaces, and let $\mathcal{U}$ be an open cover of $X \times Y$. Then for each $t = (x, y) \in X \times Y$ there is $U(t) \in \mathcal{U}$ such that t 0 $U(t)$ and so there are open neighbourhoods $V(t), W(t)$ of x, y respectively such that $V(t) \times W(t) \subseteq U(t)$. For each $x \in X$, $Y_x = \{x\} \times Y$ is homeomorphic to Y by **7.8** and is accordingly compact since Y is. Therefore, since $\{V(t) \times W(t)\colon t \in Y_x\}$ is an open cover of Y_x, it has a finite subcover $\{V(t_j) \times W(t_j)\colon j \in J(x)\}$ with $J(x)$ finite and $t_j = (x, y_j)$ for $j \in J(x)$. Observe that $\{W(t_j)\colon j \in J(x)\}$ is an open cover of Y. Also $V'(x) = \bigcap_{j \in J(x)} V(t_j)$ is an open neighborhood of x and we have, for each $x \in X$,

$$V'(x) = \bigcap_{j \in J(x)} V(t_j) \times \bigcup_{j \in J(x)} W(t_j) \subseteq \bigcup_{j \in J(x)} V(t_j) \times W(t_j) \subseteq \bigcup_{j \in J(x)} U(t_j). \qquad (*)$$

Since $\{V'(x)\colon x \in X\}$ is an open cover of X, and X is compact, it has a finite subcover $\{V'(x_k)\colon k \in K\}$. By (*) the finite subfamily $\bigcup_{k \in K} \{U(t_j) : j \in J(x_k)\}$ of $\mathcal{U}$ covers $X \times Y$. So each open cover of $X \times Y$ has a finite subcover, and $X \times Y$ is compact. ■

7.9. Corollary. The compact subsets of a Euclidean space are precisely its closed bounded subsets. (A subset of a Euclidean space is said to be *bounded* if it is a subset of a product of bounded closed intervals.)

Proof. If A is a compact subset of $\mathbb{R}^n$, then, since the latter is Hausdorff, A must be closed. Also, since projections onto coordinate spaces are continuous, the image of A in $\mathbb{R}$ under each such projection must, as the continuous image of a compact set, be a compact subset of $\mathbb{R}$, and hence, by **5.22**, contained in a bounded closed interval. It follows that A, as a subset of the product of its images under projections, is bounded.

Conversely, if A is closed and bounded, it is a closed subset of a finite product of compact intervals of $\mathbb{R}$ which, by **7.7,** must itself be a compact subset of $\mathbb{R}^n$. Therefore A is compact and the result is proved. ■

7.10. Corollary. Local compactness is a productive property.

Proof. Let X and Y be locally compact spaces and let (x, y) be a point of $X \times Y$. Then x, y have compact neighbourhoods U, V in X, Y respectively; the product $U \times V$ is by **7.7** a compact neighbourhood of (x, y). ■

7.11. Theorem. Connectedness is a productive property.

Before proceeding to the proof of this result we need

7.12. Definition. A mapping f between topological spaces is said to be *open* if the image of any open set under f is open.

7.13. Lemma. Any projection onto a coordinate space is an open mapping.

Proof. Let U be an open subset of a product $X_1 \times ... \times X_n$. Then for each point $(x_1,...,x_n)$ of U and each $i = 1, ..., n$ there is an open neighbourhood U_i of x_i such that $U_1 \times ... \times U_n \subseteq U$. But then $U_i \subseteq \pi_i[U]$, so that the latter contains a neighbourhood of each of its points and is therefore open. It follows that each π_i is an open mapping. ■

Proof of 7.11. Let X and Y be connected spaces. Suppose that $X \times Y$ is the union of two nonempty disjoint open subsets U, V. By **7.8,** $\{x\} \times Y$ is homeomorphic to Y, and therfore connected, for each $x \in X$. It follows that, for each $x \in X$, $\{x\} \times Y$ is contained either in U or in V. Therefore $x \in \pi_1[U] \Rightarrow \{x\} \subseteq Y \subseteq U$ and $x \in \pi_1[V] \Rightarrow \{x\} \times Y \subseteq V$, so that, taking the union over $x \in \pi_1[U]$ and $x \in \pi_1[V]$ respectively, we get

(*) $$\pi_1[U] \times Y \subseteq U \quad \pi_1[V] \times Y \subseteq V.$$

By **7.13**, $B_1[U]$ and $B_1[V]$ are open; since they are both nonempty and their union is X (recall that $U \cup V = X \times Y$), their intersection must be nonempty in view of the connectedness of X. It now follows from (*) that $\varnothing \neq \varnothing \neq$

$$(\pi_1[U] \cap \pi_1[V]) \times Y = (\pi_1[U] \times Y) \cap (\pi_1[V] \times Y) \subseteq U \cap V.$$

But this contradicts the assumption that U and V are disjoint. Therefore $X \times Y$ is connected.

From **5.10** and **7.11** we immediately infer

7.14. Any Euclidean space is connected.

8

Normal Spaces and Urysohn's Theorem

IN THIS CHAPTER we introduce a class of spaces—the *normal spaces*—which turn out to be richly endowed with continuous real-valued functions.

8.1. Definition. A topological space is said to be *normal* if any pair of disjoint closed subsets have disjoint neighbourhoods.

8.2. Theorem. Any compact Hausdorff space is normal.

Proof. Let A and B be disjoint closed subsets of a compact Hausdorff space. Then A and B are compact and, by **5.19**, for each b 0 B there are disjoint open neighbourhoods $U(b)$ of A and $V(b)$ of b. The family $\{V(b): b \in B\}$ is an open cover of B; since B is compact it has a finite subcover $\{V(b_i): 1 \leq I \leq n\}$. The open sets $\bigcap_{1 \leq i \leq n} U(b_i)$ and $\bigcup_{1 \leq i \leq n} V(b_i)$ are then disjoint neighbourhoods of A and B respectively. ■

8.3. Theorem. Any metric space is normal.

Proof. Let $(X, \mathbf{d})$ be a metric space and A, B closed subsets of X. Let $U = \{x: \mathbf{d}(x,A) < \mathbf{d}(x,B)\}$ and $V = \{x: \mathbf{d}(x,B) < \mathbf{d}(x,A)\}$. Since A and B are closed, it follows from **6.6.** that $A \subseteq U$ and $B \subseteq V$. And since $\mathbf{d}(x,A)$ and $\mathbf{d}(x,B)$ are both continuous in x, U and V are open. Therefore U and V are disjoint neighbourhoods of A and B, respectively; normality follows. ■

We shall need

8.4. Lemma. If B is a neighbourhood of a closed subset A of a normal space X, then there is an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq B$.

Proof. There is an open set V such that $A \subseteq V \subseteq B$, so that A and $X - V$ are disjoint closed sets, which, by the normality of X, have disjoint open neighbourhoods U and W respectively. Then $A \subseteq U \subseteq X - W \subseteq V$; since $X - W$ is closed, $A \subseteq U$ f $\overline{U} \subseteq X - W \subseteq V$. The open set U meets the requirements of the Lemma. ■

Now we prove

8.5. Urysohn's Theorem. Let A and B be disjoint closed subsets of a normal space X. Then there exists a continuous function $f: X \to [0,1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

Proof. The set $X - B$ is a neighbourhood of the closed set A, so, by the normality of X and the preceding lemma, A has an open neighbourhood $U(½)$ such that

$$A \subseteq U(½) \subseteq \overline{U(½)} \subseteq X - B.$$

Now $U(½)$ and $X - B$ are neighbourhoods of the closed sets A and $\overline{U(½)}$ respectively, so in the same way there exist open sets $U(¼)$ and $U(¾)$ such that

$$A \subseteq U(¼) \subseteq \overline{U(¼)} \subseteq U(½) \subseteq \overline{U(½)} \subseteq U(¾) \subseteq \overline{U(¾)} \subseteq X - B.$$

Continuing this process indefinitely, for each dyadic rational number $m/2^n = r$ (where $n = 1, 2, 3,...$ and $m = 1,2,...,2^n - 1$) we obtain an open set $U(r)$ such that, for $r_1 < r_2$,

$$A \subseteq U(r_1) \subseteq \overline{U(r_1)} \subseteq U(r_2) \subseteq \overline{U(r_2)} \subseteq X - B.$$

We can now define the function $f: X \to [0,1]$ by setting

$$f(x) = 0 \text{ if } x \in \bigcap_r U(r) \qquad f(x) = \sup\{r: x \notin U(r)\} \text{ if } x \notin \bigcap_r U(r).$$

Clearly $f[A] = \{0\}$ and $f[B] = \{1\}$. We have finally to show that f is continuous; and for this it suffices to show that $f^{-1}[[0,a)]$ and $f^{-1}[(a,1]]$ are open for each $a \in (0,1)$. Now $f(x) < a \Leftrightarrow x \in U(r)$ for some $r < a$; hence

$$f^{-1}[[0,a)] = \{x: f(x) < a\} = \bigcap\{U(r): r < a\},$$

and this latter set is open. Similarly, $f(x) > a \Leftrightarrow x \in \overline{U(r)}$ for some $r > a$, so that

$$f^{-1}[(a,1]] = \{x: f(x) > a\} = \bigcup\{X - \overline{U(r)} : r > a\},$$

and this last set is also open. Therefore f is continuous. ■

Let us call a space X *completely regular* if for each point x of X and each closed subset not containing x there is a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f[A] = 1$. Using the observation that singletons are closed in Hausdorff spaces, we see from Urysohn's theorem that *every normal Hausdorff space is completely regular.* In particular, we have

8.6. Corollary. Let X be a normal Hausdorff space. Then for any distinct points x, y of X there is a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. ■

In other words, for any normal Hausdorff space, there are enough real-valued continuous functions defined on it to distinguish its points.

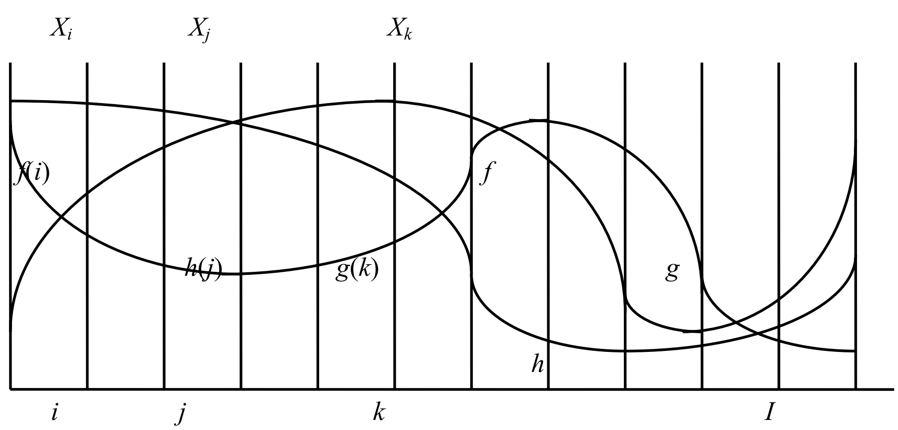
9

Generalized Product Spaces and Tihonov's Theorem

SUPPOSE WE ARE GIVEN an arbitrary—possibly infinite—family $\{X_i: i \in I\}$ of topological spaces. We show how the product of such a family may be defined. First, we need to specify what is meant by the product of the family of *sets* $\{X_i: i \in I\}$. By analogy with the finite case, this ought to be the collection of all "I-tuples" $(x_i: i \in I)$ with $x_i \in X_i$ for each $i \in I$. But what, exactly, is an "I-tuple"? A plausible definition is: "a function with domain I". In that case we make the following

9.1. Definition. Let $\{X_i: i \in I\}$ be a nonempty indexed family of nonempty sets. A *choice function* on $\{X_i: i \in I\}$ is a map $f: I \to \{X_i: i \in I\}$ such that $f(i) \in I$ for all $i \in I$. The *product* $\prod_{i \in I} X_i$ of the family is defined to be the set of all choice functions on it. If each X_i is identical with some set X, then the product is simply the set of all functions from I to X, and is denoted by X^I.

Notice that, if I is finite, say $I = \{1, ..., n\}$, then $\prod_{i \in I} X_i$ is essentially the same as $X_1 \times ... \times X_n$ defined Chapter **0.** The diagram below illustrates the general situation.



Here the horizontal axis represents I, the vertical lines the X_i, and the curves typical choice functions on $\{X_i: i \in I\}$. Just as in the case of finitely many sets, we define the *projections* $\pi_i: \prod_{i \in I} X_i \to X_i$ by setting $\pi_i(f) = f(i)$ for each $i \in I$ and $f \in \prod_{i \in I} X_i$. Thus $\pi_i(f)$ is the "i^{th} coordinate" of f in the sense that, if we write x_i for $f(i)$ and represent f as the "I-tuple" $(x_i: i \in I)$, then $\pi_i(f)$ is its "i^{th} coordinate" x_i.

Now we make the following

9.2. Definition. Let $\{X_i: i \in I\}$ be a family of topological spaces. The weak topology induced by the family $\{\pi_i: i \in I\}$ of projections is called the *product topology* on $\prod_{i \in I} X_i$. Equipped with this topology $\prod_{i \in I} X_i$ is called the *product* of the family $\{X_i: i \in I\}$, and each X_i is called a *coordinate space.*

By definition, the product topology is the smallest topology on a product which renders all projections onto coordinate spaces continuous. Basic open subsets of a product determined by the family of projections onto coordinate spaces will simply be called *basic sets.* In a product $\prod_{i \in I} X_i = X$, each basic set is of the form

(*) $$\pi_i^{-1}[U_1] \cap \ldots \cap \pi_i^{-1}[U_n] = \{f \in X: f(i_1) \in U_1 \ \& \ \ldots \ \& \ f(i_n) \in U_n\},$$

where $\{i_1,\ldots,i_n\} \subseteq I$ and each U_k is open in X_i . The set (*) is called the *basic set determined by* $U_1, \ldots, U_n$. Each *subbasic open set* in the product topology is of the form

$$\pi_i^{-1}[U] = \{f \in X: f(i) \in U\}$$

with U open in X_i.

The product topology is sometimes called the *topology of pointwise convergence* in view of the following result.

9.3. Proposition. Let $(f_n) = (f_0, f_1, \ldots)$ be a sequence of elements of the product X of a family $\{X_i: i \in I\}$ of topological spaces. Then (f_n) converges (in the product topology) to an element f of X iff the sequence $(f_n(i)) = (f_0(i), f_1(i),\ldots)$ converges to $f(i)$ for each $i \in I$.

Proof. Suppose (f_n) converges to f, and let U be a neighbourhood of $f(i)$. Then $\pi_i^{-1}[U] = V$ is a neighbourhood of f, and so, since (f_n) converges to f, there is m such that $f_n \in V$ whenever $m \geq n$. But then $f_n(i) \in U$ for all $m \geq n$ and so $(f_n(i))$ converges to $f(i)$ for each $i \in I$. Conversely, suppose that $(f_n(i))$ converges to $f(i)$ for each $i \in I$, and let U be a neighbourhood of f in X. Then U contains a basic neighbourhood V of f determined by open neighbourhoods $U_1, \ldots, U_n$ of $f(i_1),\ldots,f(i_n)$ in some coordinate spaces $X_{i1} ,\ldots,X_{in}$ respectively. Since $(f_n(i_j))$ converges to $f(i_j)$, for each $j = 1, \ldots, n$ there is a number m_j such that $f_k(i_j) \in U_j$ whenever $k \geq m_j$. If m is the largest of the numbers $m_1,\ldots,m_n$, then $f_k \in V$ whenever $k \geq m$, and therefore (f_n) converges to f. ■

We now set about proving what is probably the most important result in general topology: *Tihonov's theorem* that the product of an arbitrary family of compact spaces is compact. In order to prove this theorem we need a little more set-theoretic machinery.

9.4. Definition. A subfamily $\mathcal{C}$ of a family $\mathcal{F}$ of sets is called a *chain in* $\mathcal{F}$ if it is totally ordered by inclusion, that is, if $X \subseteq Y$ or $Y \subseteq X$ for any $X, Y \in \mathcal{C}$. $\mathcal{F}$ is said to be *closed under unions of chains* if, whenever $\mathcal{C}$ is a chain in $\mathcal{F}$, we have $\bigcup\mathcal{C} \in \mathcal{F}$. A member M of $\mathcal{F}$ is said to be *maximal* if for any $X \in \mathcal{F}$ we have $M \subseteq X \Rightarrow \quad M = X$.

The following lemma, whose proof we omit, gives a sufficient condition for a family of sets to have a maximal member.

9.5. Zorn's Lemma. Any family of sets closed under unions of chains has a maximal member.

Zorn's lemma is used in the proof of

9.6. Tihonov's Theorem. Any product of compact spaces is compact.

Proof. Let $\{X_i: i \in I\}$ be a family of compact spaces. To show that their product X is compact it suffices to show that, if $\mathfrak{F}$ is any family of closed subsets of X with the *fip*, then $\bigcap\mathfrak{F} \neq \varnothing$. So let $\mathfrak{F}$ be such a family and let $\mathfrak{Z}$ be the collection of all families of subsets of X which include $\mathfrak{F}$ and have the *fip*. It is a simple matter to verify that $\mathfrak{Z}$ is closed under unions of chains, and so by Zorn's lemma it has a maximal member $\mathcal{M}$. We show that $\bigcap_{M \in \mathcal{M}} \overline{M} \neq \varnothing$.

First of all observe that $\mathcal{M}$ satisfies the two following conditions:

(i) $M_1,...,M_n \in \mathcal{M} \Rightarrow M_1 \cap \ldots \cap M_n \in \mathcal{M}$;

(ii) $A \subseteq X$ & $A \cap M \neq \varnothing$ for all $M \in \mathcal{M} \Rightarrow A \in \mathcal{M}$.

To verify (i): If $M_1,...,M_n \in \mathcal{M}$, then clearly the family $\mathcal{M} \cup \{M_1 \cap \ldots \cap M_n\}$ is a member of $\mathfrak{Z}$; since it includes $\mathcal{M}$ and $\mathcal{M}$ is maximal in $\mathfrak{Z}$, it must coincide with $\mathcal{M}$, so that, *a fortiori*, $M_1 \cap \ldots \cap M_n$ must be a member of $\mathcal{M}$. For (ii), suppose that A is a subset of X which meets every member of $\mathcal{M}$. Then, for each finite subset $\{M_1,..., M_n\}$ of $\mathcal{M}$ we have by (i) $M_1 \cap \ldots \cap M_n \in \mathcal{M}$, so that $A \cap (M_1 \cap \ldots \cap M_n) \neq \varnothing$. Therefore $\mathcal{M} \cup \{A\}$ has the *fip*, and so is a member of $\mathfrak{Z}$ including $\mathcal{M}$. The latter's maximality implies then that $A \in \mathcal{M}$. This proves (ii).

Now for each $i \in I$ the family $\{\overline{\pi_i[M]}: M \in \mathcal{M}\}$ of closed subsets of the compact space X_i has the *fip* (since $\mathcal{M}$ itself does) and hence nonempty intersection. For each $i \in I$ choose a member x_i of this intersection. Then $x = (x_i: i \in I) \in X$ has the property that each open neighbourhood U of x_i meets $\pi_i[M]$, and so $\pi_i^{!1}[U]$ meets M, for any $M \in \mathcal{M}$. Therefore, by (ii), $\pi_i^{!1}[U] \in \mathcal{M}$. It follows now from (i) that, for any open neighbourhoods $U_1,...,U_n$ of $x_{i1}, ..., x_{in}$ respectively, $\pi_{i1}^{!1}[U_1] \cap \ldots \cap \pi_{in}^{!1}[U_n] \in \mathcal{M}$. In other words, every basic neighbourhood of x is a member of $\mathcal{M}$. Since $\mathcal{M}$ has the *fip*, each basic neighbourhood of x meets each member of $\mathcal{M}$, that is, x is in the closure of each member of $\mathcal{M}$. Thus $x \in \bigcap_{M \in \mathcal{M}} \overline{M} \neq \varnothing$.

Finally, since each member of $\mathfrak{F}$ is closed, and $\mathfrak{F} \subseteq \mathcal{M}$, it follows that $\bigcap_{M \in \mathcal{M}} \overline{M} \subseteq \bigcap\mathfrak{F}$, so that $\bigcap\mathfrak{F} \neq \varnothing$ and the result follows. ■

10

Urysohn's Metrization Theorem

IN THIS SECTION we prove a classical result of Urysohn, which gives a purely topological sufficient condition for an arbitrary space to be metrizable. Restricted to compact Hausdorff spaces, this condition is also necessary. We shall need some definitions and preliminary results.

10.1. Definition. A *cube* is a product space of the form J^X, where J is the closed unit interval [0,1] with its usual topology (and J^X has been assigned the product topology).

By the theorems of Heine-Borel and Tihonov, any cube is *compact*. Furthermore we have

10.2. Proposition. If X is countable, the cube J^X is metrizable.

Proof. If X is finite, the assertion follows from **7.5.**

Suppose now that X is countable and infinite; then we may assume without loss of generality that $X = \mathbb{N}$, the set of natural numbers. We have to find a metric on $J^{\mathbb{N}}$ which induces the product topology. To this end, define **d** by setting, for $f, g \in J^{\mathbb{N}}$,

$$\mathbf{d}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \mid f(n) - g(n) \mid$$

Using standard inequalities, it is not hard to show that **d** is a metric on $J^{\mathbb{N}}$. To show that the metric topology induced by **d** coincides with the product topology, observe first that, if V is an open 2^{-p}-sphere about a point $f \in J^{\mathbb{N}}$ and

$$U = \{g \in J^{\mathbb{N}}: *f(n) - g(n)* < 2^{-p-n-2} \text{ for all } n < p + 2\},$$

then $U \subseteq V$. For if $g \in U$, then

$$\mathbf{d}(f, g) < \sum_{n=0}^{p+2} 2^{-p-n-2} + \sum_{n=p+1}^{\infty} 2^{-n} < 2^{-p-1} + 2^{-p-1} = 2^{-p}.$$

Since U is a neighbourhood of f in the product topology on $J^{\mathbb{N}}$, it follows that each set which is open in the metric topology is also open in the product topology. Conversely, let U be a member of the defining subbase for the product topology. Then $U = \{f \in J^{\mathbb{N}}: f(n) \in W\}$ for some $n \in \mathbb{N}$ and some open set W in J. If $f \in U$, then, since W is a neighborhood of $f(n)$, for some real $\varepsilon > 0$ we have $\{x \in J: |f(n) - x| < \varepsilon\} \subseteq W$. But then the open $\varepsilon 2^{-n}$-sphere about f in $J^{\mathbb{N}}$ is contained in U. For we have $\mathbf{d}(f, g) > 2^{-n} |f(n) - g(n)|$, so that if g lies in the open $g2^{-n}$-sphere about f, then

$$|f(n) - g(n)| \leq 2^n \mathbf{d}(f, g) < 2^n . \varepsilon 2^{-n} = \varepsilon,$$

so that $g \in U$. It follows that each member of the defining subbase, and hence each member of the product topology, is open in the metric topology. ■

It follows from **10.2** that every subspace of $J^{\mathbb{N}}$ is metrizable, so that any space homeomorphic to such a subspace is metrizable.

Our aim now is to find sufficient conditions on a space for it to be homeomorphic to a subspace of $J^{\mathbb{N}}$, and so metrizable.

10.3. Definition. Let X be a topological space and $\mathscr{F}$ a family of functions to $\mathbb{R}$. $\mathscr{F}$ is said to (i)
distinguish points of X if for each pair x, y of distinct points of X there is $f \in \mathscr{F}$ such that $f(x) \neq f(y)$; (ii)
distinguish points and closed sets of X if for each nonempty closed set A of X and each $x \in X - A$ there is $f \in \mathscr{F}$ such that $f(x) = 0$ and $f[A] = \{1\}$. An *Urysohn family* on X is a family of *continuous* functions on X to J satisfying (i) and (ii).

Let $\mathscr{F}$ be a family of functions on a space X to J. For each x 0 X we define $e_x \in J^{\mathscr{F}}$ by setting $e_x(f) = f(x)$ for $f \in J^{\mathscr{F}}$. Now define $\boldsymbol{e}: X \to J^{\mathscr{F}}$ by setting $\boldsymbol{e}(x) = e_x$ for each $x \in X$. The map $\boldsymbol{e}$ is called the *evaluation map* of X into $J^{\mathscr{F}}$.

10.4. Proposition. Let $\mathscr{F}$ be a family of *continuous* functions on a space X to J. Then

(i) the evaluation map $\boldsymbol{e}$ is a continuous map of X into the cube $J^{\mathscr{F}}$;

(ii) if, in addition, $\mathscr{F}$ distinguishes points and closed sets of X, then $\boldsymbol{e}$ is an open mapping to the subspace $\boldsymbol{e}[X]$ of $J^{\mathscr{F}}$;

(iii) $\boldsymbol{e}$ is injective iff $\mathscr{F}$ distinguishes points.

Proof. (i) We may regard $J^{\mathscr{F}}$ as a product $\prod_{f \in \mathscr{F}} J_f$, where each J_f coincides with J. Then the composition of each projection π_f with $\boldsymbol{e}$ is f, a continuous map. Therefore, by **7.2(ii),** $\boldsymbol{e}$ is continuous.

(ii) To prove this it suffices to show that if $\mathscr{F}$ distinguishes points and closed sets, then for each $x \in X$ and each open neighbourhood U of x, there is a neighbourhood V of $\boldsymbol{e}(x)$ in $J^{\mathscr{F}}$ such that $V \cap \boldsymbol{e}[X] \subseteq \boldsymbol{e}[U]$. If $U = X$ the assertion is trivial, so assume $U \neq X$. Then, since $\mathscr{F}$ distinguishes points and closed sets, we can choose $f \in \mathscr{F}$ to satisfy $f(x) = 0$, $f[X - U] = \{1\}$. It is now easily verified that $V = \{$n $\in J^{\mathscr{F}}$: n$(f) \neq 1\}$ is an open neighbourhood of $\boldsymbol{e}(x)$ in $J^{\mathscr{F}}$ satisfying the required condition.

Finally ,**(iii)** is obvious. ■

This result, together with **10.2**, gives:

10.5. Corollary. If $\mathscr{F}$ is an Urysohn family on a space X, then the evaluation map $\boldsymbol{e}$ is a homeomorphism of X to a subspace of $J^{\mathscr{F}}$. In particular, if $\mathscr{F}$ is countable, then $J^{\mathscr{F}}$ is metrizable, and so is X. ■

This corollary shows that a sufficient condition for metrizability is the existence of a countable Urysohn family. We now show that each member of an extensive class of spaces possesses such a family, and is accordingly metrizable.

10.6. Urysohn's Metrization Theorem. Any normal Hausdorff space with a countable base possesses a countable Urysohn family, and is therefore metrizable.

Proof. Let X be a normal Hausdorff space with a countable base. Since singletons are closed in any Hausdorff space, to prove the theorem it suffices to construct a family $\mathscr{F}$ of continuous functions on X to J which distinguishes points and closed sets.

Let $\mathcal{B}$ be a countable base for X and let $\boldsymbol{P}$ be the set of all pairs $(U, V) \in \mathcal{B} \times \mathcal{B}$ such that $\overline{U} \subseteq V$. For each $(U, V) \in \boldsymbol{P}$, $\overline{U}$ and $X - V$ are disjoint closed sets, so by **8.5** we can choose a function $f: X \to J$ such that $f[\overline{U}] = \{0\}$ and $f[X - V] = \{1\}$. Let $\mathscr{F}$ be the family of all such functions; then $\mathscr{F}$ is certainly countable. It remains to show that $\mathscr{F}$ distinguishes points and closed sets. Suppose that A is closed in X and $x \in X - A$. Since $\mathcal{B}$ is a base, there is $V \in \mathcal{B}$ such that $x \in V \subseteq X - A$ and by **8.4** there is an open set U such that $x \in \overline{U} \subseteq V$. The fact that $\mathcal{B}$ is a base implies that we may take U to be a member of $\mathcal{B}$. Thus $(U, V) \in \boldsymbol{P}$, and so, if f is the corresponding member of $\mathscr{F}$, then $f(x) = 0$ and $f[A] = \{1\}$. So $\mathscr{F}$ distinguishes points and closed sets and the theorem is proved. ■

8.2, 10.6, and **6.11** now give

10.7. Corollary. A compact Hausdorff space is metrizable iff it possesses a countable base. ■

Thus the property of possessing a countable base characterizes metrizable spaces within the class of compact Hausdorff spaces. The problem of characterizing (by purely topological means) metrizable spaces among *all* spaces is much more difficult and will not be taken up here.

11

The Stone-Weierstrass Theorem

ONE OF THE MOST IMPORTANT tools of classical analysis is the representation of real or complex-valued functions as power series. The partial sums of the power series expansion of a function form a sequence of *polynomials* which provide approximations to the function of increasing accuracy. Now it is not true that every continuous real-valued function has a power series expansion, but a remarkable theorem of Weierstrass —his *approximation theorem* —asserts the next best thing, namely, that every such function defined on a closed interval can be approximated by a sequence of polynomials. Moreover, the polynomials can be chosen in such a way as to make the approximation *uniform* in a natural sense.

In this chapter we prove Stone's celebrated generalization of this theorem to arbitrary compact Hausdorff spaces.

Let X be a topological space and let **C**(X) be the set of all real-valued continuous functions on X. We may regard **C**(X) as a *commutative ring* by defining the sum $f + g$ and the product $f.g$ of two members f, g of **C**(X) as follows:

$$(f + g)(x) = f(x) + g(x) \quad (f.\, g)(x) = f(x).\, g(x)$$

for $x \in X$. It is easy to verify that $f + g$ and $f.g$ are in **C**(X) if both f and g are, and that defining addition and multiplication in this way turns **C**(X) into a commutative ring. For each $a \in \mathbb{R}$ we write $\boldsymbol{a}$ for the constant function on X with value a. Clearly each such constant function is a member of **C**(X).

Now suppose that X is a *compact* space. Then we know from **5.24** that each member of **C**(X) is bounded, and so we can define a *metric* **d** on **C**(X) by

$$\mathbf{d}(f, g) = \sup\{|f(x) - g(x)|: x \in X\}$$

for $f, g \in$ **C**(X). A sequence (f_n) of members of **C**(X) is said to *converge uniformly* to $f \in$ **C**(X) if for each $\varepsilon > 0$ there is a natural number n such that $|f(x) - f_m(x)| < \varepsilon$ for all $m \geq n$ and all $x \in X$. Under these conditions f is called the *uniform limit* of the sequence (f_n). Clearly (f_n) converges uniformly to f if and only if it converges to f in the metric topology on **C**(X) induced by **d**.

Let **P**(a, b) be the set of all polynomial functions with real coefficients defined on a compact closed interval $[a,b]$ in $\mathbb{R}$. Clearly each member of **P**(a, b) is continuous; that is, **P**(a, b) $\subseteq$ **C**(a, b). Now the Weierstrass approximation theorem asserts that *each member of* **C**(a, b) *is the uniform limit of a sequence of members of* **P**(a, b). In terms of the metric topology for **C**(a, b), this assertion is equivalent to saying that **P**(a, b) is a *dense* subset of **P**(a, b).

Our aim is to extract the characteristic features of **P**(a, b) which ensure its density in **C**(a, b).

We observe that **P**(a, b) has the following properties:

(i) it is a *subring* of $\mathbf{C}(a, b)$, i.e., if $f, g \in \mathbf{P}(a, b)$, then $f + g, f - g, f. g \in \mathbf{P}(a, b)$;

(ii) it *distinguishes points* of $[a,b]$, i.e., for any each pair x, y of distinct points of $[a,b]$ there is $f \in \mathbf{P}(a, b)$ such that $f(x) \neq f(y)$;

(iii) it contains all the constant functions.

These properties turn out to be the ones we seek, as we have

11.1. The Stone-Weierstrass Theorem. Let X be a compact Hausdorff space, and let A be a subring of $\mathbf{C}(X)$ which distinguishes points of X and contains the constant functions. Then A is dense in $\mathbf{C}(X)$ (with respect to the metric topology on $\mathbf{C}(X)$).

The proof of this theorem requires a sequence of lemmas. We assume throughout that X is a compact Hausdorff space.

11.2. Lemma. Let A be a subring of $\mathbf{C}(X)$. Then the closure $\overline{A}$ of A in the metric topology for $\mathbf{C}(X)$ is also a subring of $\mathbf{C}(X)$.

Proof. Let $f, g \in \overline{A}$. Then, by **3.8** and **6.7** f and g are the uniform limits of sequences (f_n), (g_n), respectively, from A. But then $f + g$ is the uniform limit of the sequence $(f_n + g_n)$ so that $f + g \in \overline{A}$. Similarly for $f - g$ and $f. g$. ■

Before proceeding to the next lemma we need another definition. Given $f, g \in \mathbf{C}(X)$, define the functions $|f|, f \vee g, f \wedge g$ on X by setting, for $x \in X$,

$$|f|(x) = |f(x)|, \quad (f \vee g)(x) = \max[f(x), g(x)], \quad (f \wedge g)(x) = \min[f(x), g(x)].$$

It is readily shown that, if $f \in \mathbf{C}(X)$, then $|f| \in \mathbf{C}(X)$. Hence, in view of the fact that

(*) $$f \vee g = \tfrac{1}{2}[f + g + |f - g|], \quad f \wedge g = \tfrac{1}{2}[f + g - |f - g|],$$

we have $f, g \in \mathbf{C}(X) \Rightarrow f \vee g, f \wedge g \in \mathbf{C}(X)$.

11.3. Lemma. Let A be a subring of $\mathbf{C}(X)$ which is closed in the metric topology and contains the constant functions. Then $f, g \in A \Rightarrow f \vee g, f \wedge g \in A$.

Proof. By (*) above, it suffices to show that $f \in A \Rightarrow |f| \in A$. Suppose $f \in A$ and assume first that $\sup\{|f(x)|: x \in X\} < 1$. A familiar result of elementary analysis asserts that the binomial expansion of $(1 - t)^{1/2}$ converges uniformly to the positive value of $(1 - t)^{1/2}$ for $|t| < 1$. Each partial sum $p_n(t)$ of this binomial expansion is a polynomial in t and, since the convergence of the sequence $(p_n(t))$ is uniform, given $\varepsilon > 0$ we can find a member $p(t)$ of this sequence for which

(**) $$|(1 - t)^{1/2} - p(t)| < \varepsilon \text{ for } |t| < 1.$$

Since $|f(x)| < 1$ for all $x \in X$, we have $|1 - f(x)^2| \leq 1$ for all $x \in X$, so that, if we substitute $1 - f(x)^2$ for t in (**), and note that the positive value of $(f(x))^2)^{1/2}$ is $|f(x)|$, we get

$$||f(x)^* - p(1 - f(x)^2| < \varepsilon$$

for all $x \in X$. Taking the supremum over $x \in X$ of the left hand side of this inequality gives

$$\mathbf{d}(|f|, p(\mathbf{1} - f^2)) \leq \varepsilon,$$

where $p(\mathbf{1} - f^2)$ is the member of $\mathbf{C}(X)$ obtained by substituting $\mathbf{1} - f^2$ for t in $p(t)$. Since A is a subring containing f as well as the constant functions, it is clear that $p(\mathbf{1} - f^2) \in A$. Therefore, for arbitrary $\varepsilon > 0$, $|f|$ is within distance ε of a member of A. Since A is closed, it follows that $|f| \in A$.

If, on the other hand, $\sup\{|f(x)|: x \in X\} = \alpha > 1$, let $g = \alpha^{-1}.f$. Clearly $g \in A$ and $\sup\{|g(x)|: x \in X\} \leq 1$, so that $|g| \in A$ by the first part of the proof. Hence $|f| = |\alpha.g| = = \alpha.|g| \in A$. ■

11.4. Lemma. Let A be a subring of $\mathbf{C}(X)$ which distinguishes points and contains the constant functions. Then for each pair x, y of distinct points of X and each pair a, b of real numbers there is a function f 0 A such that $f(x) = a$ and $f(y) = b$.

Proof. Since A distinguishes points there is $g \in A$ such that $g(x) \neq g(y)$. Define $f: X \to \mathbb{R}$ by

$$f(z) = a[g(z) - g(y)]/[g(x) - g(y)] \ + \ b[g(z) - g(x)]/[g(y) - g(x)].$$

This f meets the requirements. ■

The next lemma is crucial.

11.5. Lemma. Let A be a closed subring of $\mathbf{C}(X)$ which distinguishes points and contains the constant functions. Then $A = \mathbf{C}(X)$.

Proof. Let f 0 $\mathbf{C}(X)$; we have to show that f 0 A. This will be the case if, given any $\varepsilon > 0$, we can show that there is a function $g \in A$ such that $f(x) - \varepsilon < g(x) < f(x) + \varepsilon$ for all $x \in X$. For then $\mathbf{d}(f, g) < \varepsilon$, and so, since A is closed and ε arbitrary, it will follow that $f \in A$.

Let x be an arbitrary but fixed point of X, and let y be an arbitrary point of X different from x. By **11.4**, there is a member of A whose values at x and y coincide with those of f at these two points. Let us denote this function by f_y in order to indicate its dependence on y. Thus f_y 0 A satisfies $f_y(x) = f(x)$ and $f_y(y) = f(y)$. Now define

$$U_y = \{z \in X: f_y(z) < f(z) + \varepsilon\}.$$

Then U_y is an open set containing both x and y, so that the family $\{U_y: y \in X \ \& \ y \neq x\}$ is an open cover of X. Since X is compact, this open cover has a finite subcover $\{U_{y1}, ..., U_{yn}\}$. To indicate its dependence on x, write g_x for the function $f_{y1} \wedge \ldots \wedge f_{yn}$. By **11.3**, $g_x \in A$ and we have $g_x(x) = f(x)$, $g_x(z) < f(z) + \varepsilon$ for all $z \in X$.

Now consider the open set

$$V_x = \{z \in X: g_x(z) > f(z) - \varepsilon\}.$$

Since $x \in V_x$, the family $\{V_x: x \in X\}$ is an open cover of X which, in view of the compactness of X, has a finite subcover $\{V_{x1},...,V_{xn}\}$. Let g be the function $g_{x1} \vee ... \vee g_{xn}$. By **12.3**, $g \in A$ and it is clear from the definition of g that $f(z) - \varepsilon < g(z) < f(z) + \varepsilon$ for all $z \in X$. The lemma follows. ■

We can now complete the proof of Theorem **11.1.** Let A be a subring of **C**(X) satisfying the conditions of the theorem. Then, by Lemma **11.2**, $\overline{A}$ is a closed subring satisfying the conditions of the theorem, so that, by **11.5**, $\overline{A}$ = **C**(X), in other words, A is dense in **C**(X). ■

As an immediate consequence of **12.1** we have

11.6. Corollary. *The Weierstrass approximation theorem.* Each continuous real-valued function on a closed interval of ℝ is the uniform limit of a sequence of polynomial functions. ■

So far we have confined our attention to real-valued functions on topological spaces. What is the situation for *complex-valued* functions?

Let ℂ be the field of complex numbers with its usual (metric) topology (i.e., the topology obtained by identifying ℂ with ℝ × ℝ). Let X be a topological space and let $\mathbf{C}_\star(X)$ be the set of all continuous functions on X to ℂ. We may regard $\mathbf{C}_\star(X)$ as a ring by defining addition and multiplication pointwise -- just as we did for **C**(X). Moreover, if X is *compact*, every function in $\mathbf{C}_\star(X)$ is bounded, so we can define the usuual metric **d** on $\mathbf{C}_\star(X)$ by

$$\mathbf{d}(f, g) = \sup\{|f(x) - g(x)|: x \in X\}$$

for $f, g \in \mathbf{C}_\star(X)$. We may then assign $\mathbf{C}_\star(X)$ the metric topology induced by **d**.

Our aim now is to prove a version of the Stone-Weierstrass theorem for $\mathbf{C}_\star(X)$. That is, we want to find sufficient conditions on a subring A of $\mathbf{C}_\star(X)$ to ensure its density in the latter. Now first of all it is not hard to see that conditions (i) – (iii) on p.43 are *not* in general sufficient to ensure this. For let X be the closed unit disc $\{z: |z| \leq 1\}$ in the complex plane. Clearly X is a compact Hausdorff space. Let A be the set of all functions in $\mathbf{C}_\star(X)$ which are analytic (i.e. differentiable) in the interior of X. Evidently A satisfies conditions (i) – (iii) on p.43; moreover, A is closed in view of the fact that the uniform limit of a sequence of analytic functions on a compact set is analytic. Nevertheless A does not coincide with $\mathbf{C}_\star(X)$, because the conjugation function $z \mapsto z^*$ is in $\mathbf{C}_\star(X)$ but not in A, since it is not differentiable anywhere. (Here the *conjugate* of $z = a + ib$ is $z^* = a - ib$, where $i = \sqrt{-1}$ as usual).

What, then, is to be done? The above example shows the way: we must insist that A be closed under *conjugation* as well.

Let $f \in \mathbf{C}_\star(X)$. We define the *conjugate* f^* of f by setting $f^*(x) = f(x)^*$ for $x \in X$. We also define the *real* and *imaginary parts* of f by

$$(***) \qquad \mathrm{R}f = \tfrac{1}{2}.(f + f^*) \qquad \mathrm{I}f = -\tfrac{1}{2}\mathbf{i}.(f - f^*)$$

where, as before, constant functions are indicated by bold-face values. Observe that $f^* \in \mathbf{C}_\star(X)$, $\mathrm{R}f, \mathrm{I}f \in \mathbf{C}(X)$ and $f = \mathrm{R}f + \mathbf{i}.\mathrm{I}f$.

Let us say that a subset A of $\mathbf{C}_\star(X)$ is *self-conjugate* if $f \in A \Rightarrow f^* \in A$. Then we have

11.7. Lemma. Let X be a compact space and let $A \subseteq \mathbf{C}_\star(X)$. Then if A is self-conjugate, so is $\overline{A}$, the closure of A in the metric topology for $\mathbf{C}_\star(X)$.

Proof. Suppose $f \in \overline{A}$; we have to show that $f^* \in \overline{A}$. Since $f \in \overline{A}$, f is the uniform limit of a sequence (f_n) from A. But then, clearly, f^* is the uniform limit of the sequrnce (f_n^*). Since A is self-conjugate, each $f_n^* \in A$, so $f^* \in \overline{A}$. ■

We can now finally prove

11.8. **The Complex Stone-Weierstrass Theorem.** Let X be a compact Hausdorff space and let A be a self-conjugate subring of $\mathbf{C}_\star(X)$ which distinguishes points of X and contains the constant functions. Then A is dense in $\mathbf{C}_\star(X)$.

Proof. We have to show that $\overline{A} = \mathbf{C}_\star(X)$. By **11.7**, $\overline{A}$ is self-conjugate, and it obviously distinguishes points and contains the constant functions since A does. An argument similar to that for **11.2** shows that $\overline{A}$ is a subring of $\mathbf{C}_\star(X)$. From this it follows directly that the real valued functions in $\overline{A}$ form a closed subring B of $\mathbf{C}_\star(X)$.

We claim that $B = \mathbf{C}(X)$. By **11.5** it suffices to show that B distinguishes points and contains the (real) constant functions. Let x and y be distinct points of X. Since $\overline{A}$ distinguishes points, there is a function $f \in \overline{A}$ which takes different values at x and y. In that case either $\mathrm{R}f$ or $\mathrm{I}f$ takes different values at these points. Since $\overline{A}$ is a self-conjugate subring of $\mathbf{C}_\star(X)$, formulas (***) show that both $\mathrm{R}f$ and $\mathrm{I}f$ are in B, so that B distinguishes points. And B contains all the constant functions since $\overline{A}$ does. This proves the claim.

Knowing as we now do that $B = \mathbf{C}(X)$, we can now easily complete the proof of the theorem. Let $f \in \mathbf{C}_\star(X)$. Then $\mathrm{R}f$ and $\mathrm{I}f$ are in $\boldsymbol{C}(X)$, hence in B and so certainly in $\overline{A}$. But since $f = \mathrm{R}f + \mathbf{i}.\mathrm{I}f$ and $\overline{A}$ is a subring containing the constant functions, it follows that $f \in \overline{A}$. Hence $A = \mathbf{C}_\star(X)$ and we are through. ■

11.9. Corollary. *The Complex Weierstrass Approximation Theorem.* Let X be a compact subset of the complex plane. Then any continuous complex-valued function on X is the uniform limit of a sequence of polynomials in z and z^* with complex coefficients.

Proof. The set of such polynomials satisfies the conditions of **11.8.** ■

Appendix A

Some Fundamental Results on Linear Spaces

IN THIS APPENDIX we shall assume an acquaintance with the basic theory of linear spaces.

By a *linear* or *vector* space we shall understand a linear space over the field $\mathbb{R}$ of real numbers. We use lower case greek letters α, β, γ to denote arbitrary real numbers.

Let X be a linear space, let A, B be subsets of X, and let $a \in X$. We write

$$A + B \text{ for } \{x + y: x \in A \;\&\; y \in B\}$$

$$A \;!\; B \text{ for } \{x - y: x \in A \;\&\; y \in B\}$$

$$!A \text{ for } \{-x: x \in A\}$$

$$\alpha A \text{ for } \{\alpha x: x \in A\}$$

$$A \pm a \text{ for } A \pm \{a\}$$

$$a \pm A \text{ for } \{a\} \pm A.$$

A subset K of X is said to be *convex* if $"K + (1 - \alpha)K \subseteq K$ whenever $0 \leq \alpha \leq 1$. Evidently any subspace, and any one-point subset, of X is convex.

A1. Lemma. (i) The intersection of any family of convex sets is convex.

(ii) If K and L are convex, so are αK and $K \pm L$.

(iii) Let $x_1,\ldots,x_n$ be members of a convex set K and let $\alpha_1,\ldots, \alpha_n$ be nonnegative real numbers such that $\alpha_1 + \ldots + \alpha_n = 1$. Then $\alpha_1 x_1 + \ldots + \alpha_n x_n \in K$.

Proof. (i) and **(ii)** are left as exercises to any unlikely reader. As for **(iii)**, we argue by induction on n. If $n = 1$, the assertion is trivial, while if $n = 2$, it is true by definition. Suppose now that it is true for some $n \geq 2$. Then, writing $\beta = \alpha_2 + \ldots + \alpha_{n+1}$, and

$$y = (\alpha_2/\beta)x_2 + \ldots + (\alpha_{n+1}/\beta)x_{n+1},$$

it follows from our induction hypothesis that $y \in K$. Since $\alpha_1 + \beta = 1$, we have

$$\sum_{i=1}^{n+1} \alpha_i x_i = \alpha_1 x_1 + \beta y \in K,$$

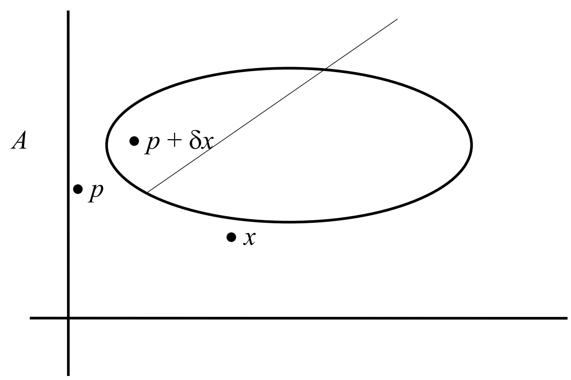
and **(iii)** follows. ■

Let A be a subset of a linear space X. Then by **(i)** above, there is a least convex set containing A. This set is denoted by A^c and called the *convex hull* of A. Using **(iii)** above, it is easy to verify that A^c consists of all *convex combinations* of members of A: that is, of elements of X of the form

$$\alpha_1 y_1 + \ldots + \alpha_n y_n$$

with $y_1,\ldots,y_n \in A$, $\alpha_1, \ldots,\alpha_n \geq 0$ and $\alpha_1 + \ldots + \alpha_n = 1$.

An element $p \in A$ is called an *internal point* of A if for each $x \in X$ there is $\varepsilon > 0$ such that $p + \delta x \in A$ whenever $|\delta| \leq \varepsilon$.



Let K be a convex subset of X and suppose that the origin 0 of X is an internal point of K. Define the map $g_K\colon X \to \mathbb{R}$ by setting, for each $x \in X$,

$$g_K(x) = \inf\{\alpha\colon \alpha > 0 \ \&\ x \in \alpha K\}.$$

g_K is called the *gauge* of K. For example, if K is the unit disc in the complex plane, then $g_K(z) = |z|$.

A.2. Lemma. Let K be a convex subset of a linear space X and suppose that the origin is an internal point of K. If g is the gauge of K, then

(i) $0 \leq g(x) < \infty$ for any $x \in X$;

(ii) $g(\alpha x) = \alpha g(x)$ for any $x \in X$ and $\alpha \geq 0$;

(iii) $g(x) \leq 1$ for $x \in K$ and $g(x) \geq 1$ for $x \notin K$;

(iv) $g(x + y) \leq g(x) + g(y)$ for all $x, y \in K$.

Proof. The proofs of **(i) - (iii)** are left as exercises to the reader. To prove **(iv)**, we notice that, if $\gamma > g(x) + g(y)$, then $\gamma = \alpha + \beta$ with $\alpha > g(x)$, $\beta > g(y)$. It now follows from the convexity of K that the point

$$\gamma^{-1}(x + y) = (\alpha + \beta)^{-1}(x + y) = \alpha(\alpha + \beta)^{-1}(\alpha^{-1}x) + \beta(\alpha + \beta)^{-1}(\beta^{-1}y)$$

is in K, since, by the choice of α and β, $\alpha^{-1}x$ and $\beta^{-1}y$ are both in K. Hence $x + y \in \gamma K$ and so $g(x + y) \leq \gamma$. This proves **(iv)**. ■

A proper linear subspace Y of a linear space X is said to be *maximal* if there is no proper subspace Z of X such that $Z \neq Y$ and $Y \subseteq Z$. A set of the form $Y + x$, with Y a maximal subspace, is called a *hyperplane* in X. For example, in Euclidean n-space, maximal subspaces are $(n-1)$-dimensional, and hyperplanes are translates of these.

A *linear functional* on a linear space X is a linear function on X to $\mathbb{R}$.

A.3. Lemma. Let Y be a subset of a linear space X.

(i) Y is a maximal subspace of X iff there is a nonzero linear functional f on X such that $Y = f^{-1}(0)$.

(ii) Y is a hyperplane in X iff Y ! y is a maximal subspace of X for each $y \in Y$.

(iii) Y is a hyperplane in X iff there is a nonzero linear functional f on X and a real number α such that $X =$ f^{-1}(").

Proof. (i) If f is a nonzero linear functional on X, then clearly $f^{-1}(0) = Y$ is a proper linear subspace of X. Moreover, if $x \notin Y$, then $Y \cup \{x\}$ generates X: for if $z \in X$, then $z - (f(z)/f(x))x \in Y$. So Y is maximal.

Conversely, suppose that Y is a maximal linear subspace of X. Choose $x \notin Y$; then since Y is maximal, $Y \cup \{x\}$ generates X, so every element of X is uniquely expressible in the form $y + \alpha x$, with $y \in Y$. Fix $\beta \neq 0$, and define $f: X \to \mathbb{R}$ by setting $f(y + \alpha x) = \alpha\beta$. Then f is a linear functional on X and $Y = f^{-1}(0)$.

The proof of **(ii)** is easy and is entrusted to the reader.

(iii) It follows from **(i)** and **(ii)** that

Y is a hyperplane in $X \Leftrightarrow Y$! y is a maximal subspace of X for any $y \in Y$

$\Leftrightarrow$ there is a nonzero linear functional on X such that $Y - y = f^{-1}(0)$

$\Leftrightarrow$ there is a nonzero linear functional f on X and an " such that $Y = f^{-1}(\alpha)$. ■

We have just shown that, for each hyperplane H, there is a linear functional f and a real number α such that

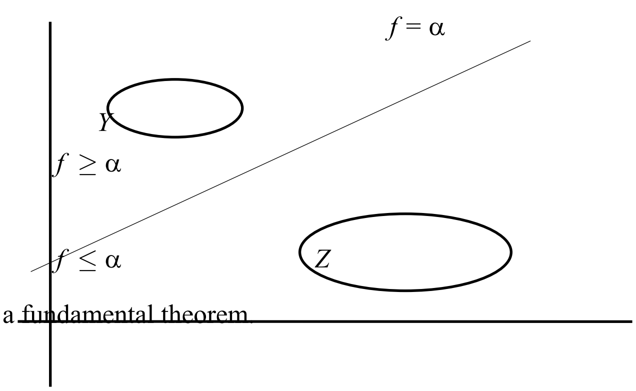
$$H = f^{-1}(\alpha).$$

An equation of this form is called a *representation* of H. It is easily shown that if $H = g^{-1}(\beta)$ is another representation of H, then there is $\lambda \neq 0$ such that $g = \lambda f$ and $\beta = \lambda\alpha$. We leave the proof of this assertion to the reader.

If Y and Z are subsets of a linear space X, we say that a nonzero linear functional f on X *separates* Y and Z if there is a real number α such that

$$Y \subseteq \{x: f(x) \leq \alpha\} \text{ and } Z \subseteq \{x: f(x) \geq "\}.$$

A hyperplane H is said to *separate* Y and Z if, for some representation $H = f^{-1}(\alpha)$ of H, the functional f separates Y and Z. Clearly, two sets can be separated by a hyperplane iff they can be separated by a linear functional.



We now prove a fundamental theorem.

A.4. The Hahn-Banach Theorem. Suppose that the real-valued function p on the linear space X satisfies

$$p(x+y) \leq p(x) + p(y),\ p(\alpha x) = \alpha p(x) \quad \text{for } \alpha \geq 0,\ x, y \in X.$$

Let f be a linear functional defined on a subspace Y of X such that

$$f(x) \leq p(x) \quad \text{for } x \in X.$$

Then there is a linear functional F on X such that

$$F|Y = f \text{ and } F(x) \leq p(x) \quad \text{for } x \in X.$$

Proof. Consider the set of all linear functionals g which extend (i.e. include) f and for which the inequality $g(x) \leq p(x)$ holds for all $x \in \text{dom}(g)$, where the latter is a subspace of X containing Y. This set is closed under unions of chains and so Zorn's lemma applies to yield a maximal member F. Thus F is a maximal linear extension of f such that $F(x) \leq p(x)$ for all $x \in \text{dom}(F)$. It remains to show that the domain Z of F is X.

For contradiction's sake, suppose that there is a point u in X which is not in Z. Then any point in the subspace U of X generated by $Z \cup \{u\}$ has a unique representation in the form $z + \alpha u$. For any constant γ, the function G defined on U by setting

$$G(z + \alpha u) = F(z) + \alpha ($$

is a linear functional properly extending F. The desired contradiction will be obtained and the proof completed if we can show that γ can be chosen in such a way that

(*) $$G(x) \leq p(x) \quad \text{for all } x \in U.$$

Let $x, y \in U$; then the inequality

$$F(y) - F(x) = F(y - x) \leq p(y - x) \leq p(y + u) + p(-u - x)$$

gives

$$-p(-u - x) - F(x) \leq p(y + u) - F(y).$$

Since the left-hand side of this last inequality is independent of y and the right hand side is independent of x, there is a constant γ such that

(i) $$\gamma \leq p(y + u) - F(y)$$

(ii) $$-p(-u - y) - F(y) \leq ($$

for $y \in Z$. For $x = z + \alpha u$ in U, the inequality

$$G(x) = F(z) + \alpha\gamma \leq p(z) + \alpha u = p(x),$$

which holds for $\alpha = 0$ by hypothesis, is obtained for $\alpha > 0$ by replacing y by $\alpha^{-1}z$ in (i), and for $\alpha < 0$ by replacing y by $\alpha^{-1}z$ in (ii).

Thus we obtain (*) in all cases, and hence the required contradiction. ■

This result enables us to prove the

A.5. Corollary. *Basic Separation Theorem.* Let M and N be disjoint convex subsets of a linear space X, and suppose that M has an internal point. Then there is a nonzero linear functional, or, equivalently, a hyperplane, separating M and N.

Proof. If a is an internal point of M, then, as is easily verified, 0 is an internal point of M ! a. Moreover, it isn't hard to see that a linear functional separates M and N iff it separates M ! a and N ! a. Thus it suffices to prove the theorem under the assumption that 0 is an internal point of M.

Let b be any point of N. Then $-b$ is an internal point of M ! N, and so 0 is an internal point of the convex set $K = M$! $N + b$. Since M and N are disjoint, $b \notin K$. Let g be the gauge of K; then $g(b) \geq 1$ by **A.2(iii)**. If, for real α, we put $f_0(\alpha b) = \alpha g(b)$, then f_0 is a linear functional defined on the one dimensional subspace of X generated by $\{b\}$. Moreover,

$$f_0(\alpha b) \leq g(\alpha b)$$

for all α, since for $\alpha \geq 0$ we have $f_0(\alpha b) = \alpha g(b) = g(\alpha b)$, while for $\alpha < 0$ we have $f_0(\alpha b) = \alpha f_0(b) < 0 \leq g(\alpha b)$. Therefore, by the Hahn-Banach theorem, f_0 can be extended to a linear functional F on X such that $F(x) \leq g(x)$ for all $x \in X$. It follows that $F(x) \leq g(x) \leq 1$ for $x \, \varepsilon \, K$, while $F(b) = g(b) \geq \lambda$ 1. Accordingly F separates K and $\{b\}$; it follows immediately that it separates M ! N and $\{0\}$, and so also M and N. ■

Normed Linear Spaces

A *norm* on a linear space X is a function $x \mapsto \| x \|$ to the nonnegative real numbers such that, for all $x, y \in X$,

(i) $\| x \| = 0 \Leftrightarrow x = 0$;

(ii) $\| x + y \| \leq \| x \| + \| y \|$

(iii) $\| \alpha x \| = |\alpha| \, \| x \|$ for all real α.

A linear space equipped with a norm is called a *normed (linear) space*.

Examples. 1. On $\mathbb{R}^n$ we can define a norm by putting

$$\| x \| = \sqrt{\sum_{i=1}^{n} x_i^2} .$$

where $x = (x_1, \ldots, x_n)$.

2. If X is a compact topological space, we can define a norm on $\mathbf{C}(X)$ by putting

$$\| f \| = \sup\{|f(x)| : x \in X\}.$$

When X is a closed interval $[a, b]$ we can define another norm on $\mathbf{C}([a,b])$ by

$$\| f \| = \int_a^b | f(x) \mathrm{d}x .$$

Each normed linear space has a natural *metric* $\mathbf{d}$ defined on it by

$$\mathbf{d}(x,y) = \| x - y \| .$$

for $x, y \in X$. The metric topology induced by **d** is called the *norm topology* on X. From now on we assume that each normed space has been assigned its norm topology.

We proceed to establish some basic facts about normed spaces. Let X be a fixed normed space.

(i) The map $(x, y) \mapsto x + y$ from $X \times X$ to X is continuous. To prove this, let $\varepsilon > 0$. Then, if $u, v \in X$ satisfy $\| x - u \| < \varepsilon/2$, $\| y - v \| < \varepsilon/2$, we have

$$\| x + y - (u + v) \| \leq \| x - u \| + \| y - v \| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad ■$$

(ii) The map $x \mapsto -x$ is a homeomorphism of X with itself. For if $\| x - y \| < \varepsilon$, then

$$\| -x - (-y) \| = \| y - x \| = \| x - y \| < \varepsilon,$$

so that the map in question is continuous. Since it is also self-inverse, it is a homeomorphism. ■

(iii) For fixed $\alpha \neq 0$, the map $x \mapsto \alpha x$ is a homeomorphism of X with itself. This is proved in a manner similar to the previous assertions.

(iv) For fixed $a \in X$, the map $x \mapsto a + x$ is a homeomorphism of X with itself. For the map in question is the composition of the maps $x \mapsto (a, x) \mapsto a + x$, and so continuous. Its inverse is $x \mapsto -a + x$ which is also continuous. ■

It follows immediately from **(iv)** that if U is a neighbourhood of x, then, for any $a \in X$, U ! a is a neighbourhood of $x - a$.

A.6. Lemma. Let f be a linear functional on a normed space X. Then the following conditions are equivalent:

(i) f is continuous;

(ii) f is continuous at 0;

(iii) f is bounded on the set $\{x \in X : \| x \| \leq 1\}$;

(iv) for some α, $|f(x)| \leq \alpha \| x \|$ for all $x \in X$.

Proof. (i) $\Leftrightarrow$ (ii). One direction is trivial. Suppose conversely that f is continuous at 0, and let U be any neighbourhood of $f(x)$ for arbitrary $x \in X$. Then U ! $f(x)$ is a neighbourhood of 0 and so, since f is continuous at 0, there is a neighbourhood V of 0 such that $f[V] \subseteq U$! $f(x)$. But $V + x$ is a neighbourhood of x and $f[V + x] = f[V] + f(x) \subseteq U$. Hence f is continuous at x, and so continuous.

(ii) $\Rightarrow$ (iv). If f is continuous at 0, there is $\varepsilon > 0$ such that $|f(x)| < 1$ if $\| x \| < \varepsilon$. For arbitrary $x \neq 0$, the point $y = (\varepsilon/2\| x \|)x$ has norm $\| y \| < \varepsilon$; so

$$(\varepsilon/2\| x \|)|f(x)| = |f(y)| < 1,$$

and hence

$$|f(x)| < 2/\varepsilon\| x \|.$$

(iv) $\Rightarrow$ (ii) is easy and is left to the reader.

(iii) $\Rightarrow$ (iv). Assume (iii) and let $\alpha = \sup\{| f(x) | : \| x \| \leq 1\}$. Then for arbitrary $x \neq 0$, we have

$$| f(x) | \;=\; \| x \| \, f(\| x \|^{-1} x) \;\le\; \alpha \| x \| .$$

Finally, (iv) $\Rightarrow$ (iii) is obvious. ■

A.7. Lemma. Any interior point of a subset Y of a normed space X is an internal point of Y.

Proof. If a is an interior point of Y, then there is $\varepsilon > 0$ such that, for all $x \in X$,

$$\| a - x \| \le \varepsilon \Rightarrow x \in Y.$$

Therefore, if $|\delta| \le \| x \|^{-1}$, we have

$$\| a + \delta x - a \| = \| \delta x \| \le \varepsilon,$$

so that $a + \delta x \in Y$, and a is an internal point of Y. ■

A.8. Lemma. If a linear functional on a normed space separates two sets, one of which has an interior point, then the functional is continuous.

Proof. Let f be a linear functional on a normed space X separating two subsets M and N, the former of which has an interior point a. Let U be a neighbourhood of 0 such that $a + U \subseteq M$. (Take $U = W$! a, where W is a neighbourhood of a included in M.) Then $f[U] \subseteq f[M]$! $f(a)$ and so, since f separates M and N, $f[M]$, and hence also $f[U]$, is contained in an interval of the form [–", →) of the real line, where $\alpha > 0$. Let $V = U \cap$!U; then $V =$!V and V is a neighbourhood of 0 such that $f[V] \subseteq [\alpha, \alpha]$. But then

$$f[\varepsilon\alpha^{-1}V] \subseteq [\varepsilon, -\varepsilon]$$

for any $\varepsilon > 0$. Since $\varepsilon\alpha^{-1}V$ is a neighbourhood of 0, it follows that f is continuous there, and so, by **A.6**, continuous. ■

From this and the two preceding lemmas we immediately infer the

A.9. Separation Theorem for Normed Spaces. In a normed space, any disjoint pair of convex sets, at least one of which has an interior point, can be separated by a nonzero continuous linear functional, and hence by a (closed) hyperplane. ■

Finally, we prove the

A.10. Continuous Extension Theorem. Any continuous linear functional defined on a subspace of a normed space X has a continuous linear extension to X.

Proof. Let f be a continuous linear functional defined on a subspace Y of X. Then by the continuity of f, there is a neighbourhood U of 0 in X of the form $\{x \in X\colon \| x \| \le \varepsilon\}$ such that $|f(x)| < 1$ for $x \in U \cap Y$. It is easy to verify that U is convex; since 0 is an interior point of U, it is also (by **A.7**) an internal point. Let p be the gauge of U.

We claim that $|f(x)| \le p(x)$ for $x \in Y$. For if $\alpha > p(x)$, then, by definition of p, $\alpha^{-1}x \in Y$, so that $|f(x)| / \alpha < 1$, i.e. $|f(x)| < \alpha$. This establishes the claim.

By the Hahn-Banach theorem, there is a linear functional F on X which extends f and satisfies $F(x) \leq p(x)$ for all $x \in X$. One verifies immediately that $p(x) = p(-x)$ for all $x \neq\in X$, so that

$$-F(x) = F(-x) \leq p(-x) = p(x).$$

Therefore $|F(x)| \leq p(x)$ for all $x \in X$. Since $p(x) \leq 1$ on U, it follows that $|F(x)| \leq 1$ for all $x \in U$, and so $|F(x)| \leq 1/\varepsilon$ whenever $\| x \| \leq 1$. Accordingly F is bounded on $\{x: \| x \| \leq 1\}$, and therefore, by **A.6,** continuous. ■

Appendix B

Complete Metric Spaces and the Baire Category Theorem

IT IS READILY seen that a sequence of points $a_0, a_1, \ldots$ in a metric space $(X, \mathbf{d})$ converges in the metric topology if there is a point $a \in X$ with the following property: for every $\varepsilon > 0$ there is an integer N such that $\mathrm{d}(a_n, a) < \varepsilon$ for $n \geq N$. The sequence is a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that $\mathbf{d}(a_m, a_n) < \varepsilon$ whenever $m, n \geq N$. A metric space is said to be *complete* if every Cauchy sequence in it converges. It is a basic fact of classical analysis that each Euclidean space $\mathbb{R}^n$ with its usual metric topology is complete.

A subset A of a topological space X is *nowhere dense* if $\overset{\circ}{\overline{A}} = \varnothing$, or equivalently, if $X - \overline{A}$ is dense. A is *meagre* or of *first category* if it is the union of a countable family of nowhere dense sets, and *nonmeagre* or of *second category* if it is not meagre. Evidently a space is of second category in itself iff it is not the union of countably many nowhere dense *closed* sets. A second category set is in some sense "large".

We now prove the

Baire Category Theorem. Every complete metric space $(X, \mathbf{d})$ is of second category in itself.

Proof. Let $F_0, F_1, \ldots$ be a sequence of nowhere dense closed subsets of X. Then $X - F_0$ is nonempty and open. Choose x_0 and $\varepsilon_0 < 1$ so that $x_0 \in X - F_0$ and

$$S(x_0, \varepsilon_0) \cap F_0 = \varnothing.$$

Now the sphere $S(x_0, \tfrac{1}{2}\varepsilon_0) \not\subseteq F_1$. Accordingly there is $x_1 \in S(x_0, \tfrac{1}{2}\varepsilon_0)$ and $\varepsilon_1 < \tfrac{1}{2}$ such that

$$S(x_1, \varepsilon_1) \cap F_1 = \varnothing \qquad S(x_1, \varepsilon_1) \subseteq S(x_0, \tfrac{1}{2}\varepsilon_0).$$

Arguing recursively, construct a sequence $x_0, x_1, \ldots$ of points of X and a sequence $\varepsilon_0, \varepsilon_1, \ldots$ of radii such that

(a) $S(x_{n+1}, \varepsilon_{n+1}) \subseteq S(x_n, \tfrac{1}{2}\varepsilon_n)$

(b) $0 < \varepsilon_n < \frac{1}{2^n}$

(c) $S(x_n, \varepsilon_n) \cap F_n = \varnothing$.

The sequence $x_0, x_1, \ldots$ is Cauchy in view of the fact that

$$\mathbf{d}(x_{n+k}, x_n) \leq \sum_{i=0}^{k-1} \mathbf{d}(x_{n+i}, x_{n+i+1}) < \sum_{i=0}^{k-1} \tfrac{1}{2}\varepsilon_{n+i} < \sum_{i=0}^{k-1} \frac{1}{2^{n+i+1}} < \frac{1}{2^n} .$$

Since X is complete, $x_0, x_1, \ldots$ converges to some point $x \in X$. It follows from (a) that $x_{n+k} \in S(x_n, \tfrac{1}{2}\varepsilon_n)$ for all k, and we can choose k sufficiently large so as to ensure that $\mathbf{d}(x_{n+k}, x) < \tfrac{1}{2}\varepsilon_n$. So we have

$$\mathbf{d}(x_n, x) \le \mathbf{d}(x_{n+k}, x) + \mathbf{d}(x_{n+k}, x_n) \le \tfrac{1}{2}\varepsilon_n + \tfrac{1}{2}\varepsilon_n \le \varepsilon.$$

Hence $x \in S(x_n, \varepsilon_n)$ for every n, so that by (c), $x \notin \bigcup_n F_n$. So X is not the union of the F_n and is therefore of second category. ■

We use this theorem to show that "most" continuous functions on $\mathbb{R}$ fail to be differentiable anywhere, thereby extending a classical result of Weierstrass.

Let $\mathbf{C}[0,1]$ be the set of continuous real-valued functions f on [0,1] for which $f(0) = f(1)$, with the metric

$$\mathbf{d}(f, g) = \sup_{x\in[0,1]} | f(x) - g(x) |.$$

It is a fact from classical analysis that, with this metric, $\mathbf{C}[0,1]$ is complete.

Now let Γ be the space obtained from $\mathbf{C}[0,1]$ by extending each $f \in \mathbf{C}[0,1]$ to the whole of $\mathbb{R}$ by periodicity and imposing the same metric. Γ is complete, and so also of second category in itself. Let $K \subseteq \Gamma$ be the set of functions such that, for some ξ, the set

$$\left\{\left|\frac{f(\xi+h)-f(\xi)}{h}\right| : h > 0\right\}$$

is bounded. Then K contains all functions in Γ which are differentiable somewhere. *We claim that K is of first category in* Γ.

To prove this, define

$$K_n = \left\{ f \in \Gamma : \exists\xi\forall h > 0 \left|\frac{f(\xi+h)-f(\xi)}{h}\right| \le n \right\}.$$

Clearly $K = \bigcup_n K_n$. We first claim that each K_n is closed.

To prove this, let $f \in \overline{K}_n$ and let $f_0, f_1, \ldots$ be a sequence from K_n converging to f. For each $k = 0, 1, \ldots$ choose $\xi_k \in [0,1]$ to satisfy, for all $h > 0$,

$$\left|\frac{f_k(\xi_k+h)-f_k(\xi_k)}{h}\right| \le n. \qquad (*)$$

Then some subsequence $\xi_{k_0}, \xi_{k_1}, \ldots$ of $\xi_0, \xi_1, \ldots$ converges to some $\xi \in [0,1]$. Since $f_0, f_1, \ldots$ converges to f, so does $f_{k_0}, f_{k_1}, \ldots$. Each f_k being continuous, we may then pass to the limit in (*) to obtain, for all h > 0,

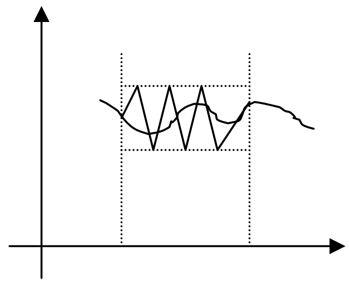
$$\left|\frac{f(\xi+h)-f(\xi)}{h}\right| \le n.$$

Hence $f \in K_n$, and K_n is accordingly closed.

We show finally that each K_n is *nowhere dense.* Since K_n is closed, it suffices to show that $\Gamma - K_n$ is dense in Γ. To this end suppose $g \in \Gamma$ and let $\varepsilon > 0$. Partition [0,1] into k equal intervals in such a way that, if x,x' are in the same interval of the partitioning, then $|g(x) - g(x')| < \tfrac{1}{2}\varepsilon$. Now over the i^{th} subinterval $[\tfrac{i-1}{k}, \tfrac{i}{k}]$, if we consider the rectangle R defined by

$$g(\tfrac{i-1}{k}) - \tfrac{1}{2}\varepsilon \le y \le g(\tfrac{i-1}{k}) + \tfrac{1}{2}\varepsilon.$$

Then the points $(\tfrac{i}{k}, g(\tfrac{i}{k}))$ and $(\tfrac{i-1}{k}, g(\tfrac{i-1}{k}))$ lie on the right and left sides, respectively, of R. By connecting the two points with a polygonal graph which remains within R and whose line segments have slopes exceeding n in absolute value, we obtain a continuous function which is within ε of g and is at the same time a member of $\Gamma - K_n$. So the latter is dense in Γ.



Exercises

1. Show that the only topology on infinite set with respect to which every infinite subset is open is the discrete topology.

2. A topological space is said to be a $\mathbf{T_0}$-*space* if for each pair of distinct points there is a neighbourhood of at least one of them which does not contain the other, and a $\mathbf{T_1}$-*space* if singletons are closed.

(i) Show that every $\mathbf{T_1}$-space is a $\mathbf{T_0}$-space, but not conversely.

(ii) Show that any Hausdorff space is a $\mathbf{T_1}$-space, but not conversely.

(iii) Show that no $\mathbf{T_0}$-space contains a finite set of distinct points $x_1,\ldots,x_n$ such that, for each $k < n$, $x_{k+1} \in \overline{\{x_k\}}$ and $x_1 \in \overline{\{x_n\}}$

(iv) Show that a space is $\mathbf{T_1}$ iff for each pair of distinct points there is a neighbourhood of each one of them which does not contain the other.

3. Let $(X, \leq)$ be a partially ordered set.

(i) Show that the family of intervals $\{y \in X: x \leq y\}$, for $x \in X$, is a base for a topology (the *order topology* determined by $\leq$) on X.

(ii) Show that, with the order topology, X is a $\mathbf{T_0}$-space in which the intersection of any family of open sets is open, and in which $\overline{\{x\}} = \{y: y \leq x\}$.

(iii) let X be a $\mathbf{T_0}$-space in which the intersection of any family of open sets is open. Show that the relation $x \in \overline{\{y\}}$ is a partial ordering on X and that the order topology on X determined by this partial ordering is the original topology on X.

(iv) Deduce **2.(iii)**.

(v) Show that a map between partially ordered sets is continuous with respect to the corresponding order topologies iff it is order preserving.

4. The *boundary* ∂A of a subset A of a topological space X is defined to be the set $\overline{A} \cap (X - A)$.

(i) Show that $\partial \overline{A} \cup \partial \overset{0}{A} \subseteq \partial A$ and $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

(ii) If $\overline{A} \cap \overline{B} = \varnothing$, show that $\partial(A \cup B) = \partial A \cup \partial B$.

5. Give an example of two topological spaces X, Y, a continuous map f: $X \to Y$, and a subset A of X such that $f[\overline{A}] \neq \overline{f[A]}$.

6. Let X, Y be topological spaces, and suppose that Y is Hausdorff. Show that, if f and g are continuous maps from X

to Y, the set $\{x \in X: f(x) = g(x)\}$ on which f and g coincide is closed. Deduce that if f and g coincide on a dense subset of X, then $f = g$.

7.(i) If A and B are connected subsets of a topological space such that $A \cap \overline{B} \neq \varnothing$, show that $A \cup B$ is connected.
(ii) Let $(A_n)_{n \geq 0}$ be an infinite sequence of connected sets such that $A_n \cap A_{n+1} \neq \varnothing$ for all $n \geq 0$. Show that $\bigcup_{n \geq 0} A_n$ is connected.

8. Let A be a subset of a topological space X. Show that any connected subset of X which meets both A and $X - A$ also meets ∂A.

9. Show that, if A is a connected subset of a topological space, then any subset B such that $A \subseteq B \subseteq \overline{A}$ is also connected.

10. A maximal connected subspace of a topological space X is called a *component* of X.
(i) Show that each component is a closed set, and that any pair of components are either disjoint or identical.
(ii) Show that the component containing a given point is contained in the intersection of the family of all clopen sets containing that point.
(iii) A space is said to be *totally disconnected* if its components are singletons. Show that the space of rational and the space of irrational numbers are both totally disconnected (but not discrete).
(iv) show that the following conditions on a topological space X are equivalent: (a) the components of X are open sets; (b) the intersection of the family of clopen sets containing any given point is an open set.

11. A space is said to be *locally connected* if each point has a local base consisting of connected neighbourhoods. Show that each component of a locally connected space is open, and give an example of a locally connected but disconnected space.

12. Show that a countable metric space with at least 2 points cannot be connected.

13. A *filter* on a set X is a family $\mathfrak{F}$ of subsets of X with the following properties: (a) every subset of X which contains a member of $\mathfrak{F}$ belongs to $\mathfrak{F}$, (b) finite intersections of members of $\mathfrak{F}$ belong to $\mathfrak{F}$, (c) the empty set is not a member of $\mathfrak{F}$. If X be a topological space, a filter $\mathfrak{F}$ on X is said to *converge* to x if every neighbourhood of x belongs to $\mathfrak{F}$.
(i) Let $\mathfrak{A}$ be a family of subsets of X. Show that $\mathfrak{A}$ is contained in some filter on X iff $\mathfrak{A}$ has the finite intersection property (**5.15**).
(ii) Show that a filter $\mathfrak{F}$ converges to x iff every member of some local base at x belongs to $\mathfrak{F}$.

(iii) Show that X is Hausdorff iff no filter on X converges to more than one point.

(iv) A *cluster point* of a filter $\mathfrak{F}$ is a point which belongs to the closure of every member of $\mathfrak{F}$. Show that x is a cluster point of $\mathfrak{F}$ iff there is a filter $\mathfrak{G} \supseteq \mathfrak{F}$ which converges to x.

(v) Show that the set of cluster points of any filter is closed.

(vi) Show that the closure of a subset A coincides with the set of cluster points of filters on A.

14. For each point x of a topological space X let $\mathfrak{N}(x)$ be the family of all neighbourhoods of x. **(i)** Show that $\mathfrak{N}(x)$ is a filter on X satisfying (a) $x \in \bigcap\mathfrak{N}(x)$, (b) for any $U \in \mathfrak{N}(x)$ there is $V \in \mathfrak{N}(x)$ such that $U \in \mathfrak{N}(y)$ for all $y \in V$.

(ii) Conversely, suppose that for each member x of a set X we are given a filter $\mathfrak{M}(x)$ on X satisfying conditions (a) and (b) above. Show that there is a unique topology $\mathfrak{T}$ on X such that, for each $x \in X$, $\mathfrak{M}(x)$ is the family of $\mathfrak{T}$-neighbourhoods of x.

15. (i) Show that any pair of (nonempty) open intervals, any pair of half-open intervals, and any pair of closed intervals in $\mathbb{R}$ (with the relativized usual topology) are homeomorphic.

(ii) Show that any nonempty open interval is homeomorphic with $\mathbb{R}$.

(iii) Show that no nonempty open interval is homeomorphic with a closed interval.

16. Show that every subset of an infinite set with the cofinite topology is compact, connected, and $\mathbf{T_1}$.

17. (i) Show that, if A is a subset, and U an open subset, of a topological space, then $U \cap \overline{A} \subseteq \overline{A \cap U}$, so that, if A is dense, then $U \subseteq \overline{A \cap U}$.

(ii) Deduce that a dense locally compact subset of a Hausdorff space is open.

18. Show that a subset of a topological space meets each dense subset iff its interior is nonempty.

19. If A is a subset of a topological space X write A^* for $\overset{0}{\overline{A}}$ and $\widetilde{A}$ for $\overline{\overset{0}{A}}$. Then $A \subseteq B$ implies $A^* \subseteq B^*$ and $\widetilde{A} \subseteq \widetilde{B}$.

(i) Show that, if A is open, then $A \subseteq A^*$, and that if A is closed, then $\widetilde{A} \subseteq A$.

(ii) Deduce that, if A is any subset of X, then $A^{**} = A^*$ and $\widetilde{\widetilde{A}} = \widetilde{A}$.

(iii) Use **17(i)** to show that, if U is open, then, for any subset A, $U \cap A^* \subseteq (U \cap A)^*$.

(iv) Deduce from this that, if U and V are open, then $(U \cap V)^* = U^* \cap V^*$.

20. Let $\mathfrak{A}$ be a family of closed compact subsets of a topological space whose intersection is contained in an open set U. Show that there is a finite subfamily of $\mathfrak{A}$ whose intersection is contained in U.

21. Let X be a set with a total ordering #. An *open interval* in X is a subset of the form $\{z: x < z < y\}$ with x, y

$\in X$. Define a topology on X by calling a subset Y of X open if Y contains an open interval about each one of its points. Show that, with this topology, X is connected iff $(X, \leq)$ is complete and contains no gaps, i.e.,if whenever $x < y$ in X, there is z for which $x < z < y$.

22. A space is said to be *regular* if the family of closed neighbourhoods of each point is a local base at that point.

(i) Show that a space is regular iff each closed set and each point not contained in it have disjoint neighbourhoods.

(ii) Show that each subspace of a regular space is regular.

(iii) Show that the closure of a compact subset of a regular space is compact.

23. Let X be a compact space satisfying the first axiom of countability. Show that each sequence in X has a convergent subsequence.

24. A *Cauchy sequence* in a metric space $(X, \mathbf{d})$ is a sequence (x_n) such that for each $\varepsilon > 0$ there is n such that $\mathbf{d}(x_p, x_q) < \varepsilon$ whenever $p, q \geq n$. A metric space is said to be *complete* if every Cauchy sequence in it converges. Show that a subspace of a complete metric space is complete iff it is closed.

25. A map of one topological space to another is said to be *closed* if it carries closed sets to closed sets.

(i) Show that the image of a normal space under a closed continuous map is normal, and give an example to show the the assumption that the map be closed can't be dropped.

(ii) Show that the projection maps $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are not closed.

26. Let $\mathbb{N}$ be assigned the cofinite topology. Show that the sequence 0, 1, 2, 3,... converges to every point of $\mathbb{N}$.

27. A metric space $(X, \mathbf{d})$ is called an *ultrametric space* if, for all $x, y, z \in X$, $\mathbf{d}(x,y) \leq \max(\mathbf{d}(x,z),\mathbf{d}(y,z))$. Prove the following assertions concerning an ultrametric space $(X,\mathbf{d})$:

(i) if $\mathbf{d}(x,z) \neq \mathbf{d}(y,z)$, then $\mathbf{d}(x,y) = \max(\mathbf{d}(x,z),\mathbf{d}(y,z))$—in other words, all triangles in X are isosceles;

(ii) any open sphere is clopen;

(iii) the space is totally disconnected.

28. Show that a space X is Hausdorff iff the set $\{(x, x): x \in X\}$ is closed in the product topology for $X \times X$.

29. Let f be a map from a locally compact space X onto a Hausdorff space Y. If f is both continuous and open, show that Y is also locally compact.

30. Recall that, if X is a topological space, $\mathbf{C}(X)$ denotes the ring of continuous real-valued functions on X. We write $\mathbf{C}^*(X)$ for the subring of all bounded continuous real-valued functions on X.

(i) Show that $f \in \mathbf{C}(X)$ has a multiplicative inverse iff it vanishes nowhere. Show also that $f \in \mathbf{C^*}(X)$ has a multiplicative inverse in $\mathbf{C^*}(X)$ iff f is *bounded away from zero*, i.e. if there is $r > 0$ for which $|f(x)| \geq r$ for all $x \in X$.

(ii) For each $f \in \mathbf{C}(X)$, write $Z(f)$ for $f^{-1}(0)$. Show that (a) $Z(f)$ is always closed; (b) $Z(f.g) = Z(f) \cup Z(g)$ and $Z(f^2 + g^2) = Z(f) \cap Z(g)$ for all $f, g \in \mathbf{C}(X)$.

(iii) An element f of $\mathbf{C}(X)$ is *idempotent* if $f^2 = f$. Show that X is connected iff the constant functions **0** and **1** are the only idempotent elements of $\mathbf{C}(X)$.

(iv) A space X is said to be *pseudocompact* if $\mathbf{C}(X) = \mathbf{C^*}(X)$. Show that any compact space is pseudocompact, and that the continuous image of a pseudocompact space is pseudocompact.

(v) For each $x \in X$ show that the set $I_x = \{f \in \mathbf{C}(X): f(x) = 0\}$ is a maximal ideal in $\mathbf{C}(X)$, and that its intersection with $\mathbf{C^*}(X)$ is a maximal ideal in the latter.

(vi) Suppose that X is compact. Show that each proper ideal in $\mathbf{C}(X)$ is included in one of the form I_x. Deduce that the maximal ideals in $\mathbf{C}(X)$ are precisely those of the form I_x.

31. Prove the *two-variable Weierstrass approximation theorem:* if $f(x, y)$ is a real-valued function defined on the closed rectangle $[a,b] \times [c,d]$ in the Euclidean plane, then f is the uniform limit of polynomials in x and y with real coefficients.