Differentiable manifolds
Math 6510 Class Notes

Mladen Bestvina

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1 Definition of a manifold

Intuitively, an n-dimensional manifold is a space that is equipped with a set of local cartesian coordinates, so that points in a neighborhood of any fixed point can be parametrized by n-tuples of real numbers.

1.1 Charts, atlases, differentiable structures

Definition 1.1. Let X be a set. A *coordinate chart* on X is a pair (U, φ) where $U \subset X$ is a subset and $\varphi : U \to \mathbb{R}^n$ is an injective function such that $\varphi(U)$ is open in $\mathbb{R}^n$. The inverse $\varphi^{-1} : \varphi(U) \to U \subset X$ is a *local parametrization.*

Of course, we want to do calculus on X, so charts should have some compatibility.

Definition 1.2. Two coordinate charts (U_1, φ_1) and (U_2, φ_2) on X are *compatible* if

(i) $\varphi_i(U_1 \cap U_2)$ is open in $\mathbb{R}^n$, $i = 1, 2$, and

(ii) $\varphi_2\varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is C^∞ with C^∞ inverse.

The function in (ii) is usually called a *transition map.*

Definition 1.3. An *atlas* on X is a collection $\mathcal{A} = \{(U_i, \varphi_i)\}$ of pairwise compatible charts with $\cup_i U_i = X$.

Example 1.4. $\mathbb{R}^n$ with atlas consisting of the single chart $(\mathbb{R}^n, id)$. Similarly for open subsets of $\mathbb{R}^n$.

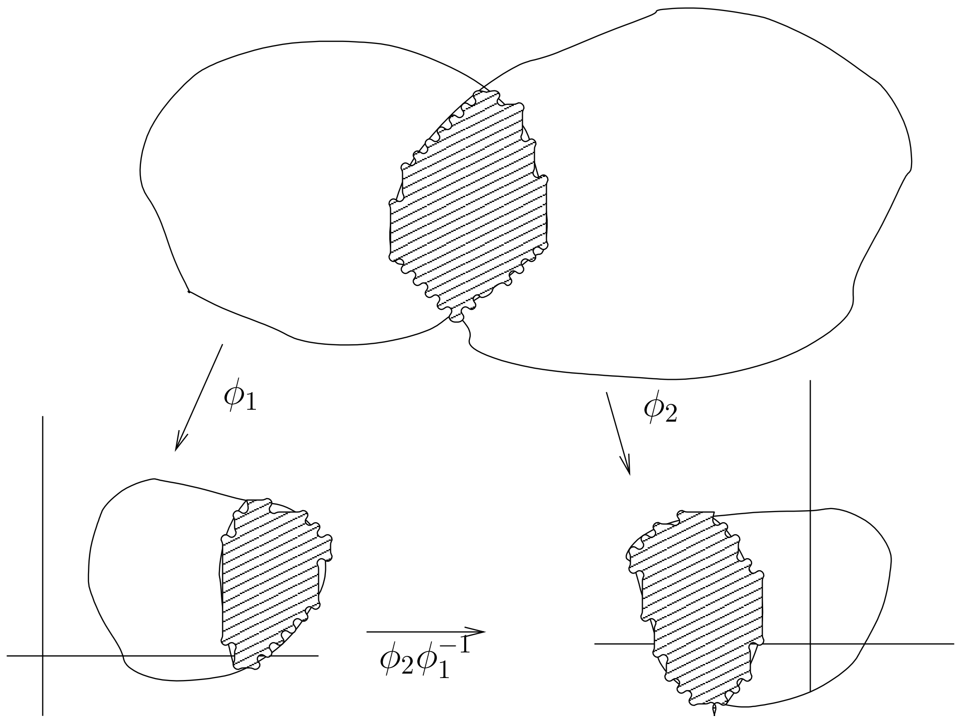

Figure 1: The obligatory transition map picture

Example 1.5. Let $S^1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Consider 4 charts $(U_\pm, \varphi_\pm)$ and $(V_\pm, \psi_\pm)$ where

$$U_+ = \{(x,y) \in S^1 | x > 0\}, \quad U_- = \{(x,y) \in S^1 | x < 0\},$$
$$V_+ = \{(x,y) \in S^1 | y > 0\}, \quad V_- = \{(x,y) \in S^1 | y < 0\}$$

and $\varphi_\pm(x,y) = x$, $\psi_\pm(x,y) = y$. These 4 charts form an atlas. For example, $\varphi_+(U_+) = \psi_+(V_+) = (-1,1) \subset \mathbb{R}$, $\varphi_+(U_+ \cap V_+) = \psi_+(U_+ \cap V_+) = (0,1)$ and $\varphi_+\psi_+^{-1}(t) = \sqrt{1-t^2}$.

Exercise 1.6. Do the same for the $(n-1)$-sphere $S^{n-1} = \{x \in \mathbb{R}^n, ||x|| = 1\}$

We would like to say that a manifold is a set equipped with an atlas. The trouble is that there are many atlases that correspond to the "same" manifold.

Definition 1.7. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are *equivalent* if their union is also an atlas.

Exercise 1.8. Check that this is an equivalence relation. Show that each equivalence relation contains a unique *maximal* atlas.

Definition 1.9. A *differentiable structure* (or a *smooth structure*) on X is an equivalence class $[\mathcal{A}]$ of atlases (or equivalently it is a maximal atlas on X).

A *differentiable manifold* (or a *smooth manifold*) is a pair $(X, [\mathcal{A}])$ where $[\mathcal{A}]$ is an equivalence class of atlases on X.

We'll be less formal and talk about a smooth manifold $(X, \mathcal{A})$, or even just X when an atlas is understood. It is also customary to denote the dimension of a manifold as a superscript, e.g. X^n.

Remark 1.10. We could have replaced C^∞ in the definition of compatibility by C^r, $r = 0, 1, 2, \cdots, \infty, \omega$ (e.g. C^0 just means "continuous", while C^ω means "analytic"). For emphasis, we can then talk about C^r-structures and C^r-manifolds.

Here are more examples of manifolds.

Example 1.11. If $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ are manifolds, then so is $(X_1 \times X_2, \mathcal{A})$ where $\mathcal{A}$ is the "product atlas"

$$\mathcal{A} = \{(U_1 \times U_2, \varphi_1 \times \varphi_2) | (U_i, \varphi_i) \in \mathcal{A}_i\}$$

where $\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is the product map. For example, $S^1 \times S^1$ is the 2-torus, and similarly $(S^1)^n$ is the n-torus.

Example 1.12. The *Riemann sphere* is

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

We consider the atlas with 2 charts: $(\mathbb{C}, j)$ and (U, i) where $j : \mathbb{C} \to \mathbb{R}^2$ is the standard identification $j(z) = (\Re z, \Im z)$, $U = \mathbb{C} \cup \{\infty\} - \{0\}$ and $i : U \to \mathbb{R}^2$ is $i(\infty) = 0$, $i(z) = j(\frac{1}{z})$. Check compatibility.

Example 1.13. Let X be the set of all straight lines in $\mathbb{R}^2$. We want to equip X with a natural differentiable structure. Let U_h be the set of all non-vertical lines and U_v the set of all non-horizontal lines. Thus $U_h \cup U_v = X$. Every line in U_h has a unique equation

$$y = mx + l$$

and we define $\varphi_h : U_h \to \mathbb{R}^2$ by sending this line to (m, l). Likewise, every line in U_v has a unique equation

$$x = my + l$$

and we define $\varphi_v : U_v \to \mathbb{R}^2$ by sending this line to (m, l). A line of the form $y = mx + l$ is non-horizontal iff $m \neq 0$ and in that case it has an equivalent equation $x = \frac{1}{m}y - \frac{l}{m}$ so that the transition map is given by $(m, l) \mapsto (\frac{1}{m}, -\frac{l}{m})$.

1.2 Topology

Every atlas $\mathcal{A}$ on X defines a topology on X. We declare that a set $\Omega \subset X$ is open iff for every chart $(U, \varphi) \in \mathcal{A}$ the set $\varphi(U \cap \Omega)$ is open in $\mathbb{R}^n$.

Exercise 1.14. Show that this really is a topology on X, i.e. that $\emptyset, X$ are open and that the collection of open sets is closed under unions and finite intersections. Also show that $\varphi : U \to \varphi(U)$ is a homeomorphism for every $(U, \varphi) \in \mathcal{A}$.

It turns out that this topology may be "bad". This is why we impose two additional conditions:

(Top1) X is Hausdorff, and

(Top2) the differentiable structure contains a representative atlas which is countable.

To see that unpleasant things can happen, consider the following examples:

Example 1.15 (Evil twins). Start with $Y = \mathbb{R} \times \{-1, 1\}$. These are two copies of the standard line. Now let X be the quotient space under the equivalence relation generated by $(t, -1) \sim (t, 1)$ for $t \neq 0$. As a set, X can be thought of as a real line, but with two origins. We define two charts: the corresponding local parametrizations are the restrictions of the quotient map $Y \to X$ to each copy of $\mathbb{R}$. Then X is a 1-manifold, but its topology is non-Hausdorff as the two origins cannot be separated by disjoint open sets.

Example 1.16. Let X be an uncountable set with the atlas that consists of charts (U, φ) where U is any 1-point subset of X and φ is the unique map to $\mathbb{R}^0$ (which is also a 1-point set). The induced topology is discrete and X is not separable, but it is a 0-manifold.

At least a discrete space is metrizable. It could be worse.

Example 1.17 (The long line). For this you need to know some set theory. Recall that ω is the smallest infinite ordinal. As a well-ordered set it is represented by the positive integers $1, 2, \cdots$. The usual closed ray $[0, \infty)$ can be thought of as the set $\omega \times [0, 1)$ with order topology with respect to the lexicographic order $(n, t) < (n', t')$ iff $n < n'$ or ($n = n'$ and $t < t'$). The usual line is obtained from the closed ray by gluing two copies at the origin. Now replace ω by Ω (the smallest uncountable ordinal) in this construction and obtain a long closed ray and a long line. The long line is a Hausdorff 1-manifold (provide an atlas!), but it isn't metrizable, or even paracompact.

Example 1.18 (Foliations). This is an elaboration on the Evil Twins, just so you can see that such examples actually show up in the real world. A *foliation* in $\mathbb{R}^2$ is a decomposition of $\mathbb{R}^2$ into subsets L (called *leaves*) which are topological lines and the decomposition is locally standard, in the sense that $\mathbb{R}^2$ is covered by charts (U, φ) with the property that $\varphi(U \cap L)$ is contained in a vertical line, for any leaf L. The standard foliation of $\mathbb{R}^2$ is the decomposition into vertical lines $x \times \mathbb{R}$. Note that in that case the *leaf space* (i.e. the quotient space where each leaf is crushed to a point) is $\mathbb{R}$. Pictured here is the so called *Reeb foliation*. The leaf space is a non-Hausdorff 1-manifold.

From now on, when I say "manifold" I will mean that (Top1) and (Top2) hold. If I really want more general manifolds I will say e.g. "non-Hausdorff manifold".

Exercise 1.19. An open subset U of a manifold X has an atlas given by restricting the charts to U. Show that the manifold topology on the U coincides with the topology induced from X.

1.3 Smooth maps, diffeomorphisms

Definition 1.20. Let $(X^n, \mathcal{A})$ and $(Y^m, \mathcal{B})$ be manifolds and $f : X \to Y$ a function. We say that f is *smooth* if for every chart $(U, \varphi) \in \mathcal{A}$ and every chart $(V, \psi) \in \mathcal{B}$ it follows that $\psi f \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \mathbb{R}^m$ is C^∞. We say that this latter map *represents f in local coordinates.*

Exercise 1.21. Show that the definition does not depend on the choice of atlases within their equivalence classes. In other words, we only need to check the definition for suitable collections $\{(U, \varphi)\}$ and $\{(V, \psi)\}$ that cover X and Y respectively.

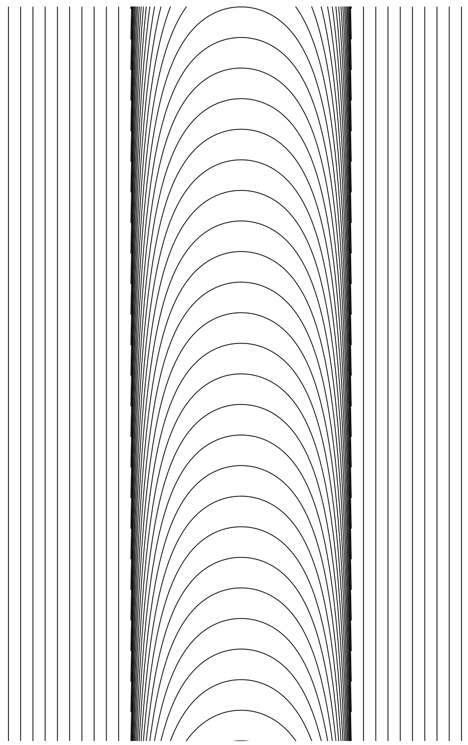

Figure 2: The Reeb foliation

Exercise 1.22. Prove that smooth functions are continuous with respect to manifold topologies.

Exercise 1.23. Denote by $C^\infty(X)$ the set of all smooth functions $f : X \to \mathbb{R}$. Show that $C^\infty(X)$ is an algebra, i.e. if $f, g : X \to \mathbb{R}$ are smooth then so are $af + bg$ and fg for any $a, b \in \mathbb{R}$.

Exercise 1.24. Let $\varphi : U \to \mathbb{R}$ be a chart on a manifold X and suppose that $f : X \to \mathbb{R}$ is a function whose support

$$supp(f) = \overline{\{x \in X | f(x) \neq 0\}}$$

is contained in U. Show that f is smooth iff $f\varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth.

Definition 1.25. $f : X \to Y$ is a *diffeomorphism* if it is a bijection and both f, f^{-1} are smooth.

Example 1.26. This one is a bit silly. Take $\mathbb{R}$ with two different differentiable structures. One is standard, given by $id : \mathbb{R} \to \mathbb{R}$, and the other is also given by a single chart, namely $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = t^3$. Then the two atlases are not compatible, but the resulting manifolds are diffeomorphic via $f : \mathbb{R}_\varphi \to \mathbb{R}_{id}$, $f(t) = t^3$.

Example 1.27. $tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a diffeomorphism. Likewise, construct a diffeomorphism between the open unit ball $\{x \in \mathbb{R}^n, ||x|| < 1\}$ and $\mathbb{R}^n$.

Example 1.28. The Riemann sphere $\hat{\mathbb{C}}$ is diffeomorphic to the 2-sphere S^2. The proof of this statement involves the *stereographic projection* $\pi : S^2 - \{N\} \to \mathbb{R}^2 = \mathbb{C}$ which extends to a bijection $\hat{\pi} : S^2 \to \hat{\mathbb{C}}$ by sending the north pole $N = (0, 0, 1)$ to ∞. Geometrically, π is defined by sending a point p to the unique intersection point between the line through N and p with the $xy-$plane. Work out an explicit formula for π and prove that $\hat{\pi}$ is a diffeomorphism.

Example 1.29. By a higher dimensional version of the stereographic projection, S^n with one point removed is diffeomorphic to $\mathbb{R}^n$.

1.4 More examples of manifolds

Example 1.30 (Group actions). Let X be a manifold and G a group of diffeomorphisms of X. Assume the following:

(F) G acts freely, i.e. $g(x) = x$ implies $g = id$,

(PD) G acts properly discontinuously, i.e. for every compact set $K \subset X$ the set $\{g \in G | g(K) \cap K \neq \emptyset\}$ is finite.

Example 1.31. Let $X = S^1$ and let $G = \mathbb{Z}$ consist of the powers of an irrational rotation $\rho : S^1 \to S^1$ (this means that ρ rotates by $\frac{a}{2\pi}$ with a irrational). This action is free, but not properly discontinuous.

When G is finite, the action is always properly discontinuous but does not have to be free.

Proper discontinuity forces the orbit $O(x) = \{g(x) | g \in G\}$ to be a discrete subset of X (for any x). However, there are examples of free actions with discrete orbits that are not properly discontinuous. One such is the action of $G = \mathbb{Z}$ on $X = \mathbb{R}^2 - \{(0, 0)\}$ by the powers of

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Let Y be the orbit set X/G and $\pi : X \to Y$ the orbit (quotient) map. Say $\mathcal{A}$ is the maximal atlas on X. Now define the following atlas on Y: Whenever $(U, \varphi) \in \mathcal{A}$ has the property that $g(U) \cap U \neq \emptyset$ implies $g = id$, then $\pi|U$ is injective and we may take $(\pi(U), \varphi\pi^{-1})$ as a chart. Compatibility is straightforward, and the fact that these charts cover Y follows from (F) and (PD) (exercise!). The verification of (Top1) and (Top2) is also left as an exercise.

As concrete examples we list the following. In each case verify (F) and (PD).

1. Consider the group $G \cong \mathbb{Z}$ of integer translations on $\mathbb{R}$. Show that the map $\mathbb{R}/\mathbb{Z} \to S^1$ given by $[t] \mapsto (cos(2\pi t), sin(2\pi t))$ is a diffeomorphism.

2. Let v_1, v_2 be two linearly independent vectors in $\mathbb{R}^2$ and consider the group $G \cong \mathbb{Z}^2$ of translations of $\mathbb{R}^2$ by vectors that are integral linear combinations of v_1 and v_2 (the collection of all such vectors is a *lattice* in $\mathbb{R}^2$). Show that regardless of the choice of v_1, v_2 the quotient manifold is diffeomorphic to the 2-torus. Likewise in n dimensions.

3. Let G consist of the powers of the *glide reflection* $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(x, y) = (-x, y+1)$. The quotient surface M is the *Moebius band*. Show that the manifold of all lines in $\mathbb{R}^2$ (Example 1.13) is diffeomorphic to M.

4. Let $G \cong \mathbb{Z}_2$ be the group whose only nontrivial element is the antipodal map $a : S^n \to S^n$, $a(x) = -x$. The quotient manifold is the *real projective n-space* $\mathbb{R}P^n$. Prove that $\mathbb{R}P^1 \to S^1$ given by $[z] \mapsto z^2$ is a diffeomorphism, if we think of S^1 as the space of complex numbers of norm 1.

Remark 1.32. If you know about covering spaces, you will recognize that $X \to Y$ is always a covering space.

1.5 The Inverse Function Theorem and submanifolds

The following theorem makes it easy to check that we have a manifold in a wide variety of situations. It is a standard theorem in calculus.

Theorem 1.33 (The Inverse Function Theorem). *Let U be an open set in $\mathbb{R}^n$ and let $F : U \to \mathbb{R}^n$ be a smooth function. Assume that $p \in U$ has the property that the derivative $D_pF : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there are neighborhoods V of p and W of $F(p)$ such that $W = F(V)$ and $F : V \to W$ is a diffeomorphism.*

Here as usual $D_pF : \mathbb{R}^n \to \mathbb{R}^n$ denotes the unique linear map such that

$$F(p+h) - F(p) = D_pF(h) + R(p,h)$$

where the remainder R satisfies $R(p,h)/||h|| \to 0$ as $h \to 0$. Recall that D_pF is represented by the *Jacobian matrix*

$$\Big(\frac{\partial F_i}{\partial x_j}(p)\Big)$$

where F_i is the ith coordinate function of F.

Sketch. The proof uses the **Contraction Principle**:
Let (M,d) be a complete metric space and $\gamma : M \to M$ a contraction, i.e. a map satisfying $d(\gamma(x),\gamma(x')) \le Cd(x,x')$ for a certain fixed $C < 1$. Then γ has a unique fixed point.[1]

First, by precomposition with an affine map $x \mapsto Ax + b$ we may assume that $p = 0$ and $DF(0) = I$. To study solutions of $F(x) = y$ we use Newton's method, as follows. Form the function $G(x) = x - F(x)$ so that $D_0G = 0$. So for some $r > 0$ the derivative D_xG has norm $< \frac{1}{2}$ in the closed ball $B(2r)$ of radius $2r$, and then from the mean value theorem we see that $||G(x)|| \le \frac{||x||}{2}$ for $x \in B(2r)$. Now show that for $y \in B(r/2)$ the function $G_y(x) = G(x) + y = x - F(x) + y$ maps $B(r)$ into itself and is a contraction. It follows that G_y has a unique fixed point in $B(r)$, i.e. that $F(x) = y$ has a unique solution in $B(r)$. So F has a unique local inverse φ defined on $B(r/2)$ and it remains to show that φ is smooth. Continuity follows from the triangle inequality plus the mean value theorem:

$$||x - x'|| \le ||F(x) - F(x')|| + ||G(x) - G(x')|| \le ||F(x) - F(x')|| + \frac{1}{2}||x - x'||$$

and hence $||x - x'|| \le 2||F(x) - F(x')||$. Next, fix $x_0 \in B(r)$, let $y_0 = F(x_0)$ and argue directly from the definition of derivative that $D_{y_0}\varphi = (D_{x_0}F)^{-1}$ using estimates much like the ones above. Finally, smoothness of φ follows from the formula $D\varphi = (DF)^{-1}$ and the smoothness of F. □

Definition 1.34. Let X^n be a manifold and $Y \subset X$ a subset. We say that Y is a *k-dimensional submanifold* of X if for every $y \in Y$ there is a chart $\varphi : U \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ of X such that $y \in U$ and $U \cap Y = \varphi^{-1}(\mathbb{R}^k \times 0)$.

Y is given the structure of a k-manifold by taking the atlas obtained by restricting the charts above to $U \cap Y \to \mathbb{R}^k$.

[1] Uniqueness is easy; to show existence choose an arbitrary $x_0 \in M$ and argue that the sequence of iterates $x_{i+1} = \gamma(x_i)$ is a Cauchy sequence.

Exercise 1.35. Check that this is really an atlas. Also check the following properties:

(i) inclusion $Y \hookrightarrow X$ is smooth, and

(ii) if M is any manifold, and $f : M \to X$ a smooth map whose image is contained in Y, then the map f viewed as a map $M \to Y$ is also smooth.

Exercise 1.36. The manifold topology on Y coincides with the topology induced from X.

Example 1.37. $\mathbb{R}^k \times \{0\}$ is a submanifold of $\mathbb{R}^k \times \mathbb{R}^{n-k}$. More generally, any open subset of $\mathbb{R}^k \times \{0\}$ is a submanifold of $\mathbb{R}^k \times \mathbb{R}^{n-k}$.

Here is the corollary of the IFT we will use:

Corollary 1.38 (The Regular Value Theorem). *Suppose $F : U \to \mathbb{R}^n$ is a smooth function defined on an open set in $\mathbb{R}^{n+m}$. Let $c \in \mathbb{R}^n$ be such that the derivative (i.e. the Jacobian matrix) $D_pF = \left(\frac{\partial F_i}{\partial x_j}(p)\right)$ has rank n (i.e. it is surjective) for every $p \in F^{-1}(c)$. Then $F^{-1}(c)$ is a submanifold of U of dimension m.*

Under the assumptions of the corollary, we also say that c is a *regular value* of F.

Proof. Let $p \in F^{-1}(c)$. After reordering the coordinates if necessary we may assume that the determinant of $\left(\frac{\partial F_i}{\partial x_j}(p)\right)$ with $1 \leq i, j \leq n$ is nonzero. Now define

$$G : U \to \mathbb{R}^n \times \mathbb{R}^m$$

by

$$G(x_1, \cdots, x_{n+m}) = (F(x_1, \cdots, x_{n+m}), x_{m+1}, \cdots, x_{n+m})$$

Then

$$D_pG = \begin{pmatrix} \left(\frac{\partial F_i}{\partial x_j}(p)\right) & 0 \\ * & I \end{pmatrix}$$

is invertible, so by the IFT there are neighborhoods V of p and W of $G(p)$ such that $G : V \to W$ is a diffeomorphism. By definition, G maps $V \cap F^{-1}(c)$ to $W \cap \{c\} \times \mathbb{R}^n$, so after composing with a translation we have the required chart. □

Example 1.39. To see that S^{n-1} is a manifold, consider the function $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(x_1, \cdots, x_n) = x_1^2 + \cdots + x_n^2$. Then $S^{n-1} = F^{-1}(1)$ so we need to check that the Jacobian D_pF has rank 1, i.e. is nonzero, for every $p \in S^{n-1}$. The derivative is $\left(\frac{\partial F}{\partial x_j}(p)\right) = (2p_1, 2p_2, \cdots, 2p_n)$. This vanishes only at $0 \in \mathbb{R}^n$ but $0 \notin S^{n-1}$.

Exercise 1.40. A *k-frame* in $\mathbb{R}^n$ is a k-tuple of vectors $(v_1, \cdots, v_k) \in (\mathbb{R}^n)^k$ that are orthonormal, i.e. $v_i \cdot v_j = \delta_{ij}$[2] Let $V_k(\mathbb{R}^n)$ denote the set of all k-frames in $\mathbb{R}^n$, so $V_k(\mathbb{R}^n)$ is a subset of $(\mathbb{R}^n)^k \cong \mathbb{R}^{nk}$. Prove that $V_k(\mathbb{R}^n)$ is a manifold by constructing a suitable function $F : \mathbb{R}^{nk} \to \mathbb{R}^m$. $V_k(\mathbb{R}^n)$ is the *Stiefel manifold*. For example, $V_1(\mathbb{R}^n) \cong S^{n-1}$. Also prove that $V_k(\mathbb{R}^n)$ is compact.

1.6 Aside: Invariance of Domain

It follows from definitions that a (nonempty!) manifold of dimension n cannot be diffeomorphic to a manifold of dimension m unless $m = n$. This is because a diffeomorphism $f : U \to V$ between open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ has derivatives $D_pf : \mathbb{R}^n \to \mathbb{R}^m$ that are necessarily isomorphisms, so $m = n$. This argument also works for C^r-manifolds, $r \neq 0$. But can a topological manifold of dimension n be homeomorphic to a topological manifold of dimension $m \neq n$? The answer is no, and it follows from the classical:

Theorem 1.41 (Invariance of Domain). *Let $f : U \to \mathbb{R}^n$ be a continuous and injective map defined on an open set $U \subset \mathbb{R}^n$. Then $f(U)$ is open and $f : U \to f(U)$ is a homeomorphism.*

The proof of this is hard, and uses methods of algebraic topology. The terminology comes from analysis, where a "domain" is traditionally an open and connected subset of Euclidean space.

Exercise 1.42. Using Invariance of Domain, prove that a nonempty C^0-manifold of dimension n cannot be homeomorphic to a C^0-manifold of dimension m unless $m = n$.

1.7 Lie groups

A *Lie group* is a manifold X that is also a group, such that the group operations

$$\mu : X \times X \to X \text{ and } inv : X \to X$$

[2] δ_{ij} is known as the *Kronecker delta*. It equals 1 when $i = j$ and otherwise it equals 0.

(multiplication and inversion) are smooth functions.

Example 1.43. $\mathbb{R}$ with addition, or more generally, $\mathbb{R}^n$ with addition.

Example 1.44. S^1 with complex multiplication.

Example 1.45. Cartesian products of Lie groups are Lie groups, e.g. the n-torus is a Lie group.

Example 1.46. Denote by $\mathcal{M}_{m\times n}$ the set of all $m \times n$ matrices with real entries. After choosing an ordering of the entries, we have an identification $\mathcal{M}_{m\times n} = \mathbb{R}^{mn}$, and in particular the set $\mathcal{M}_{m\times n}$ is a manifold of dimension mn. Now $GL_n(\mathbb{R})$, the *general linear group*, is the set of real $n \times n$ matrices of nonzero determinant, so $GL_n(\mathbb{R})$ is a subset of $\mathcal{M}_{n\times n}$. I claim that this subset is open, so that $GL_n(\mathbb{R})$ is also a manifold, of dimension n^2. To see this, recall that $\det : \mathcal{M}_{n\times n} \to \mathbb{R}$ is a polynomial map; in particular it is smooth (and hence continuous). It follows that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ is open.

The multiplication map $\mathcal{M}_{n\times k} \times \mathcal{M}_{k\times m} \to \mathcal{M}_{n\times m}$ is a polynomial, hence smooth map. The restriction of a smooth map to an open set is also smooth, so multiplication $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is also smooth. Finally, the inverse $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is also smooth, since it is given by a certain rational function with det in the denominator, and cofactors in the numerator.

Example 1.47. The *special linear group* is the group $SL_n(\mathbb{R})$ of real $n \times n$ matrices of determinant 1. In other words, $SL_n(\mathbb{R}) = \det^{-1}(1)$. To see that $SL_n(\mathbb{R})$ is a manifold, we will show that 1 is a regular value of $\det : \mathcal{M}_{n\times n} \to \mathbb{R}$.

So let's compute the partials of det at a matrix (x_{ij}), let's say with respect to x_{11}. For this, it is convenient to expand the determinant along the first row (say). Thus

$$\det((x_{ij})) = x_{11} \det A_{11} + \text{ other terms not involving } x_{11}$$

and $\frac{\partial \det}{\partial x_{11}}((x_{ij})) = \det A_{11}$, the cofactor obtained by erasing the first row and the first column. Likewise, the other partials are (up to sign) the other cofactors. From linear algebra we know that an invertible matrix cannot have all cofactors 0 (otherwise its inverse would be the zero matrix, which is absurd). This completes the argument that 1 is a regular value of det.

Now why is multiplication $SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \to SL_n(\mathbb{R})$ smooth? Here we use Exercise 1.35. According to (ii) it suffices to show that multiplication $SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is smooth. But this map is the restriction of the smooth map $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ and its smoothness follows from (i).

Example 1.48. The *orthogonal group* is the set $O(n)$ of real $n \times n$ matrices A that satisfy $AA^\top = I$. To see that $O(n)$ is a manifold, we consider the map $F : \mathcal{M}_{n\times n} \to \mathcal{S}_{n\times n}$ into the set of symmetric matrices given by $F(A) = AA^\top$. The set $\mathcal{S}_{n\times n}$ can be identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$. We will show that I is a regular value. First, compute the derivative of F, $D_AF : \mathcal{M}_{n\times n} \to \mathcal{S}_{n\times n}$:

$$D_AF(H) = \lim_{h\to 0} \frac{F(A + hH) - F(A)}{h} = AH^\top + HA^\top$$

Now we need to show that if $AA^\top = I$ and Y is an arbitrary symmetric matrix, then there is a matrix H with $AH^\top + HA^\top = Y$. The reader may verify that $H = \frac{1}{2}YA$ works. Thus $O(n)$ has dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

That group operations are smooth follows the same way as in the case of $SL_n(\mathbb{R})$.

Exercise 1.49. Show that $O(n)$ is disconnected by considering the map $\det : O(n) \to \{-1, 1\}$.

Exercise 1.50. The *special orthogonal group* $SO(n)$ is the subgroup of $O(n)$ consisting of matrices of determinant 1. Show that $SO(2)$ is diffeomorphic to S^1. Also (harder!) $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.

1.8 Complex manifolds

Most of the methods above apply to complex manifolds as well.

Example 1.51. To see that

$$\{(z_1, \cdots, z_n) \in \mathbb{C}^n | z_1^2 + \cdots + z_n^2 = 1\}$$

is a manifold, proceed as in the case of S^{n-1}. The derivative of $F : \mathbb{C}^n \to \mathbb{C}$ given by $F(z_1, \cdots, z_n) = z_1^2 + \cdots + z_n^2$ is still $(2z_1, \cdots, 2z_n)$, but this is now a linear map $\mathbb{C}^n \to \mathbb{C}$. It is still surjective unless $z_1 = \cdots = z_n = 0$, so $F^{-1}(1)$ is a manifold of (real) dimension $2n - 2$.

Exercise 1.52. Show that $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$ and $U(n)$ are Lie groups. The latter example, the *unitary group*, is the group of complex $n \times n$ matrices M which are *unitary*, i.e. satisfy $MM^* = I$, where M^* is obtained from $M^\top$ by complex-conjugating all entries. Also show that $U(1)$ is the circle.

Example 1.53. The *complex projective space* $\mathbb{C}P^n$ is the space of complex lines (i.e. 1-dimensional complex subspaces) in $\mathbb{C}^{n+1}$. A point in $\mathbb{C}^{n+1}$ will be represented by an $(n + 1)$-tuple

$$(z_0, z_1, \cdots, z_n)$$

of complex numbers, not all 0. Two such points belong to the same line iff each $(n+1)$-tuple can be obtained from the other by multiplying each coordinate by a fixed (nonzero) complex number, i.e.

$$(z_0, z_1, \cdots, z_n) \sim (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$$

for $\lambda \in \mathbb{C} - \{0\}$. It is customary to denote the equivalence class of $(z_0, z_1, \cdots, z_n)$ by

$$[z_0 : z_1 : \cdots : z_n]$$

Now we construct the charts. Let U_i be the set of equivalence classes as above with $z_i \neq 0$. This condition is independent of the choice of a representative. Each equivalence class in U_i has a unique representative with $z_i = 1$, and this gives a bijection $\varphi_i : U_i \to \mathbb{C}^n$, $i = 0, 1, \cdots, n$. For example, for $i = 0$, we have

$$\varphi_0([z_0 : z_1 : \cdots : z_n]) = \left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right)$$

We will now check that the collection $\{(U_i, \varphi_i)\}_{i=0}^n$ is an atlas. It is clear that the U_i's cover $\mathbb{C}P^n$. Let's argue that (U_0, φ_0) and (U_1, φ_1) are compatible. We have $\varphi_0(U_0 \cap U_1) = (\mathbb{C} - \{0\}) \times \mathbb{C}^{n-1}$ and

$$\varphi_1 \varphi_0^{-1}(z_1, z_2, \cdots, z_n) = \varphi_1([1 : z_1 : z_2 : \cdots : z_n]) = \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \cdots, \frac{z_n}{z_1}\right)$$

which is smooth.

Exercise 1.54. $\mathbb{C}P^1$ is diffeomorphic to the Riemann sphere via the diffeomorphism that extends φ_0 by sending $[0 : 1]$ to ∞.

1.9 Miscaleneous exercises

Exercise 1.55. Show that

$$\{(x, y, z) \in \mathbb{R}^3 | (1+z)x^2 - (1-z)y^2 = 2z(1-z^2)\}$$

is a surface in $\mathbb{R}^3$.

Exercise 1.56. Show that

$$\{(x, y, z) \in \mathbb{R}^3 | x^3 + y^3 + z^3 - 3xyz = 1\}$$

is a surface in $\mathbb{R}^3$.

Exercise 1.57. The *Grassmann manifold* (or the *Grassmannian*) $G_k(\mathbb{R}^n)$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$. The goal of this exercise is to endow $G_k(\mathbb{R}^n)$ with an atlas. There are two strategies:

(a) For any coordinate plane $P \in G_k(\mathbb{R}^n)$ let U_P be the set consisting of those $V \in G_k(\mathbb{R}^n)$ with the property that V is the graph of a linear function $P \to P^\perp$. Thus U_P can be identified with the set $Hom(P, P^\perp)$ of linear maps $P \to P^\perp$, which in turn can be identified with $\mathbb{R}^{k(n-k)}$ after choosing bases. Declare this identification $U_P \to \mathbb{R}^{k(n-k)}$ a chart and check that this collection of $\binom{n}{k}$ charts forms an atlas.

(b) (i) For any $V \in G_k(\mathbb{R}^n)$ consider the orthogonal projection $\pi_V : \mathbb{R}^n \to V \subset \mathbb{R}^n$. The matrix M_V of π_V is symmetric, has rank k, and satisfies $M_V^2 = M_V$. Conversely, any symmetric matrix M of rank k satisfying $M^2 = M$ has the form $M = M_V$ for some $V \in G_k(\mathbb{R}^n)$. Thus $G_k(\mathbb{R}^n)$ is realized as a certain subset $\mathcal{M}_k(\mathbb{R}^n)$ of $\mathcal{M}_{n\times n}$.

(ii) Now suppose
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
is a block matrix, with A nonsingular of size $k \times k$. Show that this matrix has rank k iff $D = CA^{-1}B$. (See also Problem # 13 on p.27 in Guillemin-Pollack.)

(iii) Show that the block matrix above belongs to $\mathcal{M}_k(\mathbb{R}^n)$ iff
- A is symmetric,
- $C = B^\top$,
- $D = CA^{-1}B$, and
- $A^2 + BC = A$.

(iv) Show that $\{(A, B) \in \mathcal{S}_{k\times k} \times \mathcal{M}_{k\times(n-k)} | A^2 + BB^\top = A\}$ is a manifold at $(I, 0)$.

(v) Finish the proof that $\mathcal{M}_k(\mathbb{R}^n)$ is a manifold. What is its dimension?

Exercise 1.58. Prove that the Grassmannian $G_k(\mathbb{R}^n)$ is compact by showing that the map
$$span : V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$$
from the Stiefel manifold (Exercise 1.40) is smooth and surjective.

Exercise 1.59. Show that $P \mapsto P^\perp$ is a diffeomorphism $G_k(\mathbb{R}^n) \to G_{n-k}(\mathbb{R}^n)$.

Exercise 1.60. $G_1(\mathbb{R}^n) = \mathbb{R}P^{n-1}$.

Exercise 1.61. Define a complex Grassmannian $G_k(\mathbb{C}^n)$ and show it is a manifold. E.g. $G_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$.

2 Tangent and cotangent spaces

The notion of the tangent space is fundamental in smooth topology. We all know what it means intuitively, e.g. for curves and surfaces in Euclidean space, but it takes some work to develop the notion in abstract. There are also several approaches, each with some advantages and some disadvantages.

The basic idea is that the tangent space should be associated to a point p on a manifold X, denoted $T_p(X)$, that it should be a vector space of dimension equal to that of X, and that a diffeomorphism $f : X \to Y$ should induce an isomorphism $T_p(X) \to T_{f(p)}(Y)$.

When U is an open set in $\mathbb{R}^n$ we define $T_p(U) = \mathbb{R}^n$, where we mentally visualize a vector based at p. Now, keeping in mind the above properties, when X is an arbitrary manifold and (U, φ) a chart with $\varphi(p) = 0$ (we also say the chart is *centered at* p), we would like to identify $T_p(X)$ with $T_0(\mathbb{R}^n)$ via $D_p\varphi$. To take into account different choices of charts, the formal definition is

Definition 2.1 (Tangent vectors via charts). A tangent vector at $p \in X$ is an equivalence class of triples (U, φ, v) where $\varphi : U \to \mathbb{R}^n$ is a chart centered at p and $v \in \mathbb{R}^n$. The equivalence relation is given by

$$(U, \varphi, v) \sim (U', \varphi', v') \iff D_0(\varphi'\varphi^{-1})(v) = v'$$

It is easy to see that the set of tangent vectors forms a vector space using the vector space structure of $\mathbb{R}^n$.

This definition is pretty natural, given the definition of a manifold. However, its major drawback is that it is not intrinsic, and that is what topologists strive for. Smooth topology is much more than calculus in charts.

So how should we think about a tangent vector in an intrinsic way. Well, in $\mathbb{R}^n$ a tangent vector at 0 gives rise to an operator on the space of smooth functions, namely directional derivative. If $v \in \mathbb{R}^n$ then we have an operator

$$\partial_v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

defined by

$$\partial_v(f) = (t \mapsto f(tv))'(0)$$

and this operator has the following properties:

(1) it is a linear map,

(2) it satisfies the Leibnitz rule $\partial_v(fg) = f(0)\partial_v(g) + g(0)\partial_v(f)$, and

(3) its value depends only on the value of f in a neighborhood of 0, i.e. if $f = g$ on some neighborhood of 0, then $\partial_v(f) = \partial_v(g)$.

We now make this abstract.

Definition 2.2. A *derivation* at $p \in X$ is linear map $D : C^\infty(X) \to \mathbb{R}$ that satisfies the Leibnitz rule $D(fg) = f(p)D(g) + g(p)D(f)$ and depends only on the value of the function in a neighborhood of 0.

Another way to phrase the last property is to talk about germs. A *germ* of smooth functions at p is an equivalence class of functions in $C^\infty(X)$ where $f \sim g$ if $f = g$ on a neighborhood of p. Then the last property amounts to saying that D is defined on the space of germs.

It is straightforward to see that the set of derivations at p forms a vector space.

Definition 2.3 (Tangent vectors via derivations). A tangent vector at $p \in X$ is a derivation at $p \in X$.

Now how do we reconcile the two definitions? We start with the case of $0 \in \mathbb{R}^n$. For example, at the moment it is not even clear that the vector space of derivations is finite dimensional.

Proposition 2.4. *Every derivation at $0 \in \mathbb{R}^n$ has the form ∂_v for some $v \in \mathbb{R}^n$. Therefore $v \mapsto \partial_v$ induces a vector space isomorphism from the chart definition to the derivations definition of $T_0(\mathbb{R}^n)$.*

In the proof we need the following fact from calculus.

Theorem 2.5 (Taylor's Theorem with Remainder). *Let U be a convex open set around 0 and $f : U \to \mathbb{R}$ a C^∞ function. Then for any $k \geq 0$ and any $x = (x_1, \cdots, x_n) \in U$*

$$f(x) = f(0) + \sum_i x_i \frac{\partial f}{\partial x_i}(0) + \cdots + \frac{1}{k!} \sum_{i_1,\cdots,i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}(0) +$$

$$\sum_{i_1,\cdots,i_{k+1}} \int_0^1 \frac{(1-t)^k}{k!} \frac{\partial^{k+1} f}{\partial x_{i_1} \cdots \partial x_{i_{k+1}}}(tx)dt$$

Sketch. Let $\gamma : [0,1] \to \mathbb{R}^n$ be the straight line segment from 0 to x. Then

$$f(x) = f(0) + \int_\gamma df = f(0) + \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$$

Now integrate by parts with $u = \frac{\partial f}{\partial x_i}(tx)$ and $v = 1 - t$ to obtain

$$f(x) = f(0) + \sum_i x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i_1, i_2} x_{i_1} x_{i_2} \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(tx) dt$$

and continue by induction on k. □

Example 2.6. For $k = 1$ this says

$$f(x) = f(0) + \sum_i x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j} x_i x_j \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(tx) dt$$

In particular, if $f(0) = 0$ one can collect x_i's together to obtain

$$f(x) = \sum_i x_i g_i(x) \tag{1}$$

for certain $g_i \in C^\infty(\mathbb{R}^n)$ with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof of Proposition 2.4. Let D be a derivation at $0 \in \mathbb{R}^n$. First note that $D(1) = 0$. This follows from the Leibnitz rule applied to $f = g = 1$. Thus by linearity D vanishes on all constant functions. So when calculating $D(f)$ we may assume $f(0) = 0$ by subtracting $f(0)$ – this will not change $D(f)$. From Taylor's theorem with $k = 1$ we then have

$$f(x) = \sum_i x_i g_i(x)$$

where $g_i \in C^\infty(\mathbb{R}^n)$ and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$. Now apply the Leibnitz rule:

$$D(f) = \sum_i g_i(0) D(x_i) = \sum_i D(x_i) \frac{\partial f}{\partial x_i}(0)$$

so that $D = \partial_v$ for $v = \sum_i D(x_i) e_i$. □

Now what goes wrong if one tries to run the same proof in the general case of $p \in X$? We could work via a chart $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ centered at p so that $\varphi(U)$ is convex. The coordinate functions x_i then make sense on U and we similarly obtain $f(x) = \sum_i x_i g_i(x)$ where $g_i \in C^\infty(U)$ and $g_i(p) = \frac{\partial f \varphi^{-1}}{\partial x_i}(0)$. The trouble is that the last equation of the above proof no longer makes sense since D can be applied only to functions defined on all of X, and x_i, g_i are defined only on U.

The fix is to find functions defined on all of X and agreeing with the desired functions near p.

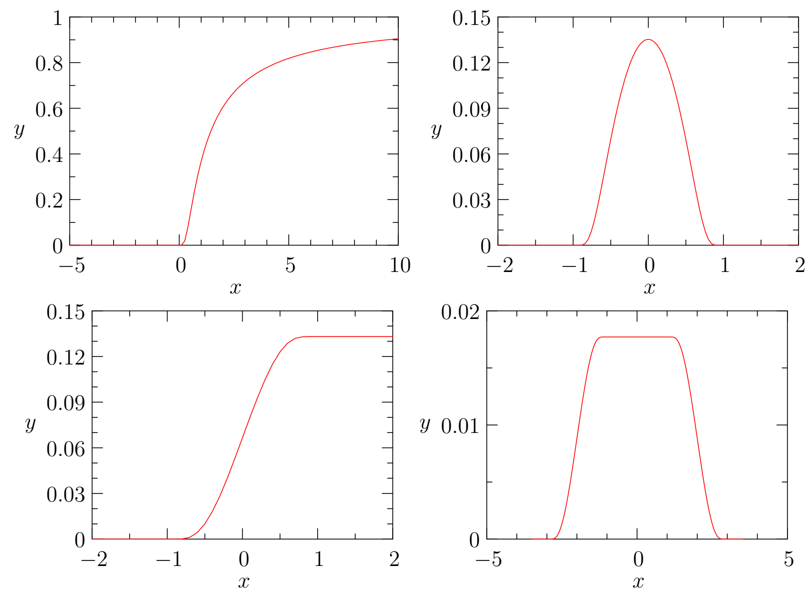

Figure 3: Functions α, β, γ, ρ_1

Lemma 2.7. *There is a smooth function* $\rho : \mathbb{R} \to \mathbb{R}$ *such that* $\rho \geq 0$, $\rho \equiv 0$ *outside* $[-3, 3]$ *and* $\rho \equiv 1$ *on* $[-1, 1]$.

Proof. The function $\alpha : \mathbb{R} \to \mathbb{R}$ given by $\alpha(x) = e^{-\frac{1}{x}}$ for $x > 0$ and $\alpha(x) = 0$ for $x \leq 0$ is smooth[3] (but not analytic!). Then $\beta(x) = \alpha(1 - x)\alpha(1 + x)$ is also smooth, vanishes outside $(-1, 1)$, and is positive on $(-1, 1)$. Thus $\gamma : \mathbb{R} \to \mathbb{R}$, $\gamma(x) = \int_{-1}^{x} \beta(t)dt$ is smooth, 0 for $x \leq -1$, constant for $x \geq 1$, and monotonically increasing. Finally, take $\rho_1(x) = \gamma(2 + x)\gamma(2 - x)$ and then rescale ρ_1 to get ρ. □

Functions such as β and ρ are called *bump functions.*[4]

Corollary 2.8. *Any smooth function* $f : U \to \mathbb{R}$ *defined on an open neighborhood* U *of* $p \in X$ *coincides in a neighborhood of* p *with a smooth function* $\tilde{f} : X \to \mathbb{R}$ *defined on all of* X.

[3]This is an exercise in calculus. The issue is infinite differentiability at 0. First show that all derivatives have the form $\alpha^{(n)}(x) = P_n(x^{-1})e^{-\frac{1}{x}}$ for $x > 0$, for a certain polynomial P_n. Then use the fact that $\lim_{t\to\infty} \frac{P(t)}{e^t} = 0$ for any polynomial P. This fact can be proved by induction on the degree of P using L'Hospital's Rule.

[4]Every time you are in southern Utah you will recall ρ.

Proof. If necessary, replace U by a smaller open set so that there is a diffeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ and $\varphi(U)$ is an open ball in $\mathbb{R}^n$, say of radius 4 (we can arrange this by rescaling). Then the function $\mu(x) = \rho(|\varphi(x)|)$ is defined for $x \in U$, is smooth, vanishes outside the φ-preimage of the ball of radius 3, and is identically 1 on the preimage of the 1-ball. Now our extension is the product of μ with f, extended by 0 outside U. □

With this corollary, the proof that all derivations are standard at $p \in X$ is complete.

Remark 2.9 (Tangent vectors via curves). There is another way of defining tangent vectors. A tangent vector at $p \in X$ is an equivalence class of curves $\gamma : (-\epsilon, \epsilon) \to X$ with $\gamma(0) = p$ where $\gamma_1 \sim \gamma_2$ if $\varphi\gamma_1$ and $\varphi\gamma_2$ have the same velocity vector at $t = 0$ for any chart φ centered at p. This definition is kind of "semi-intrinsic" since the curves themselves are intrinsic, but the equivalence relation is defined via charts. Moreover, it is not obvious that tangent vectors form a vector space. The operations of addition and scalar multiplication can be defined in charts as pointwise operations on curves.

2.1 Cotangent vectors

We've seen in Exercise 1.23 that $C^\infty(X)$ is an algebra. For $p \in X$ denote by m_p the set of functions in $C^\infty(X)$ that vanish at p. This is an ideal (i.e. it is a linear subspace and $f \in m_p, g \in C^\infty(X)$ implies $fg \in m_p$). The evaluation map $C^\infty(X) \to \mathbb{R}$, $f \mapsto f(p)$, is an isomorphism from $C^\infty(X)/m_p$ to $\mathbb{R}$ so that m_p is a maximal ideal. Now consider the ideal m_p^2 generated by (i.e. consisting of sums of) products $f_1 f_2$ with $f_1, f_2 \in m_p$. By the Leibnitz rule, any derivation at p vanishes on every function in m_p^2. Conversely, if every derivation vanishes at $f \in m_p$ then equation (1) in Example 2.6 shows that $f \in m_p^2$. To say it in another way, we have a pairing

$$T_p(X) \times m_p/m_p^2 \to \mathbb{R}$$

given by the action of derivations on functions

$$(D, [f]) \mapsto D(f)$$

This pairing is bilinear and nondegenerate (i.e. for every nonzero derivation D there is a function on which D does not vanish and for every nonzero equivalence class of functions in m_p/m_p^2 there is a derivation that does not kill it). We thus[5] have an isomorphism

$$m_p/m_p^2 \to T_p(X)^*$$

given by

$$[f] \mapsto (D \mapsto D(f))$$

The dual of the tangent space is called the *cotangent space* and we found

Proposition 2.10. $T_p(X)^* \cong m_p/m_p^2$

Why is this interesting? Well, you can talk about derivations and about m_p in a purely algebraic setting. This is how they define (co)tangent spaces of varieties in algebraic geometry, even when these are defined over fields of characteristic p and might be finite sets!

2.2 Functorial properties

Let $f : X \to Y$ be a smooth map and $p \in X$. Then there is a naturally induced linear map $T_pf : T_p(X) \to T_{f(p)}(Y)$. This linear map can be defined from any of the points of view we discussed above:

(i) (charts) Say (U, φ, v) represents a vector in $T_p(X)$, and let (V, ψ) be a chart centered at $f(p)$. After shrinking U if necessary we may assume that $f(U) \subset V$ and hence we have a map $g = \psi f \varphi^{-1} : \varphi(U) \to \psi(V)$ that represents f in local coordinates. We declare that $T_pf[(U, \varphi, v)] = [(V, \psi, D_0 g(v))]$. The verification that this is well defined and linear is painful, but straightforward, using the chain rule.

(ii) (curves) Say $\gamma : (-\epsilon, \epsilon) \to X$ represents a tangent vector at p. Declare that $T_f([\gamma]) = [f\gamma]$.

(iii) (derivations) Let Δ be a derivation representing a tangent vector at p. Define the derivation $T_pf(\Delta)$ at $f(p)$ by the formula

$$T_pf(\Delta)(\varphi) = \Delta(\varphi f)$$

[5] This is a very common situation in mathematics. A nondegenerate bilinear pairing $V \times W \to \mathbb{R}$ between two finite-dimensional vector spaces gives rise to an isomorphism $V \to W^*$ from one vector space to the dual $W^* = Hom(W, \mathbb{R})$ of the other. Why? By the definition of a non-degenerate pairing we have injections $V \to W^*$ and $W \to V^*$, so in particular $\dim V \leq \dim W^* = \dim W$ and $\dim W \leq \dim V^* = \dim V$, so $\dim V = \dim W^*$ and our injection $V \to W^*$ is necessarily an isomorphism.

We record

Proposition 2.11. *A smooth map $f : X \to Y$ induces a linear map $T_pf : T_p(X) \to T_{f(p)}(Y)$. The following holds:*

(1) If X, Y are open sets in Euclidean space then T_pf coincides with the usual derivative D_pf,

(2) If $f = id : X \to X$ then $T_pf = id : T_p(X) \to T_p(X)$, and

(3) (the chain rule) If $f : X \to Y$ and $g : Y \to Z$ are smooth maps then $T_p(g \circ f) = T_{f(p)}g \circ T_pf$.

(4) If $f : X \to Y$ is a constant map then $T_pf = 0$.

(5) If $f = g$ on a neighborhood of p then $T_pf = T_pg$.

Proof. I prove (3) from the derivations point of view and leave the rest to you. $T_p(g \circ f)(\Delta)(\phi) = \Delta(\phi g f)$ and also $T_{f(p)}g \circ T_pf(\Delta)(\phi) = T_pf(\Delta)(\phi g) = \Delta(\phi g f)$. □

2.3 Computations

The computations are particularly pleasant in the situation of Corollary 1.38. More generally, we say that $c \in Y$ is a *regular value* of a smooth map $f : X \to Y$ if for every $p \in f^{-1}(c)$ the derivative $T_pf : T_p(X) \to T_{f(p)}(Y)$ is surjective.

Proposition 2.12 (The Regular Value Theorem). *Let $f : X^{n+m} \to Y^n$ be a smooth map and $c \in Y$ a regular value of f. Then $Z = f^{-1}(c)$ is a submanifold of X of dimension m and for every $p \in Z$*

$$T_pi(T_p(Z)) = Ker[T_pf : T_p(X) \to T_{f(p)}(Y)]$$

where $i : Z \to X$ is inclusion.

Proof. We already stated the first part of the Proposition in the case that X and Y are open sets in Euclidean spaces. But the statement is local and reduces to that case via charts. Now note that $T_pi : T_p(Z) \to T_p(X)$ is an injective linear map since in suitable local coordinates it is given by $i(x_1, \cdots, x_m) = (x_1, \cdots, x_m, x_{m+1}, \cdots, x_{n+m})$. Thus the two vector spaces in the last part of the statement have the same dimension m, so it suffices to prove that $T_pi(T_p(Z)) \subseteq Ker[T_pf : T_p(X) \to T_{f(p)}(Y)]$, i.e. that $T_pf \circ T_pi = 0$. By the Chain Rule, $T_pf \circ T_if = T_p(f \circ i)$. But $f \circ i : Z \to Y$ is constant so $T_p(f \circ i) = 0$. □

In the sequel we suppress inclusion i. To "compute" the tangent space of a submanifold $Z \subset X$ means to identify $T_p(Z)$ as a subspace of $T_p(X)$.

Example 2.13. We compute the tangent space at $p = (p_1, \cdots, p_n) \in S^{n-1}$. Refer to Example 1.39. We have $T_pF = (2p_1, \cdots, 2p_n)$ so

$$T_p(S^{n-1}) = Ker(2p_1, \cdots, 2p_n) = \{v \in \mathbb{R}^n | v \cdot p = 0\} =< p >^{\perp}$$

But you knew this already, didn't you?

Example 2.14. We compute the tangent space of $SL_n(\mathbb{R})$ at I. Refer to Example 1.47. We know $SL_n(\mathbb{R}) = \det^{-1}(1)$ is a submanifold of $\mathcal{M}_{n\times n} \cong \mathbb{R}^{n^2}$ and $T_I(\mathcal{M}_{n\times n}) = \mathcal{M}_{n\times n}$. We computed $D_A(\det) : \mathcal{M}_{n\times n} \to \mathbb{R}$ for $A \in \mathcal{M}_{n\times n}$ and obtained the map

$$(x_{ij}) \mapsto \sum_{i,j} x_{ij} C_{ij}$$

where C_{ij} are the cofactors of A. When $A = I$ then $C_{ij} = \delta_{ij}$ and the map is the trace of the matrix. We conclude that $T_I(SL_n(\mathbb{R}))$ is the subspace of $\mathcal{M}_{n\times n}$ consisting of matrixes of trace 0 (such matrices are also called *traceless*).

Example 2.15. We will compute $T_I(O(n))$. Refer to Example 1.48. In the displayed formula put $A = I$. Thus we have

$$D_I F(H) = H + H^{\top}$$

and the kernel of $D_I F$ consists of skew-symmetric matrices, i.e.

$$T_I(O(n)) = \{H \in \mathcal{M}_{n\times n} | H^{\top} = -H\}$$

Exercise 2.16. Identify tangent spaces to surfaces in Exercises 1.55 and 1.56.

3 Local diffeomorphisms, immersions, submersions, and embeddings

Some smooth maps are better than others. The following is a standard and frequently used terminology. Let $f : X^n \to Y^m$ be a smooth map.

- f is a *local diffeomorphism* at $p \in X$ if $T_pf : T_pX \to T_{f(p)}Y$ is an isomorphism. Thus necessarily $n = m$. f is a *local diffeomorphism* if it is a local diffeomorphism at every $p \in X$.

- f is a *submersion* at $p \in X$ if $T_p f : T_p X \to T_{f(p)} Y$ is surjective. Thus necessarily $n \geq m$. f is a *submersion* if it is a submersion at every $p \in X$.

- f is an *immersion* at $p \in X$ if $T_p f : T_p X \to T_{f(p)} Y$ is injective. Thus necessarily $n \leq m$. f is an *immersion* if it is an immersion at every $p \in X$.

Example 3.1. The standard submersion is the projection $\mathbb{R}^n \to \mathbb{R}^m$ to the first m coordinates $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_m)$. The standard immersion is the inclusion $\mathbb{R}^n \to \mathbb{R}^m$, $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0, \cdots, 0)$.

The first thing to know about local diffeomorphisms, submersions and immersions is that they are *locally standard*.

Theorem 3.2. *Let $f : X^n \to Y^m$ be smooth and $p \in X$.*

- *If f is a local diffeomorphism at p, then there are neighborhoods $V \ni p$ and $W \ni f(p)$ such that $f : V \to W$ is a diffeomorphism. Equivalently, in suitable local coordinates near p and $f(p)$, f is represented by the identity function.*

- *If f is a submersion at p then in suitable local coordinates near p and $f(p)$ the function f represented by the projection $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_m)$.*

- *If f is an immersion at p then in suitable local coordinates near p and $f(p)$ the function f represented by the inclusion $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0, \cdots, 0)$.*

Of course, the first statement justifies the terminology.

Proof. All statements are local and we may replace X and Y by open sets in Euclidean space. The first bullet then becomes the Inverse Function Theorem, while the second bullet follows from our proof of the Regular Value Theorem: in the notation of that proof (where the function was F instead of f) we have $\pi G = F$ where π is the standard projection, so using the chart given by G around p puts F in the standard form $FG^{-1} = \pi$.

The third bullet is proved similarly, except that now we "inflate" the domain. Assume without loss that $\left(\frac{\partial f_i}{\partial x_j}(p)\right)$ has rank n when $1 \leq i, j \leq n$. Then define $G : X \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ by

$$G(x, t_1, \cdots, t_{m-n}) = f(x) + (0, \cdots, 0, t_1, \cdots, t_{m-n})$$

so that

$$D_pG = \begin{pmatrix} \left(\frac{\partial f_i}{\partial x_j}(p)\right) & * \\ 0 & I \end{pmatrix}$$

which is invertible. By the Inverse Function Theorem, G is a local diffeomorphism, and with respect to the chart given by G^{-1} around $f(p)$ the function f is represented by $G^{-1}f$, which is the standard inclusion. ☐

Corollary 3.3. *Submersions are open maps: if $f : X \to Y$ is a submersion and $U \subset X$ is open, then $f(U) \subset Y$ is also open.*

Proof. Choose a point $q \in f(U)$ so that $q = f(p)$ for some $p \in U$. In suitable local coordinates near p f is given by the standard projection, which is an open map. Thus f maps a neighborhood of p onto a neighborhood of $f(p)$. ☐

Example 3.4. Figure "8" is an immersed circle in the plane.

Example 3.5. $f : \mathbb{R} \to S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is a local diffeomorphism, and so is the restriction of f to an open interval in $\mathbb{R}$.

Example 3.6. Let $\pi : \mathbb{R}^2 \to T^2 = S^1 \times S^1$ be the standard projection from the plane to the torus,

$$\pi(x, y) = ((\cos(2\pi x), \sin(2\pi x)), (\cos(2\pi y), \sin(2\pi y)))$$

Thus π is a local diffeomorphism. Consider a line of irrational slope in $\mathbb{R}^2$, say $t \mapsto (t, at)$ with a irrational. Then the composition $\mathbb{R} \to T^2$ is given by $t \mapsto (\cos(2\pi t), \sin(2a\pi t))$. It is an injective immersion whose image is dense in T^2.

Example 3.7. $f : S^n \to \mathbb{R}P^n$ that sends x to its equivalence class $\{x, -x\}$ is a local diffeomorphism. More generally, when G is a group of diffeomorphisms acting on X freely and properly discontinuously, then the quotient map $X \to X/G$ is a local diffeomorphism. This follows from the definition of the differentiable structure on X/G: the quotient map is represented by the identity in suitable local coordinates.

Example 3.8. There is no submersion $X \to \mathbb{R}^n$ if X is a (nonempty!) compact manifold. This is because the image would have to be nonempty, compact, and open in $\mathbb{R}^n$, which is impossible.

Exercise 3.9. Let $f : X^n \to Y^m$ be a submersion. Show that all point inverses $f^{-1}(y)$ are $(n - m)$-submanifolds of X.

Example 3.10. Consider the map $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$ that sends each point $(z_0, \cdots, z_n)$ to its equivalence class $[z_0 : \cdots : z_n]$ (refer to Example 1.53). I claim that π is a submersion. To see this, fix some $p = (p_0, \cdots, p_n) \in \mathbb{C}^{n+1} - \{0\}$ and say $p_0 \neq 0$. In the local coordinates given by $\varphi_0 : U_0 \to \mathbb{C}^n$ the map π is given by

$$(z_0, \cdots, z_n) \mapsto \left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right)$$

But the latter map is clearly a submersion, even without computing anything, because the restriction to the planes $z_0 = \textit{constant}$ is a local diffeomorphism.

Example 3.11. Now consider the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ and the restriction $h : S^{2n+1} \to \mathbb{C}P^n$ of $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$. I claim that h is also a submersion. To see this, take a point $p \in S^{2n+1}$ and note that $T_p(\mathbb{C}^{n+1} - \{0\}) = T_p(S^{2n+1}) \oplus < p >$ (see Example 2.13). In addition, π sends each straight line through 0 to a single point, so that the derivative $T_p\pi$ vanishes on $< p >$. It follows that $T_p\pi$ restricted to $< p >$ is 0 and hence the restriction to $T_p(S^{2n+1})$ is surjective. But this restriction is T_ph.

The submersion $h : S^3 \to \mathbb{C}P^1 = S^2$ is famous. It is called the *Hopf map* or the *Hopf fibration*. We will discuss it further later on in the course. Can you draw a picture of this map? What are the point inverses?

3.1 Embeddings

$f : X \to Y$ is an *embedding* if $f(X)$ is a submanifold of Y and $f : X \to f(X)$ is a diffeomorphism. It is clear that an embedding is necessarily injective and an immersion. To what extent does the converse hold?

Example 3.12. Let X be the disjoint union of two lines and $Y = \mathbb{R}^2$. Let $f : X \to Y$ place one line diffeomorphically onto the x-axis, and the other line diffeomorphically onto the positive y-axis. Then f is an injective immersion, but $f(X)$ is not a submanifold of Y. A similar example is formed by figure "9" in the plane. Yet another example is given by Example 3.6.

Exercise 3.13. Show that if $f : X \to Y$ is an injective immersion and $f(X)$ is a submanifold of Y of the same dimension as X, then f is an embedding.

We will see shortly, as a consequence of Sard's theorem, that $f(X)$ couldn't possibly be a submanifold of higher dimension than X.

There is a useful condition that is often satisfied and in whose presence injective immersions are embeddings.

Definition 3.14. $f : X \to Y$ is a *proper map* if for every compact set $K \subset Y$ the preimage $f^{-1}(K)$ is compact.

For example, if X is compact then any smooth map $X \to Y$ is proper.

Observe that any proper map is *closed*, i.e. the image of any closed subset of X is closed in Y. For instance, let's argue that $f(X)$ is a closed subset of Y. Take some compact set $K \subset Y$. Then

$$f(X) \cap K = f(f^{-1}(K)) \cap K$$

is compact. But if the subset $f(X)$ of Y intersects every compact set in a compact subset, then $f(X)$ is closed (why?).

Proposition 3.15. *Let $f : X \to Y$ be a proper injective immersion. Then f is an embedding onto a closed subset (submanifold) $f(X) \subset Y$.*

Proof. Take some $q = f(p) \in f(X)$. Since f is an immersion, there is a neighborhood U of p so that $f(U)$ is a submanifold and $f : U \to f(U)$ is a diffeomorphism. It now suffices to show that $f(X - U)$ is disjoint from a neighborhood of q.

Let K be a compact subset of Y that contains q in its interior. Then $f(X - f^{-1}(K))$ misses K, hence a neighborhood of q, so we only need to show that $f(f^{-1}(K) - U)$ misses a neighborhood of q. But this set is compact and misses q, hence a neighborhood of q. □

4 Partitions of Unity

This section is about an important technique in differential topology that allows one to patch things defined chartwise. This technique is also commonly used in point-set topology, and I present both settings at the same time.

Definition 4.1. Let X be a Hausdorff topological space [a manifold]. A *partition of unity* on X is a collection of continuous [smooth] functions

$$\{\phi_\alpha : X \to \mathbb{R}\}_{\alpha \in A}$$

such that

(1) $\phi_\alpha(X) \subset [0, 1]$.

(2) The collection $\{supp(\phi_\alpha)\}_{\alpha \in A}$ of supports is locally finite.[6]

[6] This means that every $x \in X$ has a neighborhood that intersects only finitely many sets in the collection. Recall also that $supp(f)$ is the closure of the set $\{x | f(x) \neq 0\}$.

(3) $\sum_{\alpha \in A} \phi_\alpha(x) = 1$ for every $x \in X$[7]

We will say that the partition of unity $\{\phi_\alpha : X \to \mathbb{R}\}_{\alpha \in A}$ is *subordinate* to an open cover $\{U_\beta\}_{\beta \in B}$ if for every $\alpha \in A$ there exists some $\beta \in B$ such that $supp(\phi_\alpha) \subset U_\beta$.

Exercise 4.2. Any compact set in X intersects only finitely many of the supports $supp(\phi_\alpha)$.

Definition 4.3. A Hausdorff space is *paracompact* if every open cover admits a partition of unity subordinate to it.

In point-set topology there are various criteria for paracompactness. Here is a simple one.

Theorem 4.4. *Let X be a Hausdorff space that admits a countable basis of open sets $V_1, V_2, \cdots$ such that each closure $\overline{V}_i$ is compact and metrizable. Then X is paracompact.*

In the manifold world we have:

Theorem 4.5. *For every manifold X and any open cover $\{U_\beta\}$ of X there is a smooth partition of unity subordinate to the cover.*

We will give one proof for both theorems.

Proof of Theorems 4.4 and 4.5. In the manifold situation, observe that there is a basis $\{V_i\}$ as in the statement of Theorem 4.4, thanks to our standing assumption (Top 2). There are two steps.
Step 1. Construct an exhaustion

$$K_1 \subset K_2 \subset \cdots$$

of X by compact subsets K_i. This means that $K_i \subset int\ K_{i+1}$ for $i = 1, 2, \cdots$ and that $\cup_i K_i = X$. To construct an exhaustion, put $K_1 = \overline{V}_1$. Assuming K_i is constructed find $j \geq i$ such that $K_i \subset V_1 \cup \cdots \cup V_j$ and then put $K_{i+1} = \overline{V}_1 \cup \cdots \cup \overline{V}_j$.

Now define $A_i = K_i - int\ K_{i-1}$ for $i = 1, 2, \cdots$ where we take $K_0 = \emptyset$. Note that the collection $A_1, A_2, \cdots$ is locally finite. In particular, the union of any subcollection is a closed subset of X.
Step 2. Fix some $i = 1, 2, \cdots$. For every $x \in A_i$ choose a continuous [smooth] function $\phi_x : X \to \mathbb{R}$ with values in $[0, 1]$ such that $\phi_x = 1$ in a

[7]By (2) this sum has only finitely many nonzero terms for every $x \in X$.

neighborhood of x and $supp(\phi_x)$ is contained in some U_β and disjoint from any A_s with $|s-i|>1$. (This is possible since the collection $A_1, A_2, \cdots$ is locally finite and hence the union of any subcollection is a closed set.) In the manifold setting, ϕ_x is a bump function discussed earlier, while in the metric space setting one can manipulate the distance function to create continuous bump functions.[8] Now consider the open cover of A_i consisting of the sets *int* $\{y \in A_i | \phi_x(y) = 1\}$. Since A_i is compact, there is a finite subcover, say consisting of *int* $\{y \in A_i | \phi_{x_m}(y) = 1\}$, $m = 1, 2, \cdots, p_i$. For convenience, rename ϕ_{x_m} to ϕ^i_m.

The collection $\{\phi^i_m\}_{i,m}$ satisfies (1) and (2) but not (3). To achieve (3), let

$$\phi(x) = \sum_{i,m} \phi^i_m(x)$$

This is a (positive) continuous [smooth] function since in a neighborhood of any point it is equal to a finite sum of continuous [smooth] functions. Then the collection

$$\left\{\frac{\phi^i_m}{\phi}\right\}_{i,m}$$

satisfies all the requirements. □

Remark 4.6. The partition of unity we constructed is countable. In fact, any partition of unity on a manifold will necessarily have at most countably many nonzero functions.

Remark 4.7. Sometimes it is convenient to arrange that the partition of unity is indexed by the same set as the covering, so that $supp(\phi_\beta) \subset U_\beta$ for every index β. This can be easily arranged as follows. Choose a function $\sigma : A \to B$ so that $supp(\phi_\alpha) \subset U_{\sigma(\alpha)}$. Then define

$$\phi_\beta = \sum_{\sigma(\alpha)=\beta} \phi_\alpha$$

4.1 Applications

Proposition 4.8 (Proper smooth maps). *Every manifold X admits a proper smooth map $f : X \to \mathbb{R}$ with values in $[0, \infty)$.*

Proof. Choose an open covering of X by sets whose closures are compact and let $\{\phi_i\}$ be a partition of unity subordinate to this cover. We take

[8] The details of this are left as an exercise.

$1, 2, 3, \cdots$ for the index set (it is countable and we may renumber). Now define

$$f(x) = \sum_i i\phi_i(x)$$

Then f is a smooth function with values in $[0, \infty)$. Moreover, $f^{-1}[0, N] \subset supp(\phi_1) \cup \cdots \cup supp(\phi_N)$ is compact, so that f is proper. □

Theorem 4.9 (Embedding in Euclidean space). *Every manifold X^n that admits a finite atlas $\{(U_i, \varphi_i)\}_{i=1}^N$ has a proper embedding in some Euclidean space.*

Proof. Let $\{\kappa_i\}$ be a smooth partition of unity subordinate to the cover $\{U_i\}$, with the same index set (see Remark 4.7). We would like to replace κ_i with functions ω_i whose sum is not necessarily 1, but in return each $x \in X$ has a neighborhood on which some ω_i is 1. This can be accomplished as follows. Note that at every x at least one of the functions κ_i is $\geq 1/N$. Choose $\epsilon \in (0, 1/N)$ and fix a smooth function $\mu : \mathbb{R} \to \mathbb{R}$ which is 1 on $[\epsilon, \infty]$, is 0 on $[0, \epsilon/2]$ and is nonnegative everywhere. This is similar to the function γ we had earlier. Now define $\omega_i(x) = \mu(\kappa_i(x))$. Then $supp(\omega_i) \subset supp(\kappa_i) \subset U_i$ and each x has a neighborhood on which some ω_i is 1.

Now choose a proper smooth map $f : X \to \mathbb{R}$ and define

$$F : X \to (\mathbb{R}^n)^N \times \mathbb{R}^N \times \mathbb{R}$$

by

$$F(x) = (\omega_1(x)\varphi_1(x), \cdots, \omega_N(x)\varphi_N(x), \varphi_1(x), \cdots, \varphi_N(x), f(x))$$

Of course, $\varphi_1(x)$ is defined only on U_1, but the product $\omega_1(x)\varphi_1(x)$ is defined to be 0 outside the support of ω_1.

The last coordinate ensures that F is proper. To see that F is an immersion, consider $x \in X$ and a neighborhood on which some ω_i is 1. On that neighborhood, one of the coordinate functions of F is just φ_i which is an immersion at x, and so F is also an immersion at x.

Finally, we argue that F is injective. Assume $F(x) = F(y)$. In particular, $\omega_i(x) = \omega_i(y)$ for all i. Choose some i so that $\omega_i(x) > 0$. Thus $x, y \in supp(\omega_i) \subset U_i$. Since also $\omega_i(x)\varphi_i(x) = \omega_i(y)\varphi_i(y)$ it follows that $\varphi_i(x) = \varphi_i(y)$ and hence $x = y$. □

OK, so when does a manifold have a finite atlas? Of course, all compact manifolds do, by passing to a finite subcover. But it turns out that *every* manifold does.

Proposition 4.10. *Every n-manifold admits an atlas consisting of $n+1$ charts.*

I will postpone the proof until the next semester. It amounts to the basic fact in dimension theory that an n-manifold has dimension n (!). However, the proof will make more sense to you once we've talked about simplicial complexes. For now, just take my word for it.

Proposition 4.11 (Smooth approximations of continuous functions). *Let X be a manifold and $f : X \to \mathbb{R}^k$ a continuous function. Then for any $\epsilon > 0$ there is a smooth function $g : X \to \mathbb{R}$ such that*

$$||g(x) - f(x)|| < \epsilon$$

for every $x \in X$.

Proof. For every $x \in X$ choose a neighborhood U_x such that $f(U_x)$ has diameter $< \epsilon$. Let $\{\phi_i\}_{i=1}^{\infty}$ be a smooth partition of unity subordinate to the cover $\{U_x\}_{x \in X}$. For every i choose x_i so that $supp(\phi_i) \subset U_{x_i}$.

Now define $g : X \to \mathbb{R}^k$ by

$$g(x) = \sum_i \phi_i(x) f(x_i)$$

Then g is smooth and

$$||g(x) - f(x)|| = ||\sum_i \phi_i(x) f(x_i) - \sum_i \phi_i(x) f(x)|| \leq \sum_i \phi_i(x) ||f(x_i) - f(x)||$$

Whenever i is such that $\phi_i(x) > 0$, then $x \in supp(\phi_i) \subset U_{x_i}$ so that $||f(x) - f(x_i)|| < \epsilon$ and hence $||g(x) - f(x)|| < \sum_i \phi_i(x) \epsilon = \epsilon$. □

Proposition 4.12 (Local implies global for smooth extendability). *Let $X \subset \mathbb{R}^n$ be a subset and $f : X \to \mathbb{R}^k$ a function. Assume that f is locally smoothly extendable i.e. for every $x \in X$ there is an open set $U_x \ni x$ in $\mathbb{R}^n$ and a smooth function $g_x : U_x \to \mathbb{R}^n$ such that $g_x | X \cap U_x = f | X \cap U_x$.*

Then f is globally smoothly extendable i.e. there is an open set $U \supset X$ and a smooth map $g : U \to \mathbb{R}^k$ such that $g|X = f$.

Proof. Let $U = \cup_x U_x$ and let $\{\phi_x\}_{x \in X}$ be a smooth partition of unity on U subordinate to the cover $\{U_x\}_{x \in X}$ (with the same index set). Then define

$$g(y) = \sum_x \phi_x(y) g_x(y)$$

$g : U \to \mathbb{R}^k$ is smooth for the usual reasons, and when $y \in X$ then $g_x(y) = f(y)$ for all x, so that $g(y) = f(y)$. □

5 Smooth vector bundles

Definition 5.1. A *smooth n-dimensional vector bundle* ξ (or ξ^n if want to emphasise the dimension) is a triple (E, p, B) where

- E and B are smooth manifolds,
- $p : E \to B$ is a smooth map,
- each fiber $p^{-1}(b)$ is equipped with the structure of a vector space,

so that the *local triviality condition* holds:

- every $b \in B$ has a neighborhood U (called a *trivializing neighborhood*) and there is a diffeomorphism $\Phi : p^{-1}(U) \to U \times \mathbb{R}^n$ such that Φ takes each fiber $p^{-1}(x)$ to $\{x\} \times \mathbb{R}^n$ and the restriction $p^{-1}(x) \to \{x\} \times \mathbb{R}^n$ is an isomorphism of vector spaces.

B is called the *base*, and E is the *total space* of the bundle. Sometimes, to avoid confusion, we will use the notation $E(\xi)$ and $B(\xi)$. The *fiber* over $b \in B$ is the preimage $p^{-1}(b)$, also denoted $\xi(b)$.

You should think of B as a parameter space where each point $b \in B$ parametrizes a vector space $\xi(b)$. As b varies smoothly over B, so the vector space varies smoothly.

A vector bundle will be called *trivial* if the whole base can be chosen to be a trivializing neighborhood. In our minds this notion corresponds to a "constant" family of vector spaces. For example, $(B \times \mathbb{R}^n, pr_B, B)$ is a trivial bundle.

Example 5.2. Take $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$, $B = [0,1]/0 \sim 1$, $p : E \to B$ is the projection to the first coordinate. Then (E, p, B) is called the Möbius band bundle, because the total space is a Möbius band.

Definition 5.3. A *section* of ξ is a smooth map $s : B \to E$ such that $ps = id$. Thus s selects a point in each fiber $p^{-1}(b)$.

For example, the section that to each $b \in B$ assigns $0 \in p^{-1}(b)$ is called the *zero section*. Check that the zero section is smooth.

Definition 5.4. Sections $s_1, \cdots, s_k$ of ξ are *linearly independent* if for every $b \in B$ the vectors $s_1(b), \cdots, s_k(b) \in p^{-1}(b)$ are linearly independent.

For example, a single section s is linearly independent if it does not vanish anywhere. The more standard term in this case is that s is *nowhere zero*. A trivial n-dimensional bundle has n linearly independent sections.

Example 5.5. Here is a generalization $\xi^1_{\mathbb{R}P^n}$ of the Möbius band bundle. Take $B = \mathbb{R}P^n$, thought of as $S^n/x \sim -x$. The fiber over $\pm x$ will be the line λx, $\lambda \in \mathbb{R}$, in $\mathbb{R}^{n+1}$. More precisely, let

$$E = \{(\{\pm x\}, \lambda x) | x \in S^n, \lambda \in \mathbb{R}\} \subset \mathbb{R}P^n \times \mathbb{R}^{n+1}$$

Let $p : E \to \mathbb{R}P^n$ be the projection to the first coordinate. Denote by $\pi : S^n \to \mathbb{R}P^n$ the quotient map and for every open set $U \subset S^n$ that does not contain a pair of antipodal points define

$$\Phi : p^{-1}(\pi(U)) \to \pi(U) \times \mathbb{R}$$

by $\Phi(\pi(x), \lambda x) = (\pi(x), \lambda)$ for any $x \in U$. Informally, $p^{-1}(\pi(U))$ consists of pairs (line,vector on it) where line intersects U in one (and only one!) point x. Identify this point with $1 \in \mathbb{R}$ and then extend to the unique linear isomorphism of the line with $\mathbb{R}$.

It is left as an exercise to check that pairs $(p^{-1}(\pi(U)), \Phi)$ form an atlas on E, that with this differentiable structure p is smooth, and that Φ as above are local trivializations.

The bundle $\xi^1_{\mathbb{R}P^n}$ is called the *tautological line bundle* over $\mathbb{R}P^n$. When $n = 1$ this bundle is the Möbius band bundle.

Exercise 5.6. Show that every section $s : \mathbb{R}P^n \to E$ of this bundle must vanish somewhere. Hint: Consider the composition $S^n \to \mathbb{R}P^n \to E$, call it ϕ, and the map $S^n \to \mathbb{R}$ given by $x \mapsto < x, \phi(x) >$.

Exercise 5.7. In a similar fashion to $\xi^1_{\mathbb{R}P^n}$, there is a tautological k-dimensional bundle $\xi^k_{G_k(\mathbb{R}^n)}$ over the Grassmannian $G_k(\mathbb{R}^n)$, where the fiber over the k-plane $P \in G_k(\mathbb{R}^n)$ is P. Details are similar to the construction of $\xi^1_{\mathbb{R}P^n}$ and are left to the reader.

Exercise 5.8. Suppose an n-dimensional vector bundle $p : E \to B$ admits n linearly independent smooth sections. Show that this bundle is trivial. (You should argue that the map $B \times \mathbb{R}^n \to E$ constructed using these sections is a diffeomorphism.)

Definition 5.9. Let ξ^n and η^n be two bundles over the same base B. An *isomorphism* between the two bundles is a diffeomorphism $\Phi : E(\xi) \to E(\eta)$ that sends each $\xi(b)$ isomorphically onto $\eta(b)$.

Exercise 5.10. Suppose $\Phi : E(\xi) \to E(\eta)$ is a smooth map that sends each $\xi(b)$ isomorphically onto $\eta(b)$. Show that Φ is a diffeomorphism, and thus a bundle isomorphism.

Exercise 5.11. Let $C \subset B$ be a submanifold which is closed as a subset. Assume that we have a smooth section of $p : E \to B$ defined over C, i.e. we have a smooth map $s : C \to E$ such that $s(x) \in p^{-1}(x)$ for all $x \in C$. Show that s can be extended to a smooth section defined on all of B. Hint: partitions of unity.

Construction 5.12 (Restriction). Let ξ be a vector bundle over B and let B' be a submanifold of B (e.g. an open set). Then we can restrict ξ to a bundle over B', i.e. pass to ξ' with $\xi'(B) = B'$, $\xi'(E) = E' = p^{-1}(B')$ and $p' = p|E' : E' \to B'$. The restriction has the same dimension as ξ.

Construction 5.13 (Whitney sum). Let ξ_1, ξ_2 be two vector bundles over the same base B. The *Whitney sum* (or the *direct sum*) of ξ_1 and ξ_2 is the bundle $\xi = \xi_1 \oplus \xi_2$ obtained by restricting the product

$$E(\xi_1) \times E(\xi_2) \to B \times B$$

to the diagonal $\Delta = \{(b, b) \in B \times B\} \cong B$. Thus the fiber $\xi(b)$ is the direct sum $\xi_1(b) \oplus \xi_2(b)$.

5.1 Tangent bundle

Let X be a manifold and denote by TX the set of all pairs (x, v) such that $x \in X$ and $v \in T_x(X)$. There is the projection $p : TX \to X$, $p(x, v) = x$. Thus $p^{-1}(x) \cong T_x(X)$ has a natural vector space structure.

We will now equip TX with a manifold structure and show that $\tau_X = (TX, p, X)$ is a vector bundle, the *tangent bundle* of X.

Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be an atlas for X. For every chart (U_α, ϕ_α) of X we will define a chart $(p^{-1}(U_\alpha), d\phi_\alpha)$ of TX. Here

$$d\phi_\alpha : p^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$$

is defined by

$$d\phi_\alpha(x, v) = (\phi(x), T_x\phi_\alpha(v))$$

for $x \in U_\alpha$ and $v \in T_x(X)$. If $U\alpha$ and U_β overlap then the transition map $d\phi_\beta \circ d\phi_\alpha^{-1}$ is defined on

$$d\phi_\alpha(p^{-1}(U_\alpha \cap U_\beta)) = (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

and

$$d\phi_\beta \circ d\phi_\alpha^{-1}(y, w) = (\phi_\beta\phi_\alpha^{-1}(y), D_y(\phi_\beta\phi_\alpha^{-1})(w))$$

which is clearly smooth.

To see that $p : TX \to X$ is a vector bundle, note that $d\phi_\alpha$ is a local trivialization over U_α.

If $f : X \to Y$ is a smooth map we have a map $df : TX \to TY$, the differential (or the derivative) of f defined by

$$df(x, v) = (f(x), T_x f(v))$$

The map df sends a fiber $T_x(X)$ over $x \in X$ linearly to the fiber $T_{f(x)}(Y)$ over $f(x)$, and df is also smooth (exercise, write it down in charts). Any such map is called a *bundle map*.

From now on I will use notation df for the derivative, even when I mean at one point, i.e. notation $T_p f(v)$ is replaced by $df(v)$.

Example 5.14. When $U \subset \mathbb{R}^n$ is an open set, then we can identify $TU = U \times \mathbb{R}^n$. The jargon of vector bundles is really designed to make sense of patching various TU_α's together over charts.

Example 5.15. The tangent bundle TS^1 of the circle is trivial. To see this, according to Exercise 5.8, we only need to produce a nowhere zero vector field s on S^1. Recalling that $T_{(\cos t, \sin t)}S^1 = \{\lambda(-\sin t, \cos t) | \lambda \in \mathbb{R}\}$ we can take $s(\cos t, \sin t) = (-\sin t, \cos t)$.

Exercise 5.16. Show that there is a canonical diffeomorphism $T(X \times Y) = TX \times TY$ in the following sense. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be projections. Show that $dp_X \times dp_Y : T(X \times Y) \to TX \times TY$ is a diffeomorphism.

Definition 5.17. A *smooth vector field* on a manifold X is a smooth section of TX.

Exercise 5.18. Suppose that $f : X \to Y$ is a smooth map, that $Z \subset X$ is a submanifold of X closed as a subset, and that $f|Z : Z \to Y$ is a diffeomorphism. Also suppose that s is a smooth vector field on X. Show that the rule $t(f(z)) = df(s(z))$ defines a smooth vector field on Y. The vector field t is called the *push forward* of s.

Example 5.19. The tangent bundle TG of any Lie group G is trivial. This generalizes our observation about the circle, since S^1 is a Lie group. To see this, we will produce $n = \dim G$ vector fields on G that are linearly independent at every point. Start by choosing a basis $v_1, \cdots, v_n$ of $T_I G$ of the tangent space at the identity. Now we are going to "transport" each vector by left translations to each point of G. Let $g \in G$. Then we have the *left translation* $L_g : G \to G$ by g given by

$$L_g(x) = gx$$

Note that L_g is a diffeomorphism with the inverse $L_{g^{-1}}$. (Why is it smooth? It can be viewed as the restriction of the smooth multiplication $\mu : G \times G \to G$ to the submanifold $\{g\} \times G$.) In fact, the L_g's form a group isomorphic to G, the isomorphism being given by $g \mapsto L_g$.

Since $L_g(I) = g$ we can define a vector field s_i by $s_i(g) = dL_g(v_i)$. Since $dL_g : T_I G \to T_g G$ is an isomorphism, it's clear that $s_1(g), \cdots, s_n(g)$ are linearly independent. The only issue is the smoothness of the s_i's. This can be seen most easily from Exercise 5.18. The multiplication map $\mu : G \times G \to G$ is smooth and takes $G \times \{I\}$ diffeomorphically to G. Consider the vector field along $G \times \{I\}$ that to (g, I) assigns $(0, v_i) \in T_g(G) \times T_I(G) = T_{(g,I)}(G \times G)$ and show that μ pushes this vector field to s_i. Then apply Exercise 5.18.

Example 5.20. Manifolds with trivial tangent bundle are called *parallelizable*. Products of parallelizable manifolds are parallelizable. For example, tori are parallelizable.

There is a famous theorem of J.F. Adams (1962) that says that the sphere S^n is parallelizable if and only if $n = 1, 3$ or 7. The 3-sphere S^3 is a Lie group (you can think of it as $SU(2)$ or as the group of unit quaternions) and that's why it's parallelizable. The 7-sphere can be thought of as the space of unit Cayley numbers. Multiplication of Cayley numbers is not associative, but it turns out that a similar construction as in Example 5.19 shows that TS^7 is trivial (we didn't really use associativity of μ).

It's much harder to prove that S^n is not parallelizable when $n \neq 1, 3, 7$. For even n we will prove it this semester, and if at some point you take second year algebraic topology you will see a proof for $n \neq 2^k - 1$. The last case $n = 2^k - 1$ requires something called K-theory.

Example 5.21. Among compact connected surfaces (without boundary) the only parallelizable one is the torus (which is therefore the only one admitting the structure of a Lie group). We will see this later on in the course.

5.2 Subbundles, direct sums

Definition 5.22. Let $\xi = (E, p, B)$ be a vector bundle and $E' \subset E$ a submanifold. Let $p' : E' \to B$ be the restriction of p. We say that $\xi' = (E', p', B)$ is a *subbundle* of ξ if

- $E' \cap p^{-1}(b)$ is a linear subspace of $p^{-1}(b)$, and
- ξ' is a vector bundle with the subspace linear structure on fibers.

Exercise 5.23. Let $\xi = (E, p, B)$ be a smooth vector bundle and $s_1, \cdots, s_k$ are k smooth sections of ξ linearly independent at every point. Let

$$E' = \{e \in E | e \in span(s_1(p(e)), \cdots, s_k(p(e)))\}$$

Show that $p|E' : E' \to B$ is a subbundle of ξ. This is the *span* of $s_1, \cdots, s_k$. Hint: Locally construct $(n-k)$ smooth sections so that together with the s_i's they form a trivialization of ξ.

Definition 5.24. Let ξ be a vector bundle and ξ_1, ξ_2 two subbundles of ξ with the property that for every $b \in B$ we have $\xi(b) = \xi_1(b) \oplus \xi_2(b)$. We then say that ξ is the direct sum[9] of ξ_1 and ξ_2 and write $\xi = \xi_1 \oplus \xi_2$.

Exercise 5.25. Suppose $\xi = \xi_1 \oplus \xi_2$ and let $f : E(\xi) \to E(\xi_1)$ be defined by $f(e_1 + e_2) = e_1$ whenever $e_1 \in \xi_1(b)$ and $e_2 \in \xi_2(b)$. Then f is smooth. Hint: Locally find trivializations $s_1, \cdots, s_n$ so that first k are local sections of ξ_1 and the other $n-k$ of ξ_2.

Exercise 5.26. If $\xi = \xi_1 \oplus \xi_2$ then ξ is isomorphic to the Whitney sum of ξ_1 and ξ_2.

Proposition 5.27. *Let ξ_1, ξ_2, ξ_2' be subbundles of a bundle ξ. Suppose that $\xi = \xi_1 \oplus \xi_2 = \xi_1 \oplus \xi_2'$. Then $\xi_2 \cong \xi_2'$.*

Proof. Let $f : E(\xi) \to E(\xi_2')$ be the map from Proposition 5.25 and let $f' : E(\xi_2) \to E(\xi_2')$ be the restriction of f to $E(\xi_2)$. Then f' is smooth and it is an isomorphism on fibers, so f' is an isomorphism of bundles by Exercise 5.10. □

Construction 5.28 (Pull-back). Let $\xi = (E, p, B)$ be a vector bundle and $f : B' \to B$ a smooth map. We will construct a new bundle $f^*\xi = (E', p', B')$ over the base B' by "transplanting" the fibers from ξ: the fiber over $b' \in B'$ will be the fiber of ξ over $f(b')$. More precisely, define

$$E' = \{(e, b') \in E \times B' | p(e) = f(b')\}$$

[9] We already had the Whitney sum, which is really an *external* direct sum, while this is *internal*. The situation is analogous to external and internal direct sums of vector spaces.

with $p' : E' \to B'$ the projection to the second coordinate.

$$\begin{array}{ccc} E' & \to & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

Check that this is a vector bundle. It is called the *pull-back* of ξ by f. For example, when f is inclusion, $f^*\xi$ is the restriction of ξ.

Exercise 5.29. The pull-back of a trivial bundle is trivial.

Construction 5.30 (Dual bundle). Let $\xi = (E, p, B)$ be a vector bundle. We will construct a bundle $\xi^* = (E', p', B)$ over the same base with the fiber over $b \in B$ the dual vector space of $\xi(b)$. Recall that the *dual* of a vector space V (over $\mathbb{R}$) is $V^* = Hom(V, \mathbb{R})$. The operation $V \mapsto V^*$ is a contravariant functor, with $f : V \to W$ inducing $f^* : W^* \to V^*$ via

$$f^*(\phi)(v) = \phi(f(v))$$

for $\phi \in W^*$ and $v \in V$. For example, the dual $(\mathbb{R}^n)^*$ is naturally identified with $\mathbb{R}^n$ using the standard inner product: $\mathbb{R}^n \to (\mathbb{R}^n)^*$ given by $v \mapsto < v, \cdot >$ is an isomorphism. If $f : \mathbb{R}^m \to \mathbb{R}^n$ is represented by a matrix A then the dual (or adjoint) $f^* : \mathbb{R}^n \to \mathbb{R}^m$ is represented by the transpose $A^\top$.

We will define E', as a set, as the disjoint union

$$E' = \sqcup_{b \in B} (p^{-1}(b))^*$$

of duals of the fibers of p. Now we need an atlas on E'. For every trivializing open set $U \subset B$ and a trivialization $\Phi : p^{-1}(U) \to U \times \mathbb{R}^n$ we consider the fiberwise dual

$$\Phi^* : U \times (\mathbb{R}^n)^* \to p^{-1}(U)$$

and we take its inverse as a chart. It is easy to check that these charts are compatible, since the transition maps are obtained from the transition maps on E by passing to transposes in the $\mathbb{R}^n$-factor. Moreover, the inverses of Φ^* provide a local trivialization of $E' \to B$.

Example 5.31. The *cotangent bundle* of a manifold is the dual τ_X^* of the tangent bundle τ_X.

Remark 5.32. There are few other important constructions but we have to stop somewhere. If ξ_1 and ξ_2 are two bundles over the same base B one can construct the bundle $Hom(\xi_1, \xi_2)$ where the fiber over $b \in B$ is the linear space $Hom(\xi_1(b), \xi_2(b))$. For example, when ξ_2 is the trivial line bundle ϵ^1, $Hom(\xi_1, \epsilon^1)$ is just the dual ξ_1^*.

Likewise, one can construct the tensor product bundle $\xi_1 \otimes \xi_2$ whose fibers are tensor products of fibers of ξ_1 and ξ_2. For example, $Hom(\xi_1, \xi_2) \cong \xi^* \otimes \xi_2$.

5.3 Metric on a bundle

Definition 5.33. A *Euclidean* (or a *Riemannian*) metric on a smooth vector bundle $p : E \to B$ is an assignment of an inner product $< \cdot, \cdot >_b$ to each fiber $p^{-1}(b)$ in such a way that these assignments vary smoothly in the following sense: for any two smooth sections $s, s' : B \to E$ the function $B \to \mathbb{R}$ defined by $b \mapsto < s(b), s'(b) >_b$ is smooth.

Example 5.34. If $E = B \times \mathbb{R}^n$ is a trivial bundle, then we can define $g_{ij}(b) = < (b, e_i), (b, e_j) >_b$ (where the e_i's are the standard basis vectors in $\mathbb{R}^n$). Then the smoothness condition amounts to saying that the $g_{ij} : B \to \mathbb{R}$ are smooth. Of course, locally in a trivializing neighborhood we can do the same for any bundle.

Definition 5.35. A *Riemannian metric* on a manifold is a metric on its tangent bundle.

Example 5.36. A subbundle inherits a metric from the ambient bundle. A submanifold inherits a Riemannian metric from the ambient manifold. In particular, since every manifold embeds in some Euclidean space, every manifold admits a Riemannian metric.

Proposition 5.37. *Every vector bundle $p : E \to B$ admits a metric.*

Proof. Choose a metric $< \cdot, \cdot >_\alpha$ over every trivializing open set $U_\alpha \subset B$. Also choose a smooth partition of unity $\{\phi_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. Then define

$$< \cdot, \cdot >_b = \sum_\alpha \phi_\alpha(b) < \cdot, \cdot >_\alpha$$

□

Exercise 5.38. The dual bundle is isomorphic to the bundle: $\xi^* \cong \xi$.

Construction 5.39 (Gram-Schmidt). Recall the usual Gram-Schmidt construction in linear algebra that from a collection of linearly independent vectors in an inner product space produces an orthonormal collection. The same construction applies to linearly independent sections. Say $s_1, \cdots, s_k$ are linearly independent sections. Define $s'_1 = s_1/|s_1|$ and inductively, assuming $s'_1, \cdots, s'_i$ have already been defined, $a_j = < s'_j, s_{i+1} >$ (this is now a smooth function $B \to \mathbb{R}$) for $j = 1, \cdots, i$, then $t_{i+1} = s_{i+1} - a_1 s'_1 - \cdots - a_i s'_i$, and finally $s'_{i+1} = t_{i+1}/|t_{i+1}|$.

The important features of the new collection of smooth sections $s'_1, \cdots, s'_k$ are

- they are orthonormal at every point, and
- for every $1 \le i \le k$ the sections $s'_1, \cdots, s'_i$ have the same span at every point as $s_1, \cdots, s_i$.

Definition 5.40. Let $\xi = (E, p, B)$ be a smooth vector bundle equipped with a metric. Let $\xi_1 = (E_1, p, B)$ be a subbundle. The *orthogonal complement* of ξ_1 is the triple $\xi_1^\perp = (E_1^\perp, p, B)$ with

$$E_1^\perp = \cup_{b \in B}(p^{-1}(b) \cap E_1)^\perp$$

where the orthogonal complement in the fibers is taken with respect to the given metric.

Proposition 5.41. *The orthogonal complement of a subbundle is a subbundle.*

Proof. Say ξ is an n-dimensional bundle and ξ_1 is a k-dimensional subbundle. Fix some $b \in B$. In a neighborhood of b choose k sections $s_1, \cdots, s_k$ that span E_1. Then in a smaller neighborhood of b find sections $s_{k+1}, \cdots, s_n$ that together with $s_1, \cdots, s_k$ span E. Now apply Gram-Schmidt and replace these with an orthonormal family $s'_1, \cdots, s'_k, \cdots, s'_n$. Then $s'_{k+1}, \cdots, s'_n$ span $\xi_1^\perp$ so the statement follows from Exercise 5.23. □

Corollary 5.42. $\xi_1 \oplus \xi_1^\perp = \xi$. *In particular, the isomorphism type of the orthogonal complement does not depend on the choice of the metric.*

Remark 5.43. This means that one should be able to define the orthogonal complement without a choice of a metric. This is indeed possible – one can talk about the quotient bundle ξ/ξ_1 in which the fiber over $b \in B$ is the quotient vector space $\xi(b)/\xi_1(b)$. There is an induced map $E(\xi_1^\perp) \to E(\xi/\xi_1)$ which is an isomorphism of bundles.

Exercise 5.44. [10] There is an important fact that for every vector bundle $\xi = (E, p, B)$ there is another bundle $\eta = (E', p', B)$ over the same base such that $\xi \oplus \eta$ is a trivial bundle. According to the previous exercises this is the same as saying that ξ can be realized as a subbundle of a trivial bundle over B. We saw this for the case of the tangent bundle: If X is embedded in $\mathbb{R}^N$ then TX is a subbundle of the restriction of $T\mathbb{R}^N$ to X, which is a trivial bundle.

[10]This one is significantly harder than the others in this section.

(a) Show that the tautological bundle over the Grassmannian $G_k(\mathbb{R}^n)$ is a subbundle of a trivial bundle.

(b) Show that for any k-dimensional bundle ξ^k there is a smooth map $f : E(\xi) \to \mathbb{R}^N$ for some large N which linearly embeds each fiber. Hint: This is the hard part. The idea is similar to the Whitney embedding theorem. You may assume that B is covered by finitely many trivializing open sets U_i. Over each U_i we have $p^{-1}(U_i) \to \mathbb{R}^n$ which is an isomorphism on fibers. Then choose a partition of unity and combine these maps to one super-map with target a suitable cartesian product of the $\mathbb{R}^n$'s.

(c) Conclude that ξ is the pull-back of the tautological bundle and that it embeds in the trivial bundle.

5.4 Normal bundle

Definition 5.45. Let X be a submanifold of Y and fix a Riemannian metric on Y. The tangent bundle τ_X of X is a subbundle of the restriction $\tau_Y|X$ of the tangent bundle of Y to X. The *normal bundle* of X in Y is the orthogonal complement

$$\nu_{X\subset Y} = \tau_X^\perp \subset \tau_Y|X$$

Of course, the isomorphism type of $\nu_{X\subset Y}$ does not depend on the choice of the metric; it is purely a topological notion (this follows from Corollary 5.42). In particular,

$$\tau_X \oplus \nu_{X\subset Y} = \tau_Y|X$$

Example 5.46. The normal bundle $\nu_{S^n\subset\mathbb{R}^{n+1}}$ of the n-sphere in $\mathbb{R}^{n+1}$ is trivial (it is a line bundle and it admits a nowhere zero section that to each $x \in S^n$ assigns the outward unit normal at x. When τ_{S^n} is nontrivial we see a strange phenomenon that adding a trivial line bundle to a nontrivial bundle might make it trivial:

$$\tau_{S^n} \oplus \nu_{S^n\subset\mathbb{R}^{n+1}} = \tau_{\mathbb{R}^{n+1}}|S^n$$

Example 5.47. Consider the middle circle $C \subset M$ in the Möbius band $M = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$ corresponding to $t = 0$ (i.e. the 0-section of the Möbius band bundle). Thus

$$\tau_C \oplus \nu_{C\subset M} = \tau_M|C$$

Now τ_C is trivial while the other two bundles are nontrivial. The normal bundle is naturally isomorphic to the Möbius band bundle and we already saw that it is nontrivial. That $\tau_M|C$ is nontrivial will follow from the discussion of orientations.

Exercise 5.48. Show more generally that the normal bundle of the 0-section of a bundle ξ is isomorphic to ξ.