SIAM J. APPL. MATH.
Vol. 65, No. 1, pp. 113–130

APPLICATION OF SYMMETRY ANALYSIS TO A PDE ARISING IN THE CAR WINDSHIELD DESIGN*

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Abstract. A new approach to parameter identification problems from the point of view of symmetry analysis theory is given. A mathematical model that arises in the design of car windshield represented by a linear second order mixed type PDE is considered. Following a particular case of the direct method (due to Clarkson and Kruskal), we introduce a method to study the group invariance between the parameter and the data. The equivalence transformations associated with this inverse problem are also found. As a consequence, the symmetry reductions relate the inverse and the direct problem and lead us to a reduced order model.

Key words. symmetry reductions, parameter identification problems

AMS subject classifications. 58J70, 70G65, 35R30, 35R35

DOI. 10.1137/S0036139903434031

1. Introduction.

Symmetry analysis theory links differential geometry to PDEs theory [18], symbolic computation [9], and, more recently, to numerical analysis theory [3], [6]. The notion of continuous transformation groups was introduced by Sophus Lie [14], who also applied them to differential equations. Over the years, Lie's method has been proven to be a powerful tool for studying a remarkable number of PDEs arising in mathematical physics (more details can be found for example in [2], [10], and [21]). In the last several years a variety of methods have been developed in order to find special classes of solutions of PDEs, which cannot be determined by applying the classical Lie method. Olver and Rosenau [20] showed that the common theme of all these methods has been the appearance of some form of group invariance. On the other hand, parameter identification problems arising in the inverse problems theory are concerned with the identification of physical parameters from observations of the evolution of a system. In general, these are ill-posed problems, in the sense that they do not fulfill Hadamard's postulates for all admissible data: a solution exists, the solution is unique, and the solution depends continuously on the given data. Arbitrary small changes in data may lead to arbitrary large changes in the solution. The iterative approach of studying parameter identification problems is a functional-analytic setup with a special emphasis on iterative regularization methods [8].

The aim of this paper is to show how parameter identification problems can be analyzed with the tools of group analysis theory. This is a new direction of research in the theory of inverse problems, although the symmetry analysis theory is a common approach for studying PDEs. We restrict ourselves to the case of a parameter identification problem modeled by a PDE of the form

$$F(x, w^{(m)}, E^{(n)}) = 0, \tag{1.1}$$

where the unknown function $E = E(x)$ is called *parameter*, and, respectively, the arbitrary function $w = w(x)$ is called *data*, with $x = (x_1, \ldots, x_p) \in \Omega \subset R^p$ a given

*Received by the editors September 4, 2003; accepted for publication (in revised form) May 4, 2004; published electronically September 24, 2004. This work was supported by the Austrian Science Foundation FWF, Project SFB 1308 "Large Scale Inverse Problems."
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domain (here $w^{(m)}$ denotes the function w together with its partial derivatives up to order m). Assume that the parameters and the data are analytical functions. The PDE (1.1) sometimes augmented with certain boundary conditions is called *the inverse problem* associated with a *direct problem.* The direct problem is the same equation but the unknown function is the data, for which certain boundary conditions are required.

The *classical Lie method* allows us to find the *symmetry group* related to a PDE. This is a (local) Lie group of transformations acting on the space of the independent variables and the space of the dependent variables of the equation with the property that it leaves the set of all analytical solutions invariant. Knowledge of these *classical symmetries* allows us to reduce the order of the studied PDE and to determine *group-invariant solutions* (or *similarity solutions*) which are invariant under certain subgroups of the full symmetry group (for more details see [18]). Bluman and Cole [1] introduced the *nonclassical method* that allows one to find the *conditional symmetries* (also called *nonclassical symmetries*) associated with a PDE. These are transformations that leave only a subset of the set of all analytical solutions invariant. Note that any classical symmetry is a nonclassical symmetry but not conversely. Another procedure for finding symmetry reductions is the *direct method* (due to Clarkson and Kruskal [5]). The relation between these last two methods has been studied by Olver [19]. Moreover, for a PDE with coefficients depending on an arbitrary function, Ovsiannikov [21] introduced the notion of *equivalence transformations*, which are (local) Lie group of transformations acting on the space of the independent variables, the space of the dependent variables and the space of the arbitrary functions that leave the equation unchanged. Notice that these techniques based on group theory do not take into account the boundary conditions attached to a PDE.

To find symmetry reductions associated with the parameter identification problem (1.1) one can seek classical and nonclassical symmetries related to this equation. Two cases can occur when applying the classical Lie method or the nonclassical method, depending if the data w is known or not. From the symbolic computation point of view, the task of finding symmetry reductions for a PDE depending on an arbitrary function might be a difficult one, due to the lack of the symbolic manipulation programs that can handle these kind of equations. Another method to determine symmetry reductions for (1.1) might be a particular case of the direct method, which has been applied by Zhdanov [24] to certain multidimensional PDEs arising in mathematical physics. Based on this method and taking into account that (1.1) depends on an arbitrary function, we introduce a procedure to find the relation between the data and the parameter in terms of a similarity variable (see section 2). As a consequence, the equivalence transformations related to (1.1) must be considered as well. These final symmetry reductions are found by using any symbolic manipulation program designed to determine classical symmetries for a PDE system—now both the data and the parameter are unknown functions in (1.1). The equivalence transformations relate the direct problem and the inverse problem. Moreover, one can find special classes of data and parameters, respectively, written in terms of the invariants of the group action, the order of the studied PDE can be reduced at least by one, and analytical solutions of (1.1) can be found.

At the first step, the group approach of the free boundary problem related to (1.1) can be considered and, afterwards, the invariance of the boundary conditions under particular group actions has to be analyzed (see [2]). In the case of parameter identification problems we sometimes have to deal with two pairs of boundary conditions, for data and the parameter as well, otherwise we might only know the boundary conditions for the data. Thus, the problem of finding symmetry reductions for a given data can be more complicated. At least by finding the equivalence transformations related to the problem, the invariants of the group actions can be used to establish suitable domains Ω on which the order of the model can be reduced.

In this paper we consider a mathematical model arising in the car windshield design. Let us briefly explain the *gravity sag bending process*, one of the main industrial processes used in the manufacture of car windshields. A piece of glass is placed over a rigid frame, with the desired edge curvature and heated from below. The glass becomes viscous due to the temperature rise and sags under its own weight. The final shape depends on the viscosity distribution of the glass obtained from varying the temperature. It has been shown that the sag bending process can also be controlled (in a first approximation) in the terms of Young's modulus E, a spatially varying glass material parameter, and the displacement of the glass w can be described by the thin linear elastic plate theory (see [11], [16], and [17] and references from there). The model is based on the linear plate equation

$$(E(w_{xx}+\nu w_{yy}))_{xx}+2(1-\nu)(Ew_{xy})_{xy}+(E(w_{yy}+\nu w_{xx}))_{yy}=\tfrac{12(1-\nu^2)f}{h^3} \quad \text{on} \quad \Omega, \tag{1.2}$$

where $w=w(x,y)$ represents the displacement of the glass sheet (the target shape) occupying a domain $\Omega\subset R^2$, $E=E(x,y)$ is Young's modulus, a positive function that can be influenced by adjusting the temperature in the process of heating the glass, f is the gravitational force, $\nu\in\left(0,\frac{1}{2}\right]$ is the glass Poisson ratio, and h is thickness of the plate. The *direct problem* (or the *forward problem*) is the following: for a given Young modulus E, find the displacement w of a glass sheet occupying a domain Ω before the heating process. Note that the PDE (1.2) is an elliptic fourth order linear PDE for the function w. Until now, two problems related to (1.2) have been studied: the *clamped plate case* and the *simply supported plate case* (more details can be found for example in [15]). In this paper we consider the clamped case, in which the following boundary conditions are required: the plate is placed over a rigid frame, i.e.,

$$w(x,y)|_{\partial\Omega}=0, \tag{1.3}$$

and, respectively,

$$\frac{\partial w}{\partial n}|_{\partial\Omega}=0, \tag{1.4}$$

which means the (outward) normal derivative of w must be zero, i.e., the sheet of glass is not allowed to freely rotate around the tangent to $\partial\Omega$. The associated *inverse problem* consists of finding Young's modulus E for a given data w in (1.2). This is a linear second order PDE for Young's modulus that can be written as

$$\begin{aligned}(w_{xx}+\nu w_{yy})E_{xx}+2(1-\nu)w_{xy}E_{xy}+(w_{yy}+\nu w_{xx})E_{yy}\\ +2(\Delta w)_xE_x+2(\Delta w)_yE_y+(\Delta^2 w)E=1\end{aligned} \tag{1.5}$$

after the scaling transformations $w\rightarrow\frac{1}{k}w$ or $E\rightarrow\frac{1}{k}E$, with $k=\frac{12(1-\nu^2)f}{h^3}$. In (1.5), Δ denotes the Laplace operator. The main problem in the car windshield design is that the prescribed target shape w is frequent such that the discriminant

$$D=(1-\nu)^2w_{xy}^2-(w_{xx}+\nu w_{yy})(w_{yy}+\nu w_{xx})$$

of (1.5) changes sign in the domain Ω, so that we get a mixed type PDE. This is one of the reasons for which optical defects might occur during the process. Note that (1.5) would naturally call for boundaries conditions for E on $\partial\Omega$ in the purely elliptic case (when $D < 0$), and Cauchy data on a suitable (noncharacteristic part) $\Gamma \subset \partial\Omega$ in the purely hyperbolic part (for $D > 0$). There is a recent interest in studying this inverse problem (see, e.g., [13]). It is known [15] that a constant Young's modulus corresponds to a data which satisfies the nonhomogeneous biharmonic equation (2.29). A survey on this subject can be found in [23]. Salazar and Westbrook [22] studied the case when the data and the parameter are given by radial functions; Kügler [12] used a derivative free iterative regularization method for analyzing the problem on rectangular frames; and a simplified model for the inverse problem on circular domains was considered by Engl and Kügler [7].

So far it is not obvious which shapes can be made by using this technique. Hence, we try to answer this question by finding out the symmetry reductions related to the PDE (1.5) hidden by the nonlinearity that occurs between the data and the parameter. In this sense, we determine (see section 3) the group of transformations that leave the equation unchanged, and so, its mixed type form. Knowledge of the invariants of these group actions allows us to write the target shape and the parameter in terms of them, and, therefore, to reduce the order of the studied equation. We find again the obvious result that a Young's modulus constant corresponds to data which is a solution of a nonhomogeneous biharmonic equation. The circular case problem considered by Salazar and Westbrook is, in fact, a particular case of our study. We show that other target shapes which are not radial functions can be considered. We prove that (1.5) is invariant under scaling transformations. It follows that target shapes modeled by homogeneous functions can be analyzed as well. In particular, we are interested in target shapes modeled by homogeneous polynomials defined on elliptical domains or square domains with rounded corners.

The paper is structured as follows. To reduce the order of the PDE (1.5) we propose in section 2 a method for studying the relation between the data and the parameter in terms of the similarity variables. The equivalence transformations related to this equation are given in section 3. The symbolic manipulation program DESOLV, authors Carminati and Vu [4] has been used for this purpose. Table 1 contains a complete classification of these symmetry reductions. In the last section, we discuss the PDE (1.5) augmented with the boundary conditions (1.3) and (1.4), namely, how to use the invariants of the group actions (on suitable bounded domains Ω) in order to incorporate the boundary conditions. In this sense, certain examples of exact and of numerical solutions of the reduced ODEs are given.

2. Conditional symmetries. The direct method approach to a second order PDE

$$\mathcal{F}(x, y, E^{(2)}) = 0$$

consists of seeking solutions written in the form

$$E(x,y) = \Phi(x,y,F(z)), \quad \text{where} \quad z = z(x,y), \quad (x,y) \in \Omega. \tag{2.1}$$

In this case the function z is called *similarity variable* and its level sets $\{z = k\}$ are named *similarity curves*. After substituting (2.1) into the studied second order PDE, we require that the result to be an ODE for the arbitrary function $F = F(z)$. Hence, certain conditions are imposed upon the functions Φ, z and their partial derivatives.

The particular case

$$E(x,y) = F(z(x,y)) \tag{2.2}$$

consists of looking for solutions depending only on the similarity variable z. If z is an invariant of the group action then the solutions of the form (2.2) are as well. Assume that the similarity variable is such that $\|\nabla z\| \neq 0$ on $\bar{\Omega}$.

In this section we apply this particular approach to (1.5) in order to study if the parameter and the data are functionally independent, which means whether or not they can depend on the same similarity variable. Assume that Young's modulus takes the form (2.2). In this case we get the relation

$$\begin{aligned} F''(z)\left[z_x^2(w_{xx}+\nu w_{yy}) + 2z_xz_y(1-\nu)w_{xy} + z_y^2(w_{yy}+\nu w_{xx})\right] \\ + F'(z)\left[z_{xx}(w_{xx}+\nu w_{yy}) + 2(1-\nu)z_{xy}w_{xy} + +z_{yy}(w_{yy}+\nu w_{xx})\right. \\ \left. +2z_x(\Delta w)_x + 2z_y(\Delta w)_y\right] + F(z)(\Delta^2 w) = 1, \end{aligned} \tag{2.3}$$

which must be an ODE for the unknown function $F = F(z)$. This condition is satisfied if the coefficients of the partial derivatives of F are function of z only (note that these coefficients are also invariant under the same group action). Denote them by

$$\begin{aligned} \Gamma_1(z) &= z_x^2(w_{xx}+\nu w_{yy}) + 2z_xz_y(1-\nu)w_{xy} + z_y^2(w_{yy}+\nu w_{xx}), \\ \Gamma_2(z) &= z_{xx}(w_{xx}+\nu w_{yy}) + 2(1-\nu)z_{xy}w_{xy} + z_{yy}(w_{yy}+\nu w_{xx}) \\ &\quad + 2z_x(\Delta w)_x + 2z_y(\Delta w)_y, \\ \Gamma_3(z) &= \Delta^2 w. \end{aligned} \tag{2.4}$$

If these relations hold, then the PDE (1.5) is reduced to the second order linear ODE

$$\Gamma_1(z)F''(z) + \Gamma_2(z)F'(z) + \Gamma_3(z)F(z) = 1. \tag{2.5}$$

2.1. Data and parameter invariant under the same group. If the target shape is invariant under the same group action as Young's modulus, then

$$w(x,y) = G(z(x,y)), \tag{2.6}$$

where $G = G(z)$. Substituting (2.6) into the relations (2.4) we get

$$\begin{aligned} \Gamma_1 &= G''(z_x^2+z_y^2)^2 + G'\left[(z_x^2+\nu z_y^2)z_{xx} + 2(1-\nu)z_xz_yz_{xy} + (z_y^2+\nu z_x^2)z_{yy}\right], \\ \Gamma_2 &= 2G'''(z_x^2+z_y^2)^2 + G''\left\{[7z_x^2+(\nu+2)z_y^2]z_{xx} + 2(5-\nu)z_xz_yz_{xy}\right. \\ &\quad \left. + [7z_y^2+(\nu+2)z_x^2]z_{yy}\right\} + G'\left\{(\Delta z)^2 + 2(1-\nu)(z_{xy}^2 - z_{xx}z_{yy})\right. \\ &\quad \left. + 2\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right]\right\}, \\ \Gamma_3 &= G''''(z_x^2+z_y^2)^2 + 2G'''\left[(3z_x^2+z_y^2)z_{xx} + 4z_xz_yz_{xy} + (z_x^2+3z_y^2)z_{yy}\right. \\ &\quad + G''\left\{3(\Delta z)^2 + 4(z_{xy}^2 - z_{xx}z_{yy}) + 4\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right]\right\} + G'\Delta^2 z. \end{aligned} \tag{2.7}$$

Next, the coefficients of the partial derivatives of the function G, denoted by Γ_i, must depend only on z, i.e.,

$$\begin{aligned} \Gamma_1 &= \alpha^4 G'' + a_1 G', \\ \Gamma_2 &= 2\alpha^4 G''' + a_2 G'' + a_3 G', \\ \Gamma_3 &= \alpha^4 G'''' + 2a_4 G''' + a_5 G'' + a_6 G', \end{aligned}$$

where

$$
\begin{aligned}
&\alpha^2(z) = z_x^2 + z_y^2,\\
&a_1(z) = (z_x^2 + \nu z_y^2) z_{xx} + 2(1-\nu) z_x z_y z_{xy} + (z_y^2 + \nu z_x^2) z_{yy},\\
&a_2(z) = \left[7z_x^2 + (\nu+2) z_y^2\right] z_{xx} + 2(5-\nu) z_x z_y z_{xy} + \left[7z_y^2 + (\nu+2) z_x^2\right] z_{yy},\\
&a_3(z) = (\Delta z)^2 + 2(1-\nu)(z_{xy}^2 - z_{xx}z_{yy}) + 2\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right],\\
&a_4(z) = (3z_x^2 + z_y^2) z_{xx} + 4 z_x z_y z_{xy} + (z_x^2 + 3z_y^2) z_{yy},\\
&a_5(z) = 3(\Delta z)^2 + 4(z_{xy}^2 - z_{xx}z_{yy}) + 4\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right],\\
&a_6(z) = \Delta^2 z.
\end{aligned} \tag{2.8}
$$

The first relation in (2.8) is a two-dimensional (2D) eikonal equation. From this we get

$$
\begin{aligned}
&z_x^2 z_{xx} + 2 z_x z_y z_{xy} + z_y^2 z_{yy} = \alpha^3(z)\alpha'(z),\\
&z_{xx} = \alpha(z)\alpha'(z) - \tfrac{z_y}{z_x} z_{xy},\\
&z_{yy} = \alpha(z)\alpha'(z) - \tfrac{z_x}{z_y} z_{xy}.
\end{aligned}
$$

The last two equations imply

$$
z_y^2 z_{xx} - 2 z_x z_y z_{xy} + z_x^2 z_{yy} = \alpha^3(z)\alpha'(z) - \alpha^4(z)\frac{z_{xy}}{z_x z_y}. \tag{2.9}
$$

Assume that there is a function $\beta = \beta(z)$ such that

$$
z_{xy} = \beta(z) z_x z_y. \tag{2.10}
$$

Indeed, since the left-hand side in (2.9) depends only on z, one can easily check if z satisfies both the 2D eikonal equation in (2.8) and (2.10), then all the functions $a_i = a_i(z)$ defined by (2.8) are written in terms of α and β. Therefore, the problem of finding the similarity variable z is reduced to that of integrating the 2D eikonal equation and the PDE system

$$
\begin{cases}
z_{xx} = \alpha\alpha' - \beta z_y^2,\\
z_{xy} = \beta z_x z_y,\\
z_{yy} = \alpha\alpha' - \beta z_x^2.
\end{cases} \tag{2.11}
$$

The system (2.11) is compatible if the following relation holds:

$$
\alpha\alpha'' + \alpha'^2 - 3\beta\alpha\alpha' + \alpha^2\left(\beta^2 - \beta'\right) = 0.
$$

Denote $\mu = \frac{1}{2}\alpha^2$. In this case, the above compatibility condition can be written as

$$
\mu'' - 3\beta\mu' + 2\mu\left(\beta^2 - \beta'\right) = 0. \tag{2.12}
$$

On the other hand, if the function β is given by

$$
\beta(z) = -\frac{\lambda''(z)}{\lambda'(z)}, \tag{2.13}
$$

where λ is a nonconstant function, then (2.10) turns into

$$(\lambda(z))_{xy} = 0.$$

The general solution of this equation is given by

$$\lambda(z(x,y)) = a(x) + b(y), \tag{2.14}$$

with a and b being arbitrary functions. Substituting β from (2.13) into the compatibility condition (2.12) and after integrating once, we get

$$\mu'\lambda' + 2\mu\lambda'' = k, \tag{2.15}$$

where k is an arbitrary constant.

Case 1. If $k \neq 0$, then after integrating (2.15) and substituting back $\mu = \frac{1}{2}\alpha^2$, we get

$$\alpha^2(z) = \frac{2k\lambda(z) + C_1}{\lambda'^2(z)}. \tag{2.16}$$

The relation (2.14) implies $\lambda'(z)z_x = a'(x)$, and $\lambda'(z)z_y = b'(y)$. We substitute these relations, (2.14) and (2.16), into the 2D eikonal equation (see (2.8)). It follows that the functions $a = a(x)$ and $b = b(y)$ are solutions of the following respective ODEs:

$$a'^2(x) - 2ka(x) = C_2 \quad \text{and} \quad b'^2(y) - 2kb(y) = C_3,$$

with $C_2 + C_3 = C_1$ (here C_i are real constants). The above ODEs admit the nonconstant solutions

$$a(x) = \frac{1}{2k}\left[k^2(x - C_4)^2 - C_2\right] \quad \text{and} \quad b(y) = \frac{1}{2k}\left[k^2(y - C_5)^2 - C_3\right],$$

and so (2.14) takes the form

$$\lambda(z(x,y)) = \frac{k}{2}\left[(x - C_4)^2 + (y - C_5)^2\right] - \frac{C_1}{2k}. \tag{2.17}$$

Notice that $\frac{1}{k_1}\lambda$ or $\lambda + k_2$ defines the same function β as the function λ does. Moreover, since the PDE (1.5) is invariant under translations in the (x, y)-space, we can consider

$$\lambda(z(x,y)) = x^2 + y^2. \tag{2.18}$$

If $\sqrt{\lambda}$ is a bijective function on a suitable interval, and if we denote by $\Phi = (\sqrt{\lambda})^{-1}$ its inverse function, then the similarity variable written in the polar coordinates (r, θ) (where $x = r\cos(\theta)$, $y = r\sin(\theta)$) is given by

$$z(x,y) = \Phi(r). \tag{2.19}$$

For simplicity, we consider $\Phi = \mathrm{Id}$, and from that we get

$$E = F(r) \quad \text{and} \quad w = G(r), \quad \text{where} \quad z(x,y) = r. \tag{2.20}$$

Hence, the ODE (2.5) turns into

$$\left(G'' + \frac{\nu}{r}G'\right)F'' + \left(2G''' + \frac{\nu+2}{r}G'' - \frac{1}{r^2}G'\right)F' + \left(G'''' + \frac{2}{r}G''' - \frac{1}{r^2}G'' + \frac{1}{r^3}G'\right)F = 1, \tag{2.21}$$

which can be reduced to the first order ODE

$$\left(G'' + \frac{\nu}{r}G'\right)F' + \left(G''' + \frac{1}{r}G'' - \frac{1}{r^2}G'\right)F = \frac{r^2 - r_0^2}{2r} + \frac{\gamma}{r}, \tag{2.22}$$

where $r_0 \in [0,1]$ with the property that

$$\gamma = \left[(rG'' + \nu G')\, F' + \left(rG''' + G'' - \frac{1}{r}G'\right) F\right] |_{r=r_0}$$

is finite. The smoothness condition $G'(0) = 0$ implies that (2.22) can be written as [15]

$$\left(G'' + \frac{\nu}{r}G'\right)F' + \left(G''' + \frac{1}{r}G'' - \frac{1}{r^2}G'\right)F = \frac{r}{2}. \tag{2.23}$$

Case 2. If $k = 0$, similarly we get

$$z(x,y) = \Phi(k_1 x + k_2 y), \tag{2.24}$$

where k_1 and k_2 are real constants such that $k_1^2 + k_2^2 > 0$. In this case, for $\Phi = \mathrm{Id}$, the parameter and the data are written as

$$E = F(z) \quad \text{and} \quad w = G(z), \quad \text{where} \quad z(x,y) = k_1 x + k_2 y, \tag{2.25}$$

and the ODE (2.5) turns into

$$G''(z)F''(z) + 2G'''(z)F'(z) + G''''(z)F(z) = \frac{1}{(k_1^2 + k_2^2)^2}, \tag{2.26}$$

with $\{z | G''(z) = 0\}$ the associated set of singularities. Integrating the above ODE on the set $\{z | G''(z) \neq 0\}$ we obtain that Young's modulus is given by

$$E(x,y) = \frac{(k_1 x + k_2 y)^2 + C_1(k_1 x + k_2 y) + C_2}{2(k_1^2 + k_2^2)^2 G''(k_1 x + k_2 y)},$$

where C_i are arbitrary constants.

2.2. Data and parameter invariant under different groups. Consider two functionally independent functions on Ω, say, $z = z(x,y)$ and $v = v(x,y)$, and let

$$w = H(v(x,y)) \tag{2.27}$$

be the target shape. In this case, the data and the parameter do not share the same invariance. Similar to the above, substituting (2.27) into the relations (2.4) we get

(2.28)

$$\begin{aligned}
\Gamma_1 &= H'' \left[(z_x v_x + z_y v_y)^2 + \nu(z_y v_x - z_x v_y)^2\right] \\
&\quad + H' \left[z_x^2 v_{xx} + 2z_x z_y v_{xy} + z_y^2 v_{yy} + \nu\left(z_x^2 v_{yy} - 2z_x z_y v_{xy} + z_y^2 v_{xx}\right)\right], \\
\Gamma_2 &= H'''(v_x^2 + v_y^2)(z_x v_x + z_y v_y) + H'' \left[v_x^2 z_{xx} + 2v_x v_y z_{xy} + v_y^2 z_{yy}\right. \\
&\quad + \nu\left(v_y^2 z_{xx} - 2v_x v_y z_{xy} + v_x^2 z_{yy}\right) + 2z_x v_x v_{xx} + 2(z_x v_y + z_y v_x)v_{xy} + 2z_y v_y v_{yy} \\
&\quad \left. + (z_x v_x + z_y v_y)(\Delta v)\right] + H' \left[z_{xx} v_{xx} + 2z_{xy} v_{xy} + z_{yy} v_{yy} + \nu\left(z_{xx} v_{yy} - 2z_{xy} v_{xy}\right.\right. \\
&\quad \left.\left. + z_{yy} v_{xx}\right) + z_x (\Delta v)_x + z_y (\Delta v)_y\right], \\
\Gamma_3 &= H''''(v_x^2 + v_y^2)^2 + 2H''' \left[(3v_x^2 + v_y^2)v_{xx} + 4v_x v_y v_{xy} + (v_x^2 + 3v_y^2)v_{yy}\right] \\
&\quad + H'' \left[3v_{xx}^2 + 4v_{xy}^2 + 3v_{yy}^2 + 2v_{xx} v_{yy} + 4v_x (\Delta v)_x + 4v_y (\Delta v)_y\right] + H' \Delta^2 v.
\end{aligned}$$

Recall that Γ_i's are functions of $z = z(x,y)$ only. Since each right-hand side in the above relations contains the function $H = H(v)$ and its derivatives, we require that the coefficients of the derivatives of H to be functions of v. It follows that Γ_i must be constant and denote them by γ_i. Therefore, the last condition in (2.28) becomes

$$\Delta^2(w) = \gamma_3, \tag{2.29}$$

which is the biharmonic equation. According to the above assumption, we seek solutions of (2.29) that are functions of v only. Similar to section 2.1, we get

$$v(x,y) = \Psi(r), \quad \text{or} \quad v(x,y) = \Psi(k_1x + k_2y), \tag{2.30}$$

and thus, for $\Psi = \mathrm{Id}$, the target shape is written as

$$w(x,y) = H(r), \quad \text{or} \quad w(x,y) = H(k_1x + k_2y). \tag{2.31}$$

Since $z = z(x,y)$ and $v = v(x,y)$ are functionally independent, we get

$$z(x,y) = k_1x + k_2y, \quad v(x,y) = \sqrt{x^2+y^2} \tag{2.32}$$

or

$$z(x,y) = \sqrt{x^2+y^2}, \quad v(x,y) = k_1x + k_2y. \tag{2.33}$$

One can prove that if the coefficients γ_i are constant, and if z and v are given by (2.32) or (2.33), respectively, then $\gamma_1 = \gamma_2 = 0$, and $\gamma_3 \neq 0$. On the other hand, the solutions of the biharmonic equation (2.29) of the form (2.31) are the following:

$$w(x,y) = \frac{\gamma_3}{64}z^4 + C_1z^2 + C_2\ln(z) + C_3z^2\ln(z) + C_4 \quad \text{for} \quad z = \sqrt{x^2+y^2},$$

and, respectively,

$$w(x,y) = \frac{\gamma_3}{24(k_1^2+k_2^2)^2}v^4 + C_1v^3 + C_2v^2 + C_3v + C_4 \quad \text{for} \quad v = k_1x + k_2y,$$

and these correspond to the constant Young's modulus

$$E(x,y) = \frac{1}{\gamma_3}. \tag{2.34}$$

Notice that only particular solutions of the biharmonic equation have been found in this case (i.e., solutions invariant under rotations and translations). Since this PDE is also invariant under scaling transformations, which act not only on the space of the independent variables but on the data space as well, it is obvious to extend our study and to seek other types of symmetry reductions.

3. Equivalence transformations. Consider a one-parameter Lie group of transformations acting on an open set $\mathcal{D} \subset \Omega \times \mathcal{W} \times \mathcal{E}$, where $\mathcal{W}$ is the space of the data functions, and $\mathcal{E}$ is the space of the parameter functions, given by

$$\begin{cases} x^* = x + \varepsilon\zeta(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ y^* = y + \varepsilon\eta(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ w^* = w + \varepsilon\phi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \\ E^* = E + \varepsilon\psi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \end{cases} \tag{3.1}$$

where ε is the group parameter. Let

$$V = \zeta(x,y,w,E)\partial_x + \eta(x,y,w,E)\partial_y + \phi(x,y,w,E)\partial_w + \psi(x,y,w,E)\partial_E \tag{3.2}$$

be its associated general infinitesimal generator. The group of transformations (3.1) is called an *equivalence transformation* associated to the PDE (1.5) if this leaves the equation invariant. This means that the form of the equation in the new coordinates remains unchanged and the set of the analytical solutions is invariant under this transformation. The equivalence transformations can be found by applying the classical Lie method to (1.5), with E and w both considered as unknown functions (for more details see [10] and [21]). Following this method we obtain

$$\begin{cases} \zeta(x,y,w,E) = k_1 + k_5 x - k_4 y, \\ \eta(x,y,w,E) = k_2 + k_4 x + k_5 y, \\ \phi(x,y,w,E) = k_3 + k_7 x + k_6 y + (4k_5 - k_8)w, \\ \psi(x,y,w,E) = k_8 E, \end{cases} \tag{3.3}$$

where k_i are real constants. The vector field (3.2) is written as $V = \sum_{i=1}^{8} k_i V_i$, where

$$\begin{gathered} V_1 = \partial_x, \quad V_2 = \partial_y, \quad V_3 = \partial_w, \quad V_4 = -y\partial_x + x\partial_y, \quad V_5 = x\partial_x + y\partial_y + 4w\partial_w, \\ V_6 = y\partial_w, \quad V_7 = x\partial_w, \quad V_8 = -w\partial_w + E\partial_E. \end{gathered} \tag{3.4}$$

PROPOSITION 3.1. *The equivalence transformations related to the PDE* (1.5) *are generated by the infinitesimal generators* (3.4)*. Thus, the equation is invariant under translations in the x-space, y-space, w-space, rotations in the space of the independent variables (x,y), scaling transformations in the (x,y,w)-space, Galilean transformations in the (y,w) and (x,w) spaces, and scaling transformations in the (w,E)-space, respectively.*

Notice that the conditional symmetries found in section 2 represent particular cases of the equivalence transformations. Since each one-parameter group of transformations generated by V_i is a symmetry group, if ($w = G(x,y)$, $E = F(x,y)$) is a pair of known solutions of (1.5), so are the following:

$$\begin{aligned} w^{(1)} &= G(x-\varepsilon_1, y), & E^{(1)} &= F(x-\varepsilon_1, y), \\ w^{(2)} &= G(x, y-\varepsilon_2), & E^{(2)} &= F(x, y-\varepsilon_2), \\ w^{(3)} &= G(x,y) + \varepsilon_3, & E^{(3)} &= F(x,y), \\ w^{(4)} &= G(\tilde{x}, \tilde{y}), & E^{(4)} &= F(\tilde{x}, \tilde{y}), \\ w^{(5)} &= e^{4\varepsilon_5} G(e^{-\varepsilon_5}x, e^{-\varepsilon_5}y), & E^{(5)} &= F(e^{-\varepsilon_5}x, e^{-\varepsilon_5}y), \\ w^{(6)} &= G(x,y) + \varepsilon_6 y, & E^{(6)} &= F(x,y), \\ w^{(7)} &= G(x,y) + \varepsilon_7 x, & E^{(7)} &= F(x,y), \\ w^{(8)} &= e^{-\varepsilon_8} G(x,y), & E^{(8)} &= e^{\varepsilon_8} F(x,y), \end{aligned} \tag{3.5}$$

where $\tilde{x} = x\cos(\varepsilon_4) + y\sin(\varepsilon_4)$, $\tilde{y} = -x\sin(\varepsilon_4) + y\cos(\varepsilon_4)$, and ε_i are real constants. Moreover, the general solution of (1.5) constructed from a known one is given by

$$w(x,y) = e^{4\varepsilon_5-\varepsilon_8}G(e^{-\varepsilon_5}(\tilde{x}-\tilde{k}_1), e^{-\varepsilon_5}(\tilde{y}-\tilde{k}_2)) + e^{4\varepsilon_5-\varepsilon_8}\varepsilon_6 y + e^{4\varepsilon_5-\varepsilon_8}\varepsilon_7 x + e^{4\varepsilon_5-\varepsilon_8}\varepsilon_3,$$
$$E(x,y) = e^{\varepsilon_8}F(e^{-\varepsilon_5}(\tilde{x}-\tilde{k}_1), e^{-\varepsilon_5}(\tilde{y}-\tilde{k}_2)),$$

where $\tilde{k}_1 = \varepsilon_1\cos(\varepsilon_4) + \varepsilon_2\sin(\varepsilon_4)$, and $\tilde{k}_2 = \varepsilon_1\sin(\varepsilon_4) - \varepsilon_2\cos(\varepsilon_4)$.

The equivalence transformations form a Lie group $\mathcal{G}$ with an eight-dimensional associated Lie algebra $\mathcal{A}$. Using the adjoint representation of $\mathcal{G}$, one can find the optimal system of one-dimensional subalgebras of $\mathcal{A}$ (more details can be found in [18, pp. 203–209]). This optimal system is spanned by the vector fields given in Table 1. Denote by z, I, and J the invariants related to the one-parameter group of transformations generated by each vector field V_i. Here F and G are arbitrary functions, (r,θ) are the polar coordinates, and a, b, c are nonzero constants. To reduce the order of the PDE (1.5) one can also integrate the first order PDE system

$$\begin{cases} \zeta(x,y,w,E)w_x + \eta(x,y,w,E)w_y = \phi(x,y,w,E), \\ \zeta(x,y,w,E)E_x + \eta(x,y,w,E)E_y = \psi(x,y,w,E), \end{cases} \tag{3.6}$$

which defines the characteristics of the vector field (3.2). In Table 1, the associated reduced ODEs are listed. The invariance of (1.5) under the one-parameter groups of transformations generated by V_1, V_2, $V_1 + cV_6$, and $V_2 + cV_7$, respectively, leads us to the same ODE,

$$F''(z)G''(z) + 2F'(z)G'''(z) + F(z)G''''(z) = 1, \tag{3.7}$$

with the general solution

$$F(z) = \frac{z^2 + C_1 z + C_2}{2G''(z)} \tag{3.8}$$

on the set $\{z|G''(z) \neq 0\}$. The invariance under the scaling transformation generated by the vector field V_5 yields the reduced ODE

$$\begin{aligned} &\left[G''\left(z^2+1\right)^2 - 6z(z^2+1)G' + 12(z^2+\nu)G\right]F'' \\ &+ 2\left[\left(z^2+1\right)^2 G''' - 5z(z^2+1)G'' + 3(4z^2+\nu+1)G' - 12zG\right]F' \\ &+ \left[\left(z^2+1\right)^2 G'''' - 4z(z^2+1)G''' + 4(3z^2+1)G'' - 24zG' + 24G\right]F = 1. \end{aligned} \tag{3.9}$$

The ODE

(3.10)
$$\begin{aligned} &\left[\left(z^2+1\right)^2 G'' + 2(c-3)z(z^2+1)G' + (c-3)(c-4)(z^2+\nu)G\right]F'' \\ &+ \Big\{2\left(z^2+1\right)^2 G''' + 2(2c-5)z(z^2+1)G'' + 2(c-3)[z^2(c-4) + \nu(c-1) - 1]G' \\ &-2(c-3)(c-4)zG\Big\}F' + \Big\{\left(z^2+1\right)^2 G'''' + 2(c-2)z(z^2+1)G''' + \big[(c-3)(c-4)z^2 \\ &-2(c-2) + \nu c(c-1)\big]G'' - 2(c-4)(c-3)zG' + 2(c-4)(c-3)G\Big\}F = 1 \end{aligned}$$

TABLE 1

	Infintesimal generator	Invariants	$w = w(x,y)$	$E = E(x,y)$	ODE
1.	V_1	$z = y$ $I = w$ $J = E$	$w = G(z)$	$E = F(z)$	(3.7)
2.	V_2	$z = x$ $I = w$ $J = E$	$w = G(z)$	$E = F(z)$	(3.7)
3.	V_4	$z = r$ $I = w$ $J = E$	$w = G(z)$	$E = F(z)$	(2.21)
4.	V_5	$z = \frac{y}{x}$ $I = x^{-4}w$ $J = E$	$w = x^4 G(z)$	$E = F(z)$	(3.9)
5.	$cV_3 + V_4$	$z = r$ $I = w - c\theta$ $J = E$	$w = c\theta + G(z)$	$E = F(z)$	(2.21)
6.	$V_5 + cV_8$	$z = \frac{y}{x}$ $I = x^{c-4}w$ $J = x^{-c}E$	$w = x^{4-c}G(z)$	$E = x^c F(z)$	(3.10)
7.	$V_4 + cV_8$	$z = r$ $I = e^{c\theta}w$ $J = e^{-c\theta}E$	$w = e^{-c\theta}G(z)$	$E = e^{c\theta}F(z)$	(3.11)
8.	$V_4 + cV_5$	$z = re^{-c\theta}$ $I = r^{-4}w$ $J = E$	$w = r^4 G(z)$	$E = F(z)$	(3.13)
9.	$V_4 + cX_5 + bV_8$	$z = re^{-c\theta}$ $I = r^{\frac{b}{c}-4}w$ $J = r^{-\frac{b}{c}}E$	$w = r^{4-\frac{b}{c}}G(z)$	$E = r^{\frac{b}{c}}F(z)$	(3.14)
10.	$V_1 + cV_6$	$z = y$ $I = w - cxy$ $J = E$	$w = cxy + G(z)$	$E = F(z)$	(3.7)
11.	$V_2 + cV_7$	$z = x$ $I = w - cxy$ $J = E$	$w = cxy + G(z)$	$E = F(z)$	(3.7)
12.	$V_1 + cV_8$	$z = y$ $I = e^{cx}w$ $J = e^{-cx}E$	$w = e^{-cx}G(z)$	$E = e^{cx}F(z)$	(3.15)
13.	$V_2 + cV_8$	$z = x$ $I = e^{cy}w$ $J = e^{-cy}E$	$w = e^{-cy}G(z)$	$E = e^{cy}F(z)$	(3.15)

is obtained in case 6 of Table 1. The reduced equation

(3.11)
$$\left[G'' + \frac{\nu}{r}G' + \frac{\nu c^2}{r^2}G\right]F'' + \left[2G''' + \frac{\nu+2}{r}G'' + \frac{2\nu c^2 - 1}{r^2}G' - \frac{c^2(1+2\nu)}{r^3}G\right]F'$$
$$+ \left[G'''' + \frac{2}{r}G''' + \frac{c^2\nu - 1}{r^2}G'' + \frac{1 - c^2(2\nu+1)}{r^3}G' + \frac{2c^2(\nu+1)}{r^4}G\right]F = 1$$

is related to case 7. This can be written as the first order ODE

(3.12)
$$\left(G''+\frac{\nu}{r}G'+\frac{\nu c^2}{r^2}G\right)F'+\left(G'''+\frac{1}{r}G''+\frac{c^2\nu-1}{r^2}G'-\frac{c^2(1+\nu)}{r^3}G\right)F=\frac{r^2-r_0^2}{2r}+\frac{\gamma^*}{r},$$

where $r_0\in[0,1]$ with the property that

$$\gamma^*=\left[F'\left(rG''+\nu G'+\frac{\nu}{r}G\right)+F\left(rG'''+G''+\frac{c^2\nu-1}{r}G'-\frac{c^2(1+\nu)}{r^2}G\right)\right]|_{r=r_0}$$

is finite. In cases 8 and 9, after the change of the variable $z=\exp(t)$, the reduced ODEs are the following:

(3.13)
$$\begin{aligned}&\left\{(c^2+1)^2G''+(c^2+1)(\nu+7)G'+4\left[\nu(3c^2+1)+c^2+3\right]G\right\}F''\\&+\left\{2(c^2+1)^2G'''+(c^2+1)(\nu+19)G''+2[16+(c^2+1)(3\nu+13)]G'+8(\nu+7)G\right\}F'\\&+\left\{(c^2+1)^2G''''+12(c^2+1)G'''+4(5c^2+13)G''+96G'+64G\right\}F=1,\end{aligned}$$

and, respectively,

(3.14)
$$\begin{aligned}&\Bigg\{(c^2+1)^2G''+\left(\frac{1}{c}+c\right)[c(\nu+7)-2b]G'+\left(\frac{4}{c}-\frac{b}{c^2}\right)[c^3(1+3\nu)-c^2\nu b\\&+c(\nu+3)-b]G\Bigg\}F''+\Bigg\{2(c^2+1)^2G'''+\left(\frac{1}{c}+c\right)[c(\nu+19)-4b]G''\\&+2\left[\frac{b^2}{c^2}+\nu b^2+c^2(3\nu+13)-4bc(\nu+1)-12\frac{b}{c}+3\nu+29\right]G'\\&+\left(\frac{4}{c}-\frac{b}{c^2}\right)[2c(\nu+7)+b(\nu-5)]G\Bigg\}F'+\Bigg\{(c^2+1)^2G''''\\&+2\left(c+\frac{1}{c}\right)(6c-b)G'''+\left[\frac{b^2}{c^2}+\frac{b}{c}(\nu-17)+20c^2-bc(\nu+7)+\nu b^2+52\right]G''\\&+\left(\frac{6}{c}-\frac{b}{c^2}\right)[16c+b(\nu-5)]G'+2\left(\frac{4}{c}-\frac{b}{c^2}\right)[8c+b(\nu-3)]G\Bigg\}F=1.\end{aligned}$$

In cases 12 and 13 we get the same equation,

$$(G''+\nu c^2G)F''+2(G'''+\nu c^2G')F'+(G''''+\nu c^2G'')F=1, \tag{3.15}$$

with the general solution given by

$$F(z)=\frac{z^2+C_1z+C_2}{G''(z)+\nu c^2G(z)} \tag{3.16}$$

on the set $\{z|G''(z)+\nu c^2G(z)\neq 0\}$, where C_1 and C_2 are arbitrary real constants.

4. Conclusions. The data w is the function that models the target shape of a car windshield. Hence, we seek data with relevant physical and geometrical properties, such as smoothness and a positive curvature graph at least in the center of the bounded domain Ω, for which the boundary condition (1.3) is satisfied—which means that the sheet of glass is placed over a rigid frame. Moreover, if there is no free rotation of the plate around the tangent to $\partial\Omega$, then the condition (1.4) is required. Applying symmetry reductions theory to the PDE (1.5), we have shown that the data w and Young's modulus E can be expressed in terms of the invariants z, I, and J associated with a certain group action, i.e., $I = G(z)$ and $J = F(z)$, where w occurs in I, and E in J, respectively. Since now the technique of reducing the PDE (1.5) to an ODE has been applied only in the case of the radial functions ([15], [22], and [23]). Other symmetry reductions related to the studied model can be derived and these are listed in Table 1. The data given by homogeneous polynomials can be related to the invariance of the equation with respect to the scaling transformations (see cases 4 and 6, Table 1). The first two and last four cases in Table 1 allow us to construct other kind of data (see (3.5)). Since Ω must be bounded, the most interesting cases correspond to the rotational and the scaling symmetries. The problem of finding exact solutions of the reduced equations, which are second order linear nonhomogeneous ODEs, might be a difficult task depending on the form of the data. These equations are also, in general, ill-posed, as the initial problem is, and hence, regularization methods might be required in order to be studied, which is our current research.

One can make the following remarks: assume that $\partial\Omega = \{(x,y)|\ z(x,y) = k\}$ is the k-level set of the function z (here k being a nonzero constant) and $||\nabla z|| > 0$ on $\bar{\Omega}$. If the target shape is given by $w(x,y) = a(x,y)G(z(x,y))$, where $a = a(x,y)$ is a suitable function according to Table 1, then the boundary conditions (1.3) and (1.4) are equivalent to $G(k) = 0$ and $G'(k) = 0$. Therefore, the data might have the form $w(x,y) = a(x,y)\,(z(x,y) - k)^2\, H(z(x,y))$. This corresponds to the case when the data and the bounded domain Ω are invariant under the same symmetry reduction. In our case, this can be applied to rotational symmetries. For scaling invariance, we have to incorporate the noninvariant boundary conditions in invariant solutions. As a consequence, we can extend the study of the problem on elliptical domains and on square domains with rounded corners. For instance, the class of target shapes of the form $w(x,y) = z^m(x,y) - k^m$, where $m \geq 1$ is a natural number, satisfies the boundary condition (1.3). In this case, the normal derivative of the data on the boundary is $\frac{\partial w}{\partial n}|_{\partial\Omega} = mk^{m-1}||\nabla z|||_{\partial\Omega}$. If this quantity is small then the condition (1.4) is almost satisfied (i.e., there is a small free rotation of the plate around the tangent to $\partial\Omega$). In the following examples, we assume that the glass Poisson ration $\nu = 0.5$.

Example 1. *Rotational invariant data and parameter.* Consider the target shape of the form [23]

$$w(x,y) = G(r) = -\frac{1}{6}\,(r-1)^2\,(2r+1)\,, \quad r = \sqrt{x^2+y^2},$$

defined on the unit disc (see Figure 4.1) which satisfies the boundary conditions (1.3) and (1.4). Since $G'(0) = 0$, the reduced ODE is (2.23) and this has a singularity at $r = \frac{3}{5}$. Since $E > 0$, we consider the constant of integration $C_1 = 1$, and so,

$$E(x,y) = F(r) = -\frac{1}{11}\left(r + \frac{1}{2}\right) + (5r-3)^{-\frac{6}{5}}\,.$$

The PDE (1.5) is elliptic for $r \in \left(\frac{3}{5}, \frac{3}{4}\right)$, hyperbolic for $r \in \left[0, \frac{3}{5}\right) \bigcup \left(\frac{3}{5}, 1\right]$, and parabolic if $r = \frac{3}{5}$ or $r = \frac{3}{4}$, respectively.

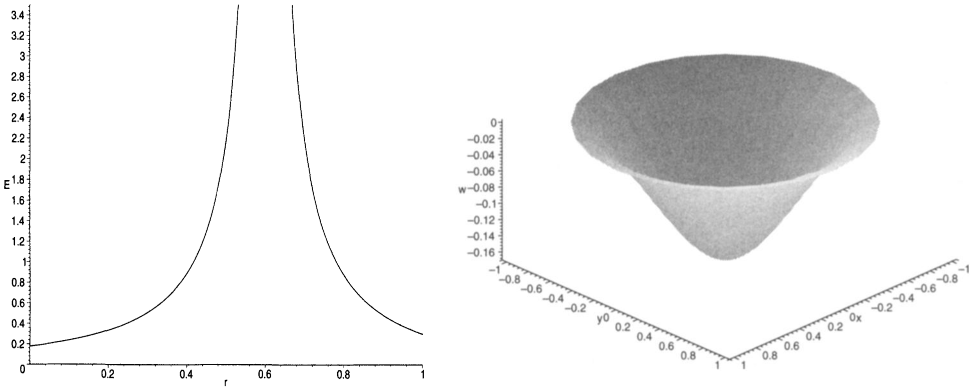

FIG. 4.1. *The parameter* $E(x,y) = F(r) = -\frac{1}{11}\left(r+\frac{1}{2}\right) + (5r-3)^{-\frac{6}{5}}$ *and the target shape* $w(x,y) = G(r) = -\frac{1}{6}(r-1)^2(2r+1)$, *with* $r = \sqrt{x^2+y^2}$, *defined on the unit disc.*

Example 2. *Particular target shapes on rounded square domains.*

(a) Suppose the Lamé oval $\partial\Omega = \{(x,y)|\ x^{2n}+y^{2n}=1\}$ is the boundary of the domain (here $n \geq 2$ is a natural number). For a target shape of the form

$$w(x,y) = \left(x^{2n}+y^{2n}\right)^m - 1, \tag{4.1}$$

$m \geq 1$ being a natural number, (1.5) is elliptic on $\Omega - \{(0,0)\}$ and parabolic in $(0,0)$. These target shapes are invariant with respect to $V_5 + cV_8 + (4-c)V_3$, where $c = 4 - 2mn$. For $x > 0$ or $x < 0$, the functions (4.1) can be written as

$$w(x,y) = x^{2mn}G(z) - 1, \quad G(z) = \left(1+z^{2n}\right)^m, \quad z = \frac{y}{x}.$$

According to case 6 in Table 1, the associated Young's modulus has the form

$$E(x,y) = x^{4-2mn}F(z), \quad z = \frac{y}{x}.$$

Since $w(x,y) = w(y,x) = w(-x,y) = w(x,-y) = w(-x,-y)$, Young's modulus also shares these discrete symmetries. Thus, the reduced ODE (3.10) can be integrated for $z \in [0,1]$. In particular, for $n = 2$ and $m = 1$, the data is a solution of the biharmonic equation and Young's modulus is $E = 48^{-1}$. For $n = 3$ and $m = 1$, the data and the numerical solution F satisfying $F(0) = 0.002$ and $F'(0) = 0$ are given in Figure 4.2.

(b) Assume that $\partial\Omega = \{(x,y)|\ x^{2n}+y^2=1\}$, where $n \geq 1$ is a natural number. Consider the class of target shapes

$$w(x,y) = x^{2n} + y^2 - 1,$$

invariant under the vector field $V_2 + 2V_6$. Hence, (1.5) is reduced to the ODE (3.7). For $n = 3$, the associated Young's modulus is given by

$$E(x,y) = F(x) = \frac{x^2 + C_1 x + C_2}{2(30x^4+1)},$$

and since $E > 0$, we can set $C_1 = 0$ and $C_2 = 2$ (see Figure 4.3). Equation (1.5) is elliptic on $\Omega - Oy$ and parabolic on the y-axis.

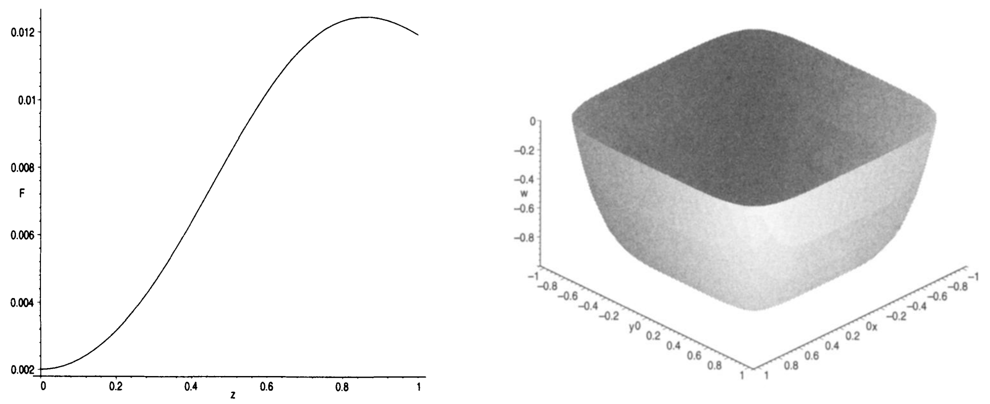

FIG. 4.2. *The parameter* $E(x,y) = x^{-2}F(z)$, *with* $z = \frac{y}{x}$, *and the data* $w(x,y) = x^6 + y^6 - 1$ *defined on the rounded square domain* $\{(x,y)|\ x^6 + y^6 < 1\}$.

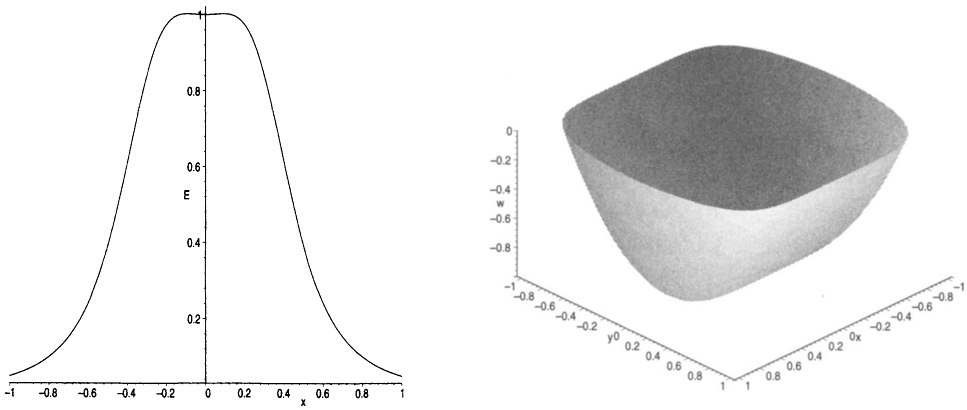

FIG. 4.3. *The parameter* $E(x,y) = \frac{x^2+2}{2(30x^4+1)}$ *and the data* $w(x,y) = x^6 + y^2 - 1$ *defined on the rounded square domain* $\{(x,y)|\ x^6 + y^2 < 1\}$.

Example 3. *Particular target shapes on elliptic domains.* Consider the data

$$w(x,y) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^m - 1, \tag{4.2}$$

on the elliptic domain $\Omega = \{(x,y)|\ \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$, where $m \geq 1$ is a natural number. These target shapes are obtained from the invariance of the studied PDE with respect to $V_5 + cV_8 + (4-c)V_3$, where $c = 4 - 2m$. The PDE (1.5) is elliptic on $\Omega - \{(0,0)\}$ and parabolic in the origin. For $x > 0$ or $x < 0$, the functions (4.2) can be written as

$$w(x,y) = x^{2m}G(z) - 1, \quad G(z) = \left(\frac{1}{a^2} + \frac{z^2}{b^2}\right)^m, \quad z = \frac{y}{x}.$$

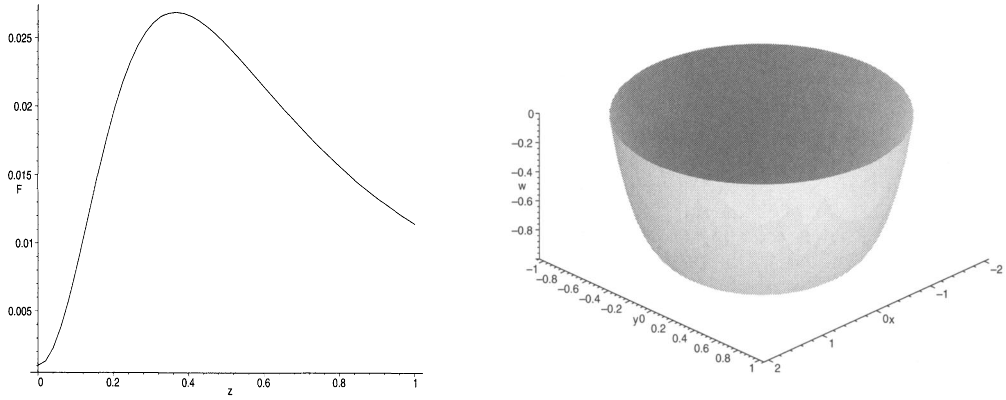

FIG. 4.4. *The parameter* $E(x,y) = x^{-2}F(z)$*, with* $z = \frac{y}{x}$*, and the data* $w(x,y) = \left(\frac{x^2}{4} + y^2\right)^3 - 1$ *defined on the elliptical domain* $\{(x,y)|\ \frac{x^2}{4} + y^2 = 1\}$.

In this case, we look for solutions to (1.5) of the form

$$E(x,y) = x^{4-2m}F(z), \quad z = \frac{y}{x}.$$

If $m = 2$, the data is a solution of the biharmonic equation, and the related Young's modulus is $E = 24\left(a^{-4} + b^{-4}\right) + 16a^{-2}b^{-2}$. If $m \geq 3$, the reduced ODE is (3.9). For $m = 3$, the data and the numerical solution F of (3.9) satisfying $F(0) = 0.001$ and $F'(0) = 0$ are plotted in Figure 4.4.

In brief, suppose that the target shape w on a domain Ω is given. In order to see if this is an invariant function with respect to the equivalence transformations related to the studied model, we should check if this is a solution of the first equation in (3.6), where the functions ζ, η, and ϕ are given by (3.3). Next, by integrating the second PDE in (3.6) we can determine the form of the parameter in terms of the similarity variables. The geometrical significance of the nonlinearity occurring between the data and the parameter in the inverse problem (1.1) is reflected by the group analysis tools. Investigating special groups of transformations connected to this equation, the order of the model can be reduced. The equation will be then written in terms of the invariants of the group actions. Another advantage of this approach is that of relating the direct and inverse problems through these symmetry reductions. It might be interesting for future study to link these results to the common approach of the inverse problems theory, especially in expressing the regularization methods in terms of the similarity variables. For other target shapes defined by functions which are not invariant under the listed symmetry reductions, the classical theory of the linear second order PDEs can be applied, but this might be quite difficult due to the form of the discriminant.

Acknowledgments. I would like to thank Prof. Heinz W. Engl and Dr. Philipp Kügler, Institute for Industrial Mathematics at Johannes Kepler University, for encouraging my research in applying the symmetry analysis to parameter identification problems, for interesting discussions, and for guiding me through this subject. I am grateful to Prof. Peter Olver, School of Mathematics, Institute of Technology at the University of Minnesota, for helpful comments which improved the presentation of this paper.

REFERENCES

[1] G. W. Bluman and J. D. Cole, *The general similarity solutions of the heat equation*, J. Math. Mech., 18 (1969), pp. 1025–1042.

[2] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Appl. Math. Sci. 81, Springer-Verlag, New York, 1989.

[3] C. J. Budd and M. D. Piggott, *Geometric integration and its applications*, in Handbook of Numerical Analysis, XI, North–Holland, Amsterdam, 2003, pp. 35–139.

[4] J. Carminati and K. Vu, *Symbolic computation and differential equations: Lie symmetries*, J. Symbolic Comput., 29 (2000), pp. 95–116.

[5] P. A. Clarkson and M. Kruskal, *New similarity reductions of the Boussinesq equation*, J. Math. Phys., 30 (1989), pp. 2201–2213.

[6] V. A. Dorodnitsyn, *Finite difference models entirely inheriting symmetry of original differential equations*, in Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, N. H. Ibragimov, M. Torrisi, and A. Valenti, eds., Kluwer, Dordrecht, The Netherlands, 1993, pp. 191–201.

[7] H. W. Engl and P. Kügler, *The influence of the equation type on iterative parameter identification problems which are elliptic or hyperbolic in the parameter*, European J. Appl. Math., 14 (2003), pp. 129–163.

[8] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, The Netherlands, 1996.

[9] W. Hereman, *Review of symbolic software for the computation of Lie symmetries of differential equations*, Euromath Bull., 1 (1994), pp. 45–82.

[10] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations, Vol.* 1: *Symmetries, Exact Solutions and Conservation Laws*, CRC Press, Boca Raton, 1994.

[11] D. Krause and H. Loch, *Mathematical Simulation in Glass Technology*, Springer-Verlag, Berlin, Heidelberg, New York, 2002.

[12] P. Kügler, *A Derivative Free Landweber Method for Parameter Identification in Elliptic Partial Differential Equations with Application to the Manufacture of Car Windshields*, Ph.D. thesis, Institute for Industrial Mathematics, Johannes Kepler University, Linz, Austria, 2003.

[13] P. Kügler, *A parameter identification problem of mixed type related to the manufacture of car windshields*, SIAM J. Appl. Math., 64 (2004), pp. 858–877.

[14] S. Lie, *Gesammelte Abhandlungen*, Band 4, B. G. Teubner, Leipzig, Germany, 1929, pp. 320–384.

[15] E. H. Mansfield, *The Bending and Stretching of Plates*, 2nd ed., Cambridge University Press, Cambridge, UK, 1989.

[16] P. S. Manservisi, *Control and optimization of the sag bending process in glass windshield design*, in Progress in Industrial Mathematics at ECMI98, L. Arkeryd, J. Bergh, P. Brenner, and R. Petterson, eds., Teubner, Stuttgart, Germany, 1999, pp. 97–105.

[17] O. S. Narayanaswamy, *Stress and structural relaxation in tempering glass*, J. Am. Ceramic Soc., 61 (1978), pp. 146–152.

[18] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Grad. Texts in Math. 107, Springer-Verlag, New York, 1986.

[19] P. J. Olver, *Direct reduction and differential constraints*, Proc. Roy. Soc. London Ser. A, 444 (1994), pp. 509–523.

[20] P. J. Olver and P. Rosenau, *The construction of special solutions to partial differential equations*, Phys. Lett. A, 114 (1986), pp. 107–112.

[21] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, W. F. Ames, trans., Academic Press, New York, 1982.

[22] D. Salazar and R. Westbrook, *Inverse problems of mixed type in linear plate theory*, European J. Appl. Math., to appear.

[23] D. Temple, *An Inverse System: An Analysis Arising from Windshield Manufacture*, M.Sc. thesis, Department of Mathematics, Oxford University, Oxford, UK, 2002.

[24] R. Z. Zhdanov, *On conditional symmetries of multidimensional nonlinear equations of quantum field theory*, in Symmetry in Nonlinear Mathematical Physics, Vol. 1, M. Shkil, A. Nikitin, and V. Boyko, eds., Natl. Acad. Sci. Ukraine, Inst. Math., Kiev, 1997, pp. 53–61.