# Notes on linear and affine connections

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**Abstract**

These notes are based on the material contained in Chapter 3 of [2]. Some topics from Chapters 1 and 2 are reviewed. The bundle of linear frames is defined and it is shown that the tangent bundle of a manifold can be considered as a bundle associated with the linear frame bundle with $\mathbb{R}^n$ as the standard fibre. Parallel transport in vector bundle is described and it is explained how it gives rise to the notion of covariant differentiation. Linear connections are defined and the properties of the corresponding covariant derivative are described. Finally, affine connections are introduced and are shown to be in a one-to-one correspondence with linear connections.

## 1. Some preliminaries

In this section we review some of the ideas and results from chapters 1 and 2 of Kobayashi and Nomizu [2] (hereafter referred to as KN) that will be relevant to us.

**1.1. Homomorphisms of principal fibre bundles.** Some remarks on notation are in order before we start reviewing the material. We shall represent a typical principal fibre bundle by $P(M, G)$. Here it is understood that $P$ and $M$ are differentiable manifolds and $G$ acts on $P$ via a *right* action $\Phi : P \times G \to P$ such that $P/G = M$. The canonical projection map is represented by $\pi : P \to M = P/G$. Given $u \in P$ and $g \in G$, we define maps $\Phi_g : P \to P$ and $\Phi_u : G \to P$ by $\Phi_g(u) := \Phi(u, g) =: \Phi_u(g) = R_g(u) = u \cdot g$ (the last bit of notation is specific to right actions; for left actions we write $\Phi_g(u) = g \cdot u$). In this sequel we shall try to be as explicit in our notation as possible although from time to time we shall point out the intuition behind the notation used in KN. To keep things under control, we shall allow ourselves some (mild) abuse of notation (like writing $ug$ instead of $u \cdot g$ for example) but (unlike KN) we won't do so without warning.

We review some material from Section 5 of Chapter 1 here. A ***homomorphism*** of a principal fibre bundle $P'(M', G')$ into another principal fibre bundle $P(M, G)$ is a triple $(f, f_1.f_0)$ such that $f$ is a fibre bundle map from $P'$ to $P$ over $f_1 : M' \to M$ and

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$f_0 : G' \to G$ is a homomorphism such that $f(R_{a'}u) = f(u'a') = R_{f_0(a)}f_1(u) = f_1(u)f_0(a)$. Since $(f, f_1)$ is a fibre bundle map, the following diagram commutes.

$$\begin{array}{ccc} P' & \xrightarrow{f} & P \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow[f_1]{} & M \end{array}$$

where $\pi' : P' \to M'$ and $\pi : P \to M$ are the natural projections. A homomorphism $(f, f_1, f_0) : P'(M', G') \to P(M, G)$ is called an ***imbedding*** if $f : P' \to P$ is an imbedding and if $f_0 : G' \to G$ is injective. In this case $f_1 : M' \to M$ is also an imbedding. We can therefore consider $f(P')(f_1(M'), f_0(G'))$ (that is, the image of $P'(M', G')$ under $(f, f_1, f_0)$) a subbundle of $P(M, G)$. If, moreover, $M' = M$ and $f_1 : M' \to M$ is the identity map of $M$, we call $(f, f_1, f_0) : P'(M', G') \to P(M, G)$ a ***reduction*** of the structure group $G$ of $P(M, G)$ to $G'$. The subbundle $f(P')(M, f_0(G'))$ is called a ***reduced subbundle***. Given $P(M, G)$ and a Lie subgroup $G'$ of $G$, we shall say that the structure group $G$ is reducible to $G'$ if there is a reduced subbundle $P'(M, G')$.

1.1 REMARKS: (i) Note that when $(f, f_1, f_0)$ is an imbedding, KN call $P'(M', G')$ itself a subbundle of $P(M, G)$. This makes sense if one identifies $P'(M', G')$ with its image $f(P')(f_1(M'), f_0(G'))$.

(ii) KN represent $f, f_0$ and $f_1$ by the same symbol $(f)$, the meaning being clear from context. This definitely saves the trouble of writing a triple for a homomorphism. We shall sometimes represent a homomorphism between principal fibre bundles simply by $f$ when the maps $f_0$ and $f_1$ are not relevant to the discussion or are clear from the context. We shall, however, never use the same notation for all of the three maps.

**1.2. Associated bundles.** Let $P(M, G)$ be a principal fibre bundle and let $F$ be a manifold on which $G$ acts on the left:

$$\begin{aligned} \Psi : G \times F &\to F \\ (g, \xi) &\mapsto \Psi(g, \xi) = g \cdot \xi. \end{aligned}$$

Define a *right* action of $G$ on $P \times F$ as follows:

$$\begin{aligned} (P \times F) \times G &\to (P \times F) \\ ((u, \xi), g) &\mapsto (\Phi(u, g), \Psi(g^{-1}, \xi)) \\ &\overset{KN}{=} (ug, g^{-1}\xi) \end{aligned}$$

Denote by $E := (P \times F)/G =: P \times_G F$ the quotient of $P \times F$ by $G$ and the projection onto $E$ by $\pi_G : P \times F \to E$. Given $(u, \xi) \in P \times F$ , we know that $\pi_G(u, \xi)$ is the equivalence class (defined by the action of $G$ on $P \times F$) containing $(u, \xi)$. We denote this equivalence class by $[u, \xi]_G$ (that is, $\pi_G(u, \xi) = [u, \xi]_G$; we follow Marsdenesque notation here). Define a map $\pi_E : E \to M$ by

$$\pi_E([u, \xi]_G) = \pi(u)$$

Now, it can be shown that $E$ has a differentiable structure that makes $\pi_E$ a surjective submersion. We shall not address this issue here and consider it a fact[1].

The upshot of this is that $\pi_E : E \to M$ is a (*locally trivial*) fibre bundle with standard fibre $F$ and we call it the ***bundle associated with*** $\boldsymbol{P(M, G)}$ ***with standard fibre*** $\boldsymbol{F}$. Following KN, we shall denote this bundle by $E(M, F, P, G)$. Sometimes (to avoid the use of excessive language) we shall call $E(M, F, P, G)$ the associated bundle (rather than "the bundle associated with $P(M, G)$ with standard fibre $F$") whenever the underlying principal fibre bundle and the standard fibre are understood to be $P(M, G)$ and $F$ respectively. This should cause no confusion. The following result is immediate once the notation is understood properly.

1.2 PROPOSITION: *Let $P(M, G)$ be a principal fibre bundle and $F$ a manifold on which $G$ acts on the left. Let $E(M, F, G, P)$ be the associated bundle. For each $u \in P$ and $\xi \in F$, write $[u, \xi]_G \overset{KN}{=} u\xi \in E$. Then each $u \in P$ is a mapping of $F$ onto $F_x = \pi_E^{-1}(x)$ where $x = \pi(u)$ and*

$$(ug)\xi = u(g\xi) \qquad \text{for } g \in G, u \in P, \xi \in F \tag{1.1}$$

1.3 REMARK: For $u \in P$ and $x = \pi(u)$, the map $u : F \to F_x$ is given by $u\xi = [u, \xi]_G$ and thus it is easy to see that $[ug, \xi]_G = [u, g\xi]_G$, which is (1.1). We shall use the notation "$u\xi$" and "$[u, \xi]_G$" interchangeably, depending on context.

**Associated vector bundles.** Let $P(M, G)$ be a principal fibre bundle and $\rho$ a representation of $G$ into $GL(n; \mathbb{R})$. Let $E(M, \mathbb{R}^n, G, P)$ be the associated bundle with standard fibre $\mathbb{R}^n$ on which $G$ acts through $\rho$. We shall call this associated bundle a ***vector bundle over*** $\boldsymbol{M}$. Each fibre $\pi_E^{-1}(x), x \in M$ has the structure of a vector space such that (see Proposition 1.2) every $u \in P$ with $\pi(u) = x$ considered as a mapping from $\mathbb{R}^n$ to $\pi_E^{-1}(x)$ is a linear isomorphism. Restating this in our notation, this means that given $[u, \xi]_G, [u, \xi_1]_G, [u, \xi_2]_G \in \pi_E^{-1}(x)$ where $\pi(u) = x$ and $c \in \mathbb{R}$, the vector space structure is given by

$$c[u, \xi]_G = [u, c\xi]_G \qquad \text{and} \quad [u, \xi_1]_G + [u, \xi_2]_G = [u, \xi_1 + \xi_2]_G$$

It clear from Proposition 1.2 that vector addition and scalar multiplication are well-defined operations. That this definition is equivalent to the "usual" definition of a vector bundle is not immediate here. We shall touch upon this issue in Section 1.4.

[1] If we ever get down to writing notes for Chapter 1, this will be discussed in detail

**Tensorial forms and associated bundles.** Recall that given a principal fibre bundle $P(M,G)$ and a representation $\rho$ of $G$ on a finite-dimensional vector space $V$, a ***pseudotensorial form of degree $r$ on $P$ of type $(\rho, V)$*** is a $V$-valued $r$-form $\varphi$ on $P$ such that

$$R_a^*\varphi = \rho(a^{-1}) \cdot \varphi, \qquad a \in G$$

A pseudotensorial form of degree $r$ on $P$ of type $(\rho, V)$ is called a ***tensorial form*** if it is horizontal in the sense that $\varphi(X_1, \ldots, X_n) = 0$ whenever $X_i = 0$ for at least one $i \in \{1, \ldots, n\}$.

Now, given $P(M,G)$ and $\rho$ on $V$, consider the associated bundle $E(M, V, G, P)$ with natural fibre $V$ on which $G$ acts by $\rho$. A tensorial form $\varphi$ of degree $r$ of type $(\rho, V)$ can be regarded as an assignment

$$M \ni x \mapsto \tilde{\varphi}_x$$

where for each $x \in M$, $\tilde{\varphi}_x : T_xM \times \ldots \times T_xM \to \pi_E^{-1}(x)$ is a $r$-multilinear skew-symmetric mapping. In particular, we define

$$\tilde{\varphi}_x(X_1, \ldots, X_r) = u(\varphi(X_1^*, \ldots, X_r^*)), \quad X_i \in T_xM \tag{1.2}$$

where $u \in P$ such that $\pi(u) = x$ and $X_i^*$ is any vector at $u$ that projects to $X_i$, that is $T_u\pi(X_i^*) = X_i$ for each $i = 1, \ldots, r$. Since $\varphi$ is a $V$-valued $r$-form, $\varphi(X_1, \ldots, X_r) \in V$. By Proposition 1.2 we know that $u : V \to \pi_E^{-1}(x)$ and thus the RHS of 1.2 is in $\pi_E^{-1}(x)$. Skew-symmetry and bilinearity properties are clear. To see that the RHS of 1.2 is independent of the choice of $X_i^*$, suppose that $Y_k^* \in T_uP$ is such that $T_u\pi(Y_k^*) = X_k = T_\pi(X_k^*)$ for some fixed $k$. This means that $X_k^* - Y_k^*$ is vertical. We compute

$$\varphi(X_1^*, \ldots, X_k^*, \ldots, X_r^*) - \varphi(X_1^*, \ldots, Y_k^*, \ldots, X_r^*) = \varphi(X_1^*, \ldots, X_k^* - Y_k^*, \ldots, X_r^*) = 0$$

since $\varphi$ is tensorial. This implies that

$$\varphi(X_1^*, \ldots, X_k^*, \ldots, X_r^*) = \varphi(X_1^*, \ldots, Y_k^*, \ldots, X_r^*)$$

which shows that definition of $\tilde{\varphi}_x$ is independent of the choice of $X_i^*$ for each $i$. Finally, we must also show that the definition is independent of the choice of $u$. To see this, let $v \in P$ such that $\pi(v) = x$. This means that $v = ua$ for some $a \in G$. Since $G$ acts on $V$ by $\rho$, by Proposition 1.2 we have

$$(ua)X = u(\rho(a)X), \quad u \in P, X \in V$$

Choose $Z_i^* \in T_{ua}P$ such that $T_{ua}\pi(Z_i^*) = X_i$. We compute

$$\begin{aligned}(ua)(\varphi(Z_1^*, \ldots, Z_r^*)) &= u(\rho(a)\varphi(Z_1^*, \ldots, Z_r^*)) \\ &= u(\varphi(T_uR_{a^{-1}}Z_1^*, \ldots, T_uR_{a^{-1}}Z_r^*)) \\ &= u(\varphi(X_1^*, \ldots, X_r^*))\end{aligned}$$

The last step follows since $T_u\pi(T_uR_{a^{-1}}Z_i^*) = T_{ua}(\pi \circ R_{a^{-1}})Z_i^* = T_{ua}\pi(Z_i^*) = X_i = T_u\pi(X_i^*)$. We have thus shown that $\tilde{\varphi}_x$ is well-defined for each $x \in M$.

Conversely, given an $r$-multilinear, skew-symmetric mapping $\tilde{\varphi}_x : T_xM \times \ldots T_xM \to \pi_E^{-1}(x)$ for each $x \in M$, we can define a $V$-valued tensorial $r$-form $\varphi$ by

$$\varphi(\bar{X}_1, \ldots, \bar{X}_r) = u^{-1}\tilde{\varphi}_x(T_u\pi(\bar{X}_1), \ldots, T_u\pi(\bar{X}_r)), \quad \bar{X}_i \in T_uP,\ \pi(u) = x \tag{1.3}$$

1.4 EXAMPLE: The above discussion shows that a tensorial 0-form of type $(\rho, V)$ on $P$ can be identified with a section $M \to E$ of $E(M, V, G, P)$. In other words, each $V$-valued function $f : P \to V$ satisfying $f(ua) = \rho(a^{-1})f(u)$ for $u \in P$ and $a \in G$ can be identified with a section of $E$. We shall have occasion to use this fact later on.

**1.3. The bundle of linear frames.** Let $M$ be an $n$-dimensional manifold. A ***linear frame at x*** is an ordered basis $u = (X_1, \ldots, X_n)$ for the tangent space $T_xM$ at $x \in M$. Let

$$L_x(M) = \{u|\ u \text{ is a linear frame at } x\}$$

and write

$$L(M) = \bigcup_{x \in M} L_x(M)$$

Define a map $\pi_L : L(M) \to M$ by

$$(\text{ a linear frame } u \text{ at } x) \mapsto x$$

The general linear group $GL(n;\mathbb{R})$ acts on $L(M)$ on the *right* in the following manner. If $a = (a^i_j) \in GL(n;\mathbb{R})$ and $u = (X_1, \ldots, X_n) \in L_x(M)$, we define $\Phi^L : L(M) \times GL(n;\mathbb{R}) \to L(M)$ by

$$(u, a) \mapsto (ua) := (a^j_1X_j, \ldots, a^j_nX_j)$$

Rather than using the elaborate notation, we write $\Phi^L(u, a) \stackrel{KN}{=} R_a(u)$ which is appropriate for right actions. Intuitively, if we think of $(X_1, \ldots, X_n)$ as a column vector, the action $R_a$ (for $a \in GL(n;\mathbb{R})$) looks like

$$R_a \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \mapsto a^T \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

One can see thus see that the action is indeed a right action.

EXERCISE 1.5 *Show that the action $\Phi^L$ is free and proper.*

This means that the quotient $L(M)/GL(n;\mathbb{R})$ possesses a differentiable structure and can be identified with the manifold $M$. Next, we show that $\pi_L : L(M) \to M$ satisfies the local-triviality condition for a principal fibre bundle. Let $(U, \phi)$ be a chart for $M$ with local coordinates $(x^1, \dots, x_n)$. Every frame $u \in L_x(M), x \in U$ can be uniquely expressed as

$$u = \left( X_1^k \frac{\partial}{\partial x^k}, \dots, X_n^k \frac{\partial}{\partial x^k} \right)$$

where $(X_i^k)$ is an invertible matrix. If we write $X_i = X_i^k \frac{\partial}{\partial x^k}$, the map $\psi : \pi_L^{-1}(U) \to U \times GL(n;\mathbb{R})$ given by

$$(X_1, \dots, X_n) \mapsto (x, (X_j^k))$$

is a diffeomorphism. We can therefore use coordinates $(x^i, X_j^k)$ on $\pi_L^{-1}(U)$ and define a differentiable structure on $L(M)$. It is also clear that the map

$$(X_1, \dots, X_n) \mapsto (X_j^k)$$

satisfies $R_b(X_1, \dots, X_n) = (Y_1, \dots, Y_n)$ where $Y_i = b_i^j X_j$ and thus defines a local bundle chart for $L(M)$. We have thus shown that $L(M)(M, GL(n;\mathbb{R}))$ is a principal fibre bundle.

There is another equivalent way to think about a linear frame. A linear frame $u = (X_1, \dots, X_n)$ at $x \in M$ can be regarded as an isomorphism $u : \mathbb{R}^n \to T_xM$ as follows. If $(e_1, \dots, e_n)$ is the standard basis for $\mathbb{R}^n$, the map $u$ is given by

$$c^i e_i \mapsto c^i X_i, \qquad c^i \in \mathbb{R}$$

The right action of $GL(n;\mathbb{R})$ on $L(M)$ is interpreted as follows. Consider $a = (a_j^i) \in GL(n;\mathbb{R})$ as a linear transformation of $\mathbb{R}^n$ which acts on $\mathbb{R}^n$ by matrix multiplication. Then $ua = R_a(u) : \mathbb{R}^n \to T_xM$ is the composition of the following two maps:

$$\mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_xM$$

**1.4. The tangent bundle as an associated bundle.** Recall that $GL(n;\mathbb{R})$ acts on $\mathbb{R}^n$ on the left by $(a, \xi) \mapsto a\xi$ (this is simply matrix multiplication). Given a manifold $M$, we write $E = L(M) \times_{GL(n;\mathbb{R})} \mathbb{R}^n$ and construct the bundle $E(M, \mathbb{R}^n, GL(n;\mathbb{R}), L(M))$ associated with $L(M)(M, GL(n;\mathbb{R}))$ with standard fibre $\mathbb{R}^n$. It is clear that this is a vector bundle over $M$ in the sense of the definition given in Section 1.2. We have the following result.

1.6 LEMMA: *The bundles $E(M, \mathbb{R}^n, GL(n;\mathbb{R}), L(M))$ and $\tau_M : TM \to M$ are naturally isomorphic as vector bundles over $M$. In particular, there a natural vector bundle isomorphism from $E$ to $TM$ over the identity mapping of $M$.*

*Proof:* Following the discussion at the end of Section 1.3, we think of a frame $u \in L_x(M)$ as an isomorphism $u : \mathbb{R}^n \to T_xM$. Thus for $\xi \in \mathbb{R}^n$, we have $u\xi \in T_xM$ where $x = \pi(u)$. We also know that $[u, \xi]_G \in \pi_E^{-1}(x)$. Now, define a map $\iota : E \to TM$ by

$$[u, \xi]_G \mapsto u\xi$$

To see that this is well-defined, for $a \in G$, consider $[ua, a^{-1}\xi]_G$ (which is equal to $[u, \xi]_G$). We have

$$\iota([ua, a^{-1}\xi]_G) = (ua)(a^{-1}\xi)$$

The right hand side is the action of the composition of the following maps on $\xi$

$$\mathbb{R}^n \ni \xi \overset{a^{-1}}{\longmapsto} \mathbb{R}^n \overset{a}{\longrightarrow} \mathbb{R}^n \overset{u}{\longrightarrow} T_xM$$

and therefore $\iota([u, \xi]_G) = \iota([ua, a^{-1}\xi]_G)$.

Next, given $v \in T_xM$, we claim that $\iota^{-1}(v) = [u, u^{-1}(v)]_G$ for any $u \in L_x(M)$. First we show that this statement is independent of the choice of $u$. Suppose that $\tilde{u} \in L_x(M)$, then $\tilde{u} = R_b(u) = ub$ for some $b \in GL(n; \mathbb{R})$. Thus we have

$$[\tilde{u}, \tilde{u}^{-1}(v)]_G = [ub, (ub)^{-1}(v)]_G = [ub.b^{-1}u^{-1}(v)]_G = [u, u^{-1}(v)]_G$$

Thus $\iota$ maps each fibre of $E$ isomorphically to a fibre of $TM$. From the discussion on associated vector bundles it is also clear that $\iota$ is linear. Finally, $\tau_M(\iota([u, \xi]_G) = x$ and thus we conclude that $\iota$ is a vector bundle isomorphism between $E$ and $TM$ over the identity on $M$. ■

1.7 REMARKS: (i) Notice that we have used the notation "$u\xi$" to represent two different objects in the sequel. In Proposition 1.2 "$u\xi$" represents the image of $\xi$ under the map $u : F \to \pi_E^{-1}(x)$. Let's call this the "first" definition. In the proof of Lemma 1.6, we have used it to represent the image of $\xi$ under the map $u : \mathbb{R}^n \to T_xM$. Call this the "second" definition. For the associated bundle $E$ considered in Lemma 1.6, the natural fibre $F = \mathbb{R}^n$ and thus according to the "First" definition we have $u : F = \mathbb{R}^n \to \pi_E^{-1}(x)$. Lemma 1.6 shows that $\pi_E^{-1}(x)$ is naturally isomorphic to $T_xM$ for every $x \in M$ and thus that the "first" and the "second" definitions are really the same (upto isomorphism). One can see the unifying power of the KN notation here.

(ii) The associated vector bundle construction described in this section is actually a special case of a general construction for arbitrary vector bundles. In the section on associated vector bundles, we presented a definition of a vector bundle over a manifold $M$. We now show how this definition is equivalent to the standard definition of a vector bundle. So suppose $\pi_E : E \to M$ is vector bundle in the usual sense, that is, it is a fibre bundle over $M$ such that the fibre $E_x$ over every $x \in M$ possess a vector space structure. Now, given $x \in M$, define $P_x := L(\mathbb{R}^n, E_x) = \{u : \mathbb{R}^n \to E_x |\ u \text{ is a linear isomorphism } \}$ and set

$$P = \bigcup_{x \in M} L(\mathbb{R}^n, E_x)$$

It is easy to see that $GL(n;\mathbb{R})$ acts on $P$ on the right and that this action is free and proper. Thus, $P(M, GL(n;\mathbb{R}))$ is a principal fibre bundle. Next, consider the usual left action of $GL(n;\mathbb{R})$ on $\mathbb{R}^n$ and form the associated bundle with total space $\tilde{E} := (P \times \mathbb{R}^n)/GL(n;\mathbb{R})$ associated with $P(M, GL(n;\mathbb{R}))$ with standard fibre $\mathbb{R}^n$. It can be seen that this associated bundle $\tilde{E}(M, \mathbb{R}^n, GL(n;\mathbb{R}), P)$ is isomorphic to $\pi_E : E \to M$. That is, there exists a bundle isomorphism from $\tilde{E}$ to $E$ over the identity map of $M$. This justifies why it makes sense to define vector bundles the way we have done in these notes.

(iii) One can think of the bundle $T^r_s(TM)$ of $(r, s)$ tensors on $M$ as an associated bundle as well. Observe that $GL(n;\mathbb{R})$ acts on $\mathbb{R}^n$ by $(A, \xi) \mapsto A\xi$ and thus it also acts on $T^r_s(\mathbb{R}^n)$ (the $(r, s)$ tensor space of $\mathbb{R}^n$) on the left by push-forward. That is

$$\begin{aligned} GL(n;\mathbb{R}) \times T^r_s(\mathbb{R}^n) &\to T^r_s(\mathbb{R}^n) \\ (A, t) &\mapsto A_* t \end{aligned}$$

It can be seen that the fibres of $T^r_s(TM)$ are isomorphic to the fibres of the bundle $E(M, T^r_s(\mathbb{R}^n), GL(n;\mathbb{R}), L(M))$ associated with $L(M)(M, GL(n;\mathbb{R}))$ with standard fibre $T^r_s(\mathbb{R}^n)$ where $E = (L(M) \times T^r_s(\mathbb{R}^n))/GL(n;\mathbb{R})$.

## 2. Connections in a vector bundle

In Section 1.2 we defined the notion of a vector bundle considered as an associated bundle $E(M, \mathbb{R}^n, G, P)$ of a principal fibre bundle $P(M, G)$ with natural fibre $\mathbb{R}^n$. In this section , we shall study the consequences of introducing a principal connection in $P(M, G)$ in this setup. In particular, we shall show that a principal connection in $P(M, G)$ gives rise to "covariant differentiation" on $E$. We first study how the notion of parallel transport can be defined on arbitrary associated bundles.

**2.1. Parallel transport in associated bundles.** Given a principal connection $\Gamma$ in a principal fibre bundle $P(M, G)$ one can define parallel transport in an associated bundle $E(M, F, G, P)$ with standard fibre $F$. Recall that we have the natural projection maps $\pi_G : P \times F \to E = (P \times_G F)$ and $\pi_E : E \to M$. Given $w \in E$, choose $(u, \xi) \in P \times F$ such that $\pi_G(u, \xi) = w$. Next, fix $\xi \in F$ and consider the mapping $\pi^\xi_G : P \to E$ given by $u \mapsto \pi_G(u, \xi)(= [u, \xi]_G)$. The ***horizontal subspace*** $H_wE$ at $w$ is defined by

$$H_wE = T_u\pi^\xi_G(H_uP) \tag{2.1}$$

where $H_uP$ is the horizontal subspace at $u$ defined by the connection $\Gamma$ in $P$.

EXERCISE 2.1 *Show that the definition of $H_wE$ in* (2.1) *is independent of the choice of* $(u, \xi) \in P \times F$.

For $w \in E$, the ***vertical subspace*** $V_wE$ at $w$ is defined in the usual manner, that is

$$V_wE = \ker(T_w\pi_E) = T_w(\pi_E^{-1}(x)), \quad x = \pi_E(w)$$

It can be seen that $T_wE = H_wE \oplus V_wE$ for each $w \in E$ and thus (2.1) defines an *Ehresmann connection* in the bundle $E(M, F, G, P)$. A curve in $E$ is ***horizontal*** if its tangent vector at each point is horizontal. Given a curve $\tau = x_t,\ 0 \leq t \leq 1$[2] in $M$ a ***horizontal lift of $\tau$ to $E$*** is a horizontal curve $\tau^*$ in $E$ such that $\pi_E(\tau^*) = \tau$. Given $w_o \in E$ such that $\pi_E(w_0) = x_0$, there is a unique horizontal lift $\tau^* = w_t$ of $\tau$ to $E$ starting from $w_0$. We prove the existence of $\tau^*$ below.

To construct the horizontal lift of $\tau = x_t$ to $E$ starting at $w_0 \in E$, we choose $(u_0, \xi) \in P \times F$ such that $\pi_G(u_0, \xi) = [u_0, \xi]_G = w_0$. Let $u_t$ be the horizontal lift of $\tau$ to $P$ (with respect to the principal connection $\Gamma$ starting at $u_0$. Then it is easy to see that the curve $w_t := [u_t, \xi]_G$ is the desired horizontal lift to $E$. Uniqueness of $\tau^*$ through $w_0$ follows from the corresponding result for lifts for principal fibre bundles. Motivated by this, we make the following definition.

2.2 DEFINITION: Let $w_0 = [u_0, \xi_0]_G \in E = P \times_G F$ and let $x_0 = \pi(u_0)$. Let $\tau = x_t,\ 0 \leq t \leq 1$, be a curve in $M$. The ***parallel transport*** of $w_0$ along the curve $\tau$ is defined to be the curve $[u_t, \xi_0]_G$ where $u_t$ is the horizontal lift of $\tau$ to $P$ starting at $u_0$.

We adopt the notation

$$\tau_t^0 : \pi_E^{-1}(x_0) \to \pi_E^{-1}(x_t)$$

for the ***parallel transport map*** along the curve $x_t$ of a point $[u_0, \xi_0]_G \in \pi_E^{-1}(x_0)$ to the corresponding point $\tau_t^0([u_0, \xi_0]_G) \in \pi_E^{-1}(x_{t+s})$. That is

$$\tau_t^0([u_0, \xi_0]_G) = [u_t, \xi_0]_G$$

where $u_t$ is the horizontal lift of $x_t$ to $P$ starting at $u_0$. In a similar manner, for $t, t+s \in [0, 1]$ we get a map

$$\tau_{t+s}^t : \pi_E^{-1}(x_t) \to \pi_E^{-1}(x_{t+s})$$

In general, it is not possible to define "covariant differentiation" on general associated bundles since there's no way to "subtract" elements in the same fibre. For associated vector bundles where the fibre possesses a vector space structure, this notion is well-defined. We discuss this case next.

**2.2. Covariant differentiation in vector bundles.** Given a principal fibre bundle $P(M, G)$ and a representation $\rho$ of $G$ on $\mathbb{R}^n$, let $E(M, \mathbb{R}^n, G, P)$ be a vector bundle over $M$. Notice that in this case the parallel transport map (between fibres of $E$) along a curve

[2]It is interesting to note that KN denote a curve as $x_t$ and not $x(t)$. This is another instance of "notational compactness" that is prevalent in the book. If we use $x(t)$ we shall have to specify the initial condition as $x(0) = x_0$ whereas the KN notation one doesn't need to.

$x_t \in M$ is a linear isomorphism. We first define covariant differentiation of a section along a curve.

2.3 DEFINITION: Let $\varphi$ be a section of $E$ defined along a curve $\tau = x_t \in M,\ t \in [a, b]$ so that $\pi_E \circ \varphi(x_t) = x_t$ for all $t \in [a, b]$. Denote the tangent vector to $\tau$ at $x_t$ by $\dot{x}_t$. Then, for fixed $t$, the ***covariant derivative*** $\boldsymbol{\nabla_{\dot{x}_t}\varphi}$ ***of*** $\boldsymbol{\varphi}$ ***with respect to*** $\boldsymbol{\dot{x}_t}$ is defined by

$$\nabla_{\dot{x}_t}\varphi = \lim_{h \to 0} \frac{1}{h}\left(\tau_t^{t+h}\varphi(x_{t+h}) - \varphi(x_t)\right)$$

2.4 REMARKS: (i) Notice that the covariant derivative $\nabla_{\dot{x}_t}\varphi$ is simply

$$\nabla_{\dot{x}_t}\varphi = \left.\frac{d}{ds}\right|_{s=0} \tau_t^{t+s}\varphi(x_{t+s})$$

(ii) Since $\tau_t^{t+h}\varphi(x_{t+h}) \in \pi_E^{-1}(x_t)$, we have $\nabla_{\dot{x}_t}\varphi \in \pi_E^{-1}(x_t)$ for every $t \in [a, b]$ and thus it defines a section of $E$ along $x_t$.

(iii) The section $\varphi$ is parallel, that is, the curve $\varphi(x_t)$ in $E$ is horizontal if and only if $\nabla_{\dot{x}_t} = 0$ for all $t \in [a, b]$.

(iv) It is clear that if $\varphi$ and $\psi$ are sections along $x_t$, then linearity of the parallel transport map the derivative maps implies that

$$\nabla_{\dot{x}_t}(\varphi + \psi) = \nabla_{\dot{x}_t}\varphi + \nabla_{\dot{x}_t}\psi$$

Also, from (ii), for an $\mathbb{R}$-valued function $\lambda$ we compute

$$\begin{aligned}
\nabla_{\dot{x}_t}(\lambda\varphi) &= \left.\frac{d}{ds}\right|_{s=0} \tau_t^{t+s}\lambda(x_{t+s})\varphi(x_{t+s}) \\
&= \left.\frac{d}{ds}\right|_{s=0} \lambda(x_{t+s})\tau_t^{t+s}\varphi(x_{t+s}) \\
&= \lambda(x_t)\left.\frac{d}{ds}\right|_{s=0} \tau_t^{t+s}\varphi(x_{t+s}) + \dot{x}_t\lambda \cdot \varphi(x_t) \\
&= \lambda(x_t)\nabla_{\dot{x}_t}\varphi + \dot{x}_t\lambda \cdot \varphi(x_t)
\end{aligned}$$

We now define covariant derivative of a section defined in a local neighborhood.

2.5 DEFINITION: For $x \in M$, let $X \in T_xM$ and $\varphi : U \to E$ be a section defined on a neighborhood $U$ containing $x$. Let $\tau = x_t,\ t \in [-\epsilon, \epsilon]$, be a curve such that $x_0 = x$ and $\dot{x}_0 := \dot{x}_t|_{t=0} = X$. Then the ***covariant derivative*** $\boldsymbol{\nabla_X\varphi}$ ***of*** $\boldsymbol{\varphi}$ ***with respect to*** $\boldsymbol{X}$ is defined by

$$\nabla_X\varphi = \nabla_{\dot{x}_0}\varphi$$

Given $\varphi$ as as above, denote by $f_\varphi$ the $\mathbb{R}^n$-valued function defined on $\pi^{-1}(U)$ corresponding to $\varphi$ (as described in Example 1.4). In particular $f_\varphi$ is defined by

$$f_\varphi(v) = v^{-1}(\varphi(\pi(v)), \quad v \in \pi^{-1}(U)$$

The following result shows that $\nabla_X \varphi$ is independent of the choice of $x_t$.

2.6 LEMMA: *Given a section $\varphi : U \to E$ where $x \in U$ and $X \in T_x M$, let $X^* \in T_u P$ be the horizontal lift of $X$ at $u \in P$. Then*

$$\nabla_X \varphi = u(X^* f_\varphi)$$

*where*

*Proof:* Let $x_t,\ t \in [-\epsilon, \epsilon]$ be a curve in $M$ such that $x_0 = x$ and $\dot{x}_0 = X$. Let $u_t$ be the horizontal lift of $x_t$ to $P$ such that $u_0 = u$ so that $X^* = \dot{u}_0$. We compute,

$$X^* f_\varphi = \lim_{h \to 0} \frac{1}{h} \left( f_\varphi(u_h) - f_\varphi(u_0) \right) = \lim_{h \to 0} \frac{1}{h} \left( u_h^{-1}(\varphi(x_h)) - u_0^{-1}(\varphi(x_0) \right) \tag{2.2}$$

and since $u_0 = u$ and $x_0 = x$ we have

$$u(X^* f_\varphi) = \lim_{h \to 0} \frac{1}{h} \left( u \circ u_h^{-1}(\varphi(x_h)) - \varphi(x) \right) \tag{2.3}$$

From Definition 2.3 and the equality (2.3), it suffices to show that

$$\tau_0^h(\varphi(x_h)) = u \circ u_h^{-1}(x_h) \tag{2.4}$$

Next, we set $\xi = u_h^{-1}(\phi(x_h))$. By definition, $u_t \xi = [u_t, \xi]_G$ is horizontal in $E$. Also, since $u_h \xi = \phi(x_h)$, we have by Definition 2.2, we have $\tau_h^0(u_0 \xi) = u_h \xi = \varphi(x_h)$ from which (2.4) follows. ■

2.7 REMARKS: (i) From Lemma 2.6 it is clear that Definition 2.5 of $\nabla_X \varphi$ is independent of the choice of the curve $x_t$.

(ii) One can also easily see that for $X \in T_x M$, $\varphi : U \to E$ and $\lambda \in \mathbb{R}$ we have $\nabla_{\lambda X} \varphi = \lambda \nabla_X \varphi$.

(iii) Given $X, Y \in T_x M$, $x \in M$ and $\varphi$ as above, let $X^*$ and $Y^*$ be the horizontal lifts of $X$ and $Y$ respectively. Then we know that $X^* + Y^*$ is the horizontal lift of $X + Y$. Thus

$$\nabla_{X+Y} \varphi = u((X+Y)^* f_\varphi) = u((X^* + Y^*) f_\varphi) = \nabla_X \varphi + \nabla_Y \varphi$$

We compile our results above in the following proposition.

2.8 PROPOSITION: (**Proposition 1.1, Chapter 3**) *Let $X, Y \in T_x M$, $x \in M$ and let $\varphi$ and $\psi$ be sections of $E$ defined in a neighborhood of $x$. Then* *(i)* $\nabla_{X+Y}\varphi = \nabla_X\varphi + \nabla_Y\varphi$*;*

*(ii)* $\nabla_X(\varphi + \psi) = \nabla_X\varphi + \nabla_X\psi$*;*

*(iii)* $\nabla_{\lambda X}\varphi = \lambda\nabla_X\varphi,\ \lambda \in \mathbb{R}$*;*

*(iv)* $\nabla_X(f\varphi) = f(x)\cdot\nabla_X\varphi + (Xf)\cdot\varphi(x)$*, where $f$ is an $\mathbb{R}$-valued function defined in a neighborhood of $x$.*

In a similar manner, we can define covariant differentiation with respect to vector fields.

2.9 DEFINITION: Let $\varphi : M \to E$ be a (global) section and $X$ a vector field on $M$. Then the ***covariant derivative of $\varphi$ with respect to $X$*** is the section $\nabla_X\varphi : M \to E$ defined by

$$(\nabla_X\varphi)(x) = \nabla_{X_x}\varphi.$$

As a direct consequence of Proposition 2.8 we have the following result.

2.10 PROPOSITION: *Let $X, Y$ be vector fields on $M$ and $\varphi$ and $\psi$ be sections of $E$. Then*

*(i)* $\nabla_{X+Y}\varphi = \nabla_X\varphi + \nabla_Y\varphi$*;*

*(ii)* $\nabla_X(\varphi + \psi) = \nabla_X\varphi + \nabla_X\psi$*;*

*(iii)* $\nabla_{\lambda X}\varphi = \lambda\nabla_X\varphi,\ \lambda \in \mathbb{R}$*;*

*(iv)* $\nabla_X(f\varphi) = f(x)\cdot\nabla_X\varphi + (Xf)\cdot\varphi(x)$*, where $f$ is an $\mathbb{R}$-valued function defined in a neighborhood of $x$.*

If $X$ is a vector field on $M$ and $X^*$ is its horizontal lift to $P$ (with respect to the principal connection that induces $\nabla$ of course). Then $\nabla_X$ corresponds to Lie differentiation $\mathfrak{L}_{X^*}$ in the following manner. It was noted earlier that there's a one-to-one correspondence[3] between sections $\varphi : M \to E$ and functions $f_\varphi : P \to \mathbb{R}^n$ that satisfy $f_\varphi(ua) = \rho(a^{-1})f_\varphi(u)$ for $u \in P$ and $a \in G$. From this and Lemma 2.6, we have the following readily verifiable result.

2.11 PROPOSITION: *If $\varphi : M \to E$ is a section and $f_\varphi : P \to \mathbb{R}^n$ be the corresponding function, then $\mathfrak{L}_{X^*}f_\varphi$ is the function corresponding to $\nabla_X\varphi$.*

**2.3. Fibre metrics in vector bundles.** To be completed later.

[3]This correspondence is explicitly defined in the paragraph below Definition 2.5 and in Example 1.4.

## 3. Linear connections

Let $L(M)(M, GL(n;\mathbb{R}))$ be the bundle of linear frames of $M$ where $n = \dim(M)$. Denote the canonical projection by $\pi_L : L(M) \to M$. We first present a definition.

3.1 DEFINITION: The ***canonical form*** of $L(M)$ is the $\mathbb{R}^n$-valued 1-form $\theta : TL(M) \to \mathbb{R}^n$ define by

$$\theta(X) = u^{-1}(T_u\pi_L X)), \quad X \in T_uL(M)$$

where $u \in L(M)$ is considered as a linear isomorphism $u : \mathbb{R}^n \to T_{\pi_L(u)}M$ as before.

3.2 PROPOSITION: *The canonical form $\theta$ of $L(M)$ is a tensorial 1-form of type $GL(n;\mathbb{R}), \mathbb{R}^n)$. It corresponds to the identity transformation of $T_xM$ at each $x \in M$.*

*Proof*: Note that $GL(n;\mathbb{R})$ acts on $\mathbb{R}^n$ by $(a, \xi) \mapsto \rho(a)\xi = a\xi,\ a \in GL(n;\mathbb{R})$ and thus we write $(ua) : \mathbb{R}^n \to T_xM,\ x = \pi_L(u)$ such that $(ua)\xi = u(a\xi)$ as usual (see the subsection **Tensorial forms and associated bundles** for details). Let $X \in T_uL(M)$ and $a \in GL(n;\mathbb{R})$. Then $T_{ua}R_aX \in T_{ua}L(M)$. We now compute

$$\begin{aligned}(R_a^*\theta) = \theta(T_{ua}R_aX) &= (ua)^{-1}(T_{ua}\pi_L(T_{ua}R_aX)) \\ &= a^{-1}u^{-1}(\theta(T_u\pi_L X)) = a^{-1}\theta(X)\end{aligned}$$

which shows that $\theta$ is pseudo-tensorial. Now, let $X \in T_uL(M)$ be vertical. Then $\theta(X) = u^{-1}(T_u\pi_L X) = 0$ and thus $\theta$ is tensorial.

For each $x \in M$, the linear map $\tilde{\varphi}_x : T_x \to T_xM$ corresponding to $\varphi$ is given by

$$\tilde{\varphi}_x(X) = u(\varphi(X^*), \quad X \in T_xM, \pi_L(u) = x$$

where $X^* \in T_uL(M)$ is such that $T_u\pi_L(X^*) = X$. Using the definition of $\varphi$ we get

$$\tilde{\varphi}_x(X) = u(u^{-1}T_u\pi_L(X^*)) = X$$

This is what we wished to show. ■

Now, we define the main object of discussion in this section.

3.3 DEFINITION: A principal connection in the bundle $L(M)(M, GL(n;\mathbb{R}))$ of linear frames over $M$ is called a ***linear connection of M***.

Given a linear connection $\Gamma$ of $M$, we associate with each $\xi \in \mathbb{R}^n$ a horizontal vector field $B(\xi)$ on $L(M)$ as follows. For each $u \in L(M), (B(\xi))_u$ is the unique horizontal vector at $u$ with the property that $T_u\pi_L(B(\xi)_u) = u\xi$. We shall call $B(\xi)$ the ***standard horizontal vector field corresponding to ξ***. Note that this vector field is only defined in the presence of a linear connection of $M$.

3.4 PROPOSITION: *The standard horizontal vector fields have the following properties:*

*(i) If $\theta$ is the canonical form of $L(M)$, then $\theta(B(\xi)_u) = \xi$ for each $\xi \in \mathbb{R}^n$ and $u \in L(M)$;*

*(ii) $T_u R_a(B(\xi)_u) = (B(a^{-1}\xi))_{ua}$, $a \in GL(n;\mathbb{R}), \xi \in \mathbb{R}^n$;*

*(iii) If $\xi \neq 0$, then $B(\xi)$ never vanishes.*

3.5 PROPOSITION: *Let $\Gamma$ be a linear connection of $M$. If $A^*$ is the fundamental vector field (infinitesimal generator) corresponding to $A \in \mathfrak{gl}(n;\mathbb{R})$ on $L(M)$ and if $B(\xi)$ is the standard horizontal vector field corresponding to $\xi \in \mathbb{R}^n$, then*

$$[A^*, B(\xi)] = B(A\xi),$$

*where $A\xi$ denotes the image of $\xi$ by $A \in \mathfrak{gl}(n;\mathbb{R})$ which acts on $\mathbb{R}^n$.*

Next, we define the ***torsion form*** $\Theta$ of a linear connection $\Gamma$ of $M$ by

$$\Theta = D\theta$$

where $D$ is the exterior covariant differential, that is $D\theta = (d\theta)(\text{hor})$. It can be shown that $\Theta$ is a tensorial 2-form on $L(M)$ of type $(GL(n;\mathbb{R}), \mathbb{R}^n)$.

3.6 THEOREM: (Structure equations) *Let $\omega, \Theta$ and $\Omega$ be the connection form, the torsion form and the curvature form respectively of a linear connection $\Gamma$ of $M$. Then the **first and second structure equations** are satisfied:*

$$d\theta(X,Y) = -\frac{1}{2}(\omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)) + \Theta(X,Y),$$
$$d\omega(X,Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X,Y)$$

*where $X, Y \in T_u L(M)$ and $u \in L(M)$.*

Considering $\theta$ as a vector-valued form and $\omega$ as a matrix-valued form, we can write the structure equations as

$$d\theta = -\omega \wedge \theta + \Theta$$
$$d\omega = -\omega \wedge \omega + \Omega$$

3.7 THEOREM: (Bianchi's identities) *For a linear connection the **first and second Bianchi identities** hold:*

$$D\Theta = \Omega \wedge \theta$$
$$D\Omega = 0$$

Recall that the collection $\{E_i^j; i, j = 1, \dots, n\}$ where $E_i^j \in \mathfrak{gl}(n;\mathbb{R})$ is an $n \times n$ matrix such that the entry in the $i$-th column and $j$-th row is 1 and the rest of the entries are zero, is a basis for $\mathfrak{gl}(n;\mathbb{R})$. Now, given a linear connection $\Gamma$ of $M$, let $B_1, \dots, B_n$ be the standard horizontal vector fields corresponding to the natural basis $\boldsymbol{e_1, \dots, e_n}$ of $\mathbb{R}^n$ and let $E_i^{j*}$ the fundamental vector fields corresponding to the basis $\{E_i^j\}$ of $\mathfrak{gl}(n;\mathbb{R})$. We have the following result.

3.8 PROPOSITION: *The $n^2 + n$ vector fields $\{B_k, E_i^{j*}; i, j = 1, \dots, n\}$ define an absolute parallelism in $L(M)$, that is, the $n^2 + n$ vectors $\{(B_k)_u, (E_i^{j*})_u; i, j = 1, \dots, n\}$ form a basis for $T_u L(M)$ for every $u \in L(M)$.*

Let $T_s^r(TM)$ be the bundle of $(r, s)$ tensors over $M$. Recall that given $x \in M$, the fibre over $x$ is given by the $(r, s)$ tensor space of $T_x M$ and denoted as $T_s^r(T_x M)$. We observed in Remarks 1.7 that $T_s^r(TM)$ is a vector bundle associated with $L(M)(M, GL(n;\mathbb{R}))$ with standard fibre $T_s^r(\mathbb{R}^n)$. Given a linear connection $\Gamma$ of $M$, there is a notion of parallel transport along a curve in $M$. Given a curve $\tau = x_t \in M$ the parallel transport along $\tau$ in $TM$ is a linear isomorphism given by $\tau_s^t : T_{x_t} M \to T_{x_s} M$ where $s, t \in \mathbb{R}$. For $\tau \in M$, we define the parallel transport along $\tau$ in $T_s^r(TM)$ as follows. For $t, s \in \mathbb{R}$, define $\tilde{\tau}_s^t : T_s^r(T_{x_t} M) \to T_s^r(T_{x_s} M)$ by

$$\tilde{\tau}_s^t(A) = (\tau_s^t)_*(A), \quad A \in T_s^r(T_{x_t} M)$$

where $(\tau_s^t)_*(A)$ is the push-forward of $A$ by $\tau_s^t$. We can now use $\tilde{\tau}$ to define covariant differentiation of sections of the tensor bundle (that is, tensor fields) in exactly the same manner in which we defined the covariant derivative of sections of a vector bundle in Section 2.3. We denote the covariant derivative of an $(r, s)$ tensor field $K$ on $M$ with respect to a vector field $X$ on $M$ by $\nabla_X K$. We have the following result.

3.9 PROPOSITION: *Let $\mathfrak{T}(M)$ be the algebra of tensor fields on $M$. Let $X$ and $Y$ be vector fields on $M$. Then the covariant derivative has the following properties:*

*(i)* $\nabla_X : \mathfrak{T}(M) \to \mathfrak{T}(M)$ *is a type-preserving derivation.*

*(ii)* $\nabla_X$ *commutes with every contraction*

*(iii)* $\nabla_X f = L_X f$ *for every function* $f : M \to \mathbb{R}$

*(iv)* $\nabla_{X+Y} = \nabla_X + \nabla_Y$

*(v)* $\nabla_{fX} K = f \cdot \nabla_X K$ *for every function $f$ on $M$ and $K \in \mathfrak{T}(M)$.*

As a consequence of this result and Proposition 2.10 we have

3.10 PROPOSITION: *If $X, Y$ and $Z$ are vector fields on $M$, then*

*(i)* $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

*(ii)* $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$

*(iii)* $\nabla_{fX} Y = f \cdot \nabla_X Y$ *for every* $f \in C^\infty(M)$

*(iv)* $\nabla_X(fY) = f \cdot \nabla_X Y + (\mathcal{L}_X f)Y$ *for every* $f \in C^\infty(M)$.

This result thus shows that given a linear connection $\Gamma$ of $M$, there exists a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ that has properties (i)-(iv). Later on we shall be able to prove that corresponding to any operator $\nabla$ that satisfies the properties (i)-(iv) in Proposition 3.10, there is a linear connection of $M$.

3.11 PROPOSITION: *Let $M$ be a manifold with a linear connection. Every derivation $D$ (preserving type and commuting with contractions) of the algebra $\mathcal{T}(M)$ of tensor fields into the tensor algebra $\mathcal{T}(T_xM)$ at $x \in M$ can be decomposed as follows:*

$$D = \nabla_X + S,$$

*where $X \in T_xM$ and $S : T_xM \to T_xM$ is a linear endomorphism.*

Given an $(r, s)$ tensor field $K$ on $M$, the ***covariant differential*** $\nabla K$ of $K$ is an $(r, s+1)$ tensor field defined by

$$(\nabla K)(X_1, \ldots, X_s; X) = (\nabla_X K)(X_1, \ldots, X_s)$$

Thus both sides of the above expression are $(r, 0)$ tensor fields. We only write the $(0, s)$ arguments since those are the only ones involved in the definition. We have the following result.

3.12 PROPOSITION: *If $K$ is an $(r, s)$ tensor field on $M$ then*

$$(\nabla K)(X_1, \ldots, X_s; X) = \nabla_X(K(X_1, \ldots, X_s)) - \sum_{i=1}^{s} K(X_1, \ldots, \nabla_X X_i, \ldots, X_s)$$

*where $X, X_i,\ i = 1, \ldots, s \in \Gamma(TM)$.*

A tensor field $K$ on $M$ is ***parallel*** if and only if $\nabla_X K = 0$ for all $X \in T_xM$ and $x \in M$. We thus have the following result.

3.13 PROPOSITION: *A tensor field $K$ on $M$ is parallel if and only if $\nabla K = 0$.*

Given an $(r, s)$ tensor field $K$ on $M$, the ***second covariant differential*** $\boldsymbol{\nabla^2 K}$ of $K$ is the $((r, s+2)$ tensor field defined by $\nabla^2 K = \nabla(\nabla K)$. We denote

$$(\nabla^2 K)(; X; Y) = (\nabla_Y(\nabla K))(; X)$$

In other words, for $X, Y, X_i,\ i = 1, \ldots, s \in \Gamma(TM)$, we have

$$(\nabla^2 K)(X_1, \ldots, X_s; X; Y) = (\nabla_Y(\nabla K))(X_1, \ldots, X_s; X)$$

This leads us to the final result of this section.

3.14 PROPOSITION: *For any tensor field $K$ and for $X, Y \in \Gamma(TM)$ we have*

$$(\nabla^2 K)(; X; Y) = \nabla_Y(\nabla_X K) - \nabla_{\nabla_Y X} K.$$

## 4. Affine connections

In this section we shall study the bundle of “affine frames”. A linear frame at a point gives an ordered basis for the tangent space at that point. If we think of this tangent space as an affine space, we can construct a new bundle by “patching together” all the “frames” corresponding to these these affine spaces. It turns out that this bundle of affine frames is a principal bundle with the group of affine automorphisms of affine Euclidean space as the structure group. One can consider principal connections in this bundle (called “affine connections”) and they naturally correspond to linear connections in $L(M)$. This explains why the terms linear and affine connections cam be (and have been ) used interchangeably.

**4.1. The bundle of affine frames.** Let $M$ be an $n$-dimensional manifold and let $L(M)(M, GL(n;\mathbb{R}))$ be the bundle of linear frames of $M$. Recall that every vector space can be thought of as an affine space modeled on itself. For each $x \in M$, when we think of $T_xM$ as an affine space modeled on itself we call it the ***affine tangent space*** at $x$ and denote it by $A_xM$. Similarly, $\mathbb{R}^n$ considered as an affine space modeled on itself is denoted by $\mathbb{A}^n$. We also denote by $A(n;\mathbb{R})$ the group of affine transformations of $\mathbb{A}^n$. It is easy to see that

$$A(n;\mathbb{R}) = \left\{ \tilde{a} = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix} \middle| \; a \in GL(n;\mathbb{R}), \xi \in \mathbb{R}^n \right\}.$$

The action of $A(n;\mathbb{R})$ on $\mathbb{A}^n$ is given by $(\eta, \tilde{a}) \mapsto a\eta + \xi$, where $\eta \in \mathbb{A}^n$. Next, we define group homomorphisms $\alpha : \mathbb{R}^n \to A(n;\mathbb{R})$ and $\beta : A(n;\mathbb{R}) \to GL(n;\mathbb{R})$ by

$$\xi \overset{\alpha}{\mapsto} \begin{pmatrix} \mathrm{id}_{GL(n;\mathbb{R})} & \xi \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix} \overset{\beta}{\mapsto} a$$

The following is a short-exact sequence of group homomorphisms.

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\alpha} A(n;\mathbb{R}) \xrightarrow{\beta} GL(n;\mathbb{R}) \longrightarrow \mathrm{id}_{GL(n;\mathbb{R})} \tag{4.1}$$

This sequence is *exact* since the kernel of each homomorphism is the image of the preceding one and thus by definition a *short-exact* sequence. Notice that $\alpha$ is injective and $\beta$ is surjective. Moreover, there is a homomorphism $\gamma : GL(n;\mathbb{R}) \to A(n;\mathbb{R})$ defined by

$$\gamma(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \in GL(n;\mathbb{R})$$

that satisfies $\beta \circ \gamma = \mathrm{id}_{GL(n;\mathbb{R})}$. This shows that the sequence is *split-exact* (see [1] for details on exact sequences). The upshot of this is that $A(n;\mathbb{R})$ is a *semidirect-product* of $GL(n;\mathbb{R})$ and $\mathbb{R}^n$. This means that for every $\tilde{a} \in A(n;\mathbb{R})$ there exists a unique pair

$(a,\xi) \in GL(n;\mathbb{R}) \times \mathbb{R}^n$ such that $\tilde{a} = \alpha(\xi)\cdot\gamma(a)$ where "$\cdot$" represents the group operation in $A(n;\mathbb{R})$.

Given $x \in M$, an ***affine frame at $x$*** is a pair $(p,u)$ where $p \in A_xM$ and $u = (X_1,\ldots,X_n)$ is a *linear frame* at $x$. That is, $u \in L_x(M)$. We denote by $\mathbf{0}$ the origin of $\mathbb{R}^n$ and by $\boldsymbol{e} = (\boldsymbol{e_1},\ldots,\boldsymbol{e_n})$ the canonical basis for $\mathbb{R}^n$ and call $(\mathbf{0},\boldsymbol{e})$ the ***canonical frame of*** $\mathbb{A}^{\boldsymbol{n}}$. Every affine frame $(p,u)$ at $x$ can be identified with an affine isomorphism $\tilde{u} : \mathbb{A}^n \to A_xM$ which maps $(\mathbf{0},\boldsymbol{e})$ into $(p,u)$. We shall find it convenient to think of $u \in L_x(M)$ as a linear isomorphism from $\mathbb{R}^n \to T_xM$. Using this notion, given an affine frame $(p,u)$ at $x \in M$ we can explicitly write down the corresponding affine isomorphism $\tilde{u}$. In particular, $\tilde{u}$ is given by

$$\tilde{u}(\tilde{\xi}) = u(\tilde{\xi} - \mathbf{0}) + p = u(\tilde{\xi}) + p$$

where we have the following picture in mind

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{\ \tilde{u}\ } & A_xM \\ {\scriptstyle I_0}\downarrow & & \downarrow{\scriptstyle I_p} \\ \mathbb{R}^n & \xrightarrow[\ u\ ]{} & T_xM \end{array}$$

where $I_0 : \mathbb{A}^n \to \mathbb{R}^n$ is the isomorphism induced once $0 \in \mathbb{A}^n$ is fixed. That is $I_0(\tilde{\xi}) = \tilde{\xi} - 0 = \tilde{\xi}$ for $\tilde{\xi} \in \mathbb{A}^n$ and similarly $I_p : A_xM \to T_xM$ is given by $I_p(X) = X - p$ for $X \in A_xM$. We let

$$A(M) := \bigcup_{x\in M} \{\text{set of affine frames at } x\}$$

and define the canonical projection $\pi_A : A(M) \to M$ by

$$(\text{an affine frame at } x) \mapsto x.$$

Now, the group $A(n;\mathbb{R})$ acts on $A(M)$ as follows:

$$\begin{aligned} A(M) \times A(n;\mathbb{R}) &\to A(M) \\ (\tilde{u},\tilde{a}) &\mapsto (\tilde{u}\tilde{a}) \end{aligned} \tag{4.2}$$

where $\tilde{u}\tilde{a} : \mathbb{A}^n \to A_xM,\ x = \pi_A(\tilde{u})$ is interpreted as the composition of the following two maps

$$\mathbb{A}^n \xrightarrow{\tilde{a}} \mathbb{A}^n \xrightarrow{\tilde{u}} A_xM$$

Exercise 4.1 *Show that*

*(1) $A(M)$ is a differentiable manifold,*

*(2) The right action* (4.2) *of $A(n;\mathbb{R})$ on $A(M)$ is free and proper.*

Using Exercise 4.1, it can be seen that $A(M)(M, A(n;\mathbb{R}))$ is a principal fibre bundle. We call it the ***bundle of affine frames*** of $M$.

There exist natural bundle homomorphisms $\tilde{\beta} : A(M) \to L(M)$ and $\tilde{\gamma} : L(M) \to A(M)$ over the identity map of $M$ given by

$$\tilde{u} = (p, u) \xmapsto{\tilde{\beta}} u \quad \text{and} \quad u \xmapsto{\tilde{\gamma}} (0_x, u)$$

where $x = \pi_L(u)$ (recall that $\pi_L : L(M) \to M$ is the natural projection) and $0_x \in T_xM$ is the zero vector. It can easily be seen that $\tilde{\beta}$ is surjective and $\tilde{\gamma}$ is injective and that $\tilde{\beta} \circ \tilde{\gamma} = \mathrm{id}_{L(M)}$. Thus, $L(M)$ can be thought of as a subbundle of $A(M)$ via $\tilde{\gamma}$. More precisely, $\tilde{\gamma}(L(M))(M, \gamma(GL(n;\mathbb{R}))$ is a subbundle of $A(M)(M, A(n;\mathbb{R}))$. Next we introduce connections in $A(M)$.

4.2 DEFINITION: A ***generalized affine connection of M*** is a principal connection in the bundle $A(M)(M, A(n;\mathbb{R}))$.

We shall now study the relationship between generalized affine connections and linear connections as promised earlier. Before we do that, let us mention that corresponding to the sequence (4.1) of group homomorphisms, there exists a natural sequence of Lie algebra homomorphisms given by

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{T_e\alpha} \mathfrak{a}(n;\mathbb{R}) \xrightarrow{T_e\beta} \mathfrak{gl}(n;\mathbb{R}) \longrightarrow 0$$

where by abuse of notation we represent the identity elements of $GL(n;\mathbb{R})$ and $A(n;\mathbb{R})$ by the same symbol $e$. This shows that the Lie algebra $\mathfrak{a}(n;\mathbb{R})$ is the semi-direct sum of $\mathfrak{gl}(n;\mathbb{R})$ and $\mathbb{R}^n$ (thought of as a Lie algebra). This means that as a vector space (that is, without considering the Lie algebra structure) $\mathfrak{a}(n;\mathbb{R}) = T_e\gamma(\mathfrak{gl}(n;\mathbb{R})) \oplus T_e\alpha(\mathbb{R}^n)$ and for $T_e\gamma \cdot A_i \oplus T_e\alpha \cdot \xi_i \in T_e\gamma(\mathfrak{gl}(n;\mathbb{R})) \oplus T_e\alpha(\mathbb{R}^n),\ i = 1, 2$ the Lie bracket is given by

$$[T_e\gamma \cdot A_1 \oplus T_e\alpha \cdot \xi_1, T_e\gamma \cdot A_1 \oplus T_e\alpha \cdot \xi_1] = T_e\gamma \cdot [A_1, A_2] \oplus T_e\alpha \cdot (A_1(\xi_2) - A_2(\xi_1)) \quad (4.3)$$

For every $\tilde{A} \in \mathfrak{a}(n;\mathbb{R})$ there exists a pair $(A, \eta) \in \mathfrak{gl}(n;\mathbb{R}) \times \mathbb{R}^n$ such that

$$\tilde{A} = T_e\gamma(A) + T_e\alpha(\eta). \quad (4.4)$$

For $\tilde{A} = \begin{pmatrix} A & \eta \\ 0 & 0 \end{pmatrix}$ the equality (4.4) simply reads

$$\begin{pmatrix} A & \eta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix}$$

4.3 Remark: We have tried to use explicit notation in this discussion. KN write (4.4) as simply

$$\tilde{A} = A + \eta$$

and similarly $\mathfrak{a}(n;\mathbb{R}) \overset{KN}{=} \mathfrak{gl}(n;\mathbb{R}) + \mathbb{R}^n$ which can lead to some confusion in understanding as to where each object lives. Since this is an important section, we shall avoid using KN notation. The intuition behind their notation is clear–they identify $\mathfrak{gl}(n;\mathbb{R})$ and $\mathbb{R}^n$ with their images $T_e\gamma(\mathfrak{gl}(n;\mathbb{R})$ and $T_e\alpha(\mathbb{R}^n)$ respectively.

Now, let $\tilde{\omega}$ be the connection 1-form of a generalized connection $\tilde{\Gamma}$ of $M$. Then the pull-back $\tilde{\gamma}^*\tilde{\omega}$ of $\tilde{\omega}$ to $L(M)$ is an $\mathfrak{a}(n;\mathbb{R})$-valued 1-form on $L(M)$. Using the semi-direct sum decomposition of $\mathfrak{a}(n;\mathbb{R})$ as in (4.4), we write

$$\tilde{\gamma}^*\tilde{\omega} = T_e\gamma \cdot \omega + T_e\alpha \cdot \varphi \tag{4.5}$$

such that $\omega$ is a $\mathfrak{gl}(n;\mathbb{R})$-valued 1-form on $L(M)$ and $\varphi$ is an $\mathbb{R}^n$-valued 1-form on $L(M)$. Before proceeding further, let us recall a result from Chapter 2 of [2].

4.4 Proposition: (Proposition 6.4, Chapter 2) *Let $Q(M,H)$ be a subbundle of a principal fibre bundle $P(M,G)$, where $H$ is a Lie subgroup of $G$. Assume that the Lie algebra $\mathfrak{g}$ of $G$ admits a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\mathrm{ad}(H)(\mathfrak{m}) = \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of $H$. For every connection 1-form $\omega$ in $P$, the $\mathfrak{h}$component $\omega'$ of $\omega$ restricted to $Q$ is a connection 1-form in $Q$.*

Exercise 4.5 *Consider the decomposition* (4.5).

*(1) Use Proposition 4.4 to show that $\omega : TL(M) \to \mathfrak{gl}(n;\mathbb{R})$ defines a connection 1-form in $L(M)(M, GL(n;\mathbb{R}))$.*

*(2) Show that $\varphi : TL(M) \to \mathbb{R}^n$ is a tensorial 1-form of type $(GL(n;\mathbb{R}), \mathbb{R}^n)$.*

As a consequence of Exercise 4.5, $\varphi$ corresponds to a $(1,1)$-tensor field $K_\varphi$ on $M$ given by[4]

$$K_\varphi(X) = u(\varphi(X^*)), \quad X \in T_xM, \;\; \pi_L(u) = x$$

where $X^* \in T_uL(M)$ is such that $T_u\pi_L(X^*) = X$. We now present the main result of this section.

4.6 Proposition: *Let $\tilde{\omega}$ be the connection 1-form of a generalized affine connection $\tilde{\Gamma}$ of $M$ and let*

$$\tilde{\gamma}^*\tilde{\omega} = T_e\gamma \cdot \omega + T_e\alpha \cdot \varphi$$

*where $\omega$ and $\varphi$ are as described before. Let $\Gamma$ be the linear connection of $M$ defined by $\omega$ and let $K$ be the $(1,1)$ tensor field on $M$ defined by $\varphi$. Then*

*(i) The correspondence between the set of generalized affine connections of $M$ and the set of pairs consisting of a linear connection of $M$ and a $(1,1)$ tensor field on $M$ given by $\tilde{\Gamma} \mapsto (\Gamma, K)$ is one-to-one.*

[4] See the subsection **Tensorial forms and associated bundles** for details.

*(ii) The homomorphism* $\tilde{\beta} : A(M) \to L(M)$ *maps* $\tilde{\Gamma}$ *into* $\Gamma$.

*Proof:* (i) We have already seen that given a generalized connection $\tilde{\Gamma}$ of $M$ there exist $\Gamma$ and $K$ corresponding to it. It therefore suffices to show that given $(\Gamma, K)$, there exists $\tilde{\Gamma}$ which induces $\Gamma$ and $K$. Let $\omega$ be the connection 1-form corresponding to $\Gamma$ and let $\varphi$ be the tensorial 1-form on $L(M)$ of type $(GL(n;\mathbb{R}), \mathbb{R}^n)$ corresponding to $K$. Given $\tilde{X} \in T_{\tilde{u}}A(M)$, choose $X \in T_uL(M)$ and $\tilde{a} \in A(n;\mathbb{R})$ such that $\tilde{u} = R_{\tilde{a}}\tilde{\gamma}(u)$ and $\tilde{X} - T_{\tilde{\gamma}(u)}R_{\tilde{a}}(T_u\tilde{\gamma}X)$ is vertical. To see that this is possible, let $u = \tilde{\beta}(\tilde{u})$, then $\pi_A(\tilde{u}) = \pi_L \circ \tilde{\beta}(\tilde{u}) = \pi_L(u)$ because $\tilde{\beta}$ is a bundle homomorphism over the identity map of $M$. Also, $\pi_A(\tilde{\gamma}(u)) = \pi_L \circ \tilde{\beta}(\tilde{\gamma}(u)) = \pi_L(u)$ since $\tilde{\beta} \circ \tilde{\gamma} = \mathrm{id}_{L(M)}$. This shows that $\tilde{\gamma}(u), \tilde{u} \in \pi_A^{-1}(x)$ where $x = \pi_A(\tilde{u})$. Thus there exists $\tilde{a} \in A(n;\mathbb{R})$ such that $\tilde{u} = R_{\tilde{a}}\tilde{\gamma}(u)$. Next, set $Y = T_{\tilde{u}}\pi_A(\tilde{X}) \in T_xM$, where $x = \pi_A(\tilde{u})$. Now, let $X \in T_uL(M)$ be the horizontal lift of $Y$ to $L(M)$ corresponding to the linear connection $\Gamma$. Then $T_{\tilde{u}}\pi_A(T_{\tilde{\gamma}(u)}R_{\tilde{a}}(T_u\tilde{\gamma}X)) = T_{\tilde{\gamma}(u)}(\pi_A \circ R_{\tilde{a}})(T_u\tilde{\gamma}Y^*) = T_u(\pi_A \circ \tilde{\gamma})X = T_u\pi_LY^* = Y$. Therefore both $\tilde{X}$ and $T_{\tilde{\gamma}(u)}R_{\tilde{a}}(T_u\tilde{\gamma}X))$ project to the same vector under $T_{\tilde{u}}\pi_A$. This means that $\tilde{X} - T_{\tilde{\gamma}(u)}R_{\tilde{a}}(T_u\tilde{\gamma}X))$ is vertical. Thus, there exists $A \in \mathfrak{a}(n;\mathbb{R})$ such that

$$\tilde{X} = T_{\tilde{\gamma}(u)}R_{\tilde{a}}(T_u\tilde{\gamma}X)) + A^*_{\tilde{u}}$$

where $A^*$ is the infinitesimal generator corresponding to $A$. Set

$$\tilde{\omega}(\tilde{X}) = \mathrm{ad}(\tilde{a}^{-1})(T_e\gamma \cdot \omega(X) + T_e\alpha \cdot \varphi(X)) + A \tag{4.6}$$

We leave it to the reader to show that this defines a connection 1-form on $A(M)$ that induces $\omega$ and $\varphi$.

(ii) Let $\tilde{X} \in T_{\tilde{u}}A(M)$ and set $u = \tilde{\beta}(\tilde{u})$ and $X = T_{\tilde{u}}\tilde{\beta}\tilde{X} \in T_uL(M)$. Now, since $\tilde{\beta} : A(M) \to L(M)$ is a bundle homomorphism with $\beta : A(n;\mathbb{R}) \to GL(n;\mathbb{R}) = A(n;\mathbb{R})/\mathbb{R}^n$ the corresponding group homomorphism, one can identify $L(M)$ with $A(M)/\mathbb{R}^n$. To see this, notice that $\mathbb{R}^n$ acts on $A(M)$ as follows

$$\begin{aligned} A(M) \times \mathbb{R}^n &\to A(M) \\ (\tilde{u}, \xi) &\mapsto \tilde{u}\alpha(\xi) \end{aligned}$$

Also, $\tilde{\beta}(\tilde{u}\alpha(\xi)) = \tilde{\beta}(\tilde{u})\beta(\alpha(\xi)) = \tilde{\beta}(\tilde{u})$ since $\beta(\alpha(\xi)) = \mathrm{id}_{GL(n;\mathbb{R})}$. It is therefore clear that $\tilde{\beta}$ maps the orbit of $\tilde{u} \in A(M)$ under the action of $\mathbb{R}^n$ on $A(M)$ to $\tilde{\beta}(\tilde{u})$ and thus we can identify $\mathrm{img}(\tilde{\beta}) = L(M)$ with $A(M)/\mathbb{R}^n$. We therefore think of $\tilde{\beta} : A(M) \to A(M)/\mathbb{R}^n$ as the natural projection. Since $T_{\tilde{u}}\tilde{\beta}\tilde{X} = X = T_u(\tilde{\beta} \circ \tilde{\gamma})X$, one can use the procedure similar to the one used in (i) to show that there exists $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$ (where $\mathbb{R}^n$ is thought of as a Lie group in the first case and as a Lie algebra in the second one) such that $\tilde{u} = u\alpha(\xi)$ and

$$\tilde{X} = T_{\tilde{\gamma}(u)}R_{\alpha(\xi)}T_u\tilde{\gamma}X + (T_e\alpha(\eta))^*_{\tilde{u}} \tag{4.7}$$

where $(T_e\alpha(\eta))^*$ is the infinitesimal generator corresponding to $T_e\alpha(\eta) \in \mathfrak{a}(n;\mathbb{R})$. Now, in order to show that $\tilde{\beta}$ maps $\tilde{\Gamma}$ to $\Gamma$ we must show that $\tilde{\beta}$ maps the horizontal subspace (with respect to $\tilde{\Gamma}$ at a point in $A(M)$ to the horizontal subspace (with respect to $\Gamma$) at the corresponding point. Suppose that $\tilde{X}$ is horizontal with respect to $\tilde{\Gamma}$. Thus

$$\begin{aligned} 0 = \tilde{\omega}(\tilde{X}) &= \tilde{\omega}(T_{\tilde{\gamma}(u)}R_{\alpha(\xi)}T_u\tilde{\gamma}X) + \tilde{\omega}((T_e\alpha(\eta))^*_{\tilde{u}}) \\ &= \mathrm{ad}(\alpha(\xi)^{-1})\tilde{\omega}(T_u\tilde{\gamma}X) + T_e\alpha(\eta) \end{aligned}$$

and therefore

$$\tilde{\omega}(T_u\tilde{\gamma}X) = (\tilde{\gamma}^*\tilde{\omega})X = -\mathrm{ad}(\alpha(\xi))(T_e\alpha(\eta))$$

Since $(\tilde{\gamma}^*\tilde{\omega})X = T_e\gamma \cdot \omega(X) + T_e\alpha \cdot \varphi(X)$ we have

$$T_e\gamma \cdot \omega(X) + T_e\alpha \cdot \varphi(X) = -\mathrm{ad}(\alpha(\xi))(T_e\alpha(\eta)).$$

Since both $T_e\alpha \cdot \varphi(X)$ and $\mathrm{ad}(\alpha(\xi))(T_e\alpha(\eta))$ are in $T_e\alpha(\mathbb{R}^n)$ and $T_e\gamma \cdot \omega(X) \in T_e\gamma(\mathfrak{gl}(n;\mathbb{R}))$, it can be seen that $\omega(X) = 0$. Thus $X$ is horizontal with respect to $\Gamma$.

■

EXERCISE 4.7 *Show that* (4.6) *defines a connection 1-form* $\tilde{\omega}$ *on* $A(M)$ *that induces* $\omega$ *and* $\varphi$.

Having proved that a generalized affine connection gets mapped to the corresponding linear connection, we now study the relationship between the corresponding curvature forms of the connections.

4.8 PROPOSITION: *In Proposition 4.6, let* $\tilde{\Omega}$ *and* $\Omega$ *be the curvature forms of* $\tilde{\Gamma}$ *and* $\Gamma$ *respectively. Then*

$$\tilde{\gamma}^*\tilde{\Omega} = T_e\gamma \cdot \Omega + T_e\alpha \cdot D\varphi \tag{4.8}$$

*where* $D$ *is the exterior covariant differentiation with respect to* $\Gamma$*, that is,* $D\varphi = (d\varphi)(\mathrm{hor})$.

*Proof:* Let $X, Y \in T_uL(M)$. To prove that

$$\tilde{\gamma}^*(\tilde{\Omega})(X,Y) = T_e\gamma \cdot \Omega(X,Y) + T_e\alpha \cdot D\varphi(X,Y), \tag{4.9}$$

it suffices to consider the following two cases: (1) at least one of $X$ or $Y$ is vertical (2) $X, Y$ are both horizontal. In the first case, since both sides of (4.8) are tensorial, they vanish identically on vertical vectors and hence (4.9) is verified for the first case. As for the second case when $X, Y$ are horizontal, we have $\omega(X) = 0 = \omega(Y)$ and therefore

$\tilde{\gamma}^*\tilde{\omega}(X) = T_e\alpha \cdot \varphi(X)$ and $\tilde{\gamma}^*\tilde{\omega}(Y) = T_e\alpha \cdot \varphi(Y)$. Using the structure equation for $\tilde{\Gamma}$, we have

$$\begin{aligned}\tilde{\gamma}^*\tilde{\omega}(X,Y) = d\tilde{\omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}Y) &= -\frac{1}{2}[\tilde{\omega}(T_u\tilde{\gamma}X), \tilde{\omega}(T_u\tilde{\gamma}X)] + \tilde{\Omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}X)\\ &= -\frac{1}{2}[T_e\alpha \cdot \varphi(X), T_e\alpha \cdot \varphi(Y)] + \tilde{\Omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}X)\\ &= \tilde{\Omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}X) \qquad (4.10)\end{aligned}$$

where the last step essentially follows from the fact that the Lie bracket on $\mathbb{R}^n$ is zero. Also, pullback commutes with the exterior derivative so that $\tilde{\gamma}^*d\tilde{\omega} = d\tilde{\gamma}^*\tilde{\omega} = T_e\gamma \cdot d\omega + T_e\alpha \cdot d\varphi$ and thus

$$\tilde{\gamma}^*d\tilde{\omega}(X,Y) = T_e\gamma \cdot d\omega(X,Y) + T_e\alpha \cdot d\varphi(X,Y) = T_e\gamma \cdot \Omega(X,Y) + T_e\alpha \cdot D\varphi(X,Y) \quad (4.11)$$

where the last equality follows since $X, Y$ are horizontal. Comparing (4.10) and (4.11) we see that

$$\tilde{\gamma}^*\tilde{\omega}(X,Y) = T_e\gamma \cdot \Omega(X,Y) + T_e\alpha \cdot D\varphi(X,Y).$$

The result now follows. ■

We are finally in a position to define affine connections.

4.9 Definition: A generalized affine connection $\tilde{\Gamma}$ of $M$ is called an ***affine connection*** if (referring to Proposition 4.6) the $\mathbb{R}^n$-valued 1-form $\varphi$ is the canonical form $\theta$ of $L(M)$.

It follows that the $(1,1)$ tensor field $K$ corresponding to an affine connection $\tilde{\Gamma}$ is the tensor field of identity transformations of tangent spaces to $M$. As a direct consequence of Proposition 4.6, we have the following result.

4.10 Proposition: *The bundle homomorphism $\tilde{\beta} : A(M) \to L(M)$ maps every affine connection $\tilde{\Gamma}$ of $M$ into a linear connection $\Gamma$ of $M$. Moreover, $\tilde{\Gamma} \mapsto \Gamma$ gives a one-to-one correspondence between the set of affine connections $\tilde{\Gamma}$ of $M$ and the set of linear connections $\Gamma$ of $M$.*

Proposition 4.10 justifies why the terms "affine connections" and "linear connections" can be used interchangeably. This is the main result of this section. It establishes a one-to-one correspondence between linear and affine connections, which, in the light of our treatment of bundles of linear and affine frames respectively, is not entirely surprising. Following KN, we shall continue to make a distinction between these two types of connections since there are other interesting features of affine connections that we would like to explore. For affine connections, Proposition 4.8 reduces to the following:

4.11 Proposition: *Let $\Theta$ and $\Omega$ be the torsion form and the curvature form of a linear connection $\Gamma$ of $M$. Let $\tilde{\Omega}$ be the curvature form of the corresponding affine connection. Then*

$$\tilde{\gamma}^*\tilde{\Omega} = T_e\gamma \cdot \Omega + T_e\alpha \cdot \Theta \qquad (4.12)$$

*where $\tilde{\gamma} : L(M) \to A(M)$ is the natural injection.*

Finally, we shall see how we can recover the two structure equations for $L(M)$ that we derived from first principles in the previous section. Consider the structure equation of an affine connection $\tilde{\Gamma}$ of $M$:

$$d\tilde{\omega} = -\frac{1}{2}[\tilde{\omega}, \tilde{\omega}] + \tilde{\Omega}$$

Restrict both sides of this equation to $\tilde{\gamma}(L(M))$, we have

$$d\tilde{\omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}Y) = -\frac{1}{2}[\tilde{\omega}(T_u\tilde{\gamma}X), \tilde{\omega}(T_u\tilde{\gamma}Y)] + \tilde{\Omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}Y) \tag{4.13}$$

Now, we use more explicit notation to represent the decomposition of $\mathfrak{a}(n;\mathbb{R})$, Let us write $\tilde{\omega}(T_u\tilde{\gamma}X) = \tilde{\gamma}^*\tilde{\omega}(X) = T_e\gamma \cdot \omega(X) \oplus T_e\alpha \cdot \theta(X)$ and $\tilde{\omega}(T_u\tilde{\gamma}Y) = T_e\gamma \cdot \omega(Y) \oplus T_e\alpha \cdot \theta(Y)$. Similarly, write

$$\tilde{\Omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}Y) = \tilde{\gamma}^*\tilde{\Omega}(X, Y) = T_e\gamma \cdot \Omega(X, Y) \oplus T_e\alpha \cdot \Theta(X, Y).$$

Using the definition of the Lie bracket on $T_e\gamma(\mathfrak{gl}(n;\mathbb{R})) \oplus T_e\alpha(\mathbb{R}^n)$ given in (4.3), we compute the Lie bracket on the RHS of (4.13)

$$\begin{aligned}[T_e\gamma \cdot \omega(X) \oplus T_e\alpha \cdot \theta(X), T_e\gamma \cdot \omega(Y) \oplus T_e\alpha \cdot \theta(Y)]& \\ = T_e\gamma \cdot [\omega(X), \omega(Y)] \oplus T_e\alpha \cdot (\omega(X)\theta(Y) - \omega(Y)\theta(X))&\end{aligned} \tag{4.14}$$

Thus the first component of the RHS of (4.13) is given by

$$-\frac{1}{2}T_e\gamma \cdot [\omega(X), \omega(Y)] + T_e\gamma \cdot \Omega(X, Y) \tag{4.15}$$

In a similar manner, we compute the second component of the RHS of (4.13) as

$$= -\frac{1}{2}T_e\alpha \cdot (\omega(X)\theta(Y) - \omega(Y)\theta(X)) + T_e\alpha \cdot \Theta(X, Y) \tag{4.16}$$

Also, from our previous observations, we have

$$d\tilde{\omega}(T_u\tilde{\gamma}X, T_u\tilde{\gamma}Y) = T_e\gamma \cdot d\omega(X, Y) \oplus T_e\alpha \cdot d\theta(X, Y) \tag{4.17}$$

Thus, comparing the first and second components of the LHS and RHS respectively, we get

$$\begin{aligned}d\omega(X, Y) &= -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y) \\ d\theta(X, Y) &= -\frac{1}{2}\left(\omega(X)\theta(Y) - \omega(Y)\theta(X)\right) + \Theta(X, Y)\end{aligned}$$

Thus we recover the first and the second structure equations for $L(M)$.

## References

[1] Thomas W. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980. Reprint of the 1974 original.

[2] Shochishi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume 1 of *Tracts in pure and applied mathematics*. Interscience Publishers, New York, London, 1964.