# UNDERGRADUATE SENIOR THESIS: THE GEOMETRY OF LIE GROUPS

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## Contents

## 1. Introduction

1.1. **About this thesis.** This is an expository thesis, primarily about Lie group geometry from a mathematical perspective, the thrust of which is motivated by applications in physics. I outline the general theory of Lie groups, providing several geometric arguments and proofs. The unitary group is the main example. Strings in one dimension make a surprise appearance in the large-$n$ limit.

Many thanks to my thesis advisor, Prof. Antal Jevicki, for helping me through all the physics, of which this thesis only scratches the surface. Also thanks to Prof. Alan Landman and Prof. Bruno Harris for helping me to connect the math and physics.

1.2. **Overview.** Lie groups have rigid geometry. In particular, for simple, compact Lie groups, there is a distinguished metric. This metric induces both a measure and a Laplacian. The measure allows one to introduce the Hilbert space of wavefunctions. The study of this function space is nothing but non-relativistic quantum mechanics: the Laplacian defines the energy of these wavefunctions, determining how they evolve in time.

Metric
(Free Lagrangian,
defines physical system)

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|---|---|
| Measure<br>(defines Hilbert<br>space) | Laplacian<br>(Free Hamiltonian,<br>defines dynamics) |

The standard physics story is as follows: the position of a particle on a Lie group is given by a group element (typically a matrix). The motion of this particle is determined by the Lagrangian, which is taken to be the metric. The action along a path $\gamma$ is then $L = \frac{1}{2}m \int \|\dot{\gamma}(t)\|^2 \, dt$, and the classical trajectories are geodesics. When we apply canonical quantization, we get the Hamiltonian $H = -\frac{\hbar^2}{2m}\nabla^2$.

We will focus on the Lie group $U_n$ defined by

$$U_n := \left\{ U \in M_{n\times n}(\mathbb{C}) | U^\dagger U = 1 \right\}.$$

Every unitary matrix is conjugate to a diagonal matrix of the form

$$\begin{pmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_n} \end{pmatrix}.$$

The set of all possible angles $(\phi_1, \ldots, \phi_n)$ defines an $n$-torus.

Rather than consider general functions on $U_n$, it will be useful to restrict our attention to *class functions*, i.e., functions that are constant on conjugacy classes. These are precisely the functions that are invariant under conjugation. Such functions are determined by their restriction to the $n$-torus. Moreover, since conjugation in $U_n$ can permute the order of the diagonal entries, class functions are symmetric functions of the $\phi_i$. A symmetric function on the $n$-torus may be interpreted as a wavefunction of $n$ bosons. Thus we have the following correspondence:

$$\begin{aligned} & \text{Conjugation-invariant functions on } U_n \\ \Longleftrightarrow\quad & \text{Symmetric functions on the } n\text{-torus} \\ \Longleftrightarrow\quad & \text{Wavefunctions of } n \text{ bosons on a circle.} \end{aligned}$$

We will also show a correspondence between bosons and fermions known as the "boson-fermion correspondence":

$$n\text{-particle bosonic QM on the circle} \iff n\text{-particle fermionic QM on the circle.}$$

This correspondence becomes even more interesting when we take the large-$n$ limit. We may consider a nested sequence of unitary groups

$$U_1 \subset U_2 \subset \cdots \subset U_\infty,$$

where $U_\infty := \cup U_k$. In this limit, we get quantum field theories. The boson-fermion correspondence then reads:

$$\text{Bosonic quantum field on the circle} \iff \text{Fermionic quantum field on the circle.}$$

The bosonic quantum field theory may be interpreted as either a theory of strings winding around a circle, or as a quantum field theory over a curved $AdS_2$ spacetime. This thesis explores the string interpretation.

The fermionic quantum field theory is a non-relativistic, free field theory on the circle. This correspondence of field theories is the simplest example of the conjectured $AdS/CFT$ correspondence, where a quantum field theory on $AdS$ spacetime corresponds to a conformal field theory.

## 2. Elementary theory of Lie groups

2.1. **Lie groups and Lie algebras.** A *Lie group* $G$ is a group with a real manifold structure such that the group multiplication $(g, h) \mapsto gh$ and the inverse map $g \mapsto g^{-1}$ are smooth with respect to this structure. Typical examples include matrix groups, such as $GL_n(\mathbb{R})$, $SO_n$, and $SU_n$.

A *Lie algebra* $\mathfrak{g}$ is a real or complex vector space with a multiplication called a *Lie bracket* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is bilinear, antisymmetric, and satisfies the Jacobi identity

$$0 = [[X, Y], Z] - [[X, Z], Y] + [[Y, Z], X].$$

Lie groups give rise to Lie algebras in the following way. Let $G$ be a Lie group, and let $\mathfrak{g}$ denote the real vector space of right-invariant vector fields on $G$. If $I$ denotes the identity element of $G$, then as vector spaces, $T_I G \cong \mathfrak{g}$ since a right-invariant vector field is determined by its value at a point. Thus, $\dim G = \dim \mathfrak{g}$. The Lie bracket is then the Lie bracket of vector fields.

One equivalent definition of the Lie bracket is the infinitesimal commutator of the flows. Thus, if $A, B \in T_I G$, then the Lie bracket $[A, B]$ is defined so that to second order in $A$ and $B$,

$$e^{[A,B]} := e^A e^B e^{-A} e^{-B}.$$

(The exponential map is defined as the flow from the origin along the right-invariant vector field.) In the case of matrix groups, we may do a power series expansion to obtain

$$e^{[A,B]} = e^{AB-BA}.$$

Thus the Lie bracket simply corresponds to the matrix commutator in $T_I G$.

One may show that if $G_1$ and $G_2$ are two connected Lie groups with isomorphic Lie algebras, then $G_1$ and $G_2$ have isomorphic universal covers. Therefore, $G_1$ and $G_2$ are locally isomorphic, and all the local information about a connected Lie group $G$ is encoded in its Lie algebra $\mathfrak{g}$.

Lie groups are typically used to describe a symmetry of a system. For example, consider a bound particle in a spherically symmetric potential with Hamiltonian $H = -\frac{\hbar^2}{2m}\nabla + U(\mathbf{x})$. Let $\mathcal{H}$ denote the Hilbert space of wavefunctions. This space has an energy spectrum decomposition into finite dimensional subspaces

$$\mathcal{H} = \oplus \mathcal{H}_E$$

such that for $\psi \in \mathcal{H}_E$, $H\psi = E\psi$. Now we have an $SO_3$ action on $\mathcal{H}$ by rotations of the wavefunctions. Furthermore, since $H$ is rotationally symmetric, this action commutes with $H$ since for any $\psi \in V$ and $r \in SO_3$, $Hr\psi = rH\psi$. Thus, if $H\psi = E\psi$, then $Hr\psi = rH\psi = Er\psi$, so the action preserves the energy spectrum, and we have an action of $SO_3$ on each $\mathcal{H}_E$. Such an action of a group on a vector space is called a representation. A *group representation* of a group $G$ on a vector space $V$ is defined as a group homomorphism $\rho : G \to GL(V)$. The rich structure of representations imposes constraints on the structure of the solutions.

Since Lie groups tend to be complicated nonlinear objects, it is advantageous to express everything in terms of Lie algebras, which have a simple linear structure. For representations we have the following procedure. Let $I \in G$ denote the identity. The derivative of a representation $\rho : G \to GL(V)$ induces a vector space map $d\rho_I : T_I G \to End(V)$. It follows that $d\rho_I([A, B]) = d\rho_I(A)d\rho_I(B) - d\rho_I(B)d\rho_I(A)$. Any such map $R : \mathfrak{g} \to End(V)$ satisfying this commutator relation is called a *Lie algebra representation*.

A Lie group representation $\rho : G \to GL(V)$ gives $V$ the structure of a left module over the group algebra of $G$. This is nothing but a fancy way of saying that we can write an expression such as $\rho(A)[v] + 2\rho(B)[v] + 3v$ in shorthand as $(A + 2B + 3I)v$. Similarly, a Lie algebra representation $R : \mathfrak{g} \to End(V)$ gives $V$ the structure of a left $\mathfrak{g}$-module. In fact, it will be useful to think of $V$ as a left module over a large ring that "includes $G$ and its derivatives." For example, if $\gamma : (0, 1) \to G$ is a smooth path, we can left-multiply not only by $\gamma(t)$ for any time $t$, but also by

$$\dot{\gamma}(t) := \lim_{u\to t} \frac{\gamma(t)-\gamma(u)}{t-u}.$$

In this case, $\gamma(t)^{-1}\dot{\gamma}(t) \in \mathfrak{g}$, so $\dot{\gamma} = \gamma(t)\left(\gamma(t)^{-1}\dot{\gamma}(t)\right) \in G\mathfrak{g}$. To go to higher derivatives

$$\ddot{\gamma}(t) := \frac{d}{dt}\dot{\gamma}(t),$$

we rewrite $\dot{\gamma}(t) = \gamma(t)g(t)$, where $g(t) := \gamma(t)^{-1}\dot{\gamma}(t) \in \mathfrak{g}$. Thus

$$\ddot{\gamma} = \dot{\gamma}g + \gamma\dot{g} = \gamma(g^2 + \dot{g}).$$

Now $\dot{g}(t) \in \mathfrak{g}$ since $\mathfrak{g}$ is a vector space. Thus, $\ddot{\gamma}(t) \in G(\mathfrak{g}^2 \oplus \mathfrak{g})$, where $\mathfrak{g}^2$ denotes a two-fold tensor product. Similarly,

$$\dddot{\gamma}(t) = \dot{\gamma}(g^2+\dot{g}) + \gamma(g\dot{g} + \dot{g}g + \ddot{g}) = \gamma(g^3 + 2g\dot{g} + \dot{g}g + \ddot{g}),$$

so $\dddot{\gamma}(t) \in G(\mathfrak{g}^3 \oplus \mathfrak{g}^2 \oplus \mathfrak{g})$. The pattern continues, and to encompass all the derivatives, we are naturally led to the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g}) := 1 \oplus \mathfrak{g} \oplus \mathfrak{g}^2 \oplus \cdots$. This is formally defined as the free algebra over $\mathfrak{g}$ modulo the relation $ab - ba = [a, b]$. Thus, a Lie group representation on a vector space $V$ makes $V$ a left $G \cdot \mathcal{U}(\mathfrak{g})$-module.

2.2. **The adjoint representation.** Each Lie group carries a special canonical representation called the *adjoint representation*, defined as follows. A Lie group acts on itself by inner automorphisms. This left action is given by the map $G \to Aut(G)$ defined by $h \mapsto (g \mapsto hgh^{-1})$, or equivalently by the rule $h \cdot g := hgh^{-1}$. We see immediately that the identity $I$ is a fixed point of this action. Let $\sigma_h$ denote the map $g \mapsto hgh^{-1}$. Then $(d\sigma_h)_I : T_IG \to T_IG$, so $(d\sigma_h)_I \in GL(\mathfrak{g})$. The map $h \mapsto (d\sigma_h)_I$ is easily verified to be a group representation $G \to GL(\mathfrak{g})$, which one takes as the definition of the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$. We have the associated Lie algebra representation $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, which is also called the adjoint representation. One verifies that $\mathrm{ad}_A \in \mathrm{End}(\mathfrak{g})$ is given by the linear map $[A, \cdot]$. More precisely, $\mathrm{ad}_A(B) = [A, B]$. The representation relation

$$\mathrm{ad}_{[A,B]} = \mathrm{ad}_A \circ \mathrm{ad}_B - \mathrm{ad}_B \circ \mathrm{ad}_A$$

is nothing but the Jacobi identity in disguise.

2.3. **The smooth function representation.** Another important canonical representation of a Lie group is the infinite-dimensional representation $\rho_{C^\infty}$ on the space of smooth functions $C^\infty(G)$. The Lie group representation is given by $\rho_{C^\infty}(g)\cdot f(x) = f(gx)$. We will determine the corresponding Lie algebra representation after introducing some terminology.

Let $l_g$ and $r_g$ respectively denote left and right multiplication by $g$, i.e., $l_g(h) = gh$ and $r_g(h) = hg$. A *left-invariant vector field* $V$ is a vector field such that $dl_g(V) = V$. Similarly, a *right-invariant vector field* satisfies $dr_g(V) = V$. A left or right invariant vector field $V$ is determined by its value at the identity $V|_I$ since at any point $g \in G$, $V|_g$ is respectively $dl_g(V|_I)$ or $dr_g(V|_I)$.

If $A \in \mathfrak{g}$, then we define $\gamma_A(t) := e^{tA}$, so $\dot{\gamma}_A(0) = A$. The corresponding Lie algebra representation is then given by

$$d\rho_I(A)\cdot f(x) = d\rho_I(\dot{\gamma}_A(0))\cdot f(x) = \left.\frac{d}{dt}\right|_{t=0} f(e^{tA}x).$$

This is nothing but the right-invariant vector field corresponding to $A$ acting on $f$ by partial differentiation:

$$r_x(\gamma) = e^{tA}x \implies dr_x(A) = \left.\frac{d}{dt}\right|_{t=0} e^{tA}x,$$

so

$$dr_x(A)\cdot f = \left.\frac{d}{dt}\right|_{t=0} f\left(e^{tA}x\right).$$

For $A \in \mathfrak{g}$, we introduce the notation $\partial_A$ for the right-invariant vector field corresponding to $A$. Thus, $d\rho_I(A) = \partial_A$, and

$$\partial_A \cdot f(x) = \left.\frac{d}{dt}\right|_{t=0} f(e^{tA}x).$$

2.4. **The Killing form.** We want to be able to do geometry on our Lie group, so we search for suitable metrics on $G$. The natural condition to impose is invariance. Suppose we have a metric form $\langle\cdot,\cdot\rangle_I$ defined on $T_IG$, that is bilinear and symmetric. For left invariance, we demand

$$\left\langle (dl_g)_I(v), (dl_g)_I(w)\right\rangle_g = \langle v, w\rangle_I,$$

and similarly for right invariance,

$$\left\langle (dr_g)_I(v), (dr_g)_I(w)\right\rangle_g = \langle v, w\rangle_e.$$

Note that by invariance, the metric form $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_I$ on $\mathfrak{g}$ determines the metric form on all of $G$.

If left and right invariance are to simultaneously hold, we must have

$$\begin{aligned} \langle v, w \rangle &= \left\langle (dr_g)_I^{-1} \circ (dl_g)_I (v), (dr_g)_I^{-1} \circ (dl_g)_I (w) \right\rangle \\ &= \langle d\sigma_g(v), d\sigma_g(w) \rangle , \end{aligned}$$

where once again $\sigma_g$ denotes conjugation by $g$. If $g = e^{tA}$ for $A \in \mathfrak{g}$, then

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \langle d\sigma_{e^{tA}}(v), d\sigma_{e^{tA}}(w) \rangle \\ &= \left\langle \left. \frac{d}{dt} \right|_{t=0} d\sigma_{e^{tA}}(v), w \right\rangle + \left\langle v, \left. \frac{d}{dt} \right|_{t=0} d\sigma_{e^{tA}}(w) \right\rangle \\ &= \langle \mathrm{ad}_A v, w \rangle + \langle v, \mathrm{ad}_A w \rangle . \end{aligned}$$

Thus, the invariance requirement on $\mathfrak{g}$ is

$$0 = \langle [A, V], W \rangle + \langle V, [A, W] \rangle .$$

Suppose $\alpha(X, Y)$ is an invariant symmetric bilinear form. First consider the situation in which $\alpha$ is degenerate, i.e., there is some $N \in \mathfrak{g} - \{0\}$ such that for all $X \in \mathfrak{g}$, $\alpha(N, X) = 0$. Then by invariance, for any $Y \in \mathfrak{g}$, $\alpha([N, Y], X) = \alpha(N, [Y, X]) = 0$ for all $X$. Thus we see that $\{N \in \mathfrak{g} : \alpha(N, X) = 0 \text{ for all } X \in \mathfrak{g}\}$ forms a linear subspace $\mathfrak{i}$ of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$. Such a subspace is called an *ideal*. An ideal not equal to $\{0\}$ or $\mathfrak{g}$ is called a *proper ideal*.

A Lie algebra representation $\mathfrak{g}$ on $V$ is called *irreducible* if it has no proper subspaces $V_0$ such that $\mathfrak{g}V_0 \subset V_0$. A Lie algebra is called *simple* if the adjoint representation is irreducible. (For technical reasons, we separately designate that the one-dimensional Lie algebra is *not* simple.) Equivalently, a Lie algebra is called *simple* if it has no proper ideals and is not abelian. Therefore, for a simple Lie algebra $\mathfrak{g}$, $\{N \in \mathfrak{g} : \alpha(N, X) = 0 \text{ for all } X \in \mathfrak{g}\}$ is either $\{0\}$ or $\mathfrak{g}$.

Simple Lie algebras have the fabulous property that any invariant symmetric bilinear form on $\mathfrak{g}$ is either zero or nondegenerate. As a consequence, any such form is uniquely defined up to a scalar. To see why, suppose that $\alpha$ and $\beta$ are linearly independent symmetric bilinear forms over a simple Lie algebra $\mathfrak{g}$. Then $\det(\alpha - \lambda\beta)$ is a nonconstant polynomial in $\lambda$, and so by the fundamental theorem of algebra, there is a $\lambda_0 \in \mathbb{C}$ such that $\det(\alpha - \lambda_0\beta) = 0$.

Thus $\alpha - \lambda_0\beta$ is degenerate, and since $\mathfrak{g}$ is assumed simple, $\alpha - \lambda_0\beta = 0$. This contradicts the linear independence of $\alpha$ and $\beta$. Thus, up to a scalar multiple, there can be at most one symmetric bilinear form on a simple Lie algebra $\mathfrak{g}$.

Define on $\mathfrak{g}$ a bilinear form $\kappa(X,Y) := \mathrm{Tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. By cyclic invariance of trace, $\kappa$ is symmetric. To see that $\kappa$ is invariant, we compute

$$\begin{aligned}
\kappa([X,Y],Z) &= \mathrm{Tr}\left(\mathrm{ad}_{[X,Y]} \circ \mathrm{ad}_Z\right) \\
&= \mathrm{Tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{ad}_Z - \mathrm{ad}_Y \circ \mathrm{ad}_X \circ \mathrm{ad}_Z) \\
&= \mathrm{Tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{ad}_Z - \mathrm{ad}_X \circ \mathrm{ad}_Z \circ \mathrm{ad}_Y) \\
&= \mathrm{Tr}\left(\mathrm{ad}_X \circ \mathrm{ad}_{[Y,Z]}\right) \\
&= \kappa(X,[Y,Z]).
\end{aligned}$$

This form $\kappa$ is called the *Killing form,* and it can be shown to be nonzero. Therefore, up to a scalar multiple, the Killing form is the unique invariant symmetric bilinear form.

Now for any representation of $\mathfrak{g}$ on $V$ we can define

$$\tilde{\kappa}(X,Y) = \mathrm{Tr}(v \mapsto XYv).$$

This is also an invariant symmetric bilinear form, and therefore $\tilde{\kappa}$ is proportional to $\kappa$.

Suppose $G \subset GL_n(\mathbb{R})$ is a matrix group with a simple Lie algebra $\mathfrak{g}$. Then we have the *standard representation* of $\mathfrak{g}$ on $\mathbb{R}^n$ in which we identify $\mathfrak{g}$ as a subspace in $M_{n\times n}(\mathbb{R})$, the space of $n \times n$ matrices. Now if $X, Y \in \mathfrak{g}$, then

$$\tilde{\kappa}(X,Y) = \mathrm{Tr}(XY).$$

Now suppose $M \in G$ is a matrix. Then the tangent space is $T_M G \cong dl_M(T_I G) = M\mathfrak{g}$. Similarly, $\mathfrak{g} \cong M^{-1} \cdot (T_M G)$. Furthermore, for $X, Y \in T_M G$ invariance implies

$$\tilde{\kappa}(X,Y) = \tilde{\kappa}(dl_{M^{-1}}X, dl_{M^{-1}}Y) = \mathrm{Tr}\left(M^{-1}XM^{-1}Y\right) = \mathrm{Tr}\left(XM^{-1}YM^{-1}\right).$$

In particular, the line element $(dM)^2$ is given by $\kappa(M^{-1}\dot{M}, M^{-1}\dot{M}) \propto \mathrm{Tr}\left(\left(M^{-1}\dot{M}\right)^2\right)$.

As an example, in the case that $G$ is a subgroup of the orthogonal group, $M^T M = 1 \implies M^T \dot{M} = -\dot{M}^T M$. Using $M^{-1} = M^T$ and cyclic invariance of trace, we get

$$\kappa(M^{-1}\dot{M}, M^{-1}\dot{M}) \propto \mathrm{Tr}(M^T \dot{M} M^T \dot{M}) = -\mathrm{Tr}(\dot{M}^T M M^T \dot{M}) = -\mathrm{Tr}(\dot{M}^T \dot{M}).$$

We recognize $\mathrm{Tr}(\dot{M}^T \dot{M})$ as the standard metric on $M_{n\times n}(\mathbb{R})$. Therefore, for subgroups of the orthogonal group, the invariant metric on $G$ is the restriction of the Euclidean metric on $M_{n\times n}(\mathbb{R})$ to $G$!

We can take this a step further and realize $U_n$ as a subgroup of $O_{2n}$ and deduce that $\kappa(\dot{M}, \dot{M}) \propto \mathrm{Tr}(\dot{M}^\dagger \dot{M})$.

2.5. **Cartan-Weyl basis.** Let $\mathfrak{g}$ denote a finite dimensional Lie algebra. It will be helpful to work over an algebraically closed field, so we complexify $\mathfrak{g}$ by considering $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. It is important to note that different algebras may have the same complexification. For example, $(\mathfrak{sl}_n\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{su}_n \otimes_{\mathbb{R}} \mathbb{C}$. The relationship between these two algebras become more clear if one considers the Lie group $SL_n(\mathbb{C})$. This is a complex manifold since it is given by the holomorphic constraint $\det = 1$. Thus the Lie algebra has a natural complex structure. The groups $SL_n(\mathbb{R})$ and $SU_n$ occur as **real** submanifolds, and one may verify that $(\mathfrak{sl}_n\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{su}_n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_n\mathbb{C}$. In the next section, we will study this phenomenon in more detail for the case $\mathfrak{sl}_2\mathbb{C}$.

When we complexify a real Lie algebra $\mathfrak{g}$, it will be useful to have a concept of complex conjugation on $\mathfrak{g} \otimes \mathbb{C}$. There is potential for confusion since $X \in \mathfrak{g}$ could represent a matrix with complex entries, but as an abstract vector in $\mathfrak{g}$, the complex structure of such a matrix representation is invisible. Thus the desirable notion of complex conjugation $\sigma$ for $X \otimes z$ is $\sigma(X \otimes z) := X \otimes \bar{z}$. Therefore, the real subalgebra of $\mathfrak{g} \otimes \mathbb{C}$ is the subspace fixed by $\sigma$.

We now consider a *Cartan subalgebra,* which is defined to be a maximal abelian subalgebra. One can show that all Cartan subalgebras are conjugate, and that we can find a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g} \otimes \mathbb{C}$. Typically in matrix groups, one uses the maximal subalgebra of diagonal matrices. Given some fixed choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} \otimes \mathbb{C}$, we have an eigenspace decomposition

$$\mathfrak{g} \otimes \mathbb{C} \cong \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

such that for $H \in \mathfrak{h}$ and $E^\alpha \in \mathfrak{g}_\alpha$, $\mathrm{ad}_H(E^\alpha) = \alpha(H)E^\alpha$. Note that $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{g}_0$. The $\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq \{0\}$ are called *root vectors*, and are denoted $\Phi$.

The operators $\mathrm{ad}_{E^\alpha}$ act as shifting operators on the eigenspace decomposition. More precisely, suppose $E^\alpha \in \mathfrak{g}_\alpha$ and $E^\beta \in \mathfrak{g}_\beta$. Then for $H \in \mathfrak{h}$,

$$\begin{aligned} \mathrm{ad}_H\left(\mathrm{ad}_{E^\alpha}(E^\beta)\right) &= [H, [E^\alpha, E^\beta]] \\ &= [[H, E^\alpha], E^\beta] + [E^\alpha, [H, E^\beta]] \\ &= \alpha(H)[E^\alpha, E^\beta] + \beta(H)[E^\alpha, E^\beta] \\ &= (\alpha+\beta)(H) \cdot \mathrm{ad}_{E^\alpha}(E^\beta). \end{aligned}$$

Therefore, $\mathrm{ad}_{E^\alpha} : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$. This has implications for the Killing form. Since $\mathrm{ad}_{E^\alpha} \circ \mathrm{ad}_{E^\beta} : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+(\alpha+\beta)}$, the map $\mathrm{ad}_{E^\alpha} \circ \mathrm{ad}_{E^\beta}$ is traceless unless $\alpha + \beta = 0$. Thus, if $\alpha + \beta \neq 0$, $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$. In particular, $\mathfrak{g}_\alpha \perp \mathfrak{g}_\alpha$ for $\alpha \neq 0$, so the $E^\alpha$ are null with respect to $\kappa$.

Cartan subalgebras are most useful for analyzing simple Lie algebras. From $\mathfrak{sl}_2(\mathbb{C})$ theory, one may show that for any simple Lie algebra $\mathfrak{g} \otimes \mathbb{C}$, root vectors occur in isolated pairs: $\mathbb{C}\alpha \cap \Phi = \pm\alpha$, and that each $\mathfrak{g}_\alpha$ is one-dimensional. Thus we may choose a basis of the form $\{H^i \in \mathfrak{h}\} \cup \{E^\alpha : \alpha \in \Phi\}$, and this is called a *Cartan-Weyl basis.* Moreover, for any $\alpha \in \Phi$, $E^{\pm\alpha}$ generates a $\mathfrak{sl}_2\mathbb{C}$ subalgebra.

For each pair of roots $\pm\alpha$ there is a method for designating one positive and the other negative. We denote the set of positive roots $\Phi^+$, and we have the disjoint union $\Phi = \Phi^+ \amalg \Phi^-$.

2.6. **The geometry of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{su}_2$.** We saw in the previous section that $\mathfrak{sl}_n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{su}_n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_n(\mathbb{C})$. Thus $\mathfrak{sl}_n(\mathbb{C})$ contains non-isomorphic real subalgebras.

In the case $\mathfrak{sl}_2(\mathbb{C})$ we have three generators:

$$E^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The commutation relations are

$$[H, E^-] = -2E^- \qquad [H, E^+] = 2E^+ \qquad [E^-, E^+] = -H.$$

Rewriting these relations in the ordered basis $(E^-, H, E^+)$, we have the following adjoint representation:

$$\mathrm{ad}_{E^-} := \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_H := \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \mathrm{ad}_{E^+} := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}.$$

We explicitly compute the Killing form as the matrix

$$\kappa = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Alternatively, upon noting that $\langle H, H\rangle = 8$ while $\mathrm{Tr} H^2 = 2$, we find the proportionality factor $\kappa/\tilde{\kappa} = 4$, where $\tilde{\kappa}$ is associated to the standard representation on $\mathbb{C}^2$ by $\tilde{\kappa}(A, B) = \mathrm{Tr}(AB)$. (Note that it's not $\mathrm{Tr}(A^\dagger B)$ since $\mathfrak{sl}_2\mathbb{C}$ is not unitary.) Therefore, $\kappa(A, B) = 4\mathrm{Tr}(AB)$, so we may also compute the matrix of $\kappa$ via $2 \times 2$ matrices.

The real subalgebra generated by $E^-$, $H$, and $E^+$ is $\mathfrak{sl}_2\mathbb{R}$. The Killing form is indefinite since we have the null vectors $E^-$ and $E^+$. The eigenvalues $(-2, 0, 2)$ of $H$ are all real.

In contrast, consider the real $\mathfrak{su}_2$ subalgebra generated by

$$A := \frac{2^{-3/2}}{i}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \tilde{H} := \frac{2^{-3/2}}{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B := \frac{2^{-3/2}}{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The commutation relations are

$$[\tilde{H}, A] = \frac{B}{\sqrt{2}}, \qquad [\tilde{H}, B] = -\frac{A}{\sqrt{2}}, \qquad [A, B] = \frac{H}{\sqrt{2}}.$$

The adjoint representation in the basis $(A, \tilde{H}, B)$ is

$$\mathrm{ad}_A := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \mathrm{ad}_{\tilde{H}} := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_B := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Killing form for this subalgebra is negative-definite:

$$\kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and the eigenvalues $(-i, 0, i)$ of $\tilde{H}$ are purely imaginary. We will see that purely imaginary eigenvalues in $\mathfrak{h}$ and a negative-definite Killing form are properties of compact Lie groups.

## 3. The geometry of compact Lie groups

3.1. **Case study: the unitary group.** We begin our analysis of $U_n$ by finding "radial" and "angular" coordinates for $U_n$ that decouple. This will lead to a biinvariant measure known as the Haar measure. We will then be able to compute the Laplacian. (The term "radial" coordinates is slightly misleading since they are actually angles on the maximal $n$-torus.)

The Killing form associated to the standard representation of $U$ is

$$\tilde{\kappa}(U_1, U_2) = \mathrm{Tr}(U_1^\dagger U_2).$$

The associated Lagrangian is

$$L = \frac{1}{2}\tilde{\kappa}(\dot{U}, \dot{U}) = \frac{1}{2}\mathrm{Tr}(\dot{U}^\dagger \dot{U}).$$

The nondegenerate invariant symmetric bilinear form $\tilde{\kappa}$ is not necessarily unique. The Lie algebra $\mathfrak{u}_n$ of $U_n$ is the space of anti-Hermitian matrices. (Physicists traditionally consider the space $\frac{1}{i}\mathfrak{u}_n$ of Hermitian matrices. If $H$ is Hermitian, then $iH \in \mathfrak{u}_n$.) Now $\mathfrak{u}_n$ is not simple since the diagonal matrix $iI$ generates an ideal $i\mathbb{R}I \subset \mathfrak{u}_n$ since $[\mathfrak{u}_n, i\mathbb{R}I] = 0 \subset i\mathbb{R}I$. We will proceed as normal but return to this issue later.

We may diagonalize $U$ as $VDV^\dagger$ for unitary $U$. Then

$$\begin{aligned} \dot{U} &= \dot{V}DV^\dagger + V\dot{D}V^\dagger + VD\dot{V}^\dagger, \\ \dot{U}^\dagger &= \dot{V}D^\dagger V^\dagger + V\dot{D}^\dagger V^\dagger + VD^\dagger \dot{V}^\dagger. \end{aligned}$$

The line element, a.k.a. the free Laplacian, may be evaluated using $V^\dagger V = D^\dagger D = 1$, $\dot{V}^\dagger V = -V^\dagger \dot{V}$, cyclic invariance of trace, and commutativity of diagonal matrices, to obtain

$$L = \frac{1}{2}\mathrm{Tr}(\dot{U}^\dagger \dot{U}) = \frac{1}{2}\mathrm{Tr}(\dot{D}^\dagger \dot{D}) + \mathrm{Tr}(\dot{W}^2) - \mathrm{Tr}(\dot{W} D^\dagger \dot{W} D),$$

where $\dot{W} := \frac{1}{i} V^\dagger \dot{V}$ is a Hermitian matrix. The matrix $\dot{W}$ may be interpreted as $\frac{1}{i} dl_{V^{-1}} \dot{V} \in \frac{1}{i}\mathfrak{g}$, where the vector $\dot{V}$ is being pulled back by left multiplication by $V^{-1}$ to the Lie algebra. Thus we have decomposed the velocity $\dot{U}$ into "radial" velocity $\dot{D}$ and "angular" velocity $\dot{W}$. Since $U$ is unitary, we have $D_{jk} = \delta_{jk} e^{i\phi_j}$. We compute

$$\begin{aligned}
\mathrm{Tr}(\dot{W}^2) - \mathrm{Tr}(\dot{W} D^\dagger \dot{W} D) &= \dot{W}_{jk}\dot{W}_{kl}\delta_{jl} - \dot{W}_{jk}\delta_{kl} e^{-i\phi_k} \dot{W}_{lm}\delta_{mn} e^{i\phi_m}\delta_{nj} \\
&= \dot{W}_{jk}\dot{W}_{kj} - \dot{W}_{jk} e^{-i\phi_k} \dot{W}_{kj} e^{i\phi_j} \\
&= \left|\dot{W}_{jk}\right|^2 (1 - e^{i(\phi_k - \phi_j)}) \\
&= 2\sum_{j<k} \left|\dot{W}_{jk}\right|^2 (2 - e^{i(\phi_k - \phi_j)} - e^{i(\phi_j - \phi_k)}) \\
&= 4\sum_{j<k} \left|\dot{W}_{jk}\right|^2 \sin^2 \frac{\phi_j - \phi_k}{2}.
\end{aligned}$$

Therefore,

$$L = \frac{1}{2}\sum_j \dot{\phi}_j^2 + 4\sum_{j>k} \left(\dot{W_{jk}^{\Re}}^2 + \dot{W_{jk}^{\Im}}^2\right) \sin^2 \frac{\phi_j - \phi_k}{2},$$

where $\dot{W_{jk}^{\Re}}$ and $\dot{W_{jk}^{\Im}}$ are the independent real and imaginary components of $\dot{W}_{jk}$. We have succeeded in diagonalizing the metric. Writing the Lagrangian in metric form, we have

$$(ds)^2 = 2L = (d\phi_j)^2 + 8\sum_{j>k} \left(\left(dW_{jk}^{\Re}\right)^2 + \left(dW_{jk}^{\Im}\right)^2\right) \sin^2 \frac{\phi_j - \phi_k}{2}.$$

This corresponds to a metric of the form

$$g = \begin{pmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 8\sin^2\frac{\phi_2-\phi_1}{2} & & & & & \\ & & & & & 8\sin^2\frac{\phi_2-\phi_1}{2} & & & & \\ & & & & & & 8\sin^2\frac{\phi_3-\phi_1}{2} & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 8\sin^2\frac{\phi_n-\phi_{n-1}}{2} & \\ & & & & & & & & & 8\sin^2\frac{\phi_n-\phi_{n-1}}{2} \end{pmatrix}.$$

Now we compute the invariant measure $d\mu$.

$$8^{\frac{n(n-1)}{2}} d\mu := \sqrt{\det g} = 8^{\frac{n(n-1)}{2}} \left(\prod_{j>k} \sin^2 \frac{\phi_j - \phi_k}{2}\right) d\phi_1 \wedge \cdots \wedge d\phi_n \wedge dW_{21}^{\Re} \wedge \cdots \wedge dW_{n\,n-1}^{\Im}.$$

The *maximal torus* is defined as $\mathbb{T}^n := \left\{ \begin{pmatrix} e^{i\phi_1} & & \\ & \ddots & \\ & & e^{i\phi_n} \end{pmatrix} \right\} \cong \mathbb{R}^n/2\pi\mathbb{Z}^n$. In particular, for any function of the "radial" coordinates $\phi$ and independent of the "angular" coordinates,

$$\int f(\phi)\, d\mu = \int f(\phi) \left(\prod_{j>k} \sin^2 \frac{\phi_j - \phi_k}{2}\right) d\phi_1 \wedge \cdots \wedge d\phi_n.$$

Thus the invariant measure of the maximal torus differs from the standard measure on the maximal torus by the Jacobian $\Delta^2$, where

$$\Delta := \pm \prod_{j>k} \sin \frac{\phi_j - \phi_k}{2}.$$

The symbol $\Delta$ is not to be confused with the Laplacian, which we will denote $\nabla^2$. We note that $\Delta$ is a two-valued function, but it may be lifted to a smooth single-valued function on the $2^n$-sheeted cover $\mathbb{R}^n/4\pi\mathbb{Z}^n$.

The free Hamiltonian is the negative of the Laplacian which, for a metric $g_{ij}$, is given by

$$H = -\nabla^2 = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^j}.$$

Substituting the group metric, we find

$$H = -\sum_i \frac{1}{\Delta^2}\frac{\partial}{\partial\phi_i}\Delta^2\frac{\partial}{\partial\phi_i} - \frac{1}{8}\sum_{j,k}\csc^2\frac{\phi_j-\phi_k}{2}\left(\frac{\partial^2}{\left(\partial W^{\Re}_{jk}\right)^2}+\frac{\partial^2}{\left(\partial W^{\Im}_{jk}\right)^2}\right).$$

We will be concerned with only the "radial" component of $H$, so we restrict to functions dependent only on the $\phi_i$ by setting $\frac{\partial}{\partial W^{\Re}_{jk}} = \frac{\partial}{\partial W^{\Im}_{jk}} = 0$ to obtain

$$H_{\mathbb{T}^n} := -\sum_i \frac{1}{\Delta^2}\frac{\partial}{\partial\phi_i}\Delta^2\frac{\partial}{\partial\phi_i} = -\sum_i\frac{1}{\Delta}\frac{\partial^2}{\partial\phi_i^2}\Delta + \sum_i\frac{1}{\Delta}\frac{\partial^2\Delta}{\partial\phi_i^2}.$$

3.2. **General theory of compact simple groups.** If $\mathfrak{g}$ is a simple Lie algebra, let $\{H^i \in \mathfrak{h}\}\cup\{E^\alpha : \alpha\in\Phi\}$ be a Cartan-Weyl basis. Since $\mathfrak{g}$ is simple, the roots occur in pairs and we have the decomposition

$$\mathfrak{g}\otimes\mathbb{C} = \mathfrak{h}\oplus\left(\bigoplus_{\alpha\in\Phi^+}\mathbb{C}E^\alpha\oplus\mathbb{C}E^{-\alpha}\right).$$

Two examples that are helpful to keep in mind are $\mathfrak{u}_n$ or $\mathfrak{su}_n$. We have the root vectors, which are matrices indexed by $1\le j\ne k\le n$ given by $E^{jk} = i\delta_{jk}$. The corresponding roots $\alpha_{jk}$ are then $\alpha_{jk}(\phi) = i(\phi_j-\phi_k)$.

If we restrict $\kappa$ to $\mathfrak{h}$, we discover that $\kappa(H^i,H^j) = \mathrm{Tr}(X\mapsto[H^i,[H^j,X]])$. Computing this trace in the Cartan-Weyl basis, we see that $X = H^k$ contributes nothing, but $X = E^\alpha$ contributes $\alpha(H^i)\alpha(H^j)$. Thus $\kappa(H^i,H^j) = \sum_{\alpha\in\Phi}\alpha(H^i)\alpha(H^j)$.

In the case of $\mathfrak{su}_n$, the restriction of $\kappa$ to $\mathfrak{h}$ is negative-definite: $\|\phi_iH^i\|^2 = -\sum_{j>k}(\phi_j-\phi_k)^2$. However, if we evaluate $\kappa$ on $\mathfrak{u}_n$, which is not simple, we see that $\kappa$ is degenerate: $\langle\sum_i H^i,\mathfrak{h}\rangle = 0$. This is a consequence of $\mathfrak{u}_n$ not being simple. We may recover our previous analysis of $\mathfrak{u}_n$ by noting that $-\sum_{j>k}(\phi_j-\phi_k)^2-(\phi_1+\cdots+\phi_n)^2 = -n\sum\phi_j^2$. Since $(\phi_1+\cdots+\phi_n)^2$ vanishes on $\mathfrak{su}_n$, we can augment $\kappa$ to be negative-definite on the Cartan subalgebra of $\mathfrak{u}_n$ by including this term.

Another way to think of $\mathfrak{u}_n$ is as an extension of $\mathfrak{su}_n$. Note that by factoring out the determinant, we may write $U_n = SU_n\times U_1$. Thus $\mathfrak{u}_n = \mathfrak{su}_n\oplus\mathfrak{u}_1$, where $\mathfrak{u}_1$ is just the

1-dimensional Lie algebra. The vector space of symmetric invariant forms on $\mathfrak{u}_n$ is therefore two-dimensional, spanned by $\kappa$ on $\mathfrak{su}_n$ and by $(d\phi_1 + \cdots + d\phi_n)^2$ on $\mathfrak{u}_1$.

We now return to the general situation, in which we assume that $\mathfrak{g}$ is simple. To determine the Killing form on the remainder of the Cartan-Weyl basis, the shifting property of $\mathrm{ad}_{E^\alpha}$ gives $\langle E^\alpha, \mathfrak{h}\rangle = 0$ and $\left\langle E^\alpha, E^\beta\right\rangle = \delta_{\beta,-\alpha}\left\langle E^\alpha, E^{-\alpha}\right\rangle$. Thus the metric decomposes orthogonally between $\mathfrak{h}$ and each pair of opposite roots.

We will now examine the consequences of compactness. Let $G$ be a compact simple Lie group with real Lie algebra $\mathfrak{g}$. As with any paracompact manifold, we may construct a (not necessarily invariant) positive-definite metric on $G$ via partitions of unity. Now since $G$ is compact, we may average this metric over the group to get an invariant positive-definite metric $\tilde{\kappa}$.

With respect to $\tilde{\kappa}$, for any $B \in \mathfrak{g}$ we have $(\mathrm{ad}_B)^T = -\mathrm{ad}_B$ since

$$\tilde{\kappa}(A, \mathrm{ad}_B C) = \tilde{\kappa}(A, [B, C]) = \tilde{\kappa}([A, B], C) = \tilde{\kappa}(-\mathrm{ad}_B A, C).$$

Since $\mathrm{ad}_B$ is skew-adjoint with respect to the positive-definite inner product $\tilde{\kappa}$, it follows that the eigenvalues of $\mathrm{ad}_B$ are purely imaginary. For any $B$, let $i\phi_1, \ldots, i\phi_n$ denote the eigenvalues of $\mathrm{ad}_B$. If $B \neq 0$, then $\mathrm{ad}_B \neq 0$, so there exists some $\phi_j \neq 0$. We then evaluate the Killing form $\kappa(B, B) = \mathrm{Tr}(\mathrm{ad}_B \circ \mathrm{ad}_B) = \sum_j (i\phi_j)^2 = -\sum_j \phi_j^2 < 0$. This proves that the Killing form $\kappa$ is negative-definite. Thus we may choose $H^i$ such that $\langle H^i, H^j\rangle = -\delta_{ij}$.

We can deduce even more from compactness. Suppose $\alpha \in \Phi$. Then for any $H \in \mathfrak{h}$, $\alpha(H)$ is an eigenvalue of $\mathrm{ad}_H$ on the eigenvector $E^\alpha$. Since the eigenvalues are purely imaginary, it follows that $\bar{\alpha} = -\alpha$.

Although $E^\alpha \in \mathfrak{g} \otimes C$, unfortunately $E^\alpha \notin \mathfrak{g}$ because $E^\alpha$ is not fixed under conjugation. Applying $\sigma$ to the expression $[H, E^\alpha] = \alpha(H)E^\alpha$, we get $[H, \sigma(E^\alpha)] = -\alpha(H)\sigma(E^\alpha)$, so $\sigma(E^\alpha) \in \mathfrak{g}_{-\alpha}$. Without loss of generality, we may renormalize the $E^\alpha$ so that $\sigma(E^\alpha) = E^{-\alpha}$.

Since $H^\alpha := \frac{i}{\sqrt{2}}[E^{-\alpha}, E^\alpha]$ is fixed by $\sigma$, $H^\alpha \in \mathfrak{g}$. Moreover, since $H^\alpha \in \mathfrak{h} \otimes C$, it follows that $H^\alpha \in \mathfrak{h}$. We may completely determine the $E^{\pm\alpha}$ (up to sign) by renormalizing such that both $\sigma(E^\alpha) = E^{-\alpha}$ and

$$\|H^\alpha\|^2 = -1.$$

(Since $\kappa$ is negative-definite on $\mathfrak{h}$, we could have normalized $H^\alpha$ to any negative number.) Thus $-1 = (\alpha(H^\alpha))^2 + (-\alpha(H^\alpha))^2$, so $\alpha(H^\alpha) = \pm\frac{i}{\sqrt{2}}$. By possibly sending $H^\alpha \mapsto -H^\alpha$, we may assume $\alpha(H^\alpha) = \frac{i}{\sqrt{2}}$. Therefore, $[H^\alpha, E^\alpha] = \frac{i}{\sqrt{2}} E^\alpha$ and $[H^\alpha, E^{-\alpha}] = -\frac{i}{\sqrt{2}} E^{-\alpha}$. By definition of $H^\alpha$, $[E^{-\alpha}, E^\alpha] = -i\sqrt{2} H^\alpha$. Setting $A^\alpha = \frac{1}{2}(E^\alpha + E^{-\alpha})$ and $B^\alpha = \frac{i}{2}(E^\alpha - E^{-\alpha})$, we find $[H^\alpha, A^\alpha] = \frac{1}{\sqrt{2}} B^\alpha$, $[H^\alpha, B^\alpha] = -\frac{1}{\sqrt{2}} A^\alpha$, and $[A^\alpha, A^\beta] = \frac{1}{\sqrt{2}} H^\alpha$. These are the generators of $\mathfrak{su}_2$ from section 2.6. Note that $\sigma$ preserves $A^\alpha$ and $B^\alpha$, and therefore $A^\alpha$ and $B^\alpha$ belong to $\mathfrak{g}$. (For more details on compact groups, see Fulton+Harris §26.1.)

In summary, for a compact simple group we can choose a Cartan-Weyl basis so that

$$\left\langle H^i, H^j \right\rangle = -\delta_{ij}, \qquad \left\langle H^i, E^\alpha \right\rangle = 0, \qquad \left\langle E^\alpha, E^\beta \right\rangle = -2\delta_{\alpha,-\beta}.$$

For each $\alpha \in \Phi^+$ we define

$$A^\alpha = \frac{1}{2}\left(E^\alpha + E^{-\alpha}\right) \text{ and } B^\alpha = \frac{i}{2}\left(E^\alpha - E^{-\alpha}\right),$$

which gives a negative-definite orthonormal set

$$\left\{H^j\right\} \cup \bigcup_{\alpha\in\Phi^+} \left\{A^\alpha, B^\alpha\right\}.$$

3.3. **The metric of a compact simple Lie group.** We now wish to study the geometry not just at the identity, but at an arbitrary point $X$ of $G$. Let $\dot{X}$ be a tangent vector at $X$. Then

$$\left\langle \dot{X}, \dot{X} \right\rangle_X = \left\langle X^{-1}\dot{X}, X^{-1}\dot{X} \right\rangle.$$

If the eigenvalues of $X$ are distinct, then $X$ may be brought into the diagonal form $X = VDV^{-1}$. Since $D$ is diagonal, $D = e^{\phi_j H^j}$ for some parameters $\phi_j$. Suppose we set $V = e^{\gamma_\alpha E^\alpha}$. We will show that the $\phi_j$ and $\gamma_\alpha$ form local coordinates when the eigenvalues of $X$ are distinct.

For $D$ and $V$ to be legitimate elements of $G$ we need to make sure that $\phi_j H^j$ and $\gamma_\alpha E^\alpha$ are in $\mathfrak{g}$ and not just in $\mathfrak{g}\otimes\mathbb{C}$, or equivalently that they are fixed by conjugation $\sigma$. By definition of the Cartan-Weyl basis, $H^j \in \mathfrak{g}$ so the first condition is $\phi_j H^j = \sigma(\phi_j H^j) = \overline{\phi_j} H^j$. Thus $\phi_j$ must be real. The second condition is $\gamma_\alpha E^\alpha = \sigma(\gamma_\alpha E^\alpha) = \overline{\gamma_\alpha} E^{-\alpha} = \overline{\gamma_{-\alpha}} E^\alpha$. Therefore, $\gamma_\alpha = \overline{\gamma_{-\alpha}}$.

Using the relation $X = VDV^{-1}$ we compute

$$X^{-1}\dot{X} = -VD^{-1}V^{-1}\dot{V}DV^{-1} + VD^{-1}\dot{D}V^{-1} + \dot{V}V^{-1}.$$

We evaluate $\left\langle X^{-1}\dot{X}, X^{-1}\dot{X}\right\rangle$ as the sum of six terms:

$$\begin{aligned}
\left\langle -VD^{-1}V^{-1}\dot{V}DV^{-1}, -VD^{-1}V^{-1}\dot{V}DV^{-1}\right\rangle &= \left\langle V^{-1}\dot{V}, V^{-1}\dot{V}\right\rangle. \\
\left\langle VD^{-1}\dot{D}V^{-1}, VD^{-1}\dot{D}V^{-1}\right\rangle &= \left\langle D^{-1}\dot{D}, D^{-1}\dot{D}\right\rangle. \\
\left\langle \dot{V}V^{-1}, \dot{V}V^{-1}\right\rangle &= \left\langle V^{-1}\dot{V}, V^{-1}\dot{V}\right\rangle. \\
2\left\langle -VD^{-1}V^{-1}\dot{V}DV^{-1}, VD^{-1}\dot{D}V^{-1}\right\rangle &= -2\left\langle V^{-1}\dot{V}, \dot{D}D^{-1}\right\rangle. \\
2\left\langle -VD^{-1}V^{-1}\dot{V}DV^{-1}, \dot{V}V^{-1}\right\rangle &= -2\left\langle V^{-1}\dot{V}, DV^{-1}\dot{V}D^{-1}\right\rangle. \\
2\left\langle VD^{-1}\dot{D}V^{-1}, \dot{V}V^{-1}\right\rangle &= 2\left\langle V^{-1}\dot{V}, \dot{D}D^{-1}\right\rangle.
\end{aligned}$$

Adding these up, we get

$$\left\langle X^{-1}\dot{X}, X^{-1}\dot{X}\right\rangle = \left\langle D^{-1}\dot{D}, D^{-1}\dot{D}\right\rangle + 2\left\langle V^{-1}\dot{V}, V^{-1}\dot{V} - DV^{-1}\dot{V}D^{-1}\right\rangle.$$

We now set $D = e^{\phi_j H^j}$, and $\dot{W} := V^{-1}\dot{V}$ which is the pullback $dl_{V^{-1}}(\dot{V}) \in \mathfrak{g}$. We then find

$$\begin{aligned}
\left\langle X^{-1}\dot{X}, X^{-1}\dot{X}\right\rangle &= \left\langle \dot{\phi}_j H^j, \dot{\phi}_j H^j\right\rangle + 2\left\langle \dot{W}, \dot{W} - D\dot{W}D^{-1}\right\rangle \\
&= -\dot{\phi}_j^2 + 2\left\langle \dot{W}, \dot{W} - D\dot{W}D^{-1}\right\rangle.
\end{aligned}$$

To compute $D\dot{W}D^{-1}$ we will use the following result.

**Lemma.** *For $A, B \in \mathfrak{g}$, $e^A B e^{-A} = \left(e^{ad_A}\right)(B)$.*

*Proof.* Let $F(t) = e^{tA}Be^{-tA}$. Then $\frac{dF}{dt} = AF(t) - F(t)A = \mathrm{ad}_A F(t)$. Now $\mathrm{ad}_A$ is a constant linear operator on the space of matrices, so this differential equation is solved by $F(t) = \left(e^{t\mathrm{ad}_A}\right)(F(0))$, and so $F(1) = \left(e^{\mathrm{ad}_A}\right)(B)$. □

If we write $\dot{W} = \dot{W}_k H^k + \dot{W}_\alpha E^\alpha$, then applying this lemma we find that

$$\begin{aligned}
D\dot{W}D^{-1} &= \left(e^{\mathrm{ad}_{\phi_j H^j}}\right)(\dot{W}) \\
&= \dot{W}_k \left(e^{\mathrm{ad}_{\phi_j H^j}}\right)(H^k) + \dot{W}_\alpha \left(e^{\mathrm{ad}_{\phi_j H^j}}\right)(E^\alpha).
\end{aligned}$$

Since the $H^j$ commute,

$$\begin{aligned}\left(e^{\mathrm{ad}_{\phi_j H^j}}\right)(H^k) &= (1+\mathrm{ad}_{\phi_j H^j}+\frac{1}{2}\mathrm{ad}_{\phi_j H^j}\circ\mathrm{ad}_{\phi_j H^j}+\cdots)(H^k)\\ &= (1+0+0+\cdots)(H^k)\\ &= H^k.\end{aligned}$$

Now we compute

$$\begin{aligned}\left(e^{\mathrm{ad}_{\phi_j H^j}}\right)(E^\alpha) &= (1+\mathrm{ad}_{\phi_j H^j}+\frac{1}{2}\mathrm{ad}_{\phi_j H^j}\circ\mathrm{ad}_{\phi_j H^j}+\cdots)(E^\alpha)\\ &= (1+\alpha(\phi_j H^j)+\frac{1}{2}\left(\alpha(\phi_j H^j)\right)^2+\cdots)E^\alpha\\ &= e^{\alpha(\phi_j H^j)}E^\alpha.\end{aligned}$$

We conclude that $D\dot{W}D^{-1} = \dot{W}_k H^k + \dot{W}_\alpha e^{\alpha(\phi_j H^j)}E^\alpha$, and so $\dot{W} - D\dot{W}D^{-1} = \dot{W}_\alpha(1 - e^{\alpha(\phi_j H^j)})E^\alpha$. Now we compute

$$\begin{aligned}\left\langle \dot{W}, \dot{W} - D\dot{W}D^{-1}\right\rangle &= \sum_{\alpha,\beta\in\Phi}\left\langle \dot{W}_\beta E^\beta, \dot{W}_\alpha\left(1-e^{\alpha(\phi_j H^j)}\right)E^\alpha\right\rangle\\ &= -2\sum_{\alpha\in\Phi}\dot{W}_{-\alpha}\dot{W}_\alpha\left(1-e^{\alpha(\phi_j H^j)}\right)\\ &= -2\sum_{\alpha\in\Phi^+}\left|\dot{W}_\alpha\right|^2\left(2-e^{\alpha(\phi_j H^j)}-e^{-\alpha(\phi_j H^j)}\right)\\ &= -8\sum_{\alpha\in\Phi^+}\left(\left(\dot{W}_\alpha^{\Re}\right)^2+\left(\dot{W}_\alpha^{\Im}\right)^2\right)\sin^2\left(\frac{\alpha(\phi_j H^j)}{2i}\right).\end{aligned}$$

For each $\alpha \in \Phi^+$, recall $A^\alpha := \frac{1}{2}(E^\alpha + E^{-\alpha})$ and $B^\alpha := \frac{i}{2}(E^\alpha - E^{-\alpha})$ are orthonormal. Thus $\left(\dot{W}_\alpha^{\Re}\right)^2 + \left(\dot{W}_\alpha^{\Im}\right)^2 = \left\|\dot{W}_\alpha^{\Re}A^\alpha\right\|^2 + \left\|\dot{W}_\alpha^{\Im}B^\alpha\right\|$, and we think of and $\dot{W}_\alpha^{\Re}$ and $\dot{W}_\alpha^{\Im}$ as the respective components of $\dot{A}^\alpha$ and $\dot{B}^\alpha$.

Thus we have shown that if $X = Ve^{\phi_j H^j}V^{-1}$, and $\dot{W} = V^{-1}\dot{V}$, then

$$\left\langle \dot{X}, \dot{X}\right\rangle_X = -\sum_j \dot{\phi}_j^2 - 8\sum_{\alpha\in\Phi^+}\left(\left(\dot{W}_\alpha^{\Re}\right)^2+\left(\dot{W}_\alpha^{\Im}\right)^2\right)\sin^2\left(\sum_j \phi_j\frac{\alpha(H^j)}{2i}\right),$$

or equivalently in differential notation,

$$\left(d\dot{X}\right)^2 = -\sum_j\left(dH^j\right)^2 - 8\sum_{\alpha\in\Phi^+}\left((dA^\alpha)^2+(dB^\alpha)^2\right)\sin^2\left(\sum_j \phi_j\frac{\alpha(H^j)}{2i}\right).$$

Therefore, we have succeeded in diagonalizing the Killing form:

$$\kappa = \begin{pmatrix} -1 & & & & & & & & \\ & -1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & -1 & & & & & \\ & & & & 8\sin^2\left(\frac{\alpha_1(\phi_j H^j)}{2i}\right) & & & & \\ & & & & & 8\sin^2\left(\frac{\alpha_1(\phi_j H^j)}{2i}\right) & & & \\ & & & & & & 8\sin^2\left(\frac{\alpha_2(\phi_j H^j)}{2i}\right) & & \\ & & & & & & & 8\sin^2\left(\frac{\alpha_2(\phi_j H^j)}{2i}\right) & \\ & & & & & & & & \ddots \end{pmatrix}.$$

If we denote the radial coordinates on the torus as $\phi := \sum \phi_j H^j$, and $d\phi = d\phi_1 \wedge \cdots \wedge d\phi_r$, then the invariant measure on the torus is given by

$$d\mu := (\text{const}) \cdot \sqrt{\det(-\kappa)} d\phi = \left( \prod_{\alpha \in \Phi^+} \sin^2 \frac{\alpha(\phi)}{2i} \right) d\phi.$$

Denote $\Delta := \prod_{\alpha\in\Phi^+} \sin \frac{\alpha(\phi)}{2i}$. The Laplacian on the torus is

$$\begin{aligned} \nabla^2 &= \frac{1}{\sqrt{\det(-\kappa)}} \frac{\partial}{\partial x^i} \sqrt{\det(-\kappa)} \left(-\kappa^{ij}\right) \frac{\partial}{\partial x^j} \\ &= \frac{1}{\Delta^2} \frac{\partial}{\partial \phi_i} \Delta^2 \frac{\partial}{\partial \phi_i} \\ &= \sum_i \frac{1}{\Delta} \frac{\partial^2}{\partial \phi_i^2} \Delta - \frac{1}{\Delta} \sum_i \frac{\partial^2 \Delta}{\partial \phi_i^2}. \end{aligned}$$

In the case of $\mathfrak{u}_n$ or $\mathfrak{su}_n$, we have roots $\alpha_{jk}$ defined as $\alpha_{jk}(\phi) = i(\phi_j - \phi_k)$, and conjugation gives $\sigma(\alpha_{jk}) = \alpha_{kj}$. We may take as positive roots $\alpha_{jk}$ with $j > k$.

3.4. **The curvature of a Lie group.** Recall that any element $X$ in the Lie algebra $\mathfrak{g}$ corresponds to a right-invariant vector field. Suppose $\{e_i\} \in \mathfrak{g}$ is a negative-definite orthonormal basis, and let $\partial_{e_i}$ denote the corresponding right-invariant vector fields. Then $\partial_{e_i}$ is orthonormal at each point, and the $\partial_{e_i}$ act on $C^\infty(G)$ by partial differentiation along the $e_i$ direction.

By invariance, we know that each $\partial_{e_i}$ is a Killing field. Hence the integral curves of the $\partial_{e_i}$ are geodesics. It follows that for the connection, $\nabla_{\partial_{e_i}}\partial_{e_i} = 0$ (Here $\nabla$ is the connection, not the gradient!). More generally, $\nabla_X X = 0$ for any right-invariant vector field $X$. In particular, for any right-invariant vector fields $X$, $Y$ and $Z$,

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X Y + \nabla_Y X = 2\nabla_X Y + [Y, X].$$

Therefore,

$$\nabla_Y X = \frac{1}{2}[X, Y].$$

Curvature is then

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = -\frac{1}{4}[[X,Y],Z].$$

Ricci curvature is

$$\mathrm{Ric}(X,Y) = -\frac{1}{4}\sum_i \langle [[\partial_{e_i}, X], Y], \partial_{e_i}\rangle = -\frac{1}{4}\mathrm{Tr}(\mathrm{ad}_Y \circ \mathrm{ad}_X) = -\frac{1}{4}\kappa.$$

Thus, a compact simple Lie group is an Einstein manifold of positive curvature, since $\kappa$ is negative-definite.

3.5. **The Laplacian as a Casimir operator.** It will be useful to have an expression for the Laplacian $\nabla^2$ in terms of the negative-definite orthonormal right-invariant vector fields $\partial_{e_i}$. We will almost use the standard formula

$$\nabla^2 f \overset{?}{=} g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k}\right),$$

however, the the Laplacian for a positive-definite metric will differ by a sign from the Laplacian for a negative-definite metric. The standard notion of Laplacian on a compact surface is with respect to the positive-definite metric, so we will adjust our definition by a sign:

$$\nabla^2 f = -g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k}\right).$$

In the normal coordinates induced by the $\partial_{e_i}$ at a point $p$, we have $g^{ij}(p) = -\delta_{ij}$. Since $\nabla_{\partial_{e_i}}\partial_{e_i} = 0$ we have $\Gamma^k_{ii}(p) = 0$, so

$$\left(\nabla^2 f\right)(p) = \sum_i \frac{\partial^2 f}{\partial x_i^2}(p).$$

Thus

$$\nabla^2 f = \sum_i \partial_{e_i}(\partial_{e_i}(f)) = \left(\sum_i \partial_i^2\right) f.$$

We may realize the Laplacian as an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ as follows. Define $\mathcal{C}_2 := \sum_i e_i^2$. Then for any Lie algebra representation $R$, we have $R(\mathcal{C}_2) = \sum_i R(e_i)^2$. In particular, for the representation $R_{C^\infty}$ of $\mathfrak{g}$ on $C^\infty(G)$ we have

$$R_{C^\infty}(\mathcal{C}_2) = \sum_i R_{C^\infty}(e_i)^2 = \sum_i \partial_{e_i}^2 = \nabla^2.$$

One nice property of $\mathcal{C}_2$ is that it is independent of the choice of of orthonormal basis $\{e_i\}$. Suppose $\{\tilde{e}_i\}$ is another orthonormal basis. Then by completeness of the $\tilde{e}_i$ basis,

$$e_i = \sum_j - \langle e_i, \tilde{e}_j \rangle \, \tilde{e}_j.$$

Thus,

$$\mathcal{C}_2 = \sum_i e_i^2 = \sum_{i,j,k} \langle e_i, \tilde{e}_j \rangle \, \tilde{e}_j \, \langle e_i, \tilde{e}_k \rangle \, \tilde{e}_k = \sum_{i,j,k} \tilde{e}_j \, \langle e_i \, \langle e_i, \tilde{e}_j \rangle \, , \tilde{e}_k \rangle \, \tilde{e}_k = -\sum_{j,k} \tilde{e}_j \, \langle \tilde{e}_j, \tilde{e}_k \rangle \, \tilde{e}_k = \sum_i \tilde{e}_i^2.$$

Hence, for any simple Lie algebra, we may refer to the unique element $\mathcal{C}_2$ without reference to a basis.

Another important property of $\mathcal{C}_2$ is that $[\mathcal{C}_2, X] = 0$ for all $X \in \mathfrak{g}$. To prove this, we use the identity $[AB, C] = A[B, C] + [A, C]B$ to get

$$[\mathcal{C}_2, X] = \sum_i [e_i^2, X] = \sum_i \left(e_i[e_i, X] + [e_i, X]e_i\right).$$

Using the completeness of the $e_i$ basis,

$$[e_i, X] = -\sum_j \langle [e_i, X], e_j \rangle \, e_j.$$

Thus,

$$\sum_i e_i[e_i,X] = -\sum_{i,j} e_i \langle [e_i,X],e_j\rangle e_j = -\sum_{i,j} e_i \langle e_i,[X,e_j]\rangle e_j = \sum_j [X,e_j]e_j = -\sum_i [e_i,X]e_i,$$

so

$$[\mathcal{C}_2,X] = \sum_i e_i[e_i,X] + \sum_i [e_i,X]e_i = 0.$$

As a corollary, for any right-invariant vector field $\partial_V$, we have

$$\partial_V\left(\nabla^2 f\right) = \nabla^2\left(\partial_V f\right).$$

## 4. The quantum mechanics of $U_n$

4.1. **Eigenfunctions of the Laplacian.** Recall the $U(N)$ Laplacian

$$H_{\mathbb{T}^n} := -\sum_i \frac{1}{\Delta^2}\frac{\partial}{\partial\phi_i}\Delta^2\frac{\partial}{\partial\phi_i} = -\sum_i \frac{1}{\Delta}\frac{\partial^2}{\partial\phi_i^2}\Delta + \sum_i \frac{1}{\Delta}\frac{\partial^2\Delta}{\partial\phi_i^2}.$$

We will recognize the term $\sum_i \frac{1}{\Delta}\frac{\partial^2\Delta}{\partial\phi_i^2}$ as an eigenvalue of the Laplacian. Indeed, we have the Slater determinant expression $\Delta = \pm 2^{\frac{n(1-n)}{2}}\left[e^{-i\frac{n-1}{2}\phi}, e^{-i\frac{n-3}{2}}, \ldots, e^{i\frac{n-1}{2}\phi}\right]_{AS}$, which may be verified with the Vandermonde determinant formula:

$$\begin{aligned}
\left[e^{-i\frac{n-1}{2}\phi}, e^{-i\frac{n-3}{2}}, \ldots, e^{i\frac{n-1}{2}\phi}\right]_{AS} &= e^{-i\frac{n-1}{2}\sum\phi_j}\left[1, e^{i\phi}, \ldots, e^{i(n-1)\phi}\right]_{AS} \\
&= e^{-i\frac{n-1}{2}\sum\phi_j}\prod_{1\le j<k\le n}\left(e^{i\phi_k} - e^{i\phi_j}\right) \\
&= \prod_{1\le j<k\le n} e^{\frac{-\phi_j-\phi_k}{2}}\left(e^{i\phi_k} - e^{i\phi_j}\right) \\
&= 2^{\frac{n(n-1)}{2}}\prod_{1\le j<k\le n}\sin\frac{\phi_k-\phi_j}{2}.
\end{aligned}$$

Since the Slater determinant is antisymmetric, we have verified that $\Delta$ is the wavefunction of $n$ fermions on a circle. For example, if $n=5$ we have $\left[e^{-2i\phi}, e^{-i\phi}, 1, e^{i\phi}, e^{2i\phi}\right]_{AS}$, which we recognize as the ground state. In the case $n$ is even, $\Delta$ is a fermionic wavefunction on the double-cover of the circle. For example when $n=2$, $\Delta = \left[e^{-\frac{1}{2}i\phi}, e^{\frac{1}{2}i\phi}\right]_{AS} = 2\sin\frac{\phi_2-\phi_1}{2}$. The double-cover is necessary since the expression involves half-angles. We won't worry about this minor pathology, and will assume $n$ is odd to avoid this.

The energy of fermions is simply the sum of the energy of each particle. If a particle has momentum $k$, its wavefunction is $e^{ik\phi}$ and its energy is $k^2$, so

$$-\sum_i \frac{1}{\Delta}\frac{\partial^2 \Delta}{\partial \phi_i^2} = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} k^2 = \frac{n^3-n}{12}.$$

Therefore,

$$H = -\sum_i \frac{1}{\Delta}\frac{\partial^2}{\partial \phi_i^2}\Delta - \frac{n^3-n}{12}.$$

We recognize $\frac{n^3-n}{12}$ as a normalization factor so that if $\chi$ is a constant wavefunction, $H\chi = 0$.

We will now look for eigenfunctions $\chi$ of the Laplacian. Since we are interested in class functions, we will now stipulate that wavefunctions $\chi$ are not just arbitrary functions of the $\phi_i$, but *symmetric* functions of the $\phi_i$, i.e. $\chi = \chi(\phi_1, \dots, \phi_n) = \chi(\phi_{\pi(1)}, \dots, \phi_{\pi(n)})$ for any permutation $\pi$. Thus $\chi$ represents $n$ bosons on a circle.

On the $L^2$ Hilbert space of wavefunctions, we have the inner product with respect to the invariant measure

$$\langle \chi_1 | \chi_2 \rangle_B = \int \chi_1^* \chi_2 \, d\mu = \int \chi_1^* \chi_2 \, \Delta^2 \, d\phi^n = \int (\Delta \chi_1)^* \, (\Delta \chi_2) \, d\phi^n.$$

This suggests the substitution $\psi := \Delta\chi$ to get

$$\langle \psi_1 | \psi_2 \rangle_F := \int \psi_1^* \psi_2 \, d\phi^n = \langle \psi_1 | \psi_2 \rangle_B$$

I claim that the map $F(\chi) := \Delta\chi$ is a unitary isomorphism of Hilbert spaces $F : L^2_{Sym}(\mathbb{T}^n, d\mu) \longrightarrow L^2_{AS}(\mathbb{T}^n, d\phi^n)$, where $L^2_{Sym}$ denotes symmetric functions, and $L^2_{AS}$ denotes antisymmetric functions. We have already shown that $F$ preserves the inner product, so it remains to prove that $F$ is surjective. It suffices to show that the image of $F$ is dense in $L^2_{AS}(\mathbb{T}^n, d\phi^n)$, since then $F^{-1}$ then extends continuously from $\mathrm{Im}F$ to all of $L^2_{AS}(\mathbb{T}^n, d\phi^n)$.

Finite Fourier series are dense in $L^2(\mathbb{T}^n, d\phi^n)$. The subspace of finite Fourier series is simply the ring $\mathbb{C}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}]$, where $\alpha_j := e^{i\phi_j}$. The orthogonal projection $L^2(\mathbb{T}^n, d\phi^n) \longrightarrow L^2_{AS}(\mathbb{T}^n, d\phi^n)$ corresponds to antisymmetrization of polynomials in $\mathbb{C}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}]$. Thus, the subspace of antisymmetric polynomials in $\mathbb{C}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}]$ is dense in $L^2_{AS}(\mathbb{T}^n, d\phi^n)$.

But all antisymmetric polynomials $p(\alpha)$ are divisible by $\Delta$, since $\Delta$ is essentially the Vandermonde determinant. Thus $p(\alpha)/\Delta$ is a symmetric polynomial in $L^2_{Sym}(\mathbb{T}^n, d\phi^n) \subset L^2_{Sym}(\mathbb{T}^n, d\mu)$, so $p(\alpha) = F(p(\alpha)/\Delta)$. Thus the image is dense, and $F$ is an isomorphism of Hilbert spaces.

Therefore, multiplication by $\Delta$ amounts to a change of basis from bosons to fermions. We may compute the action of $H$ on fermions:

$$\begin{aligned}
\left\langle \chi_1 | H_{\mathbb{T}^n} + \frac{n^3 - n}{12} | \chi_2 \right\rangle_I &= -\int \frac{\chi_1^*}{\Delta} \sum_i \frac{\partial^2}{\partial \phi_i} \Delta \chi_2 \, \Delta^2 \, d\phi^n \\
&= -\int \psi_1^* \sum_i \frac{\partial^2}{\partial \phi_i} \psi_2 \, d\phi^n \\
&= \left\langle \psi_1 | - \sum_i \frac{\partial^2}{\partial \phi_i} | \psi_2 \right\rangle_F .
\end{aligned}$$

We have discovered that the fermions are free particles!

If $k_1, \ldots, k_n$ are distinct integers representing momenta, We note that the energy of $\left[\alpha^{k_1}, \ldots, \alpha^{k_n}\right]_{AS} = \left[e^{ik_1\phi}, \ldots, e^{ik_n\phi}\right]_{AS}$ is given by $H_{\mathbb{T}^n}\left[e^{ik_1\phi}, \ldots, e^{ik_n\phi}\right]_{AS} = \sum_j k_j^2 - \frac{n^3-n}{12}$. Furthermore, the $\left[\alpha^{k_1}, \ldots, \alpha^{k_n}\right]_{AS}$ are orthogonal and complete. Thus we have solved the eigenfunction problem by switching to fermions.

4.2. **Weyl character formula.** We have shown that an orthonormal fermionic basis for the eigenfunctions of the Laplacian is given by $\left[\alpha^{k_1}, \ldots, \alpha^{k_n}\right]_{AS}$, where $k_1, \ldots, k_n$ are distinct integers. We wish to find the corresponding bosonic wavefunctions. These are given by

$$F^{-1}\left(\left[\alpha^{k_1}, \ldots, \alpha^{k_n}\right]_{AS}\right) \propto = \frac{\left[e^{ik_1\phi}, \ldots, e^{ik_n\phi}\right]_{AS}}{\left[e^{-i\frac{n-1}{2}\phi}, e^{-i\frac{n-3}{2}\phi}, \ldots, e^{-i\frac{n-1}{2}\phi}\right]_{AS}}.$$

This is nothing but the Weyl character formula.

We will now show that for any simple Lie group, characters of irreducible representations are eigenfunctions of the Laplacian. Suppose we have an irreducible Lie group representation $\rho : G \to \mathrm{Aut}(V)$ for some complex vector space $V$. Then we have the corresponding action $\mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$. Since $[\mathcal{C}_2, X] = 0$ for all $X \in \mathfrak{g}$, each $\mathcal{C}_2$-eigenspace of $V$ will be $\mathfrak{g}$-invariant. Since $V$ is complex, there is some nonempty $\mathcal{C}_2$-eigenspace which, by irreducibility, must be all of $V$. Therefore, $\mathcal{C}_2$ acts as a constant on the representation.

Under a choice of a negative-definite orthonormal basis $\{v_i\}$ for $V$, $\rho(g)$ is a matrix

$$g \cdot v_i = \rho_{ij}(g) v_j,$$

and thus

$$\rho_{ij}(g) = -\langle g \cdot v_i, v_j \rangle .$$

Our representation $\rho$ induces a $G$-action on $\rho_{ij}$ given by

$$h \cdot \rho_{ij}(g) := -\langle h \cdot (g \cdot v_i), v_j \rangle = -\langle (hg) \cdot v_i, v_j \rangle = \rho_{ij}(hg) v_j.$$

We recognize this action as the smooth function representation. Hence for $X \in \mathfrak{g}$, the corresponding $\mathfrak{g}$-action is

$$X \cdot \rho_{ij} = \partial_X \rho_{ij},$$

and therefore,

$$\mathcal{C}_2 \cdot \rho_{ij} = \nabla^2 \rho_{ij}.$$

Now suppose $\mathcal{C}_2 v = \lambda v$ for all $v \in V$. Then

$$\mathcal{C}_2 \cdot \rho_{ij}(g) := -\langle \mathcal{C}_2 \cdot (g \cdot v_i), v_j \rangle = -\langle \rho_{ij}(g)\, \mathcal{C}_2 \cdot v_j, v_j \rangle = \lambda \rho_{ij}(g).$$

Thus,

$$-\nabla^2 \rho_{ij} = \mathcal{C}_2 \cdot \rho_{ij} = \lambda \rho_{ij},$$

so $\rho_{ij}$ is an eigenfunction of $\nabla^2$ with eigenvalue $-\lambda$. In particular, the character $\mathrm{Tr}\rho(G) = \sum_i \rho_{ii}$ is an eigenfunction of $\nabla^2$ with eigenvalue $-\lambda$.

4.3. **Geometry of End**$(V)$**.** A nice property of $\mathfrak{u}_n$ is that $\mathfrak{u}_n \otimes \mathbb{C} = \mathfrak{gl}_n(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^n)$. Since $\mathrm{Aut}(\mathbb{C}^n) \subset \mathrm{End}(\mathbb{C}^n)$, we will actually be able to express information about the group representation of $U_n$ in terms of the algebra $\mathfrak{u}_n$! To proceed with such computations, we will need some lemmas about $\mathrm{End}(V)$, where $V$ is an $n$-dimensional complex vector space.

It's well-known that there is no canonical inner product on an abstract complex vector space $V$. This is not so for $\mathrm{End}(V)$. We generalize the notion of the Killing form to define a bilinear inner product on $\mathrm{End}(V)$ by $\langle X, Y \rangle := \mathrm{Tr}(XY)$. Note that this is not sesquilinear, so $\langle X, X \rangle$ need not even be real. We will prove the identity that if $e^\alpha$ is a positive-definite orthonormal basis of End$(V)$, then

$$\sum_{\alpha} e^{\alpha} X e^{\alpha} = \mathrm{Tr}(X) I.$$

It will actually be easier to prove a slightly more general theorem. Suppose that $s_1, \ldots, s_{n^2}$ is a basis of End$(V)$, and $t_1, \ldots, t_{n^2}$ is a dual basis, so that $\langle s_i, t_j \rangle = \delta_{ij}$. We will show that

$$\sum_{\alpha=1}^{n^2} s_{\alpha} X t_{\alpha} = \mathrm{Tr}(X) I.$$

First we will show that this expression is independent of the choice of dual bases $\{s_\alpha\}$ and $\{t_\alpha\}$. Let $\mathbf{s}$ denote the matrix $(s_1, \ldots, s_m)^T$, and let $\mathbf{t}$ denote $(t_1, \ldots, t_m)^T$. We can now rewrite the desired formula as

$$\mathbf{s}^T X \mathbf{t} = \mathrm{Tr}(X) I.$$

Now consider the matrix of endomorphisms

$$\mathbf{s}^T \mathbf{t} = \begin{pmatrix} s_1 t_1 & \cdots & s_m t_1 \\ \vdots & \ddots & \vdots \\ s_1 t_m & \cdots & s_m t_m \end{pmatrix}.$$

For a matrix $M$ of endomorphisms, define the matrix trace $\langle M \rangle$ to be the trace of each entry. Thus, by definition of a dual basis, $\left\langle \mathbf{s}^T \mathbf{t} \right\rangle$ is the $n^2 \times n^2$ identity matrix $I_{\mathrm{End}}$.

Now suppose that $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{t}}$ are another pair of dual bases of $\mathcal{A}$. Then there are $m \times m$ matrices $S$ and $T$ such that $\tilde{\mathbf{s}} = S\mathbf{s}$ and $\tilde{\mathbf{t}} = T\mathbf{t}$. Therefore, $I_{\mathrm{End}} = \left\langle \tilde{\mathbf{s}}^T \tilde{\mathbf{t}} \right\rangle = \left\langle \mathbf{s}^T S^T T \mathbf{t} \right\rangle$, so $\left\langle \mathbf{s}^T S^T T \mathbf{t} \right\rangle = \left\langle \mathbf{s}^T I_{\mathrm{End}} \mathbf{t} \right\rangle$. By the completeness of the $s_i$ and $t_i$, $S^T T = I_{\mathrm{End}}$. Now we compute

$$\tilde{\mathbf{s}}^T X \tilde{\mathbf{t}} = \mathbf{s}^T S^T X T \mathbf{t} = \mathbf{s}^T S^T T X \mathbf{t} = \mathbf{s}^T X \mathbf{t}.$$

We used the fact that $XT = TX$, since $T$ is a matrix of scalars, which commutes with the endomorphism $X$. Thus the expression

$$\mathbf{s}^T X \mathbf{t}$$

is independent of the choice of dual bases.

Now let $E^{(ab)}$ denote the matrix $E^{(ab)}_{ij} = \delta_{ai}\delta_{bj}$. Then $E^{(ab)}$ is dual to $E^{(ba)}$. Thus,

$$\begin{aligned}\mathbf{s}^T X\mathbf{t} &= \sum_{a,b} E^{(ab)} X E^{(ba)} = \sum_{a,b,c,d} E^{(ab)} X_{cd} E^{(cd)} E^{(ba)} = \sum_{a,b,c,d} X_{cd}\delta_{bc} E^{(ad)} E^{(ba)} \\ &= \sum_{a,b,c,d} X_{cd}\delta_{bc}\delta_{bd} E^{(aa)} = \sum_{a,b,c} X_{cb}\delta_{bc} E^{(aa)} = \sum_{a,b} X_{bb} E^{(aa)} = \mathrm{Tr}(X) I.\end{aligned}$$

In particular, for a positive-definite self-dual basis $\{e^\alpha\}$,

$$\sum_\alpha e^\alpha X e^\alpha = \mathrm{Tr}(X) I.$$

For a negative-definite orthonormal basis,

$$\sum_\alpha e^\alpha X e^\alpha = -\mathrm{Tr}(X) I.$$

4.4. **"Collective" variables and string theory.** We have deduced that the Weyl character formula gives a basis for the eigenfunctions of the Hamiltonian on $U_n$. Our expression for the Hamiltonian is unsatisfying since it contains the antisymmetric factor $\Delta$, while the wavefunctions are symmetric. We will now give a symmetric expression for the Hamiltonian, which we will connect with string theory.

A convenient spanning set for symmetric functions on $\mathbb{T}^n$ will be the "power sums," or "collective" variables

$$W_k := e^{ik\phi_1} + \cdots + e^{ik\phi_n} = \alpha_1^k + \cdots + \alpha_n^k = \mathrm{Tr}(U^k).$$

To span all symmetric functions, one must take sums and products of the $W_k$. For finite $n$ the $W_k$'s are dependent. For example, when $n = 2$,

$$W_1^3 - 3W_1W_2 + 2W_3 = (\alpha_1 + \alpha_2)^3 - 3(\alpha_1 + \alpha_2)(\alpha_1^2 + \alpha_2^2) + 2(\alpha_1^3 + \alpha_2^3) = 0.$$

However, as $n \to \infty$, the $W_i$ become independent. For instance when $n = 3$,

$$W_1^3 - 3W_1W_2 + 2W_3 = 6\alpha_1\alpha_2\alpha_3.$$

To write the Hamiltonian in terms of this basis, we will use the expression $H = -\nabla^2 = -\sum_i \left(\partial_{e^\alpha}\right)^2$, where $e^\alpha$ is a negative-definite orthonormal basis of $\mathfrak{g}$. We then use the chain rule to write the Hamiltonian in terms of the $W_k$.

The chain rule $\frac{\partial f}{\partial x_i} = \frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j}$ can be written in operator form as

$$\frac{\partial}{\partial \mathbf{x}} = \left[\frac{\partial}{\partial \mathbf{x}}, \mathbf{y}\right]\frac{\partial}{\partial \mathbf{y}}.$$

To derive the new Laplacian, I will give an overview in invariant notation before writing out the expression with indices.

$$\begin{aligned}
\nabla^2 &= \nabla \cdot \nabla = \nabla \cdot [\nabla, W]\partial_W \\
&= [\nabla, [\nabla, W]]\partial_W + [\nabla, W]\nabla \cdot \partial_W \\
&= [\nabla, [\nabla, W]]\partial_W + [\nabla, W][\nabla, W]\partial_W^2.
\end{aligned}$$

In tensor notation,

$$\begin{aligned}
\sum_{\alpha} (\partial_{e^\alpha})^2 &= \sum_{\alpha,r} \partial_{e^\alpha}[\partial_{e_\alpha}, W_r]\frac{\partial}{\partial W_r} \\
&= \sum_{\alpha,r}[\partial_{e_\alpha}, [\partial_{e_\alpha}, W_r]]\frac{\partial}{\partial W_r} + \sum_{\alpha,r,s}[\partial_{e_\alpha}, W_r][\partial_{e_\alpha}, W_s]\frac{\partial^2}{\partial W_r \partial W_s}.
\end{aligned}$$

By right invariance, $\partial_{e^\alpha}$ acts on the coordinate functions $U_{ij}$ by $[\partial_{e^\alpha}, U_{ik}] = e^\alpha_{ij}U_{jk}$. Thus

$$\partial_{e^\alpha} = e^\alpha_{ij}U_{jk}\frac{\partial}{\partial U_{ik}},$$

so $[\partial^\alpha, U] = e^\alpha U$. It follows that

$$\begin{aligned}
[\partial^\alpha, W_k] &= [\partial^\alpha, \mathrm{Tr}(U^k)] \\
&= \mathrm{Tr}([\partial^\alpha, U^k]) \\
&= \mathrm{Tr}(e^\alpha U^k + U e^\alpha U^{k-1} + \cdots + U^{k-1}e^\alpha U) \\
&= k\mathrm{Tr}(e^\alpha U^k).
\end{aligned}$$

Note that $[\partial^\alpha, e^\alpha] = 0$ since $e^\alpha$ is constant over $U$. Now we have

$$\sum_{\alpha}[\partial^\alpha, W_r][\partial^\alpha, W_s] = rs\mathrm{Tr}(e^\alpha U^r)\mathrm{Tr}(e^\alpha U^s).$$

Recall that since $\mathfrak{u}_n \otimes \mathbb{C} = M_{n\times n}(\mathbb{C})$, a negative-definite orthonormal basis $\{e^\alpha\} \subset \mathfrak{g}$ is a negative-definite orthonormal basis for $\mathrm{End}(\mathbb{C}^n)$ when taken with complex coefficients. Thus, for any $A \in M_{n\times n}(\mathbb{C})$, we have the completeness relation $A = \sum_i -\langle A, e^\alpha\rangle e^\alpha$, where the coefficients $-\langle A, e^\alpha\rangle$ are allowed to be complex. Thus, for any matrices $A, B \in M_{n\times n}(\mathbb{C})$,

$$\begin{aligned}\mathrm{Tr}(AB) &= \langle A, B\rangle = \sum_{\alpha,\beta}\left\langle \langle A, e^\alpha\rangle e^\alpha, \left\langle B, e^\beta\right\rangle e^\beta\right\rangle \\ &= -\sum_i \langle A, e^\alpha\rangle \langle B, e^\alpha\rangle = -\sum_\alpha \mathrm{Tr}(e^\alpha A)\mathrm{Tr}(e^\alpha B).\end{aligned}$$

Therefore,

$$\sum_\alpha [\partial^\alpha, W_r][\partial^\alpha, W_s] = rs\mathrm{Tr}(e^\alpha U^r)\mathrm{Tr}(e^\alpha U^s) = -rs\mathrm{Tr}(U^{r+s}) = -rsW_{r+s}.$$

For the factor $[\partial_{e_\alpha}, [\partial_{e_\alpha}, W_r]]$, we compute

$$\begin{aligned}\sum_\alpha [\partial^\alpha, [\partial^\alpha, W_r]] &= r\sum_\alpha [\partial^\alpha, \mathrm{Tr}(e^\alpha U^r)] \\ &= r\sum_\alpha \mathrm{Tr}([\partial^\alpha, e^\alpha U^r]) \\ &= |r|\sum_\alpha \mathrm{Tr}(e^\alpha e^\alpha U^r + e^\alpha U e^\alpha U^{r-1} + \cdots + e^\alpha U^{r-1} e^\alpha U).\end{aligned}$$

Now we invoke our identity from Section 4.3:

$$\sum_\alpha e^\alpha X e^\alpha = -\mathrm{Tr}(X)I.$$

It follows that

$$\sum_\alpha e^\alpha U^a e^\alpha U^b = -\mathrm{Tr}(U^a)U^b.$$

Taking the trace, we obtain

$$\sum_\alpha \mathrm{Tr}(e^\alpha U^a e^\alpha U^b) = -\mathrm{Tr}(U^a)\mathrm{Tr}(U^b).$$

Therefore,

$$\begin{aligned}
\sum_{\alpha}[\partial^\alpha,[\partial^\alpha,W_r]] &= |r|\sum_{\alpha}\mathrm{Tr}(e^\alpha e^\alpha U^r + e^\alpha U e^\alpha U^{r-1} + \cdots + e^\alpha U^{r-1} e^\alpha U) \\
&= -|r|\left(\mathrm{Tr}(I)\mathrm{Tr}(U^r) + \mathrm{Tr}(U)\mathrm{Tr}(U^{r-1}) + \cdots \mathrm{Tr}(U^{r-1})\mathrm{Tr}(U)\right) \\
&= -|r|\sum_{m=0}^{r-1}\mathrm{Tr}(U^m)\mathrm{Tr}(U^{r-m}) \\
&= -|r|\sum_{m=0}^{r-1}W_m W_{r-m}.
\end{aligned}$$

Putting this all together, we get

$$\begin{aligned}
H &= -\sum_{\alpha,r}[\partial_{e_\alpha},[\partial_{e_\alpha},W_r]]\frac{\partial}{\partial W_r} - \sum_{\alpha,r,s}[\partial_{e_\alpha},W_r][\partial_{e_\alpha},W_s]\frac{\partial^2}{\partial W_r \partial W_s} \\
&= \sum_{r=-\infty}^{\infty}\sum_{m=0}^{r-1}|r|\,W_m W_{r-m}\frac{\partial}{\partial W_r} + \sum_{r,s=-\infty}^{\infty} rsW_{r+s}\frac{\partial^2}{\partial W_r \partial W_s}.
\end{aligned}$$

We see that the Hamiltonian consists of two types of terms: splitting terms and joining terms. The first term acts on $W_r$ by splitting it into the superposition $\sum_{m=0}^{r-1}|r|\,W_m W_{r-m}$. The second term acts on $W_r W_s$ by joining it into $rsW_{r+s}$. Loosely speaking, a monomial $W_{i_1}W_{i_2}\cdots W_{i_k}$ represents a state consisting of $k$ strings. Each string has a winding number $i_k$. The strings can split and join, but the total winding degree $i_1 + \cdots + i_k$ is conserved.

At this point, it is beneficial to make the assumption that each winding number is non-negative, i.e., there are no factors involving $\mathrm{Tr}(U^{-k}) = \mathrm{Tr}(U^{\dagger k})$. The resulting system is qualitatively the same, but far less cumbersome. Dropping the negative winding numbers from the Hamiltonian, we get

$$\begin{aligned}
H &= \sum_{r=1}^{\infty}\sum_{m=0}^{r-1} rW_m W_{r-m}\frac{\partial}{\partial W_r} + \sum_{r,s=1}^{\infty} rsW_{r+s}\frac{\partial^2}{\partial W_r \partial W_s} \\
&= n\sum_{k=1}^{\infty} kW_k\frac{\partial}{\partial W_k} + \sum_{r=1}^{\infty}\sum_{m=1}^{r-1} rW_m W_{r-m}\frac{\partial}{\partial W_r} + \sum_{r,s=1}^{\infty} rsW_{r+s}\frac{\partial^2}{\partial W_r \partial W_s} \\
&=: H_0 + H_S + H_J.
\end{aligned}$$

We recognize the $H_0$ term as a standard kinetic term, and $H_S$ and $H_J$ are respectively splitting and joining terms.

We change to creation-annihilation operators by

$$W_n \to \sqrt{n} a_n^\dagger, \qquad \frac{\partial}{\partial W_n} \to \frac{1}{\sqrt{n}} a_n$$

to get

$$\begin{aligned} H_0 &= n \sum_{k=1}^{\infty} k a_k^\dagger a_k, \\ H_S &= \sum_{r=1}^{\infty} \sum_{m=1}^{r-1} \sqrt{rm(r-m)} a_m^\dagger a_{r-m}^\dagger a_r = \sum_{i,j=1}^{\infty} \sqrt{ij(i+j)} a_i^\dagger a_j^\dagger a_{i+j}, \\ H_J &= \sum_{i,j=1}^{\infty} \sqrt{ij(i+j)} a_{i+j}^\dagger a_i a_j. \end{aligned}$$

We have the Fock space representation with states generated by

$$a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_k}^\dagger \left|0\right\rangle.$$

The eigenstates will be superpositions of these states that are at equilibrium with respect to the joining and splitting interaction.

4.5. **Schur polynomials.** An explicit way to construct these eigenstates is through Schur polynomials. First consider

$$Q_i := \frac{[\alpha^{i+n-1}, \alpha^{n-2}, \alpha^{n-3}, \ldots, \alpha^0]_{AS}}{[\alpha^{n-1}, \alpha^{n-2}, \alpha^{n-3}, \ldots, \alpha^0]_{AS}}.$$

For $n = 3$, we have

$$\begin{aligned} Q_0 &= 1, \\ Q_1 &= \alpha_1 + \alpha_2 + \alpha_3, \\ Q_2 &= \frac{1}{2}\left((\alpha_1+\alpha_2+\alpha_3)^2 + (\alpha_1^2+\alpha_2^2+\alpha_3^2)\right), \\ Q_3 &= \frac{1}{6}\left((\alpha_1+\alpha_2+\alpha_3)^3 + 3(\alpha_1+\alpha_2+\alpha_3)(\alpha_1^2+\alpha_2^2+\alpha_3^2) + 2(\alpha_1^3+\alpha_2^3+\alpha_3^3)\right), \\ &\vdots \end{aligned}$$

This motivates the definition of

$$W_i := \alpha_1^i + \cdots + \alpha_n^i,$$

so that

$$\begin{aligned}
Q_0 &= 1, \\
Q_1 &= W_1, \\
Q_2 &= \frac{1}{2}\left(W_1^2 + W_2\right), \\
Q_3 &= \frac{1}{6}\left(W_1^3 + 3W_1W_2 + 2W_3\right), \\
&\vdots
\end{aligned}$$

If $n$ is taken to be arbitrarily large, then the expression of $Q_i$ in terms of $W_i$ are uniquely determined. Now consider the generating function

$$\begin{aligned}
\exp\left(\sum_{i=0}^{n} \phi_i k^i\right) &= \sum_{i=0}^{\infty} k^i S_i(\phi_1, \ldots, \phi_n) \\
&= k^0(1) \\
&+ k^1(\phi_1) \\
&+ k^2 \frac{1}{2}\left(\phi_1^2 + 2\phi_2\right) \\
&+ k^3 \frac{1}{6}\left(\phi_1^3 + 6\phi_1\phi_2 + 6\phi_3\right) \\
&\vdots
\end{aligned}$$

The $S_i$ are called the *Schur polynomials.* Making the substitution $\phi_i \mapsto \frac{1}{i}W_i$, we see that the Schur polynomials coincide with the $Q_i$.

For a general partition $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$, we may compute

$$Q_\lambda := \frac{\left[\alpha^{\lambda_1+n-1}, \alpha^{\lambda_2+n-2}, \alpha^{\lambda_3+n-3}, \ldots, \alpha^0\right]_{AS}}{\left[\alpha^{n-1}, \alpha^{n-2}, \alpha^{n-3}, \ldots, \alpha^0\right]_{AS}}$$

by means of the formula

$$S_\lambda := \begin{vmatrix} S_{\lambda_1} & S_{\lambda_1+1} & S_{\lambda_1+2} & \cdots \\ S_{\lambda_2-1} & S_{\lambda_2} & S_{\lambda_2+1} & \cdots \\ S_{\lambda_3-2} & S_{\lambda_3-1} & S_{\lambda_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

For example,

$$S_{\{2,1\}} = \begin{vmatrix} S_2 & S_3 \\ S_0 & S_1 \end{vmatrix} = \frac{\phi_1^3}{3} - = \frac{1}{3}\left(W_1^3 - W_3\right).$$

The corresponding eigenstate is

$$\left(\left(a_1^\dagger\right)^3 - a_3^\dagger\right)|0\rangle .$$

## 5. References


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