PHYSICAL REVIEW D
PARTICLES AND FIELDS

THIRD SERIES, VOLUME 29, NUMBER 4 15 FEBRUARY 1984

# Extension of the classical Cartan form

David E. Betounes
*Department of Mathematics, University of Southern Mississippi, Hattiesburg, Mississippi 39406-9207*
(Received 21 July 1983)

For a given Lagrangian $L$ a differential form $\theta_L^l$ is introduced which generalizes the classical notion of the Cartan form $\theta_L^1$. This extended Cartan form $\theta_L^l$ still gives rise to the same field equations (Euler-Lagrange equations), but has the added feature that $d\theta_L^l=0$ if and only if the Euler-Lagrange equations for $L$ vanish identically. This has the important consequence that the symmetries of a given Lagrangian $L$ are precisely those diffeomorphisms which transform $d\theta_L^l$ into itself, a property not shared by the classical form $d\theta_L^1$.

## I. INTRODUCTION

A rather surprising result, discussed in a recent paper of Hojman,[1] is that the characterization of those Lagrangians $L(x,u,u')$ for which the Euler-Lagrange equations vanish identically depends on the number $p$ of variables $x=(x_1,x_2,\ldots,x_p)$ and the number $q$ of fields $u=(u^1,u^2,\ldots,u^q)$. More precisely the result depends on the number $l=\min(p,q)$. Hojman's result is surprising not only because it was overlooked for so long (perhaps because $p=1$ or $q=1$ are the most commonly occurring cases) but also because it implies that the theory of symmetries of Lagrangians needs to be re-examined. Indeed in this theory the classical Cartan form $\theta_L^1$ determines the symmetries of $L$ according to the following line of reasoning:

$$\begin{aligned} g \text{ is a symmetry of } L &\Longleftrightarrow \text{the Euler-Lagrange equations for } L-g(L) \text{ vanish identically} \\ &\Longleftrightarrow d\theta^1_{L-g(L)}=0 \\ &\Longleftrightarrow g^{1*}(d\theta_L^1)=d\theta_L^1 \,. \end{aligned}$$

The first equivalence is essentially the definition of a symmetry. The third equivalence holds because the definition of $g(L)$ is rigged so that it does. Existing proofs[2] of the implication $\Longrightarrow$ in the second equivalence are invalid, and in fact the assertion is false, unless $p=1$ or $q=1$ (the reverse implication $\Longleftarrow$ is true, however, regardless of the values of $p$ and $q$). It is the purpose of this paper to show that this defect can be eliminated by introducing a new Cartan form $\theta_L^l$ in place of the classical one $\theta_L^1$.

Specifically a sequence $\theta_L^k$ $(k=1,2,\ldots,l)$ of differential forms is introduced and the following is shown:

(a) The Euler-Lagrange equations for an extremal $\gamma$ of $L$ are

$$j^1(\gamma)^*(X^1 \lrcorner d\theta_L^k)=0 \quad \forall X^1\in V(J^1(E)) \tag{1}$$

(notation will be explained later).

(b) $d\theta_L^k=0 \Longrightarrow d\theta_L^{k+1}=0$.

(c) $d\theta_L^k=0 \Longrightarrow$ the Euler-Lagrange equations for $L$ vanish identically.

(d) The Euler-Lagrange equations for $L$ vanish identically $\Longrightarrow d\theta_L^l=0$.

Equation (1) is the jet-bundle formulation of the Euler-Lagrange equations and in some ways is preferable to the classical partial differential equations for extremals. For example, it makes assertion (c) obvious, and also proves to be indispensable in discussion of symmetries. Saying that the Euler-Lagrange equations for $L$ vanish identically just means that Eq. (1) holds for every $\gamma$, or equivalently, and more simply, that equations (E1) and (E2) in Theorem D below hold.

Assertion (d) is the difficult one to prove. To prove it I first derive, in Sec. III, the formulas for the strict components of $\theta_L^k$ and $d\theta_L^k$, which are important in their own right.

Corollary 4 in Sec. IV shows that when $l>1$ the classical Cartan form $\theta^1$ does not determine all the symmetries of a given Lagrangian. The impact of this on the theory is outlined in Sec. V, with a full discussion of the details deferred to another paper.

## II. THE CARTAN FORM AND THE EULER-LAGRANGE EQUATIONS

This section introduces the Cartan form for a given Lagrangian and shows that it gives rise to the correct extremal equations. A sophisticated approach to these topics in the calculus of variations uses the notion of a fiber bundle of contact elements, a detailed account of which may be found in several of Hermann's books.[3] A large part of the work in this paper is done in local coordinates and so only a minimal amount of Hermann's fiber-bundle notation is included here.

One starts with a manifold $E$ of dimension $p+q$ which is a fiber bundle over some base space $N$ of dimension $p$. The cross sections $\gamma: N \to E$ have 1-jets $j^1(\gamma): N \to C^1(E,p)$ which are cross sections of $C^1(E,p)$, the bundle of first-order contact elements of $p$-dimensional submanifolds of $E$. The union of the images of all such 1-jets forms the jet bundle $J^1(E)$ and a Lagrangian $L$ is then any (smooth) map $L: J^1(E) \to R$. Locally one can introduce the natural jet-bundle coordinates $(x,u,u')$ where $x=(x_1,x_2,\ldots,x_p)$, $u=(u^1,u^2,\ldots,u^q)$, and $u'=\{u_i^\alpha\}_{i=1,\ldots,p}^{\alpha=1,\ldots,q}$. In a natural way one can think of $x$ as a coordinate system on $N$ and $(x,u)$ as a coordinate system on $E$. The coordinates $u'$ give the first-order partial derivatives of 1-jets: $u_i^\alpha(j^1(\gamma))=\partial\gamma^\alpha/\partial x^i$.

The Cartan form for $L$ is a certain differential $p$-form on $J^1(E)$, which is defined locally using the following differential forms on $J^1(E)$:

$$\pi = dx_1 dx_2 \cdots dx_p \ ,$$

$$\Phi^\alpha = du^\alpha - \sum_{j=1}^{p} u_j^\alpha dx_j \ .$$

Here and in the sequel the exterior product $\omega\wedge\xi$ of forms $\omega$ and $\xi$ is written as $\omega\xi$. One should also note that for any $\gamma$ the pullback of $\Phi^\alpha$ by $j^1(\gamma)$ to a form on $N$ is zero:

$$j^1(\gamma)^*(\Phi^\alpha)=0 \ . \tag{2}$$

Next let $l=\min(p,q)$ and suppose $1\le n\le l$. For indices $i_1,\ldots,i_n\in\{1,\ldots,p\}$ introduce the $(p-n)$-form

$$\pi_{i_1\cdots i_n}=e_{i_n}\lrcorner e_{i_{n-1}}\lrcorner\cdots e_{i_1}\lrcorner\pi \ ,$$

where $e_{i_r}=\partial/\partial x_{i_r}$ and the symbol $\lrcorner$ denotes contraction. This form is antisymmetric under permutation of the $i$'s and if $i_1<\cdots<i_n$ then

$$\pi_{i_1\cdots i_n}=\pm dx_1\cdots\overline{dx_{i_1}}\cdots\overline{dx_{i_n}}\cdots dx_p \ ,$$

where the overbar denotes deletion of the particular factor from the product and the $\pm$ sign depends on whether $i_1+\cdots+i_n-n(n-1)/2$ is even or odd. For indices $\alpha_1,\ldots,\alpha_n\in\{1,\ldots,q\}$ the $n$-form $\Phi^{\alpha_1}\Phi^{\alpha_2}\cdots\Phi^{\alpha_n}$ formed from the exterior product of the respective 1-forms $\Phi^\alpha$ is antisymmetric in the indices $\alpha$ and zero when pulled back by any $j^1(\gamma)$.

Finally, by introducing the $p$-forms

$$M_0=L\pi \ ,$$

$$M_n=\frac{1}{(n!)^2}\sum_{\substack{\alpha_1\cdots\alpha_n\\ i_1\cdots i_n}}\frac{\partial^n L}{\partial u_{i_1}^{\alpha_1}\cdots\partial u_{i_n}^{\alpha_n}}\Phi^{\alpha_1}\cdots\Phi^{\alpha_n}\pi_{i_1\cdots i_n} \tag{3}$$

one can define the $k$th Cartan form for $L$ by

$$\theta_L^k=M_0+M_1+\cdots+M_k \ , \tag{4}$$

where $1\le k\le l$. It should be noted that the association of $L$ with $\theta_L^k$ defines a linear mapping $\theta^k$ from the 0-forms on $J^1(E)$ (the Lagrangians) into the $p$-forms on $J^1(E)$ and so $\theta^k$ is naturally called the $k$th Cartan form. The 1st Cartan form $\theta^1$ is the classical one and the $l$th Cartan form $\theta^l$ will be referred to as *the Cartan form.* In the sequel the properties of $\theta^k$ for any $1\le k\le l$ will be described.

The first property of concern deals with extremals. Recall that the action integral with density $L$ is the functional on the cross sections $\gamma: N\to E$ defined by

$$\underline{L}(\gamma)=\int_N j^1(\gamma)^*(L\pi) \ .$$

Because of (2) one sees that

$$j^1(\gamma)^*(\theta_L^k)=j^1(\gamma)^*(L\pi)$$

and so the action integral is determined by $\theta_L^k$. The purpose of the first theorem is to show that the Euler-Lagrange equations for the extremals of $L$ may be expressed in terms of $\theta_L^k$. A well-known result, especially useful in the theory of symmetries, is that the classical Euler-Lagrange equations for an extremal $\gamma$ of $L$ may be expressed by

$$j^1(\gamma)^*(X^1\lrcorner d\theta_L^1)=0 \quad \forall X^1\in V(J^1(E)) \ ,$$

where $V(J^1(E))$ denotes a certain set of vector fields on $J^1(E)$.[4]

*Theorem A.* Suppose $k\ge 2$, then

$$j^1(\gamma)^*(X^1\lrcorner d\theta_L^k)=j^1(\gamma)^*(X^1\lrcorner d\theta_L^1)$$

for any Lagrangian $L$, any cross section $\gamma$, and any vector field $X^1$.

*Proof.* The idea of the proof is simple: For any $n\ge 2$, $M_n$ as given by Eq. (3) involves at least two $\Phi^\alpha$'s in $\Phi^{\alpha_1}\cdots\Phi^{\alpha_n}$ and consequently

$$j^1(\gamma)^*(X^1\lrcorner dM_n)=0 \ .$$

To show that this holds it suffices, because of the linear properties of the operators involved, to consider a typical summand of $M_n$. Such a summand has the form

$$G=\Phi^{\alpha_1}\Phi^{\alpha_2}\cdots\Phi^{\alpha_n}R \ .$$

The Lie derivative $\mathscr{L}_{X^1}$ may be used to write

$$X^1\lrcorner d=-d(X^1\lrcorner)+\mathscr{L}_{X^1} \ .$$

Now $j^1(\gamma)^*\mathscr{L}_{X^1}$ applied to $G$ gives zero for the following reasons: $\mathscr{L}_{X^1}$ is a derivation and so applying it to $G$ gives the sum of terms each of which contains at least one $\Phi^\alpha$. But then applying $j^1(\gamma)^*$ and using property (2) of $\Phi^\alpha$ as well as the property

$$j^1(\gamma)^*(\omega\wedge\xi)=j^1(\gamma)^*\omega\wedge j^1(\gamma)^*\xi$$

gives the desired conclusion. Next, since $X^1\lrcorner$ is an antiderivation similar reasoning (and commutability of $d$ with pullbacks) gives

$$j^1(\gamma)^*d(X^1\lrcorner)G=dj^1(\gamma)^*(X^1\lrcorner)G=0\ .$$

This proves the theorem.

*Corollary 1.* The Euler-Lagrange equations for an extremal $\gamma$ of $L$ are

$$j^1(\gamma)^*(X^1\lrcorner d\theta_L^k)=0\quad \forall X^1\in V(J^1(E))\ .$$

## III. EXPRESSIONS FOR $\theta_L^k$ AND $d\theta_L^k$

In this section the strict components of $\theta_L^k$ and $d\theta_L^k$ are computed. While the expressions derived are rather complicated notationally, the pattern they reflect is quite simple and is a necessary prerequisite for proving Theorem D in Sec. IV. The expressions involve the quantities

$$C_{l_1\cdots l_n}^{\beta_1\cdots\beta_n}=\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\frac{\partial^n L}{\partial u_{l_{\sigma 1}}^{\beta_1}\cdots\partial u_{l_{\sigma n}}^{\beta_n}}\ , \tag{5}$$

where $S_n$ is the group of permutations on the set $\{1,2,\ldots,n\}$ and $(-1)^\sigma$ is $\pm1$ according to whether $\sigma$ is an even or odd permutation.

*Theorem B.* The following formula holds:

$$\theta_L^k=\sum_{m=1}^{k+1}\sum_{\substack{\beta_m<\cdots<\beta_k\\ l_m<\cdots<l_k}}H_{l_m\cdots l_k}^{\beta_m\cdots\beta_k}du^{\beta_m}\cdots du^{\beta_k}\pi_{l_m\cdots l_k}\ , \tag{6}$$

where

$$H_{l_m\cdots l_k}^{\beta_m\cdots\beta_k}=\sum_{r=0}^{m-1}\frac{(-1)^r}{r!}\times\sum_{\substack{\beta_1\cdots\beta_r\\ l_1\cdots l_r}}(C_{l_1\cdots l_r l_m\cdots l_k}^{\beta_1\cdots\beta_r\beta_m\cdots\beta_k})u_{l_1}^{\beta_1}\cdots u_{l_r}^{\beta_r}\ . \tag{7}$$

In these expressions the following *notational conventions* hold:

(i) When $m=k+1$, $H_{l_m\cdots l_k}^{\beta_m\cdots\beta_k}=H$, $du^{\beta_m}\cdots du^{\beta_k}=1$, $\pi_{l_m\cdots l_k}=\pi$, and there is no sum over $\beta_m<\cdots<\beta_k$ and $l_m<\cdots<l_k$.

(ii) In $C_{l_1\cdots l_r l_m\cdots l_k}^{\beta_1\cdots\beta_r\beta_m\cdots\beta_k}$ one deletes $\beta_1\cdots\beta_r$ and $l_1\cdots l_r$ when $r=0$ and deletes $\beta_m\cdots\beta_k$ and $l_m\cdots l_k$ when $m=k+1$.

(iii) $C=L$.

(iv) $u_{l_1}^{\beta_1}\cdots u_{l_r}^{\beta_r}=1$ when $r=0$.

*Proof.* The proof consists of expanding the expression (3) for $M_n$, simplifying, and then collecting coefficients of like differentials in the sum (4). For notational convenience let

$$\begin{bmatrix}\alpha_1\cdots\alpha_n\\ i_1\cdots i_n\end{bmatrix}=\frac{1}{(n!)^2}\frac{\partial^n L}{\partial u_{i_1}^{\alpha_1}\cdots\partial u_{i_n}^{\alpha_n}}\ . \tag{8}$$

Also for $0<r<n$ let $T_{n,r}$ be the subset of $S_n$ consisting of those permutations $\tau$ such that

$$\tau 1<\cdots<\tau r\ \text{and}\ \tau(r+1)<\cdots<\tau n\ .$$

By convention take $T_{n,0}$ and $T_{n,n}$ to consist of only the identity permutation. Further, let $S_{n,r}=\{\phi\mid\phi^{-1}\in T_{n,r}\}$.

As a first step in the proof one can obtain the following formula by induction on $n$:

$$\Phi^{\alpha_1}\Phi^{\alpha_2}\cdots\Phi^{\alpha_n}=\sum_{r=0}^{n}(-1)^{nr}\sum_{\substack{j_1\cdots j_r\\ \tau\in T_{n,r}}}(-1)^\tau u_{j_1}^{\alpha_{\tau 1}}\cdots u_{j_r}^{\alpha_{\tau r}}(du^{\alpha_{\tau(r+1)}}\cdots du^{\alpha_{\tau n}})dx^{j_1}\cdots dx^{j_r}\ . \tag{9}$$

Then substituting this in expression (4) for $M_n$ and reindexing the sum on the $\alpha$'s by $\beta_1=\alpha_{\tau 1},\ldots,\beta_n=\alpha_{\tau n}$ one arrives at

$$M_n=\sum_{r=0}^{n}(-1)^{nr}\sum_{\substack{\beta_1\cdots\beta_n j_1\cdots j_r\\ i_1\cdots i_n}}\sum_{\phi\in S_{n,r}}(-1)^\phi\begin{bmatrix}\beta_{\phi 1}\cdots\beta_{\phi n}\\ i_1\cdots i_n\end{bmatrix}u_{j_1}^{\beta_1}\cdots u_{j_r}^{\beta_r}du^{\beta_{r+1}}\cdots du^{\beta_n}(dx^{j_1}\cdots dx^{j_r})\pi_{i_1\cdots i_n}\ . \tag{10}$$

Next the sum over $i_1\cdots i_n$, $j_1\cdots j_r$ in (10) can be simplified since $dx^{j_1}\cdots dx^{j_r}\pi_{i_1\cdots i_n}=0$ unless $i_1\cdots i_n$ are distinct and $\{j_1,\ldots,j_r\}$ is a subset of $\{i_1,\ldots,i_n\}$. In this situation the $j$'s have the form $j_1=i_{\tau\sigma 1},\ldots,j_r=i_{\tau\sigma r}$ for some $\tau\in T_{n,r}$ and some $\sigma\in S_r$. Then

$$(dx^{j_1}\cdots dx^{j_r})\pi_{i_1\cdots i_n}=(-1)^\sigma(dx^{i_{\tau 1}}\cdots dx^{i_{\tau r}})\pi_{i_1\cdots i_n}=[(-1)^{\sigma+\tau+nr+r}]\pi_{i_{\tau(r+1)}\cdots i_{\tau n}}\ . \tag{11}$$

Substituting this into (10) the sum on the $j$'s is replaced by sums on $\tau$ and $\sigma$. Then reindexing the sum on the $i$'s by $l_1=i_{\tau 1},\ldots,l_n=i_{\tau n}$ one arrives at

$$M_n=\sum_{r=0}^{n}(-1)^r\sum_{\substack{\beta_1\cdots\beta_n\\ l_1\cdots l_n}}\sum_{\substack{\phi,\psi\in S_{n,r}\\ \sigma\in S_r}}(-1)^{\phi+\psi+\sigma}\begin{bmatrix}\beta_{\phi 1}\cdots\beta_{\phi n}\\ l_{\psi 1}\cdots l_{\psi n}\end{bmatrix}(u_{l_{\sigma 1}}^{\beta_1}\cdots u_{l_{\sigma r}}^{\beta_r})(du^{\beta_{r+1}}\cdots du^{\beta_n})\pi_{l_{r+1}\cdots l_n}\ . \tag{12}$$

Now the sums on $l_1 \cdots l_r$ and $\sigma \in S_r$ can be switched and then for a given $\sigma$ the sum over $l_1 \cdots l_r$ can be reindexed $l_{\sigma 1} \to l_1, \ldots, l_{\sigma r} \to l_r$. Then $l_{\psi 1} \cdots l_{\psi n}$ becomes $l_{\hat{\sigma}^{-1}\psi 1} \cdots l_{\hat{\sigma}^{-1}\psi n}$. Here an expression like $\hat{\tau}$ denotes the natural extension of a $\tau \in S_r$ to a $\hat{\tau} \in S_n$, namely, take $\hat{\tau} = \tau$ on $\{1, \ldots, r\}$ and $\hat{\tau}$=identity map elsewhere. Having reindexed and noting that a sum over $\sigma \in S_r$ is the same as a sum over $\sigma^{-1} \in S_r$ one can see that the latter part of (12) may be replaced by

$$\begin{bmatrix} \beta_{\phi 1} \cdots \beta_{\phi n} \\ l_{\hat{\sigma}\psi 1} \cdots l_{\hat{\sigma}\psi n} \end{bmatrix} (u_{l_1}^{\beta_1} \cdots u_{l_r}^{\beta_r})(du^{\beta_{r+1}} \cdots du^{\beta_n})\pi_{l_{r+1} \cdots l_n} . \tag{13}$$

Now because of the antisymmetry of $\pi_{l_{r+1} \cdots l_n}$ the sum of (13) over all $l_{r+1} \cdots l_n$ is equivalent to the sum of

$$(-1)^{\tau} \begin{bmatrix} \beta_{\phi 1} \cdots \beta_{\phi n} \\ l_{\hat{\tau}\hat{\sigma}\psi 1} \cdots l_{\hat{\tau}\hat{\sigma}\psi n} \end{bmatrix} (u_{l_1}^{\beta_1} \cdots u_{l_r}^{\beta_r})(du^{\beta_{r+1}} \cdots du^{\beta_n})\pi_{l_{r+1} \cdots l_n} \tag{14}$$

over all $l_{r+1} < \cdots < l_n$ and all $\tau \in S_n^r$=the set of permutations of $\{r+1, \ldots, n\}$. However, one can show that each $\Delta \in S_n$ has a unique factorization of the form $\Delta = \hat{\tau}\hat{\sigma}\psi$ where $\tau \in S_n^r$, $\sigma \in S_r$, and $\psi \in S_{n,r}$. Consequently (14) together with these observations gives

$$M_n = \sum_{r=0}^{n} (-1)^r \sum_{\substack{\beta_1 \cdots \beta_n \\ l_1 \cdots l_r,\, l_{r+1} < \cdots < l_n}} \sum_{\substack{\phi \in S_{n,r} \\ \Delta \in S_n}} (-1)^{\phi + \Delta} \begin{bmatrix} \beta_{\phi 1} \cdots \beta_{\phi n} \\ l_{\Delta 1} \cdots l_{\Delta n} \end{bmatrix} (u_{l_1}^{\beta_1} \cdots u_{l_r}^{\beta_r})(du^{\beta_{r+1}} \cdots du^{\beta_n})\pi_{l_{r+1} \cdots l_n} . \tag{15}$$

However, the inner sum in (15) is the same as

$$\sum_{\substack{\phi \in S_{n,r} \\ \Delta \in S_n}} (-1)^{\phi + \Delta} \begin{bmatrix} \beta_1 \cdots \beta_n \\ l_{\Delta\phi^{-1}1} \cdots l_{\Delta\phi^{-1}n} \end{bmatrix} = \frac{n!}{r!(n-r)!} \sum_{\lambda \in S_n} (-1)^{\lambda} \begin{bmatrix} \beta_1 \cdots \beta_n \\ l_{\lambda 1} \cdots l_{\lambda n} \end{bmatrix}$$

$$= \frac{1}{r!(n-r)!} C_{l_1 \cdots l_n}^{\beta_1 \cdots \beta_n} . \tag{16}$$

Finally, substituting (16) into (15) and replacing the sum over $\beta_{r+1} \cdots \beta_n$ by sums over $\beta_{r+1} < \cdots < \beta_n$ and $\mu \in S_n^r$ one arrives at

$$M_n = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \sum_{\substack{\beta_1 \cdots \beta_r,\, \beta_{r+1} < \cdots < \beta_n \\ l_1 \cdots l_r,\, l_{r+1} < \cdots < l_n}} C_{l_1 \cdots l_n}^{\beta_1 \cdots \beta_n} (u_{l_1}^{\beta_1} \cdots u_{l_r}^{\beta_r})(du^{\beta_{r+1}} \cdots du^{\beta_n})\pi_{l_{r+1} \cdots l_n} . \tag{17}$$

The formula for $\theta_L^k$ now results from summing (17) as $n$ ranges from 0 to $k$ and collecting together all terms for which $n-r$ has the same value.

Having derived the strict components for $\theta_L^k$ it is now relatively straightforward to compute the strict components of $d\theta_L^k$ and so the proof of Theorem C is omitted for the sake of brevity. The components need to be expressed in a suitable form for use in Theorem D, so let the following quantities be introduced:

$$D_\alpha C_{l_1 \cdots l_n}^{\beta_1 \cdots \beta_n} = \frac{\partial C_{l_1 \cdots l_n}^{\beta_1 \cdots \beta_n}}{\partial u^\alpha} - \sum_{s=1}^{n} \frac{\partial C_{l_1 \cdots l_s \cdots l_n}^{\beta_1 \cdots \alpha \cdots \beta_n}}{\partial u^{\beta_s}} .$$

*Theorem C.* The following formula holds:

$$d\theta_L^k = \sum_{m=1}^{k+1} \left[ \sum_{\substack{\alpha, \beta_m < \cdots < \beta_k \\ i, l_m < \cdots < l_k}} A_{i\, l_m \cdots l_k}^{\alpha\beta_m \cdots \beta_k} du_i^\alpha du^{\beta_m} \cdots du^{\beta_k} \pi_{l_m \cdots l_k} + \sum_{\substack{\alpha < \beta_m < \cdots < \beta_k \\ l_m < \cdots < l_k}} B_{l_m \cdots l_k}^{\alpha\beta_m \cdots \beta_k} du^\alpha du^{\beta_m} \cdots du^{\beta_k} \pi_{l_m \cdots l_k} \right] , \tag{18}$$

where

$$A_{i\, l_m \cdots l_k}^{\alpha\beta_m \cdots \beta_k} = \sum_{r=0}^{m-1} \frac{(-1)^r}{r!} \sum_{\substack{\beta_1 \cdots \beta_r \\ l_1 \cdots l_r}} \left[ \frac{\partial C_{l_1 \cdots l_r l_m \cdots l_k}^{\beta_1 \cdots \beta_r \beta_m \cdots \beta_k}}{\partial u_i^\alpha} - {}^*C_{i\, l_1 \cdots l_r l_m \cdots l_k}^{\alpha\beta_1 \cdots \beta_r \beta_m \cdots \beta_k} \right] u_{l_1}^{\beta_1} \cdots u_{l_r}^{\beta_r} \tag{19}$$

and

$$B^{\alpha\beta_m\cdots\beta_k}_{\ l_m\cdots l_k}=\sum_{r=0}^{m-1}\frac{(-1)^r}{r!}\sum_{\substack{\beta_1\cdots\beta_r\\ l_1\cdots l_r}}\left[D_\alpha C^{\beta_1\cdots\beta_r\beta_m\cdots\beta_k}_{l_1\cdots l_r l_m\cdots l_k}-{}^*\sum_i\frac{\partial C^{\alpha\beta_1\cdots\beta_r\beta_m\cdots\beta_k}_{i\,l_1\cdots l_r l_m\cdots l_k}}{\partial x_i}-{}^*\sum_{\beta,i}\frac{\partial C^{\alpha\beta_1\cdots\beta_r\beta_m\cdots\beta_k}_{i\,l_1\cdots l_r l_m\cdots l_k}}{\partial u^\beta}u_i^\beta\right]\times u_{l_1}^{\beta_1}\cdots u_{l_r}^{\beta_r}\ . \tag{20}$$

The notational conventions from Theorem B are assumed to hold here, as well as the convention that the terms marked with an asterisk in (19) and (20) are to be omitted when $r=m-1$.

## IV. THE EQUATION $d\theta_L^k=0$

A great amount of effort has been spent in deriving the component expression for $d\theta_L^k$ in Sec. III, but now the main theorem, from which the desired results follow, can be easily proved.

*Theorem D.* The following two assertions are equivalent: (I) $d\theta_L^k=0$ and (II) the Euler-Lagrange equations for $L$ vanish identically, i.e.,

$$\frac{\partial^2 L}{\partial u_i^\alpha\partial u_j^\beta}+\frac{\partial^2 L}{\partial u_j^\alpha\partial u_i^\beta}=0\ , \tag{E1}$$

$$\frac{\partial L}{\partial u^\alpha}-\sum_i\frac{\partial^2 L}{\partial x_i\partial u_i^\alpha}-\sum_{\beta i}\frac{\partial^2 L}{\partial u^\beta\partial u_i^\alpha}u_i^\beta=0\ , \tag{E2}$$

and $L$ has nullity $k$, i.e.,

$$\frac{\partial^{k+1}L}{\partial u_i^\alpha\partial u_{l_1}^{\beta_1}\cdots\partial u_{l_k}^{\beta_k}}=0\ . \tag{E3}$$

*Proof.* (I) $\Longrightarrow$(II): This is the obvious part, at least if one works with the jet-bundle formulation, namely, $\gamma$ is an extremal if

$$j^1(\gamma)^*(X^1\lrcorner d\theta_L^k)=0\quad\forall X^1\in V(J^1(E)) \tag{21}$$

and so clearly every $\gamma$ is an extremal since $d\theta_L^k=0$. Thus, (E1) and (E2) hold. Since (E1) says that $\partial^2L/\partial u_i^\alpha\partial u_j^\beta$ is antisymmetric under an interchange $i$ and $j$ one has according to (5) that for every $n$

$$C^{\beta_1\cdots\beta_n}_{l_1\cdots l_n}=\partial^n L/(\partial u_{l_1}^{\beta_1}\cdots\partial u_{l_n}^{\beta_n})\ . \tag{22}$$

In Eq. (19) from Theorem C the $m=1$ coefficient is

$$0=A_{i\,l_1\cdots l_k}^{\alpha\beta_1\cdots\beta_k}=\partial C^{\beta_1\cdots\beta_k}_{l_1\cdots l_k}/\partial u_i^\alpha\ . \tag{23}$$

Hence, (E3) holds. A direct and very concrete proof of this part of the theorem can be obtained without using the abstraction of the jet bundle, Eq. (21), by merely writing out a few of the coefficients $A$, $B$ of $d\theta_L^k$ for $m=1,2,3\ldots$ . By looking at the $A$'s and working down to $m=k$ one finds that (E1) holds. Then (22) above holds and so by (23) one can see that (E3) holds. By looking at the $B$'s and working down to $m=k+1$ one finds that (E2) holds.

(II) $\Longrightarrow$(I): This is the unobvious part of the theorem. It says that if every $\gamma$ satisfies Eq. (21) and if (E3) holds then $d\theta_L^k=0$. The author knows of no way to prove this which circumvents first computing the strict components of $d\theta_L^k$ as was done in Theorem C. However, having these components one sees that they are zero as follows. Because (E1) holds one can see that (22) above holds and hence the expression in the square brackets in Eq. (19) from Theorem C is zero. In the exceptional case when $r=m-1$ the expression in the square brackets consists of only the first term, but this is zero because of the nullity assumption (E3). Hence, all the $A$ components of $d\theta_L^k$ are zero. To see that all the $B$ components are zero, let $F^\alpha$ stand for the expression on the left-hand side of Eq. (E2). By taking derivatives and using Eq. (22) one finds that

$$0=\frac{\partial^n F^\alpha}{\partial u_{l_1}^{\beta_1}\cdots\partial u_{l_n}^{\beta_n}}=D_\alpha C^{\beta_1\cdots\beta_n}_{l_1\cdots l_n}-\sum_i\partial C^{\alpha\beta_1\cdots\beta_n}_{i\,l_1\cdots l_n}/\partial x_i-\sum_{\beta,i}(\partial C^{\alpha\beta_1\cdots\beta_n}_{i\,l_1\cdots l_n}/\partial u^\beta)u_i^\beta\ . \tag{24}$$

Thus, the expression in the square brackets in Eq. (20) from Theorem C is zero. In the exceptional case when $r=m-1$ the expression in the square brackets consists of only the first term, but this is zero [to see this take $n=k$ in Eq. (24) and note that because of (E3), $C^{\alpha\beta_1\cdots\beta_k}_{i\,l_1\cdots l_k}=0$ and so the latter two terms on the right-hand side are zero].

*Corollary 2.* If $d\theta_L^k=0$ then $d\theta_L^{k+1}=0$ (assuming of course that $k+1\le l$).

*Corollary 3.* $d\theta_L^l=0$ if and only if the Euler-Lagrange equations for $L$ vanish identically.

*Proof.* Applying Theorem D with $k=l=\min(p,q)$ one sees that the nullity condition (E3) is redundant since it is implied by the antisymmetry condition (E1), namely, $\partial^{l+1}L/\partial u_i^\alpha\partial u_{i_1}^{\alpha_1}\cdots\partial u_{i_l}^{\alpha_l}$ is antisymmetric under an interchange of either two $\alpha$'s or two $i$'s and so it must be zero since either $\alpha,\alpha_1,\ldots,\alpha_l$ or $i,i_1,\ldots,i_l$ has at least one element listed twice.

From an intuitive standpoint one can see (by inspecting the proof of Theorem D) how the Euler-Lagrange equations for $L$ can vanish identically, yet $d\theta_L^k$ is not necessarily zero for any $k<l$. Namely, (E1) and (E2) yield Eqs.

(22) and (24) as direct consequences. Because of this the components of $d\theta_L^k$ given by Eqs. (19) and (20) reduce to just the $r=m-1$ terms and these depend only on $\partial^{k+1}L/(\partial u_i^\alpha \partial u_{l_1}^{\beta_1} \cdots \partial u_{l_k}^{\beta_k})$ and on $u_{l_1}^{\beta_1} \cdots u_{l_{m-1}}^{\beta_{m-1}}$. Thus, $d\theta_L^k$ is not necessarily zero since its components do not involve the partial derivatives of $L$ to a high enough order.

It also seems appropriate to point out here why the proof in Ref. 2 (showing that $d\theta_L^1=0$ whenever the Euler-Lagrange equations vanish identically) is invalid. The mistake is that instead of (E1) the stronger form of it,

$$\frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta}=0 ,$$

is incorrectly derived as a consequence of the identical vanishing of the Euler-Lagrange equations.

*Corollary 4.* Suppose $l>1$ and $1 \le k < l$. Then there exist Lagrangians $L$ for which the Euler-Lagrange equations vanish identically and yet $d\theta_L^k \neq 0$.

*Proof.* By inspecting Hojman's characterization of those Lagrangians $L$ for which the Euler-Lagrange equations vanish identically one can easily pick such an $L$ which does *not* have nullity $k$ and hence by Theorem D one has that $d\theta_L^k \neq 0$. (Actually Hojman's work requires that $L$ *not* depend on $x$, but his results can be extended to cover this case.)

Corollaries 3 and 4 show that $\theta^l$ is the correct choice for *the* Cartan form. Another way to phrase these corollaries is to say that the set of Lagrangians characterized by Hojman is the kernel of the linear map $L \to d\theta_L^l$ but is not the kernel of $L \to d\theta_L^k$ for any $k<l$. Of course this latter kernel is easy to identify using Theorem D and Hojman's results.

*Example 1.* As an example for the case $p=2=q$ consider the Lagrangian

$$L=u_1^1 u_2^2 - u_2^1 u_1^2 .$$

One can easily check that the Euler-Lagrange equations for $L$ vanish identically. For such a simple example let $\theta_L^1$, $\theta_L^2$ be computed directly from the definition in Eq. (4) and then $d\theta_L^1$, $d\theta_L^2$ follow easily from the differential calculus. (For more complicated examples the computations become so tedious that one might as well do them in general as in Theorems B and C.) First

$$\theta_L^1=\left[L-\frac{\partial L}{\partial u_1^1}u_1^1-\frac{\partial L}{\partial u_2^1}u_2^1-\frac{\partial L}{\partial u_1^2}u_1^2-\frac{\partial L}{\partial u_2^2}u_2^2\right]dx^1dx^2+\frac{\partial L}{\partial u_1^1}du^1dx^2+\frac{\partial L}{\partial u_1^2}du^2dx^2-\frac{\partial L}{\partial u_2^1}du^1dx^1-\frac{\partial L}{\partial u_2^2}du^2dx^1$$

$$=-(u_1^1u_2^2-u_2^1u_1^2)dx^1dx^2+u_2^2du^1dx^2-u_2^1du^2dx^2+u_1^2du^1dx^1-u_1^1du^2dx^1 .$$

Consequently

$$d\theta_L^1=(-u_1^1du_2^2-u_2^2du_1^1+u_2^1du_1^2+u_1^2du_2^1)dx^1dx^2+du_2^2dx^1dx^2-du_2^1du^2dx^2+du_1^2du^1dx^1-du_1^1du^2dx^1$$

$$\neq 0 .$$

Next, to compute $\theta_L^2$ first compute

$$M_2=\frac{1}{4}\sum_{\substack{\alpha_1\alpha_2\\ i_1i_2}}\frac{\partial^2 L}{\partial u_{i_1}^{\alpha_1}\partial u_{i_2}^{\alpha_2}}\left[du^{\alpha_1}-\sum_j u_j^{\alpha_1}dx^j\right]\left[du^{\alpha_2}-\sum_k u_k^{\alpha_2}dx^k\right]\pi_{i_1i_2} .$$

After multiplying this out and using $\pi_{12}=1$, $\pi_{21}=-1$, and otherwise $\pi_{i_1i_2}=0$ one gets $M_2=du^1du^2-\theta_L^1$. Thus, $\theta_L^2=du^1du^2$ and consequently $d\theta_L^2=0$.

## V. CONCLUSION

Arguments have been put forth to indicate that $\theta^l$ is the correct generalization of the classical Cartan form $\theta^1$. Indeed the whole sequence $\theta^k$ ($k=1,2,\ldots,l$) proves to be useful in the theory. The main results consisted of (a) deriving formulas for the strict components of $\theta_L^k$, $d\theta_L^k$ and (b) showing that the kernel of the linear map $L \to d\theta_L^k$ is precisely the set $Z_k$ of Lagrangians satisfying conditions (E1), (E2), and (E3) in Theorem D. Hojman's result explicitly determines the form of the Lagrangians in $Z_l$ and this can be extended to $Z_k$. These results are interesting *per se*, but their real importance is in regard to symmetries and conservation laws. An extension of the classical theory ($l=1$) on symmetries will be detailed in a sequel to this paper, but an outline of the prominent features of this extension can be predicted here.

The symmetry group $G$ for a given Lagrangian $L$ consists of those transformations (bundle maps) $g: E \to E$ such that the Euler-Lagrange equations for $L-g(L)$ vanish identically. The Cartan forms $\theta_L^k$ determine a natural decomposition of $G$ as follows. Let $\theta_L^0=L\pi$ and for $k=0,1,2,\ldots,l$ let $G_k$ be the group of transformations $g$ for which $g^{1*}(d\theta_L^k)=d\theta_L^k$. Then one gets a chain of subgroups

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_l = G$$

in which $G_1$ consists of the symmetries determined by the classical Cartan form and $G_0$ consists of symmetries which leave $L$ form invariant. To verify these assertions using the results of this paper one must first prove that

$$g^{1*}(\theta_L^k)=\theta_{g(L)}^k \tag{25}$$

for any $k$ and any $g$. For $k=0$ this is just the definition of $g(L)$ and for $k=1$ it is the classical result proved by Hermann.[5] Using (25) one sees that saying $g^{1*}d\theta_L^k=d\theta_L^k$ is equivalent to saying

$$d\theta_{L-g(L)}^k=0 \; .$$

The infinitesimal symmetries and conserved currents are determined much like those for the classical case: A vector field of the form

$$X=\sum \xi^i(x)\frac{\partial}{\partial x^i}+\sum \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

on $E$ is a $k$th-order infinitesimal symmetry of $L$ if its flow $\{g_t\}_{t\in R}$ has each $g_t$ belonging to $G_k$. In terms of Lie derivatives this means that

$$\mathscr{L}_{X^1}(d\theta_L^k)=0 \; . \tag{26}$$

However, one can see, after first proving the identity

$$\mathscr{L}_{X^1}\theta_L^k=\theta^k_{\mathscr{L}_{X^1}(L)+\mathrm{div}(\xi)L} \; ,$$

that condition (26) is the same as

$$d\theta^k_{\mathscr{L}_{X^1}(L)+\mathrm{div}(\xi)L}=0$$

and this, according to the results of this paper, is equivalent to the condition

$$\mathscr{L}_{X^1}(L)+\mathrm{div}(\xi)L\in Z_k \; . \tag{27}$$

Conserved currents for such infinitesimal symmetries $X$ are determined in the customary fashion, namely, because of Eq. (26) there exists (via Poincaré's lemma) a $(p-1)$-form $\omega'$ such that

$$d\omega'=\mathscr{L}_{X^1}(\theta_L^k)=d(X^1\lrcorner\theta_L^k)+X^1\lrcorner d\theta_L^k \; . \tag{28}$$

Then $\omega\equiv\omega'-X^1\lrcorner\theta_L^k$ is a conserved current for $L$ since if $\gamma$ is an extremal of $L$ then

$$dj^1(\gamma)^*\omega=j^1(\gamma)^*d\omega=j^1(\gamma)^*(X^1\lrcorner d\theta_L^k)=0$$

(the last equation being the Euler-Lagrange equation).

One can thus see that for $l>1$ there is a broader class of symmetries, infinitesimal symmetries, and conservation laws than that predicted by the classical Cartan form. This can be illustrated by examples (such as example 1 above) which are very simple computationally but regrettably are not very meaningful physically. On the other hand, the following example for $l=4$ is quite important but the computations and details are somewhat lengthy (and so will only be summarized here).

*Example 2.* Consider the electromagnetic Lagrangian

$$\begin{aligned}L=\tfrac{1}{2}[&(A_t^1+\phi_x)^2+(A_t^2+\phi_y)^2+(A_t^3+\phi_z)^2\\&-(A_y^3-A_z^2)^2-(A_z^1-A_x^3)^2-(A_x^2-A_y^1)^2\\&-(A_x^1+A_y^2+A_z^3+\phi_t)^2] \; ,\end{aligned}$$

which (with a slight abuse of notation) could be rewritten using $x=(x_0,x_1,x_2,x_3)\;=(t,x,y,z)$, $u=(A^0,A^1,A^2,A^3)=(\phi,A^1,A^2,A^3)$, and $u_i^\alpha=A_{x_i}^\alpha$. To look for infinitesimal ($k$th-order) symmetries $X$ of $L$ one should take the route indicated above to reduce the problem to condition (27). Suppose the components of $X$ are assumed to have the form $\xi(x)=Rx$ and $\eta(x,u)=Su$ where $R$ and $S$ are $4\times4$ matrices. It can be shown that in this case $X$ is an infinitesimal symmetry if and only if $R=\lambda I+T$ and $S=-\lambda I+V$ where $\lambda$ is a real number, $I$ the $4\times4$ identity matrix, and $T$, $V$ are matrices in the Lie algebra of the Lorentz group. Furthermore

$$\begin{aligned}L^\#&\equiv\mathscr{L}_{X^1}L+\mathrm{div}(\xi)L\\&=\sum_{i\neq j}\sum_\alpha(V-T)_{ij}(u_\alpha^\alpha u_j^i-u_j^\alpha u_\alpha^i) \; ,\end{aligned} \tag{29}$$

where $(V-T)_{ij}$ denotes the $ij$th entry of the matrix $V-T$. This [together with Eq. (27) and an extension of Hojman's work to cover $Z_k$] shows that an infinitesimal symmetry $X$ of this type is either zeroth order (when $T=V$) or second order (when $T\neq V$). Of course the symmetry group generated by $X$ corresponds to a transformation of the variables $x$ and fields $u$ by Lorentz transformations $\widetilde{T}$ and $\widetilde{V}$. The zeroth-order group ($\widetilde{T}=\widetilde{V}$) leaves $L$ form invariant and leads to familiar conservation laws of the form

$$\mathrm{div}[W(\gamma)]=0 \; , \tag{30}$$

where $W(\gamma)$ is the vector field on $N$ such that

$$W(\gamma)\lrcorner\pi=j^1(\gamma)^*(X^1\lrcorner\theta_L^k)=j^1(\gamma)^*(X^1\lrcorner\theta_L^1) \; .$$

The second-order symmetries ($\widetilde{T}\neq\widetilde{V}$) allow for the variables and fields to be transformed independently of one another and lead to conservation laws of the form

$$\mathrm{div}[W(\gamma)]=\mathrm{div}[\Omega(\gamma)] \; , \tag{31}$$

where $\Omega(\gamma)$ is the vector field on $N$ such that

$$\Omega(\gamma)\lrcorner\pi=j^1(\gamma)^*(\omega')$$

and $\omega'$ is an antidifferential of the extended Cartan form, i.e., $d\omega'=\theta^2_{L^\#}$ with $L^\#$ given by Eq. (29). An alternative expression for the conservation law (31) is

$$\mathrm{div}[W(\gamma)]=L^\#(\gamma) \; . \tag{32}$$

Here $L^\#(\gamma)=j^1(\gamma)^*(L^\#)$ is just expression (29) evaluated at the particular extremal $\gamma$. Previous treatments[6] have derived conservation laws of the form (31), but have incorrectly deduced (as Hojman pointed out) that $\Omega(\gamma)$ depended on $x$ and $u$ but not the $u_i^\alpha$'s (as would be the case if the classical Cartan form determined all the symmetries). Under these circumstances $L^\#(\gamma)=\mathrm{div}[\Omega(\gamma)]$ would be linear in the $u_i^\alpha$'s. However, the example at hand explicitly exhibits second-order symmetries for which $L^\#(\gamma)$ is quadratic in the $u_i^\alpha$'s.

[1]S. Hojman, Phys. Rev. D 27, 451 (1983).
[2]See, for example, R. Hermann, *Geometry, Physics, and Systems* (Marcel Dekker, New York, 1973), Theorem 3.1, p. 181.
[3]See Ref. 2; R. Hermann, *Differential Geometry and the Calculus of Variations* (Academic, New York, 1968). The ideas and notation from Ref. 2, pp. 173—199 are especially pertinent for this paper.
[4]See Ref. 2, p. 181.
[5]See Ref. 2, p. 185, Eq. (4.6).
[6]E. L. Hill, Rev. Mod. Phys. 23, 253 (1951), see Eq. (43).